Investigations into the structure of infinite matroids

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Introduction

This thesis started out as a research project on infinite matroids. The current research in infinite matroids is based on the definition of an infinite matroid presented in [**13**], of which a preprint was available before the actual publication in 2013. Since then, several topics in infinite matroids have been researched. This includes extending two basic notions from finite matroids to infinite ones, namely (finite) connectivity and representability.

Of particular interest to this thesis are matroids arising from graph-like spaces and Ψ-matroids. Finite graphs naturally induce matroids, the most obvious among these are the cycle matroid and the bond matroid, which are dual to each other. There is an interpretation of the graph as a topological space such that the circuits correspond to homeomorphic images of the unit circle. In this interpretation, every edge of the graph is associated with a subset of the topological space such that for every homeomorphic image of the unit circle there is a well-defined set of edges of the graph which are "contained" in the homeomorphic image. In this sense, the circuits of the cycle matroid of a finite graph G are the edge sets of homeomorphic images of the unit circle in the interpretation of G as a topological space.

Given an infinite graph, there are several possible generalisations of the cycle matroid, among them the finite cycle matroid, the topological cycle matroid and (if it exists) the algebraic cycle matroid. As in the finite case, for each of these matroids there is a topological space with subspaces associated with edges of the underlying graph such that the circuits of the matroid are the edge sets of homeomorphic images of the unit circle. These topological spaces share quite a few properties. So one possible generalisation of the finite cycle matroids are matroids for which there is a suitable topological space with subspaces associated with edges such that the circuits of the matroid are the edge sets of the homeomorphic images of the unit circle. This approach is taken in [**8**], where a definition of suitable topological spaces, called graph-like spaces, is given and a definition of when a matroid is induced by a graph-like space. In that paper, it is shown that a matroid is induced by a graph-like space if and only if every finite minor of the matroid is the cycle matroid of a graph and every intersection of a circuit with a co-circuit is finite.

Another construction of matroids are Ψ-matroids. The basic construction consists of a tree, a matroid at every node of the tree, a finite set of edges of the matroids at every edge of the tree, and a set of ends of the tree. The matroids are then thought to be glued together along the edge sets associated with the edges of the tree, and the set of ends puts a structure on in which ways the circuits may contain edges of infinitely many of the original matroids. In [**6**] and [**7**] it is shown that if the set of ends is not too weird, and either the gluing corresponds to an infinite two-sum of matroids or all matroids are representable over the same field and have some extra property (being finite is enough) then the result of the gluing is indeed a matroid. In [**6**] it is shown that for locally finite graphs there is a tree of matroids whose ends correspond to the ends of the locally finite graph such that gluing along the tree of matroids with respect to a set of ends corresponds to a construction from the locally finite graph with respect to the same set of ends, and that the topological cycle matroid and the finite cycle matroid are special cases of this construction with respect to the set of all ends and the set of no ends.

Other areas in which research has been made include matroid union and matroid intersection, the structure of infinite matroids with respect to separations of low order (this is connected to the second part of this thesis), more connections to matroids arising from graphs (including gammoids) and of course Menger's theorem.

This thesis has two parts, which are described in detail in their respective introductions. The first part mainly consists of a generalisation of a statement, that was shown in [**28**] to hold for cofinitary matroids, to two other classes of matroids. The second part is about the theory of flowers as developed in [**17**] and [**5**] and is mainly concerned with the generalisation of this theory to infinite matroids. The theory in [**17**] holds in a more general context than matroids, and similarly many of the statements proved in the second part of this thesis turn out to hold in more general context than infinite matroids.

CHAPTER 1

Basics

1.1. Sets, maps and intervals

Disjoint union is indicated by the symbol $\mathbin{\dot\cup}$. The set $\mathbb{N}$ contains 0. For a set E, its power set is denoted by $\mathcal{P}(E)$. Unless stated explicitly otherwise, partitions do not contain empty partition classes. Given a set X, a subset Y of X is called a *non-trivial* subset of X if $\emptyset \neq Y \neq X$. For three sets I, x and y, the set $I \cup \{x\}$ is denoted by $I + x$ and the set $I \backslash \{y\}$ is denoted by $I - y$. Intervals of the form $\{c\colon a \leq c < b\}$ are denoted by $[a, b[$ and the intervals $]a, b]$ and $]a, b[$ analogously. A maximal element of a pre-order $(X, \leq)$ is an element $x \in X$ such that for all y in Y with $x \leq y$ also $y \leq x$ holds.

Lemma 1.1. [**19**] *Let G be a graph and ω an end of G such that there are arbitrarily large finite families of pairwise disjoint rays contained in ω. Then there is an infinite family of pairwise disjoint rays contained in ω.*

1.1.1. Chains and limit-closed maps.

Definition 1.2. A map $\mu : X \to \mathbb{N}$ is called *limit-closed* if for every $k \in \mathbb{N}$ and every chain Y of X with $\mu(y) \leq k$ for all y in Y there is a supremum x of Y in X and that supremum satisfies $\mu(x) \leq k$.

Lemma 1.3. *Let X be a partially ordered set with a limit-closed map $\lambda : X \to \mathbb{N}$. Also let $k \in \mathbb{N}$ and let D be a directed subset of X such that $\lambda(d) \leq k$ for all d in D. Then D has a supremum x in X, and that supremum satisfies $\lambda(x) \leq k$.*

Proof. The proof is by transfinite induction on the cardinality of D. If D is finite, then it has a biggest element x and the lemma holds, so assume otherwise. Count D as $(d_\mu)_{\mu<\nu}$, where ν is the cardinality of D.

If D is countable, then let d'_0 be d_0 and recursively define d'_{i+1} to be an element of D such that $d'_i \leq d'_{i+1}$ and $d_{i+1} \leq d'_{i+1}$. Then $(d'_i)_{i<\omega}$ is a chain and its elements satisfy $\lambda(d'_i) \leq k$. As λ is limit-closed, the chain has a supremum x in X which satisfies $\lambda(x) \leq k$.

If D is uncountable, then let $f : D \times D \to D$ be a map such that $f(a, b) \geq a$ and $f(a, b) \geq b$ for all elements a and b of D. For every $\mu < \nu$ let D_μ be the closure of $\{d_\kappa\colon \kappa \leq \mu\}$ under f. Then every D_μ with finite μ is finite or countable and every D_μ with infinite μ has cardinality at most μ. Thus every D_μ with $\mu < \nu$ is a directed set of cardinality less than ν, so by the induction hypothesis there is a supremum d'_μ of D_μ that satisfies $\lambda(d'_\mu) \leq k$. Also $\mu \leq \mu' < \nu$ implies that $D_\mu \subseteq D_{\mu'}$ and hence $d'_\mu \leq d'_{\mu'}$. So $(d'_\mu)_{\mu<\nu}$ is a chain, and its supremum x in X, which satisfies $\lambda(x) \leq k$, is the supremum of D. $\square$

Corollary 1.4. *Let E be a set and $\lambda : \mathcal{P}(E) \to \mathbb{N}$ a limit-closed map. If there is $k \in \mathbb{N}$ such that $\lambda(S) \leq k$ for all finite $S \subseteq E$, then also $\lambda(S) \leq k$ for all subsets S of E.* $\square$

1.2. Cyclic orders

This section starts with a short collection of basics about cyclic orders and their connection to linear orders. In this section, quite often distinct linear or cyclic orders on the same ground set are considered. Because of this, and in contrast to the rest of the thesis, in this section cyclic orders and linear orders are not implicit but introduced formally correct as relations on the ground set. The notation of $\leq$ is kept for a linear order, but the linear order in question is added as an index where necessary, for example in $s \leq_L t$ for a linear order L. Similarly, for intervals of cyclic orders (introduced later), the cyclic order in question may be indicated by an index.

In the second part of this thesis, cyclically ordered sets are index sets for flowers. In particular in Chapter 5, for reasons explained there, the index sets have a particular form: they are the cycle completion of another cyclic order. Intuitively, the following happens: Let I be some cyclically ordered set. If one envisions I as boxes arranged in a circle according to the cyclic order (see also Fig. 1), then it is possible to cut up the circle at two places without cutting through boxes, thereby dividing the set of boxes into two intervals. If I is finite, then every one of these "cut points" is between two boxes. So I and the set of possible cut points form together another cyclically ordered set. If I is infinite, then not every cut point is between two boxes, but still I and the set of possible cut points form together a cyclically ordered set, the cycle completion of I.

In Chapter 5 there is a partial order on the set of k-pseudoflowers for which a comparison of distinct index sets is needed. To do this formally, also homomorphisms of cyclic orders and their interactions with cycle completions are analysed in this section. Both the cycle completion and this section's statements about the interaction of homomorphisms of cyclic orders and cycle completion are formally complex and take up most of this section, but should intuitively be clear. This thesis' way of formalising the cycle completion via cuts is just one possibility among many.

Definition 1.5. [**35**, Definition 1.1] Given two linear orders A and B on disjoint ground sets, the linear order $A \oplus B$ is the linear order defined on the union of the ground sets of A and B by letting $x \leq y$ if $x \leq_A y$ or $x \leq_B y$ or $x \in A$ and $y \in B$.

Definition 1.6. [**35**] A set of triples Z in $S \times S \times S$ is a *cyclic order* of the set S if it has the following four properties:

- (cyclic) $\forall a, b, c \in S : (a, b, c) \in Z \Rightarrow (b, c, a) \in Z$
- (antisymmetric) $\forall a, b, c \in S : (a, b, c) \in Z \Rightarrow (c, b, a) \notin Z$
- (linear) $\forall a, b, c \in S$ pairwise distinct $: (a, b, c) \notin Z \Rightarrow (c, b, a) \in Z$
- (transitive) $\forall a, b, c, d \in S : (a, b, c) \in Z \wedge (a, c, d) \in Z \Rightarrow (a, b, d) \in Z$.

Note that what is called a cyclic order here is sometimes (also in [**35**]) called a linear cyclic order or a complete cyclic order, with a cyclic order not necessarily being linear. The distinction is not made here as all cyclic orders under consideration are linear.

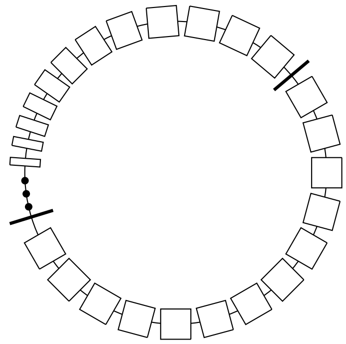

FIGURE 1. A set (whose elements are indicated by boxes) which is cyclically ordered (indicated by the arrangement of the boxes on a circle) and two "cutting points" dividing the cyclically ordered set into two intervals.

For two elements a and b of S, the set of elements $c \in S$ satisfying $(a, c, b) \in Z$ is denoted by $]a, b[$. The sets $]a, b[+a$, $]a, b[+b$ and $]a, b[+a+b$ are denoted by $[a, b[$, $]a, b]$ and $[a, b]$ respectively. As every cyclic order Z uniquely determines its ground set S, it is not necessary to distinguish between intervals of (S, Z) and intervals of Z, and similarly for linear orders.

An *interval* of the cyclic order is a subset I of S such that for all s and t in S either $[s, t] \subseteq I$ or $[t, s] \subseteq I$.

Remark 1.7. [**34**, Lemma 1.4] The definition of a cyclic order implies that if Z is a cyclic order on a set S and (s, s', t) is an element of Z then s, s' and t are pairwise distinct.

Definition 1.8. [**35**, Lemma 1.11,Definition 2.1, Theorem 2.3] Given a linear order L on a set S, the cyclic order Z *induced by* L consists of those triples (s, s', t) of elements of S such that in L one of the equations $s < s' < t$, $s' < t < s$ and $t < s < s'$ holds. Given a cyclic order Z on a set S, a *cut of* Z is a linear order L on S such that Z is the cyclic order induced by L.

Lemma 1.9. [**34**, Theorem 3.1] *For every cyclic order Z on set S and every s in S there is a cut of Z whose smallest element is s.*

Remark 1.10. By Lemma 1.9, every non-trivial interval of a cyclic order Z is also an interval of a cut L of Z. Also, such an interval inherits a linear order from every cut of which it is an interval, and that linear order does not depend on the chosen cut.

Observation 1.11. *Let Z be a cyclic order on a set S, L a cut of Z and s, s' and t elements of S such that $(s, s', t) \in Z$. If $s \leq t$ in L, then $s < s' < t$ in L.* $\square$

Definition 1.12. Let Z be a cyclic order on a set S and Z' a cyclic order on a set S'. In analogy to homomorphisms of linear orders, a *homomorphism* from Z to Z' is a map $f : S \to S'$ such that $(s, s', t) \in Z$ for all elements s, s' and t of S with $(f(s), f(s'), f(t)) \in Z'$. A *strong homomorphism* from Z to Z' is a homomorphism from Z to Z' such that $f^{-1}(s')$ is an interval of Z for all $s' \in S'$. A *homomorphism of cyclic orders* is a map for which there are cyclic orders Z and Z' such that the map is a homomorphism from Z to Z', and similarly for strong homomorphisms.

The cyclic order $\{(t,s,r) \colon (r,s,t) \in Z\}$ on S is the *mirror* of Z.

If the image of a homomorphism of cyclic orders contains at least three elements, then it is already a strong homomorphism. But every map between cyclically ordered sets whose image has at most two elements is a homomorphism of cyclic orders, and not all such maps should be considered as homomorphisms in this thesis.

The inverse of any bijective homomorphism of cyclic orders is also a homomorphism of cyclic orders, so a bijective homomorphism of cyclic orders is already an isomorphism of cyclic orders. Also, every injective homomorphism of cyclic orders is a strong homomorphism of cyclic orders, so when deriving a definition of isomorphism of cyclic orders, it does not matter whether that is derived from homomorphisms or strong homomorphisms.

Strong homomorphisms of cyclic orders have the property that pre-images of intervals are again intervals. Maps with that property are nearly strong homomorphisms again:

Lemma 1.13. *Let Z be a cyclic order on a set S and Z' a cyclic order on a set S'. Let $f: S \to S'$ be a map such that for all intervals I of Z' the set $f^{-1}(I)$ is an interval of Z. Then f is a strong homomorphism from Z to Z' or a strong homomorphism from Z to Z'', where $Z'' = \{(s,s',t) \colon (t,s',s) \in Z'\}$.*

Proof. If for all elements r, s and t of S the implication

$$(f(r), f(s), f(t)) \in Z' \Rightarrow (t,s,r) \in Z$$

holds, then f is a strong homomorphism from Z'' to Z'. So assume that there are elements r, s and t of S such that $(f(r), f(s), f(t)) \in Z'$ and $(r,s,t) \in Z$. First consider the case that there is $u \in S$ such that $(f(r), f(u), f(t)) \in Z'$. Then $f^{-1}([f(t), f(r)])$ is an interval of S which contains t and r but not s, so $[t,r]$ is a subset of $f^{-1}([f(t), f(r)])$. As also $u \notin f^{-1}([f(t), f(r)])$, $u \notin [t,r]$ and thus $(r,u,t) \in Z$.

Now let r', s' and t' be elements of S such that $(f(r'), f(s'), f(t')) \in Z'$. In order to show that $(r', s', t') \in Z$, first consider the case that the number n of elements in $\{f(r'), f(s'), f(t')\}$ which are not contained in $\{f(r), f(s), f(t)\}$ is zero. Assume, by renaming if necessary, that $f(r') = f(r)$, $f(s') = f(s)$ and $f(t') = f(t)$. Then by three applications of the previous paragraph, $(r, s', t) \in Z$ and thus $(s', t', r) \in Z$ and hence $(r', s', t') \in Z$.

Next consider the case that $n = 1$. Assume, again by renaming if necessary, that $(f(r), f(r'), f(s)) \in Z'$ (see also the left cyclic order of Fig. 2). By the first paragraph of this proof, $(r', s, t) \in Z$ and $(r', t, r) \in Z$ and hence also $(r', s, r) \in Z$. Then there are three cases: Either $f(s') = f(s)$ and $f(t') = f(t)$ or $f(s') = f(t)$ and $f(t') = f(r)$ or $f(s') = f(s)$ and $f(t') = f(r)$. In all three cases, by the case $n = 0$ also $(r', s', t') \in Z$.

Next consider the case that $n = 2$, and that one of the intervals $]f(r), f(s)[$, $]f(s), f(t)[$, and $]f(t), f(r)[$ contains both elements of $\{f(r'), f(s'), f(t')\}$ which are not contained in $\{f(r), f(s), f(t)\}$. Assume, by renaming if necessary, that both $f(r')$ and $f(s')$ are contained in $]f(r), f(s)[$. In that case $(f(r'), f(s'), f(t')) \in Z'$ implies that $f(r') \in]f(r), f(s')[$ (see also the middle cyclic order in Fig. 2). Also, by the case $n = 1$, (r', s, t), (s', s, t) and (t, r, r') are all contained in Z. As $f^{-1}([f(t), f(r')])$ contains t and r' but neither s or s', $(r', s, t) \in Z$ implies $[t, r'] \subseteq f^{-1}([f(t), f(r')])$. Thus s' is not contained in $[t, r']$ and hence $(r', s', t) \in Z$.

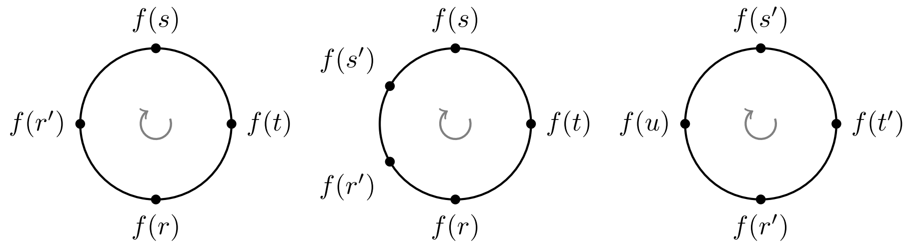

FIGURE 2. Three of the cases in the proof of Lemma 1.13

Because (s', s, t) and (t, r, r') are contained in Z, also (r', s', s) and (r', s', r) are contained in Z. By the case $n = 0$, also $(r', s', t') \in Z$.

Next consider the case that $n = 2$ and none of the intervals $]f(r), f(s)[$, $]f(s), f(t)[$ or $]f(t), f(s)[$ contains two elements of $\{f(r'), f(s'), f(t')\}$. Assume, by renaming if necessary, that $f(t') \in \{f(r), f(s), f(t)\}$ and that there is $u \in \{r, s, t\}$ such that $(f(r'), f(u), f(s')) \in Z'$ (see also the right cyclic order of Fig. 2). In this case (t', r', u) and (u, s', t') are both contained in Z' by the case $n = 1$ and thus (r', s', t') is also contained in Z'.

The only case left is the case $n = 3$. Assume, by renaming if necessary, that $(f(r'), f(s), f(s')) \in Z'$. Then (s, s', t') and (s, t', r') are contained in Z by the case $n = 2$ and thus $(r', s', t') \in Z$. □

Definition 1.14 (Remark 2.3)**.** [**34**] Given a cyclic order Z of a set S and a subset S' of S, the set of triples in Z which only contain elements of S' is a cyclic order on S', the *induced cyclic order* on S'.

Theorem 1.15. [**35**, Theorem 3.6] *Let Z be a cyclic order on set S and let K and L be distinct cuts of Z. Then there are non-empty disjoint subsets A and B of S such that $A \cup B = S$, $K|A = L|A$, $K|B = L|B$, $K = K|A \oplus K|B$ and $L = K|B \oplus K|A$.*

The following construction of a linear order $D(L)$ starting from a linear order L is very similar to both the Dedekind completion of L and the pseudo-line $L(L)$ as in [**8**, Definition 4.1]. Similarly to the Dedekind completion, $D(L)$ consists of initial segments of L and of the elements of L itself. But here an element l of L is not identified with an initial segment of L. The construction of $D(L)$ can be obtained from the pseudo-line $L(L)$ by replacing all the intervals $(0, 1) \times \{l\}$ by just l. The topology of the pseudo-line is not needed in the context of this thesis.

Example 1.16. (See also [**8**]) Let L be a linear order on a set S and let $V(L)$ be the set of *initial segments of* L, i.e. subsets S' of S which satisfy that if s is an element of S' and t is an element of S with $t < s$ then also $t \in S'$. The subset relation is a natural linear order on $V(L)$. Define a linear order on the disjoint union of S and $V(L)$ by letting $x \leq y$ if either both x and y are contained in S and $x \leq y$ in L or both are contained in $V(L)$ and $x \leq y$ in $V(L)$ or $x \in y$ or $y \in S \backslash x$. Denote the resulting linear order on $S \cup V(L)$ by $D(L)$. The smallest element of $D(L)$ is the empty set and the biggest element of $D(L)$ is S. Denote $S \cup (V(L) \backslash \{S\})$ by $V'(L)$, the restriction of $D(L)$ to $V'(L)$ by $D'(L)$ and the cyclic order induced by $D'(L)$ by $Z(L)$. For every element s of S, the set $\{t \in S \colon t < s\}$ is the predecessor and the set $\{t \in S \colon t \leq s\}$ is the successor of s in $D(L)$.

Every subset of $D(L)$ has a supremum and an infimum in $D(L)$, which can be seen as follows: Given a subset V' of $V(L)$, the set $\bigcup V'$ is an initial segment of S and is the supremum of V' both in $D(L)|V(L)$ and in $D(L)$. Similarly the set $\bigcap V'$ is the infimum of V' in $D(L)|V(L)$ and $D(L)$. So in order to show that every subset of $S \cup V(L)$ has a supremum and an infimum in $D(L)$, it suffices to consider subsets S' of S, and by symmetry it suffices to show that S' has a supremum in $D(L)$. The set $\{s \in S | \exists t \in S' : s \leq t\}$, denoted by S'', is an initial segment of S which is an upper bound of S'. Also, no proper subset of S'' is an upper bound of S'. So if S' has an upper bound in $D(L)$ which is less than S'', then that upper bound is contained in S. In particular, as $D(L)$ contains between any two elements of S at least one element of $V(L)$, there is at most one upper bound of S' which is less than S''. Thus S' has a supremum in $D(L)$.

Lemma 1.17. *Let L and K be cuts of a cyclic order Z on a set S such that $K = (L|(S\backslash S')) \oplus (L|S')$ for an initial segment S' of L. Then the map*

$$V'(L) \to V'(K), \quad x \mapsto \begin{cases} x & x \in S \\ x \cup (S\backslash S') & x \in V'(L)\backslash S,\ x \subsetneq S' \\ x\backslash S' & x \in V'(L)\backslash S,\ S' \subseteq x \end{cases}$$

is the unique bijective strong homomorphism from $Z(L)$ to $Z(K)$ which preserves S.

Proof. The map $F_1 : V'(L) \to V'(L|S') \cup V'(L|(S\backslash S'))$ which maps elements of S to themselves, initial segments which are properly contained in S' to themselves and initial segments I containing S' to $I\backslash I'$ is an isomorphism of the linear orders $D'(L)$ and $D'(L|S') \oplus D'(L|(S\backslash S'))$. Similarly the map $F_2 : V'(K) \to V'(L|S') \cup V'(L|(S\backslash S'))$ which maps every elements of S to themselves, initial segments properly contained in $S\backslash S'$ to themselves and initial segments I containing $S\backslash S'$ to $I \cap S'$ is an isomorphism of the linear orders $D'(K)$ and $D'(L|(S\backslash S')) \oplus (L|S')$. Thus the map given in the lemma, which equals $F_2^{-1} \circ F_1$, is a bijective strong homomorphism from $Z(L)$ to $Z(K)$.

Let F and G be two bijective strong homomorphisms from $Z(L)$ to $Z(K)$ which preserve S. Assume for a contradiction that there is $v \in V'(L)$ such that $F(v)$ is less than $G(v)$ in $D'(K)$. As F and G both are bijective and preserve S, $F(v)$ and $G(v)$ are both contained in $V'(K)\backslash S$. Thus there are elements $s \in G(v)\backslash F(v)$ and $t \in S\backslash G(v)$. Then $(F(t), F(v), F(s))$ equals $(t, F(v), s)$ and is thus contained in $Z(K)$. Because F is a strong homomorphism from $Z(L)$ to $Z(K)$, this implies that $(t, v, s) \in Z(L)$. But similarly $(s, G(v), t) \in Z(K)$ and thus $(s, v, t) \in Z(L)$, a contradiction. □

Corollary 1.18. *Let Z be a cyclic order on set S and let L and K be cuts of Z. Then there is a unique bijective strong homomorphism from $Z(L)$ to $Z(K)$ which preserves S.*

Proof. Let $S' \subseteq S$ such that $K = (L|(S\backslash S')) \oplus (L|S')$ and such that S' is an initial segment of L. Such a set exists by Theorem 1.15. Then the statement follows from Lemma 1.17. □

Lemma 1.19. *Let Z by a cyclic order on a non-empty set S and let $\mathcal{V}$ be the set of cuts of Z. For every cut $L \in \mathcal{V}$ denote the map $V'(L) \to S \cup \mathcal{V}$ which maps every element of S to itself and every initial segment S' to $(L|(S\backslash S')) \oplus (L|S')$ by*

η_L. Also denote $\{(\eta_L(a), \eta_L(b), \eta_L(c))\colon (a,b,c) \in Z(L)\}$ by T_L. Then T_L is a cyclic order on $S \cup \mathcal{V}$ which does not depend on the choice of L.

Proof. By Theorem 1.15 the maps η_L are surjective, so they are bijections between $V'(L)$ and $S \cup \mathcal{V}$. Thus every T_L arises from $Z(L)$ by renaming the elements of $V'(L)$ and thus is a cyclic order on $S \cup \mathcal{V}$, and η_L is a bijective strong homomorphism from $Z(L)$ to T_L. Let L and K be elements of $\mathcal{V}$ and let S' be an initial segment of L such that $K = (L|(S\backslash S')) \oplus (L|S')$ (such a segment exists by Theorem 1.15). Denote the unique bijective strong homomorphism from $Z(L)$ to $Z(K)$ preserving S, which exists by Lemma 1.17, by F. Then $\eta_K \circ F(s) = \eta_L(s)$ for all $s \in S$. Also, for all initial segments I of L which are properly contained in S',

$$\begin{aligned}\eta_K \circ F(I) = \eta_K(I \cup (S\backslash S')) &= (K|(S'\backslash I)) \oplus (K|(I \cup (S\backslash S')))\\ &= (L|(S'\backslash I)) \oplus (L|(S\backslash S')) \oplus (L|I)\\ &= (L|(S\backslash I)) \oplus (L|I) = \eta_L(I),\end{aligned}$$

and similarly for all initial segments I of L which contain S'

$$\begin{aligned}\eta_K \circ F(I) = \eta_K(I\backslash S') &= (K|(S\backslash(I\backslash S'))) \oplus (K|(I\backslash S'))\\ &= (L|(S\backslash I))) \oplus (L|S') \oplus (L|(I\backslash S'))\\ &= (L|(S\backslash I)) \oplus (L|I) = \eta_L(I).\end{aligned}$$

So $\eta_K \circ F = \eta_L$ and thus $\eta_K \circ F \circ \eta_L^{-1}$ is the identity. But $\eta_K \circ F \circ \eta_L^{-1}$ is also a composition of bijective strong homomorphisms of cyclic orders and thus the identity is a bijective strong homomorphism from T_L to T_K, so $T_L = T_K$. $\square$

Definition 1.20. For a cyclic order Z on set S, denote the union of S with all cuts of Z by $\mathcal{S}(Z)$. If S is non-empty, then the *cycle completion* of Z, denoted by $\mathcal{Z}(Z)$, is the cyclic order on $\mathcal{S}(Z)$ given in Lemma 1.19. If S is empty, then $\mathcal{S}(Z)$ only contains the empty cut, and thus $\mathcal{Z}(Z)$ is the empty cyclic order. A subset X of $\mathcal{S}(Z)\backslash S$ is *closed* if for all elements x and y of $\mathcal{S}(Z)\backslash S$ either $X \cap [x,y] = \emptyset$ or the supremum and the infimum of $X \cap [x,y]$ in the linear order of $[x,y]$ are contained in X.

Lemma 1.21. *Let Z be a cyclic order on set S. Then for every non-trivial interval I of S there are unique elements v and w of $\mathcal{S}(Z)\backslash S$ such that $I = [v,w] \cap S$.*

Proof. Let L be a cut of Z such that I is an interval of L and such that some element of S is bigger than all elements of I in L. By construction of $D(L)$ there are unique elements v and w of $V'(L)\backslash S$ such that $I = [v,w] \cap S$ in $D(L)$. Then v and w are also the unique elements of $V'(L)\backslash S$ such that $I = [v,w] \cap S$ in $Z(L)$, and thus $\eta_L(v)$ and $\eta_L(w)$ are the unique elements of $\mathcal{S}(Z)$ such that $I = [\eta_L(v), \eta_L(w)] \cap S$ in $\mathcal{Z}(Z)$. $\square$

In Chapter 5 cycle completions will play an important role as the index sets of flowers. Given two cyclic orders Z and Z' on sets S and S', concatenation from a flower on index set $\mathcal{Z}(Z)$ to a flower on index set $\mathcal{Z}(Z')$ is defined in terms of a surjective strong homomorphism $F : \mathcal{S}(Z) \to \mathcal{S}(Z')$ such that $F(S) \subseteq S'$. The following lemmas establish a few facts about such strong homomorphisms.

Lemma 1.22. *Let Z and Z' be cyclic orders on sets S and S'. Let $F : \mathcal{S}(Z) \to \mathcal{S}(Z')$ be a surjective strong homomorphism of cyclic orders such that $F(S) \subseteq S'$.*

Then there is for every $v' \in \mathcal{S}(Z')\backslash S'$ exactly one $v \in \mathcal{S}(Z)$ with $F(v) = v'$, and there is for every $s' \in S'$ some $s \in S$ with $F(s) = s'$.

PROOF. Every interval of $\mathcal{Z}(Z)$ with at least two elements contains at least one element of S. As $F^{-1}(v')$ does not contain elements of S, it has at most one element. Because F is surjective, $F^{-1}(v')$ contains exactly one element v. Let t' and q' be the predecessor and successor of s' in $\mathcal{Z}(Z')$. Then $[t', q']$ taken in $\mathcal{Z}(Z')$ consists of t', q' and s'. Let t and q be the unique elements of $\mathcal{S}(Z)$ such that $F(t) = t'$ and $F(q) = q'$, and let s be an element of $[t, q]$ taken in $\mathcal{Z}(Z)$ which is contained in S. As F is a strong homomorphism of cyclic orders and $F(s)$, q' and t' are pairwise disjoint, $F(s) \in [t', q']$ in $\mathcal{Z}(Z')$. Thus $F(s) = s'$. □

So F naturally induces two other strong homomorphisms of cyclic orders: The restriction of F to S is a surjective strong homomorphism from Z to Z' and $g : \mathcal{S}(Z')\backslash S' \to \mathcal{S}(Z)\backslash S$ which maps every element to its unique pre-image under F is an injective strong homomorphism of cyclic orders. In the other direction, all surjective strong homomorphisms f from Z to Z' are derived from a surjective strong homomorphism from $\mathcal{Z}(Z) \to \mathcal{Z}(Z')$ with $F(S) \subseteq S'$.

Lemma 1.23. *Let Z and Z' be cyclic orders on set S and S' respectively, and let $F : \mathcal{S}(Z) \to \mathcal{S}(Z')$ be a surjective strong homomorphism from $\mathcal{Z}(Z)$ to $\mathcal{Z}(Z')$ with $F(S) \subseteq S'$. Then for all elements v and w of $\mathcal{S}(Z)$ such that $F(v)$ and $F(w)$ are distinct elements of $\mathcal{S}(Z')\backslash S'$ the equations $F^{-1}(]F(v), F(w)[) =]v, w[$ and $F^{-1}([F(v), F(w)]) = [v, w]$ hold.*

PROOF. By Lemma 1.22, v is the only element of $\mathcal{S}(Z)$ which is mapped to $F(v)$ by F and similarly for w. So for all $x \in \mathcal{S}(Z) - v - w$, $(v, x, w) \in \mathcal{Z}(Z)$ if and only if $(F(v), F(x), F(w)) \in \mathcal{Z}(Z')$ and thus the two equations hold. □

Lemma 1.24. *Let Z and Z' be cyclic orders on sets S and S' and let f be a surjective strong homomorphism from Z to Z'. If S' has at least two elements, then there is a unique surjective strong homomorphism F from $\mathcal{Z}(Z)$ to $\mathcal{Z}(Z')$ such that the restriction of F to S equals f.*

PROOF. Let L' be some cut of Z' and let L be the cut of Z where $x <_L y$ if $f(x) <_{L'} f(y)$ or if $f(x) = f(y)$ and $x < y$ in the interval $f^{-1}(f(x))$. For $v \in V'(L)\backslash S$ let $A_v = \{f(s) : s <_L v\}$ and $B_v = \{f(s) : s >_L v\}$. Then $A_v \cap B_v$ contains at most one element, and if it contains an element then denote that element by s_v. Define $F : V'(L) \to V'(L')$ to map x to itself if $x \in S$, to s_v if that is defined and to A_v otherwise. Then F is a surjective map from $V'(L)$ to $V'(L')$ whose restriction to S is f and which satisfies $\forall x, y \in dS : F(x) < F(y) \Rightarrow x < y$. Furthermore $F^{-1}(v')$ contains exactly one element for all $v' \in V'(L)\backslash S'$. So F induces a surjective strong homomorphism F' from $\mathcal{Z}(Z)$ to $\mathcal{Z}(Z')$ whose restriction to S is f.

Let G be a surjective strong homomorphism from $\mathcal{Z}(Z)$ to $\mathcal{Z}(Z')$ whose restriction to S is f. By Lemma 1.21 there are unique elements v' and w' of $\mathcal{S}(Z')\backslash S'$ such that $\{s'\} = [v', w'] \cap S'$ and by Lemma 1.22 there are unique elements v and w of $\mathcal{S}(Z)$ such that $G(v) = v'$ and $G(w) = w'$. Then by Lemma 1.23

$$f^{-1}(s') = G^{-1}(s') \cap S = G^{-1}(]G(v), G(w)[) \cap S =]v, w[\cap S = [v, w] \cap S.$$

So v and w are the unique elements of $\mathcal{S}(Z)\backslash S$ such that $[v, w] \cap S = f^{-1}(s')$, and thus $G^{-1}(s')$ is determined by f. Thus if G differs from F', then there is $v \in \mathcal{S}(Z)$

such that $F'(v)$ and $G(v)$ are distinct elements of $\mathcal{S}(Z')\backslash S'$. But then there are elements s' and t' of S' with $(s', F'(v), t') \in \mathcal{Z}(Z')$ and $(t', G(v), s') \in \mathcal{Z}(Z')$ and elements s and t of S with $f(s) = s'$ and $f(t) = t'$. As F' and G are strong homomorphisms from $\mathcal{Z}(Z)$ to $\mathcal{Z}(Z')$, this implies $(s, v, t) \in \mathcal{Z}(Z)$ and $(t, v, s) \in \mathcal{Z}(Z)$, a contradiction. So G does not differ from F'. $\square$

In Lemma 1.24, the requirement that S' have at least two elements is only needed for the uniqueness of F, F also does exists if S' has only one element. And if S' does not have any elements, then also $S = \emptyset$ and hence $\mathcal{S}(Z)$ and $\mathcal{S}(Z')$ have only one element each. In this case, the unique map between them is again a surjective strong homomorphism of cyclic orders.

The following example shows that not every injective strong homomorphism $f : \mathcal{S}(Z')\backslash S' \to \mathcal{S}(Z)\backslash S$ is derived from some suitable $F : \mathcal{S}(Z) \to \mathcal{S}(Z')$:

Example 1.25. Let S be the non-negative integers, L the usual linear order and Z the cyclic order induced by L. Then every element v of $V'(L)\backslash S$ has a successor $\nu(v)$ in the restriction of $D'(L)$ to $V'(L)\backslash S$. Then $\nu : V'(L)\backslash S \to V'(L)\backslash S$ is an injective strong homomorphism of cyclic orders. But there is no surjective strong homomorphism F from $\mathcal{Z}(Z)$ to itself which satisfies $F(S) \subseteq S$ and $F \circ f = id$, as the smallest element of L cannot be mapped anywhere suitable.

Lemma 1.27 shows that some subsets of $\mathcal{S}(Z)\backslash S$ are isomorphic to the restriction of $\mathcal{Z}(Z'')$ to $\mathcal{S}(Z'')\backslash S''$ for a suitable cyclic order Z'' on set S''.

Lemma 1.26. *Let Z and Z' be cyclic orders on sets S and S' respectively and let F be a surjective strong homomorphism from $\mathcal{Z}(Z)$ to $\mathcal{Z}(Z')$ with $F(S) \subseteq S'$. Then every $v \in \mathcal{S}(Z)\backslash S$ satisfies $F(v) \in S'$ if and only if there are elements s and t of $F^{-1}(F(v)) \cap S$ such that $v \in [s, t]$.*

Proof. The "if" direction is clear. In order to show the "only if" direction, let v be an element of $\mathcal{S}(Z)\backslash S$ such that $F(v) \in S'$. In the case that S' has only one element, by Lemma 1.22 there is a unique element $w \in \mathcal{S}(Z)$ with $F(w) \notin S'$, and necessarily $w \notin S$. So there are elements s and t of S such that $v \in [s, t]$ and $w \in [t, s]$. Then $w \notin [s, t]$ and thus $[s, t] \subseteq F^{-1}(F(v))$. So assume that S' has at least two elements. Let u' and w' be the predecessor and successor of $F(v)$ in $\mathcal{S}(Z')$. Again by Lemma 1.22 there are unique elements u and w of $\mathcal{S}(Z)$ with $F(u) = u'$ and $F(w) = w'$. As neither u nor w is contained in S there are elements s and t of S such that $s \in [u, v]$ and $t \in [v, w]$, so $v \in [s, t] \subseteq [u, w]$. Furthermore

$$F^{-1}(F(v)) = F^{-1}(]F(u), F(w)[) =]u, w[$$

where the last equation holds by Lemma 1.23. $\square$

Lemma 1.27. *Let Z be a cyclic order on set S and x and y distinct elements of $\mathcal{S}(Z)$. Let C be a non-empty subset of $[x, y]\backslash S$ such that for all $C' \subseteq C$ the supremum and infimum of C' in $[x, y]$ are contained in C. Then there is a cyclic order Z' on set S' and a surjective strong homomorphism F from $\mathcal{Z}(Z)$ to $\mathcal{Z}(Z')$ such that $C = F^{-1}(\mathcal{S}(Z')\backslash S')$.*

Proof. If C has only one element, then the lemma holds, so assume otherwise. Define an equivalence relation on S where $s \sim t$ if there is an interval of $\mathcal{S}(Z)$ containing both s and t but no element of C. Let S' be the set of equivalence classes, f the projection from S to S' and Z' the cyclic order on S' which contains all triples $(f(s), f(s'), f(t))$ where $(s, s', t) \in Z$ and $f(s)$, $f(s')$ and $f(t)$ are pairwise disjoint. Then f is a surjective strong homomorphism from Z to Z' and thus by Lemma 1.24 there is a surjective strong homomorphism F from $\mathcal{Z}(Z)$ to $\mathcal{Z}(Z')$ whose restriction to S is f. Thus by Lemma 1.22, for every element v of $\mathcal{S}(Z')\backslash S'$ the set $F^{-1}(v)$ contains exactly one element $g(v)$. Then $g : \mathcal{S}(Z')\backslash S' \to \mathcal{S}(Z)$ is an injective strong homomorphism from $\mathcal{Z}(Z')|(\mathcal{S}(Z')\backslash S')$ to $\mathcal{Z}(Z)$.

Assume for a contradiction that there is $v \in C$ such that $F(v) \in S$. Then there are by Lemma 1.26 elements s and t of S such that $v \in [s, t] \subseteq F^{-1}(F(v))$. Thus $f(s) = f(t)$, so $[t, s]$ is an interval of $\mathcal{Z}(Z)$ which avoids C. But then the image of f has only one element, contradicting the fact that C has at least two elements. Thus C is a subset of the image of g.

Let v be an element of the image of g. As C contains at least two elements, there is $s \in S$ with predecessor y' and successor x' in $\mathcal{Z}(Z)$ such that both $[x', v]$ and $[v, y']$ contain an element of C. As for every subset of C its supremum and infimum in $[x, y]$ are contained in C, also the supremum p of $[x', v] \cap C$ in $[x', v]$ and the infimum q of $[v, y'] \cap C$ in $[v, y']$ are contained in C. Assume for a contradiction that neither p nor q equals v. Then there are elements s and t of S such that $s \in [p, v]$ and $t \in [v, q]$. Then $[s, t] \cap C = \emptyset$, so $[s, t] \cap S \subseteq f^{-1}(s))$. Then $F^{-1}(f(s))$ is an interval of $\mathcal{Z}(Z)$ which contains s and t but not v, so it contains $[t, s]$. But then all elements of S are mapped to $f(s)$ by f, contradicting the fact that C and S' have both at least two elements. So v equals p or q and is thus contained in C. $\square$

1.3. (Infinite) matroids

Terminology for infinite matroids is mostly taken from [**13**]. In [**14**] the finite connectivity is explored and the definitions of finitarisation, nearly finitary and k-nearly finitary can be found in [**3**]. Alternatively, these definitions and statements (and many more in this and the next section) can be found in [**10**].

Definition 1.28. [**13**] Given a ground set E, a set $\mathcal{C}$ of subsets of E is the set of circuits of an infinite matroid M if it satisfies

(C1) The empty set is not contained in $\mathcal{C}$.
(C2) Elements of $\mathcal{C}$ are not proper subsets of each other.
(C3) For all $C \in \mathcal{C}$, all $z \in C$ and all families $(C_x)_{x \in X}$ such that $z \notin X$ and $C_x \cap (X + z) = \{x\}$ for all x in X there is $C' \in \mathcal{C}$ containing z such that $C' \subseteq (C \cup \bigcup_{x \in X} C_x)\backslash X$.
(CM) The set $\mathcal{I} := \{I \subseteq E : I$ does not contain elements of $\mathcal{C}\}$ satisfies that for all $I \in \mathcal{I}$ and all $F \subseteq E$ with $I \subseteq F$ the set $\{J \in \mathcal{I} : I \subseteq J \subseteq F\}$ has a maximal element.

A subset of E is called *dependent* if it contains a circuit and *independent* otherwise. For a subset X of E, a maximal element of the set of independent sets contained in X is a *base* of X, and bases of E are *bases of* M. If there is a finite bound on the size of bases of X, then the biggest size of a base of X is the *rank* of X, and otherwise the rank of X is ∞.

The *dual* of M, denoted M^*, is the matroid whose set of bases is the set $\{E\backslash X : X$ is a base of $M\}$. Given a subset X of E, the *restriction* of M to X, denoted by $M{\restriction}X$, is the matroid on ground set X whose independent sets are the independent sets of M that are contained in X. The *deletion* of X, denoted by $M\backslash X$, is defined as $M{\restriction}(E\backslash X)$. The *contraction* of X, denoted M/X, is defined as

$(M^*\backslash X)^*$, and the *contraction* of M to X, denoted by $M.X$, is $M/(E\backslash X)$. *Cobases* and *cocircuits* of M are bases and circuits of the dual of M, respectively.

Lemma 1.29. [**13**, Lemma 3.5] *Let M be a matroid on ground set E. The following statements are equivalent for all sets $I \subseteq X \subseteq E$:*

- *I is a base of M/X.*
- *There exists a base I' of $M\backslash X$ such that $I \cup I'$ is a base of M.*
- *$I \cup I''$ is a base of M for all bases I'' of $M\backslash X$.*

Lemma 1.30. [**13**, Lemma 3.7] *Let B and B' be bases of some matroid M. If $B\backslash B'$ is finite, then $|B\backslash B'| = |B'\backslash B|$.*

Lemma 1.31. [**1**, Corollary 2.5] *Let M be a matroid on ground set $E = C \cup\!\!\!\cdot\, \{e\} \cup\!\!\!\cdot\, D$. Then either there is a circuit C' with $e \in C \subseteq C + e$ or a cocircuit D' with $e \in D \subseteq D + e$ but not both.*

Definition 1.32. [**12**] In a matroid M, a union of circuits is a *scrawl*. A scrawl of M^* is a *coscrawl* of M.

Lemma 1.33. [**12**, Lemma 2.6] *Let M be a matroid and $S \subseteq E(M)$. Then the following are equivalent:*

- *S is a scrawl of M.*
- *$|S \cap D| \neq 1$ for all cocircuits D of M.*
- *$|S \cap D| \neq 1$ for all coscrawls D of M.*

Corollary 1.34. [**12**, Corollary 2.7] *Let M be a matroid with ground set $E = C \cup\!\!\!\cdot\, X \cup\!\!\!\cdot\, D$, and let $S \subseteq X$. Then S is a scrawl of $M/C\backslash D$ if and only if there is a scrawl S' of M such that $S \subseteq S' \subseteq S \cup X$.*

If a matroid M is *tame*, i.e. the intersection of any circuit with any cocircuit is finite, then a stronger version of (C3) holds. The version and its proof are due to Bowler and Carmesin [**11**].

Lemma 1.35 (tame circuit elimination)**.** [**11**] *Let M be a tame matroid. Let C be a circuit, $z \in C$ and $X \subset C$ a linearly ordered set not containing z. Let $(C_x)_{x\in X}$ be a set of circuits not containing z such that $x \in C_x \cap X \subseteq \lceil x \rceil$ for all $x \in X$. Then there is a circuit C' such that $z \in C' \subseteq (C \cup \bigcup_{x\in X} C_x)\backslash X$.*

Proof. Denote the set $(C \cup \bigcup_{x\in X} C_x)\backslash X$ by Y. Assume for a contradiction that there is no circuit C' such that $z \in C' \subseteq Y$. Then by Lemma 1.31 there is a cocircuit D of M with $D \cap Y = \{z\}$. As M is tame, the intersection of D with X, which is contained in $D \cap C$, is finite and thus has a least element x. Then C_x is a subset of $Y \cup \lceil x \rceil$. As $D \cap Y = \{z\}$ and $D \cap \lceil x \rceil = \{x\}$, the intersection of D with C_x is a subset of $\{x, z\}$. Because z is not contained in C_x but x is contained in both D and C_x, the intersection of D and C_x consists of the one element x, a contradiction to Lemma 1.31. $\square$

Definition 1.36. [**14**, Section 4] Let M be a matroid and X a subset of the ground set of M. If there is a finite set F such that $(B \cup B')\backslash F$ is a base of M for some bases B of $M\restriction X$ and B' of $M\backslash X$, then the minimal size of such a set F is the *connectivity* of X. If no such finite F exists, then the connectivity of X is ∞.

Lemma 1.37. [**14**, Lemma 14] *Let M be a matroid and X a subset of $E(M)$ of finite connectivity. For every base B of $M\restriction X$, every base B' of $M\backslash X$ and every set*

$F \subseteq B \cup B'$ such that $(B \cup B') \backslash F$ is a base of M, the size of F is the connectivity of X.

Note that by the previous lemma, if B is a base of $M \restriction X$ and B' is a base of $M.X$ such that $B' \subseteq B$, then the size of $B \backslash B'$ equals $\lambda(X)$.

Lemma 1.38. [**14**, Section 5] *For every matroid M, the connectivity function is submodular and identical to the connectivity function of M^*. Furthermore, for every increasing sequence X_i of subsets of $E(M)$, if there is $k \in \mathbb{N}$ such that all X_i have connectivity at most k, then the union of all X_i also has connectivity at most k.*

Lemma 1.39. *Let M be a matroid and C and D disjoint subsets of the ground set. Then $\lambda_M(C \cup D) = \lambda_{M/C}(D) + \lambda_{M \backslash D}(C)$.*

Proof. Let B_1 be a base of $M \backslash (C \cup D)$, B_2 a base of $M/C \backslash D$ and B_3 a base of $M/(C \cup D)$ such that $B_3 \subseteq B_2 \subseteq B_1$. Denote the ground set of M without $C \cup D$ by A. Then

$$\begin{aligned} \lambda_M(C \cup D) &= \lambda_M(A) = |B_1 \backslash B_3| = |B_1 \backslash B_2| + |B_2 \backslash B_3| \\ &= \lambda_{M \backslash D}(A) + \lambda_{M/C}(A) = \lambda_{M \backslash D}(C) + \lambda_{M/C}(D). \end{aligned}$$ □

Definition 1.40. [**36**] Let M be a finite matroid. For a subset X of the ground set of M, the *nullity* of X is the difference between the size of X and its rank. For two disjoint sets X and Y of the ground set of Y, the *local connectivity* $\sqcap_M(X, Y)$ of X and Y is $r(X) + r(Y) - r(X \cup Y)$.

The definition of nullity can easily be extended to infinite matroids as follows: Let M be an infinite matroid, X a subset of the ground set of M and B a base of $M \restriction X$. Define the nullity of X to be the size of $X \backslash B$ if that is finite and ∞ otherwise. As $X \backslash B$ is a base of $(M \restriction X)^*$, the nullity of X is the rank of $(M \restriction X)^*$ and thus does not depend on the choice of B by Lemma 1.30.

In a finite matroid M, the local connectivity of two sets X and Y equals $\lambda_{M \restriction (X \cup Y)}(X, Y)$. As the connectivity of infinite matroids coincides with the usual connectivity on matroids that happen to be finite, the straightforward generalisation of local connectivity to infinite matroids is simply $\sqcap_M(X, Y) = \lambda_{M \restriction (X \cup Y)}(X)$.

Lemma 1.41. *Let M be a matroid and $X \subseteq E(M)$ a set of finite connectivity. Then there are disjoint subsets C and D of X such that for $F := X \backslash (C \cup D)$ and $N := M/C \backslash D$ the equations $M/X = N/F$, $M \backslash X = N \backslash F$ and $|F| = \lambda_M(X)$ hold.*

Proof. Let B be a base of $M \restriction X$ and C a base of $M.X$ such that $C \subseteq B$. Denote $X \backslash B$ by D. Then, by the definition of connectivity, $X \backslash (C \cup D)$ has size $\lambda_M(X)$. Every element d of D is spanned by B in M and thus a loop of M/B. Hence D is a union of connected components of M/B, so

$$M/X = M/B \backslash D = M/C \backslash D/F.$$

Dually, D is a base of $M^*.X$ and $X \backslash C$ is a base of $M^* \restriction X$, so

$$M \backslash X = (M^*/X)^* = (M^*/D \backslash C/F)^* = M/C \backslash D \backslash F.$$ □

Lemma 1.42. *In the situation of Lemma 1.41, let Y and Z be disjoint subsets of $E(M) \backslash X$. Then the following equations hold:*

- $\lambda_M(Y) = \lambda_N(Y)$.

- $\sqcap_M(Y,Z) = \sqcap_N(Y,Z)$.
- $\sqcap_M(Z, X\cup Y) = \sqcap_N(Z, F\cup Y)$.

Proof. For every subset S of $E(M)\backslash X$, the equalities $M\restriction S = N\restriction S$ and $M.S = N.S$ hold. In particular $M\restriction(Y\cup Z) = N\restriction(Y\cup Z)$, so

$$\sqcap_M(Y,Z) = \lambda_{M\restriction(Y\cup Z)}(Y) = \lambda_{N\restriction(Y\cup Z)}(Y) = \sqcap_N(Y,Z).$$

Also the equations

$$(M\restriction(X\cup Y\cup Z))\restriction Z = M\restriction Z = N\restriction Z = (N\restriction(F\cup Y\cup Z))\restriction Z$$

and

$$(M\restriction(X\cup Y\cup Z)).Z = (M/X/Y)\restriction Z = (N/F/Y)\restriction Z = (N\restriction(F\cup Y\cup Z)).Z$$

hold and thus

$$\sqcap_M(Z, X\cup Y) = \sqcap_N(Z, F\cup Y).$$

In particular

$$\lambda_M(Y) = \sqcap_M(Y, E(M)\backslash Y) = \sqcap_N(Y, E(N)\backslash Y) = \lambda_N(Y). \qquad \square$$

Lemma 1.43. *Let M be a matroid and $X\subseteq E(M)$ a set of finite connectivity in M. Then there is a union C of finitely many circuits such that*

$$M/X = M/(X\cap C)\backslash(X\backslash C).$$

Proof. Let B be a base of $M\restriction X$ and B' a base of $M.X$ such that $B'\subseteq B$. Denote $B\backslash B'$ by I. By definition of connectivity, I is finite. For every $e\in B\backslash B'$, the fundamental circuit of e in B' in M can be extended to a circuit C_e of M by only adding edges not contained in X. Let C be the union of the circuits C_e, then C is a finite union of circuits. Denote $M/(C\cap B')$ by N. Then I is independent in N and spanned by $E(M)\backslash X$ in N, so $\lambda_N(I) = |I|$. Thus by Lemma 1.39

$$\begin{aligned} 0 \le \lambda_{N/I}(X\backslash C) &= \lambda_N((X\backslash C)\cup I) - \lambda_{N\backslash(X\backslash C)}(I) \\ &= \lambda_N(E(M)\backslash X) - |I| \le \lambda_M(E(M)\backslash X) - |I| = 0 \end{aligned}$$

and thus $\lambda_{N/I}(X\backslash C) = 0$. As $N/I = M/(C\cap X)$, the set $X\backslash C$ is a union of components of $M/(C\cap X)$ so $M/(C\cap X)\backslash(X\backslash C) = M/X$. $\square$

Lemma 1.44. *Let X be a subset of the ground set of a matroid M. Let B be a base of $M\restriction X$ and B' a base of $M.X$ such that $B\backslash B'$ is finite. Then the connectivity of X is $|B\backslash B'| - |B'\backslash B|$.*

Proof. Let A be a base of $M.X$ which is contained in B. Then $A\backslash B'$ is a subset of $B\backslash B'$ and hence finite, so $|B'\backslash A| - |A\backslash B'|$ is defined and 0. Thus

$$\begin{aligned} |B\backslash B'| &= |B\backslash B'| + |B'\backslash A| - |A\backslash B'| \\ &= |B\backslash A| + |B'\backslash B| + |A\backslash B'| - |A\backslash B'| \\ &= |B\backslash A| + |B'\backslash B| \end{aligned}$$

where the second equality holds because

$$(B\backslash B') \mathbin{\dot\cup} (B'\backslash A) = (B\backslash A) \mathbin{\dot\cup} (B'\backslash B) \mathbin{\dot\cup} (A\backslash B').$$

As $B\backslash B'$ is finite, also $B'\backslash B$ has to be finite and thus

$$\lambda(X) = |B\backslash A| = |B\backslash B'| - |B'\backslash B|. \qquad \square$$

Definition 1.45. [**3**] Let M be a matroid. It is called *finitary* if all its circuits are finite. The set of finite circuits is the set of circuits of a finitary matroid M_{fin}. Every base of M is contained in a base of M_{fin}, and M is called *nearly finitary* if for all bases B of M contained in a base B_{fin} of M_{fin} the set $B_{\text{fin}} \backslash B$ is finite. It is *k-nearly finitary* for some $k \in \mathbb{N}$ if even $|B_{\text{fin}} \backslash B| \leq k$ for all such bases B and B_{fin}.

The following observation will be used frequently in the first part of this thesis.

Observation 1.46. *Let M be a matroid. For every $k \in \mathbb{N}$ the following statements are equivalent:*

- *M is not k-nearly finitary.*
- *M contains a scrawl of nullity at least $k+1$ that is independent in M_{fin}.*
- *M contains a family $(C_x)_{x \in X}$ of infinite circuits such that $\bigcup_{x \in X} C_x$ is independent in M_{fin}, $|X| \geq k+1$ and $x \in C_y \Leftrightarrow x = y$.*

Similar statements can be made for M not being nearly finitary.

Lemma 1.47. *Let M be a matroid and let C be a scrawl of finite nullity which is independent in M_{fin}. Then there is for every finite set K a cobase of C which is disjoint from K.*

Proof. Let B be a cobase of C such that the intersection of B with K has minimal size. Assume for a contradiction that there is some edge $e \in B \cap K$. Then the unique circuit C' contained in $(C \backslash B) + e$ is infinite and thus contains an edge f which is not contained in K. So $B - e + f$ is a cobase of C which contains less elements of K, a contradiction. □

Theorem 1.48. [**28**] *Let M be a cofinitary matroid. If M is nearly finitary then it is also k-nearly finitary for some $k \in \mathbb{N}$.*

Lemma 1.49. *Any matroid in which a component of its finitarisation has infinite connectivity is not nearly finitary.*

Proof. Let M be a matroid and K a component of M_{fin} such that $\lambda_M(K) = \infty$. Let B_1 be a base of M_{fin}. As K is a component of M_{fin}, the sets $B_1 \backslash K$ and $B_1 \cap K$ are bases of $M \backslash K$ and $M \restriction K$ respectively. Thus there are subsets B_2 and B_3 of B which are bases of M such that $B_2 \backslash K$ is a base of $M \backslash K$ and $B_3 \cap K$ is a base of $M \restriction K$. As $\lambda_M(K) = \infty$, the set $B_3 \backslash B_2$ is infinite. As $B_3 \backslash B_2$ is contained in $B_1 \backslash B_2$, the latter set is infinite aswell. So B_1 and B_2 witness that M is not nearly finitary. □

Tree decompositions (T, τ) of matroids are introduced in [**32**] and the separations induced by the edges of T are defined in [**24**].

Definition 1.50. A tree-decomposition of a matroid M on ground set E is a tree T together with a map $\tau : E \to V(T)$ such that to every leaf some element of the ground set is mapped. Deleting a directed edge from T yields two subtrees. If X is the set of edges mapped to nodes of the subtree containing the tail of the edge, and similarly Y is the set of edges mapped to nodes of the subtree containing the head of the edge, then (X, Y) is the ordered bipartition[1] *induced by* the oriented edge, and $\{X, Y\}$ is the unoriented bipartition induced by the unoriented version

[1]This actually is a separation. Oriented and unoriented separations are introduced in Section 1.5.

of the same edge. Similarly, if $v \in V(T)$ is a node of T such that $\tau^{-1}(v) = \emptyset$, then the trees $T - v$ induce a partition (with possibly empty partition classes) of E.

1.3.1. Psi-matroids. The formal definition of a Ψ-matroid M with finite matroids at the nodes is quite complex. Intuitively, what is important for Chapter 2 is the following: There is a tree set (definition see Definition 1.73) of separations of M which cuts up the matroid into finite parts, and if $(P_i, Q_i)_{i \in \mathbb{N}}$ is a chain of the tree set then either for every circuit C there is $i \in \mathbb{N}$ with $C \subseteq P_i$ or for every cocircuit D there is $i \in \mathbb{N}$ with $D \subseteq P_i$.

Definition 1.51. [**7**, Section 5] Let k be a field and E a set. For a subspace V of k^E denote the set of all supports of elements of V by $S(V)$. Two subspaces V and W of k^E are *orthogonal* if for all $v \in V$ and all $w \in W$ the intersection of the two supports is finite and $\sum_{e \in E} v(e)w(e) = \emptyset$. A *presentation* Π on E is a pair (V, W) of orthogonal subspaces of k^E such that $S(V)$ and $S(W)$ satisfy the following: For every partition $E = P \cup Q \cup \{e\}$ either there is $S \in S(V)$ with $e \in S \subseteq P + e$ or there is $S \in S(W)$ with $e \in S \subseteq Q + e$. Π *presents* M if the circuits of M are the minimal non-empty elements of $S(V)$ and the cocircuits of M are the minimal non-empty elements of $S(W)$.

Definition 1.52. [**7**, Definition 6.1] A *tree of presentations* $\mathcal{T}$ consists of a tree $\mathcal{T}$, together with functions $\overline{V}, \overline{W}$ assigning to each node t of T a presentation $\Pi(t) = (\overline{V}(t), \overline{W}(t))$ on the ground set $E(t)$, such that for any two nodes t and t' of T, $E(t) \cap E(t')$ is finite and if $E(t) \cap E(t')$ is non-empty then tt' is an edge of T. For any edge tt' the set $E(tt')$ is defined to be $E(t) \cap E(t')$, and the *ground set* $E(\mathcal{T})$ of $\mathcal{T}$ is $\left(\bigcup_{t \in V(T)} E(t)\right) \setminus \left(\bigcup_{tt' \in E(T)} E(tt')\right)$.

Definition 1.53. [**7**, Definition 6.2] Let $\mathcal{T} = (T, \overline{V}, \overline{W})$ be a tree of presentations and Ψ a set of ends of T. A pair $(S, \overline{v})$ is a Ψ*-pre-vector* if

- S is a subtree of T whose ends are contained in Ψ
- $\overline{v}$ maps every node t of S to some element of $\overline{V}(t)$
- If st is an edge of S then $\overline{v}(s)$ and $\overline{v}(t)$ agree on $E(st)$ and are non-zero on $E(st)$
- If st is an edge from a node of S to a node of T not in S then $\overline{v}(s)$ maps all elements of $E(st)$ to 0

Then every Ψ-pre-vector $(S, \overline{v})$ defines an *underlying vector* in $k^{E(\mathcal{T})}$ which maps e to $\overline{v}(t)(e)$ if there is a node t of S such that $e \in E(t)$ and maps e to 0 otherwise. The vectorspace of all linear combinations of underlying vectors is denoted by $V_\Psi(\mathcal{T})$. Similarly, a Ψ^C*-pre-covector* is a pair $(S, \overline{w})$ where

- S is a subtree of T whose ends are not contained in Ψ
- $\overline{w}$ maps every node t of S to some element of $\overline{W}(t)$
- If st is an edge of S then $\overline{w}(s)$ and $-\overline{w}(t)$ agree on $E(st)$ and are not identically zero on $E(st)$
- If st is an edge from a node of S to a node of T not in S then $\overline{w}(s)$ maps all elements of $E(st)$ to 0

Every pre-covector $(S, \overline{w})$ defines an *underlying covector* as above, and the vectorspace of all linear combinations of underlying covectors is denoted by $W_\Psi(\mathcal{T})$. Then $(V_\Psi(\mathcal{T}), W_\Psi(\mathcal{T}))$ is a presentation $\Pi_\Psi(\mathcal{T})$, and if it presents a matroid M then M is a Ψ*-matroid.*

In the situation of the last definition: If $E(t)$ is finite for every node t of T, then M is *a Ψ-matroid with finite parts*.

Remark 1.54. Remark 6.3 from [**7**] implies that any minor of a Ψ-matroid is again a Ψ-matroid. It also implies that if a matroid is presented by $\Pi_\Psi(\mathcal{T})$ for some tree of presentations $\mathcal{T}$ in which every $E(t)$ is finite, then all minors of M also have this property.

In a tree of presentations $\mathcal{T}$ with underlying tree T, every element of the ground set is assigned to a unique node of T, so every edge of T induces a bipartition of the ground set. Call these bipartitions the (unoriented) *tree-induced* separations[2] of $\mathcal{T}$. If $\mathcal{T}$ induces a matroid M, then the *tree-induced* separations of M are those of $\mathcal{T}$.

Lemma 1.55. *If a tree of presentations induces a matroid M, then $\lambda(P)$ is finite for all tree-induced separations $\{P, Q\}$.*

Proof. Let M be a matroid which is induced by a tree of presentations $\mathcal{T}$ with underlying tree T. Let $\{P, Q\}$ be a tree-induced separation of M which arises from an edge st of T. Let B be a base of $M \restriction P$ and B' a base of $M.P$. Then for each $f$in $B \backslash B'$ the fundamental circuit of f in B' with respect to $M.P$ can be extended to a circuit C_f of M by adding edges of Q. There is a linear combination $\sum_{1 \le i \le m_f} \lambda_i v_{fi}$ of underlying vectors of pre-vectors $(S_{fi}, \overline{v}_{fi})$ such that C_f is the support of the linear combination. Define w_f to be the projection of $\sum_{1 \le i \le m_f} \lambda_{fi} \overline{v}_{fi}(s)$ to $E(st)$.

In order to show that $(w_f)_{f \in B \backslash B'}$ is linearly independent, let $\sum_{f \in F} \mu_f w_f = 0$ be a linear combination where $F \subseteq B \backslash B'$ is finite. Delete those edges $s't'$ where the projection of $\sum_{f \in F} \sum_{1 \le i \le m_f} \mu_f \overline{v}_{fi}(s')$ to $E(s't')$ is zero from the forest $\bigcup_{f \in F, 1 \le i \le m_f} S_{fi}$ and let S be the resulting component containing s. Then S does not contain the edge st. Now $\left(S, \sum_{f \in F, 1 \le i \le m_f} \mu_f \lambda_{fi} \overline{v}_{fi}\right)$ is a pre-vector and the support of its underlying vector v is contained in B. As the support of v is a scrawl of M and B is independent, $v = 0$. In particular for every $g \in F$

$$v(g) = \sum_{f \in F} \mu_f v_f(g) = \mu_g v_g(g)$$

and thus $\mu_g = 0$ because $v_g(g) \neq 0$. So $(w_f)_{f \in B \backslash B'}$ is linearly independent. Because the vectors w_f are vectors in a vector space of dimension $E(st)$, the set $B \backslash B'$ has at most $|E(st)|$ many elements and thus is finite. $\square$

1.4. Graph-like spaces

Definitions about graph-like spaces are mainly taken from [**8**].

Definition 1.56. [**8**, Definitions 3.1, 3.3] A *graph-like space* G is a topological space (also denoted G) together with a vertex set $V = V(G)$, an edge set $E = E(G)$ and for each $e \in E$ a continuous map $\iota_e^G : [0, 1] \to G$ (the superscript may be omitted if G is clear from the context) such that:

- the underlying set of G is $V \sqcup [(0, 1) \times E]$;
- $\iota_e(x) = (x, e)$ for any $x \in]0, 1[$ and $e \in E$;
- ι_e and $\iota_e(1)$ are vertices (called the *endvertices* of e);
- $\iota_e \restriction]0, 1[$ is an open map; and

[2] Oriented and unoriented separations are introduced in Section 1.5.

- for any two distinct $v, v' \in V$ there are disjoint open subsets U, U' of G partitioning $V(G)$ and with $v \in U$ and $v' \in U'$.

The *inner points of the edge* e are the elements of $]0,1[\times\{e\}$.

Definition 1.57. [**8**, Section 3] Let G be a graph-like space and $R \subseteq E(G)$. The *restriction* of G to R, denoted $G\restriction R$, is the graph-like space whose point set is the set of vertices of G together with inner points of edges in R, and whose topology is the subspace topology. The vertex set of $G\restriction R$ is the same as the vertex set of G, the set of edges is R and for an edge e in R the map ι_e is the same as for G. The *deletion* of R, denoted $G\backslash R$, is the restriction of G to $E\backslash R$.

Definition 1.58. [**8**] For an edge set F, the topological closure of the set $]0,1[\times F$ is denoted by $\overline{F}$ and called the *standard subspace* of F in G.

Definition 1.59. [**8**, Definition 3.3] For two disjoint open sets U and W such that $U \cap V$ and $W \cap V$ partition V, the set of edges with one end vertex in U and one in W is a *topological cut* of G.

Definition 1.60 (Contraction)**.** [**8**, Section 3] Let G be a graph-like space and $C \subseteq E(G)$. Let $\sim$ be the equivalence relation on $V(G)$ where $u \sim v$ if every topological cut arising from open sets containing exactly one of u and v each meets C. The *contraction* of C, denoted G/C, is a graph-like space whose point set is the set of equivalence classes of $\sim$ together with the set of inner points of edges not in C. For an edge $e \notin C$, its new end vertices are the equivalence classes of its end vertices in G. There is a map of graph-like spaces f_C from G to G/C which maps every vertex to its equivalence class, every inner point of an edge not in C to itself and every inner point of an edge in C to the equivalence class of its end vertices. Via this map, the point set of G/C is a quotient of the point set of G, and the topology of G/C is the quotient topology. The *contraction of* G *onto* C, denoted $G.C$, is $G/(E(G)\backslash C)$.

Definition 1.61. [**8**, Definition 4.1] A *pseudo-line* is a graph-like space whose edge set P is totally ordered, whose vertex set is the set of initial segments of P and whose topology has $\{S(p,r)^+ : p \in P, r \in (0,1)\} \cup \{S(p,r)^- : p \in P, r \in (0,1)\}$ as a subbasis, where $S(p,r)$ consists of the vertices not containing p, inner points of edges which are smaller than p and $(0,r) \times p$ and $S(p,r)^-$ is defined accordingly.

Definition 1.62. [**8**, section 3] A *map of graph-like spaces* is a continuous map ϕ from the point set of some graph-like space G to the point set of a graph-like G' such that there are maps $\phi_V : V(G) \to V(G')$ and $\phi_E : E(G) \to E(G') \times \{+,-\} \cup V(G')$ with the property that $\phi(x))\phi_V(x)$ if $x \in V(G)$, $\phi(x) = \phi_E(e)$ if $x = (e,r)$ and $\phi_E(e)$ is a vertex, $\phi(x) = (f,r)$ if $x = (e,r)$ and $\phi_E(e) = (f,+)$, and $\phi(x) = (f, 1-r)$ if $x = (e,r)$ and $\phi_E(e) = (f,-)$.

As is usual with arcs, in [**8**] there are two objects called pseudo-arcs: injective maps of graph-like spaces from a pseudo-line to a graph-like space and images thereof. In this thesis only the second type of *pseudo-arcs* is relevant: subspaces of G for which there is an injective map of graph-like spaces which has a pseudo-line as domain and the subspace as image. Every pseudo-arc P inherits a linear order $<_P$ from its pseudo-line and has a smallest vertex $\mathrm{in}(P)$ and a biggest vertex $\mathrm{ter}(P)$. In [**8**] $\mathrm{in}(L)$ and $\mathrm{ter}(L)$ are called start-vertex and end-vertex, however, in order to avoid confusion, here they are called *initial vertex* and *terminal vertex* or *first* *vertex* and *last vertex* and the term *endvertices* is used to refer to both the initial and terminal vertex of a pseudo-arc. A *pseudo-circle* arises from a pseudo-line by identifying the smallest and biggest vertex. Graph-like spaces have two structures at the same time: on one hand they are topological spaces, on the other hand they are a collection of edges and vertices, just as a graph. Thus also every pseudo-arc and every pseudo-circle is a set of edges and vertices.

As a graph-like space is a topological space, it has (topological) components and arc-connected components. Additionally, in analogy to the arc-connected components, pseudo-arc connected components can be defined.

Lemma 1.63. [**8**, Corollary 4.7] *If F is the edge set of a pseudo-arc A, then $A = \overline{F}$ where A is considered as a point set in G.*

Lemma 1.64. [**8**, Corollary from Lemma 4.13] *Let X be a closed subset of the point set of a pseudo-arc A. Then in the order induced by A, X has a smallest and a biggest element.*

Lemma 1.65. [**8**, Lemma 4.12] *Let G be a graph-like space and A and B two pseudo-arcs of G such that $\mathrm{ter}(A) = \mathrm{in}(B)$. Then there is a pseudo-arc C which is a subset of $A \cup B$ (as point set and thus also as collection of edges and vertices of G) such that $\mathrm{in}(C) = \mathrm{in}(A)$ and $\mathrm{ter}(C) = \mathrm{ter}(B)$.*

The following terminology is defined for pseudo-arcs in analogy to paths in graphs: For a vertex v, a *v-fan* in G is a family of pseudo-arcs which start in v and are otherwise disjoint, and a *v-in-fan* is a family of pseudo-arcs whose terminal vertex is v and which are disjoint otherwise. A non-trivial pseudo-arc is a pseudo-arc with more than one vertex. Two pseudo-arcs are internally disjoint if their sets of edges are disjoint and if the intersection of their vertex sets contains only vertices which are end vertices of both pseudo-arcs.

Lemma 1.66. *Let G be a graph-like space inducing a matroid M and V' a finite vertex set. Let F be the edge set of a union of pseudo-arc components of $G - V'$. Then F has finite connectivity.*

Proof. Assume for a contradiction that $\lambda(F) > |V'|$. Then there is a set $X \subseteq F$ of size at least $|V'|$ and a family of pseudo-circles $(C_x)_{x\in X}$ such that $\bigcup_{x\in X} C_x \cap F$ is independent, $x \in C_x$ for all $x \in X$ and $x \notin C_{x'}$ for all $x \neq x' \in X$. Define an auxiliary multigraph H on vertex set V' as follows: For every $x \in X$ there is a shortest pseudo-arc in C_x with end vertices in V', connect these two endvertices in H with an edge. Then H has at least as many edges as vertices, so it contains a circle. This circle can be translated into a subset of $\bigcup_{x\in X} C_x \cap F$ containing a pseudo-circle, contradicting the fact that $\bigcup_{x\in X} C_x \cap F$ is independent. □

Lemma 1.67. *Let G be a graph-like space inducing a matroid M and X an edge set of finite connectivity. Then there is an edge set $C \subseteq X$ such that $M/X = M/C\backslash(X\backslash C)$ and such that $\overline{C}$ is compact.*

Proof. By Lemma 1.43 there is a union C' of finitely many circuits such that $M/(C' \cap X)\backslash(X\backslash C') = M/X$. Let $C = C' \cap X$. For every pseudo-circle C'' the set $\overline{C''}$ is compact. As C' is a finite union of circuits, the topological closure of $]0, 1] \times C'$ is a finite union of pseudo-circles and thus compact. So $\overline{C}$, which is a closed subset of a compact set, is compact as well. □

Lemma 1.68. *Let G be a graph-like space inducing a matroid and X an edge set such that $\overline{X}$ is compact. Then every vertex which gets identified with another vertex under the projection $G \to G/X$ is contained in $\overline{X}$.*

Proof. Let v be a vertex which is not contained in $\overline{X}$. For every vertex w in $\overline{X}$ there are disjoint open subsets U_w and W_w of G partitioning $V(G)$ such that $v \in U_w$ and $w \in W_w$. As $\overline{X}$ is compact, also $V(G) \cap \overline{X}$ is compact, and the sets W_w form an open cover of $V(G) \cap \overline{X}$. Thus there is a finite set V_X of vertices in $\overline{X}$ such that $\bigcup_{w \in V_X} W_w$ contains $V(G) \cap \overline{X}$. Denote $\bigcap_{w \in V_X} U_w$ by U and $\bigcup_{w \in V_X} W_w$ by W. Then U and W are disjoint open sets partitioning $V(G)$ such that $v \in U$ and $\overline{X} \cap V(G) \subseteq W$. In particular every edge in X has both its end vertices in W, and the topological cut induced by V and W contains no edges of X. So the topological cut induced by V and W witnesses that v is not identified with any vertex in $\overline{X}$ under the contraction of X.

Let u be a vertex not contained in $\overline{X}$. Let U' and W' be disjoint open sets partitioning $V(G)$ such that $v \in U'$ and $u \in W'$. Let $U'' = U \cap U'$ and $W'' = W \cup W'$. Then U'' and W'' are disjoint open sets partitioning $V(G)$ such that $v \in U''$ and $\overline{X} + u \subseteq W''$. In particular the topological cut induced by U'' and W'' does not contain edges of X, so it witnesses that v and u are not identified under the contraction of X. □

The idea for the following lemma is taken from [**28**, Lemma 3.4], which in turn does similar things to the proof of Proposition 1.4 of [**3**].

Lemma 1.69. *Let G be a graph-like space which induces a matroid M and let $v_1, \ldots, v_n$ be finitely many vertices of G. Also let C be a scrawl with cobase I such that no edge in I has some v_i as an end vertex and such that all pseudo-circles whose edge set is contained in C contain also contain some v_j. Then there is a set $(\mathcal{P}_j)_{1 \leq j \leq n}$ of v_j-in-fans such that*

- *No $\mathcal{P}_j$ contains a trivial pseudo-arc;*
- *The edge set of every pseudo-arc contained in one of the $\mathcal{P}_j$ is contained in $C \backslash I$; and*
- *The sum of the cardinalities of the $\mathcal{P}_j$ is at least $|I| + 1$.*

Proof. For each $i \in I$ let C_i be a pseudo-circle whose edge set is a circuit contained in $(C \backslash I) + i$. Then C_i contains two pseudo-arcs P_i^1 and P_i^2 whose first vertex is an end vertex of i, whose last vertex is some v_j and which do not contain further vertices v_l. Because no edge i has a v_j as an end vertex, all those pseudo-arcs are non-trivial. Let two pseudo-arcs P_i^x and P_j^y be related if they share a vertex which is not some v_j, and extend this relation to an equivalence relation. Let H be an auxiliary multi-graph on the set of equivalence classes with edge set I where the end vertices of $i \in I$ are the equivalence classes of P_i^1 and P_i^2. If H contains a finite circle, then there is a pseudo-circle whose edges are contained in C and which does not contain any of the vertices v_j. Thus H is a forest and the number of equivalence classes is at least $|I| + 1$. Let $\mathcal{P}$ be a set of representatives of the equivalence classes. If two distinct elements of $\mathcal{P}$ meet in some vertex, then that vertex is some v_j. So $\mathcal{P}$ can be organised into v_j-in-fans as required for this lemma. □

1.5. Separation systems

Definition 1.70. [**20**] A *separation system* $(\vec{S}, \leq, {}^*)$ is a set $\vec{S}$ together with a partial order $\leq$ and an involution * which is order-reversing, i.e. $\vec{s} \leq \vec{t} \Leftrightarrow \vec{s}^* \geq \vec{t}^*$ for all elements $\vec{s}$ and $\vec{t}$ of $\vec{S}$. For an element $\vec{s}$ of $\vec{S}$, $\vec{s}^*$ is also denoted as $\overleftarrow{s}$ and called the *inverse* of $\vec{s}$. The *orientations* of $\vec{s}$ are $\vec{s}$ and $\overleftarrow{s}$.

A *universe* $(\vec{S}, \leq, {}^*, \vee, \wedge)$ is a separation system in which all elements $\vec{s}$ and $\vec{t}$ of $\vec{S}$ have a supremum $\vec{s} \vee \vec{t}$ and an infimum $\vec{s} \wedge \vec{t}$. A universe is *submodular* if it comes with an order function $|\cdot|$, i.e. a symmetric submodular function with values in the non-negative integers together with ∞. For any non-negative integer k, the set of all separations in $\mathcal{U}$ of order less than k forms, together with the partial order and involution inherited from $\mathcal{U}$, a separation system S_k.

Note that this thesis' definition of a submodular universe differs from the definition in [**20**] in that the order function is not only allowed to have integer values but additionally can take the value infinity. It is important that arbitrary positive reals are not allowed: this property ensures that if for some $k \in \mathbb{N}$ two separations whose infimum and supremum exist have order less than k, then their infimum having order at least $k-1$ implies their supremum having order less than k as well. The latter fact is used frequently, e.g. in Lemma 4.33. As opposed to [**20**], this thesis' emphasis lies with infinite separation systems, and some come with a natural order function which does take the value ∞, for example the connectivity function from Example 1.76 or the connectivity function of an infinite matroid. Depending on the context, the value ∞ can be avoided by taking the subuniverse of separations of finite order.

Example 1.71. Assume that $\mathcal{U}$ is a subuniverse of a universe $\mathcal{U}'$ and that the order function of $\mathcal{U}$ is denoted by σ. Then σ can be extended to an order function σ' of $\mathcal{U}'$ by letting $\sigma'(\vec{s}) = \sigma(\vec{s})$ if $\vec{s}$ is contained in $\mathcal{U}$ and $\sigma'(\vec{s}) = \infty$ otherwise. In particular, σ' is submodular and symmetric. If additionally σ is limit-closed and chains of separations in $\mathcal{U}$ of bounded order have the same supremum in $\mathcal{U}'$ as they have in $\mathcal{U}$, then σ' is limit-closed, too.

On the other hand, the set of separations in $\mathcal{U}$ which have finite order is closed under joins and suprema and thus is the set of separations of a subuniverse $\mathcal{U}'$ of $\mathcal{U}$. If the order function of $\mathcal{U}$ is limit-closed, then so is the induced order function of $\mathcal{U}'$, which additionally does not contain ∞ in its image.

Definition 1.72. [**20**] Let $\vec{S}$ be a separation system. An element $\vec{s}$ of $\vec{S}$ is *degenerate* if $\vec{s} = \overleftarrow{s}$. A separation $\vec{s}$ is *trivial* if there is a separation $\vec{t}$ in $\vec{S}$ such that $\vec{s} < \vec{t}$ and $\vec{s} < \overleftarrow{t}$. A separation $\vec{s}$ is *small* if $\vec{s} \leq \overleftarrow{s}$. The inverse of a small separation is *co-small* and the inverse of a trivial separation is *co-trivial*. A separation system is *essential* if it none of its elements are degenerate or trivial; and it is *regular* if none of its elements are small.

Note that a separation system $\vec{S}$ which is a subsystem of some other separation system $\vec{S}'$ may be essential while containing elements which are trivial in $\vec{S}'$: whether a separation is trivial or not depends on the existence of a witness of the triviality, and after the deletion of all such witnesses the separation is not trivial any more. On the other hand, being small or degenerate does not depend on the existence of a witness, and thus if an element is small (or degenerate) in some separation system, then it also is small (or degenerate) in all subsystems which still contain that element.

Definition 1.73. [**20**] Two elements of a separation system $\vec{S}$ are *nested* if they have orientations $\vec{s}$ and $\vec{t}$ such that $\vec{s} \leq \vec{t}$, and they *cross* if they are not nested. A separation $\vec{s}$ points towards a separation $\vec{t}$ if $\vec{s} \leq \vec{t}$ or $\vec{s} \leq \overleftarrow{t}$. A separation system is *nested* if its elements are pairwise nested, and a *tree set* if it is additionally essential. A set of separations is a *star* if its elements are non-degenerate and all distinct elements $\vec{s}$ and $\vec{t}$ satisfy $\vec{s} \leq \overleftarrow{t}$.

In [**20**], officially it is only defined when a separations points towards an unoriented separation (not introduced here). But the concepts of oriented and unoriented separations are closely related, and, as is also the case with other definitions involving separations, pointing towards an oriented separation and pointing towards an unoriented separation does not really make a difference.

Example 1.74. [**21**, Chapter 3] Let T be a tree. Let $\vec{E}$ be the set of orientations of edges of T. Define a partial order on $\vec{E}$ where an edge from a vertex x to a vertex y is strictly less than an edge from a vertex u to a vertex v if $\{x, y\} \neq \{u, v\}$ and the unique path in T from y to u avoids both x and v. Then $\vec{E}$, together with this partial order and the involution which maps every edge to its other orientation, is a tree set,the *edge tree set* of T.

Definition 1.75. [**9**] A *homomorphism* of two separation systems $(\vec{S}, \leq, ^*)$ and $(\vec{S}', \leq', ^{*\prime})$ is a map $\phi : \vec{S} \to \vec{S}'$ such that for all elements $\vec{s}$ and $\vec{t}$ of $\vec{S}$, $\phi(\vec{s}^*) = \phi(\vec{s})^*$ and $\vec{s} \leq \vec{t} \Rightarrow \phi(\vec{s}) \leq \phi(\vec{t})$. A *isomorphism* of separation systems $(\vec{S}, \leq, ^*)$ and $(\vec{S}', \leq', ^{*\prime})$ is a bijective homomorphism of separation systems whose inverse is also a homomorphism of separation systems.

A *homomorphims* of universes $(\vec{S}, \leq, ^*, \vee, \wedge)$ and $(\vec{S}', \leq', ^{*\prime}, \vee', \wedge')$ is a map $\phi : \vec{S} \to \vec{S}'$ such that for all elements $\vec{s}$ and $\vec{t}$ of $\vec{S}$, $\phi(\vec{s}^*) = \phi(\vec{s})^*$ and $\phi(\vec{s} \vee \vec{t}) = \phi(\vec{s}) \vee \phi(\vec{t})$[3]. An *isomorphism* of universes is a bijective homomorphism of universes[4].

In this thesis, only separation systems of the form S_k for some universe $\mathcal{U}$ are considered. Suprema and infima are always taken in the surrounding universe: So they are always defined, but not always contained in the separation system.

There are two main examples of submodular universes:

Example 1.76 (Definitions from [**20**] and notation from [**9**])**.** Let V be a set. A *separation of* V is a pair (A, B) such that $A \cup B = V$, and its *separator* is $A \cap B$. The relation $\leq$ where $(A, B) \leq (C, D)$ if and only if $A \subseteq C$ and $D \subseteq B$ is a partial order in which any two separations (A, B) and (C, D) have supremum $(A \cup C, B \cap D)$ and infimum $(A \cap C, B \cup D)$. If the involution mapping (A, B) to (B, A) is denoted by *, then the set of separations of V together with * and $\leq$ and the join and meet induced by $\leq$ is a universe of separations $\mathcal{U}(V)$. The only degenerate separation of $\mathcal{U}(V)$ is (V, V) and all small separations of $\mathcal{U}(V)$ are of the form (A, V).

The universe $\mathcal{U}(V)$ comes with a natural submodular function with values in $\mathbb{N} + \infty$: Let the order of a separation be the size of its separator if that is finite

[3] this automatically implies that $\phi(\vec{s} \wedge \vec{t}) = \phi(\vec{s}) \wedge \phi(\vec{t})$ and $\vec{s} \leq \vec{t} \Rightarrow \phi(\vec{s}) \leq \phi(\vec{t})$

[4] this already implies that the inverse is also a homomorphism of universes

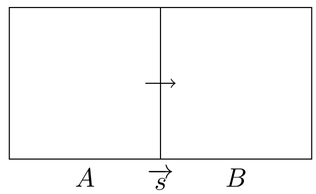

FIGURE 3. A separation $\overrightarrow{s} = (A, B)$ in $\mathcal{UB}(V)$ is depicted by the little arrow from A to B on the line separating A from B.

and ∞ otherwise. In this thesis, whenever submodular subuniverses of $\mathcal{U}(V)$ are considered then the order function is tacitly assumed to also be inherited from the natural order function of $\mathcal{U}(V)$.

In particular, the set of oriented separations of a graph is a subuniverse of $\mathcal{U}(V)$ where V is the set of vertices of G.

There are several possible ways to extend the notion of separation from a graph to a graph-like space. The one used in this thesis is the following, which also ensures that if G is a graph-like space on vertex set V then the set of oriented separations of G is a subset of $\mathcal{U}(V)$.

Definition 1.77. A separation of a graph-like space on vertex set V is an element (A, B) of $\mathcal{U}(V)$ such that every pseudo-arc which contains vertices of both A and B also contains a vertex of $A \cap B$.

Lemma 1.78. *Let V be a set, (A, V) a small separation in $\mathcal{U}(V)$ and (C, D) a separation of finite order. If (C, D) and $(C, D) \vee (A, V)$ have the same order, then they are equal.*

PROOF. The separation $(A, V) \vee (C, D)$ equals $(A \cup C, D)$ and thus has separator $(A \cup C) \cap D$. That $(A \cup C) \cap D$ and $C \cap D$ have the same size implies that they are equal. So

$$A \backslash C = (D \backslash C) \cap A = (D \cap A) \backslash C \subseteq (D \cap (A \cup C)) \backslash C = (C \cap D) \backslash C = \emptyset$$

and hence $(A, V) \vee (C, D) = (C, D)$. $\square$

Example 1.79 ([**20**] and [**9**])**.** Let V be a set. Let $\mathcal{UB}(V)$ be the subuniverse of $\mathcal{U}(V)$ which consists of all the separations (A, B) with $A \cap B = \emptyset$. There is only one small separation of $\mathcal{UB}(V)$, namely $(\emptyset, V)$, and no degenerate separation. Just on its own, $\mathcal{UB}(V)$ does not have a natural order function, but if V is e.g. the ground set of a matroid, then the connectivity function of that matroid is an example of an order function on $\mathcal{UB}(V)$. The universe $\mathcal{UB}(V)$ is isomorphic to the lattice of subsets of V via $(A, B) \mapsto A$. Via this isomorphism, all subsets of V are separations of $\mathcal{UB}(V)$.

Definition 1.80. [**24**] Let $(S, ^*, \leq)$ be a separation system. A subset O of S such that $O \cap \overrightarrow{s}, \overleftarrow{s}$ has exactly one element for all $\overrightarrow{s} \in S$ is an *orientation* of S. It is *consistent* if it does not contain two elements $\overleftarrow{s}$ and $\overrightarrow{t}$ which are not orientations of each other and $\overrightarrow{s} < \overrightarrow{t}$. A *profile* is a consistent orientation P with the property that for any two elements $\overrightarrow{s}$ and $\overrightarrow{t}$ the separation $(\overrightarrow{s} \vee \overrightarrow{t})^*$ is not contained in P (possibly because it does not exist in S). The latter property is also called the *profile property*. A profile is *regular* if it does not contain a co-small separation. Given a submodular universe $\mathcal{U}$, a k-profile P of $\mathcal{U}$ is a profile of the separation system S_k. Two profiles that contain distinct orientations of a separation $\vec{s}$ are *distinguished* by that separation, and a set S' of separations distinguishes a set $\mathcal{P}$ of profiles if any two distinct profiles in $\mathcal{P}$ are distinguished by some separation in S'.

Theorem 1.81. [**21**, Section 3, in particular Theorem 3.3 (i)] *Every finite regular tree set $\mathcal{E}$ is isomorphic via an isomorphism ϕ to the edge tree set of some tree whose vertices are the consistent orientations of $\mathcal{E}$. Given a consistent orientation O of $\mathcal{E}$, an orientation of an edge of the tree is contained in $\phi(O)$ if and only if it points towards the vertex O.*

Note that [**21**] contains the stronger statement that, given a finite regular tree set τ, there is a tree whose edge tree set *is* τ. This clashes with the convention that the edge set of a graph G on vertex set V consists of two-element subsets of V. Of course that problem can be easily circumvented by an appropriate definition of the edge set, but that is not necessary in this context.

Definition 1.82. [**29**] Let $\mathcal{U}$ be a submodular universe and k and l elements of $\mathbb{N}$ such that $l \leq k$. For a k-profile P, the *truncation of P to an l-profile* is the intersection of P with the separation system S_l. If $k \neq 0$, then the *truncation of P*, without further mention of a second integer l, is the truncation of P to a $k-1$-profile.

Definition 1.83. Let $k \in \mathbb{N}$ and let $\mathcal{U}$ be a submodular universe whose order function is limit-closed. A k-profile P of $\mathcal{U}$ is called *limit-closed* if for every chain of elements of P the supremum of that chain in $\mathcal{U}$, which exists and has order less than k, is also contained in P.

Definition 1.84. [**22**] A set of separations is called *strongly consistent* if it does not contain elements $\vec{r}$ and $\vec{s}$ with $\overleftarrow{r} < \vec{s}$. Thus, an orientation is strongly consistent if and only if for every $\vec{s} \in O$ and every separation $\vec{r}$ with $\vec{r} \leq \vec{s}$ also $\vec{r} \in O$.

Lemma 1.85. [**22**, Lemma 7] *An orientation of a separation system is strongly consistent if and only if it is consistent and contains all small separations.*

1.6. Finite flowers

1.6.1. In (poly-)matroids. This subsection contains definitions and facts about k-flowers from [**5**]. The setting of that paper is that of a *polymatroid* on a finite set E: an increasing submodular map $f : \mathcal{P}(E) \to \mathbb{N}$ such that $f(\emptyset) = 0$. The set f induces a connectivity function $\lambda_f(X) = f(X) + f(E \backslash X) - f(E)$ on subsets X of E and a local connectivity function $\sqcap_f(X, Y) = f(X) + f(Y) - f(X \cup Y)$ on disjoint subsets X and Y of E.

Definition 1.86. [**5**] Let f be a polymatroid on ground set E and $k \in \mathbb{N}$. A *k-flower* of M is a partition $(P_1, \dots, P_n)$ where all partition sets P_i have connectivity $k-1$ and the union of two adjacent partition sets also has connectivity $k-1$, where P_n is adjacent to P_1. For $I' \subseteq \{1, \dots, n\}$ denote the union of all partition sets with index in I' by $P_{I'}$. A k-flower is a *k-daisy* if the non-trivial subsets I' of $\{1, \dots, n\}$ with $\lambda(P_{I'}) = k-1$ are exactly those which are intervals of $\{1, \dots, n\}$. A k-flower is a *k-anemone* if $\lambda(P_{I'}) = k-1$ holds for all non-trivial subsets I' of $\{1, \dots, n\}$.

Lemma 1.87. [**5**] *Every k-flower is a k-daisy or a k-anemone.*

Lemma 1.88. [**5**, part of Theorem 1.3 and Lemma 3.6] *Let k be an integer, $k \geq 1$, and let $(P_1, \dots, P_n)$ be a k-flower with at least five petals. Denote $\sqcap(P_1, P_2)$ by c and $\sqcap(P_1, P_3)$ by d. Then the local connectivity of any two adjacent petals is the same as c and the local connectivity of any two non-adjacent petals is d. Also $(P_1, \dots, P_n)$ is a k-anemone if and only if $c = d$. Furthermore, the local connectivity of a non-trivial interval I' of I and a non-trivial subset I'' of $I \backslash I'$ is*

$$\sqcap_M(P_{I'}, P_{I''}) = \begin{cases} d & \textit{if no element of } I'' \textit{ is adjacent to an element of } I' \\ 2c - d & \textit{if two elements of } I'' \textit{ are adjacent to elements of } I' \\ c & \textit{otherwise} \end{cases}$$

1.6.2. In connectivity systems with a tangle. This subsection is about the k-flowers from [**17**]. The setting in [**17**] includes a connectivity system: a finite set E and a symmetric submodular integer-valued function defined on the power set of E. In that paper, k-flowers are defined with respect to a k-tangle T. Note that the definition defines k-flowers to have petals of connectivity k, whereas in [**5**] k-flowers have petals of connectivity $k - 1$. This is due to the fact that the papers define the connectivity function of a matroid in slightly different terms: In [**5**], the connectivity function λ of a matroid is defined as in this thesis, while in [**17**] the connectivity of a set X is defined to be $\lambda(X) + 1$. So if some partition of a matroid is a k-flower in the sense of [**5**] and an l-flower with respect to an l-tangle T in the sense of [**17**], then $k = l$.

Definition 1.89. [**17**] Let (E, λ) be a connectivity system, k an element of $\mathbb{N}$ and T a k-tangle. A k-flower in T is a partition $(P_1, \dots, P_n)$, whose elements are the *petals* of the flower, such that every petal and every union of two adjacent petals has connectivity k and such that no petal is contained in an element of T. A k-flower is a k-*daisy* if the non-trivial sets I' with $\lambda(P_{I'}) = k$ are exactly those which are an interval of I, and a k-*anemone* if all non-trivial unions of petals have connectivity k.

Lemma 1.90. [**17**] *Every k-flower is a k-daisy or a k-anemone.*

Part 1

Nearly finitary matroids

Introduction to nearly finitary matroids

Because of their similarity to finite matroids, finitary matroids are a class of infinite matroids which are particularly easy to deal with. For example, some results on finite matroids can be extended to finitary matroids simply by compactness. For every infinite matroid M, its set of finite circuits is the set of circuits of a finitary matroid M_{fin}, its finitarisation. Clearly every independent set of M is also independent in the finitarisation, so every base of M can be extended to a base of M_{fin}. A matroid can be very close to its finitarisation, in other words nearly finitary, in that every base can be extended to a base of the finitarisation by only adding finitely many elements of the ground set. There may even be an integer $k \in \mathbb{N}$ such that every base can be extended to a base of the finitarisation by adding at most k many elements of the ground set. Matroids with the latter property are called k-nearly finitary.

It is shown in [**2**] that for two nearly finitary matroids on the same ground set the matroid union exists and is nearly finitary. In order to be able to apply this result, in [**3**] it is shown that if M is the algebraic cycle matroid of a graph G or the topological-cycle matroid of a locally finite 2-connected graph G then M is nearly finitary if and only if G does not have an infinite set of vertex-disjoint rays. Here, Halin's theorem (see e.g. [**18**]) that every graph containing, for every $k \in \mathbb{N}$, a set of k many pairwise disjoint rays also contains an infinite set of pairwise disjoint rays can be applied. Thus in these two cases, M happens to be nearly finitary if and only if there is some $k \in \mathbb{N}$ such that M is k-nearly finitary. The statement that an algebraic cycle matroid is nearly finitary if and only if it is k-nearly finitary for some $k \in \mathbb{N}$ is equivalent to Halin's theorem, and is thus (and also because rays are hard to define in matroids) a generalisation of Halin's theorem. The authors of [**3**] then ask whether all nearly finitary matroids are also k-nearly finitary for some $k \in \mathbb{N}$.

Conjecture. [**2**] *Given a nearly finitary matroid M, is there some $k \in \mathbb{N}$ such that M is also k-nearly finitary?*

This question is about a phenomenon which frequently arises when dealing with infinite structures: Does the presence of arbitrarily large finite families of something also imply the presence of an infinite family?

So far, no counterexample to the conjecture is known. In [**3**] the conjecture was shown for algebraic cycle matroids and for nearly finitary topological cycle matroids of locally finite graphs. In [**28**] the conjecture was shown for cofinitary matroids. This thesis shows the conjecture for two new classes of matroids. The first class is that of Ψ-matroids with finite parts in the sense of [**6**].

Theorem (Lemma 2.14)**.** *Every nearly finitary Ψ-matroid with finite parts is k-nearly finitary for some $k \in \mathbb{N}$.*

The second class is a subclass of the class of matroids arising from a graph-like space in the sense of [**8**].

Theorem (Theorem 3.24)**.** *Let $G = (V, E)$ be a graph-like space inducing a matroid M such that there is a finite vertex set which meets all infinite pseudo-circles. If M is nearly finitary then it is k-nearly finitary for some $k \in \mathbb{N}$.*

At a first glance, it is not quite clear why this theorem should apply to a wide range of matroids. But there is a much more technical condition on graph-like spaces, such that problem of proving the conjecture for matroids arising from such graph-like spaces can be reduced to the problem of proving the conjecture for graph-like spaces in which some finite vertex set meets all infinite pseudo-circles. This technical condition is a generalisation of the following property of graph-like spaces, reminiscent of a separability condition.

Theorem (Corollary 3.25)**.** *Let $G = (V, E)$ be a graph-like space inducing a matroid M in which for all distinct vertices $v, w \in V$ there is a finite vertex set $V' \subseteq V - v - w$ such that v and w are contained in different topological components of $G - V' - E_{vw}$, where E_{vw} is the set of edges from v to w. If M is nearly finitary, then there is $k \in \mathbb{N}$ such that M is k-nearly finitary.*

Both proofs depend on a thorough analysis of the effect the deletion or contraction of a set of finite connectivity has on a matroid being (k-)nearly finitary, to be found in Section 2.1, and a reduction to the case that for every set of finite connectivity either the deletion of this set or the restriction to this set is finitary, to be found in Section 2.2. The proof for matroids arising from certain graph-like spaces also relies on a simplification of pseudo-arcs in Section 3.1. This simplification is applied in circumstances where it is necessary to trace how a pseudo-arc passes though the different components of the finitarisation of a matroid, and the exact behaviour inside such a component is irrelevant. The word simplification here refers to the fact that, as far as possible, the behaviour inside a component of the finitarisation is ignored.

CHAPTER 2

Nearly finitary Ψ-matroids

2.1. Deletion or contraction of sets of finite connectivity

This section presents several results on how the property of a matroid being (k-)nearly finitary interacts with the deletion or contraction of sets of finite connectivity. These results can be phrased a lot more concisely with the following definition:

Definition 2.1. For a matroid M let $\mathcal{F}(M)$ be the collection of all $F \subseteq E$ such that there are bases B of M and B_{fin} of M_{fin} with $B \subseteq B_{\text{fin}}$ and $F \subseteq B_{\text{fin}} \backslash B$.

Lemma 2.2. *Let M be a matroid, $F \in \mathcal{F}(M)$ and (X, Y) a separation of M of connectivity $k \in \mathbb{N}$. Then there is a set $G \subseteq F$ of size at most k such that $(F \cap X) \backslash G \in \mathcal{F}(M/Y)$.*

Proof. Let B and B_{fin} be bases of M and M_{fin} respectively such that $B \subseteq B_{\text{fin}}$ and $F \subseteq B_{\text{fin}} \backslash B$. Let $B_Y \in \mathcal{B}(M \restriction Y)$, $B'_Y \in \mathcal{B}(M.Y)$ such that $B'_Y \subseteq B_Y$ and define $N = M/B'_Y \backslash (Y \backslash B_Y)$, $Y' = B_Y \backslash B'_Y$. Then $M \backslash Y = N \backslash Y'$ and $M/Y = N/Y'$. Let $G' \subseteq B \backslash Y$ such that G' is spanned by $B \backslash (Y \cup G')$ in $(M/Y)_{\text{fin}}$ and let $G \subseteq B_{\text{fin}} \backslash Y$ be a set containing G' which is spanned by $B_{\text{fin}} \backslash (Y \cup G)$ in $(M/Y)_{\text{fin}}$. As $B_{\text{fin}} \backslash Y$ does not contain finite circuits in $M \backslash Y = N \backslash Y'$ but Y' only contains k edges, G also contains at most k edges. So there are a maximal such set G' and a maximal such set G containing this G'. Additionally, G can be chosen such that $G \cap B = G'$. Then G contains at most k edges. Furthermore $B_{\text{fin}} \backslash (Y \cup G)$ is a base of $B_{\text{fin}} \backslash Y$ in $(M/Y)_{\text{fin}}$, so there is a base B'_{fin} of $(M/Y)_{\text{fin}}$ containing $B_{\text{fin}} \backslash (Y \cup G)$. Because $B \backslash (Y \cup G) = B \backslash (Y \cup G')$ is spanning in M/Y, there is a base B' of M/Y which is contained in $B \backslash (Y \cup G)$. Then

$$B' \subseteq B \backslash (Y \cup G) \subseteq B_{\text{fin}} \backslash (Y \cup G) \subseteq B'_{\text{fin}}$$

and every edge of $B_{\text{fin}} \backslash (Y \cup B \cup G)$ is contained in $B'_{\text{fin}} \backslash B'$. $\square$

Corollary 2.3.
- *Let M be a matroid and (X, Y) a separation of finite connectivity of M. If both M/X and M/Y are k-nearly finitary for some $k \in \mathbb{N}$, then also M is k-nearly finitary for some (possibly different) $k \in \mathbb{N}$.*
- *Let M be a matroid and (X, Y) a separation of M of finite connectivity. If both M/X and M/Y are nearly finitary, then M itself is also nearly finitary.* $\square$

Lemma 2.4. *Let M be a matroid, $F \in \mathcal{F}(M)$ and (X, Y) a separation of M of connectivity $k \in \mathbb{N}$. Then there is a set $G \subseteq F$ of size at most k such that $(F \cap X) \backslash G \in \mathcal{F}(M \backslash Y)$.*

Proof. Let B and B_{fin} be bases of M and M_{fin} respectively such that $B \subseteq B_{\text{fin}}$ and $F \subseteq B_{\text{fin}} \backslash B$. Let B'_{fin} be a base of $(M{\restriction}X)_{\text{fin}}$ containing $B_{\text{fin}} \cap X$. Then B'_{fin} is spanning in $M{\restriction}X$, so there is a base B' of $M{\restriction}X$ such that $B \cap X \subseteq B' \subseteq B'_{\text{fin}}$. Because of

$$\kappa_{M\backslash(X\backslash B')}(B') = \kappa_{M\backslash(X\backslash B')}(Y) \leq \kappa_M(Y) = k,$$

the set $B'\backslash(B \cap X) = B'\backslash B$ contains at most k elements. Let $G = (B'\backslash B) \cap B_{\text{fin}}$, then G has also at most k elements and $(F \cap X)\backslash G \subseteq B'_{\text{fin}}\backslash B'$. □

Corollary 2.5.

- *Let M be a matroid and (X, Y) a separation of M of finite connectivity. If both $M\backslash X$ and $M\backslash Y$ are k-nearly finitary for some $k \in \mathbb{N}$, then also M is k-nearly finitary for some (possibly different) $k \in \mathbb{N}$.*
- *Let M be a matroid and (X, Y) a separation of M of finite connectivity. If both $M\backslash X$ and $M\backslash Y$ are nearly finitary, then M itself is also nearly finitary.* □

Lemma 2.6. *Let M be a matroid and (X, Y) be a separation of M. Then for all sets $F_X \in \mathcal{F}(M\backslash Y)$ and $F_Y \in \mathcal{F}(M/X)$ their union $F_X \cup F_Y$ is contained in $\mathcal{F}(M)$.*

Proof. Let $B_X \in \mathcal{B}(M\backslash Y)$, $B'_X \in \mathcal{B}((M\backslash Y)_{\text{fin}})$, $B_Y \in \mathcal{B}(M/X)$ and $B'_Y \in \mathcal{B}((M/X)_{\text{fin}})$ be bases such that $B_X \subseteq B'_X$, $B_Y \subseteq B'_Y$, $F_X \subseteq B'_X\backslash B_X$ and $F_Y \subseteq B'_Y\backslash B_Y$. Then $B_X \cup B_Y$ is a base of M. Assume for a contradiction that $B'_X \cup B'_Y$ contains a finite circuit C. Then C cannot be a subset of B'_X, hence $C \cap B'_Y$ is non-empty. But $C \cap B'_Y$ is a finite scrawl in $(M/X)_{\text{fin}}$ which is a subset of B'_Y, so it is empty. Hence there is no such circuit C and there is a base B_{fin} of M_{fin} which contains $B'_X \cup B'_Y$. Thus $B_X \cup B_Y$ and B_{fin} witness that $F_X \cup F_Y \in \mathcal{F}(M)$. □

Corollary 2.7. *Let N be a minor of M. Then $\mathcal{F}(N) \subseteq \mathcal{F}(M)$. In particular if N is not nearly finitary then M is not nearly finitary and if N is not k-nearly finitary for any $k \in \mathbb{N}$, then M is not k-nearly finitary for any $k \in \mathbb{N}$.* □

Corollary 2.8. *Let $X_1, \dots, X_n$ be sets of finite connectivity such that each $M{\restriction}X_i$ is k-nearly finitary for some $k \in \mathbb{N}$. Then the restriction of M to the union of the X_i is also k-nearly finitary for some $k \in \mathbb{N}$.*

Proof. The proof is by induction on n, and the case $n = 0$ is clear, so assume $n > 0$. Denote the union of all X_i with $i < n$ by X. Then by submodularity of the connectivity function both X and $X_n\backslash X$ have finite connectivity. Also by Corollary 2.7 the restriction of M to $X_n\backslash X$ is k-nearly finitary for some $k \in \mathbb{N}$, so by Corollary 2.5 the restriction of M to $X \cup X_n$ is k-nearly finitary for some $k \in \mathbb{N}$. □

Lemma 2.9. *Let M be a matroid and (X, Y) a separation of M of connectivity $k \in \mathbb{N}$. Then for all $F \in \mathcal{F}(M\backslash Y)$ there is a set G of size at most k such that $F\backslash G \in \mathcal{F}(M/Y)$. Similarly for all $F \in \mathcal{F}(M/Y)$ there is a set G of size at most k such that $F\backslash G \in \mathcal{F}(M\backslash Y)$.*

Proof. Let $F \in \mathcal{F}(M\backslash Y)$. Then by Corollary 2.7 F is an element of $\mathcal{F}(M)$ and by Lemma 2.2 there is a set G of size at most k such that $F\backslash G \in \mathcal{F}(M/Y)$. Similarly let $F' \in \mathcal{F}(M/Y)$. Then by Corollary 2.7 F is an element of $\mathcal{F}(M)$ and by Lemma 2.4 there is a set G' of size at most k such that $F'\backslash G' \in \mathcal{F}(M\backslash Y)$. □

Corollary 2.10. *Let M be a matroid and (X, Y) a separation of M of finite connectivity. Then there is $k \in \mathbb{N}$ such that M/X is k-nearly finitary if and only if there is $k \in \mathbb{N}$ such that $M \backslash X$ is k-nearly finitary. Furthermore M/X is nearly finitary if and only if $M \backslash X$ is nearly finitary.* □

Lemma 2.11. *Let M be a matroid, C an infinite circuit of M and (X, Y) a separation of M of finite connectivity. If $C \cap X$ is infinite, then it contains an infinite circuit of M/Y.*

Proof. Let B be a base of $C \cap X$ in $M \backslash Y$. Then $B = C \cap X$ or, if C is a subset of X, there is an edge e such that $B = C \cap X - e$. In particular B contains all but finitely many edges of $C \cap X$. Let B' be a base of $C \cap X$ in M/Y which is contained in B. Then

$$\kappa_{M \backslash (X \backslash C)}(X \cap C) = \kappa_{M \backslash (X \backslash C)}(Y) \leq \kappa_M(Y),$$

so B' contains all but finitely many edges of $C \cap X$. Furthermore $C \cap X$ is a scrawl of M/Y, so $C \cap X \subseteq \bigcup_{e \in B \backslash B'} C_e^{B'}$. As $B \backslash B'$ is finite and $C \cap X$ is infinite, one of the fundamental circuits $C_e^{B'}$ has to be infinite as well. □

2.2. Reduction to matroids with extra property

This section reduces the problem of which nearly finitary matroids are also k-nearly finitary for some $k \in \mathbb{N}$ to nearly finitary matroids with the following extra property:

(∗) For all separations (P, Q) with $\kappa(P) < \infty$ either $M \backslash P$ or $M \backslash Q$ is finitary.

Lemma 2.12. *Let M be a matroid. If there is a sequence $(P_i, Q_i)_{i \in \mathbb{N}}$ of separations of finite connectivity and infinite circuits C_i of $M/P_i \backslash Q_{i+1}$ such that $P_i \subseteq P_{i+1}$ then M is not nearly finitary.*

Proof. For each non-negative integer $i \in \mathbb{N}$ pick an edge $f_i \in C_i$ and a base B_i of $M/P_i \backslash Q_{i+1}$ containing $C_i - f_i$. Then every set $B_0 \cup B_1 \cup \cdots \cup B_{i-1}$ is a base of $M/P_0 \backslash Q_i$, so C_i can be extended to a circuit C_i' of $M/P_0 \backslash Q_{i+1}$ by adding edges from $B_0 \cup \cdots \cup B_{i-1}$. Assume for a contradiction that $B' := \bigcup_{i \in \mathbb{N}} C_i'$ contains a finite circuit C. Then C is a finite set contained in $\bigcup_{i \in \mathbb{N}}(B_i + f_i)$, so there is a smallest index $j \in \mathbb{N}$ such that $C \subseteq \bigcup_{i=0}^{j}(B_i + f_i)$. For this index j, $C \backslash P_j$ is a non-empty finite scrawl of $M/P_j \backslash Q_{j+1}$. But $C \backslash P_j$ is also a subset of $B_j + f_j$, so $C \backslash P_j$ has to be the fundamental circuit of f_j in $M/P_j \backslash Q_{j+1}$, a contradiction to the fact that C_j is that fundamental circuit and is infinite. So there is no such finite circuit C, hence there is a base B_{fin} of M_{fin} which contains B'. Let $F = \{f_i | i \in \mathbb{N}\}$. Then F is spanned by $B' \backslash F$ in M, thus there is a base B of M which is contained in B_{fin} and does not contain an edge of F. Then $B \subseteq B_{\text{fin}}$ and $B_{\text{fin}} \backslash B$ contains F and is thus infinite. So B and B_{fin} witness that M is not nearly finitary. □

Corollary 2.13. *Let M be a matroid which is not k-nearly finitary for any $k \in \mathbb{N}$. Then M is not nearly finitary or there is a set X of finite connectivity such that M/X is not k-nearly finitary for any $k \in \mathbb{N}$ and satisfies (∗).*

Proof. Let $P_0 = \emptyset$ and $Q_0 = E(M)$. As long as this is possible, define recursively separations (P_i, Q_i) of finite connectivity such that $P_{i-1} \subseteq P_i$, M/P_i is not k-nearly finitary for any $k \in \mathbb{N}$ and $M/P_{i-1} \backslash Q_i$ contains an infinite circuit. If this process does not stop after finitely many steps, then by Lemma 2.12 M is not nearly finitary and the lemma holds. So assume that there is an index i such that (P_{i+1}, Q_{i+1}) cannot be defined. Let $N = M/P_i$. Then N is not k-nearly finitary for any $k \in \mathbb{N}$. Let (P, Q) be a separation of N.

First consider the case that $N/P = M/P_i/P$ is not k-nearly finitary for any $k \in \mathbb{N}$. The separation $(P \cup P_i, Q)$ of M has finite connectivity, so because it was not a possible choice for (P_{i+1}, Q_{i+1}) the matroid $N \backslash Q = M/P_i \backslash Q$ is finitary. Now consider the case that there is $k \in N$ such that N/P is k-nearly finitary for some $k \in \mathbb{N}$. Then by Corollary 2.3 N/Q is not k-nearly finitary for any $k \in \mathbb{N}$, so symmetrically to the previous case $N \backslash P$ is finitary. $\square$

2.3. Proof for Ψ-matroids

Lemma 2.14. *Every nearly finitary Ψ-matroid with finite parts is k-nearly finitary for some $k \in \mathbb{N}$.*

Proof. Let M be a Ψ-matroid with finite parts which is not k-nearly finitary for any $k \in \mathbb{N}$. By Corollary 2.7 it is enough to show that M has a minor which is not nearly finitary. Let $\mathcal{T}$ be a tree of presentations with finite parts which induces M. By Corollary 2.13 and Remark 1.54 it suffices to consider the case that for every separation (P, Q) of finite connectivity there is one side such that M restricted to that side is finitary. By Corollary 2.5 M restricted to the other side is not k-nearly finitary for any $k \in \mathbb{N}$. So every edge of $\mathcal{T}$ can be directed uniquely toward the side on which M is not k-nearly finitary for any $k \in \mathbb{N}$.

By repeated applications of Corollary 2.5, as $E(t)$ is finite for every node t of $\mathcal{T}$, no node of $\mathcal{T}$ has only ingoing edges. Also, if $P' \subseteq P$ and $M \restriction P$ is finitary, then also $M \restriction P'$ is finitary, so no two oriented edges point away from each other. Hence there is an end ω of $\mathcal{T}$ towards which all oriented edges point. As M is not k-nearly finitary for any $k \in \mathbb{N}$, there is an infinite circuit C of M. For all tree-induced separations (P, Q) such that $M \restriction P$ is finitary C is not a subset of P. So $\omega \in \Psi$.

Recursively define a sequence of tree-induced separations as follows. Pick some tree-induced separation (P_0, Q_0) such that $M \backslash Q_0$ is finitary. If (P_i, Q_i) has already been defined and such a separation exists, let (P_{i+1}, Q_{i+1}) be a tree-induced separation such that $P_i \subseteq P_{i+1}$, the restriction of M to P_{i+1} is finitary and $M \backslash P_i / Q_{i+1}$ contains an infinite circuit. If (P_{i+1}, Q_{i+1}) does not exist then the recursion stops.

First consider the case that the recursion never stops. For each $i \in \mathbb{N}$ let $\overrightarrow{e_i}$ be the edge of the tree whose deletion yields (P_i, Q_i). Because $M \backslash P_i$ is not finitary, $\overrightarrow{e_i}$ points towards ω. For every $i \in \mathbb{N}$ let $f_i \in C_i$ be an edge and B_i a base of $M \backslash P_i / Q_{i+1}$ containing $C_i - f_i$. Then $B := \bigcup_{i \in \mathbb{N}} B_i$ is independent in M and because $\omega \in \Psi$, B is also a base of $M \backslash P_0$. Let $F = \{f_i | i \in \mathbb{N}\}$. Assume for a contradiction that $B \cup F$ contains a finite circuit C. Let $j \in \mathbb{N}$ be the smallest index such that $C \cap P_j$ is non-empty. Then $j > 0$ and C does not meet P_{j-1}. Thus $C \cap P_j$ is a finite non-empty scrawl of $M \backslash P_{j-1} / Q_j$. But $C \cap P_j$ is also a subset of $B_j + f_j$ and the fundamental circuit of f_j in B_j with respect to $M \backslash P_{j-1} / Q_j$ equals C_j and is thus infinite. So there is no such finite circuit C. Let B_{fin} be a base of M_{fin} which contains $B \cup F$. Then B is a subset of B_{fin} and $B_{\text{fin}} \backslash B$ contains F and is thus infinite. So B and B_{fin} are bases which witness that M is not nearly finitary.

So the only case which is left to consider is that the recursion stops at some $i \in \mathbb{N}$. Similarly to the proof of Corollary 2.13 there is a tree-induced separation (P, Q) of M such that all tree-induced separations (P', Q') of M with $P \subseteq P'$

satisfy that $M\backslash P/Q'$ contracted to one side is finitary. It now suffices to show that $M\backslash P$ is not nearly finitary, assume without loss of generality that $M = M\backslash P$.

Let R be a ray of T converging to ω. Let $\overrightarrow{e_1}$ be its first edge in forward direction, $\overrightarrow{e_2}$ its second edge and so on. Let (P_i, Q_i) be the separation which is induced by deleting $\overrightarrow{e_i}$. Then by Lemma 2.11, $C \cap P_i$ is finite for all infinite circuits C of M and indices $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ the matroid $M\backslash P_i$ is not k-nearly finitary for any $k \in \mathbb{N}$, so it is in particular not i-nearly finitary and there are infinite circuits $C_0^i, \dots, C_i^i$ of $M\backslash P_i$ such that $C_0^i \cup \dots \cup C_i^i$ does not contain finite circuits and has nullity at least $i+1$. Let $X := \bigcup_{i\in\mathbb{N}} C_0^i \cup \dots \cup C_i^i$ and $M' = M \restriction X$. Then M' is not k-nearly finitary for any $k \in \mathbb{N}$. Furthermore for every $i \in \mathbb{N}$ the set

$$\begin{aligned} X \cap P_j &= \left(\bigcup_{i\in\mathbb{N}} C_0^i \cup \dots \cup C_i^i \right) \cap P_j \\ &= \left(\bigcup_{i=0}^{j-1} C_0^i \cup \dots \cup C_i^i \right) \cap P_j \\ &= \bigcup_{i=0}^{j-1} \left(C_0^i \cap P_j\right) \cup \dots \cup \left(C_i^i \cap P_j\right) \end{aligned}$$

is a finite union of finite sets and thus finite. Let D be a cocircuit of M'. Then there is a cocircuit D' of M such that $D' \cap X = D$. As $\omega \in \Psi$ there is an edge $\overrightarrow{e}$ of T pointing towards the part of T containing ω such that for the separation (P', Q') induced by $\overrightarrow{e}$ the cocircuit D' is contained in P'. So there also is an edge e_j of R such that $D' \subseteq P_j$. Hence $D \subseteq P_j \cap X$, but this is a finite set hence D is finite. As D was an arbitrary cocircuit of M', M' is a cofinitary matroid which is not k-nearly finitary for any $k \in \mathbb{N}$, so by Theorem 1.48 it is not nearly finitary. By Corollary 2.7 M is not nearly finitary. $\square$

CHAPTER 3

Nearly finitary graphic matroids

3.1. Simplification of pseudo-arcs

For this section, let G be a graph-like space which induces a matroid M such that some set V_f of vertices not incident with any edges of G meets all infinite pseudo-circles. Also consider all pseudo-arcs to be linearly ordered sets consisting of edges and vertices. As a consequence, it makes sense to consider intervals of pseudo-arcs and paths.

The goal of this section is to show that two pseudo-arcs not passing through vertices of V_f can be shortened to non-trivial pseudo-arcs so as to either traverse through completely different components of M_{fin} or to traverse in the same way between the components of M_{fin}. For that, given a pseudo-arc not containing vertices of V_f as inner points, the information which vertices and edges within a component of M_{fin} the pseudo-arcs contains and in which order can be forgotten. Relevant is only to which components the edges belong and which vertices are passed between the components. This information can be extracted via the following construction:

Definition 3.1. Let A be a pseudo-arc containing no vertices of V_f as inner vertices. Denote the set of components of M_{fin} by $\mathbf{Comp}$. Define a map $\sigma : A \to V \cup \mathbf{Comp}$ via

$$\sigma(x) = \begin{cases} K & x \in K \in \mathbf{Comp} \\ K & x \in V \wedge K \in \mathbf{Comp} \wedge \exists e, f \in K \cap A : e <_A x <_A f \\ x & \text{otherwise.} \end{cases}$$

Then the image of σ is the *simplification* of A, denoted by $\sigma(A)$, and carries a linear order induced by A.

It may not be obvious from the definition, but $\sigma(A)$ and its linear order are well defined, as the next few lemmas show.

Lemma 3.2. *Let K be a component of M_{fin} and e, f two edges of A in K. Then all edges between e and f in A are also contained in K.*

Proof. By symmetry it suffices to consider the case $e <_A f$. Because K is a component of M_{fin}, there is a finite path P whose first edge is f and whose last edge is e. If all edges of A are edges of P, then the lemma holds. Otherwise, following A from e to f and then going back to e via P includes a pseudo-circle C. Because the vertices in V_f are not incident with edges, no vertex of V_f is a vertex of P, and in particular the end vertices of e and f are not contained in V_f. So by the choice of A the pseudo-arc $]e, f[_A$ does not contain any vertices of V_f. Thus neither does C, so C has to be finite and all its edges belong to K. □

Corollary 3.3. *The map $\sigma : A \to V \cup$ **Comp** is well-defined.*

Corollary 3.4. *The order of A induces a linear order of $\sigma(A)$.*

Proof. Corollary 3.3 shows that $\sigma^{-1}(K)$ is an interval of A for every K in **Comp**, and for every vertex v the set $\sigma^{-1}(v)$ has at most one element by definition of σ. □

The following Lemma shows that every component contained in a simplification has "end vertices" in that simplification. Pseudo-arcs share this property with simplifications of pseudo-arcs, which might be justification to think of simplifications in $V \uplus \mathbf{Comp}$ as playing a similar role to pseudo-arcs in G, with the components taking the role of edges. The biggest difference here is that the end vertices of an edge in a pseudo-arc are determined by the edge, which is not true for components in simplifications.

Lemma 3.5. *Let A be a pseudo-arc. For every component K contained in $\sigma(A)$ there are vertices u and w such that u is the predecessor and w is the successor of K in $\sigma(A)$.*

Proof. Let I be the set of elements of A which are mapped to K. By Corollary 3.4 I is an interval of the linear order of A, so its point set is also an interval of the point set of A. The point set of I has an infimum u in the point set of A. Whenever an inner point of an edge is contained in the point set of I, that edge is contained in I and thus all its inner points are contained in the point set of I. So u is not an inner point of an edge and thus is a vertex.

For every component K' in $\sigma(A)$, the set $\sigma^{-1}(K')$ is equal to I or disjoint from I and thus does not contain u as an inner vertex. Hence $\sigma(u)$ is not a component of M_{fin}, so $u = \sigma(u) \in \sigma(A)$. □

The next two lemmas show that for any two pseudo-arcs avoiding V_f their simplifications are equal between any two common elements. These results are then applied to interior parts of pseudo-arcs meeting V_f only in their end vertices in order to establish the behaviour of such pseudo-arcs near vertices of V_f.

Lemma 3.6. *Let A and B be two pseudo-arcs not meeting V_f with the same starting vertex x and the same end vertex y. Then $\sigma(A) = \sigma(B)$ as linearly ordered sets.*

Proof. The first step is to show that $\sigma(A) \cap \mathbf{Comp} \subseteq \sigma(B) \cap \mathbf{Comp}$. Assume not, then there is a component F in $\sigma(A) \setminus \sigma(B)$. Let $e \in F$ be an edge which is contained in A and denote its end vertices by u, w such that $u <_A e <_A w$. Then the walk $uAxByAw$ contains a pseudo-arc P from u to w which does not contain the edge e. So P can be closed to a circuit by e. Because P does not meet V_f, the circuit has to be finite. So all its edges are in the same component of M_{fin}, thus all the edges of P are in F which is not contained in $\sigma(B)$. But this implies that all edges of P are edges of A, as is e, so A contains a circuit in contradiction to its being a pseudo-arc. So by symmetry also $\sigma(A) \cap \mathbf{Comp} = \sigma(B) \cap \mathbf{Comp}$.

The second step is to show that all vertices contained in $\sigma(A)$ are contained in B. Assume not, so there is a vertex u contained in $\sigma(A) \setminus B$. Because B considered as a point set is closed there is an open interval I of the point set of A containing u. Let e and f be edges meeting I and a and b end vertices of e and f such that $a <_A e <_A u <_A f <_A b$. Then the walk $aAxByAb$ contains a pseudo-arc P from a to b. Because I is disjoint from the point set of B, P meets aAb only in a and b, so P can be closed to a circuit by bAa. This circuit does not meet V_f and is thus finite. Because aAb contains e and f, these edges belong to the same component of M_{fin}, which contradicts $u \in \sigma(A)$.

The third step is to show that all vertices in $\sigma(A)$ are also contained in $\sigma(B)$. Assume not, so there is $u \in \sigma(A) \backslash \sigma(B)$. As $u \in B$ by the previous step, this implies that there are $K \in \mathbf{Comp}$ and edges $e, f \in B \cap K$ such that $e <_B u <_B f$. Thus $\sigma(xBu)$ and $\sigma(uBy)$ both contain K. But by the first step $\sigma(xBu) \cap \mathbf{Comp} = \sigma(xAu) \cap \mathbf{Comp}$, so also the simplification of xAu contains K and similarly the simplification of uAy contains K, which together show that u cannot appear in the simplification of A. By symmetry $\sigma(A) = \sigma(B)$ as sets.

The last step is to show that the linear order of $\sigma(A)$ and $\sigma(B)$ are the same. Let u be a vertex and K a component of $\sigma(A)$. Then the simplifications of uAy and uBy are the same sets, so $u <_{\sigma(A)} K$ if and only if $u <_{\sigma(B)} K$. Now let K and K' both be components of $\sigma(A)$ such that $K <_{\sigma(A)} K'$. Then there is a vertex $u \in \sigma(A)$ such that $K <_{\sigma(A)} u <_{\sigma(A)} K'$. This implies $K <_{\sigma(B)} u <_{\sigma(B)} K'$, thus $K <_{\sigma(B)} K'$. Similarly, if u and w are vertices of $\sigma(A)$ such that u is less than v, then there is a component K between u and v in A which witnesses that u is also less than v in $\sigma(B)$. □

Lemma 3.7. *Let A and B be two pseudo-arcs not meeting V_f. If there are $X, Y \in \sigma(A) \cap \sigma(B)$ such that $X <_{\sigma(A)} Y$ and $X <_{\sigma(B)} Y$ then $[X, Y]_{\sigma(A)} = [X, Y]_{\sigma(B)}$.*

Proof. If $X = Y$ then this is true trivially, so assume otherwise. If X and Y are both vertices of G, then this is true by Lemma 3.6 applied to XAY and XBY. So to prove the lemma, it suffices to find vertices which can substitute X and Y if necessary. If X is a vertex, let $x_A = X = x_B$ and P_x the trivial path consisting of that vertex. Otherwise let e be an edge of A which is contained in X and let x_A be the end vertex of e with $x_A <_A e$. Then there is a finite path P_x with edges in X from x_A to an end vertex x_B of an edge of B. Similarly, if Y is a vertex let $y_A = Y = y_B$ and let P_y be the trivial path consisting of that vertex. Otherwise let f be an edge of A contained in Y and y_A the end vertex of f with $f <_A y_A$. Then there is a finite path P_y with edges in Y from y_A to an end vertex y_B of an edge of B. In all four cases $x_A P_x x_B B y_B P_y y_A$ contains a pseudo-arc B' from x_A to y_A. By Lemma 3.6 $\sigma(x_A A y_A) = \sigma(B')$, so

$$[X, Y]_{\sigma(A)} \subseteq \sigma(x_A A y_A) = \sigma(B') \subseteq [X, Y]_{\sigma(B)} + x_A + y_A$$

and hence $[X, Y]_{\sigma(A)} \subseteq [X, Y]_{\sigma(B)}$. By symmetry, equality holds. □

Lemma 3.8. *Let A and B be two pseudo-arcs which have the same vertex $v \in V_f$ as end vertex and otherwise do not meet V_f. Also assume that $\sigma(A) \cap \sigma(B) = \{v\}$. Then $A \cap B$ contains at most two elements, one of which is v.*

Proof. Assume that $A \cap B$ contains at least two elements u and w which are not v. As $\sigma(A) \cap \sigma(B) = \{v\}$, u and w are vertices. Assume without loss of generality that $u <_A w$. By Lemma 3.6 $\sigma(uBw)$ equals $\sigma(uAw)$. Because uAw contains at least one edge, $\sigma(uAw)$ contains at least one component K. This component is then also contained in $\sigma(uBw)$, so it is contained in $\sigma(B)$. Thus K is contained in $\sigma(A) \cap \sigma(B)$ but is not equal to v, which is a contradiction. □

Lemma 3.9. *Let A and B be two pseudo-arcs meeting V_f only in their end vertices, if at all. Assume that the biggest vertex v of B is contained in V_f and that all edges of B are contained in the same component K of M_{fin}. If $A \cap (B - v)$ has no upper* *bound in $B - v$, then A contains a non-trivial pseudo-arc A' such that $\sigma(A')$ also contains v and no other components than K.*

Proof. Assume that $A \cap B$ has no upper bound in $B - v$. For any edge of $A \cap B$, both vertices incident with it are also contained in $A \cap B$. So, as v is not incident with any edge, $B \cap A \cap V$ also has no upper bound in $B - v$, and v is contained in the closure of $V \cap B - v$. Let x and y be two distinct vertices of $A \cap (B - v)$ such that $x <_A y$. Then Lemma 3.6 implies that $\sigma(xAy)$ contains K as its only component and in particular that $\sigma(A)$ contains K. So by Lemma 3.5 K has a predecessor p and a successor s in $\sigma(A)$. All edges of xAy are contained in K and thus in pAs, so xAy is a subset of pAs. Hence all vertices in $A \cap B - v$ are contained in pAs, so also $v \in pAs$. Because A only meets V_f in its end vertices, v equals p or s and pAs can be chosen as A'. $\square$

Lemma 3.10. *Let A and B be two non-trivial pseudo-arcs which share their last vertex $v \in V_f$ and are otherwise disjoint from V_f. Then one of the following holds:*

- *There are vertices $u_A \in A \cap V - v$ and $u_B \in B \cap V - v$ such that $\sigma(u_A A) \cap \sigma(u_B B) = \{v\}$*
- *There are vertices $u_A \in A \cap V - v$ and $u_B \in B \cap V - v$ such that $\sigma(u_A A) = \sigma(u_B B)$ as ordered sets*
- *v has a predecessor in $\sigma(A)$ and one in $\sigma(B)$ and those two predecessors are the same component of M_{fin}.*

Proof. First consider the case that in one of $\sigma(A)$, $\sigma(B)$ there is a predecessor K of v. Assume, by renaming A and B if necessary, that this happens in $\sigma(A)$. Then K is a component of M_{fin}, so there is a predecessor u_K of K in $\sigma(A)$, and $\sigma(u_K Av)$ consists only of the three elements u_K, K and v. Also because v is not incident with an edge, the ordered set $u_K Av - v$ has no biggest element.

In this paragraph, consider the case that the set $\{w \in u_K Av \cap V - v : w \in B\}$ has no upper bound in $u_K Av$. Then this set is infinite and for each two elements u and w of it Lemma 3.6 shows that the segment of B between u and w only uses edges from K. In particular $K \in \sigma(B)$ and K has a predecessor u_B and a successor w_B in $\sigma(B)$, and all vertices of $\{w \in u_K Av \cap V - v : w \in B\}$ are contained in $u_B B w_B$. Because v is not incident with an edge but it is the last vertex of $u_X Av$, it is contained in the closure of $u_K Av \cap (V - v)$. As $u_B B w_B$ is closed as a point set, it has to contain v, so $v = w_B$. So K is also the predecessor of v in B and the third option from the lemma holds.

So consider the case that the set $\{w \in u_K Av \cap V - v : w \in B\}$ has an upper bound in $u_X Av$. Let u_A be a vertex of $u_K Av$ such that the only vertex of $u_A Av$ which is contained in B is v. If $K \notin \sigma(B)$, then $\sigma(B) \cap \sigma(u_A Av) = \{v\}$ and the first option of the lemma holds, so assume otherwise. Likewise if K is the predecessor of v in B, then the third option of the lemma holds, so assume otherwise. Thus the successor w_B of K in B is not v, and $\sigma(w_B Bv)$ and $\sigma(u_A Av)$ share only v, so in this case the first option of the lemma holds.

Now the case that v has a predecessor in one of the simplifications is done. So assume that there is no such predecessor. If $\sigma(A) \cap \sigma(B) - v$ is empty, then the $\sigma(A) \cap \sigma(B) = \{v\}$ and the first option of the lemma holds. So assume that there is some element X of $\sigma(A) \cap \sigma(B) - v$. Let S be the set of elements $Z \in \sigma(A) \cap \sigma(B) - v$ such that $X \leq_{\sigma(A)} Z$ and $X \leq_{\sigma(B)} Z$. Again there are two cases: If S has an upper bound in $\sigma(A) - v$, then let u be a vertex of $\sigma(A) - v$ which is bigger than all elements of S. Also let w be a vertex bigger than X in $\sigma(B) - v$. Then $\sigma(uAv) \cap \sigma(wBv) = \{v\}$ and the first option of the lemma holds. If the set S does not have an upper bound in $\sigma(A) - v$, then for all $Z \in S$ Lemma 3.7 shows that $[X, Z]_{\sigma(A)} = [X, Z]_{\sigma(B)}$. Because S does not have an upper bound in $\sigma(A) - v$, this implies that $[X, v]_{\sigma(A)} - v$ is an interval of $\sigma(B)$ and has thus a vertex w which is its biggest element or successor in $\sigma(B)$. The fact that v is in the topology a limit of the set of vertices of $[X, v]_{\sigma(A)} - v$ implies that $v = w$ and thus $[X, v]_{\sigma(A)} = [X, v]_{\sigma(B)}$. Let t be a vertex of $[X, v]_{\sigma(A)} - v$, then $\sigma(tAv) = \sigma(tBv)$ and the second option of the lemma holds. $\square$

This section closes with the observation that pseudo-arcs meeting V_f only in their end vertices, if at all, and only using edges from one component of M_{fin} are very similar to usual paths.

Observation 3.11. *Let A be a pseudo-arc which meets V_f only in its end vertices, if at all. If $\sigma(A)$ contains exactly one component K of M_{fin}, then deleting no, two or one end vertex of A turns it into a path, double ray, ray or inverse of a ray respectively.*

Proof. Denote the end vertices of A by v and w. Every non-trivial path P such that $A \cap P$ consists of the end vertices x and y of P can be closed by a pseudo-arc contained in A to a pseudo-circle C. Because neither vertices of P nor inner vertices of A are contained in V_f, also C does not meet V_f. So C is finite and thus also xAy is finite.

Now let x and y be two vertices which are incident with distinct edges of A. Because these two edges are both contained in K, there is a finite path P from x to y. This path is a finite union of paths P_i which meet A exactly in their end vertices x_i and y_i, and paths Q_j which are subpaths of A. Then xAy equals the union of the Q_j and the paths $x_i A y_i$ and is thus finite. So for all distinct edges $e, f \in A$ the set $[e, f]_A$ is finite. $\square$

3.2. Finding a few important vertices

As already mentioned in the introduction, the condition that a graph-like space has a finite set of vertices meeting all infinite pseudo-circles can be obtained from two more natural-looking conditions. The content of this section is to establish those conditions and to deduce that it suffices to consider graph-like spaces in which some finite vertex set meets all infinite pseudo-circles. The conditions are as follows:

(S1) For all distinct vertices $v, w \in V$ there is a finite vertex set $V' \subseteq V - v - w$ such that v and w are contained in different topological components of $G - V' - E_{vw}$, where E_{vw} is the set of edges from v to w.

(S2) For all vertices $v \neq w \in V$ there are edge sets $F_v, F_w \subseteq E$ of finite connectivity such that $v \notin \overline{F_w}$, $w \notin \overline{F_v}$ and $M \backslash (F_v \cup F_w)$ is k-nearly finitary for some $k \in \mathbb{N}$.

First of all, (S1) is the most natural-looking condition on graph-like spaces as it resembles finite separability. This is a stronger condition than (S2):

Lemma 3.12. *Every graph-like space which induces a matroid and satisfies (S1) also satisfies (S2).*

PROOF. Let G be a graph-like space which induces a matroid M and satisfies (S1). Let v and w be two different vertices of G. Then there is a finite set $V' \subseteq V - v - w$ of vertices such that v and w are contained in different components of $G - V' - E_{vw}$. Let F_v be the set of edges of the topological component of $G - V' - E_{vw}$ containing v and let $F_w = E\backslash(F_v \cup E_{vw})$. Then $v \notin \overline{F_w}$ and $w \notin \overline{F_v}$. Also $M\backslash(F_v \cup F_w)$ is the matroid on edge set E_{vw} in which all edges are parallel. So $M\backslash(F_v, F_w)$ has only circuits of size 2 and is hence finitary. The sets F_v and F_w have finite connectivity by Lemma 1.66. $\square$

By Corollary 2.7 the second condition is clearly preserved under deletion of any edge sets. The following lemma shows that it can be preserved under contraction of a set of finite connectivity in the matroid:

Lemma 3.13. *Let G be a graph-like space which induces a matroid M and satisfies (S2). Then there is for every edge set X of finite connectivity a graph-like space G' which induces M/X and also satisfies (S2).*

PROOF. By Lemma 1.67 there is a set $C \subseteq X$ such that $M/X = M/C\backslash(X\backslash C)$ and such that $\overline{C}$ is compact. Denote $G/C\backslash(X\backslash C)$ by G'. For the rest of the proof, standard subspaces are taken in G' unless stated otherwise. Let v and w be vertices of G such that $\pi(v)$ and $\pi(w)$ are distinct. The proof will show in three cases that (S2) holds for $\pi(v)$ and $\pi(w)$ in M/X. In each of the three cases, it will be shown that the stronger conditions of the following claim are met.

Claim. Let F and F' be edge sets of finite connectivity in M such that $M\backslash(F \cup F')$ is k-nearly finitary for some $k \in \mathbb{N}$. Also let $\{Y_v, Y_w\}$ be a partition of $\overline{C} \cap V(G) + v + w$ into closed vertex sets such that

- $v \in Y_v$ and $\pi^{-1}(\pi(Y_v)) = Y_v$ and $Y_v \cap \overline{F'} = \emptyset$; and
- $w \in Y_w$ and $\pi^{-1}(\pi(Y_w)) = Y_w$ and $Y_w \cap \overline{F} = \emptyset$.

Then (S2) holds in M/X for $\pi(v)$ and $\pi(w)$.

PROOF OF CLAIM. Let $E = F\backslash X$ and $E' = F'\backslash X$. By symmetry it suffices to show that E has finite connectivity in M/X, that $\pi(w)$ is not contained in the standard subspace of E in G' and that $M/X\backslash(E \cup E')$ is k-nearly finitary for some $k \in \mathbb{N}$. As E has finite connectivity in M, it also has finite connectivity in M/X.

By Lemma 1.68, any vertices u of G such that $\pi(u) \in \pi(\overline{F})$ are either contained in $\overline{F}$ or in $\overline{C}\backslash Y_w$. In the latter case $u \in Y_v$. Thus $\pi^{-1}(\pi(\overline{F} \cup Y_v))$ equals $\overline{F} \cup Y_v$, so $\pi(\overline{F} \cup Y_v)$ is a closed set which is disjoint from $\pi(Y_w)$. So the standard subspace of E in G' is contained in $\pi(\overline{F} \cup Y_v)$ and thus disjoint from $\pi(Y_w)$. In particular the standard subspace of E in G' does not contain $\pi(w)$.

Denote the set $(F \cup F') \cap X$ by Z. Because Z has finite connectivity in M, it also has finite connectivity in $M\backslash((F \cup F')\backslash X)$. So by Corollary 2.10, the fact that $M\backslash(F \cup F')$ is k-nearly finitary for some $k \in \mathbb{N}$ implies that also $M\backslash((F \cup F')\backslash X)/Z$ is k-nearly finitary for some $k \in \mathbb{N}$. Now $M/X\backslash(E \cup E')$ is a minor of $M\backslash((F \cup F')\backslash X)/Z$, so by Corollary 2.7 also $M/X\backslash(E \cup E')$ is k-nearly finitary for some $k \in \mathbb{N}$. $\Diamond$

As (S2) holds for v and w in G, there are candidates for F and F' in the previous claim. Also if $v \notin \overline{C}$, then $\{v\}$ and $\overline{C} + w$ are possible candidates for Y_v and Y_w, but the candidates do not yet interact the way they need to in order to apply the previous claim. Thus other candidates for F and F' are needed which interact better with the sets Y_v and Y_w, which are supplied by the following claim:

> **Claim.** Let Y be a closed subset of $V(G) \cap \overline{C}$ and v a vertex which is not contained in Y. Then there are sets F and F' of finite connectivity in M such that v is not contained in $\overline{F'}$, the intersection of $\overline{F}$ and Y is empty and $M \backslash (F \cup F')$ is k-nearly finitary for some $k \in \mathbb{N}$.

Proof of claim. As Y is a closed subset of a compact set, it is compact. For every $u \in Y$ let F_u and F'_u be edge sets of finite connectivity such that $v \notin \overline{F'_u}$ and $u \notin \overline{F_u}$ and $M \backslash (F_u \cup F'_u)$ is not k-nearly finitary for any $k \in \mathbb{N}$. Then the complements of the sets $\overline{F_u}$ form an open cover of Y, so there is a finite set $Y_f \subseteq Y$ such that the complements of the sets $\overline{F_u}$ with $u \in Y_f$ cover Y. Let $F = \bigcap_{u \in Y_f} F_u$ and $F' = \bigcup_{u \in Y_f} F'_u$. By submodularity of the connectivity function both F and F' have finite connectivity. Also $E(M) \backslash (F \cup F')$ is a subset of $\bigcup_{u \in Y_f} E(M) \backslash (F_u \cup F'_u)$, and by Corollary 2.8 the restriction of M to the latter set is k-nearly finitary for some $k \in \mathbb{N}$. So by Corollary 2.7 the restriction of M to $E(M) \backslash (F \cup F')$ is k-nearly finitary for some $k \in \mathbb{N}$. Furthermore $v \notin \overline{F'}$ and $\overline{F}$ is disjoint from Y. $\Diamond$

Now to the three cases. The first case is that one of v and w is contained in $\overline{C}$ and the other is not Assume that $v \notin \overline{C}$. Then for $Y_w = V(G) \cap \overline{C}$ there are edge sets F and F' of finite connectivity in M such that $v \notin \overline{F'}$, the intersection of $\overline{F}$ and Y_w is empty and $M \backslash (F \cup F')$ is k-nearly finitary for some $k \in \mathbb{N}$. Let $Y_v = \{v\}$, then by Lemma 1.68 the first claim of this proof can be applied and (S2) holds for $\pi(v)$ and $\pi(w)$ in G'.

The next case is the case where neither v nor w is contained in $\overline{C}$. Then there are for v and $Y = \overline{C} \cap V(G)$ edge sets F and F' of finite connectivity in M such that $M \backslash (F \cup F')$ is k-nearly finitary for some $k \in \mathbb{N}$, the intersection of $\overline{F}$ and $\overline{C} \cap V(G)$ is empty and $v \notin \overline{F'}$. Also (S2) holds for M, so there are edge sets H and H' of finite connectivity in M such that $M \backslash (H \cup H')$ is k-nearly finitary for some $k \in \mathbb{N}$, and $v \notin \overline{H'}$ and $w \notin \overline{H}$. Let $E = F \cap H$ and $E' = F' \cup H'$. Then $M \backslash (E \cup E')$ is a minor of $M \backslash ((F \cup F') \cap (H \cup H'))$ and the latter matroid is k-nearly finitary for some $k \in \mathbb{N}$ by Corollary 2.8, so by Corollary 2.7 also $M \backslash (E \cup E')$ is k-nearly finitary for some $k \in \mathbb{N}$. Furthermore v is neither contained in $\overline{F'}$ nor in $\overline{H'}$, so $v \notin \overline{E'}$. Also $\overline{F}$ is disjoint from $\overline{C} \cap V(G)$ and $\overline{H}$ does not contain w, so $\overline{E}$ is disjoint from $\overline{C} \cap V(G) + w$. So the first claim of this proof can be applied to E, E', $Y_v = \{v\}$ and $Y_w = \overline{C} \cap V(G) + w$, and (S2) holds for $\pi(v)$ and $\pi(w)$ in G'.

The last case is the case that both v and w are contained in $\overline{C}$. As $\pi(v) \neq \pi(w)$ there are disjoint open sets U and W partitioning $V(G)$ such that $v \in U$ and $w \in W$ and such that no edge of C has one end vertex in U and one in W. Then there is no vertex of G' whose pre-image under π contains both vertices of U and of W. Let $Y_v = U \cap \overline{C} \cap V(G)$ and $Y_w = W \cap \overline{C} \cap V(G)$. Then Y_v equals $(\overline{C} \cap V(G)) \backslash W$ and is thus closed. Also $\pi^{-1}(\pi(Y_v))$ contains by Lemma 1.68 only vertices in $Y_v \cup \overline{C}$, but it does not contain elements of Y_w. So $\pi^{-1}(\pi(Y_v)) = Y_v$. Similarly Y_w is a closed set such that $\pi^{-1}(\pi(Y_w)) = Y_w$. For every vertex $u \in Y_v$ apply the second claim of this proof to u and Y_w to obtain edge sets F_u and F'_u of finite connectivity in M such that $M \backslash (F \cup F')$ is k-nearly finitary for some $k \in \mathbb{N}$ and such that $u \notin \overline{F'_u}$ and $\overline{F_u} \cap Y_w = \emptyset$. Then the complements of the sets $\overline{F'_u}$ form an open cover of Y_v. As Y_v is a closed subset of a compact set, it is itself compact, so there is a finite set $Z \subseteq Y_v$ such that the complements of the sets $\overline{F'_u}$ with $u \in Z$ cover Y_v. Let $F = \bigcup_{u \in Z} F_u$ and $F' = \bigcap_{u \in Z} F'_u$. Then $M \backslash (F \cup F')$ is a minor of $M \backslash \bigcap_{u \in Z}(F_u \cup F'_u)$ and the latter matroid is k-nearly finitary by Corollary 2.8, so by Corollary 2.7 also $M \backslash (F \cup F')$ is k-nearly finitary for some $k \in \mathbb{N}$. Also $Y_v \cap \overline{F'} = \emptyset$ and $\overline{F} \cap Y_w = \emptyset$. So the second claim of this proof can be applied and (S2) holds for $\pi(v)$ and $\pi(w)$. □

As the reduction in Section 2.2 to matroids satisfying $(*)$ happens by contraction of a set of finite connectivity, this reduction can be applied without losing (S2). So in order to reduce the case of graphic matroids satisfying (S2) to graphic matroids in which a finite vertex set meets every infinite pseudo-circle, $(*)$ can be assumed additionally.

Lemma 3.14. *If M satisfies $(*)$ and arises from a graph-like space with property (S2), then M is k-nearly finitary for some $k \in \mathbb{N}$ or there is a vertex v meeting all infinite pseudo-circles.*

Proof. Assume that M satisfies $(*)$ and arises from a graph-like space with property (S2) but is not k-nearly finitary for any $k \in \mathbb{N}$. Let S be the set of all those subsets Q of $E(M)$ of finite connectivity such that $M \backslash Q$ is finitary. Define $V_f = \bigcap_{Q \in S} \overline{Q}$. For every edge e the set $E - e$ is an element of S and $\overline{E - e}$ does not contain inner points of e. So V_f contains only vertices.

Assume for a contradiction that there is an infinite pseudo-circle C whose image in G does not meet V_f. Then the complements of the sets $\overline{Q}$ with $Q \in S$ form an open cover of the point set of $\mathrm{im}(C)$. Because the point set of $\mathrm{im}(C)$ is compact, there is a finite set $Q_1, \dots, Q_n$ of elements of S such that the complements of the sets $\overline{Q_1}, \dots, \overline{Q_n}$ cover the point set of $\mathrm{im}(C)$. Then the point set of $\mathrm{im}(C)$ does not meet $\overline{Q_1} \cap \dots \cap \overline{Q_n}$, hence in particular it does not meet $\overline{Q_1 \cap \dots \cap Q_n}$. As $Q_1 \cap \dots \cap Q_n =: Q$ has finite connectivity, several applications of Corollary 2.5 show that $M \restriction Q$ is not k-nearly finitary for any $k \in \mathbb{N}$, which implies that $M \backslash Q$ is finitary. This is a contradiction to the fact that the edge set of $\mathrm{im}(C)$ is disjoint from Q.

As M is not finitary, V_f is non-empty. Let v be an element of V_f and w a vertex which can be separated from v, as witnessed by F_v and F_w. Because v is contained in V_f but not in $\overline{F_w}$, the set F_w is not contained in S. So $M \backslash F_w$ is not finitary. But then $M \backslash F_w$ is not k-nearly finitary for any $k \in \mathbb{N}$ and thus by Corollary 2.5 $M \restriction F_v$ is also not k-nearly finitary for any $k \in \mathbb{N}$, implying that $M \backslash F_v$ is finitary and $(F_v, E \backslash F_i v) \in S$. Hence $w \notin V_f$. So V_f contains only one vertex. □

Corollary 3.15. *Let G be a graph-like space satisfying (S2) and inducing a matroid M which is not k-nearly finitary for any $k \in \mathbb{N}$. Then M is not nearly finitary or there is a graph-like space G' inducing a minor N of M such that N is not k-nearly finitary for any $k \in \mathbb{N}$ and such that some vertex set of G' meets every infinite pseudo-circle of G'.*

Proof. By Corollary 2.13 either M is not nearly finitary or there is a set X of finite connectivity such that M/X is not k-nearly finitary for any $k \in \mathbb{N}$ and satisfies $(*)$. In the first case the corollary holds, so assume the second case. Then by Lemma 3.13 there is a graph-like space G' which induces M/X and satisfies (S2). Thus by Lemma 3.14 G' has a vertex meeting every infinite pseudo-circle of G'. □

Remark 3.16. In the proof of Lemma 3.14, a vertex set V_f is constructed that meets every infinite pseudo-circle of the graph from which M arises. It is then shown that if two vertices v and w are separated in the sense of (S2), that is that there are edge sets F_v and F_w of finite connectivity such that $v \notin \overline{F_w}$, $w \notin \overline{F_v}$ and $M \backslash (F_v \cup F_w)$ is k-nearly finitary for some $k \in \mathbb{N}$, then at least one of the vertices v and w is not contained in V_f. So if the assumption (S2) that any two vertices can be separated is weakened to the assumption that every vertex is separated from all but finitely many other vertices, then the proof shows that there is a finite vertex set V_f meeting all infinite pseudo-circles. The corollary of Lemma 3.14 can be weakened similarly. And the other sections of this chapter work with the weaker condition that there is a finite set of vertices meeting all pseudo-circles.

Remark 3.17. Let M be a matroid arising from a graph-like space G such that some finite vertex set V_f meets all infinite cycles. If M' is a restriction of M, then M' arises from a graph-like space G' which is a subspace of G, so $V_f \cap V(G)$ meets all infinite cycles in G.

3.3. Proof for graph-like spaces

This section contains the proof of Theorem 3.24, which was mentioned in the introduction. The proof of the theorem starts with a matroid M which is induced by a graph-like space in which some finite vertex set meets all infinite pseudo-circles and which is not k-nearly finitary for any $k \in \mathbb{N}$. The main proof idea is the following: Every scrawl independent in M_{fin} with large nullity induces a large family of pseudo-arcs which meet V_f exactly in their end vertices and are internally disjoint. These pseudo-arcs come in several types: they may have a last edge, which is then incident with a vertex of V_f and contained in some component of M_{fin}, or they may not have a last edge but still have a last component in the simplification, or have neither. The proof then has several case distinctions where in one case there are arbitrarily large such families of pseudo-arcs of the same type and M can be shown to not be nearly finitary and in the other case (a minor of) M has (essentially) no pseudo-arcs of the type under consideration. As the proof is quite long, several of the cases in which M is shown to not be nearly finitary are proven in separate lemmas which form the rest of this section. **In all these lemmas, G is a graph-like space inducing a matroid M which is not k-nearly finitary for any $k \in \mathbb{N}$. Also V_f is a finite vertex set meeting every infinite pseudo-circle.**

Lemma 3.18. *Assume that M is connected and that there are infinitely many components K of M_{fin} which contain an edge which is incident with a vertex of V_f. Then M is not nearly finitary.*

Proof. By the pigeonhole principle there is a vertex $v \in V_f$ which is incident with edges of infinitely many components of M_{fin}. Count a subset of these components by $K_0, K_1, \ldots$ and in each K_i pick an edge e_i which is incident with v. Recursively define a family $(C_i)_{i \in \mathbb{N}}$ of pseudo-circles such that $\bigcup_{i \in \mathbb{N}} C_i$ does not contain a finite pseudo-circle and each C_i contains e_{i+1} but no e_j with $j > i+1$ as follows: Because M is connected, there is a pseudo-circle C_0 containing e_0 and e_1, and because e_0 and e_1 are in different components of M_{fin}, C_0 is infinite. Now assume that $C_0, \ldots, C_{n-1}$ have already been defined such that each C_i with $i \leq n-1$ is a pseudo-circle containing e_0 and e_{i+1}. As M is connected, there is a pseudo-circle C'_n which contains e_0 and e_{n+1}. No C_j with $j < n$ contains e_{n+1} and $\overline{C_0 \cup \cdots \cup C_{n-1}}$ is closed, so C'_n contains a pseudo-arc A which contains e_{n+1} and meets $\overline{C_0 \cup \cdots \cup C_{n-1}}$ only in its end vertices. One of these end vertices is v, because that is an end vertex of e_{n+1}. Denote the other end vertex of A by w. There is $j < n$ such that $w \in \overline{C_j}$. Then $\overline{C_j}$ contains a pseudo-arc from v to w, complete A by this pseudo-arc to obtain a circuit C_n. Every circuit contained in $C_0 \cup \cdots \cup C_n$ contains two edges which have v as an end vertex. These two edges are e_i and e_j for distinct indices i and j and thus not contained in a common finite circuit. Any finite circuit contained in $\bigcup_{i\in\mathbb{N}} C_i$ is contained in $C_0 \cup \cdots \cup C_n$ for some index n, so there is no such circuit. Also the set $\{e_1, e_2, \dots\}$ witnesses that $\bigcup_{i\in\mathbb{N}} C_i$ has infinite nullity. So M is not nearly finitary. $\square$

Lemma 3.19. *Let $v \in V_f$ be a vertex such that only finitely many components of M_{fin} contain an edge incident with v. Then there is a restriction of M which is also not k-nearly finitary for any $k \in \mathbb{N}$ and in which there are no edges incident with v.*

Proof. Let two edges incident with v be equivalent if they are contained in the same component of M_{fin} and denote the equivalence classes by $K_1, \dots, K_n$. From each K_j pick an edge e_j. Let $(C_i)_{i\in I}$ be a finite family of circuits such that $\bigcup_{i\in I} C_i$ is independent in M_{fin} and has nullity $|I|$. Because every circuit C_i can meet at most twice as many edges from $K_1 \cup \cdots \cup K_n$ as there are vertices in V_f, the set $F := \bigcup_{i\in I} C_i \cap \bigcup_{j=1}^{n} K_j \backslash \{e_1, \dots, e_n\}$ is finite, too.

Now $(C_i)_{i\in I}$ will be modified in several steps until the only edges of $K_1 \cup \cdots \cup K_n$ which are contained in $\bigcup_{i\in I} C_i$ are edges e_i. As long as F is non-empty, let e be an edge in it. Then $e \in K_j$ for some index j, and $e \neq e_j$. So there is a finite circuit C containing e and e_j. Let K be the component of $M_{\text{fin}} \restriction \bigcup_{i\in I} C_i \cup C$ containing e. Then K has finite nullity in M_{fin} because $\bigcup_{i\in I} C_i$ is independent in M_{fin}, hence K is a finite join of finite circuits and thus finite. Because $\bigcup_{i\in I} C_i$ has finite nullity in M, there is by Lemma 1.47 a cobase $\{f_1, \dots, f_n\}$ of it which does not contain edges of K. So the set $\{f_1, \dots, f_n\} + e$ is coindependent in $M \restriction \bigcup_{i\in I} C_i \cup C$ and there is a base B of $M \restriction \bigcup_{i\in I} C_i \cup C$ which is disjoint from $\{f_1, \dots, f_n\} + e$. Then every edge f_i has a fundamental circuit C'_i in B. Because every finite circuit of $\bigcup_{i\in I} C_i \cup C$ contains edges of the finite circuit C, every such circuit is a subset of K. But as no edge f_i is contained in K, this implies that every finite circuit contained in $\bigcup_{i\in I} C'_i$ is a subset of B, so there is no such finite circuit and the family $(C_i)_{i\in I}$ is independent concerning finite circuits. So $M \backslash (K_0 \cup \cdots \cup K_n)$ is not k-nearly finitary for any $k \in \mathbb{N}$. $\square$

Lemma 3.20. *Assume that there are no edges incident with vertices in V_f. Also assume that there is $k \in \mathbb{N}$ such that every scrawl C that is independent in M_{fin} either has nullity at most k or contains at least two vertices of V_f in $\overline{C}$. Then M is not nearly finitary.*

Proof. Let C be a scrawl which is independent in M_{fin} and which has a finite cobase I. For every $v \in V_f$ let I_v be the set of those edges $i \in I$ such that the fundamental circuit C_i of i in $(C \backslash I) + i$ contains no vertices of V_f other than v in its topological closure. Then the union C_v of the fundamental circuits C_i for $i \in I_v$ is a finite union and thus its topological closure also contains no vertices of V_f other than v. Hence for every $v \in V_f$ the set I_v contains at most k many elements. Let

I' be the remained of I after deletion of the sets I_v. Then $|I'| \geq |I| - k * |V_f|$. Also for every edge $i \in I'$ the fundamental circuit C_i contains at least two vertices of V_f in its topological closure, so the pseudo-circle with edge set C_i contains a pseudo-arc R_i which contains i, whose inner vertices are not contained in V_f and whose end vertices are contained in V_f. The set $(R_i)_{i \in I'}$ is a family of internally disjoint pseudo-arcs which meet V_f precisely in their end vertices and such that $\bigcup_{i \in I'} R_i$ contains no finite circuits. As M is not k-nearly finitary for any $k \in \mathbb{N}$, there are arbitrarily large finite such families. Because V_f is finite, there distinct vertices v and w in V_f such that there are arbitrarily large finite families of internally disjoint pseudo-arcs with first vertex v and last vertex w such that the union of their edge sets does not contain a finite vertex.

In order to show that there is an infinite family of internally disjoint pseudo-arcs with first vertex v and last vertex w such that the union of their edge sets does not contain a finite circuit, it suffices to show that every finite such family $(R_j)_{j \in J}$ can be extended by some pseudo-arc. For that let $(R_l)_{l \in L}$ be a finite family of internally disjoint pseudo-arcs with first vertex v and last vertex w such that the union of the edge sets does not contain a finite circuit and such that $|L| > (k+1)|J| + 1$. Let $j \in J$. Let L_j be the set of indices L in L such that R_l shares an internal vertex with R_j. Connect the pseudo-arcs R_l with $l \in L_j$ by finitely many sub-pseudo-paths of R_j such that the result does not contain w, does not contain pseudo-circles which do not contain v, but contains for every $l \in L_j$ a pseudo-circle which contains a non-trivial sub-pseudo-path of R_l with last vertex v. Then the edge set of this result is independent in M_{fin} and contains a scrawl whose nullity is at least $|L_j| - 1$. This scrawl does not contain w in its topological closure, so its nullity is bounded by k. Hence $|L_j| \leq k + 1$. So every R_j with $j \in J$ contains inner vertices of at most $k+1$ many R_l with $l \in L$. Thus there are two indices l_1 and l_2 in L such that no R_j with $j \in J$ contains inner vertices of R_{l_1} or R_{l_2}. As the union of R_{l_1} and R_{l_2} does not contain a finite pseudo-circle, one of them is infinite, so this infinite pseudo-arc can be added to $(R_j)_{j \in J}$. □

For $v \in V_f$ say a component K of M_{fin} *captures* v if there is a non-trivial pseudo-arc which contains v and whose edges are all contained in K. Note that in the context where simplifications of pseudo-arcs exist, a component K captures v if and only if there is a pseudo-arc with last vertex v in whose simplification K is the predecessor of v.

Lemma 3.21. *Assume that M is connected and that no edge has a vertex of V_f as end vertex. If for some $v \in V_f$ there are infinitely many components of M_{fin} capturing v then M is not nearly finitary.*

Proof. Let $v \in V_f$ such that there are infinitely many components of M_{fin} capturing v. For every pseudo-circle C denote by $\mathcal{K}(C)$ the set of components K of M_{fin} such that C contains a non-trivial pseudo-arc with last vertex v whose edges are contained in K. Recursively define pseudo-circles $C_0, C_1, \ldots$ such that every $\mathcal{K}(C_i)$ contains two elements, one of which is not contained in any $\mathcal{K}(C_j)$ with $j < i$ and such that for every $i \in \mathbb{N}$ every pseudo-circle contained in the union of the C_j with $j \leq i$ contains v. Note that the last condition implies that every circuit contained in the union of the edge sets of the C_j with $j \leq i$ is infinite.

Assume that $C_0, \ldots, C_{i-1}$ have already been defined but C_i is not yet defined. For every $j < i$, the set $\mathcal{K}(C_j)$ can contain at most two components, so let K and

K' be distinct components of M_{fin} capturing v which are not contained in $\mathcal{K}(C_j)$ for any $j < i$. Let P and P' be non-trivial pseudo-arcs with last vertex v whose edge sets are contained in K and K', respectively.

Let $j < i$. As $\mathcal{K}(C_j)$ has two elements K and K', C_j contains two non-trivial pseudo-arcs Q and Q' with last vertex v whose edge sets are contained in K_j and K'_j, respectively. Then C_j is the union of Q, Q' and a pseudo-arc Q'' which does not contain v. As Q'' does not contain v, by shortening them if necessary P and P' can be assumed to not contain vertices of Q''. Also, by shortening them if necessary, without loss of generality none of the pseudo-arcs P, P', Q and Q' contains a vertex of $V_f - v$. As the components K, K', K_j and K'_j are pairwise distinct, by Lemma 3.9 the pseudo-arcs P, P', Q and Q' can be shortened further such that they are internally disjoint. Thus P and P' can be assumed to not contain inner vertices which are also contained in C_j. Repeating this argument for all $j < i$, P and P' can be assumed to not contain inner vertices which are also contained in some C_j with $j < i$.

As M is connected, there is a pseudo-circle containing an edge of P and an edge of P'. This pseudo-circle contains a pseudo-arc which contains an edge of P and an edge of P' but not v. Thus there also is a pseudo-arc R not containing v whose first vertex is a vertex of P, whose last vertex is a vertex of P', and whose interval vertices are neither contained in P nor in P'. If R is disjoint from all pseudo-circles C_j with $j < i$, then $P \cup P' \cup R$ contains a unique pseudo-circle C_i. This pseudo-circle meets the other pseudo-circles C_j with $j < i$ exactly in v. Also in this case $\mathcal{K}(C_i) = \{K, K'\}$. So assume that R is not disjoint from the pseudo-circles C_j with $j < i$, and let u be the first vertex of R which is also contained in some C_j with $j < i$. Then the pseudo-arc which starts in u, then contains a pseudo-arc contained in R ending in a vertex of P and then continuing along P until v can be extended to a pseudo-circle C_i by adding a pseudo-arc contained in C_j. Then $P \in \mathcal{K}(C_i)$ contains K. Also every circuit in the union of the edge sets of the C_j with $j \leq i$ contains v. Thus in both cases C_i has been defined as required.

Let C be the union of the edge sets of the pseudo-circles C_i with $i \in \mathbb{N}$. Then C is a scrawl with infinite nullity which does not contain finite circuits. Thus M is not nearly finitary. □

Lemma 3.22. *Assume that no edge is incident with a vertex of V_f. Also assume that there is a component K of M_{fin} for which there are arbitrarily large finite v-in-fans of pseudo-arcs whose edges are contained in K. Then M is not nearly finitary.*

Proof. Let G' be the graph whose set of vertices is the set of vertices of G which are incident with edges of K and whose edge set is K. Because K is connected in M_{fin}, also G' is connected. By Observation 3.11 every pseudo-arc which only uses edges of K is a path, double ray, ray or inverse thereof of G'. Say a ray of G' converges to a vertex $w \in V_f$ if there is a pseudo-arc whose edge set is the set of edges of the ray and whose vertex set is the set of vertices of the ray together with w. So by Observation 3.11 there are arbitrarily large families of pairwise disjoint rays which converge to v.

First consider the case that there are infinitely many ends of G' containing a ray which converges to a vertex of V_f. Then there is $w \in V_f$ such that infinitely many ends of G' contain a ray converging to w, and thus there is an infinite family of pairwise disjoint rays which all converge to w. These can be connected to a tree T. Then the edge set of T is a scrawl of M which does not contain finite circuits. A double ray which is the union of two rays converging to w can be turned into a pseudo-circle by adding w. Deleting a finite number of edges from T cannot separate all rays of T converging to w, so the edge set of T has infinite nullity. So M is not nearly finitary.

So assume that only finitely many ends of G' contain a ray converging to a vertex of V_f. Thus one end ω of these finitely many ends of G' contains arbitrarily large finite families of pairwise disjoint rays converging to v. The end ω can be separated from the finitely many other ends containing rays converging to a vertex of V_f by finitely many vertices $w_1, \dots, w_n$. Let G'' be the component of $G' - \{w_1, \dots, w_n\}$ containing ω. Then all pseudo-circles whose edges are contained in G'' are by Observation 3.11 finite unions of double rays together with a subset of V_f, and every tail of one of these double rays is contained in ω.

The following two claims show that all rays of G'' in ω which are disjoint from some other ray in G'' of ω converge to v. In particular from now on all rays are rays of G''.

Claim. Let R_1, R_2 and R_3 be three rays in ω such that R_1 and R_3 converge to v and R_3 is disjoint from R_2 and shares only its first vertex with R_1. Then R_2 converges to v.

Proof of claim. First consider the case that R_1 and R_2 intersect infinitely often. Let $(C_x)_{x \in X}$ be the set consisting of the edge sets of the circles of the form $axbPcQ$ where a and b are the end vertices of the edge x which is an edge of R_1 but not of R_2, c is a vertex, P is a (possibly trivial) subpath of R_1 meeting R_2 only in c and Q is a subpath of R_2 such that cQa traverses R_2 in forward direction. Apply infinite circuit elimination where C is the union of the edge sets of R_1 and R_3, z is an edge of R_3 and $(C_x)_{x \in X}$ is as above. The resulting circuit is the edge set of a double ray which contains R_3 and a tail of R_2. This shows that R_2 converges to v.

So now consider the case that R_1 and R_2 intersect only finitely often, so assume, by shortening R_2 if necessary, that R_1 and R_2 are disjoint. Because R_1 and R_2 belong to the same end of G', there are infinitely many pairwise disjoint paths from R_2 to R_1. Shortening them yields either infinitely many pairwise disjoint paths from R_2 to R_1 not meeting R_3 or infinitely many pairwise disjoint paths from R_2 to R_3 not meeting R_1. In both cases infinite circuit elimination yields a circuit which is the edge set of a double ray which contains a tail of R_2. This implies that R_2 converges to v. ◊

Claim. Let R_1 be a ray in ω which is disjoint from another ray in ω. Then R_1 converges to v.

Proof of claim. As G'' is connected, there is a ray R_2 in ω whose first vertex is a vertex of R_1 and which is otherwise disjoint from R_1. Assume, by shortening R_1 if necessary, without loss of generality that the first vertex of R_2 is also the first vertex of R_1. As there is a double ray which can be turned into a pseudo-circle by adding v, there are two rays R_3 and R_4 converging to v which share their first vertex and are otherwise disjoint. Because ω is the only end of G'' containing rays converging to v, both R_3 and R_4 are contained in ω. Consider the case that one of R_1 and R_2 meets one of R_3 and R_4 only finitely many times, say R_1 meets R_3 only finitely many times. Let R_1' be a tail of R_1 which is disjoint from R_3. Then the previous claim can be applied to R_4, a tail of R_1 and R_3 to show that R_1 converges to v. As G'' is connected, there is a ray R_1' containing a tail of R_1 which shares it first vertex with R_3 and is otherwise disjoint from R_3. Then the previous claim can be applied to a tail of R_3, a tail of R_2 and R_1' to show that also R_2 converges to v.

So assume that R_1 meets both R_3 and R_4 infinitely many times and so does R_2. Let $E_1 = E(R_1) \cup E(R_2)$ and $E_2 = E(R_3) \cup E(R_4)$. As all four rays are contained in ω, there are infinitely many paths which are pairwise disjoint, start in a vertex of R_1 or R_2, end in a vertex of R_3 or R_4 and do not have inner vertices which are contained in one of the four rays. Thus by adjusting initial segments of R_1 and R_2 if necessary, without loss of generality there is an edge $z \in E_1 \cap E_2$. The set of edges in $E_2 \backslash E_1$ is countable, count it by $\{e_1, e_2, \ldots\}$. Let X be the empty set. For each index $i \in \mathbb{N}$ do the following. If there is a circuit $C_i \subseteq E_1 \cup E_2$ containing e_i and not meeting $\{z, e_1, \ldots, e_{i-1}\}$ then add e_i to X and let $C_{e_i} = C_i$. Otherwise do nothing and go on to the next index. Put on X the order $e_i \leq e_j :\Leftrightarrow j \leq i$. Applying Lemma 1.35 to E_2, z and $(C_x)_{x \in X}$ yields a circuit C such that $z \in C \subseteq (E_1 \cup E_2) \backslash X$. In order to prove the claim, it suffices to show that $C = E_1$.

Assume for a contradiction that $(E_1 \cup E_2) \backslash X$ contains a circuit C' (finite or edge set of a double ray) which is not E_1. Then C' contains an edge $e_i \in E_2 \backslash E_1$, and there is a shortest path P with end vertices in R_1 or R_2 containing e_i. Let j be the smallest index of such that e_j is an edge of P. Then no edge in $\{z, e_1, \ldots, e_{j-1}\}$ is an edge of P. If the end vertices of P are not separated by z in E_2 then the edge set of P can be closed to a finite circuit by adding edges of E_2. Otherwise there are, because R_1 meets R_3 and R_4 infinitely often, infinitely many pairwise disjoint paths whose first vertex is a vertex of R_3, whose last vertex is a vertex of P_4 and which do not contain vertices of R_3 and R_4 as inner vertices. Let R_5 and R_6 be the rays obtained from the union of R_3 and R_4 by deleting z. Of the infinitely many paths, only finitely many contain an edge of $\{z, e_1, \ldots, e_{j-1}\}$, only finitely many meet R_5 or R_6 before that ray meets P, and only finitely many do not have both an end vertex in R_5 and an end vertex in R_6. So one of these paths P' does not contain an edge of $\{z, e_1, \ldots, e_{j-1}\}$ and has one end vertex in R_5 which is later in R_5 than the end vertex of P in R_5 and has one end vertex in R_6 which is later in R_6 than the end vertex of P in R_6. Then the edge sets of P and P' form together with finitely many edges of R_3 and R_4 a circuit. In both cases there is a circuit which should have been chosen as C_{e_j} such that $e_j \in X$, a contradiction.

So $(E_1 \cup E_2) \backslash X$ contains exactly one circuit, and that is E_1. Hence the tails of the double ray with edge set E_1 converge to v, so R_1 converges to v. $\Diamond$

As there are arbitrarily large finite pairwise disjoint families of rays in ω, there is by Lemma 1.1 also an infinite such family. By the previous claim, all rays in this infinite family converge to v. As G'' is connected, these rays can be connected to a tree, and the edge set of the tree contains a scrawl of infinite nullity. Thus M is not nearly finitary. $\square$

Lemma 3.23. *Assume that there is $v \in V_f$ such that the simplification of every pseudo-arc containing v is infinite and such that there are arbitrarily large v-fans of pseudo-arcs which are intervals of pseudo-circles. Then M is not nearly finitary.*

Proof. The following basic result about the interaction of two pseudo-arcs which both end in v will be used in the rest of the proof without further mention.

Let A and B be two non-trivial pseudo-arcs which have v as last vertex. Assume, by shortening them if necessary, that A and B do not meet $V_f - v$. By applying Lemma 3.10, shortening A and B further if necessary implies that their simplifications are either equal or disjoint up to v. In the first case, because the simplifications of A and B are infinite there are infinitely many vertices in $\sigma(A) \cap \sigma(B)$ and all these vertices are also contained in $A \cap B$. In the second case Lemma 3.8 states that A and B meet in at most one other element than v, and shortening them again if necessary yields $A \cap B = \{v\}$. The two cases are mutually exclusive.

Recursively define families $(C_i)_{0 \leq i \leq n}$ with the following properties: Every C_i is a pseudo-circle, the union of the edge sets of the C_i is independent in M_{fin} and the union of the pseudo-circles contains a v-fan of at least $n+1$ many pseudo-arcs whose simplifications are pairwise disjoint except at v. The first pseudo-circle C_0 can simply be chosen as a circuit containing a pseudo-arc of one of the v-fans. Because v is not incident with any edge, C_0 is infinite. Assume that $C_0, \dots, C_{n-1}$ are already constructed. Each C_i gives rise to two non-trivial pseudo-arcs P_i and Q_i which have v as last vertex and are otherwise disjoint. Let $\mathcal{P}$ be the set consisting of the P_i and Q_i. Assume, by shortening them where necessary, that the elements of $\mathcal{P}$ do not contain vertices of V_f other than v and that pairwise they are either disjoint except at v or have equal simplifications. Let $\mathcal{P}'$ be a maximal subset of elements of $\mathcal{P}$ which are pairwise disjoint except for v.

Let $\mathcal{Q}$ be a v-in-fan of non-trivial pseudo-arcs which are intervals of circuits such that $\mathcal{Q}$ contains at least one more element than $\mathcal{P}'$. Assume, by shortening them if necessary, that the elements of $\mathcal{Q} \cup \mathcal{P}'$ are disjoint except for v or have the same simplification. Then no element of $\mathcal{P}'$ can have the same simplification as two elements of $\mathcal{Q}$, as those two elements of $\mathcal{Q}$ then would have the same simplification, contradicting the fact that the only vertex they share is v. Thus every pseudo-arc in $\mathcal{P}'$ is internally disjoint from every pseudo-arc in $\mathcal{Q}$ except for one pseudo-arc. So $\mathcal{Q}$ contains a pseudo-arc Q which together with the elements of $\mathcal{P}'$ forms a v-in-fan. By construction of $\mathcal{P}'$, the pseudo-arc Q can be shortened such that the only vertex which it shares with any C_i is v. Let C'_n be a pseudo-circle which contains Q. Then C'_n contains a pseudo-arc Q' which contains Q, whose first vertex is a vertex of some C_i, whose last vertex is v and whose inner vertices are not vertices of any C_i with $i < n$. Also Q' can be extended to a pseudo-circle C_n by adding a pseudo-arc contained in some C_i.

When all C_i are defined, their edge sets form together a scrawl witnessing that M is not nearly finitary. □

Theorem 3.24. *Let $G = (V, E)$ be a graph-like space inducing a matroid M such that there is a finite vertex set which meets all infinite pseudo-circles. If M is nearly finitary then it is k-nearly finitary for some $k \in \mathbb{N}$.*

Proof. Let V_f be a finite vertex set of G meeting all infinite pseudo-circles. Assume that M is not k-nearly finitary for any $k \in \mathbb{N}$. It suffices to show that M has a minor which is not nearly finitary, as then by Corollary 2.7 also M is not nearly finitary. By Lemma 1.49 either M is not nearly finitary and the theorem holds for M or there is a component K of M such that the restriction to that component is not k-nearly finitary for any $k \in \mathbb{N}$. In the second case the restriction of G to K, in which V_f also meets every infinite pseudo-circle, induces the restriction of M to K. So assume without loss of generality that M equals $M{\restriction}K$ and thus that M is connected.

If there are infinitely many components of M_{fin} which contain an edge with an end vertex in V_f, then by Lemma 3.18 M is not nearly finitary and the theorem holds. So assume that there are only finitely many such components, so by applying Lemma 3.19 to each $v \in V_f$ there is a restriction of M which is not k-nearly finitary for any k in $\mathbb{N}$. Again this restriction is induced by a restriction of G in which V_f meets all infinite pseudo-circles, so assume without loss of generality that no edge is incident with any vertex in V_f.

Once again, by restricting M to one of its components if necessary, assume that M is connected while still not being k-nearly finitary for any $k \in \mathbb{N}$. Even after the restriction, V_f meets all infinite pseudo-circles and no vertex of V_f is incident with an edge. Recall that a component K of M_{fin} captures a vertex $v \in V_f$ if there is a non-trivial pseudo-arc with last vertex v whose edges are contained in v. By Lemma 3.21 if some vertex v of V_f is captured by infinitely many components of M_{fin} then M is not nearly finitary and the theorem holds, so assume that the set $\mathcal{D}$ of components of M_{fin} which capture a vertex of V_f is finite.

Consider the case that the restriction of M to some $D \in \mathcal{D}$ is not k-nearly finitary for any $k \in \mathbb{N}$. Thus in $M{\restriction}D$ there are scrawls independent in $(M{\restriction}D)_{\text{fin}}$ with arbitrarily large finite nullity. By Lemma 1.69 applied to these scrawls, any cobase of the scrawl and V_f, there is some $v \in V_f$ such that there are arbitrarily large finite v-in-fans whose edge sets are contained in D. Thus by Lemma 3.22 M is not nearly finitary and the theorem holds.

So assume that for every $D \in \mathcal{D}$ the restriction of M to D is k-nearly finitary for some $k \in \mathbb{N}$. If some element of $\mathcal{D}$ has infinite connectivity in M, then by Lemma 1.49 M is not nearly finitary and the theorem holds. So assume that every element of $\mathcal{D}$ has finite connectivity in M. Thus by submodularity of the connectivity function, also all unions of subsets of $\mathcal{D}$ have finite connectivity. Let N be the matroid arising from M by deletion of $\bigcup \mathcal{D}$. As M is not k-nearly finitary for any $k \in \mathbb{N}$, by repeated application of Corollary 2.5 also N is not k-nearly finitary for any $k \in \mathbb{N}$. Additionally, N is induced by the graph-like space $G \setminus \bigcup \mathcal{D}$ in which every pseudo-circle meets V_f, no edge is incident with a vertex of V_f, and no component of N_{fin} captures a vertex of V_f. As N is a minor of M and it suffices to show that N is not nearly finitary, assume that $M = N$. In particular the simplification of every non-trivial pseudo-arc whose last vertex is contained in V_f is infinite.

As M is not k-nearly finitary for any $k \in \mathbb{N}$, there are scrawls independent in M_{fin} of arbitrarily large finite nullity. If for every $v \in V_f$ there is a finite bound on the nullity of a scrawl independent in M_{fin} whose topological closure meets V_f only in v, then the maximum of these bounds is an upper bound on the nullity of a scrawl independent in M_{fin} whose topological closure contains only one element of V_f. In this case by Lemma 3.20 M is not nearly finitary and the lemma holds for M. So assume otherwise. Thus there is $v \in V_f$ such that there is no finite bound on the nullity of a scrawl independent in M_{fin} whose topological closure contains no vertices of V_f other than v. By Lemma 1.69 there are arbitrarily large finite v-in-fans. Thus by Lemma 3.23 M is not nearly finitary. $\square$

Corollary 3.25. *Let $G = (V, E)$ be a graph-like space inducing a matroid M in which for all distinct vertices $v, w \in V$ there is a finite vertex set $V' \subseteq V - v - w$ such that v and w are contained in different topological components of $G - V' - E_{vw}$,* *where E_{vw} is the set of edges from v to w. If M is nearly finitary, then there is $k \in \mathbb{N}$ such that M is k-nearly finitary.*

PROOF. Assume that M is not k-nearly finitary for any $k \in \mathbb{N}$. Then by Corollary 2.13 there is a minor G' of G such that the matroid induced by G' is not k-nearly finitary for any $k \in \mathbb{N}$ and satisfies $(*)$. So by Lemma 3.12 G' satisfies (S2) and thus by Corollary 3.15 there is a finite vertex set V_f meeting all pseudo-circles. Now Theorem 3.24 can be applied to show that the matroid induced by G' is not nearly finitary, so also M is not nearly finitary. $\square$

Part 2

Flowers and trees

Introduction to flowers and trees

In [**17**], given some connectivity function λ on a finite ground set E, $k \in \mathbb{N}$ and a k-tangle $\mathcal{T}$, Clark and Whittle describe the structure of the separations of connectivity k that belong to the highly connected region encoded by $\mathcal{T}$. For their structure theorem, they define an equivalence relation on the set of subsets of connectivity k that depends on $\mathcal{T}$. The equivalence relation is a refinement of the equivalence relation "contained in the same tangles with truncation $\mathcal{T}$". As the subsets of the ground set can be seen as elements of the separation system of bipartitions of the ground set as in Example 1.76, in the language of this thesis Clark and Whittle are indeed talking about separations. Their goal is to find a tree decomposition such that every edge induces a separation of connectivity k and such that for every equivalence class some representative is induced by an edge. The separations induced by an edge are said to be *displayed* by the tree decomposition. As the displayed separations are necessarily nested and distinct equivalence classes need not contain nested representatives, it is in general not possible to induce representatives of all equivalence classes. In order to remedy that, Clark and Whittle introduce the concept of k-flowers. These are partitions of the ground set, together with a cyclic order, such that (among other properties) certain unions of partition classes are separations of connectivity k, these are the separations displayed by the k-flower. Most separations displayed by a k-flower cross. Clark and Whittle describe a tree decomposition with k-flowers assigned to some nodes such that (with two exceptions) every equivalence class contains a separation that is displayed either by the tree decomposition or by one of the k-flowers at the nodes.

Let $\mathcal{T}'$ be a $k+1$-tangle. Then $\mathcal{T}'$ orients the separations displayed by the tree decomposition and thus corresponds to a node. As the equivalence relation studied by Clark and Whittle is a refinement of "contained in the same tangles with truncation $\mathcal{T}$", distinct tangles with truncation $\mathcal{T}$ correspond to distinct nodes of the tree and are thereby distinguished by the tree. Tree structures like this are called *trees of tangles*. As intuitively tangles are the highly connected regions of a graph or matroid, finding trees of tangles in general means displaying the structure of a graph or matroid in a tree-like way. There are already several papers (for example [**16**], [**23**], [**24**] and [**27**]) on ways to find trees of tangles in several settings, often establishing trees of tangles with additional properties like canonicity. Some of these papers also consider the infinite cases. It is known for example that for every locally finite graph G and every $k \in \mathbb{N}$ there is a canonical tree-decomposition distinguishing all robust k-profiles (see [**16**, Theorem 7.3]). The transition to k-profiles instead of k-tangles is natural and is done in this thesis, too: In graphs as well as in matroids every k-tangle is a robust k-profile, and the definition of k-profiles translates better to the more general setting of separation systems.

In [**4**], Aigner-Horev, Diestel and Postle show that every (possibly infinite) connected matroid with at least three elements has a tree-decomposition of uniform adhesion 2 such that every torso is 3-connected or a circuit or a cocircuit. In this thesis' terminology, in particular taking into account the slightly different definition of connectivity, that means that every separation induced by an edge of the tree-decomposition has connectivity 1. And replacing, in a torso that is a circuit or a cocircuit, every torso edge by the corresponding subset of the ground set E of the whole matroid yields a partition of E in which every union of partition classes is $\emptyset$ or E or has connectivity 1. Together, this means that every separation of connectivity 1 is either displayed by the tree decomposition or by one of the partitions. So the structure theory of trees with flowers has, in the case $k = 1$ ($k = 2$ with the other definition of connectivity), already been extended to infinite matroids.

The goal of this part is to translate the work of [**17**] to the infinite setting. The original goal was to translate the work to infinite matroids, but it turns out that most of the results mainly use the fact that the connectivity function of matroids has a property referred to as limit-closed in this thesis. So the results are considered in the more general context of universes of bipartitions which are closed under unions of chains and whose connectivity function is limit-closed like the connectivity function of matroids is, see Chapter 6. In addition to that setting, in Chapter 5 a setting of vertex separations with a similar condition on limits of chains of separations is investigated, among others this covers the cases of vertex separations of graphs and graph-like spaces, and Chapter 4 investigates another setting of bipartitions, to be explained later. These choices of particular separation systems are natural, as separation systems of vertex separations and separation systems of bipartitions are still the main examples of separation systems, and because tangles and trees of tangles were invented for graphs before they were translated to matroids.

Strategy. The approach taken in this thesis differs in some respects from that of [**17**]. A minor but important technicality is that in this thesis, in contrast to [**17**], the definition of the connectivity function of a matroid does not have the additional $+1$. Thus (as in [**5**]) the k-flowers display separations of connectivity $k - 1$ instead of k. And, as already stated, k-profiles instead of k-tangles will be considered. Also, for some $k - 1$-profile Q, it is not the set of all profiles whose truncation is Q is considered but only a subset $\mathcal{P}$. This allows for proving theorems in the special case that all profiles in $\mathcal{P}$ are closed under taking unions of chains of separations. As finite profiles do not contain infinite chains of separations, such a distinction between different types of k-profiles does not arise in the finite setting. The last, and most important, difference is to take as equivalence relation "contained in the same profiles in $\mathcal{P}$".

Section 4.3 translates the equivalence classes of separations into a subsystem of a universe of bipartitions that is closed under taking unions. The earlier sections of the same chapter show that the bipartitions in such a subsystem can be arranged into a tree-like structure with flower-like objects at some nodes, very much like the tree decomposition with flowers in [**17**]. For finite subsets this tree-like structure is indeed a tree decomposition. Working with the coarser equivalence class avoids some of the technical difficulties of [**17**] where there is an equivalence class that is not closed under taking unions of separations. This simplifies and sometimes makes possible statements and proofs, but of course the cost is that the structure of these equivalence classes is a coarser representation of the structure of all separations.

By the results of Chapter 4, the equivalence classes have a tree-like structure. But the tree decompositions obtained in [**17**] do not display equivalence classes of separations but separations themselves. To translate the structure of the equivalence classes into a tree decomposition with k-flowers displaying separations, it is in particular necessary to be able to turn one flower-like structure into a k-flower of the original separation system. To do so is the main goal of Chapters 5 and 6 for two special cases of separation systems. Here, the first question to solve is what the definition of a k-flower should be. Clark and Whittle show that with their definition, there are only two types of k-flowers: k-anemones, for which all non-trivial unions of partition classes have connectivity k, and k-daisies, for which the unions of petals with connectivity k are exactly the unions of non-trivial intervals. The definition of k-flowers and the proof that they are all k-anemones or k-daisies rely highly on the fact that in every cyclic order with at least three elements, every element has two neighbours. As the tree of equivalence classes may have infinite flower-like structures, and it thus is necessary to have a definition that allows for infinite k-flowers, even in the case of bipartitions it is not possible to just take the definition of k-flowers by Clark and Whittle as is.

After having found a good definition of k-flowers, the next step is to try and find, for a flower-like structure of the equivalence classes, a k-flower that displays representatives of all the equivalence classes of the flower-like structure. That is essentially the same as looking for k-flowers displaying representatives from as many equivalence classes as possible. Phrased differently, if $\preccurlyeq$ is the pre-order in which a k-flower is less than a second one if all equivalence classes represented by a separation displayed by the first k-flower are also represented by a separation displayed by the second k-flower, then the goal is to find maximal elements of that pre-order. A natural approach here would be to use Zorn's Lemma, for which it would be necessary to show that every chain has an upper bound. The upper bound is usually obtained by a limit process from the elements of the chain. Unfortunately, in the setting of both Chapter 5 and Chapter 6, in general two $\preccurlyeq$-comparable finite k-flowers cannot be easily combined into one. The problem becomes only more difficult for $\preccurlyeq$-chains of k-flowers. The solution to this problem taken in this thesis is to take the detour via another partial order $\leq$, where essentially a k-flower is less than a second one if all separations (instead of equivalence classes) displayed by the first k-flower are also displayed by the second one. This partial order is better suited to constructing upper bounds, and its maximal elements essentially are $\preccurlyeq$-maximal. But, at least with the definitions of k-flowers under consideration in this thesis, $\leq$-chains of k-flowers still need not have a k-flower as an upper bound, which is why in Chapters 5 and 6 even more general definitions of k-pseudoflowers are introduced. These definitions will be discussed in detail in the respective sections.

Details and results. It is shown in Section 4.3 that, given a submodular universe $\mathcal{U}$ and a set $\mathcal{P}$ of robust k-profiles with the same truncation, the equivalence classes are a subsystem of a separation system of bipartitions. In order to do so it is shown that the map Φ that maps every separation to the set of k-profiles not containing it is a homomorphism of separation systems. It is also shown that for any two non-nested elements of the image of ϕ their join is also contained in the image of ϕ.

Up to deletion of the empty set and the whole ground set if necessary, the image of ϕ is a separation system that is a subsystem of the set of bipartitions of $\mathcal{P}$, contains neither $\emptyset$ nor $\mathcal{P}$, and has the extra property that for every two non-nested elements their join is also contained in the separation system. The first two sections of Chapter 4 analyse such separation systems $\mathcal{B}$. Thus out of the three settings in which tree-structures with flowers are investigated (the other two being the vertex separations in Chapter 5 and the bipartitions with limit-closed connectivity function in Chapter 6), this can be seen as the most general setting: From every suitable set of k-profiles in any separation system, such a separation system of bipartitions arises. On the other hand, in the theory of k-flowers (and even more so in the related theory of tangle-tree-theorems), much of the work lies in showing that certain suprema of separations are permissible in some way (are contained in the separation system or have the correct connectivity). From this point of view, the assumption that the join of any two non-nested separations is also contained in the separation system under consideration is a very strong one, as it guarantees that any joins (and thus also meets) that might ever become interesting are permissible. Also, among separation systems, separation systems of bipartitions are particularly easy to deal with, not least because they are distributive. Together, at least for finite ground sets, these observations explain why the setting of Chapter 4 can be seen as the setting with the strongest assumptions, that is, the least general setting. This is supported by the fact that, as described earlier, all the separations are displayed by one tree decomposition with flowers. For infinite separation systems, even stronger results on the structure of $\mathcal{B}$ than presented in Section 4.2 could immediately obtained if $\mathcal{B}$ had the additional property that for every infinite chain of separations in $\mathcal{B}$, their union was also contained in $\mathcal{B}$ or the whole ground set. If $\mathcal{B}$ arises from an underlying separation system together with a set of k-profiles as described in Section 4.3, then this extra property can be guaranteed by additional assumptions of limit-closedness on either the underlying separation system (for example the assumptions of limit-closedness satisfied by the special separation systems under investigation in Chapters 5 and 6) or on the set of k-profiles.

Even without the additional limit-closedness property, the separation systems of bipartitions in Chapter 4 have a lot of structure, which is obtained in the first two sections of Chapter 4 as follows. Denote the ground set of the separation system of bipartitions by E and the separation system under consideration by $\mathcal{B}$. For the finest equivalence relation on $\mathcal{B}$ such that any two non-nested separations are contained in the same equivalence class, denote the set of equivalence classes with more than one element by $\mathcal{V}$. For $V \in \mathcal{V}$, denote the coarsest partition of E such that all elements of V are unions of partition classes by $\partial(V)$. If V is finite, then it is either anemone-like in that every non-trivial union of elements of $\partial(V)$ is contained in V, or it is daisy-like in that there is a cyclic order on $\partial(V)$ (unique up to mirroring) such that the elements of V are exactly the non-trivial unions of intervals of $\partial(V)$. If V is infinite, then finite subsets of it are considered that are suitable in that they behave like finite elements of V, and either all suitable finite subsets are anemone-like or all suitable finite subsets are daisy-like. If the suitable finite subsets are daisy-like, then the cyclic orders of those subsets can be combined into a cyclic order of $\partial(V)$ such that all elements of V are unions of intervals of $\partial(V)$.

If $V \in \mathcal{V}$ is finite, then the elements of $\partial(V)$ are contained in the set $\mathcal{E}$ of separations in $\mathcal{B}$ that are nested with all other elements of $\mathcal{B}$. As a consequence, if $\mathcal{B}$ is finite, then $\mathcal{E}$ is a tree set that distinguishes all elements of $\mathcal{V}$ from each other. Thus if $\mathcal{B}$ is finite then its elements are displayed by a tree decomposition of E, and if some node has a flower-like structure attached then that structure arises from an element of $\mathcal{V}$ corresponding to that node. This structure theorem is, modulo some translation work, the same as [**26**, Theorem 4] but was obtained independently. The profiles of $\mathcal{B}$ correspond naturally to the nodes of the tree decomposition, and no node can correspond to more than one profile or element of $\mathcal{V}$. It is then shown in Section 4.3.1 that if $\mathcal{B}$ arises from some underlying separation system, then every profile of $\mathcal{B}$ induces a profile of the separation system.

Chapter 6 starts with the definitions of k-flowers and k-pseudoflowers in the setting of separation systems of bipartitions with a limit-closed connectivity function. It turns out that the two most obvious candidates for infinite anemones are the same. A detailed analysis of how the connectivity of some separations in a k-pseudoanemone determines the connectivity of other separations in the same k-pseudoanemone shows that also the two most obvious candidates for the definition of a k-pseudoanemone (one with a cylcic order and the other without) are essentially the same. By continuing that analysis it can be shown that for an anemone Φ with at least $k+1$ many petals, all k-pseudoanemones extending Φ can be combined to a single $\leq$-maximal k-pseudoanemone. Every other k-pseudoflower also extends to a $\leq$-maximal one, though here uniqueness is not necessarily given.

In Section 6.5 the partial order $\preccurlyeq$ is under consideration. One problem to solve here is the following: In finite k-pseudoflowers, every k-profile has to contain the inverse of one of the partition classes (in this chapter, k-pseudoflowers are partitions with a cyclic order), and is thus considered to be pointing to this partition class. That is not necessarily the case in infinite k-pseudoflowers. But every k-profile that, for a given k-pseudoflower, does not point towards a partition class of that k-pseudoflower is not limit-closed in the sense that it contains a chain of separations whose supremum is not contained in the k-profile. And for a k-profile P that is not limit-closed, all l-profiles with $l \geq k$ that extend P can be combined into a unique profile P' of all separations of finite connectivity. Hence if the set of k-profiles to be distinguished contains at least two k-profiles, then their common truncation P_0 is limit-closed. The obvious corollary is that if a k-pseudoflower contains so many separations of connectivity less than $k-1$ that some of them can be organised into an infinite $k-1$-pseudoflower, then P_0 points towards a partition class of the $k-1$-pseudoflower. With a little more effort, it even follows that a k-pseudoflower whose unions of intervals fail to distinguish two profiles with truncation P_0 must be an extension of an infinite anemone and can be extended to a k-pseudoanemone of which a union of partition classes does distinguish the two profiles. This implies that most $\leq$-maximal k-pseudodaisies are also $\preccurlyeq$-maximal and is one of the reasons why for k-pseudoanemones the partial order $\preccurlyeq_A$ is considered, where $\Phi \preccurlyeq_A \Psi$ if all profiles distinguished by unions of partition classes of Φ (instead of unions of intervals) are also distinguished by unions of partition classes of Ψ. The other reason is that if only unions of intervals are considered, then there are $\leq$-maximal anemones distinguishing infinitely many k-profiles that are not maximal in the partial order of distinguishing profiles. But with $\preccurlyeq_A$, most $\leq$-maximal k-pseudoanemones are $\preccurlyeq_A$-maximal. In the special case that the separation system of bipartitions arises from a matroid, there are no infinite k-daisies and no interesting infinite chains of k-daisies, as is shown in Chapter 8.

Chapter 6 ends with a section relating the k-pseudoflowers back to the abstract setting of Section 4.3. For a k-pseudoflower Φ, every separation of Φ is contained in an equivalence class of separations. If Φ distinguishes at least four profiles, then it displays separations whose equivalence classes are not nested, and thus are elements of some V in $\mathcal{V}$. This V is uniquely determined by Φ (still assuming that Φ distinguishes four profiles) and is anemone-like if and only if Φ is an extension of an anemone. Of course the nestedness of distinct elements of $\mathcal{V}$ translates back to nestedness of k-pseudoflowers belonging to them.

In Chapter 5, separation systems of vertex separations are considered that satisfy a condition of limit-closedness, for example the separations of finite connectivity of a graph-like space. Seen from the perspective of graphs, for a definition of k-pseudoflower in this context it does not suffice to put edges in partition classes, but now vertices have to be placed between the partition classes. That leads to a more complex definition of k-pseudoflower than in Chapter 6, but one that still has so-called petals taking the role of the partition classes of the k-pseudoflowers. In this definition, every k-pseudoflower is either inherently daisy-like, and thus called a k-pseudodaisy, or inherently anemone-like and called a k-pseudoanemone. Also the partial order $\leq$ has to be defined in a more complicated way. Two k-pseudoflowers are defined to be comparable via $\leq$ if there is a specific type of map from one k-pseudoflower to the other, and it is true but non-obvious that in most cases this map is unique. Also, in this setting (as opposed to Chapter 6) it is possible that a k-profile does not point towards a petal of a finite k-pseudoflower, so it is shown that in the relevant cases every k-profile points towards a petal.

From then on, the chapter mostly deals with k-pseudodaisies. It is shown that if all the profiles under consideration are limit-closed, then most $\leq$-maximal k-pseudodaisies are $\preccurlyeq$-maximal, where again $\Phi \preccurlyeq \Psi$ if all profiles distinguished by Φ are also distinguished by Ψ. Section 5.3 contains the proof that there are $\leq$-maximal k-pseudodaisies.

Recall that, given some finite separation system $\mathcal{S}$ and a set of k-profiles $\mathcal{P}$ with the same truncation, the equivalence classes of separations are displayed by a tree decomposition with flowers. Chapter 7 explores what of the tree decomposition with flowers can be recovered if one does not have full information of the separation system. The first section of that chapter discusses the case that there is no (easily accessible) list of all separations of connectivity $k-1$, and only a few of them, $\vec{s}_1$, $\vec{s}_2, \dots$ are given. In this case, an algorithm is described that iteratively computes as much of the structure of the equivalence classes as possible from the first l of the separations. For this routine to work it is necessary to be able to determine whether two equivalence classes are nested. The second section of the chapter is entirely speculative and contains a few ideas on what might work when it is not possible to determine whether two equivalence classes are nested.

CHAPTER 4

Bipartitions with the corner property

Of the various examples of separation systems in which flowers are explored in this thesis, one seems to be particularly easy to deal with because it has so much extra structure: A subsystem of the universe of bipartitions of some ground set where all corners of crossing elements of the separation system are again contained in the separation system. Such separation systems are the topic of this chapter, and Section 4.3 will explain why such separation systems are still sufficiently general to be of interest and that there is a close connection to the topic of tangle-tree-theorems.

4.1. Finite universes

For this section fix a ground set E and a set $\mathcal{B}$ of bipartitions[1] of E such that $\mathcal{B}$ is a separation system with the following property:

If $\vec{s}$ and $\vec{t}$ are crossing elements of $\mathcal{B}$, then $\vec{s} \vee \vec{t}$ is also contained in $\mathcal{B}$.

This section is about finite sets $\mathcal{B}$, but some lemmas are formulated such that they also hold for infinite sets $\mathcal{B}$, which are considered in Section 4.2.

The goal of this section is to organise the elements of $\mathcal{B}$ into maximal flowers and a tree set which interacts nicely with the set of flowers. Clearly any two crossing elements of $\mathcal{B}$ should belong to the same maximal flower, and belonging to the same maximal flower should be an equivalence relation. So consider the finest partition of $\mathcal{B}$ in which elements of different partition classes do not cross. The separations of $\mathcal{B}$ which are nested with $\mathcal{B}$ are exactly those whose partition class only contains this one separation. The set of these separations, denoted by $\mathcal{E}$, is a tree set and will indeed be the tree set associated with $\mathcal{B}$. Denote the set of partition classes with more than one element by $\mathcal{V}$. As described in the introduction to this part of the thesis, the elements of $\mathcal{V}$ behave quite similarly to sets of separations displayed by a flower. If they happen to be finite, then by Lemmas 4.13 and 4.14 they can be organised into structures which are very similar to sets of separations displayed by flowers as in e.g. [**17**], [**5**] or Chapter 6, but for infinite elements of $\mathcal{V}$ that is not necessarily the case. Therefore, given an element V of $\mathcal{V}$, suitable finite subsets of V, called pre-flowers, will be considered as finite approximations of V, and there will be no extra definition of what a flower is in this setting.

Definition 4.1. A *pre-flower* is a finite subset $\mathcal{F}$ of $\mathcal{B}$ with at least two elements such that for all elements $\vec{s}$ and $\vec{t}$ of $\mathcal{F}$ there are $\vec{u_0}, \dots, \vec{u_n} \in \mathcal{F}$ with $a = \vec{u_0}$, $b = \vec{u_n}$ such that $\vec{u_i}$ and $\overrightarrow{u_{i+1}}$ are not nested for $0 \leq i \leq n-1$.

The following notation will be used for both elements of $\mathcal{V}$ and pre-flowers:

[1] In contrast to the definition of the universe of bipartitions, the set $\mathcal{B}$ cannot contain the empty set nor the whole ground set.

Definition 4.2. Given a set S of bipartitions of E let $\sim_S$ be the equivalence relation on E where $e \sim_S f$ if and only if no element of S distinguishes e and f. Denote the set of equivalence classes of $\sim_S$ by $\partial(S)$, the set of elements of $\mathcal{B}$ which are unions of elements $\partial(S)$ by $\overline{S}$, and the set of elements of $\overline{S}$ which are not orientations of elements in $\partial(S)$ by $\mathring{S}$.

Every pre-flower is a subset of some element of $\mathcal{V}$, and if V is finite, then it is itself a pre-flower. Also every pre-flower $\mathcal{F}$ contains two crossing separations, so $\partial(\mathcal{F})$ has at least four elements. Furthermore, the bipartitions to which the elements of $\partial\mathcal{F}$ correspond form a star; and the separations of $\mathcal{B}$ towards which all elements of the star point are exactly the separations contained in $\overline{\mathcal{F}}$. Also $\mathcal{F} \subseteq \mathring{\mathcal{F}} \subseteq \overline{\mathcal{F}}$, and every element of $\mathring{\mathcal{F}}$ contains at least two elements of $\partial(\mathcal{F})$ and is disjoint from at least two elements of $\partial(\mathcal{F})$. As every corner of elements of $\overline{\mathcal{F}}$ corresponds again to a union of elements of $\partial(\mathcal{F})$, also $\overline{\mathcal{F}}$ is closed under unions of crossing elements.

The following two lemmas establish basic facts about the interaction of elements of some $V \in \mathcal{V}$ and elements of $\mathcal{B}\backslash V$.

Lemma 4.3. *Let V be an element of $\mathcal{V}$ and let $\overrightarrow{s}$ be an element of $\mathcal{B}\backslash V$. Then some orientation of $\overrightarrow{s}$ points towards all elements of V.*

Proof. Let P be the set of elements of V towards which $\overrightarrow{s}$ points and let Q be the set of elements of V towards which $\overleftarrow{s}$ points. As $\overrightarrow{s}$ is nested with every element of V, $P \cup Q = V$. Assume for a contradiction that there is an element $\overrightarrow{t} \in P \cap Q$. Then both orientations of $\overrightarrow{s}$ point towards $\overrightarrow{t}$, so either $\overrightarrow{s}$ is an orientation of $\overrightarrow{t}$ or $\overrightarrow{t}$ is not regular. But $\overrightarrow{t}$ is regular because it is an element of $\mathcal{B}$. Furthermore every element of V crosses some element of V, so V is closed under taking inverses and in particular $\overrightarrow{s}$ is not an orientation of $\overrightarrow{t}$. So $P \cap Q = \emptyset$. By the definition of $\mathcal{V}$, this implies that one of P and Q is empty. $\square$

Lemma 4.4. *Let V be an element of $\mathcal{V}$ and let $\overrightarrow{s}$ be a separation in $\mathcal{B}\backslash V$. Then there is a unique element $\overrightarrow{a}$ of $\partial(V)$ such that $\overleftarrow{a}$ points towards $\overrightarrow{s}$.*

Proof. Existence: By Lemma 4.3 there is some orientation $\overrightarrow{t}$ of $\overrightarrow{s}$ which points towards all elements of V. Then no two elements of $\overrightarrow{t}$ are on different sides of separations in V, thus all elements of $\overrightarrow{t}$ are contained in the same element $\overrightarrow{a}$ of $\partial(V)$. So $\overleftarrow{a}$ points towards $\overrightarrow{t}$ and hence also towards $\overrightarrow{s}$.

Uniqueness: Assume for a contradiction that for some other element $\overrightarrow{b}$ of $\partial(V)$ also $\overleftarrow{b}$ points towards $\overrightarrow{s}$. As $\overrightarrow{a} \leq \overleftarrow{b}$, $\overrightarrow{a}$ points towards $\overrightarrow{s}$ as well, and similarly $\overrightarrow{b}$ points towards $\overrightarrow{s}$, too. That both orientations of $\overrightarrow{a}$ point towards $\overrightarrow{s}$ implies that either $\overrightarrow{a}$ is an orientation of $\overrightarrow{s}$ or $\overrightarrow{s}$ is small or cosmall. As $\overrightarrow{s}$ is an element of $\mathcal{B}$, it is neither small nor cosmall, so $\overrightarrow{a}$ is an orientation of $\overrightarrow{s}$. Similarly because both orientations of $\overrightarrow{b}$ point towards $\overrightarrow{s}$, $\overrightarrow{b}$ is an orientation of $\overrightarrow{s}$. Hence $\overrightarrow{b}$ is an orientation of $\overrightarrow{a}$, which is not possible for distinct elements of $\partial(V)$. $\square$

Now comes a more detailed analysis of pre-flowers. For that, fix a pre-flower $\mathcal{F}$ for the following lemmas and corollaries, up to and including Lemma 4.14.

For any two elements $\overrightarrow{s}$ and $\overrightarrow{t}$ of $\mathcal{F}$ there is a witnessing sequence $\overrightarrow{u_0}, \dots, \overrightarrow{u_n}$ of elements of $\mathcal{F}$ such that $\overrightarrow{s} = \overrightarrow{u_0}$ and $\overrightarrow{u_n} = \overrightarrow{t}$. The following lemma implies by induction that such a sequence can be kept short if one allows the $\overrightarrow{u_i}$ to be elements of $\mathring{\mathcal{F}}$:

Lemma 4.5. *Let $\vec{p}$, $\vec{q}$, $\vec{r}$ and $\vec{s}$ be elements of $\mathcal{B}$ such that $\vec{p}$ crosses $\vec{q}$, $\vec{q}$ crosses $\vec{r}$ and $\vec{r}$ crosses $\vec{s}$. Then $\vec{p}$ crosses $\vec{s}$ or there is $\vec{t} \in \mathcal{B}$ which crosses both $\vec{p}$ and $\vec{s}$.*

Proof. If $\vec{p}$ and $\vec{s}$ are equal or inverses of each other, then $\vec{q}$ crosses both $\vec{p}$ and $\vec{s}$. So assume that $\vec{p}$ and $\vec{s}$ are nested but neither equal nor inverses of each other. If $\vec{q}$ crosses $\vec{s}$ or is an orientation of $\vec{s}$ then the lemma holds, so assume that some orientation $\vec{q'}$ of $\vec{q}$ points towards $\vec{s}$. Similarly assume there is an orientation $\vec{r'}$ of $\vec{r}$ which points towards $\vec{p}$. As $\vec{p}$ does not cross $\vec{s}$, $\vec{q'} \vee \vec{r'}$ crosses both $\vec{p}$ and $\vec{s}$. $\square$

Thus any two nested elements of $\mathcal{F}$ cross a common separation in $\mathring{\mathcal{F}}$. To extend this statement to all separations in $\mathring{\mathcal{F}}$, it suffices to show that all separations in $\mathring{\mathcal{F}}$ cross some separation in $\mathcal{F}$.

Lemma 4.6. *Every element of $\mathring{\mathcal{F}}$ crosses some element of $\mathcal{F}$.*

Proof. Let $\vec{s}$ be an element of $\overline{\mathcal{F}}$ which does not cross any element of $\mathcal{F}$. Then for every element of $\mathcal{F}$ there is some orientation of $\vec{s}$ which points towards that element. Also, for two crossing elements of $\mathcal{F}$ there is one orientation of $\vec{s}$ pointing towards both those separations. By choice of $\mathcal{F}$ there is some orientation $\vec{t}$ of $\vec{s}$ which points towards all elements of $\mathcal{F}$. Then the elements of E contained in $\vec{t}$ are not distinguished by elements of $\mathcal{F}$, so $\vec{t}$ contains only one element of $\partial(\mathcal{F})$, with which it coincides. Thus $\vec{s}$ is not contained in $\mathring{\mathcal{F}}$. $\square$

Corollary 4.7. *Let $\vec{s}$ and $\vec{t}$ be nested separations in $\mathring{\mathcal{F}}$. Then there is an element of $\mathring{\mathcal{F}}$ which crosses both $\vec{s}$ and $\vec{t}$.* $\square$

In order to show that every $\vec{s}$ in $\partial(\mathcal{F})$ is also contained in $\overline{\mathcal{F}}$, it is useful to have a minimal element of $\mathring{\mathcal{F}}$ at hand towards which $\vec{s}$ points. Such a minimal element can easily be chosen with the following extra property:

Lemma 4.8. *Let $\vec{t}$ be an element of $\mathring{\mathcal{F}}$ and let $\vec{a}$ be an element of $\partial(\mathcal{F})$ such that $\vec{a} \leq \vec{t}$. Then there is a minimal element $\vec{s}$ of $\mathring{\mathcal{F}}$ which has the additional property that $\vec{a} \leq \vec{s} \leq \vec{t}$.*

Proof. Let $\vec{s}$ be minimal among all elements of $\mathring{\mathcal{F}}$ which satisfy $\vec{a} \leq \vec{s} \leq \vec{t}$. Assume for a contradiction that $\vec{s}$ is not minimal in $\mathring{\mathcal{F}}$. By Lemma 4.6 there is an element of $\mathcal{F}$ which crosses $\vec{s}$, and one of its orientations $\vec{q}$ is bigger than $\vec{a}$. Then $\vec{q} \wedge \vec{s}$ is not contained in $\mathring{\mathcal{F}}$ and thus an element of $\partial(\mathcal{F})$, which then has to be $\vec{a}$. Hence $\vec{s} \wedge \overleftarrow{q}$ is the set $\vec{s} \setminus \vec{a}$. Because $\vec{s}$ is not minimal in $\mathring{\mathcal{F}}$, it is the union of at least three elements of $\partial(\mathcal{F})$. So $\vec{s} \wedge \overleftarrow{q}$ is the union of at least two elements of $\partial(\mathcal{F})$ and thus contained in $\mathring{\mathcal{F}}$.

By Corollary 4.7 there is a separation in $\mathring{\mathcal{F}}$ which crosses both $\vec{s} \wedge \overleftarrow{q}$ and $\vec{s}$. This separation has an orientation $\vec{u}$ such that $\vec{a} \leq \vec{u}$. Then $\vec{a} \leq \vec{u} \wedge \vec{s} \leq \vec{t}$. As $\vec{u}$ crosses $\vec{s} \wedge \overleftarrow{q}$, there is an element of $\partial(\mathcal{V})$ which is contained in both $\vec{u}$ and in $\vec{s} \wedge \overleftarrow{q}$ and which is thus not $\vec{a}$. Hence $\vec{u} \wedge \vec{s}$ is an element of $\mathring{\mathcal{F}}$ with $\vec{a} \leq \vec{u} \wedge \vec{s} \leq \vec{t}$, contradicting the choice of $\vec{s}$. $\square$

Corollary 4.9. *Let $\vec{a}$ be an element of $\partial(\mathcal{F})$. Then at least two minimal elements of $\mathring{\mathcal{F}}$ contain $\vec{a}$.* $\square$

Corollary 4.10. *$\partial(\mathcal{F})$ is a subset of $\overline{\mathcal{F}}$.*

Proof. Let $\vec{a} \in \partial(\mathcal{F})$ and let $\vec{s}$ be a minimal element of $\mathring{\mathcal{F}}$ containing $\vec{a}$. Then $\vec{s}$ crosses some separation in $\mathcal{F}$ by Lemma 4.6 and some orientation $\vec{f}$ of that separation contains $\vec{a}$. Then $\vec{s} \wedge \vec{f}$ is an element of $\overline{\mathcal{F}}$ which is not an element of $\mathring{\mathcal{F}}$ and contains $\vec{a}$, so it is equal to $\vec{a}$. □

Just as all definitions of flower in this thesis, every pre-flower corresponds to a daisy or an anemone. Those two types are distinguished by the behaviour of minimal sets: A pre-flower $\mathcal{F}$ is a *pre-anemone* if some element of $\partial(\mathcal{F})$ is contained in at least three minimal elements of $\mathring{\mathcal{F}}$ and a *pre-daisy* otherwise. The following two lemmas establish facts about minimal elements of $\mathring{\mathcal{F}}$ which are then used to show that pre-anemones and pre-daisies indeed have structures reminiscent of anemones and daisies:

Lemma 4.11. *Let $\vec{s}$ be a minimal element of $\mathring{\mathcal{F}}$. Then $\vec{s}$ is the union of exactly two elements of $\partial(\mathcal{F})$.*

Proof. Let $\vec{s}$ be a minimal element of $\mathring{\mathcal{F}}$. Then $\vec{s}$ contains at least two elements $\vec{a}$ and $\vec{b}$ of $\partial(\mathcal{F})$. Let $\vec{f}$ be an element of $\mathcal{F}$ which distinguishes elements of $\vec{a}$ from elements of $\vec{b}$. Then $\overleftarrow{s}$ does not point towards $\vec{f}$ by minimality of $\vec{s}$, and $\vec{a}$ and $\vec{b}$ witness that $\vec{s}$ does not point towards $\vec{f}$. So $\vec{s}$ and $\vec{f}$ cross. Now $\vec{s} \wedge \vec{f}$ and $\vec{s} \wedge \overleftarrow{f}$ are both contained in $\overline{\mathcal{F}}$ but not in $\mathring{\mathcal{F}}$, hence they contain only one element of $\partial(\mathcal{F})$ each. Thus $\vec{s} = \vec{a} \cup \vec{b}$. □

Lemma 4.12. *The only partition of $\partial(\mathcal{F})$ in which every minimal element of $\mathring{\mathcal{F}}$ is contained in some partition class is the partition with only one element.*

Proof. Assume for a contradiction that there is a partition of $\partial(\mathcal{F})$ in which every minimal element of $\mathring{\mathcal{F}}$ is contained in some partition class and which has two distinct partition classes. Consider a finest such partition. Because every partition class contains at least two elements by Corollary 4.9 and Lemma 4.11, $\bigcup P$ is an element of $\mathring{\mathcal{F}}$ for every partition class P. Let $\vec{s}$ be a an element of $\mathring{\mathcal{F}}$ minimal with respect to the property that it crosses $\bigcup P$ and $\bigcup Q$ for two distinct partition classes P and Q ($\vec{s}$ exists by Corollary 4.7). Let $\vec{a}$ be an element of P which is less than $\vec{s}$ and let $\vec{b}$ be an element of P which is less than $\overleftarrow{s}$ such that $\vec{a} \cup \vec{b}$ is an element of $\mathring{\mathcal{F}}$. Then $\vec{s} \setminus \vec{a}$, which is a corner of $\vec{s}$ and $\vec{a} \cup \vec{b}$, is contained in $\mathring{\mathcal{F}}$ and thus by minimality of $\vec{s}$ does not cross $\bigcup R$ for any partition class other than Q. In particular $\vec{a}$ is the only element of P which is contained in $\vec{s}$. By Lemma 4.8 there is a minimal element $\vec{t}$ of $\mathring{\mathcal{F}}$ such that $\vec{a} \leq \vec{t} \leq \vec{s}$. Because $\vec{a}$ is the only element of P which $\vec{t}$ contains, the other element of $\partial(\mathcal{F})$ which $\vec{t}$ contains has to be from a different partition class, a contradiction. □

Lemma 4.13. *If $\mathcal{F}$ is a pre-anemone, then $\overline{\mathcal{F}}$ is the set of all unions of elements of $\partial(\mathcal{F})$ that are neither $\emptyset$ nor E.*

Proof. Recall that $\partial(\mathcal{F})$ has at least four elements. If $\partial(\mathcal{F})$ has exactly four elements, then there is one element $\vec{a}$ in $\partial(\mathcal{F})$ such that for every other element $\vec{b}$ of $\partial(\mathcal{F})$ the set $\vec{a} \cup \vec{b}$ is an element of $\mathring{\mathcal{F}}$. As $\mathcal{B}$ is closed under taking inverses, so is $\mathring{\mathcal{F}}$, and thus all six possible unions of two elements of $\partial(\mathcal{F})$ are contained in $\mathring{\mathcal{F}}$. Thus if $\partial(\mathcal{F})$ contains exactly four elements, the lemma holds, so assume otherwise.

Let $\vec{a}$ be an element of $\partial(\mathcal{F})$ which is contained in three minimal elements of $\mathring{\mathcal{F}}$. Let $\vec{b}$, $\vec{c}$ and $\vec{d}$ be elements of $\partial(\mathcal{F})$ such that $\vec{a}\cup\vec{b}$, $\vec{a}\cup\vec{c}$ and $\vec{a}\cup\vec{d}$ are all minimal elements of $\mathring{\mathcal{F}}$. Then $\vec{a}\cup\vec{b}\cup\vec{c}$ is a corner of the crossing separations $\vec{a}\cup\vec{b}$ and $\vec{a}\cup\vec{c}$, so $\vec{a}\cup\vec{b}\cup\vec{c}\in\mathring{\mathcal{F}}$. Also $\vec{b}\cup\vec{c}$ is a corner of the crossing separations $\vec{a}\cup\vec{b}\cup\vec{c}$ and $\vec{a}\cup\vec{d}$ and thus $\vec{b}\cup\vec{c}$ is an element of $\mathring{\mathcal{F}}$. Similarly $\vec{c}\cup\vec{d}$ and $\vec{b}\cup\vec{d}$ are elements of $\mathring{\mathcal{F}}$, implying that each of $\vec{b}$, $\vec{c}$ and $\vec{d}$ also is contained in three minimal elements of $\mathring{\mathcal{F}}$ and that the union of any two of them is an element of $\mathring{\mathcal{F}}$. Thus by Lemma 4.12 every element of $\partial(\mathcal{F})$ is contained in at least three minimal elements of $\mathring{\mathcal{F}}$ and the union of any two elements of $\partial(\mathcal{F})$ is an element of $\mathring{\mathcal{F}}$. Let P be any non-trivial subset of $\partial(\mathcal{F})$. If P or $\partial(\mathcal{F})\backslash P$ has only one element, then the union of all elements of P is an element of $\partial(\mathcal{F})$ or the inverse of an element of $\partial(\mathcal{F})$ and thus contained in $\overline{\mathcal{F}}$ by Corollary 4.10. If both P and $\partial(\mathcal{F})\backslash P$ have at least two elements, then the union of all elements of P is an element of $\overline{\mathcal{F}}$ as corners of crossing elements of $\mathring{\mathcal{F}}$ are contained in $\overline{\mathcal{F}}$. $\square$

Lemma 4.14. *If $\mathcal{F}$ is a pre-daisy, then there is a cyclic order on $\partial(\mathcal{F})$ such that the elements of $\overline{\mathcal{F}}$ are exactly the unions of non-trivial intervals of $\partial(\mathcal{F})$. This cyclic order is unique up to mirroring.*

Proof. Let C be a cyclic order on $\partial(\mathcal{F})$ such that for every element of $\partial(\mathcal{F})$ its two neighbours in C are the two elements with which it is contained in a minimal element of $\vec{\mathcal{G}}$. Such a cyclic order exists by Lemma 4.12. As every union of two crossing elements of $\overline{\mathcal{F}}$ is again contained in $\overline{\mathcal{F}}$, every union of elements of $\partial(\mathcal{F})$ which form a proper interval of C is an element of $\overline{\mathcal{F}}$. Assume for a contradiction that there is an element $\vec{s}$ of $\mathring{\mathcal{F}}$ which is not the union of a proper interval of C. Choose $\vec{s}$ such that it is the union of a minimal number of intervals of C. Let $\vec{a}$ and $\vec{b}$ be elements of $\partial(\mathcal{F})$ contained in $\vec{s}$ such that the interval I of C from $\vec{a}$ to $\vec{b}$ only contains $\vec{a}$, $\vec{b}$ and at least one element of $\partial(\mathcal{F})$ which is not contained in $\vec{s}$. Because $\vec{s}$ is not an interval of C, there is an element of $\partial(\mathcal{F})$ which is neither contained in $\vec{s}$ not in I. Thus $\vec{s}$ and I cross and the corner $\vec{a}\cup\vec{b}$ is an element of $\mathring{\mathcal{F}}$. But then $\vec{a}\cup\vec{b}$ is a minimal element of $\mathring{\mathcal{F}}$, contradicting the fact that $\vec{a}$ is contained in exactly two different minimal elements of $\mathring{\mathcal{F}}$. $\square$

Corollary 4.15. *Let $\mathcal{F}$ be a pre-flower. Then one of the following happens:*

- *The elements of $\overline{\mathcal{F}}$ are exactly the unions of elements of $\partial(\mathcal{F})$ that are neither $\emptyset$ nor E.*
- *There is a cyclic order on $\partial(\mathcal{F})$, unique up to mirroring, such that the elements of $\overline{\mathcal{F}}$ are exactly the unions of non-trivial intervals of $\partial(\mathcal{F})$.*

Definition 4.16. For a pre-flower $\mathcal{F}$ such that not all non-trivial union of partition classes are contained in $\overline{\mathcal{F}}$, the *cyclic orders of* $\mathcal{F}$ are the cyclic orders on $\partial(\mathcal{F})$ such that the subsets of $\partial(\mathcal{F})$ whose unions are contained in $\overline{\mathcal{F}}$ are exactly the non-trivial intervals.

Thus the general analysis of pre-flowers is concluded. Now follows, for finite $\mathcal{B}$, the construction of a tree displaying as much of the structure of $\mathcal{B}$ as possible. The edges of that tree are going to be the elements of $\mathcal{E}$. The elements of $\mathcal{V}$ relate to the separations in $\mathcal{E}$ as follows:

Lemma 4.17. *Let V be a finite equivalence class in $\mathcal{V}$. Then $V = \mathring{V}$ and all elements of $\partial(V)$ are contained in $\mathcal{E}$.*

Proof. As V is a pre-flower, $V \subseteq \mathring{V}$. Let $\overrightarrow{s}$ be an element of $\mathring{V}$. By Lemma 4.6 there is an element $\overrightarrow{t}$ of V which crosses $\overrightarrow{s}$, so $\overrightarrow{s} \in V$ by the definition of $\mathcal{V}$.

Let $\overrightarrow{a}$ be an element of $\partial(V)$, which by Corollary 4.10 is contained in $\mathcal{B}$. In order to show that $\overrightarrow{a}$ does not cross any element $\overrightarrow{s}$ of $\mathcal{B}$, consider two cases. If $\overrightarrow{s} \in \overline{V}$, then $\overrightarrow{s} \in \mathring{V}$ or $\overrightarrow{s} \in \partial(V)$, and in both cases $\overrightarrow{s}$ and $\overrightarrow{a}$ are nested. If $\overrightarrow{s} \notin \overline{V}$, then by Lemma 4.3 some orientation of $\overrightarrow{s}$ points towards all elements of V and is thus less than some element of $\partial(V)$. Also in this case $\overrightarrow{s}$ and $\overrightarrow{a}$ are nested. $\square$

Note that by Theorem 1.81, if $\mathcal{E}$ is finite then there is a tree T whose edge tree set is isomorphic to $\mathcal{E}$ and whose vertices are the consistent orientations of $\mathcal{E}$. For every $V \in \mathcal{V}$ and every $\overrightarrow{s} \in \mathcal{E}$ there is by Lemma 4.3 an orientation of $\mathcal{E}$ which points towards all elements of V. So the set of elements of $\mathcal{E}$ which point towards the elements of V is an orientation O_V of $\mathcal{E}$, and it is a consistent orientation. Thus every element of $\mathcal{V}$ naturally corresponds to a vertex of T. Similarly for every element e of the ground set E, the set of elements in $\mathcal{E}$ which do not contain e is a consistent orientation O_e of $\mathcal{E}$. These orientations are all distinct:

Lemma 4.18. *Assume $\mathcal{B}$ is finite and let X and Y be distinct elements of $\mathcal{V} \cup E$. Then either the consistent orientations O_X and O_Y of $\mathcal{E}$ are distinct or X and Y are both elements of E and there is no element of $\mathcal{B}$ which contains only one of X and Y.*

Proof. First consider the case that one of X and Y, X say, is contained in E and the other is not. Then X is contained in a petal $\overrightarrow{s}$ in $\partial(Y)$. As by Lemma 4.17 every petal of every element in $\mathcal{V}$ is contained in $\mathcal{E}$, also $\overrightarrow{s} \in \mathcal{E}$ and thus $\overrightarrow{s} \in O_Y$ while $\overleftarrow{s} \in O_X$.

Next consider the case that both X and Y are contained in E. If there is no element of $\mathcal{B}$ which contains only one of X and Y, then there is nothing to show. So assume that there is an element $\overrightarrow{s}$ of $\mathcal{B}$ which contains X but not Y. If $\overrightarrow{s} \in \mathcal{E}$, then $\overrightarrow{s} \in O : Y$ and $\overleftarrow{s} \in O_X$. If $\overrightarrow{s} \in V$ for some $V \in \mathcal{V}$, then X and Y are contained in distinct petals $\overrightarrow{s_X}$ and $\overrightarrow{s_Y}$ of V and as petals are contained in $\mathcal{E}$, $O_X \neq O_Y$ follows.

Last consider the case that both X and Y are contained in $\mathcal{V}$. As X and Y are distinct, there is some element of $\mathcal{B}$ in $X \backslash Y$. By Lemma 4.3 some orientation $\overrightarrow{s}$ of that element of $\mathcal{B}$ points towards the elements of Y. By Lemma 4.4 there is an petal $\overrightarrow{p}$ in $\partial(Y)$ such that $\overrightarrow{s} \leq \overleftarrow{p}$. Then also some orientation of $\overrightarrow{p}$ points towards all elements of X, and $\overrightarrow{s} \leq \overleftarrow{p}$ implies that $\overleftarrow{p}$ points towards the elements of X. Because all petals are elements of $\mathcal{E}$, $\overrightarrow{p} \in O_Y$ while $\overleftarrow{p} \in O_X$. $\square$

Now that the consistent orientations O_V have been introduced, Lemma 4.17 can be strengthened as follows:

Corollary 4.19 (of Lemma 4.17)**.** *Let V be a finite element of $\mathcal{V}$. Then $V = \mathring{V}$ and $\partial(V)$ is the set of maximal elements of O_V.*

Proof. By Lemma 4.17, the elements of $\partial(V)$ are elements of $\mathcal{E}$ and therefore contained in O_V. By Lemma 4.4 for every element $\overrightarrow{s}$ of O_V, the inverse of some element $\overrightarrow{p}$ of $\partial(V)$ points towards $\overrightarrow{s}$. As O_V is consistent, either $\overrightarrow{s}$ is an orientation of $\overrightarrow{p}$, in which case they are equal, or $\overrightarrow{s} < \overrightarrow{p}$. Thus every maximal element of O_V is contained in $\partial(V)$. As the elements of $\partial(V)$ are pairwise incomparable, all of them must be maximal elements of O_V. $\square$

So if $\mathcal{B}$ is finite, then the following structure theorem holds.

Theorem 4.20. *Let E be a set and $\mathcal{B}$ a finite subsystem of the separation system of bipartitions on ground set E such that neither orientation of $\{\emptyset, E\}$ is contained in $\mathcal{B}$ and such that the union of any two non-nested elements of $\mathcal{B}$ is also contained in $\mathcal{B}$. Then there is a tree decomposition of E in which some nodes of the tree T are labelled with either A or D such that the following statements hold.*

- *Every separation induced by an oriented edge of T is an element of $\mathcal{B}$.*
- *If a node v of T is labelled with A, then no element of E is assigned to this node, and thus v induces a partition of E. All non-trivial unions of partition classes of this partition are contained in $\mathcal{B}$.*
- *If a node v is labelled with D, then no element of E is assigned to this node, and thus v induces a partition of E. There is a cyclic order of the partition that is unique up to mirroring such that among the unions of partition classes, the ones contained in $\mathcal{B}$ are exactly those that are unions of non-trivial intervals.*
- *All elements of $\mathcal{B}$ are of one of the three types above.*

Proof. Let T be a tree whose edge tree set is isomorphic to $\mathcal{E}$. Mapping every element e of E to the node of T corresponding to O_e turns T into a tree decomposition such that the separations induced by the oriented edges are exactly the elements of $\mathcal{E}$. For $V \in \mathcal{V}$ label the node of T corresponding to O_V with A if all non-trivial unions of elements of $\partial(V)$ are contained in V and with D otherwise. Then the edges of T pointing to that node induce $\partial(V)$ by Corollary 4.19, so no element of E is mapped to this node. The second part of the third item and the fourth item then follow from Corollary 4.15. $\square$

Remark 4.21. The structure of a tree with flowers whose existence is stated in Theorem 4.20 is very close to the tree with flowers whose existence is shown in [**17**], but the theorem here does not follow from the results in that paper because in this section there is no order function and no equivalence relation.

Theorem 4.20 can be translated to [**26**, Theorem 4], even if it has been obtained independently. The translation is as follows: By forgetting about the orientations and deleting bipartitions in which one partition class has only one element, $\mathcal{B}$ can be turned into a split system as defined in [**26**]. That split system induces a so-called decomposition frame that has the intersection property. The nodes labelled D correspond to semi-brittle elements of the decomposition stated in [**26**, Theorem 4], and the nodes labelled A correspond to the brittle elements.

Also the profiles of $\mathcal{B}$ can be related to the tree. For every profile P of $\mathcal{P}$, the intersection of P with $\mathcal{E}$ is a consistent orientation O_P of $\mathcal{E}$, and thus corresponds to a vertex of the tree. That vertex cannot simultaneously belong to an element of $\mathcal{V}$.

Lemma 4.22. *Assume that $\mathcal{B}$ is finite and let P be a profile of $\mathcal{B}$. Then $P \cap \mathcal{E}$ is not of the form O_V for any $V \in \mathcal{V}$.*

Proof. Assume for a contradiction that there is $V \in \mathcal{V}$ such that $P \cap \mathcal{E} = O_V$. Let $\vec{s}$ be a maximal element of $P \cap V$. By Corollary 4.19 $\partial(V)$ is a subset of P.

And by Corollary 4.15 there is an element $\vec{p}$ of $\partial(V)$ such that $\vec{p} \vee \vec{x}$ is contained in $\overline{V}$ but not equal to $\vec{x}$. By the profile property $\vec{x} \vee \vec{p}$ is contained in P, hence it is not contained in V. Thus $\vec{x} \vee \vec{p}$ is the inverse of an element of $\partial(V)$, a contradiction to $\partial(V) \subseteq P$. □

The following lemma shows that every vertex of T with degree at least four that does not correspond to O_V for any $V \in \mathcal{V}$ is of the form $P \cap \mathcal{E}$ for a unique profile P of $\mathcal{B}$. The lemma will also be used in Section 4.3 in order to show that there are circumstances under which such a vertex corresponds to an element of $\mathcal{V} \cup E$.

Lemma 4.23. *Assume $\mathcal{B}$ is finite and let O be a consistent orientation of $\mathcal{E}$ such that $O \neq O_V$ for all $V \in \mathcal{V}$. Then there is a unique consistent orientation of $\mathcal{B}$ which contains O, and if $O = O_e$ for some $e \in E$ or the degree of O in T is at least 4, then the consistent orientation is in fact a profile.*

Proof. For $V \in \mathcal{V}$ there is $\overrightarrow{s_V} \in O \backslash O_V$, and every element of $\mathring{V}$ has a unique orientation which points towards $\overrightarrow{s_V}$. This orientation does not depend on the choice of $\overrightarrow{s_V}$. The union O' of O and, for all V in $\mathcal{V}$, the set of elements of $\mathring{V}$ which point towards $\overrightarrow{s_V}$, is the unique consistent orientation of $\mathcal{B}$ which contains O.

In order to show that O' is a profile it suffices to show that for any three elements $\vec{p}$, $\vec{q}$ and $\vec{r}$ of O' the union is not the whole ground set. Let $\vec{s}_1, \ldots, \vec{s}_n$ be the maximal elements of O'. These are also the maximal elements of O. Then $\vec{p} \vee \vec{q} \vee \vec{r} \leq \vec{s}_i \vee \vec{s}_j \vee \vec{s}_l$ for some, not necessarily distinct, indices i, j and l. If $O = O_e$ for some $e \in E$, then no $\vec{s}_m$ contains e and hence $\vec{s}_i \vee \vec{s}_j \vee \vec{s}_l \neq E$. If the degree of O in T is at least 4, then $\vec{s}_i \vee \vec{s}_j \vee \vec{s}_l$ is less than or equal to the inverse of another maximal element of O'. As O' is a subset of $\mathcal{B}$ and hence only contains non-trivial separations, $\vec{s}_i \vee \vec{s}_j \vee \vec{s}_l \neq E$. In both cases $\vec{p} \vee \vec{q} \vee \vec{r} \neq E$ and thus O' is a profile. □

So the profiles can be related to the tree-decomposition of $\mathcal{B}$ as follows.

Lemma 4.24. *In the situation of Theorem 4.20: For every profile $\mathcal{P}$ label the node of the tree decomposition that corresponds to $P \cap \mathcal{E}$ with P. Then no node is labelled with two profiles, and if $E \neq \emptyset$ then the nodes labelled with profiles are exactly the nodes not labelled with A or D that either have degree at least 4 or to which some element of the ground set is mapped.*

Proof. By Lemma 4.22, no node can be labelled with one of A or D and with a profile of $\mathcal{B}$ simultaneously. Thus if a node v is labelled with a profile P, then, as every element of $\mathcal{B}$ can be found somewhere in the tree, the set of maximal elements of P is a subset of $\mathcal{E}$. Therefore, P is determined by $P \cap \mathcal{E}$ and hence no node can be labelled with two distinct profiles of $\mathcal{B}$.

Let v be a node labelled with a profile P. If v has degree 3, then let $\vec{p}$, $\vec{q}$ and $\vec{r}$ be the separations induced by the three edges whose head is v. As P is a profile and P contains all three separations, $\vec{p} \wedge \vec{q}$ cannot be equal to $\overleftarrow{s}$. This must be witnessed by an element e of the ground set, and then e must be mapped to v in the tree decomposition. Similarly if v has degree 2, then the separations induced by the edges with head v cannot be inverses of each other, and if v has degree 1 then the separation induced by the edge with head v cannot be cotrivial, so also in these cases there must be an element e of the ground set such that e is mapped to v in the tree decomposition.

In the other direction, let v be a node that has degree at least four or to which an element e of the ground set is mapped. Then by Lemma 4.23 there is a profile P such that $P \cap \mathcal{E}$ corresponds to v, and thus v is labelled by P. $\square$

4.2. Infinite universes

If $\mathcal{B}$ is infinite, then $\mathcal{E}$ might be infinite. In that case, $\mathcal{E}$ is still a tree-set, but it is not isomorphic to the edge tree set of a finite tree and not necessarily isomorphic to the edge tree set of some infinite tree. Furthermore $\mathcal{V}$ might have infinite elements. For such an infinite element V, the elements of $\partial(V)$ are not necessarily elements of $\mathcal{E}$, and Lemmas 4.13 and 4.14 only hold in weaker forms. This section introduces additional structure on the set of pre-flowers of an element of $\mathcal{V}$, in order to prove weaker analogues of those two lemmas. For that, throughout this section let V be a fixed element of $\mathcal{V}$.

Let $\Phi(V)$ be the set of pre-flowers which are subsets of V. The subset relation naturally induces a partial order on $\Phi(V)$. Given two comparable pre-flowers $\mathcal{F} \subseteq \mathcal{G}$ of $\Phi(V)$, the partition $\partial(\mathcal{G})$ is a refinement of the partition $\partial(\mathcal{F})$, denote the map which is the inclusion from $\partial(\mathcal{G})$ to $\partial(\mathcal{F})$ by $\pi_{\mathcal{F}\mathcal{G}}$ or just π. Thus $\overline{\mathcal{F}} \subseteq \overline{\mathcal{G}}$ and $\mathring{\mathcal{F}} \subseteq \mathring{\mathcal{G}}$. In particular, $\mathcal{F}$ is a pre-anemone if and only if $\mathcal{G}$ is one:

Lemma 4.25. *Let $\mathcal{F}$ and $\mathcal{G}$ be pre-flowers in $\Phi(V)$ such that $\mathcal{F} \subseteq \mathcal{G}$. Then $\mathcal{F}$ is a pre-anemone if and only if $\mathcal{G}$ is a pre-anemone.*

Proof. First consider the case that $\mathcal{F}$ is a pre-anemone, and let $\vec{p}$, $\vec{q}$, $\vec{r}$ and $\vec{s}$ be elements of $\partial(\mathcal{F})$ such that the union of $\vec{p}$ with any of the other three elements of $\partial(\mathcal{F})$ is an element of $\overline{\mathcal{F}}$. Then there is no cyclic order on $\partial(\mathcal{G})$ which turns all of $\pi^{-1}(\vec{p} \cup \vec{q})$, $\pi^{-1}(\vec{p} \cup \vec{r})$ and $\pi^{-1}(\vec{p} \cup \vec{s})$ into intervals of $\partial(\mathcal{G})$. So by Lemma 4.14 $\mathcal{G}$ is not a pre-daisy, thus it is a pre-anemone.

Now consider the case that $\mathcal{G}$ is a pre-anemone, and let $\vec{p}$, $\vec{q}$, $\vec{r}$ and $\vec{s}$ be distinct elements of $\partial(\mathcal{F})$. Then $\bigcup \pi^{-1}(\vec{p} \cup \vec{q})$, $\bigcup \pi^{-1}(\vec{p} \cup \vec{r})$ and $\bigcup \pi^{-1}(\vec{p} \cup \vec{s})$ are all elements of $\overline{\mathcal{G}}$ by Lemma 4.13, so $\bigcup(\vec{p} \cup \vec{q})$, $\bigcup(\vec{p} \cup \vec{r})$ and $\bigcup(\vec{p} \cup \vec{s})$ are also elements of $\overline{\mathcal{F}}$. As there is no cyclic order on $\partial(\mathcal{F})$ which turns these three sets into intervals of $\partial(\mathcal{F})$, $\mathcal{F}$ is not a pre-daisy by Lemma 4.14 and thus is a pre-anemone. $\square$

The union of two pre-flowers with non-empty intersection is a pre-flower, as well. As furthermore every two separations of V are contained in some common pre-flower in $\Phi(V)$, the latter is a directed set. Thus either all pre-flowers in $\Phi(V)$ are pre-anemones, or they are all pre-daisies. If the pre-flowers are pre-daisies, then their cyclic orders can be chosen such that the maps $\pi_{\mathcal{F}\mathcal{G}}$ are homomorphisms of cyclic orders as follows.

Lemma 4.26. *Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be elements of $\Phi(V)$ such that $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$. Then for every cyclic order of $\mathcal{F}$ there is a unique cyclic order of $\mathcal{G}$ such that $\pi_{\mathcal{F}\mathcal{G}}$ is a homomorphism of cyclic orders. Also, if $T_{\mathcal{F}}$, $T_{\mathcal{G}}$ and $T_{\mathcal{H}}$ are cyclic orders of their respective pre-flowers, and two of the maps $\pi_{\mathcal{F}\mathcal{G}}$, $\pi_{\mathcal{G}\mathcal{H}}$ and $\pi_{\mathcal{F}\mathcal{H}}$ are homomorphisms of cyclic orders, then all three are homomorphisms of cyclic orders.*

Proof. Every cyclic order of $\mathcal{G}$ induces a unique cyclic order of $\mathcal{F}$ such that $\pi_{\mathcal{F}\mathcal{G}}$ is a homomorphism of cyclic orders. As there are two cyclic orders of $\mathcal{G}$, which induce different cyclic orders of $\mathcal{F}$, and there are two cyclic orders of $\mathcal{F}$, the two cyclic orders of $\mathcal{F}$ are necessarily induced by the two cyclic orders of $\mathcal{G}$.

For the second statement, if $\pi_{\mathcal{F}\mathcal{G}}$ is a homomorphism of cyclic orders from $T_{\mathcal{G}}$ to $T_{\mathcal{F}}$ and $\pi_{\mathcal{G}\mathcal{H}}$ is a homomorphism of cyclic orders from $T_{\mathcal{H}}$ to $T_{\mathcal{G}}$, then $\pi_{\mathcal{F}\mathcal{H}}$ is their concatenation and thus a homomorphism of cyclic orders from $T_{\mathcal{H}}$ to $T_{\mathcal{F}}$. If $\pi_{\mathcal{F}\mathcal{G}}$ is a homomorphism of cyclic orders from $T_{\mathcal{G}}$ to $T_{\mathcal{F}}$ but $\pi_{\mathcal{G}\mathcal{H}}$ is not a homomorphism of cyclic orders from $T_{\mathcal{H}}$ to $T_{\mathcal{G}}$, then $\pi_{\mathcal{G}\mathcal{H}}$ is a homomorphism of cyclic orders from the mirror of $T_{\mathcal{H}}$ to $T_{\mathcal{G}}$. So in this case, $\pi_{\mathcal{F}\mathcal{H}}$ is a homomorphism of cyclic orders from the mirror of $T_{\mathcal{H}}$ to $T_{\mathcal{F}}$ and thus is not a homomorphism of cyclic orders from $T_{\mathcal{H}}$ to $T_{\mathcal{F}}$. Similarly, if $\pi_{\mathcal{F}\mathcal{G}}$ is not a homomorphism of cyclic orders from $T_{\mathcal{G}}$ to $T_{\mathcal{F}}$ but $\pi_{\mathcal{G}\mathcal{H}}$ is a homomorphism of cyclic orders from $T_{\mathcal{H}}$ to $T_{\mathcal{G}}$, then $\pi_{\mathcal{F}\mathcal{H}}$ is not a homomorphism of cyclic orders from $T_{\mathcal{H}}$ to $T_{\mathcal{F}}$. □

Putting the cyclic orders of the pre-daisies together, in analogy to Lemma 4.14 there is a cyclic order, unique up to mirroring, in which the elements of $\overline{V}$ are unions of intervals:

Lemma 4.27. *If the elements of $\Phi(V)$ are pre-daisies, then there is a cyclic order on $\partial(V)$ such that all elements of $\overline{V}$ are unions of intervals of $\partial(V)$, and that cyclic order is unique up to mirroring.*

Proof. Let T_V be a cyclic order on $\partial(V)$ such that all elements of $\overline{V}$ are unions of intervals. Then for every pre-flower $\mathcal{F} \in \Phi(V)$ there is a unique cyclic order $T_{\mathcal{F}}$ on $\partial(\mathcal{F})$ such that the inclusion $\partial(V) \to \partial(\mathcal{F})$ is a homomorphism of cyclic orders. As every element of $\overline{\mathcal{F}}$ is the union of an interval of $T_{\mathcal{F}}$, by Lemma 4.14 $T_{\mathcal{F}}$ is a cyclic order of $\mathcal{F}$. So T_V induces a family $(T_{\mathcal{F}})_{\mathcal{F}\in\Phi(V)}$ where every $T_{\mathcal{F}}$ is a cyclic order of $\mathcal{F}$ and such that the inclusions $\partial(\mathcal{G}) \to \partial(\mathcal{F})$ for pseudo-flowers $\mathcal{F} \subseteq \mathcal{G}$ are homomorphisms of cyclic orders. For this proof, call such a family a consistent family of cyclic orders.

In the other direction, every consistent family of cyclic orders $(T_{\mathcal{F}})_{\mathcal{F}\in\Phi(V)}$ induces a set of triples T_V of $\partial(V)$ containing all those triples $(\overrightarrow{p}, \overrightarrow{q}, \overrightarrow{r})$ for which there is $\mathcal{F} \in \Phi(V)$ such that $(\pi_{\mathcal{F}}(\overrightarrow{p}), \pi_{\mathcal{F}}(\overrightarrow{q}), \pi_{\mathcal{F}}(\overrightarrow{r})) \in T_{\mathcal{F}}$. Then T_V is cyclic as every $T_{\mathcal{F}}$ is cyclic, and it is linear because every element of V is contained in some pre-flower. In order to show that T_V is antisymmetric and transitive it suffices to show that for any two triples $(\overrightarrow{p}, \overrightarrow{q}, \overrightarrow{r})$ and $(\overrightarrow{p}', \overrightarrow{q}', \overrightarrow{r}')$ in T_V there is some $\mathcal{G} \in \Phi(V)$ such that both $(\pi_{\mathcal{F}}(\overrightarrow{p}), \pi_{\mathcal{F}}(\overrightarrow{q}), \pi_{\mathcal{F}}(\overrightarrow{r}))$ and $(\pi_{\mathcal{F}}(\overrightarrow{p}'), \pi_{\mathcal{F}}(\overrightarrow{q}'), \pi_{\mathcal{F}}(\overrightarrow{r}'))$ are contained in $T_{\mathcal{F}}$. But that is true as the union of any two pre-flowers in Φ is a subset of another pre-flower in Φ and the maps $\pi_{\mathcal{F}\mathcal{G}}$ are homomorphisms of cyclic orders.

Let $\mathcal{F}$ be any element of $\Phi(V)$. Then every consistent family of cyclic orders can be constructed from, and is thus determined by, its cyclic order of $\mathcal{F}$. In order to see that, let $T_{\mathcal{F}}$ be a cyclic order of $\mathcal{F}$. For every $\mathcal{G} \in \Phi(V)$ with $\mathcal{F} \subseteq \mathcal{G}$ there is by Lemma 4.26 a unique cyclic order $T_{\mathcal{G}}$ of $\mathcal{G}$ such that the map $\pi : \partial(\mathcal{G}) \to \partial(\mathcal{F})$ is a homomorphism of cyclic orders. By Lemma 4.26, for pre-flowers $\mathcal{G}$ and $\mathcal{H}$ in $\Phi(V)$ such that $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$, the map $\pi_{\mathcal{G}\mathcal{H}}$ then is a homomorphism of cyclic orders. For a pre-flower $\mathcal{G} \in \Phi(V)$ that does not contain $\mathcal{F}$, let $\mathcal{H} \in \Phi(V)$ be a pre-flower that contains both $\mathcal{F}$ and $\mathcal{G}$ and let $T_{\mathcal{G}}$ be the cyclic order on $\mathcal{G}$ such that $\pi_{\mathcal{G}\mathcal{H}}$ is a homomorphism of cyclic orders. Also by Lemma 4.26 this definition does not depend on the choice of $\mathcal{H}$. Furthermore, all maps $\pi_{\mathcal{G}\mathcal{H}}$ where $\mathcal{G}$ and $\mathcal{H}$ are pre-flowers of V are homomorphisms of cyclic orders.

So in summary, the cyclic orders of $\partial(V)$ such that every element of $\overline{V}$ is the union of an interval correspond to the cyclic orders of $\mathcal{F}$. As there are exactly two of the latter, and the mirror of a suitable cyclic order of $\partial(V)$ is again suitable, there is a suitable cyclic order of $\partial(V)$ and that is unique up to mirroring. $\square$

If the pre-flowers in $\Phi(V)$ are pre-anemones, then the following weaker version of Lemma 4.13 holds:

Lemma 4.28. *If the pre-flowers contained in $\Phi(V)$ are pre-anemones, then for any two elements $\vec{r}$ and $\vec{s}$ of V, the intersection is contained in $\overline{V} \cup \{\emptyset\}$ and the union is contained in $\overline{V} \cup \{E\}$.*

Proof. By symmetry it suffices to show that the union of $\vec{r}$ and $\vec{S}$ is contained in $\overline{V} \cup \{E\}$. By the definition of $\mathcal{V}$, there is a pre-flower $\mathcal{F} \subseteq V$ that contains both $\vec{r}$ and $\vec{s}$. Thus by Lemma 4.13 both $\vec{r}$ and $\vec{s}$ are unions of petals of $\mathcal{F}$. So also the union of $\vec{r}$ and $\vec{s}$ a union of petals of $\mathcal{F}$. If the union is not E, then by the same lemma the union is contained in $\overline{\mathcal{F}}$, and as $\overline{\mathcal{F}} \subseteq \overline{V}$ the lemma follows. $\square$

So by choosing a suitable cyclic order, all elements of V are unions of intervals of $\partial(V)$, but in general it is not true that all unions of intervals of $\partial(V)$ are also contained in V or even in $\overline{V}$. Still, the intervals whose union is contained in V have an additional structure as follows. Recall the cycle completion of V from the section on cyclic orders, and denote the set $\mathcal{S}(\partial(V))\backslash\partial(V)$ by $\mathcal{Z}$. Then the non-trivial intervals of $\partial(V)$ correspond to the sets $[v, w] \cap \partial(V)$ where v and w are distinct elements of $\mathcal{Z}$. Let $Z \subseteq \mathcal{Z}$ be the set of all vertices $v \in \mathcal{Z}$ for which there is an element $w \in \mathcal{Z}$ such that the union of $[v, w] \cap \partial(V)$ is contained in V.

Lemma 4.29. *For all distinct elements v and w of Z, the union of $[v, w] \cap \partial(V)$ is contained in $\overline{V}$.*

Proof. For elements x and y of Z denote the union of $[x, y] \cap \partial(V)$ by $E(x, y)$. Let x and y be elements of Z such that $E(v, x)$ and $E(w, y)$ are contained in V. If $x = w$ or $v = y$, then $E(v, w) \in V$ and the lemma holds, so assume otherwise. If $E(v, x)$ and $E(w, y)$ do not cross, then by Corollary 4.7 there is an element $\vec{r} \in V$ which crosses both $E(v, x)$ and $E(w, y)$. In that case, one of the corners of $E(w, y)$ and $\vec{r}$ crosses $E(v, x)$ and is of the form $E(w, z)$ or $E(z, w)$ for some $z \in \mathcal{Z}$. As that corner crosses $E(v, x)$, it is not only contained in $\overline{V}$ but also in V. Thus by replacing y with z if necessary, the separations $E(v, x)$ and $E(w, y)$ can be assumed to be crossing. If $x \in]v, w[$, then $y \in]v, x[$ and $E(v, w) = E(v, x) \cup E(y, w)$. If $x \in]w, v[$, then $y \in]x, v[$ and $E(v, w) = E(v, x) \cap E(y, w)$. In both cases $E(v, w)$ is contained in $\overline{V}$. $\square$

If $z \in \mathcal{Z}\backslash Z$, then the set $\{E(v, w) \colon v, w \in Z \wedge z \in [w, v]\}$ is a profile of V which, as the following lemma shows, induces a profile of $\mathcal{B}$. This profile will be used in Section 4.3 in order to show that in some cases $\mathcal{Z}\backslash Z$ is comparably small.

Lemma 4.30. *Let $V \in \mathcal{V}$ and let P be a profile of $\overline{V}$. If P contains no element $\vec{s}$ with $\overleftarrow{s} \in \partial(V)$, then P together with the set of all separations in $\mathcal{B}\backslash V$ which point towards V is a profile of $\mathcal{B}$.*

Proof. Let P' be the union of P and all separations in $\mathcal{B}\backslash V$ which point towards V. As $\mathcal{B}$ is a regular separation system and P contains all elements of $\partial(V) \cap \mathcal{B}$, P' is an orientation of $\mathcal{B}$. In order to show that P' is consistent, let $\vec{s}$ and $\vec{t}$ be elements of $\mathcal{B}$ with $\vec{s} \leq \vec{t} \in P'$. If both $\vec{s}$ and $\vec{t}$ are contained in V, then $\vec{s} \in P'$ by consistency of P. If $\vec{t}$ is an element of V but $\vec{s}$ is not, then $\vec{s}$

points towards V and is thus contained in P'. If neither $\vec{s}$ nor $\vec{t}$ is contained in V, then $\vec{t}$ points towards V and so does $\vec{s}$.

In order to show the profile property, let $\vec{s}$ and $\vec{t}$ be elements of P' such that $\vec{s} \vee \vec{t} \in B$. If both $\vec{s}$ and $\vec{t}$ are contained in $\overline{V}$, then $\vec{s} \vee \vec{t} \in P'$ by the profile property of P. Next consider the case that neither $\vec{s}$ or $\vec{t}$ are contained in V. By Lemma 4.4 there are petals $\vec{p}$ and $\vec{q}$ of V such that $\vec{s} \le \vec{p}$ and $\vec{t} \le \vec{q}$. Assume for a contradiction that $\vec{p} \neq \vec{q}$. Let $\vec{r}$ be an element of V such that $\vec{p} \le \vec{r} \le \overleftarrow{q}$. Then $\vec{s} \vee \vec{t}$ crosses $\vec{r}$ and this thus contained in V. But then $\vec{s} = (\vec{s} \vee \vec{t}) \wedge \vec{r}$, so $\vec{s}$ is contained in $\overline{V}$, a contradiction to the choice of $\vec{s}$. So $\vec{p} = \vec{q}$ and thus $\vec{s} \vee \vec{t} \le \vec{p}$, thus $\vec{s} \vee \vec{t}$ points towards V and is thus contained in P'.

Last consider the case that one of $\vec{s}$ and $\vec{t}$ is contained in V and the other is not contained in $\overline{V}$, say $\vec{s} \in V$ and $\vec{t} \notin \overline{V}$. In particular $\vec{s}$ points towards $\vec{t}$, so $\vec{s} \le \vec{t}$ or $\vec{s} \le \overleftarrow{t}$. In the first case $\vec{s} \vee \vec{t}$ is certainly contained in P', so consider the other case. Let $\vec{r}$ be a separation in V crossing $\vec{t}$ such that $\vec{s} \le \overleftarrow{r}$. As $\vec{s}$ is not contained in $\overline{V}$, it is not equal to $\vec{t} \vee \vec{r}$ and thus $\vec{s} \vee \vec{t}$ crosses $\vec{r}$. In particular the separation $\vec{q}$ defined as $(\vec{s} \vee \vec{t}) \wedge \overleftarrow{r}$ is an element of B and strictly bigger than $\vec{s}$. As $\vec{q}$ crosses $\vec{t}$ and is thus contained in V, also $\vec{q} \wedge \overleftarrow{t}$ is an element of $\overline{V}$. But the latter separation equals $\vec{s}$, a contradiction to the choice of $\vec{s}$. So this case cannot happen. □

4.3. As abstractions of underlying separation systems

Let $\mathcal{U}$ be a submodular universe, $k \in \mathbb{N}$ and S the set of separations of $\mathcal{U}$ of order less than k. Let $\mathcal{P}$ be a non-empty set of regular k-profiles which all have the same truncation Q. Declaring two separations in S to be equivalent if and only if they are contained in the same elements of $\mathcal{P}$ induces a natural equivalence relation on S. This section shows that the set of equivalence classes has the structure of a separation system of bipartitions that is closed under finite unions. Most of the statements in this section may not have been formulated like this and with this terminology before, but describe nevertheless phenomena known and used often in tree-of-tangles theory. In particular, the nested set obtained in [**24**, Theorem 3.6] applied to S and $\mathcal{P}$ is very close to the set of equivalence classes that do not cross other equivalence classes, as will be explained in Example 4.37.

The first lemma of this section shows that the involution, partial order and join of S induce natural maps on the set of equivalence classes which turn the latter into a separation system naturally isomorphic to a separation system of bipartitions of $\mathcal{P}$. Intuitively, that separation system of bipartitions condenses from S the information of how it distinguishes the elements of $\mathcal{P}$.

Definition 4.31. Let $\phi : S \to \mathcal{UB}(\mathcal{P})$ map every separation $\vec{p}$ to the set of elements of $\mathcal{P}$ which contain $\overleftarrow{p}$.

Lemma 4.32. *The map ϕ respects $*$, $\le$ and $\vee$.*

Proof. For all $\vec{s} \in S$ the complement of $\phi(\vec{p})$ in E equals $\phi(\overleftarrow{p})$. If $\vec{p}$ and $\vec{q}$ are elements of S such that $\vec{p} \le \vec{q}$ then, as the elements of $\mathcal{P}$ are regular consistent orientations, all elements of $\mathcal{P}$ which contain $\overleftarrow{p}$ also contain $\overleftarrow{q}$, so $\phi(\vec{p}) \le \phi(\vec{q})$. If $\vec{p}$ and $\vec{q}$ are elements of S such that $\vec{p} \vee \vec{q}$ is also contained in S, then $\phi(\vec{p}) \vee \phi(\vec{q}) \le \phi(\vec{p} \vee \vec{q})$ as ϕ respects the partial order. Let P be a profile in $\mathcal{P}$ which is contained in $\phi(\vec{p} \vee \vec{q})$. Then P contains $(\vec{p} \vee \vec{q})^*$ and thus by the profile property P also contains $\overleftarrow{p}$ or $\overleftarrow{q}$. Thus P is also contained in $\phi(\overrightarrow{p}) \vee \phi(\overrightarrow{q})$. So $\phi(\overrightarrow{p} \vee \overrightarrow{q})$ is contained in $\phi(\overrightarrow{p}) \vee \phi(\overrightarrow{q})$ and thus the two sets are equal. $\square$

Part of Lemma 4.32 is a statement about elements of S whose join is contained in S as well. Lemma 4.33 shows that in many cases that join does exist, a fact that will later on also reveal more structure of the image of ϕ. Both the following and the previous lemma of course imply similar statements for the meet operation.

Lemma 4.33. *Let $\overrightarrow{p}$ and $\overrightarrow{q}$ be elements of S such that $\phi(\overrightarrow{p}) \wedge \phi(\overrightarrow{q}) \notin \{\emptyset, \mathcal{P}\}$. Then $\overrightarrow{p} \vee \overrightarrow{q} \in S$.*

Proof. As $\phi(\overrightarrow{p}) \wedge \phi(\overrightarrow{q}) \neq \emptyset$, there is a profile in $\mathcal{P}$ which contains both $\overleftarrow{p}$ and $\overleftarrow{q}$. If $\overrightarrow{p} \wedge \overrightarrow{q}$ is contained in S, then the profile containing both $\overleftarrow{p}$ and $\overleftarrow{q}$ also contains $\overleftarrow{p} \vee \overleftarrow{q}$. In this case $\overrightarrow{p} \wedge \overrightarrow{q}$ distinguishes two profiles in $\mathcal{P}$ and thus has order $k-1$. So in each case $\overrightarrow{p} \wedge \overrightarrow{q}$ has order at least $k-1$ and by submodularity $\overrightarrow{p} \vee \overrightarrow{q}$ has order at most $k-1$ and is thus contained in S. $\square$

If two elements of S satisfy $\phi(\overrightarrow{p}) \leq \phi(\overrightarrow{q})$ then $\overrightarrow{p} \leq \overrightarrow{q}$ does not necessarily hold. But Lemma 4.34 shows for nested elements of the image of ϕ that there do exist pre-images that are nested.

Lemma 4.34. *Let $\overrightarrow{s}$ and $\overrightarrow{t}$ be elements of the image of ϕ such that $\overrightarrow{s} \leq \overrightarrow{t}$. Then there are elements $\overrightarrow{p}$ and $\overrightarrow{q}$ of S such that $\phi(\overrightarrow{p}) = \overrightarrow{s}$, $\phi(\overrightarrow{q}) = \overrightarrow{t}$ and $\overrightarrow{p} \leq \overrightarrow{q}$.*

Proof. If $\overrightarrow{s} = \overrightarrow{t}$, then it suffices to pick $\overrightarrow{p} = \overrightarrow{q}$, so assume otherwise. Let $\overrightarrow{p}$ and $\overrightarrow{q}$ be elements of S such that $\phi(\overrightarrow{p}) = \overrightarrow{s}$ and $\phi(\overrightarrow{q}) = \overrightarrow{t}$. Also if $\overrightarrow{s} = \emptyset$ and $\overrightarrow{t} = \mathcal{P}$, then at least one of $\overrightarrow{p} \wedge \overrightarrow{q}$ and $\overrightarrow{p} \vee \overrightarrow{q}$ is contained in S. In the first case, replace $\overrightarrow{p}$ by $\overrightarrow{p} \wedge \overrightarrow{q}$ and in the second case replace $\overrightarrow{q}$ with $\overrightarrow{p} \vee \overrightarrow{q}$ and the lemma holds. So assume also that $\overrightarrow{s} \neq \emptyset$ or $\overrightarrow{t} \neq \mathcal{P}$.

So one of $\overrightarrow{s} \wedge \overrightarrow{t}$ and $\overrightarrow{s} \vee \overrightarrow{t}$ is not contained in $\{\emptyset, \mathcal{P}\}$. Thus by Lemma 4.33, one of $\overrightarrow{p} \wedge \overrightarrow{q}$ and $\overrightarrow{p} \vee \overrightarrow{q}$ is contained in S, and replacing one of $\overrightarrow{p}$ and $\overrightarrow{q}$ with the existing corner as above shows that the lemma holds. $\square$

Denote the image of ϕ after deleting $\emptyset$ and $\mathcal{P}$ by $\mathcal{B}$. Note that the empty set is contained in the image of ϕ if and only if S contains a separation which is contained in all profiles in $\mathcal{P}$, and similarly for $\mathcal{P}$. Also $\mathcal{B}$ is a regular separation system which is a sub-system of the universe of bipartitions of $\mathcal{P}$. Furthermore, by Lemma 4.33 in $\mathcal{B}$ many joins are defined:

Corollary 4.35 (of Lemma 4.33)**.** *Let $\overrightarrow{s}$ and $\overrightarrow{t}$ be elements of $\mathcal{B}$ with $\overrightarrow{s} \wedge \overrightarrow{t} \neq \emptyset$ and $\overrightarrow{s} \vee \overrightarrow{t} \neq \mathcal{P}$. Then both $\overrightarrow{s} \wedge \overrightarrow{t}$ and $\overrightarrow{s} \vee \overrightarrow{t}$ are elements of $\mathcal{B}$.*

Proof. Let $\overrightarrow{s} = \phi(\overrightarrow{p})$ and $\overrightarrow{t} = \phi(\overrightarrow{q})$. By Lemma 4.33 both $\overrightarrow{p} \wedge \overrightarrow{q}$ and $\overrightarrow{p} \vee \overrightarrow{q}$ are contained in S, and so by Lemma 4.32 both $\overrightarrow{s} \wedge \overrightarrow{t}$ and $\overrightarrow{s} \vee \overrightarrow{t}$ are contained in $\mathcal{B}$. $\square$

Hence $\mathcal{B}$ is closed under unions of crossing elements, so the theory of Sections 4.1 and 4.2 can be applied. Also, if S is finite, then most equivalence classes have a biggest and a smallest element.

Corollary 4.36 (of Lemma 4.33)**.** *If S is finite, then for every $\overrightarrow{s} \in \mathcal{B}$ the set $\phi^{-1}(\overrightarrow{s})$ has a biggest and smallest element.*

Example 4.37 (Tree of tangles)**.** Let $\mathcal{U}$ be a finite submodular universe, $k \in \mathbb{N}$ and $\mathcal{P}$ a set of regular l-profiles with $l \leq k$ which is closed under taking truncations. A common strategy for the proof of tree-of-tangles theorems is to find, recursively from $l = 1$ to $l = k$, for every $l-1$-profile Q in $\mathcal{P}$ a tree (set) which distinguishes all the profiles in $\mathcal{P}$ whose truncation is Q, and to combine all the tree sets into a large one. To a certain extent, this strategy can be mimicked within the current theory: Given an l-profile $Q \in \mathcal{P}$ with $l \leq k-1$, let $\mathcal{P}_Q$ be the set of profiles in $\mathcal{P}$ whose truncation to an l-profile is Q. Then ϕ_Q, $\mathcal{B}_Q$ and $\mathcal{E}_Q$ can be defined as in this section, where $\mathcal{P}_Q$ takes the role of the set of $l+1$-profiles whose truncation is Q. Call the tree (unique up to isomorphism) whose edge tree set is isomorphic to $\mathcal{E}_Q$ the *abstract tree* of Q. If Q is not the empty set, then $k-1 \neq 0$ and thus the truncation Q' of Q to a $k-2$-profile exists. In that case, the orientation O_Q of $\mathcal{E}$ which Q induces is a vertex of the abstract tree of Q'. Thus the set of abstract trees has itself a tree-structure. For an example see Fig. 1. Also for every tree set $\mathcal{E}_Q$ there is by Lemma 4.33 a tree set T_Q contained in S which is mapped isomorphically to $\mathcal{E}_Q$ by ϕ_Q.

It is not always possible to combine the tree sets T_Q into one tree set, as elements of distinct tree sets T_Q need not be nested. Less abstract, the following can happen: There are integers $k \leq l$ and two $k+1$-profiles P_1 and P_2 which are distinguished by exactly two separations $\vec{r}$ and $\overleftarrow{r}$ of S that have order k. Also, there are two $l+1$-profiles Q_1 and Q_2 which are distinguished by exactly two separations $\vec{s}$ and $\overleftarrow{s}$ of S that have order l, and such that $\vec{s}$ crosses $\vec{r}$. As $k \leq l$, one of the orientations of $\vec{r}$ is contained in both Q_1 and Q_2, say $\vec{r} \in Q_1 \cap Q_2$. Under these assumptions, $\vec{r} \vee \vec{s}$ has order at least $l+1$, as otherwise it would distinguish Q_1 from Q_2. Thus $\vec{r} \wedge \vec{s}$ has order less than k and it does not distinguish P_1 from P_2, thus it must be contained in $P_1 \cap P_2$. Likewise, $\vec{r} \wedge \overleftarrow{s}$ has order less than k and is contained in $P_1 \cap P_2$. Thus one of P_1 and P_2 contains not only $\vec{r} \wedge \vec{s}$ and $\vec{r} \wedge \overleftarrow{s}$ but also $\overleftarrow{r}$. [**15**] contains in Section 6 a graph with k-blocks (special cases of $k+1$-profiles) and $k+1$-blocks behaving exactly as described here, and introduces the notion of robustness.

In [**24**] that same notion of robustness is formulated in terms better suited to the context of this thesis: A k-profile P of a universe is robust if for all $\vec{r} \in P$ and all separations $\vec{s}$, if both $\overleftarrow{r} \wedge \vec{s}$ and $\overleftarrow{r} \wedge \overleftarrow{s}$ have order less than the order of $\vec{r}$, then they are not both contained in P. Profiles being robust has the following implication: A separation distinguishes two profiles P_1 and P_2 *efficiently* if it distinguishes them but no separation of lesser order than $\vec{s}$ distinguishes P_1 and P_2. Separations contained in any T_Q efficiently distinguish some profiles in $\mathcal{P}$, and typically separations in trees of tangles also efficiently distinguish two profiles of the set of profiles under consideration. Robustness of the profiles in $\mathcal{P}$ now ensures that, given two crossing separations of different orders which each distinguish two profiles efficiently, there is a corner of the two profiles that distinguishes all those profiles efficiently that are distinguished efficiently by the one separation of the original ones that has the bigger order. This property is used in the proof of [**24**, Theorem 3.6] to show that, under a slightly weaker notion of robustness, every robust set of profiles in a submodular separation system has a tree of tangles. In the same way, if all elements of $\mathcal{P}$ are robust then it can be shown that the tree sets T_Q can be chosen in such a way that the union of all T_Q is a nested set.

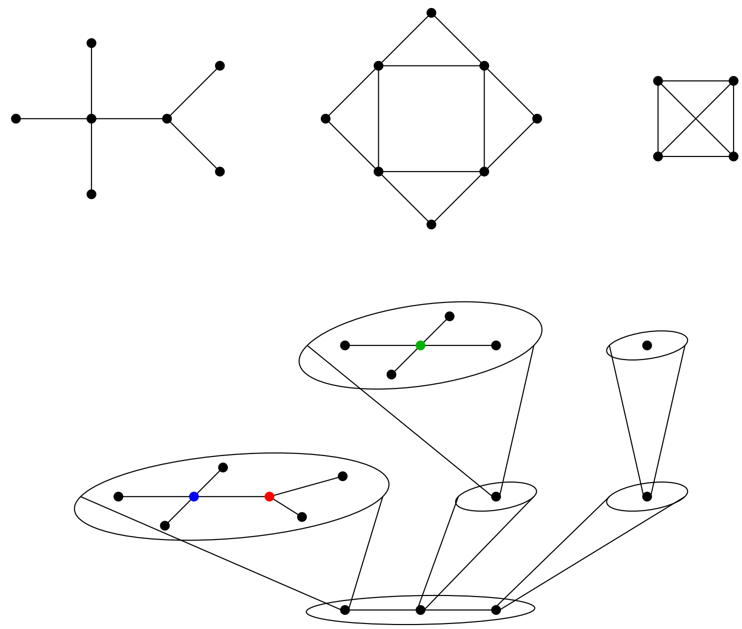

FIGURE 1. A graph with three components and its collection of abstract trees for the set of all k-profiles with $k \leq 3$. The black vertices correspond to profiles, the blue vertex corresponds to an anemone, the green vertex corresponds to a daisy and the red vertex corresponds to nothing.

Theorem 3.3 of [**24**] makes a statement about the existence of canonical trees of tangles. The proof constructs a tree set τ such that the restriction of ϕ to τ has $\mathcal{E}$ as its image and is nearly injective or injective. The proof can be translated into a construction of τ from $\mathcal{E}$ as follows: Let T be a tree whose edge tree set is isomorphic to $\mathcal{E}$. Let v be a vertex of T such that the maximal distance from v to a leaf is minimized. If v is unique, then let $O \subseteq \mathcal{E}$ be the set of separations whose inverse is contained in the consistent orientation v. If v is not unique, then there are only two possible choices v and v' which are joined by an edge of T, let O be the set of separations whose inverse is contained in $v \cup v'$. By Corollary 4.36 there is for every $\overrightarrow{r} \in O$ a biggest element of $\phi^{-1}(\overrightarrow{r})$. The tree set consisting of all these biggest elements equals τ. Thus if v is unique, then there is a canonical tree set contained in S which is mapped to $\mathcal{E}$ isomorphically by ϕ.

4.3.1. Properties of the set of profiles which induce additional properties of the abstract setting. The following sections contain further properties of the abstract setting which can be obtained by making more assumptions on $\mathcal{U}$ and its order function. This subsection contains a few results which are obtained by making further assumptions on $\mathcal{P}$. Recall the following consistent orientations of $\mathcal{E}$ which were defined in Section 4.1: Let $O_P = \{\overrightarrow{s} \in \mathcal{E} \colon P \notin \overrightarrow{s}\}$ for all $P \in \mathcal{P}$ and $O_V = \{\overrightarrow{s} \in \mathcal{E} \colon \overrightarrow{s}$ points towards the elements of V $\}$ for all $V \in \mathcal{V}$.

Lemma 4.38. *Let P' be a profile of $\mathcal{B}$. Then*

$$\{\vec{s} \in S \colon \phi(\vec{s}) \in P' \cup \{\emptyset\}\}$$

is a profile of S whose truncation to a $k-1$-profile is Q.

Proof. Denote $\{\vec{s} \in S \colon \phi(\vec{s}) \in P' \cup \{\emptyset\}\}$ by Q'. In order to prove consistency, let $\vec{s}$ and $\vec{t}$ be elements of S such that $\vec{s} \le \vec{t}$ and $\vec{t} \in Q'$. Then $\phi(\vec{s}) \le \phi(\vec{t})$ and $\phi(\vec{t}) \in P'$, so by consistency of P' also $\phi(\vec{s}) \in P'$, implying $\vec{s} \in Q'$.

In order to show the profile property, let $\vec{s}$ and $\vec{t}$ be elements of Q' such that $\vec{s} \vee \vec{t} \in S$. Then $\phi(\vec{s})$ and $\phi(\vec{t})$ are both contained in P', so also $\phi(\vec{s}) \vee \phi(\vec{t})$ is an element of P'. But the latter bipartition equals $\phi(\vec{s} \vee \vec{t})$, so also $\vec{s} \vee \vec{t} \in Q'$. ☐

Corollary 4.39. *Assume that $\mathcal{B}$ is finite and $\mathcal{P}$ is the set of all k-profiles with truncation Q. Let T be a tree whose edge tree set is isomorphic to $\mathcal{E}$ and whose vertices are consistent orientations of $\mathcal{E}$. Then every vertex of T with degree at least 4 is of the form O_X for some $X \in \mathcal{V} \cup \mathcal{P}$.*

Proof. Let v be a vertex of T which has degree at least 4 and which is not of the form O_V for some $V \in \mathcal{V}$. Then by Lemma 4.23 there is a profile P of $\mathcal{B}$ such that $P \cap \mathcal{E} = v$. The set $\{\vec{s} \in S \colon \phi(\vec{s}) \in P \cup \{\emptyset\}\}$ is a profile P' of S with truncation Q, so it is contained in $\mathcal{P}$. Also $O_{P'} = v$. ☐

For the next corollary, recall that for a cyclic order C the ground set of the cycle completion is denoted by $\mathcal{S}(C)$. Furthermore for an element V of $\mathcal{V}$ which contains pre-daisies the set $\mathcal{S}(\partial(V)) \backslash \partial(V)$ is denoted by $\mathcal{Z}$. Also $Z \subseteq \mathcal{Z}$ contains all elements v for which there is $w \in \mathcal{Z}$ such that the union of $[v, w] \cap \partial(V)$ is contained in V.

Corollary 4.40. *Assume that $\mathcal{P}$ is the set of all k-profiles with truncation Q and let $V \in \mathcal{V}$ such that all pre-flowers contained in V are pre-daisies. Then every $v \in \mathcal{Z} \backslash Z$ has exactly one neighbour in $\mathcal{S}(\partial(V))$.*

Proof. Assume for a contradiction that v has neighbours $\vec{p}$ and $\vec{q}$ in $\mathcal{S}(\partial(V))$. As $\vec{p}$ and $\vec{q}$ are distinct elements of $\partial(V)$, there is an element $\vec{s}$ of V such that $\vec{p} \subseteq \vec{s}$ but $\vec{q} \cap \vec{s} = \emptyset$. By choice of the cyclic order on $\partial(V)$ there is $w \in \mathcal{Z}$ such that $\vec{s} = \bigcup([v, w] \cap \partial(V))$ or $\vec{s} = \bigcup([w, v] \cap \partial(V))$, which is a contradiction to $v \notin Z$. In order to show that v has some neighbour, consider the set of all elements of $\overline{V}$ of the form $\bigcup([u, w] \cap \partial(V))$ where $v \notin [u, w]$. This subset of $\overline{V}$ is a profile of $\overline{V}$ which can by Lemma 4.30 be extended to a profile of $\mathcal{B}$. Let P be the set of elements of S which are mapped to the profile of $\mathcal{B}$ by ϕ. Also P is a profile, this time of S, by Lemma 4.38, and P is contained in some petal $\vec{p}$ of V. Then $\vec{p}$ has to be adjacent to v in $\mathcal{S}(\partial(V))$. ☐

One reason why the structural results on $\mathcal{B}$ are so much weaker in the infinite case is the fact that so far there are no extra assumptions on the closure of $\mathcal{B}$ under infinite operations. A possible extra assumption is for $\mathcal{B}$ to be closed under infinite unions of chains, of course unless that union is the whole ground set of $\mathcal{B}$. This extra assumption can be obtained by asking that all profiles in $\mathcal{P}$ have this property:

Lemma 4.41. *Assume that for all $P \in \mathcal{P}$, all chains $(\vec{s}_j)_{j\in J}$ of elements of P have a supremum in $\mathcal{U}$ and that supremum is contained in P. Then every chain $(\vec{s}_j)_{j\in J}$ of elements of $\mathcal{B}$ has the property that $\bigcup_{j\in J} \vec{s_j}$ either equals $\mathcal{P}$ or is contained in $\mathcal{B}$.*

Proof. By Zorn's Lemma there is a $\subseteq$-maximal chain $(\vec{r}_j)_{j\in J'}$ such that ϕ restricted to this chain is injective and has its image contained in $\{\vec{s_j}_{j\in J}\}$. Denote the supremum of $\{\vec{r_j} : j \in J'\}$ by $\vec{r}$ and the union of all $\vec{s_j}$ with $j \in J$ by $\vec{s}$. As the profiles are all closed under taking suprema of chains, every profile which is not contained in $\vec{s}$ contains all $\vec{r_j}$ and thus also $\vec{r}$. So $\phi(\vec{r}) \le \vec{s}$. Assume for a contradiction that $\phi(\vec{r}) < \vec{s}$, so there is $j \in J$ such that $\vec{s_j} \nleq \phi(\vec{r})$. Then $\phi(\vec{r_j'}) < \vec{s_j}$ for all $j' \in J'$, so similarly to $\phi(\vec{r}) \le \vec{s}$ also $\phi(\vec{r}) \le \vec{s_j}$. But then by Lemma 4.33 there is some $\vec{p} \in S$ such that $\phi(\vec{p}) = \vec{s_j}$ and $\vec{r} \le \vec{p}$, and $\mathcal{P}$ can be added to the chain $(\vec{r_j})_{j\in J'}$, contradicting the maximality of that chain. So $\phi(\vec{r}) = \vec{s}$, so $\vec{s}$ is contained in the image of ϕ. Hence the only reason for $\vec{s}$ not to be contained in $\mathcal{B}$ is if $\vec{s} = \mathcal{P}$. □

The results in this chapter progress from the separation system to the separation system of equivalence classes: From the entirety of the separation system together with $\mathcal{P}$, the separation system of the equivalence classes is extracted and then that separation system can be analysed further, leading (at least in the finite case) to a tree structure with flowers. In Chapter 7 another approach will be taken, in the setting where the underlying separation system is a finite subsystem of a universe of bipartitions: Given only some separations of the underlying separation systems of order $k - 1$, how much of the tree structure can be recovered?

CHAPTER 5

Vertex separations with limits

Definition 5.1. A *limit-closed universe of vertex separations* $\mathcal{U}$ on ground set V is a limit-closed sub-universe of $\mathcal{U}(V)$ such that if $k \in \mathbb{N}$ and $(A_i, B_i)_{i \in I}$ is a chain in $\mathcal{U}$ of separations of order at most k then the supremum of the chain is of the form $(A \cup X, B)$ where $A = \bigcup_{i \in I} A_i$ and $B = \bigcap_{i \in I} B_i$.

Note that a sub-universe of $\mathcal{U}(V)$ whose induced order function is still limit-closed need not be a limit-closed universe of vertex separations.

In [**25**], an oriented separation of a graph on vertex set V is said to be a separation $(A, B) \in \mathcal{U}(V)$ such that every path meeting both A and B also meets $A \cap B$. The set of oriented separations of the graph is clearly a limit-closed universe. Recall that a separation of graph-like space on vertex set V is an element (A, B) of $\mathcal{U}(V)$ such that every pseudo-arc meeting both A and B also meets $A \cap B$. For a given graph-like space, these separations clearly form a universe, due to some very basic properties of paths. Less obviously, the order function of the universe is limit-closed, which can be shown for example by applying Menger's theorem. In [**30**], Hendrik Heine generalised the concept of directed paths to so-called dipath spaces, in such a way that the directed pseudo-arcs of a graph-like space on ground set V turn into dipaths of a dipath space on the same ground set. In particular, defining a separation of a dipath space to be an element (A, B) of $\mathcal{U}(V)$ such that every dipath starting in A and ending in B also meets $A \cap B$ ensures that the separations of a graph-like space on ground set V are exactly the separations of the induced dipath space. Hence the following statement also applies to graph-like spaces:

Theorem 5.2. [**30**, Version of Theorem 4.5 explained in the paragraph after the proof] *Let $\mathcal{G}$ be a dipath space, $A, B \subseteq V(\mathcal{G})$ and $k \in \mathbb{N}$. Then either there exists a set of size less than k meeting every path from A to B or a set of k disjoint paths from A to B.*

The same author also showed a version of the following corollary for dipath spaces, the proof works in both cases:

Corollary 5.3. [**31**] *Let $k \in \mathbb{N}$, let G be a graph-like space and let $(A_i, B_i)_{i \in I}$ be a chain of separations of order $\leq k$. Let $A = \bigcup_{i \in I} A_i$ and $B = \bigcap_{i \in I} B_i$. Then there exists $X \subseteq V(\mathcal{P})$ such that $(A \cup X, B)$ is a separation of order at most k.*

PROOF. If there are $k+1$ many disjoint pseudo-arcs containing vertices both of A and B, then there is $i \in I$ such that these disjoint pseudo-arcs all contain vertices of both A_i and B_i, a contradiction. Thus any set of disjoint pseudo-arcs containing vertices of both A and B has at most k elements. Let G' be the graph-like space obtained from G by deleting the elements of $A \cap B$ and all incident edges. As every element of $A \cap B$ is a trivial pseudo-arc which meets both A and B, in G' every set of disjoint pseudo-arcs containing vertices of both $A \backslash B$ and $B \backslash A$ has at most $k - |A \cap B|$ many elements, hence by Theorem 5.2 there is a set X of at most $k - |A \cap B|$ many vertices that meets every pseudo-arc which contains vertices of both $A \backslash B$ and $B \backslash A$. So $((A \backslash B) \cup X, B)$ is a separation of G' of order at most $k - |A \cap B|$ and $(A \cup X, B)$ is a separation of G of order at most k. □

In the situation of Corollary 5.3, for any two separations of the form $(A \cup X, B)$ and $(A \cup Y, B)$, the meet $(A \cup (X \cap Y), B)$ is also a separation of the graph-like space. Thus there is a smallest such X, and this X satisfies that $(A \cup X, B)$ is the supremum of the chain $(A_i, B_i)_{i \in I}$ in the universe of separations of the graph-like space. In the special case of the separations of a graph, the set X is always empty. In graph-like spaces, that is not necessarily the case:

Example 5.4. Let P be the partial order arising from $\mathbb{N}$ by the addition of a biggest element. Consider the pseudoline $L(P)$, which is a graph-like space. For every $n \in \mathbb{N}$ let v_n be the vertex of $L(P)$ which is the supremum of the edge n and let $S(n)$ be the separation (A_n, B_n) where A_n consists of the vertices w of $L(P)$ with $w \leq v_n$ and B_n consists of the vertices w of $L(P)$ with $w \geq v_n$. Then $(S_n)_{n \in \mathbb{N}}$ is a chain of separations in the universe of separations of $L(P)$. Let u_1 be the biggest vertex of $L(P)$ and U_2 the second biggest vertex of $L(P)$. Then the supremum of $(S_n)_{n \in \mathbb{N}}$ is of the form (A, B) where A contains all vertices of $L(P)$ but u_1, and B consists of u_1 and u_2. In particular u_2 is contained in A but not contained in any A_n with $n \in \mathbb{N}$.

For this chapter, fix some limit-closed universe of vertex separations $\mathcal{U}$ and some $k \in \mathbb{N}$. Additionally, if $k \neq 0$ then fix some $k-1$-profile P and let $\mathcal{P}$ be a set of k-profiles which all have truncation P if P exists. Just as in Section 4.3 let two separations in $\mathcal{S}_k$ be equivalent if and only if they are contained in the same elements of $\mathcal{P}$. See Remark 5.9 for a remark on the choice of a set of profiles instead of tangles. For a cyclically ordered set I, denote its cycle completion by $C(I)$. The elements of $C(I) \backslash I$ are also called the *cutpoints* of $C(I)$.

5.1. Definition of k-pseudoflowers and k-flowers

For separation systems of bipartitions, it is feasible to let flowers be partitions of the ground set such that certain unions of partition classes are elements of the separation system. For $\mathcal{U}$, this approach is not feasible any more, as separations are not any more in direct correspondence to subsets of the ground set. At least for finite flowers, this problem can be overcome by allowing the "partition classes" to overlap. Then the separation associated to a subset of partition classes can have elements on both sides. Such structures behave very much like flowers in matroids:

Lemma 5.5. *Let $(P_i)_{i \in I}$ be a finite family of subsets of some ground set V and $k \in \mathbb{N}$. For all $I' \subseteq I$ denote $\bigcup_{i \in I'} P_{i'}$ by $V(I')$ and $(V(I'), V(I \backslash I'))$ by $S(I')$. Assume that*

- *I is cyclically ordered and has at least three elements*
- *$S(i)$ is a separation of $\mathcal{U}$ of order $k-1$ for all $i \in I$*
- *$S(\{i, j\})$ is a separation of $\mathcal{U}$ of order $k-1$ for all adjacent indices i and j in I.*

Then every $v \in V$ is contained in all P_i, in exactly two P_i with adjacent indices, or in exactly one P_i. Furthermore, the number of elements which two P_i with adjacent indices share does not depend on the P_i.

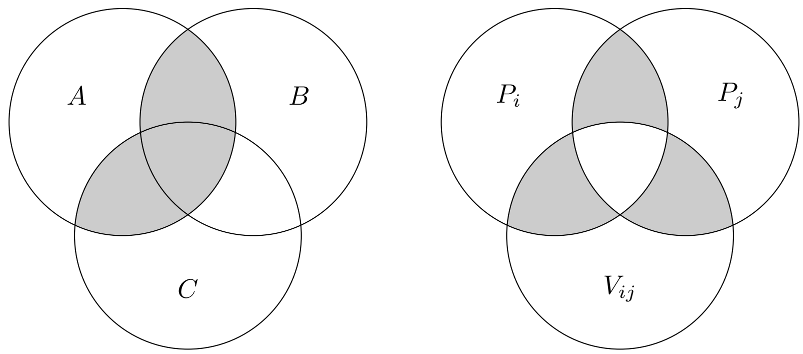

FIGURE 1. To the left: Three sets A, B and C. The shaded area is $A \cap (B \cup C)$. To the right: Three sets P_i, P_j and V_{ij}. The three shaded areas contain, in the circumstances of the proof of Lemma 5.5, the same amount of vertices.

PROOF. First consider three sets A, B and C with pairwise finite intersection such that $|A \cap (B \cup C)| = |B \cap (C \cup A)| = |C \cap (A \cup B)|$. The set $A \cap (B \cup C)$ is indicated in Fig. 1, the other two sets are symmetrical. Then the following equations hold:

$$\begin{aligned} A \cap (B \cup C) &= (A \cap (B \backslash C)) \uplus (A \cap (C \backslash B)) \uplus (A \cap B \cap C) \\ &= ((A \cap B) \backslash C) \uplus ((A \cap C) \backslash B) \uplus (A \cap B \cap C) \\ B \cap (C \cup A) &= ((A \cap B) \backslash C) \uplus ((B \cap C) \backslash A) \uplus (A \cap B \cap C) \\ C \cap (A \cup B) &= ((A \cap C) \backslash B) \uplus ((B \cap C) \backslash A) \uplus (A \cap B \cap C). \end{aligned}$$

As a result, $|(A \cap B) \backslash C| = |(A \cap C) \backslash B| = |(B \cap C) \backslash A|$.

Let j be some index of I and let i and l be its neighbours in I. Denote $V(I \backslash \{i, j\})$ by V_{ij} and $V(I \backslash \{j, l\})$ by V_{jl}. The sets P_i, P_j and V_{ij} are depicted in Fig. 1. Applying the previous paragraph to the sets V_{ij}, P_i and P_j yields

$$\left|(P_i \cap P_j) \backslash V_{ij}\right| = \left|(P_j \cap V_{ij}) \backslash P_i\right|.$$

Applying the previous paragraph to the sets V_{jl}, P_j and P_l yields

$$\left|(P_j \cap P_l) \backslash V_{jl}\right| = \left|(P_j \cap V_{jl}) \backslash P_l\right|.$$

Furthermore $(P_j \cap P_l) \backslash V_{jl} \subseteq (P_j \cap P_l) \backslash P_i \subseteq (P_j \cap V_{ij}) \backslash P_i$ and $(P_i \cap P_j) \backslash V_{ij} \subseteq (P_i \cap P_j) \backslash P_l \subseteq (P_j \cap V_{jl}) \backslash P_l$, so these sets all have the same size. In particular P_j shares as many vertices with only P_i as it shares with only P_l.

Also,

$$\begin{aligned} P_j \cap V(I \backslash \{j\}) &= \big((P_j \cap V_{ij}) \backslash P_i\big) \uplus \big((P_j \cap P_i) \backslash V_{ij}\big) \uplus \big(P_j \cap P_i \cap V_{ij}\big) \\ &= \big((P_j \cap P_l) \backslash V_{jl}\big) \uplus \big((P_j \cap P_i) \backslash V_{ij}\big) \uplus \big(P_j \cap P_i \cap V_{ij}\big) \end{aligned}$$

so every $v \in V$ which is contained in P_j and at least one other element of the family $(P_i)_{i \in I}$ is either contained in exactly two elements of the family (namely P_j and one of P_i and P_l) or it is contained in at least three elements of the family, one of those elements being P_i. As this, by symmetry, holds for all indices in I, the lemma holds. $\square$

By Lemma 5.5, for all $I' \subseteq I$ the order of $S(I')$ can be expressed in terms of the number of arcs of I', the number a of vertices contained in all P_i and the number d of vertices which two adjacent P_i share with each other but with no other P_i. So it is immediate that the separations $S(I')$ of order $k-1$ are either all intervals or all such separations. A little more complicated is the question of which separations $S(I')$ are contained in $\mathcal{U}$:

Lemma 5.6. *Let I' be a non-trivial subset of I. Then $S(I')$ is an element of $\mathcal{U}$.*

Proof. The proof is by induction on the size of I'. The cases where I' has only one element or has exactly two elements and those are adjacent are done by the requirements on $(P_i)_{i\in I}$. So consider the case that I' contains two non-adjacent elements i and j. Then $V(I'-i) \cup V(I'-j) = V(I')$ and

$$V((I\backslash I')+i) \cap V((I\backslash I')+j) = V(I\backslash I') \cup (P_i \cap P_j) = V(I\backslash I')$$

as all vertices in $P_i \cap P_j$ are contained in all vertex sets. Hence $S(I') = S(I'-i) \vee S(I'-j)$. By induction $S(I'-i)$ and $S(I'-j)$ are elements of $\mathcal{U}$, so also $S(I')$ is an element of $\mathcal{U}$. □

Just defining families of vertex sets as in Lemma 5.5 to be flowers does not extend well to the infinite case. In the following example there is a set of separations such that every subset is displayed by some common partition in the sense above, but for which it is not possible to place all vertices of V into suitable sets P_i.

Example 5.7. Let C be the pseudocycle obtained from the pseudoline on edge set $\mathbb{Z}$ with the usual order. Denote the one vertex which is not incident with an edge by $\tilde{v}$. For all distinct vertices v and w of C there are two separations which have $\{v, w\}$ as separator. Denote the set of all such separations by $\mathcal{S}$. For any finite subset of $\mathcal{S}$ there is for $k = 3$ a partition with cyclic order as in Lemma 5.5 such that all separation of that finite subset are of the form $S(I')$. But there is no partition with cyclic order as in Lemma 5.5 such that all separations in $\mathcal{S}$ are of the form $S(I')$, as can be seen as follows: Let $(P_i)_{i\in I}$ be a partition with cyclic order as in Lemma 5.5 such that some separation in $\mathcal{S}$ with $\tilde{v}$ in its separator is of the form $S(I')$. In particular $k = 3$ and there are at least two sets P_i which contain $\tilde{v}$. Let $i' \in I$ such that $\tilde{v} \in P_{i'}$. Then $S(i')$ contains $\tilde{v}$ in its separator and thus there are infinitely many separations in $\mathcal{S}$ towards which $S(i')$ points and infinitely many separations in $\mathcal{S}$ towards which $S(i')^*$ points. But $S(i')$ points towards all but finitely many separations of the form $S(I')$, so there are infinitely many separations in $\mathcal{S}$ which are not of the form $S(I')$.

But it is clear where the spare vertices belong. So the following, slightly more complicated definition arises, which is also very close to the definition of k-pseudoflowers from Chapter 6:

Definition 5.8. Let I be a set of size at least 2 with a cyclic ordering. Let $C(I)$ be the cycle completion of I and $(P_z)_{z\in C(I)}$ a family of vertex sets. Define $X = \bigcap_{i\in I} P_i$ and for all distinct $v, w \in C(I)\backslash I$ let $V(v, w) = \bigcup_{z\in[v,w]} P_z$. Then $(P_z)_{z\in C(I)}$ is a *k-pseudoflower* if

- For every $v \in C(I)\backslash I$ the set P_v has size $(k-1-|X|)/2$ and is disjoint from X;
- For all distinct $v, w \in C(I)\backslash I$ the pair $S(v, w) = (V(v, w), V(w, v))$ is a separation of order at most $k-1$ and $V(v, w) \cap V(w, v) = P_v \cup P_w \cup X$;

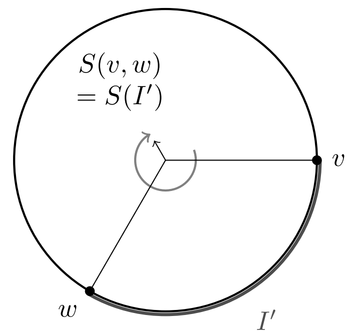

FIGURE 2. This figure shows a k-pseudoflower: The big cycle depicts the index set I, the grey arc with the arrow tip indicates that in this case the cyclic order of I is depicted clockwise. Also I' is a non-trivial interval of I of the form $[v,w] \cap I$ for two elements v and w of $C(I)\backslash I$. The interval separation $S(I')$ is shown, too.

- For every $i \in I$ the set $V(p(i), s(i))$ equals P_i and

$$P_{p(i)} = P_{s(i)} \Rightarrow P_i \nsubseteq P_{s(i)} \cup X \tag{$*$F}$$

  where $p(i)$ and $s(i)$ are the predecessor and successor of i in $C(I)$ respectively.

A k-pseudoflower is *finite* if the index set is finite. A *k-flower* is a k-pseudoflower in which all separations $S(v,w)$ are regular separations with order exactly $k-1$.

For $v \neq w \in C(I)\backslash I$ the set $V(v,w)$ is the *interval set* of $[v,w]$ and the separation $S(w,v)$ is the *interval separation* of $[v,w]$. For a non-trivial interval $I' \subseteq I$ let v and w be the unique elements of $C(I)\backslash I$ such that $I' = [v,w] \cap I$. Then the *interval set* $V(I')$ of I' is $V(v,w)$ and the *interval separation* $S(I')$ of I' is $S(v,w)$. Also define $V(\emptyset) = \emptyset$ and $V(I) = V(G)$. For each index $i \in I$ the *petal* of i is its interval set $V(i)$ and the *petal separation* of i is its interval separation $S(i)$. The sets P_v with $v \in C(I)\backslash I$ are the *gluing sets* of the k-pseudoflower. A separation (A,B) is *displayed* by a k-pseudoflower if it is an interval separation of that k-pseudoflower. An equivalence class $\mathcal{E}$ of separations is displayed by a k-pseudoflower if an element of it is displayed by the k-pseudoflower. A k-pseudoflower *distinguishes* two profiles if it displays a separation distinguishing those two profiles. Any k-pseudoflower Φ induces an equivalence relation on $\mathcal{P}$ where two profiles are equivalent if they are not distinguished by Φ. The term that a k-pseudoflower *distinguishes at least n profiles* means that the equivalence relation on $\mathcal{P}$ induced by the k-pseudoflower has at least n many equivalence classes. A separation that distinguishes two profiles in $\mathcal{P}$ is called *proper*. Two separations *cross properly* if for all of the four possibilities to choose one orientation of the first separation and one orientation of the second separation there is an element of $\mathcal{P}$ containing those two orientations.

Remark 5.9. It would have been possible to define k-tangles for limit-closed universes of vertex separations (or even more specifically for graph-like spaces) and then work with a set of k-tangles instead of $\mathcal{P}$. For graph-like spaces this definition could have been generalised from the notion of a k-tangle of a graph, and for the

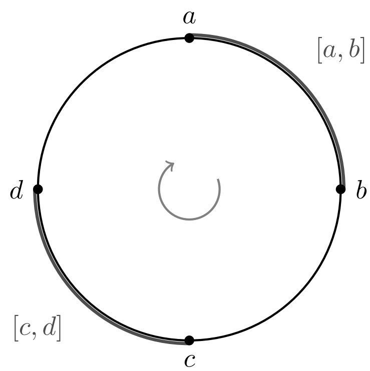

FIGURE 3. A cyclic order with two intervals. This figure depicts some of the notation used in Lemma 5.11 and Corollary 5.12.

limit-closed universes of vertex separations the definition of a k-tangle for an arbitrary separation system could have been used. With any of these generalisations, a k-tangle T is certainly a k-profile and has the property that if u, v and w are distinct cutpoints of the index set of a k-pseudoflower Φ such that $v \in [u, w]$, then

$$\big(S(u,v) \in T \wedge S(v,w) \in T\big) \Rightarrow S(u,w) \in T.$$

The circumstance that k-profiles do not necessarily satisfy this condition is the reason why not every k-profile is located somewhere in every k-pseudoflower. This is also the reason why several theorems in this chapter require a k-pseudoflower which is an extension of a k-flower with at least four petals in addition to distinguishing at least three profiles, in contrast to just requiring the k-pseudoflower to distinguish at least four (or possibly only three) profiles.

5.2. Basic properties of k-pseudoflowers

Definition 5.10. A k-pseudoflower is called a *k-pseudoanemone* if the sets P_v for $v \in C(I)\backslash I$ are all empty and a *k-pseudodaisy* otherwise. Call the vertices in X its *anemone vertices* and the size of the P_v its *daisy number*.

Note that in an anemone all sets P_i with $i \in I$ are disjoint.

Lemma 5.11. *Let Φ be a k-pseudoflower on index set I. Let a, b, c and d be elements of $C(I)\backslash I$ such that a, b and c are pairwise distinct and $b \in [a, c]$. Also assume that if $d \in \{a, b, c\}$, then $d = a$, and if $d \notin \{a, b, c\}$ then $d \in [c, a]$. Then $V(a,b) \cap V(c,d) = (P_a \cap P_d) \cup (P_b \cap P_c) \cup X$ and $V(a,c) \cap V(b,d) = V(b,c) \cup (P_a \cap P_d)$.*

PROOF. Clearly $V(a,b) \cap V(c,d) \subseteq V(a,c) \cap V(c,a) = P_a \cup P_c \cup X$. Similarly $V(a,b) \cap V(c,d)$ is a subset of $P_a \cup P_b \cup X$, $P_d \cup P_b \cup X$ and $P_d \cup P_c \cup X$. Together these subsetrelations imply

$$\begin{aligned} V(a,b) \cap V(c,d) &\subseteq (P_a \cup P_b \cup X) \cap (P_a \cup P_c \cup X) \cap (P_d \cup P_b \cup X) \cap (P_d \cup P_c \cup X) \\ &= (P_a \cap P_d) \cup (P_b \cap P_c) \cup X. \end{aligned}$$

As also $(P_a \cap P_d) \cup (P_b \cap P_c) \cup X \subseteq V(a,b) \cap V(c,d)$, these two sets are equal. Thus

$$\begin{aligned} V(a,c) \cap V(b,d) &= V(b,c) \cup (V(a,b) \cap V(c,d)) \\ &= V(b,c) \cup (P_a \cap P_d) \cup (P_b \cap P_c) \cup X \\ &= V(b,c) \cup (P_a \cap P_d). \end{aligned}$$ □

Corollary 5.12. *If $P_a \cap P_d = \emptyset$ then $S(b,c) = S(a,c) \wedge S(b,d)$.*

Lemma 5.13. *Let $(P_i)_{i \in C(I)}$ be a k-flower on index set I. Then $P_v \cap P_w = \emptyset$ for all $v, w \in C(I)\backslash I$.*

Proof. Let v and w be distinct elements of $C(I)\backslash I$. Let i and i' be elements of I such that $v \in [i, i']$ and $w \in [i', i]$. In $C(I)$, denote the predecessor and successor of i by p and s respectively and the predecessor and successor of i' by p' and s' respectively. Then by Lemma 5.11

$$P_v \cap P_w \subseteq V(s, p') \cap V(s', p) = X \cup (P_{p'} \cap P_{s'}) \cup (P_p \cap P_s) = X$$

where the last equality holds because all petal separations are separations of order $k-1$. As P_v and P_w are disjoint from X, $P_v \cap P_w = \emptyset$. □

Lemma 5.14. *For all $x \in \bigcup_{v \in C(I)\backslash I} P_v$ the set $\{w \in C(I)\backslash I : x \in P_w\}$ is an interval of $C(I)\backslash I$.*

Proof. Assume otherwise. Then there are pairwise distinct $t, u, v, w \in C(I)\backslash I$ such that $t \in [v, w]$, $u \in [w, v]$, $x \in P_t \cap P_u$ and $x \notin P_v \cup P_w$. But then $x \in V(v,w) \cap V(w,v)$, which contradicts the fact that $x \notin P_v \cup P_w \cup X$. □

Definition 5.15. A k-pseudoflower Φ' *extends* another k-pseudoflower Φ, written $\Phi \leq \Phi'$, if the sets X and X' coincide and there is a map $F : C(I') \to C(I)$ respecting the cyclic ordering such that $F(I') = I$ and $V(v,w) = V(F^{-1}(v), F^{-1}(w))$ for all $v, w \in C(I)\backslash I$. If Φ' extends Φ, then Φ is a *concatenation* of Φ'.

The construction of the completion of an index set implies that for distinct index sets I and I' the sets $C(I)\backslash I$ and $C(I')\backslash I'$ will almost certainly be disjoint. Thus even if $\Phi \leq \Phi'$ are k-pseudoflowers which are otherwise very similar, e.g. because I is a subset of I' with only one element less than I' and F is the identity on I—in which case also the sets of interval separations are nearly the same— the sets $C(I)\backslash I$ and $C(I')\backslash I'$ completely different as sets. But in that case, $C(I)\backslash I$ and $C(I')\backslash I'$ are in close correspondence to each other via F, and so it is natural to simply identify corresponding elements as follows: The definition of $\leq$ implies that if Φ and Φ' are k-pseudoflowers and F is a map witnessing that $\Phi \leq \Phi'$ then identifying every element of $C(I)\backslash I$ with its unique pre-image under F keeps interval sets $V(v,w)$ and interval separations $S(v,w)$ well-defined. If Φ is an extension of a k-flower with at least three petals, then the identification does not even depend on the choice of F (because there is only one such choice) and keeps the definition of gluing sets well-defined, as the following lemma shows:

Lemma 5.16. *Let $\Phi \leq \Phi'$ be k-pseudoflowers. If Φ is an extension of a k-flower with three petals then the witness F that $\Phi \leq \Phi'$ is unique and satisfies $P_v = P_{F^{-1}(v)}$ for all $v \in C(I)\backslash I$.*

Proof. In order to show that F is unique, let $F : C(I') \to C(I)$ and $F' : C(I') \to C(I)$ be witnesses that $\Phi \leq \Phi'$. Then F and F' are surjective. Let i be an element of I' and assume for a contradiction that $F(i) \neq F'(i)$. Because Φ is an extension of a k-flower with three petals, there are three elements of $C(I)\backslash I$ such that their petals are pairwise disjoint. Thus by changing the roles of F and F' if necessary it is possible to assume that $[F(i), F'(i)] \cap C(I)\backslash I$ contains two elements v and w such that P_v and P_w are disjoint. Let $t \in [F'(i), F(i)] \cap C(I)\backslash I$ and let x be an element of $P_i\backslash X$. If x is not contained in P_v, then because it is contained in both sides of the separation $S(v,t)$ it is also contained in P_t. Otherwise x is contained in P_v and thus not contained in P_w, again implying that $x \in P_t$. But now $P_i\backslash X \subseteq P_t$, which is a contradiction to $|P_i\backslash X| > |P_t|$ which is implied by $Eq.$ (*F). Thus F and F' are surjective strong homomorphisms from $C(I')$ to $C(I)$ which agree on I' and map I' to I, so by Lemma 1.24 $F = F'$.

The second step is to show that $P_v = P_{F^{-1}(v)}$ for all $v \in C(I)\backslash I$. This is clear for a k-pseudoanemone, so assume that Φ is a k-pseudodaisy. For that consider first the case that there are elements w and u of $C(I)\backslash I$ such that P_v, P_w and P_u are pairwise disjoint and such that $v \in [u,w]$. Then Lemma 5.11 implies

$$\begin{aligned} P_v \cup X &= V(u,v) \cap V(v,w) \\ &= V(F^{-1}(u), F^{-1}(v)) \cap V(F^{-1}(v), F^{-1}(w)) \\ &= X \cup P_{F^{-1}(v)} \cup (P_{F^{-1}(u)} \cap P_{F^{-1}(w)}) \end{aligned}$$

so because P_v and $P_{F^{-1}(v)}$ have the same size and are both disjoint from X, they are equal.

Next consider the case that there are elements s, t and u of $C(I)\backslash I$ such that P_s, P_t and P_u are pairwise disjoint and such that $t \in [s,u]$ and $v \in]s,t[$. Then, again by applying Lemma 5.11,

$$\begin{aligned} P_v &= (V(s,v) \cap V(v,t))\backslash X \\ &= (V(F^{-1}(s), F^{-1}(v)) \cap V(F^{-1}(v), F^{-1}(t))\backslash X \\ &= P_{F^{-1}(v)} \cup (P_{F^{-1}(s)} \cap P_{F^{-1}(t)}) \\ &= P_{F^{-1}(v)} \cup (P_s \cap P_t) = P_{F^{-1}(v)}. \end{aligned}$$ □

So if Φ' is an extension of Φ, then the elements of $C(I)\backslash I$ can be seen as elements of $C(I')\backslash I'$. But the other way around is also possible: If Φ' is a k-pseudoflower on index set I', then for some sets $D \subseteq C(I')\backslash I'$ there is a k-pseudoflower Φ such that D essentially is $C(I)\backslash I$:

Example 5.17 (extension via cutpoints)**.** Let Φ be a k-pseudoflower and D a subset of $C(I)\backslash I$ such that there is a surjective strong homomorphism F from $C(I)$ to some $C(I')$ such that $F(I) \subseteq I'$ and $F^{-1}(C(I')\backslash I') = D$. By Lemma 1.27, in particular finite sets and subsets of $C(I)\backslash I$ which miss only one cutpoint which has two neighbours in $C(I)$ have that property. For all $i \in I'$ with predecessor p and successor s in $C(I')$ define $P_i = V(F^{-1}(p), F^{-1}(s))$ and for all $v \in C(I')\backslash I'$ define $P_v = P_{F^{-1}(v)}$. Then $(P_z)_{z \in C(I')}$ is a k-pseudoflower and F witnesses that $(P_z)_{z \in C(I')} \leq \Phi$. Denote $(P_z)_{z \in C(I')}$ by $\Phi(D)$.

Example 5.18 (extension via splitting a petal)**.** Let Φ and Φ' be k-pseudoflowers such that $F : C(I') \to C(I)$ witnesses that $\Phi \leq \Phi'$. Also assume that there is exactly one element i of I such that $F^{-1}(i)$ has more than one element, and that F maps exactly two elements of I' to i. Then Φ' *arises from* Φ *by splitting the petal* i. Denote the elements of $F^{-1}(i)$ by i_1 and i_2 such that i_1 is the predecessor of i_2

in I'. Then there is exactly one element $m \in]i_1, i_2[$ and all interval separations of Φ' which are not interval separations of Φ are of the form $S(m, v)$ or $S(v, m)$ for some $v \in C(I)\backslash(I + m)$.

Given a k-profile P and a k-pseudoflower Φ on index set I, P orients all separations displayed by Φ. Drawing all these orientations as arrows, in most cases these arrows all point towards the same place in $C(I)$. The following definition makes this precise:

Definition 5.19. Let Φ be a k-pseudoflower on index set I. A k-profile P is *located at* $(v, +)$ in Φ for some cutpoint $v \in C(I)\backslash I$ if $S(w, v) \in P$ for all w in $C(I)\backslash(I+v)$. Similarly P is located at $(v, -)$ if $S(v, w) \in P$ for all w in $C(I)\backslash(I + v)$.

Remark 5.20. If two profiles are located in Φ but not at the same location or locations, then Φ distinguishes the two profiles.

Let Φ be a k-pseudoflower on index set I and P a profile. If P is located somewhere, then there are two cases: The first is the case that P is located at two distinct pairs (a, b) and (c, d). In this case $\{(a, b), (c, d)\} = \{(v, +), (w, -)\}$ for two elements v and w of $C(I)\backslash I$ such that w is the successor of v. If i is the unique element of I in $]v, w[$, then $S(i)^*$ is an element of P. In this case, say that P is located *at a petal*.

The second case is that P is located at exactly one pair (a, b). In this case P contains all petal separations. Also, if $b = +$ then a does not have a successor in $C(I)\backslash I$ and if $b = -$ then a does not have a predecessor in $C(I)\backslash I$. In particular I is infinite. In this case, say that P is located *at a non-petal*.

There is a second way to define locations: As elements of $C(C(I)\backslash I)\backslash(C(I)\backslash I)$. In this case, P is located at some location u if and only if $S(w, v) \in P$ for all distinct elements v and w of $C(I)\backslash I$ such that $u \in [w, v]$. Let ι be the map which maps every location of the form $(v, +)$ to the successor of v in $C(C(I)\backslash I)$ and every location of the form $(v, -)$ to the predecessor of v in $C(C(I)\backslash I)$. Then P is located at (v, a) if and only if it is located at $\iota(v, a)$. In particular, ι is surjective and two locations (v, a) and (w, b) are mapped to the same element of $C(C(I)\backslash I)\backslash(C(I)\backslash I)$ if and only if any profile located at those locations is located at the same petal. The second way of defining locations is probably nicer in that locations are unique, but it is also less nice in that it involves the cycle completion of a cycle completion, which is not the most intuitive of concepts (and does not have nice notation).

Lemma 5.21. *In every finite k-flower with at least four petals, every k-profile is located at some pair.*

PROOF. Let Φ be a finite k-flower with at least four petals and P a k-profile. Let S be the set of interval separations $S(I')$ where both I' and $I\backslash I'$ have at least two elements. As Φ has at least four petals, S is non-empty, and as Φ is finite there is a maximal element $\vec{s}$ of $S \cap P$. Let $\vec{t}$ be an interval separation of Φ which crosses $\vec{s}$ and is contained in P. Then $\vec{s} \vee \vec{t}$ is an interval separation of Φ by Corollary 5.12 and it is contained in P by the profile property. So by maximality of $\vec{s}$, $\vec{s} \vee \vec{t}$ is not contained in S and thus the inverse of a petal separation $S(i)$. So P is located at $(v, +)$ where v is the predecessor of i in $C(I)$. $\square$

Lemma 5.22. *In every k-pseudoflower which is an extension of a k-flower with four petals, every profile is located at some pair.*

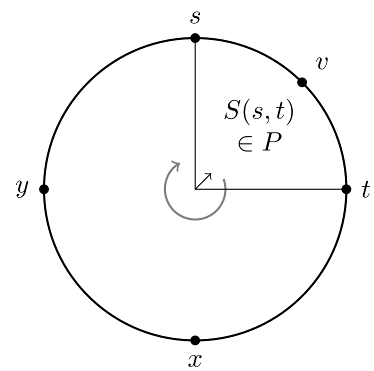

FIGURE 4. $S(s,t)$ is an interval separation of a k-pseudoflower and is contained in a profile P. See also Lemma 5.22.

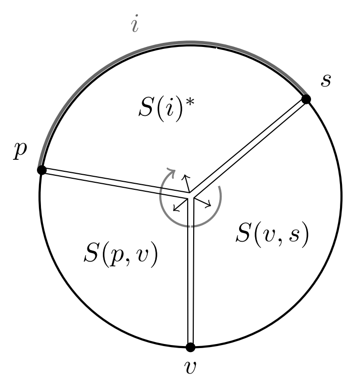

FIGURE 5. A separation (C,D) which properly crosses a petal separation $S(i)$ and is anchored at a cutpoint v.

PROOF. Let Φ be a k-pseudoflower which is an extension of a k-flower with four petals, $\Phi(\{s,t,x,y\})$ say, and let P be a k-profile. Assume that $t \in [s,x]$ and $y \in [x,s]$. By Lemma 5.21 P is located at some pair in $\Phi(\{s,t,x,y\})$, and as Φ is finite P contains the inverse of some petal separation of Φ, $S(s,t)$ say. In the interval $[s,t]$ let v be the supremum of $\{w \in [s,t[\setminus I \colon S(t,w) \in P\}$. There are three cases: $v = t$, $S(v,x) \in P$ and $S(x,v) \in P$. In the first case, $v = t$, P is located at $(v,-)$.

Consider the second case: $v \neq t$ and $S(v,x) \in P$. Then $S(v,w) = S(v,x) \vee S(t,w)$ for all $w \in [s,v[$ by Lemma 5.13 and Corollary 5.12 and thus $S(v,w) \in P$. Hence P is located at $(v,-)$.

In the last case, $v \neq t$ and $S(x,v) \in P$. Again $S(t,v) = S(t,s) \vee S(x,s)$ and thus $S(t,v) \in P$. If t is the successor of v in $C(I) \setminus I$, then $S(v,t) \in P$ implies that P is located at $(v,+)$. Otherwise let $w \in]v,t[\setminus I$. By the definition of v, P does not contain $S(t,w)$, so $S(t,w) = S(t,s) \vee S(y,w)$ implies that $S(y,w)$ is not an element of P. Thus P contains $S(w,y)$, and $S(w,v) = S(w,y) \vee S(x,v)$ implies that $S(w,v) \in P$. As this is true for all $w \in]v,t[\setminus I$, P is located at $(v,+)$. $\square$

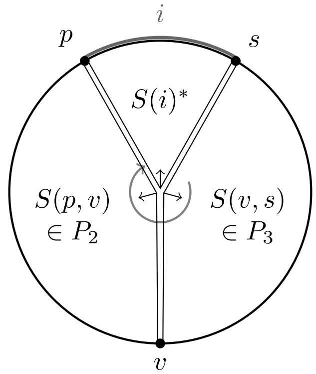

FIGURE 6. A k-pseudoflower with petal i and cutpoints p, s and v of the index set together with three separations. This figure depicts some of the notation of Lemma 5.24.

The profiles of $\mathcal{P}$ induce a pre-order $\preccurlyeq$ on the set of k-pseudoflowers as follows: Recall that a k-pseudoflower Φ distinguishes two profiles in $\mathcal{P}$ if the profiles are distinguished by a separation that is displayed by Φ. Let $\Phi \preccurlyeq \Psi$ if any profiles in $\mathcal{P}$ distinguished by Φ are also distinguished by Ψ. The rest of this section leads to the fact that if a k-pseudoflower Φ has a petal that properly crosses a separation, then there is a k-pseudoflower Ψ such that not only $\Phi \leq \Psi$ but also $\Phi \preccurlyeq \Psi$. As a result, if $\mathcal{P}$ only contains limit-closed profiles then any $\leq$-maximal k-pseudodaisy satisfying some mild assumptions is also $\preccurlyeq$-maximal.

Definition 5.23 (see Fig. 5)**.** Let Φ be a k-pseudoflower and i a petal of Φ with predecessor p and successor s in $C(I)$. A separation (A,B) of order $k-1$ properly crossing $S(i)$ is *anchored* at $v \in C(I)\backslash I$ if some orientation (C,D) of it satisfies $(C,D) \vee S(i) = S(v,s)$ and $(D,C) \vee S(i) = S(p,v)$. If $(A,B) = (C,D)$, then (A,B) is *positively anchored*, otherwise *negatively anchored*.

Lemma 5.24. *Let Φ be a k-pseudoflower which distinguishes at least three profiles and is an extension of a k-flower with at least four petals. Let (C,D) be a separation of order $k-1$ which properly crosses some petal separation $S(i)$ of Φ. Then there is a separation (C',D') properly crossing $S(i)$ such that*

- $(C,D) \wedge S(i) = (C',D') \wedge S(i)$ *and* $(D,C) \wedge S(i) = (D',C') \wedge S(i)$.
- (C',D') *is anchored at some* $v \in C(I)\backslash I$.

PROOF. Let P_1 be a profile which contains both (C,D) and $S(i)^*$. If Φ distinguishes two profiles which contain both (C,D) and $S(i)$ then let P_2 be a profile which contains both (D,C) and $S(i)$. In this case some profile P_3 which contains both (C,D) and $S(i)$ is distinguished from P_2 by Φ. Otherwise let P_3 be a profile which contains both (C,D) and $S(I)$. As Φ distinguishes at least three profiles, it distinguishes some profile P_2 from both P_1 and P_3. Because P_2 is distinguished from P_1, it contains $S(i)$. Because P_2 is also distinguished from P_3, it does not contain (C,D), so it contains (D,C). In both cases, P_2 contains $S(i)$ and (D,C), while P_3 contains $S(i)$ and (C,D). Furthermore Φ distinguishes P_2 and P_3.

Denote the predecessor of i in $C(I)$ by p and the successor by s. By Lemma 5.22 both P_2 and P_3 are located somewhere in Φ. As Φ distinguishes P_2 and P_3, these are different locations, and so there is a cutpoint $v \in C(I)\backslash I$ such that $S(p,v)$ is contained in exactly one of the profiles P_2 and P_3. The other profile then contains $S(v,s)$. If P_2 contains $S(p,v)$, then let $(C',D') = ((C,D) \vee S(v,p)) \wedge S(v,s)$, otherwise let $(C',D') = ((C,D) \vee S(s,v)) \wedge S(p,v)$. From here on assume that P_2 contains $S(p,v)$, the other case is symmetric.

As P_3 contains all three separations (C,D), $S(v,p)$ and $S(v,s)$, while P_2 contains none of these separations, (C',D') is a separation of order $k-1$ which is contained in P_3 and not in P_2. Let P_4 be a profile which contains $S(i)^*$ and (D,C). Then P_1 contains (C',D') and P_4 contains (D',C'), so together the four profiles show that (C',D') properly crosses $S(i)$. Also

$$\begin{aligned}(C',D') \wedge S(i) &= ((C,D) \vee S(v,p)) \wedge S(v,s) \wedge S(i)\\ &= ((C,D) \vee S(v,p)) \wedge S(i)\\ &= ((C,D) \wedge S(i)) \vee (S(v,p) \wedge S(i)).\end{aligned}$$

Because both (C,D) and (C',D') properly cross $S(i)$, the separations $(C,D) \wedge S(i)$ and $(C',D') \wedge S(i)$ both have order $k-1$. As $S(v,p) \wedge S(i)$ is trivial, $(C,D) \wedge S(i)$ and $(C',D') \wedge S(i)$ are equal by Lemma 1.78. Similarly $(D,C) \wedge S(i) = (D',C') \wedge S(i)$.

The fact that $S(v,p) \leq (C',D') \leq S(v,s)$ implies that

$$\begin{aligned}S(v,s) &\geq (C',D') \vee S(i)\\ &\geq S(v,p) \vee S(i)\\ &= (V(v,p) \cup V(p,s), V(p,v) \cap V(s,p))\\ &= (V(v,s), V(s,v) \cup P_p)\end{aligned}$$

where the last equality holds by Lemma 5.11. As (C',D') properly crosses $S(i)$, the separation $(C',D') \vee S(i)$ has order $k-1$ and thus $S(v,s) = (C',D') \vee S(i)$. Similarly $S(p,v) = (D',C') \vee S(i)$. □

Lemma 5.25. *Let Ψ be a k-pseudoflower and i a petal of Ψ with predecessor p and successor s. If a separation (C,D) of order $k-1$ properly crossing $S(i)$ is positively anchored at $v \in C(I)\backslash I$, then $C \cap P_s = \emptyset$, $D \cap P_p = \emptyset$, $X \subseteq C \cap D$ and $(C \cap D)\backslash V(p,s) = P_v$.*

Proof. Since P_s is contained in the separator of $S(i)$, but does not meet $V(v,p)$ it cannot meet C. Similarly $(D,C) \wedge S(\{i\}) = S(s,v)$ implies that P_p does not meet D. The third statement is immediate from the fact that X occurs in the separators of both $S(s,v)$ and $S(v,p)$.

For the last equation, note that $(C,D) \wedge S(i)^* = S(v,p)$ implies both $C \cap V(s,p) = V(v,p)$ and $D \cup V(p,s) = V(p,v)$. Taking the intersection gives $(C \cap D \cap V(s,p)) \cup (C \cap V(s,p) \cap V(p,s)) = V(v,p) \cap V(p,v)$. Deleting $V(p,s)$ on both sides and simplifying gives the result. □

Lemma 5.26. *Let Φ be a finite k-pseudoflower distinguishing at least three profiles which is an extension of a k-flower with at least four petals. Let (C,D) be a separation of order $k-1$ which properly crosses some petal separation $S(i)$ of Φ and is positively anchored at $v \in C(I)\backslash I$. Then some k-flower Φ' arises from Φ by splitting i (notation as in Example 5.18), witnessed by F, such that*

- $P_m = (C \cap D)\backslash V(I-i)$.
- $V(i_1) = C \cap V(i)$ *and* $V(i_2) = D \cap V(i)$.

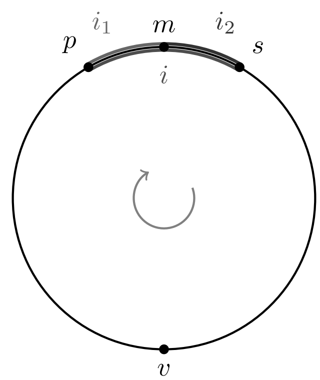

FIGURE 7. A cyclic order with two adjacent elements i_1 and i_2, and some cutpoints of the cycle completion of I, as they appear in Lemma 5.26

- *(C, D) is an interval separation of Φ'.*

PROOF. Denote the predecessor and successor of i in $C(I)$ by p and s, respectively. Let Φ' be obtained from Φ by replacing $i \in I$ with i_1, m and i_2 in this order and setting $P_m = (C \cap D)\backslash V(s,p)$, $P_{i_1} = C \cap P_i$ and $P_{i_2} = D \cap P_i$.

Φ' clearly satisfies the third condition of a k-pseudoflower. By Lemma 5.25 the separator $C \cap D$ consists of the disjoint sets P_m, X and P_v, so $|P_m| = k - |P_v| - |X| = (k - |X|)/2$. To show that Φ' is a k-pseudoflower, it thus remains to show the second condition. Let $x, y \in C(I)$ be arbitrary. Because Φ is a k-pseudoflower it suffices to consider the case that $y = m$. Since P_{i_1} and P_{i_2} meet only in $P_m \cup X$, indeed $V(x,m) \cap V(m,x) = P_x \cup P_m \cup X$.

So to show that Φ is a k-pseudoflower it suffices to show that $((V(x,m), V(m,x))$ is a separation. Without loss of generality v occurs after x in $[s,p]$, the other case is analogous. Note that $S(p,x) \vee (C,D) = (V(x,p) \cup V(v,m), V(p,x) \cap V(m,v)) = (V(x,m), V(m,x))$, where the last step uses Lemma 5.11. It follows that $((V(x,m), V(m,x))$ is a separation, since it occurs as a corner between two separations.

Thus Φ' is a k-pseudoflower. By construction, (C, D) appears as the interval separation $S(m, v)$. □

Lemma 5.27. *Let Φ be a k-pseudoflower which is a k-pseudodaisy, distinguishes at least three profiles, and is an extension of a k-flower with four petals. If Φ is $\leq$-maximal and $\mathcal{P}$ only contains limit-closed profiles, then Φ is also $\preccurlyeq$-maximal.*

PROOF. See Fig. 8 for a depiction of some of the notation. Let Φ be a k-pseudoflower which distinguishes at least three profiles and is an extension of a k-flower with at least four petals. Also assume that Φ is not $\preccurlyeq$-maximal among all k-pseudoflowers, as witnessed by Φ'. Let P_1 and P_2 be two limit-closed profiles which are distinguished by Φ' but not by Φ.

By Lemma 5.22 P_1 and P_2 are located at some pair (v, x). Assume for the rest of this proof that the profiles are located at $(v, -)$, the other case is symmetric. Because Φ distinguishes at least three profiles there is an interval separation $S(v, w)$ of Φ whose inverse is contained in two profiles P_3 and P_4 which are distinguished

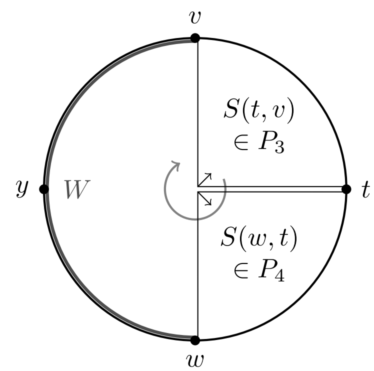

FIGURE 8. A k-pseudoflower with two of its interval separations, and an interval of its ground set. This figure depicts some of the notation used in the proof of Lemma 5.27.

by Φ. As Φ distinguishes P_3 and P_4, there is some interval separation of Φ which distinguishes P_3 and P_4. By several applications of the profile property there is $t \in]v, w[$ such that $S(v,t)$ distinguishes P_3 and P_4. Assume, by swapping the names of P_3 and P_4 if necessary, that $S(t,v) \in P_3$. Again by the profile property also $S(t,w)$ distinguishes P_3 and P_4 with $S(w,t) \in P_4$.

The k-pseudoflower Φ' distinguishes P_1, P_2, P_3 and P_4 pairwise, so there is an interval separation (C,D) of Φ' which is contained in P_3 but not P_4 and which distinguishes P_1 and P_2. By swapping the names of P_1 and P_2 if necessary, assume that (C,D) is contained in P_1 and P_3 and that its inverse is contained in P_2 and P_4.

If v has a predecessor v' in $C(I)\setminus I$, then (C,D) properly crosses $S(v,v')$ and by Lemmas 5.24 and 5.26 there is an extension Φ'' of Φ which distinguishes P_1 and P_2. Assume for a contradiction that v has no predecessor in $C(I)\setminus I$. Let W be the interval $]w,v[$. For all $x \in W$ let S_x be the separation $((C,D) \vee S(t,x)) \wedge S(t,v)$. All three separations (C,D), $S(t,x)$ and $S(t,v)$ are contained in P_3, not contained in P_4, and have order at most $k-1$, so S_x also has order at most $k-1$. Because the universe is a limit-closed universe of vertex separations, there is a unique supremum (A,B) of $(S_x)_{x\in W}$. As the union of all sets $V(v,x)$ with $x \in W$ is the whole ground set V, there is some $y \in W$ such that $V(v,y)$ contains $A \cap B$. But P_1 is limit-closed so (A,B) is still contained in P_1 and thus properly crosses $S(v,y)$. Hence applying Lemma 5.25 to the concatenation of Φ on cutpoints v, t and y shows that $A \cap B$ contains a vertex not in $V(v,y)$. Again because (A,B) properly crosses $V(v,y)$, $A \cap B$ also contains a vertex not in $V(y,v)$, a contradiction to the choice of y. $\square$

5.3. Limits of chains of k-daisies

Let $(\Phi_j)_{j\in J}$ be a $\leq$-chain of k-pseudoflowers which are k-pseudodaisies, extensions of k-flowers with at least four petals and distinguish at least three profiles. This section consists of the proof that there is a k-pseudoflower which is an upper bound of the chain $(\Phi_j)_{j\in J}$. If the chain contains a maximal Φ_j, then Φ_j already is an upper bound of the chain, so assume otherwise

5.3.1. Taking the inverse limit. By Lemma 5.16, for all indices $l \leq j$ in J there is a unique map F_{lj} witnessing that $\Phi_l \leq \Phi_j$. Because they are unique, $F_{lj} \circ F_{jm} = F_{lm}$ for all indices $l \leq j \leq m$ in J. So there is an inverse limit with projections $(\pi_j)_{j \in J}$. Because all F_{lj} respect the cyclic order, the inverse limit also has a cyclic order which is respected by the projections. Define I to contain all elements i of the inverse limit such that $\pi_j(i)$ is contained in I_j for all $j \in J$. Let V_F be the set of elements of the inverse limit which are not contained in I. For $i \in I$ define P_i to be $\bigcap_{j \in J} P_{\pi_j(i)}$.

Similarly to identifying cutpoints of the index set of two k-pseudoflowers, one extending the other, also cutpoints of the $C(I_j)$ can be identified with each other and elements of V_F as follows: Let $v \in V_F$ and $j \in J$ such that $\pi_j(v)$ is a cutpoint of $C(I_j)$. Then for all $l \in J$ which are bigger than j there is by Lemma 5.16 a unique element w_l of $C(I_j)$ such that $F_{lj}(w_l) = \pi_j(v)$, and furthermore $w_l \in C(I_j) \backslash I_j$. Hence v is the only element of $I \cup V_F$ mapped to $\pi_j(v)$ by π_j, $\pi_l(v)$ is a cutpoint for all sufficiently large $l \in J$ and $P_{\pi_l(v)}$ does not depend on $l \in J$ as long as $\pi_l(v)$ is a cutpoint of $C(I_l)$. Identify v with all $\pi_l(v)$ which are a cutpoint, implicitly defining P_v. As $V(v, w)$ does not depend on $j \in J$ as long as it is defined, this identification did not make the term $V(v, w)$ ambiguous. And intervals are taken in $I \cup V_F$ unless otherwise stated. Let Ψ be the family of vertex sets $(P_z)_{\in I \cup V_F}$. If $I \cup V_F$ is not isomorphic to $C(I)$, then Ψ is not a k-pseudoflower. Apart from possibly not having a suitable index set, Ψ is quite close to being a k-pseudoflower:

Lemma 5.28. *$V(v, w) = \bigcup_{z \in [v,w]} P_z$ for all distinct v and w in $C(I) \backslash I$, where the interval is taken in $I \cup V_F$.*

Proof. "$\subseteq$" Let $u \in V(v, w)$. If also $u \in V(w, v)$ then $u \in P_v \cup P_w \cup X$, which is a subset of $\bigcup_{z \in [v,w]} P_z$ as $[v, w]$ contains an element of I. So assume that $u \notin V(w, v)$. If there is $z \in V_F$ such that $u \in P_z$, then $u \notin V(w, v)$ implies that $z \notin [w, v]$ and thus $z \in [v, w]$. If there is no $z \in V_F$ such that $u \in P_z$, then for all $j \in J$ there is a unique $i_j \in C_j$ such that $u \in P_{i_j}$. In this case all i_j are contained in I_j and $F_{jl}(i_j) = i_l$ for all $l \leq j \in J$. So there is $i \in I$ such that $\Pi_j(i) = i_j$ for all $j \in J$, and $u \in P_i$. Again $u \notin V(v, w)$ implies $i \in [v, w]$.

"$\supseteq$" Let $j \in J$ be sufficiently large such that C_j contains v, w and z if $z \in V_F$. Then Φ_j witnesses that $P_z \subseteq V(v, w)$. Furthermore all P_z are disjoint from X. $\square$

Corollary 5.29. *For all distinct element v and w of V_F the pair*

$$\Big(\bigcup_{z \in [v,w]} P_z, \bigcup_{z \in [w,v]} P_z \Big),$$

where intervals are taken in $I \cup V_F$, is a separation of order at most $k - 1$ with separator $P_v \cup P_w \cup X$.

5.3.2. Completing the index set. In general $V_F \neq C(I) \backslash I$ and thus Ψ is not a k-pseudoflower. Denote the set of cutpoints of $C(I)$ which are not in V_F by V_N. By Lemma 1.24, the projections π_j extend uniquely to projections $\Pi_j : C(I) \to C_j$. For all distinct $v, w \in C(I) \backslash I$ define $V'(v, w)$ to be $P_v \cup P_w \cup \bigcup_{t \in [v,w] \backslash V_N} P_t$. In order for $V'(v, w)$ to be defined for elements of V_N, it is still necessary to define P_v for elements v of V_N.

Lemma 5.30. *Every $v \in V_N$ has a unique neighbour in $C(I)$, and that neighbour is an element of I.*

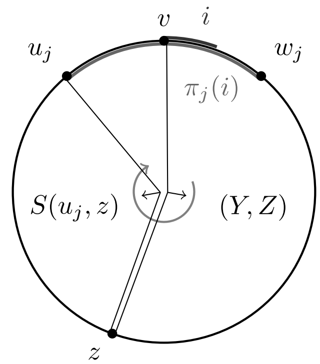

FIGURE 9. Some of the notation fixed until here, in the case that v is the predecessor of i.

PROOF. As $v \notin V_F$, $\sigma_j(v)$ is an element of I_j for all $j \in J$. Thus $(\sigma_j(v))_{j\in J}$ is an element i of I, and it is a neighbour of v. Assume for a contradiction that v has another neighbour i' in $C(I)\backslash I$. Then also $i' \in I$. As i' is a neighbour of v, for every $j \in J$ either $\sigma_j(i') = \sigma_j(v)$ or $\sigma_j(i')$ is a neighbour of $\sigma_j(v)$ in C_j. Both $\sigma_j(i')$ and $\sigma_j(v) = \sigma_j(i)$ are contained in I_j, so they cannot be neighbours in C_j, implying that $\sigma(i) = \sigma(i')$ for all $j \in J$. Because $(\sigma_j)_{j\in J}$ restricts to $(\pi_j)_{j\in J}$ on I, this implies that $i = i'$, so v indeed has only one neighbour in $C(I)$. □

Let $v \in V_N$ and i its unique neighbour in $C(I)$ which exists by Lemma 5.30. For every $j \in J$ let u_j be the predecessor and w_j the successor of $\pi_j(i)$ in C_j. Let $z \in V_F$ be a cutpoint such that for all sufficiently large $j \in J$, both $S(z, w_j)$ and $S(z, u_j)$ distinguish two profiles in $\mathcal{P}$. If v is the predecessor of i in $C(I)$, then let (Y, Z) be the supremum of the set $\{S(z, w) \colon w \in]z, v[\cap V_F\}$. If v is the successor of i in $C(I)$, then let (Y, Z) be the supremum of the set $\{S(w, z) \colon w \in]v, z[\cap V_F\}$. In both cases (Y, Z) exists and has order at most k because the order function is limit-closed. By definition, P_z is disjoint from all P_{u_j} and P_{w_j} where $j \in J$ is sufficiently large, and thus P_z is disjoint from all $V(u_j, w_j)$.

Define $P_v := ((Y \cap Z)\backslash(P_z \cup X)$. Let $j \in J$ be a sufficiently large index such that $P_z \cap V(u_j, w_j) = \emptyset$. Then both $V(w_j, z)$ and $V(z, u_j)$ contain P_z, so also both Y and Z contain P_z, implying that P_v has at most as many elements as P_z. Furthermore (Y, Z) distinguishes two profiles in $\mathcal{P}$ and thus P_v has exactly as many elements as P_z.

Lemma 5.31. *$P_v \subseteq V(u_j, w_j)$ for all $j \in J$.*

PROOF. It suffices to show the claim for all $j \in J$ such that $z \in C_j$. Let y be a vertex contained in $(Y \cup S) \cap Z$ but not in $V(u_j, w_j)$. In the case that v is the predecessor of i, $S(z, u_j) \le (Y \cup S, Z) \le S(z, w_j)$. So in this case $y \in Y \cup S \subseteq V(z, w_j)$, which implies $y \in V(z, u_j)$ because $u \notin V(u_j, w_j)$. Furthermore $y \in Z \subseteq V(u_j, z)$ and thus y is contained in the separator of $S(z, u_j)$, which is $X \cup P_z \cup P_{u_j}$. As y is not contained in P_{u_j}, it is contained in $X \cup P_z$, implying $y \notin P_v$. The case that v is the successor of i is symmetric. □

Corollary 5.32. *For every $y \in P_v$ either $y \in P_{u_j} \cup P_{w_j}$ for all sufficiently large $j \in J$ or $y \in P_i$.*

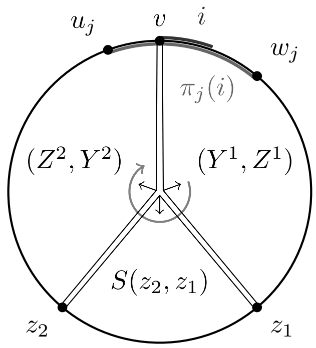

FIGURE 10. Some notation of Lemma 5.33 in the case that v is the predecessor of i.

Lemma 5.33. *The set P_v does not depend on the choice of z.*

PROOF. Let z_1 and z_2 be two possible choices for z. Assume $z_1 \in [v, z_2]$ if v is the predecessor of i and $z_2 \in [v, z_1]$ otherwise. Denote the sets defined by z_l by a superscript index l. Denote $Y^2 \cup V(z_1, z_2)$ by A and $\big(Z^2 \cap V(z_2, z_1)\big)\backslash(P_{z_2}\backslash P_{z_1})$ by B. Then $A \supseteq V(z_2, u_j) \cup V(z_1, z_2) = V(z_1, u_j)$ for all $j \in J$ with $u_j \in [z_2, v]$. Furthermore

$$\begin{aligned}(Z^2 \cap V(z_2, z_1))\backslash(P_{z_2}\backslash P_{z_1}) &\subseteq (V(u_j, z_2) \cap V(z_2, z_1))\backslash(P_{z_2}\backslash P_{z_1})\\ &= (V(u_j, z_1) \cup P_{z_2})\backslash(P_{z_2}\backslash P_{z_1})\\ &= V(u_j, z_1)\end{aligned}$$

where the second to last equality is true by Lemma 5.11. So $(A, B) \geq S(z_1, u_j)$ for all $j \in J$ with $u_j \in [z_2, v]$ and thus $(A, B) \geq (Y^1, Z^1)$. As a result $Y^1 \subseteq Y^2 \cup V(z_1, z_2)$.

Because P_{v_1} is contained in $V(z_2, z_1)$ and disjoint from both P_{z_l}, it is also disjoint from $V(z_1, z_2)$. So P_{v_1} is contained in Y^2. Also $(Y^1, Z^1) \geq (Y^2, Z^2)$. So $P_{v_1} \subseteq Z^1 \subseteq Z^2$ and thus $P_{v_1} \subseteq Y^2 \cap Z^2$. Furthermore $P_{v_1} \subseteq V(u_j, w_j)$ for all $j \in J$ implies that P_{v_1} is disjoint from P_{z_2} by choice of z_2, hence $P_{v_1} \subseteq P_{v_2}$. Because both P_{v_l} have the same size, $P_{v_1} = P_{v_2}$. □

Lemma 5.34. *Some orientation of $(V'(z, v), V'(v, z))$ is (Y, Z).*

PROOF. This is the proof that $(V'(z, v), V'(v, z)) = (Y, Z)$ if i is the successor of v. The proof that $(V'(z, v), V'(v, z)) = (Z, Y)$ if v is the successor of i is symmetric.

The set $\bigcup_{t \in [z,v]\backslash V_N} P_t$ equals $\bigcup_{t \in [z,v]\backslash V_N} V(z, t)$ which is a subset of Y and P_v also a subset of Y, hence $V'(z, v) \subseteq Y$. Also $Y\backslash V'(z, v) \subseteq Y\backslash V(z, u_j) \subseteq V(u_j, z)$ for all $j \in J$, so $Y\backslash V'(z, v) \subseteq Z$ and hence $Y\backslash V'(z, v) \subseteq Y \cap Z = P_z \cup P_v \cup X$. As the latter is a subset of $V'(z, v)$, $Y = V'(z, v)$.

For all sufficiently large $j \in J$, $\bigcup_{t \in [v,z]\backslash V_N} P_t$ is a subset of $V(u_j, z)$ and thus $V'(v, z)\backslash P_v \subseteq Z$. Also $P_v \subseteq Z$ and thus $V'(v, z) \subseteq Z$. Let $y \in Z\backslash X$ and let $w \in I \cup V_F$ be some element such that $y \in P_w$. If $w \in [v, z]$, then also $y \in V'(v, z)$. Otherwise $w \in [z, v]$, so $w \in [z, u_j]$ for some $j \in J$ and thus $y \in V(z, u_j) \subseteq Y$. In this case $y \in Y \cap Z = P_v \cup P_z \cup X \subseteq V'(v, z)$. Also $X \subseteq V'(v, z)$, thus $Z \subseteq V'(v, z)$. □

Corollary 5.35. *$(V'(z,v), V'(v,z))$ is a separation with separator $P_v \cup P_z \cup X$.*

Lemma 5.36. *If v is the predecessor of i and a vertex y of G is contained in P_{u_j} for all sufficiently large $j \in J$, then $y \in P_v$.*

PROOF. By construction of Y and Z the vertex y is contains in $Y \cap Z$, so it suffices to show that $y \notin X \cup P_z$. Because all P_{v_j} are disjoint from X also $y \notin X$. Furthermore P_z is disjoint from all sufficiently large $V(u_j, w_j)$, but $V(u_j, w_j)$ contains y for all sufficiently large $j \in J$. □

5.3.3. The interval separations are indeed k-separations.

Lemma 5.37. *$V'(v,w) = \bigcup_{t\in[v,w]} P_t$ for all distinct $v, w \in V_F \cup V_N$.*

PROOF. "$\subseteq$" is clear by definition of $V'(v,w)$. Let t be an element of $[v,w]$ and $y \in P_t$. If $y \in P_z$ for some $z \in [v,w]\backslash V_N$ or $t \in \{v,w\}$ then $y \in V'(v,w)$ follows immediately, so assume otherwise. In particular $t \in V_N$, so t has a neighbour $i \in I$, and $y \notin P_i$ and $i \in [v,w]$.

Consider the case that t is the predecessor of i, the other case is symmetric. Then $u_j \in [v,w]$ for all sufficiently large $j \in J$, implying that $y \notin P_{u_j}$ for all sufficiently large $j \in J$ by the assumption that $u \notin P_z$ for $z \in [v,w]\backslash V_N$. Because $y \notin P_i$ also $u \notin P_{\pi_j(i)}$ for all sufficiently large $j \in J$. So by Corollary 5.32 $y \in P_{w_j}$ for all sufficiently large $j \in J$. Again by assumption $w_j \notin [v,w]$ for all sufficiently large j and thus that w is the successor of i. By Lemma 5.36 also $y \in P_w$ and hence $y \in V'(v,w)$. □

Lemma 5.38. *See also Fig. 3. Let a, b, c and d be elements of $V_F \cup V_N$ such that*

- *$b \in [a,c]$ and $d \in [c,a]$*
- *$(V'(a,c), V'(c,a))$ is a separation with separator $X \cup P_a \cup P_c$*
- *$(V'(b,d), V'(d,b))$ is a separation with separator $P_b \cup P_d \cup X$.*
- *$P_a \cap P_d = \emptyset$*

Then $(V'(b,c), V'(c,b))$ is a separation with separator $P_b \cup P_c \cup X$.

PROOF. As $P_a \subseteq V'(c,b)$, Lemma 5.37 implies

$$V(G) \subseteq X \cup P_a \cup P_c \cup \bigcup_{t\in I\cup V_F} P_t \subseteq V'(b,c) \cup V'(c,b).$$

Let u be a vertex of $V'(c,b)\backslash(P_b \cup P_c \cup X)$. Because $P_a \cap P_d$ is empty, either $u \in V'(c,a)\backslash(P_a \cup P_c \cup X)$ or $u \in V'(d,b)\backslash(P_b \cup P_d \cup X)$. In the first case, because $(V'(a,c), V'(c,a))$ is a separation with separator $P_a \cup P_c \cup X$, u is an element of $V'(c,a)\backslash V'(a,c)$. Similarly, if $u \in V'(d,b)\backslash(P_b \cup P_d \cup X)$ then $u \in V'(d,b)\backslash V'(b,d)$. Hence every neighbour of u is contained in $V'(c,a)$ or in $V'(d,b)$, so every neighbour of u is contained in $V'(c,b)$ by Lemma 5.37 and thus $(V'(b,c), V'(c,b))$ is a separation. Furthermore, the fact that either $u \notin V'(a,c)$ or $u \notin V'(b,d)$ implies that u is not contained in the intersection and thus not in $V'(b,c)\backslash(P_b \cup P_c)$. Hence $V'(b,c) \cap V'(c,b) = P_b \cup P_c \cup X$ and thus the separator of $(V'(b,c), V'(c,b))$ is $P_b \cup P_c \cup X$. □

Lemma 5.39. *For all distinct $v, w \in V_F \cup V_N$ the pair $(\bigcup_{t\in[v,w]} P_t, \bigcup_{t\in[w,v]} P_t)$ is a separation with separator $X \cup P_v \cup P_w$.*

PROOF. By Lemma 5.37 it suffices to show that $(V'(v,w),V'(w,v))$ is a separation with the correct separator for all distinct elements v and w of $V_F \cup V_N$. If both v and w are contained in V_F then this is true by Corollary 5.29. Consider first the case that exactly one is contained in V_F, assume by switching the names if necessary that $w \in V_F$. Call a separation *relevant* if it distinguishes two profiles in $\mathcal{P}$. If $S(w,p(\pi_j(v)))$ and $S(s(\pi_j(v)),w)$ are both relevant for all sufficiently large $j \in J$, then w is a suitable candidate for z in the definition of P_v and thus $(V'(w,v),V'(v,w))$ is a separation with separator $P_v \cup P_w \cup X$ by Corollary 5.35. So it suffices to consider the case that there are arbitrarily large $j \in J$ for which $S(w,p(\pi_j(v)))$ is not relevant, the case where $S(s(\pi_j(v)),w)$ is not relevant is symmetric.

Let $z \in V_F$ with which P_v might have been defined. Then by choice of z there is $t \in V_F$ such that $S(t,z)$ is relevant and such that $t \in [v,z]$ if $w \in [z,v]$ and $t \in [z,v]$ if $w \in [v,z]$. Also, $(V'(z,v),V'(v,z))$ is a separation with separator $P_z \cup P_v \cup X$ by Corollary 5.35 and as $t \in V_F$, $(V'(w,t),V'(t,w))$ is a separation with separator $P_w \cup P_t \cup X$. Furthermore $S(t,z) = (V'(t,z),V'(z,t))$, and because $S(t,z)$ is relevant and thus has order $k-1$ this implies $P_t \cap P_z = \emptyset$. If $t \in [v,z]$ then apply Lemma 5.38 for $a = z$, $b = w$, $c = v$ and $d = t$. Otherwise apply the lemma for $a = t$, $b = v$, $c = w$ and $d = z$. In both cases $(V'(w,v),V'(v,w))$ is a separation with separator $P_v \cup P_w \cup X$.

Now assume that both v and w are contained in V_N. Because every Φ_j distinguishes at least three profiles, assume, by swapping the names of v and w if necessary, that there are t and u in V_F such that $t \in [v,w]$ and $u \in [t,w]$ and $S(t,u)$ is relevant. As u and t are contained in V_F, $(V'(v,u),V'(u,v))$ and $(V'(t,w),V'(w,t))$ are separations of order $k-1$ with separator $P_v \cup P_u \cup X$ and $P_t \cup P_w \cup X$ respectively. Furthermore $P_t \cap P_u = \emptyset$ because $S(t,u)$ is relevant, thus Lemma 5.38 can be applied to $a = u$, $b = w$, $c = v$ and $t = d$ also in this case $V'(v,w),V'(w,v))$ is a separation with separator $X \cup P_v \cup P_w$. $\square$

5.3.4. Deleting redundant petals. Φ is now nearly a k-pseudoflower, the only property missing is Eq. (∗F). That property can be obtained by deleting troublesome elements from I. Let I_N be the set of elements of I for which $P_i \backslash X = P_{s(i)} = P_{p(i)}$. For every $i \in I_N$ there is a neighbour $N(i)$ of i in $C(I)$ which is contained in V_N.

Lemma 5.40. *Every $i \in I_N$ has exactly one neighbour v in $C(I)$ which is not contained in $N(I_N)$.*

PROOF. Let i be an element of I_N and v its neighbour in $C(I)$ which is not $N(i)$. If $v \in V_F$, then $v \notin N(I_N)$ because $N(I_N) \subseteq V_N$. Otherwise $v \in V_N$ so v has exactly one neighbour in $C(I)$, and that neighbour is i. So if $v = N(i')$ for some $i' \in I_N$, then $i' = i$. But that contradicts the choice of v. $\square$

Lemma 5.41. *The identity on $I \backslash I_N$ extends to an isomorphism of cyclic orders $\tilde{F} : C(I)\backslash(I_N \cup N(I_N)) \to C(I\backslash I_N)$.*

PROOF. It suffices to show that every non-trivial interval I' of $I\backslash I_N$ is of the form $[v,w]_C \cap (I\backslash I_N)$ for some elements $v,w \in C(I)\backslash(I \cup N(I_N))$. Let I'' be the set of all elements of I which are either contained in I' or contained in $[i,i']_I$ for two indices $i,i' \in I'$ such that $[i,i']_{I'} \subseteq I'$. Then I'' is a non-trivial interval of I, so it is of the form $[v,w]_C \cap I$ for two elements v and w of $C(I)\backslash I$. If v is not contained in $N(I_N)$, then define v' to be v. Otherwise $v = N(i)$ for the unique element $i \in I$ which is a neighbour of v in $C(I)$. In this case let v' be the other neighbour of i in $C(I)$, which is not contained in $N(I_N)$. Define w' similarly. Now $I' = [v', w'] \cap (I \backslash I_N)$. □

Identify $C(I \backslash I_N)$ with $C(I) \backslash (I_N \cup N(I_N))$ via $\tilde{F}$. Let Φ' denote the family $\big(P_{\tilde{F}^{-1}(z)}\big)_{z \in C(I \backslash I_N)}$.

Lemma 5.42. *For all distinct elements v and w of $(C(I) \backslash I) \backslash N(I_N)$*

$$\bigcup_{z \in [v,w]_C} P_z = \bigcup_{z \in [v,w]_{C(I \backslash I_N)}} P_z.$$

PROOF. The inclusion "$\supseteq$" is clear. In order to show the other direction, let $u \in [v,w]_C$. If $u \in C(I) \backslash (I_N \cup N(I_N))$ then u is identified with an element of $C(I \backslash I_N)$ via $\tilde{F}$. If $u \in I_N$, then u has a unique neighbour u' in $C(I)$ such that $\tilde{F}(u')$ exists, and then $P_u = X \cup P_{u'}$. If u is of the form $N(i)$ for some $i \in I_N$, then i has a unique neighbour u' in $C(I)$ such that $F(u')$ exists, and $P_u = P_{u'}$. So in all cases $P_u \subseteq \bigcup_{z \in [v,w]_{C(I \backslash I_N)}} P_z$. □

Lemma 5.43. *Φ' is a k-pseudoflower such that $\Phi_j \leq \Phi'$ for all $j \in J$.*

PROOF. For all distinct elements v and w of $C(I \backslash I_N) \backslash (I \backslash I_N)$

$$\begin{aligned} \bigcup_{z \in [v,w]_{C(I \backslash I_N)}} P_z &= \bigcup_{z \in [v,w]_C} P_z && \text{by Lemma 5.42} \\ &= V'(v,w) && \text{by Lemma 5.37} \end{aligned}$$

so by Lemma 5.39 the separation $S(v,w)$ taken in Φ' is a separation with separator $P_v \cup P_w \cup X$ and also $X = V(G) \backslash \bigcup_{z \in C(I \backslash I_N)} P_z$. Furthermore if v and w are both contained in V_F, then by Lemma 5.28 the separation $S(v,w)$ taken in Φ' equals the separation $S(v,w)$ taken in any Φ_j in which it is defined. Thus the projections $\Pi_j : C(I) \to C(I_j)$ witness that $\Phi_j \leq \Phi'$ for all $j \in J$.

In order to show that Φ' satisfies Eq. (∗F), let i be an index of $I \backslash I_N$ such that its predecessor $p(i)$ and its successor $s(i)$ in $C(I \backslash I_N)$ satisfy $P_{s(i)} = P_{p(i)}$. Because all Φ_j distinguish at least three profiles and are less than Φ', there are $v, w \in V_F$ such that $P_v \cap P_w = \emptyset$ and $\{v,w\} \cap \{p(i), s(i)\} = \emptyset$. For all $x \in P_{s(i)}$ by Lemma 5.14 either $x \in P_z$ for all $z \in [p(i), s(i)] \cap (C(I) \backslash I)$ or $x \in P_z$ for all $z \in [s(i), p(i)] \cap (C(I) \backslash I)$. But as v and w are both contained in $[s(i), p(i)] \cap (C(I) \backslash I)$, for all x in $P_{s(i)}$ $\forall z \in [p(i), s(i)] \cap (C(I) \backslash I) : x \in P_z$. As a result $P_z = P_{s(i)}$ for all $z \in [p(i), s(i)] \cap (C(I) \backslash I)$. So for the neighbours t and u of i in $C(I)$ the set $P_{s(i)}$ equals both P_t and P_u. Because $i \notin I_N$, this implies that $P_i \nsubseteq P_t$, and thus $P_i \nsubseteq P_{s(i)}$. So Φ' satisfies Eq. (∗F) and is hence a k-pseudoflower. □

So in this section it was shown that if $(\Phi_j)_{j \in J}$ is a $\leq$-chain of k-pseudoflowers which are k-pseudodaisies, extensions of k-flowers with at least four petals and distinguish at least three profiles, then there is a k-pseudodaisy which is an upper bound of the chain $(\Phi_j)_{j \in J}$. In particular, if Ψ is a k-pseudodaisy which distinguishes at least three profiles in $\mathcal{P}$ and is an extension of a k-flower with at least four petals, then in the set of k-pseudoflowers Φ' with $\Psi \leq \Phi'$ every $\leq$-chain has an upper bound. Thus the following theorem follows by Zorn's Lemma.

Theorem 5.44. *Let Φ be a k-pseudodaisy which distinguishes at least three profiles in $\mathcal{P}$ and is an extension of a k-flower with at least four petals. Then there is a $\leq$-maximal k-pseudoflower Ψ such that $\Phi \leq \Psi$.*

CHAPTER 6

Bipartitions with connectivity function and limits

For this chapter, fix a ground set E, a limit-closed order function λ on the universe $\mathcal{UB}(E)$ of bipartitions of E and some $k \in \mathbb{N}$. This chapter focuses on S_k, and the fact that for every chain $(\overrightarrow{s_i})_{i\in I}$ of separations of bounded order, viewed as subsets of E, the supremum is $\bigcup_{i\in I} \overrightarrow{s_i}$ is important throughout this chapter. Note that this is in contrast to Chapter 5, where a subuniverse of $\mathcal{U}(V)$ is considered in which suprema of chains of separations are explicitly allowed to deviate a bit from the supremum of the same chain in $\mathcal{U}(V)$.

6.1. Definition of k-pseudoflowers and k-flowers

Given the definitions of k-flowers in [**5**] for finite polymatroids and [**17**] for finite connectivity systems with a k-tangle (both are cited in Section 1.6), the following is a good provisional definition of finite k-flowers for (possibly infinite) connectivity systems: A finite k-flower is an ordered partition $(P_1, \dots, P_n)$ of the the ground set such that every partition class and the union of any two adjacent partition classes (where P_1 is adjacent to P_n) has connectivity exactly $k-1$. Of course, asking that all these sets have connectivity k instead of $k-1$ would work just as well. As explained in the introduction, it is necessary to find a definition of k-flowers which display infinitely many separations, and if such k-flowers still are defined to have petals then they necessarily have infinitely many petals. At least in the case that the infinite flower is more daisy-like than anemone-like, these petals have to be arranged in a cyclic order. As opposed to finite cyclic orders, infinite cyclic orders need not be isomorphic just because they have the same size, and furthermore elements do not necessarily have adjacent elements. To resolve this, the definition of a k-flower for infinite connectivity systems has a partition with a cyclically ordered index set (whereas the finite k-flowers are partitions on index set $\{1, \dots, n\}$) and instead of asking for the union of adjacent petals to have connectivity exactly $k-1$, any union of a non-trivial interval of petals must have connectivity exactly $k-1$. The definition is a special case of the following definition of a k-pseudoflower.

Definition 6.1. A *k-pseudoflower* is a partition $(P_i)_{i\in I}$ with a cyclically ordered index set I such that the union of any interval of petals has order at most $k-1$. The sets P_i are the *petals* of the k-pseudoflower $(P_i)_{i\in I}$. Given a subset I' of I, the set $\bigcup_{i\in I'} P_i$ is denoted by $V(I')$ and the *separation* $S(I')$ *of* I' is the separation $(V(I'), V(I\setminus I'))$. The separations $S(I')$ where $\emptyset \subsetneq I' \subsetneq I$ are the separations *displayed* by the k-pseudoflower.

A *concatenation* of a k-pseudoflower $(P_i)_{i\in I}$ is k-pseudoflower $(Q_i)_{i\in I'}$ such that for every $i \in I$ there is an index $f(i) \in I'$ with $P_i \subseteq Q_{f(i)}$ and such that the map $f : I \to I'$ is a strong homomorphism of cyclic orders. If k-pseudoflower Φ is a concatenation of a k-pseudoflower Ψ, then this is denoted as $\Phi \leq \Psi$, and Ψ is called an *extension* of Φ.

A k-pseudoflower is a *k-flower* if it has at least four petals[1] and the union of any non-trivial interval of petals has order exactly $k-1$. It is a *finite k-flower* if it is a k-flower with finitely many petals.

As already mentioned in the introduction, it would be good for $\leq$-chains of k-flowers to have upper bounds in the order $\leq$. The standard procedure to construct these upper bounds is to take the common refinement of all the partitions of the k-flowers in the chain and combine the cyclic orders into a cyclic order of the common refinement. The resulting partition with cyclic order need not be a k-flower, as there might be separations displayed by the partition that are not displayed by any k-flower in the chain and thus cannot be guaranteed to have order k. But every separation displayed by the resulting partition is a limit of separations that are displayed by k-flowers in the chain and, as the connectivity function is limit-closed, thus has order at most $k-1$. That is why k-pseudoflowers are defined, and indeed in Lemma 6.14 it is shown that $\leq$-chains of k-pseudoflowers have upper bounds.

The definition of a finite k-flower is the same as the provisional definition given earlier in this section. As this definition is so close to the already existing definitions of k-flowers, it is not surprising that finite k-flowers are either k-daisies or k-anemones. Indeed, the following analogon of [**5**, Theorem 1.1] holds and the proof, which only uses the facts that every petal has order $k-1$, every union of two adjacent petals has order $k-1$ and that the order function is submodular, also applies for this chapter's definition of a k-flower.

Lemma 6.2. *In a finite k-flower, either all non-trivial unions of petals have order $k-1$ or the non-trivial unions of petals of order $k-1$ are exactly those whose index set is an interval of I.*

Proof. This lemma can be shown by using the proof of [**5**, Theorem 1.1], which works even though the setting of that theorem is slightly different. $\square$

Note that an infinite partition with cyclic order is a k-flower if and only if all its finite concatenations are finite k-flowers. The finite concatenations can also be used to determine infinite k-daisies and k-anemones as follows. Every finite concatenation of a k-flower is either a daisy, or an anemone. For every two finite concatenations Ψ and Ψ' of a k-flower Φ, there is a third concatenation Φ' of Φ that extends both Ψ and Ψ', and so Ψ is a k-daisy if and only if Ψ' is a k-daisy. Thus the following definition arises.

Definition 6.3. A k-flower is a *k-anemone* if all non-trivial unions of petals of finite concatenations have order $k-1$. A k-flower is a k-daisy if every finite concatenation of it has the property that the non-trivial unions of petals which have order $k-1$ are exactly those where the indices of the petals form an interval of I.

Given that there are two types of finite k-flowers, and that the definition of infinite k-flowers is closely related to the characterisation of finite k-daisies, one might think that maybe there should be a definition of infinite k-anemones that is closer to the characterisation of finite k-anemones. The most obvious choice here would

[1] see the last paragraph before the next section for a remark about the lower bound on the number of petals

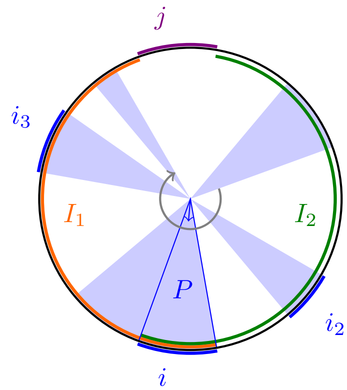

Figure 1. Some intervals and petals of a k-flower as in the proof of Lemma 6.4. The union of petals depicted in light blue is the set R.

be to let a k-anemone be a partition such that every non-trivial union of partition classes has order $k-1$. It turns out that that is not a different possible definition of infinite k-anemone, but a property of the current definition of k-anemones.

Lemma 6.4. *Every non-trivial union of petals of a k-anemone has order $k-1$.*

Proof. Let Φ be a k-anemone on index set I. As λ is limit-closed, the order of all unions of petals is at most $k-1$. Assume for a contradiction there is a non-trivial subset I' of I such that the order of $V(I')$ is less than $k-1$. Let $i \in I'$ and let $\mathcal{Q}$ be the set of sets $V(I'')$ where $i \in I'' \subseteq I'$ and $\lambda(V(I'')) < k-1$. Again as λ is limit-closed, by Zorn's Lemma there is a minimal element R of $\mathcal{Q}$. As R has order less than $k-1$, it is not a union of petals of a finite concatenation of Φ. In particular there are indices i_2, i_3 and j in I such that $V(i_2)$ and $V(i_3)$ are contained in R, $V(j)$ is not contained in R and $i \in]i_2, i_3[$, $j \in]i_3, i_2[$ (these are represented in Fig. 1). Let I_1 be the interval $[i, j[$ and I_2 the interval $]j, i]$. Then $R \wedge V(I_1)$ is a proper subset of R and contains P_i, so by minimality of R in $\mathcal{Q}$ the order of $R \wedge V(I_1)$ is at least $k-1$. Thus

$$\lambda(R \vee V(I_1)) \leq \lambda(R) + \lambda(V(I_1)) - \lambda(R \wedge V(I_1)) \leq \lambda(R)$$

and similarly $\lambda(R \vee V(I_2)) \leq \lambda(R)$. As a result

$$\begin{aligned} k-1 &= \lambda(V(I-j)) = \lambda(R \vee V(I_1) \vee V(I_2)) \\ &\leq \lambda(R \vee V(I_1)) + \lambda(R \vee V(I_2)) - \lambda((R \vee V(I_1)) \wedge (R \vee V(I_2))) \\ &\leq \lambda(R) + \lambda(R) - \lambda(R \wedge (V(I_1) \vee V(I_2))) \\ &= 2\lambda(R) - \lambda(R) = \lambda(R) < k-1 \end{aligned}$$

which is a contradiction. □

Similarly, one might want to give distinct definitions of k-pseudoanemones and k-pseudodaisies and give a definition of k-pseudoanemones that is closer to the characterisation of finite k-anemones. In particular, there are the following possibilities of k-pseudoflowers that are anemone-like:

Definition 6.5. A k-pseudoanemone is a k-pseudoflower that can be concatenated into a k-anemone. A strong k-pseudoanemone is a k-pseudoflower for which all unions of partition classes have order at most $k-1$. For two strong k-pseudoanemones Φ and Ψ denote $\Phi \leq_A \Psi$ if the partition of Φ is coarser than the partition of Ψ.

For a strong k-pseudoanemone, replacing the cyclic order of the partition with any other cyclic order yields again a strong k-pseudoanemone. Furthermore, given two strong k-pseudoanemones Φ and Ψ, the relation $\Phi \leq_A \Psi$ holds if and only if there is a k-pseudoflower Ψ' with the same partition as Ψ such that $\Phi \leq \Psi'$. In this sense for strong k-pseudoanemones the cyclic order does not really have a meaning, and therefore strong k-pseudoanemones should be compared by $\leq_A$ in addition to $\leq$. Considering two strong k-pseudoanemones with the same partition to be the same strong k-pseudoanemone turns $\leq_A$ into a partial order.

One of the main results of the next section is that every k-pseudoanemone that can be concatenated into a k-anemone with $k+1$ many petals is a strong k-pseudoanemone (see Corollary 6.12) and thus that, as infinite k-flowers are the focus, these two definitions are essentially the same. As a strong k-pseudoanemone cannot be concatenated into a k-daisy, a k-pseudoflower that can be concatenated into an anemone with sufficiently many petals is clearly anemone-like.

In [**5**] and [**17**] k-flowers are allowed to have less than four petals. That allows for k-flowers which are not unambiguously classified as daisies or anemones. As the focus of this part of the thesis is to translate the existing theory of k-flowers to infinite k-flowers, it seems reasonable to simplify the presentation of this chapter by restricting the definition of a k-flower to partitions with at least four petals.

6.2. The order of different unions of petals in k-pseudoanemones

As by definition every k-pseudoanemone can be concatenated into a k-anemone, the following lemma implies a statement about the order of unions of petals of the k-pseudoanemone which nearly form a union of petals of the k-anemone.

Lemma 6.6. *Let Φ be a k-anemone and Q a petal of Φ. Let R be a non-empty union of petals of Φ such that $R \cap Q = \emptyset$ and $R \cup Q \neq E$. Then for any subset S of Q, the connectivity of $S \cup R$ does not depend on the choice of R.*

Proof. Let S be a subset of Q and let R_1 and R_2 be candidates for R. In order to show $\lambda(R_1 \cup S) = \lambda(R_2 \cup S)$ it suffices to consider the case that R_1 is a subset of R_2. By Lemma 6.4 every non-trivial union of petals of Φ has order $k-1$, so

$$\begin{aligned}\lambda(R_2 \cup S) &\leq \lambda(R_1 \cup S) + \lambda(R_2) - \lambda(R_1) = \lambda(R_1 \cup S) = \lambda(E \backslash (R_1 \cup S)) \\ &= \lambda((E \backslash (R_1 \cup Q)) \cup (Q \backslash S)) \\ &\leq \lambda(E \backslash (R_1 \cup Q)) + \lambda((E \backslash (R_2 \cup Q)) \cup (Q \backslash S)) - \lambda(E \backslash (R_2 \cup Q)) \\ &= \lambda((E \backslash (R_2 \cup Q)) \cup (Q \backslash S)) = \lambda(R_2 \cup S).\end{aligned}$$

So $\lambda(R_1 \cup S) = \lambda(R_2 \cup S)$. □

So within a petal of an anemone, another connectivity function is induced.

Lemma 6.7. *Let Φ be a k-anemone with distinct petals Q and R. Then the map $\mu : \mathcal{P}(Q) \to \mathbb{N}$ defined by $\mu(S) = \lambda(S \cup R)$ is submodular, symmetric, limit-closed, bounded from below by $k-1$, and does not depend on the choice of R.*

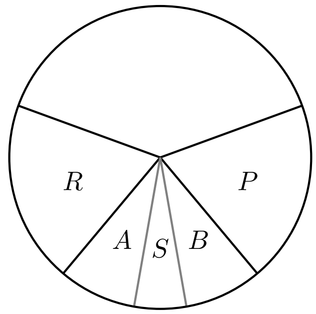

FIGURE 2. Notation from the proof of Lemma 6.8.

PROOF. As λ is submodular and limit-closed, μ is also submodular and limit-closed. By Lemma 6.6 μ does not depend on the choice of R and, for all $S \subseteq Q$,

$$\mu(S) = \lambda(S \cup R) = \lambda((E \backslash (R \cup Q)) \cup S) = \lambda(R \cup (Q \backslash S)) = \mu(Q \backslash S)$$

so μ is symmetric. Also

$$\begin{aligned} 2\mu(S) &= \mu(S) + \mu(Q\backslash S) = \lambda(R \cup S) + \lambda(R \cup (Q\backslash S)) \\ &\geq \lambda(R) + \lambda(R \cup Q) = 2k - 2, \end{aligned}$$

so $\mu(S) \geq k - 1$. $\square$

When S is not any subset of a petal of an anemone, but a petal of a k-pseudoanemone extending the anemone, then $\mu(S) = k - 1$, as the next lemma shows.

Lemma 6.8. *Let Φ' be a k-pseudoanemone which has a concatenation into a k-anemone Φ. Then the union of any petal S of Φ' and any petal R of Φ not containing S has order $k - 1$.*

PROOF. Denote the petal of Φ which contains S by Q. By Lemma 6.6 it suffices to consider the case that R is adjacent to Q. Denote the neighbour of Q in Φ which is not R by P. Deleting S from Q yields two, possibly empty, unions of intervals of petals of Φ'; denote them by A and B such that A is adjacent to R and B is adjacent to P. Then

$$\begin{aligned} \lambda(R \cup S) &\leq \lambda(R \cup A \cup S) + \lambda(R \cup S \cup B) - \lambda(R \cup A \cup S \cup B) \\ &\leq \lambda(R \cup S \cup B) = \lambda(S \cup B \cup P) \leq k - 1 \end{aligned}$$

where the equality holds by Lemma 6.6. Thus $\lambda(R \cup S) = k - 1$ by Lemma 6.7. $\square$

In order to deduce from Lemma 6.8 that every k-pseudoanemone extending an anemone with sufficiently many petals is a strong k-pseudoanemone, the following two elementary properties of submodular functions are needed.

Lemma 6.9. *Let E be a finite set and $\lambda : \mathcal{P}(E) \to \mathbb{Z}$ a submodular function such that $\lambda(\emptyset) \geq 0$. Let $k \in \mathbb{N}$ and let A be minimal with the property that $\lambda(A) \geq k$. Then A has at most k elements.*

PROOF. This proof shows by induction on the size of $A \backslash A'$ that for all proper subsets A' of A the inequality $\lambda(A') \leq k - |A \backslash A'|$ holds. This is true if $A \backslash A'$ has only one element, because $\lambda(A') < k$ implies $\lambda(A') \leq k-1$. So assume that $A \backslash A'$ has at least two elements and let e be an element of $A \backslash A'$. Then

$$\begin{aligned}\lambda(A') &= \lambda((A'+e) \cap (A-e)) \\ &\leq \lambda(A'+e) + \lambda(A-e) - \lambda(A) \\ &\leq k - |A \backslash (A'+e)| + k - 1 - k \\ &= k - |A \backslash A'|\,.\end{aligned}$$

Thus $0 \leq \lambda(\emptyset) \leq k - |A \backslash \emptyset|$ and thus A has size at most k. $\square$

Lemma 6.10. *Let E be a finite set and $\lambda : \mathcal{P}(E) \to \mathbb{Z}$ submodular. Let X and Y be subsets of E such that $X \subseteq Y$ and $\lambda(X) < \lambda(Y)$. Then there is $e \in Y \backslash X$ such that $\lambda(X) < \lambda(X+e)$.*

Proof. Define a function $\mu : \mathcal{P}(Y \backslash X) \to \mathbb{Z}$ via $\mu(A) = \lambda(A \cup X) - \lambda(X)$. Then μ is submodular and satisfies $\mu(\emptyset) = 0$. Furthermore $\mu(Y \backslash X) \geq 1$, so there is a minimal set $X' \subseteq Y \backslash X$ such that $\mu(X') \geq 1$. By Lemma 6.9 the set X' has size 1, so it contains exactly one element e. Hence $\lambda(X+e) = \mu(e) + \lambda(X) \geq \lambda(X) + 1$. $\square$

Lemma 6.11. *Let Φ' be a k-pseudoflower which has a concatenation Φ into a k-anemone. Every union of petals of Φ' which either contains a petal of Φ or is disjoint from a petal of Φ has order at most $k-1$.*

Proof. By symmetry of λ it suffices to consider unions of petals of Φ' which contain a petal of Φ.

Let R be a petal of Φ. Denote the set of indices i of Φ' with $P_i \nsubseteq R$ by I' and let μ' be the map defined on $\mathcal{P}(I')$ via $\mu'(T) = \lambda(V(T) \cup R)$. Then $\mu'(\emptyset) = k-1$ and $\mu'(i) = k-1$ for all $i \in I'$ by Lemma 6.8, so $\mu'(T) \leq k-1$ for all finite subsets T of I' by Lemma 6.10. As λ is limit-closed, also μ' is limit-closed and thus $\mu'(T) \leq k-1$ for all subsets T of I' by Corollary 1.4. $\square$

Corollary 6.12. *Let Φ' be a k-pseudoflower which can be concatenated into a k-anemone Φ with at least $k+1$ many petals. Then Φ' is a strong k-pseudoanemone.*

Proof. All unions of at most k many petals of Φ' are disjoint from a petal of Φ and thus have order at most $k-1$ by Lemma 6.11. Thus by Lemma 6.9 all finite unions of petals of Φ' have order at most $k-1$, so by Corollary 1.4 all unions of petals of Φ' have order at most $k-1$. $\square$

Remark 6.13. In particular, every k-pseudoflower which can be concatenated into a k-anemone with at least $k+1$ many petals cannot be concatenated into a k-daisy.

6.3. Finding maximal k-pseudoflowers and maximal strong k-pseudoflowers

As already mentioned earlier, for a $\leq$-chain of k-pseudoflowers an upper bound can be found by taking the common partition and defining a suitable cyclic order.

Lemma 6.14. *Every $\leq$-chain of k-pseudoflowers with cyclic orders has an upper bound.*

Proof. Let $(\Phi_j)_{j\in J}$ be a $\leq$-chain of k-pseudoflowers. For every $e \in E$ let P_e be the intersection of all petals of the Φ_j which contain e. The sets P_e are the petals of Ψ. In order to define a cyclic order on them, let P_e, P_f and P_g be distinct petals of Ψ and let j be an index of J such that e, f and g are contained in distinct petals P'_e, P'_f, and P'_g of Φ_j. Then the cyclic order of P'_e, P'_f and P'_g does not depend on the choice of j, so it is well-defined to put P_e, P_f and P_g in the same order as P'_e, P'_f and P'_g and this induces a cyclic order on the set of petals of Ψ. Also clearly every Φ_j is a concatenation of Ψ. $\square$

Corollary 6.15. *For every k-pseudoflower Φ there is a $\leq$-maximal k-pseudoflower Ψ such that $\Phi \leq \Psi$.* $\square$

Just as there are $\leq$-maximal k-pseudoflowers, there also are $\leq_A$-maximal strong k-pseudoanemones. That fact does not follow immediately from Corollary 6.15, as $\Phi \leq \Psi$ for a strong k-pseudoanemone Φ does not necessarily imply that Ψ is a strong k-pseudoanemone as well. Thus the proof will take a detour via certain subsets of the power set of E, of which there are maximal ones by Zorn's Lemma, and then show that the resulting set can be transformed back into a strong k-pseudoanemone. The transformation back is done separately in Lemma 6.16.

Lemma 6.16. *Let $\mathcal{A}$ be a subset of $\mathcal{P}(E)$ containing $\emptyset$ such that for all elements A and B of $\mathcal{A}$ the sets $E\backslash A$ and $A\cap B$ are contained in $\mathcal{A}$ and $\lambda(A) \leq k-1$. Then there is a partition of E such that every union of partition classes has order at most $k-1$ and every element of $\mathcal{A}$ is a union of partition classes.*

Proof. Note that $\mathcal{A}$ is closed under finite unions and finite intersections of its elements. For a finite subset F of E define $\mathcal{S}_F = \{A \in \mathcal{A} \colon A \cap F = \emptyset\}$. Then by Lemma 1.3 the set $\mathcal{S}_F$ has a supremum S_F in $\mathcal{P}(E)$ whose order is at most $k-1$, and here supremum means $S_F = \bigcup \mathcal{S}_F$. If $F = \{e\}$, then denote $S_{\{e\}}$ by S_e and let $\mathcal{Q} = \{E\backslash S_e \colon e \in E\}$. As $F \subseteq S_F$ for all finite subsets F of E, in order to show that $\mathcal{Q}$ is a partition of E it suffices to show that for elements e and f of E either $S_e = S_f$ or $E = S_e \cup S_f$. If $e \in A \Leftrightarrow f \in A$ holds for all A in $\mathcal{A}$, then $\mathcal{S}_{\{e\}} = \mathcal{S}_{\{f\}}$ and hence $S_e = S_f$. Otherwise there is a set A in $\mathcal{A}$ that contains f but not e. In this case, $A \subseteq S_e$ and $E\backslash A \subseteq S_f$, so $E = A \cup (E\backslash A) \subseteq S_e \cup S_f$.

In order to show that every union of elements of $\mathcal{Q}$ has order at most $k-1$, it suffices by Lemma 1.3 to show that every finite union of elements in $\mathcal{Q}$ has order at most $k-1$. By the definition of $\mathcal{Q}$ that is the same as to show for every finite subset F of E that $\bigcup_{e\in F}(E\backslash S_e)$ has order at most $k-1$. For this let X and Y be sets whose disjoint union is F. Then $\mathcal{S}_F \subseteq \mathcal{S}_X \cap \mathcal{S}_Y$, so $S_F \subseteq S_X \cap S_Y$. Also for $e \in S_X \cap S_Y$ there are elements A_1 and A_2 of $\mathcal{A}$ such that $e \in A_1 \in \mathcal{S}_X$ and $e \in A_2 \in \mathcal{S}_Y$. Then $A_1 \cap A_2 \in \mathcal{A}$ and $e \in A_1 \cap A_2 \in \mathcal{S}_F$, so $e \in S_F$. Thus $S_F = S_X \cap S_Y$. By induction this implies $S_F = \bigcap_{e\in F} S_e$. So

$$\lambda\Big(\bigcup_{e\in F}(E\backslash S_e)\Big) = \lambda\Big(E\backslash \bigcap_{e\in F} S_e\Big) = \lambda(E\backslash S_F) \leq k-1. \qquad \square$$

Lemma 6.17. *For every strong k-pseudoanemone Φ there is a $\leq_A$-maximal strong k-pseudoanemone Ψ such that $\Phi \leq_A \Psi$.*

Proof. The set of separations displayed by Φ only has elements of order at most $k-1$ and is closed under taking finite unions, finite intersections and complements. By Zorn's Lemma there is a maximal set $\mathcal{A}$ of subsets of E which has these properties and contains all separations displayed by Φ. By its maximality $\mathcal{A}$ contains both $\emptyset$ and E. Then by Lemma 6.16 there is a partition of E such that every union of partition classes has order at most $k-1$ and every element of $\mathcal{A}$ is a union of partition classes. Choosing an arbitrary cyclic order turns the partition into a strong k-pseudoanemone, and by maximality of $\mathcal{A}$ that strong k-pseudoanemone is $\leq_A$-maximal. $\square$

6.4. Combining distinct extensions of an anemone

Extensions of a k-anemone can in general be quite different. But is has already been shown in Section 6.2 that k-pseudoanemones that can be concatenated into a k-anemone with at least $k+1$ many petals have additional properties. This section shows another property of k-anemones with at least $k+1$ many petals: All their extensions can be combined into one strong k-pseudoanemone. The next two lemmas show that for extensions that subdivide only one selected petal of the k-anemone.

Lemma 6.18. *Let Φ be a k-anemone and Q a petal of Φ. There is a partition of Q such that the subsets S of Q with $\mu(S) = k-1$ are exactly the unions of partition classes.*

Proof. By Lemma 6.7, $\mu(S) \geq k-1$ for all subsets S of Q. Thus if S_1 and S_2 are subsets of Q with $\mu(S_1) = \mu(S_2) = k-1$, then by submodularity of μ also $\mu(S_1 \cup S_2) = \mu(S_1 \cap S_2) = k-1$.

Lemma 6.16 can be applied to the limit-closed function μ and the set $\mathcal{A}$ of all $S \subseteq Q$ with $\mu(S) = k-1$. As μ is limited from below by $k-1$, all unions of partition classes of the obtained partition $\mathcal{Q}$ have order exactly $k-1$, so $\mathcal{A}$ is the set of unions of partition classes. $\square$

Lemma 6.19. *Let Φ be an anemone, Q a petal and $\mathcal{Q}$ a partition of Q. Denote the common refinement of $\mathcal{Q} \cup \{E \backslash Q\}$ and the set of petals of Φ by $\mathcal{Q}'$. Then $\mathcal{Q}'$ is the set of petals of a k-pseudoflower if and only if $\mathcal{Q}'$ is the set of petals of a strong k-pseudoanemone if and only if every element S of $\mathcal{Q}$ satisfies $\mu(S) = k-1$.*

Proof. If $\mathcal{Q}'$ is the set of petals of a k-pseudoflower, then by Lemma 6.11 it is the set of petals of a strong k-pseudoanemone. If $\mathcal{Q}'$ is the set of petals of a strong k-pseudoanemone, then by Lemma 6.8 also $\mu(S) = k-1$ for all elements S of $\mathcal{Q}$. Now consider the case that $\mu(S) = k-1$ for all elements S of $\mathcal{Q}$. Pick a cyclic order of $\mathcal{Q}'$ which can be concatenated to Φ. In order to show that $\mathcal{Q}'$ together with this cyclic order is a k-pseudoflower, it suffices by Lemma 6.7 and the symmetry of λ to show that $\lambda(S) = k-1$ for all unions S of elements of Q. By Lemma 6.18 $\mu(S) = k-1$. Let R_1 and R_2 be distinct petals of Φ which are distinct from Q. Then by Lemma 6.7

$$\lambda(S) \leq \lambda(S \cup R_1) + \lambda(S \cup R_2) - \lambda(S \cup R_1 \cup R_2) = \mu(S) = k-1. \qquad \square$$

Corollary 6.20. *Let Φ be a k-anemone and Q a petal of Φ. Let $\mathcal{S}$ be the set of partitions of k-pseudoflowers Ψ such that $\Phi \leq_A \Psi$ and all petals of Φ except possibly Q are also petals of Ψ. Then all elements of $\mathcal{S}$ are partitions of strong k-pseudoanemones and $\mathcal{S}$ has a $\leq_A$-biggest element.*

Proof. By Lemma 6.18 and Lemma 6.19. $\square$

These refinements of the individual petals can be combined.

Lemma 6.21. *Let Φ be a k-anemone with at least $k+1$ many petals and denote its partition by $\mathcal{Q}$. For each petal P of Φ let $\mathcal{Q}_P$ be a partition of P such that the common refinement of $\mathcal{Q}_P \cup \{E \backslash P\}$ and $\mathcal{Q}$ is a strong k-pseudoanemone. Then the common refinement of all partitions $\mathcal{Q}_P \cup \{E \backslash P\}$ is a strong k-pseudoanemone.*

PROOF. By Corollary 6.12 it suffices to show that there is a cyclic order which turns the common refinement of all partitions $\mathcal{Q}_P \cup \{E \backslash P\}$ into a k-pseudoflower which is an extension of Φ. For that, it suffices to show that for all distinct petals P and P' of Φ the common refinement of $\mathcal{Q}_P \cup \{E \backslash P\}$, $\mathcal{Q}_{P'} \cup \{E \backslash P'\}$ and $\mathcal{Q}$ is a strong k-pseudoanemone. In order to show the latter, let S be a union of elements of $\mathcal{Q}_P$, S' a union of elements of $\mathcal{Q}_{P'}$ and Q a non-empty union of petals of $\mathcal{A}$ which contains neither P nor P'. Then

$$\lambda(Q \cup S \cup S') \leq \lambda(Q \cup S) + \lambda(Q \cup S') - \lambda(Q) \leq k-1$$

where $\lambda(Q) = k-1$ by Lemma 6.4 and

$$\lambda(S \cup S') \leq \lambda(P \cup S') + \lambda(P' \cup S) - \lambda(P \cup P') \leq k-1. \qquad \square$$

Theorem 6.22. *For every k-anemone Φ with at least $k+1$ many petals there is a strong k-pseudoanemone Ψ such that $\Phi \leq_A \Psi$ and $\Phi \leq \Psi' \Rightarrow \Psi' \leq_A \Psi$ for all k-pseudoflowers Ψ'.*

PROOF. For every petal Q there is by Corollary 6.20 a finest partition into which a k-pseudoflower can split that petal and by Lemma 6.21 all these partitions can be combined into a strong k-pseudoanemone Ψ which has the required properties. $\square$

6.5. Distinguishing profiles

Let $\mathcal{P}$ be a set of k-profiles which have the same truncation P_0 to a $k-1$-profile. Just as in Chapter 5, define two separations of order $k-1$ to *cross properly* if all four corners distinguish elements of $\mathcal{P}$. This goal of this section is to show that there are k-pseudoflowers distinguishing as many elements of $\mathcal{P}$ as possible, that is to find maximal elements of the following pre-order. Recall that for a pre-order $\preccurlyeq$, a maximal element Φ is one where $\Phi \preccurlyeq \Psi$ implies $\Psi \preccurlyeq \Phi$.

Definition 6.23. For k-pseudoflowers Φ and Ψ let $\Phi \preccurlyeq \Psi$ if every two profiles in $\mathcal{P}$ that are distinguished by the union of an interval of Φ is also distinguished by the union of an interval of Ψ.

Assume that there is a $\leq$-maximal k-pseudoflower Φ that is not $\preccurlyeq$-maximal. Φ can be assumed to distinguish a minimal number of profiles: If it does not, then it either is $\preccurlyeq$-maximal or there is a k-pseudoflower Φ' distinguishing more profiles such that $\Phi \preccurlyeq \Phi'$. Given a finite number, repeating this step sufficiently often guarantees that Φ distinguishes that number many profiles. As Φ is not $\preccurlyeq$-maximal, there is a k-pseudoflower Ψ such that $\Phi \preccurlyeq \Psi$, and such that there are two profiles P_1 and P_2 in $\mathcal{P}$ that are distinguished by Ψ but not by Φ. If both P_1 and P_2 point to a petal of Φ in the sense that they contain the inverse of the petal, then they point to the same petal of Φ and it can be shown, just as in the finite case and in Chapter 5, that Φ can be extended to a k-pseudoflower distinguishing P_1 and P_2.

Lemma 6.24. *Let Φ be a k-pseudoflower distinguishing at least three profiles. Also let $\vec{s}$ be a separation which properly crosses some petal $S(i)$ of Φ. Then there is* *an extension of* Φ *which arises from* Φ *by splitting* i *and has* $S(i) \wedge \overrightarrow{s}$ *and* $S(i) \wedge \overleftarrow{s}$ *as petals.*

Proof. The definition of a separation being anchored somewhere as in Definition 5.23 also makes sense in the context of this lemma. The proof of Lemma 5.24, which shows that there is a separation $\overrightarrow{t}$ which properly crosses $S(i)$, is anchored in Φ and satisfies $\overrightarrow{t} \wedge S(i) = \overrightarrow{s} \wedge S(i)$ and $\overleftarrow{t} \wedge S(i) = \overleftarrow{s} \wedge S(i)$, also works in the context of bipartitions. The only difference is that neither Lemma 1.78 nor Lemma 5.11 are needed as there is only one small separation, which is the empty set, and the intersection of unions of petals is also a union of petals in bipartitions. So $\overrightarrow{s}$ can be assumed to be anchored in Φ, and by taking the inverse of $\overrightarrow{s}$ if necessary it can be assumed to be positively anchored in Φ. Let I' be an interval such that $(\overrightarrow{s} \vee S(i))^* = V(I')$ such that $i \notin I'$ and $I' + i$ is also an interval. As $\overrightarrow{s}$ properly crosses $S(i)$, the separations $S(I')$ and $S(I' + i)$ distinguish two profiles which have P_0 as their truncation, so they both have order $k - 1$.

The second part of the proof is very similar to the proof of Lemma 5.26 but much simpler. Let Φ' be the cyclically ordered partition obtained from the set of petals of Φ by replacing $V(i)$ by $S(i) \wedge \overrightarrow{s}$ and $S(i) \wedge \overleftarrow{s}$ such that $S(i) \wedge \overrightarrow{s}$ is the predecessor of $S(i) \wedge \overleftarrow{s}$. Denote the index of the petal $S(i) \wedge \overleftarrow{s}$ in Φ' by j. In order to show that Φ' is a k-pseudoflower, let J be an interval containing j such that $V(J)$ does not contain all of $V(i)$.

If $J = I' + j$, then $V(J)$ equals $\overleftarrow{s}$ and thus has order $k - 1$. If J properly contains $I' + j$, then $J \backslash I$ can be uniquely written as the union of a non-empty interval J' and the index j of $S(i) \wedge \overleftarrow{s}$. As $S(I')^* = \overrightarrow{s} \vee S(i)$ and $\overrightarrow{s}$ properly crosses $S(i)$, the separation $S(I')$ distinguishes two k-profiles which truncate to P_0 and thus has order $k - 1$. As $S(I') = S(J - j) \wedge S(I' + j)$, by submodularity $S(J)$ has order at most $k - 1$.

If J does not contain $I' + j$, then $J - j$ is contained in I'. Similarly to the case where J properly contains $I' + j$, $V(I' + i)$ has order $k - 1$ and $V(I' + i) = V(I' + j) \cup V(J - j + i)$, so by submodularity $V(J)$ has order at most $k - 1$. $\square$

As opposed to the vertex separations of Chapter 5, in this chapter, for every finite k-pseudoflower and every profile in $\mathcal{P}$ the profile has to point towards a petal of the k-pseudoflower. But it can still happen that in an infinite k-pseudoflower, some element of $\mathcal{P}$ does not point towards a petal. In theory, even worse, if a k-pseudoflower contains an infinite $k-1$-pseudoflower as a concatenation, then the common truncation of the elements in $\mathcal{P}$ need not point towards a petal of the $k-1$-pseudoflower.

Fortunately, this particular problem does not occur. Just as for vertex separations, every k-profile can be located somewhere in the cyclic order of the set of petals (locations of profiles in k-flowers are defined and work just the same as in Chapter 5, with the additional property that every profile is located somewhere in every k-pseudoflower). So a k-profile that does not point to a petal of some k-pseudoflower contains a chain of separations whose supremum is the whole ground set and thus not contained in the profile, thus the k-profile is not limit-closed. And, remarkably, a k-profile that is not limit-closed is induced by a unique profile of all finite-order separation and cannot be the truncation of two distinct $k+1$-profiles, as will be shown now. In particular, if $\mathcal{P}$ contains at least two k-profiles, then their truncation is limit-closed.

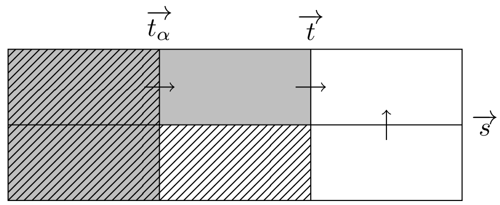

FIGURE 3. Some notation of the proof of Lemma 6.26. The rectangle is all of E, the set $\vec{q}$ is depicted with stripes and the set $\vec{r}$ is depicted in light grey.

Lemma 6.25. *Let P be a k-profile and $(\vec{t_\alpha})_{\alpha<\kappa}$ an increasing chain of separations in P such that its supremum $\vec{t}$ is not contained in P. Then for every separation $\vec{s}$ of finite order there is a cofinal set $F \subseteq \kappa$ such that $\lambda((\vec{s} \wedge \vec{t}) \vee \vec{t_\alpha}) \leq k-1$ and $\lambda((\overleftarrow{s} \wedge \vec{t}) \vee \vec{t_\alpha}) \leq k-1$ for all $\alpha \in F$.*

PROOF. As $\lambda(\vec{s} \wedge \vec{t_\alpha})$ is bounded by $\lambda(\vec{s})+k-1$, there is a cofinal set $G \subseteq \kappa$ such that $\lambda(\vec{s} \wedge \vec{t_\alpha})$ does not depend on $\alpha \in G$. Then for all $\alpha, \beta \in G$

$$\lambda((\vec{s} \wedge \vec{t_\alpha}) \vee \vec{t_\beta}) \leq \lambda(\vec{s} \wedge \vec{t_\alpha}) + \lambda(\vec{t_\beta}) - \lambda(\vec{s} \wedge \vec{t_\alpha} \wedge \vec{t_\beta}) = \lambda(\vec{t_\beta}) \leq k-1.$$

Thus for every $\alpha \in G$ the separation $(\vec{s} \wedge \vec{t}) \vee \vec{t_\alpha}$ is the supremum of the separations $(\vec{s} \wedge \vec{t_\beta}) \vee \vec{t_\alpha}$ where $\beta \in G$, so the order of $(\vec{s} \wedge \vec{t}) \vee \vec{t_\alpha}$ is at most $k-1$. Similarly there is a cofinal subset F of G such that $\lambda((\overleftarrow{s} \wedge \vec{t}) \vee \vec{t_\alpha}) \leq k-1$ for all $\alpha \in F$. $\square$

Lemma 6.26. *Let P be a k-profile and let $(\vec{t_\alpha})_{\alpha<\kappa}$ be an increasing chain of separations in P such that its supremum $\vec{t}$ is not contained in P. Then the set*

$$\{\vec{s} \in \mathcal{U} \colon \lambda(\vec{s}) \in \mathbb{N} \wedge \exists \alpha < \kappa : (\vec{s} \wedge \vec{t}) \vee \vec{t_\alpha} \in P\}$$

is a profile of the set of finite order separations and induces every l-profile which induces P.

PROOF. Let $Q = \{\vec{s} \in \mathcal{U} \colon \lambda(\vec{s}) \in \mathbb{N} \wedge \exists \alpha < \kappa : (\vec{s} \wedge \vec{t}) \vee \vec{t_\alpha} \in P\}$. In order to show that Q contains some orientation of every separation of $\mathcal{U}$ of finite order, let $\vec{s} \in \mathcal{U}$ with $\lambda(\vec{s}) \in \mathbb{N}$. By Lemma 6.25 there is $\alpha < \kappa$ such that both $\vec{q} := (\vec{s} \wedge \vec{t}) \vee \vec{t_\alpha}$ and $\vec{r} := (\overleftarrow{s} \wedge \vec{t}) \vee \vec{t_\alpha}$ (see Fig. 3) have order at most $k-1$. By submodularity one of $\vec{q} \vee \overleftarrow{r}$ and $\vec{q} \wedge \overleftarrow{r}$ has order at most $k-1$, assume without loss of generality that it is $\vec{q} \vee \overleftarrow{r}$. If $\vec{q} \vee \overleftarrow{r} \in P$, then by consistency also $\vec{q} \in P$ which implies $\vec{s} \in Q$. So assume otherwise, thus $(\vec{q} \vee \overleftarrow{r})^* \in P$. Because the join of $(\vec{q} \vee \overleftarrow{r})^*$ and $\vec{t_\alpha}$ is again $\vec{r}$, by the profile property $\vec{r} \in P$ and thus $\overleftarrow{s} \in Q$.

The separation which is the whole ground set is not contained in Q. So, in order to show that Q is a profile, it suffices to show that for every two elements $\vec{r}$ and $\vec{s}$ of Q their join is also contained in Q: Then Q cannot contain both $\vec{s}$ and $\overleftarrow{s}$ for any $\vec{s}$ of finite order, and furthermore Q is consistent and a profile. In order to show that the join of any two elements $\vec{r}$ and $\vec{s}$ is again contained in Q, apply Lemma 6.25 several times to obtain a cofinal set $F \subseteq \kappa$ for $\vec{s}$, a cofinal subset $G \subseteq F$ for $\vec{r}$ and a cofinal set $H \subseteq G$ for $\vec{s} \vee \vec{r}$. Let $\alpha < \kappa$ such that $(\vec{r} \wedge \vec{t}) \vee \vec{t_\alpha} \in P$. Then for all $\beta \in H$ with $\beta > \alpha$, the separation $(\vec{r} \wedge \vec{t}) \vee \vec{t_\beta}$ has order at most $k-1$ and is the join of $(\vec{r} \wedge \vec{t}) \vee \vec{t_\alpha}$ and $\vec{t_\beta}$. So by the profile property

$(\vec{r} \wedge \vec{t}) \vee \vec{t_\beta} \in P$. Similarly for all sufficiently large β in H, $(\vec{s} \wedge \vec{t}) \vee \vec{t_\beta} \in P$. As $((\vec{r} \vee \vec{s}) \wedge \vec{t}) \vee \vec{t_\beta}$ is for all $\beta < \kappa$ the join of $(\vec{r} \wedge \vec{t}) \vee \vec{t_\beta}$ and $(\vec{s} \wedge \vec{t}) \vee \vec{t_\beta}$, it is for all sufficiently large $\beta \in H$ contained in P by the profile property. Thus $\vec{r} \vee \vec{s}$ is contained in Q. □

Corollary 6.27. *The common truncation of any two distinct k-profiles to a $k-1$-profile is limit-closed.* □

This fact can now be used to show that if in a k-pseudoflower two distinct elements of $\mathcal{P}$ are not distinguished and do not point towards a petal, then the k-pseudoflower has to be a k-pseudoanemone.

Lemma 6.28. *Let Φ be a k-pseudoflower and let P_1 and P_2 be two k-profiles with truncation P_0 which are located at the same non-petal of Φ. Then Φ can be concatenated into an infinite k-anemone and some union of partition classes of the $\leq_A$-maximal partition*[2] *extending Φ distinguishes P_1 and P_2.*

Proof. Let I be the index set of Φ and $v \in C(I)\backslash I$ the location of the two profiles. Also assume that $S(w, v) \in P_1$ for all $w \in C(I)\backslash(I + v)$, the other case is symmetric. Denote the common truncation of P_1 and P_2 by P_0. By Corollary 6.27 P_0 is limit-closed.

As an intermediate step of finding the infinite k-anemone, let $x \in C(I)\backslash(I+v)$. Show that there is $y \in]x, v[\backslash I$ such that $\lambda(S(x, w)) = k - 1$ for all $w \in [y, v[\backslash I$ as follows: Let $C_x = \{w \in]x, v[\backslash I \colon \lambda(S(x, w)) < k - 1\}$. If C_x is empty, then y can be chosen arbitrarily from $]x, v[\backslash I$. So assume that C_x is non-empty and denote its supremum in $[x, v]$ by v'. Then every separation of the form $S(x, w)$ with $w \in C_x$ is contained in P_0 and the supremum of all those separations is $S(x, v')$. As P_0 is limit-closed, $S(x, v')$ is contained in P_0 and thus in P_1. Because $S(x, v) \notin P_1$, this implies that $v \neq v'$. Thus y can be chosen arbitrarily from $]v', v[\backslash I$.

Let x_0 be an element of $C(I)\backslash(I+v)$. Define recursively a (possibly transfinite) sequence $(x_\alpha)_{\alpha<\nu}$ as follows: For a limit α, if the supremum v' of $(x_\beta)_{\beta<\alpha}$ in $[x, v]$ is v, then terminate the construction. Otherwise let $x_\alpha = v'$. For a successor ordinal $\alpha+1$ there is by the previous paragraph some $x_{\alpha+1} \in]x_\alpha, v[$ such that $\lambda(S(x_\alpha, w)) = k - 1$ for all $w \in [x_{\alpha+1}, v[$. As a result, $x_\beta \in]x_\alpha, v[$ and $\lambda(S(x_\alpha, x_\beta)) = k - 1$ for all $\alpha < \beta < \nu$ and the supremum in $[x_0, v]$ of all x_α is v. Assume for a contradiction that $S(v, x_\alpha)$ has order less than $k - 1$ for some $0 < \alpha < \nu$. Then for all ordinals β with $\alpha < \beta < \nu$

$$\lambda(S(v, x_\beta)) \leq \lambda(S(v, x_\alpha)) + \lambda(S(x_0, x_\beta)) - \lambda(S(x_0, x_\alpha)) = \lambda(S(v, x_\alpha)) < k - 1.$$

Thus all $S(v, x_\beta)$ with $\alpha < \beta < \nu$ are contained in P_0. But the supremum of all these separations is the separation identified with the whole ground set and thus not contained in P_0, contradicting the fact that P_0 is limit-closed.

Let $\vec{s}$ be a separation of order at most k which distinguishes P_1 and P_2. Applying Lemma 6.25 to the chain $(S(v, x_\alpha))_{0<\alpha<\nu}$ yields a cofinal $F \subseteq \nu$ such that $\lambda(S(v, x_\alpha) \vee \vec{s}) \leq k - 1$ and $\lambda(S(v, x_\alpha) \vee \overleftarrow{s}) \leq k - 1$ for all $\alpha \in F$. As the sets $S(v, x_\alpha) \vee \vec{s}$ and $S(v, x_\alpha) \vee \overleftarrow{s}$ distinguish P_1 and P_2, they have order exactly $k - 1$. Let $\mu < \nu$ be a limit-ordinal such that the supremum of $F \cap \mu$ is μ. Then $\vec{s} \vee S(v, x_\mu)$ is the supremum of the separations $(\vec{s} \vee S(v, x_\alpha))_{\alpha\in\mu\cap F}$ and thus has order at most $k - 1$. Similarly the order of $\overleftarrow{s} \vee S(v, x_\mu)$ is at most $k - 1$.

[2] Recall that by Theorem 6.22 this is a strong k-pseudoanemone that is unique up to the choice of the cyclic order

Let G be the union of $F-0$ and all limit ordinals $\mu < \nu$ for which the supremum of $F \cap \mu$ is μ. Then G is closed in ν and thus by Lemma 1.27 there is an index set I' such that there is an isomorphism $f : C(I') \backslash I' \to \{x_\alpha \colon \alpha \in G\}$. So by Lemma 1.27 there is a concatenation $\Phi(\{x_\alpha \colon \alpha \in G\} + v)$ of Φ. That concatenation is an infinite k-flower.

Let α_1 be the smallest element of G, α_2 the smallest element but one of G and so on. Denote x_{α_j} by y_j for all $j \geq 1$. Let l be the maximum of k and 3. For every $\alpha \in G$ with $\alpha_l < \alpha$

$$\begin{aligned}
&\lambda(S(y_{l-1}, y_l) \vee (S(y_l, x_\alpha) \wedge \overrightarrow{s})) \\
&\qquad \leq \lambda(S(v, y_l) \vee (S(y_l, x_\alpha) \wedge \overrightarrow{s})) + \lambda(S(y_{l-1}, x_\alpha)) - \lambda(S(v, x_\alpha)) \\
&\qquad = \lambda(S(v, y_l) \vee (S(y_l, x_\alpha) \wedge \overrightarrow{s})) + k - 1 - (k-1) \\
&\qquad \leq \lambda(S(v, y_l) \vee (S(y_{l-1}, v) \wedge \overrightarrow{s})) + \lambda(S(v, y_l) \vee S(y_{l-1}, x_\alpha)) \\
&\qquad\quad - \lambda(S(v, y_l) \vee S(y_{l-1}, x_\alpha) \vee \overrightarrow{s}) \\
&\qquad = \lambda(S(v, y_l) \vee \overrightarrow{s}) + \lambda(S(v, x_\alpha)) - \lambda(S(v, x_\alpha) \vee \overrightarrow{s}) = k - 1.
\end{aligned}$$

Denote $S(y_l, v) \wedge \overrightarrow{s}$ by $\overrightarrow{p}$ and $S(y_l, v) \wedge \overleftarrow{s}$ by $\overrightarrow{q}$. By the previous paragraph, $S(y_{l-1}, y_l) \vee \overrightarrow{p}$, which is the supremum of all the sets $S(y_{l-1}, y_l) \vee (S(y_l, x_\alpha) \wedge \overrightarrow{s})$ with $\alpha \in G$ and $\alpha_l < \alpha$, has order at most $k-1$. Symmetrically $S(y_{l-1}, y_l) \vee \overrightarrow{q}$ has order at most $k-1$. Then

$$\begin{aligned}
\lambda(S(y_{l-2}, y_{l-1}) \vee \overrightarrow{p}) &\leq \lambda(S(y_{l-2}, y_l) \vee \overrightarrow{p}) + \lambda(S(v, y_{l-1}) \vee \overrightarrow{p}) - \lambda(S(v, y_l) \vee \overrightarrow{p}) \\
&= \lambda(S(y_{l-2}, y_l) \vee \overrightarrow{p}) + \lambda(S(y_{l-1}, y_l) \vee \overrightarrow{q}) - \lambda(S(v, y_l) \vee \overrightarrow{s}) \\
&\leq \lambda(S(y_{l-2}, y_l)) + \lambda(S(y_{l-1}, y_l) \vee \overrightarrow{p}) - \lambda(S(y_{l-1}, y_l)) \\
&\quad + (k-1) - \lambda(S(v, y_l) \vee \overrightarrow{s}) \\
&= \lambda(S(y_{l-1}, y_l) \vee \overrightarrow{p}) \leq k - 1
\end{aligned}$$

and symmetrically $\lambda(S(y_{l-2}, y_{l-1}) \vee \overrightarrow{q}) \leq k-1$. Thus by submodularity also $\lambda(S(y_{l-2}, y_{l-1}) \vee S(y_l, v)) \leq k-1$ and hence $\Phi(v, y_1, \ldots, y_l)$ is a k-anemone with at least $k+1$ many petals. As a result, also $\Phi(\{x_\alpha : \alpha \in G\} + v)$ is a k-anemone. By Lemma 6.7, $\lambda(S(y_{l-1}, y_l) \vee \overrightarrow{p}) = \lambda(S(y_{l-1}, y_l) \vee \overrightarrow{q}) = k-1$ and thus by Lemma 6.19 there is an extension of $\Phi(v, y_1, \ldots, y_l)$ which is a strong k-pseudoanemone, contains $\overrightarrow{p}$ as a petal and thus distinguishes P_1 and P_2. This latest strong k-pseudoanemone, $\Phi(\{x_\alpha : \alpha \in G\} + v)$ and Φ are all extensions of $\Phi(v, y_1, \ldots, y_l)$ and thus the finest extension Ψ of $\Phi(v, y_1, \ldots, y_l)$ which exists by Theorem 6.22 is also the $\leq_A$-maximal partition extending Φ and distinguishes P_1 and P_2. $\square$

So most $\leq$-maximal k-pseudoflowers that extend a daisy are also $\preccurlyeq$-maximal.

Lemma 6.29. *Let Φ be a $\leq$-maximal k-pseudoflower which has a concatenation into a k-daisy and distinguishes at least three profiles. Then Φ is $\preccurlyeq$-maximal.*

Proof. Assume for a contradiction that there is a k-pseudoflower Φ' such that $\Phi \preccurlyeq \Phi'$ and such that Φ' distinguishes two profiles P_1 and P_2 from $\mathcal{P}$ which are not distinguished by Φ. As all profiles are located somehwere in Φ and Φ does not distinguish P_1 and P_2, these two profiles are located at the same place in Φ. By Lemma 6.28 that location has to belong to a petal i. As Φ distinguishes sufficiently many profiles, there is a separation $S(v, w)$ of Φ' which not only distinguishes P_1 from P_2, but also distinguishes two profiles P_3 and P_4 which are distinguished in

Φ from P_1 as well as from each other. Then $S(v,w)$ properly crosses $S(i)$, which is by Lemma 6.24 a contradiction to the fact that Φ is $\leq$-maximal. $\square$

Unfortunately, it is not true that a $\leq$-maximal k-pseudoanemone (with sufficiently many petals and distinguishing sufficiently many k-profiles) is also necessarily $\preccurlyeq$-maximal. This is illustrated by the next example, Example 6.32. In this example there is a k-anemone whose partition is finer that all the partitions of k-pseudoflowers. But there are many possible choices for the cyclic order on the set of petals, and which profiles are distinguished depends on the cyclic order.

The example makes use of the notion of ultrafilters, to be found for example in [**33**].

Definition 6.30. An *ultrafilter* of a set X is a non-empty set $\mathcal{F}$ of subsets of X with the following properties:

- The empty set is not contained in $\mathcal{F}$.
- The intersection of any two elements of $\mathcal{F}$ is again contained in $\mathcal{F}$.
- Given subsets Y and Z of X such that $Y \subseteq Z \subseteq X$ and $Y \in \mathcal{F}$, then also $Z \in \mathcal{F}$.
- If a subset Y of X has a non-empty intersection with all elements of $\mathcal{F}$, then it is contained in $\mathcal{F}$.

An ultrafilter is *free* if it does not contain finite sets.

Lemma 6.31. *Let X be a set and Y an infinite subset of X. Then there is a free ultrafilter of X which contains Y.*

Proof. This well-known result from topology can for example be shown by applying [**33**, Theorem 8.17] to the set of subsets Z of X for which $Y \setminus Z$ is finite. $\square$

Example 6.32. Let E be an infinite set and k an integer bigger than 1. Define an order function λ on the set of subsets of E via

$$\lambda(X) = \begin{cases} 0 & X = \emptyset \text{ or } X = E \\ k-1 & \text{otherwise.} \end{cases}$$

Let $\mathcal{P}$ be the set of k-profiles of E and λ. Then $\mathcal{P}$ is the set of ultrafilters of E and every cyclic order turns E into a $\leq$-maximal k-anemone in which every petal has exactly one element.

Claim. For E, λ and $\mathcal{P}$ there is no $\preccurlyeq$-maximal k-anemone.

Proof of claim. It suffices to show that if C is a cyclic order of E, then the k-anemone Φ whose petals contain only one element and whose petals are cyclically ordered according to C is not $\preccurlyeq$-maximal.

Let L be a linear order such that closing it to a cyclic order yields C. As E is infinite, there is a sequence $e_1, e_2, \ldots$ of elements of E such that in the linear order L either $e_i < e_{i+1}$ for all indices i or $e_i > e_{i+1}$ for all indices i. Assume that there is such a sequence such that $e_i < e_{i+1}$ for all indices i, the other case is symmetric. Denote the set of elements of E which are of the form e_i with an odd index i by R. Define a new linear order L' on E where $e < f$ if one of the following happens:

- both e and f are contained in R and $e < f$ in L;
- only e is contained in R; or
- neither e nor f is contained in R and $e < f$ in L.

Let C' be the cyclic order obtained from closing L' to a cyclic order, and denote the $\leq$-maximal k-anemone arising from C' by Ψ. Then Ψ distinguishes all elements of $\mathcal{P}$ which are distinguished by Φ, so $\Phi \preccurlyeq \Psi$. But there are free ultrafilters P_1 and P_2 such that P_1 contains R and P_2 contains the set of all e_i with even index. Then Ψ distinguishes P_1 from P_2, but Φ does not. Hence Φ is not $\preccurlyeq$-maximal. $\Diamond$

So in this setting there are many $\leq$-maximal k-anemones, each of them distinguishing infinitely many profiles, but no $\preccurlyeq$-maximal k-pseudoflowers.

Because of the previous example, strong k-pseudoanemones are compared by the following pre-order, instead of by $\preccurlyeq$.

Definition 6.33. Define a relation $\preccurlyeq_A$ on the set of strong k-pseudoanemones where $\Phi \preccurlyeq_A \Psi$ if all profiles in $\mathcal{P}$ which can be distinguished by a union of petals of Φ can be distinguished by a union of petals of Ψ.

So essentially, $\preccurlyeq$ and $\preccurlyeq_A$ mean a k-pseudoflower is less than another if all profiles distinguished by a separation displayed by the first k-pseudoflower are also distinguished by a separation displayed by the second k-pseudoflower. The two pre-orders just disagree on which separations count as displayed by a k-pseudoflower. The separate definition of $\preccurlyeq_A$ is also in line with the separate definition of $\leq_A$ for strong k-pseudoanemones and the observation made earlier that for strong k-pseudoanemones the cyclic order of the partition is unimportant. Now $\leq_A$-maximal strong k-pseudoanemones can be shown to be $\preccurlyeq_A$-maximal.

Lemma 6.34. *Let Φ be a $\leq_A$-maximal strong k-pseudoanemone which has a concatenation into a k-anemone and distinguishes at least three profiles. Then Φ is $\preccurlyeq_A$-maximal.*

Proof. Assume for a contradiction that Φ is not $\preccurlyeq_A$-maximal. So there is a strong k-pseudoanemone Ψ such that $\Phi \preccurlyeq_A \Psi$ but not $\Psi \preccurlyeq_A \Phi$. Let P_1 and P_2 be two profiles which are distinguished by a union of petals of Ψ but not by a union of petals of Φ. As no union of petals of Φ distinguishes P_1 from P_2, they are in particular located at the same location in Φ. If that location is a non-petal, then by Lemma 6.28 there is a strong k-pseudoanemone Φ' which can be concatenated into an infinite k-anemone such that $\Phi \leq_A \Phi'$ and such that some union of petals of Φ distinguishes P_1 and P_2. Thus Φ' is a strong k-pseudoanemone and some cyclic order on it gives an extension of Φ. So there is an extension of Φ of which some union of petals distinguishes P_1 and P_2, contradicting the fact that Φ is $\leq_A$-maximal and that none of its unions of petals distinguishes P_1 from P_2.

If the location of P_1 and P_2 is a petal i of Φ, let P_3 and P_4 be profiles which are distinguished from each other and from P_1 in Φ. Let $\vec{s}$ be a separation displayed by Ψ which distinguishes P_1 from P_2 and P_3 from P_4. Then $\vec{s}$ properly crosses $S(i)$, so by Lemma 6.24 there is an extension of Φ which has $S(i) \wedge \vec{s}$ and $S(i) \wedge \overleftarrow{s}$ as petals and thus distinguishes P_1 from P_2. This is a contradiction to the fact that Φ is $\leq_A$-maximal. $\square$

6.6. Relation to abstract flowers

This section uses the results and notation from Chapter 4. Just as in Chapter 5, a separation is *relevant* if it distinguishes two elements of the set $\mathcal{P}$, that is, if its image under ϕ is not contained in $\{\emptyset, \mathcal{P}\}$.

Lemma 6.35. *Let Φ be a k-pseudoflower which distinguishes at least four profiles. Then there is a unique flower $V(\Phi) \in \mathcal{V}$ such that for every separation $\vec{s}$ displayed by Φ the set $\phi(\vec{s})$ is contained in $\overline{V(\Phi)} \cup \{\emptyset, \mathcal{P}\}$.*

Proof. Let $\mathcal{S}$ be the set of separations displayed by Φ which are contained in two profiles which are distinguished by Φ and whose inverses are also contained in two profiles which are distinguished by Φ. As Φ distinguishes at least four profiles, the set $\mathcal{S}$ contains at least two elements which properly cross, and as Φ is a k-pseudoflower the set $\mathcal{S}$ is closed under crossing. Thus there is a flower $V(\Phi)$ in $\mathcal{V}$ such that $\phi(\vec{s}) \in V(\Phi)$ for all separations $\vec{s}$ in $\mathcal{S}$. Let $\vec{s}$ be a relevant separation displayed by Φ such that no two profiles of $\mathcal{P}$ which contain $\vec{s}$ are distinguished by $\vec{s}$. As Φ distinguishes at least four profiles, there are elements $\vec{s_1}$ and $\vec{s_2}$ in Φ such that $\vec{s} = \vec{s_1} \vee \vec{s_2}$ and thus $\phi(\vec{s})$, which equals $\phi(\vec{s_1}) \cup \phi(\vec{s_2})$, is contained in $\overline{V(\Phi)}$. Similarly $\phi(\vec{s}) \in \overline{V(\Phi)}$ for all relevant separations $\vec{s}$ displayed by Φ such that no two profiles in $\mathcal{P}$ which contain $\overleftarrow{s}$ are distinguished by Φ. Thus $\phi(\vec{s}) \in \overline{V(\Psi)} \cup \{\emptyset, \mathcal{P}\}$ for all separations $\vec{s}$ displayed by Φ. □

Lemma 6.36. *Every k-pseudoflower Φ which distinguishes at least four profiles has a concatenation into an anemone if and only if $V(\Phi)$ is a k-anemone. Similarly, Φ has a concatenation into a k-daisy if and only if $V(\Phi)$ is a daisy.*

Proof. Let Φ' be a k-flower with exactly four petals P_1, P_2, P_3, and P_4 (such that the cyclic order is induced by the linear order on $\{1,2,3,4\}$) which is a concatenation of Φ and distinguishes four profiles. Then V' defined as $\{\phi(P_1 \cup P_2), \phi(P_2 \cup P_3), \phi(P_3 \cup P_4), \phi(P_4 \cup P_1)\}$ is a pre-flower which is contained in $V(\Phi)$. It suffices to show that Φ' is a k-anemone if and only if V' is a pre-anemone. If Φ' is a k-anemone, then the set $P_1 \cup P_3$ has order $k-1$ and thus $\phi(P_1 \cup P_3)$ witnesses that V' is a pre-anemone. So assume that V' is a pre-anemone and let $\vec{s}$ be an element of S such that $\phi(\vec{s}) = \phi(P_1) \cup \phi(P_3)$. By repeated application of Lemma 4.33 there is a separation $\vec{t}$ in S which is equivalent to $\vec{s}$ and satisfies $P_1 \subseteq \vec{t}$, $P_3 \subseteq \vec{t}$, $P_2 \subseteq \overleftarrow{t}$ and $P_4 \subseteq \overleftarrow{t}$. But then $\vec{t} = P_1 \cup P_3$ and thus Φ' is a k-anemone. □

Lemma 6.37. *Let Φ be a k-pseudoflower which distinguishes at least four profiles and let $\vec{s}$ be a relevant separation of order $k-1$ such that $\phi(\vec{s})$ points towards all elements of $V(\Phi)$ and such that $\phi(\vec{s})$ has at least two elements*[3]*. Then there is a petal $\vec{p}$ of Φ such that $\phi(\vec{s}) \leq \phi(\vec{p})$.*

Proof. Assume without loss of generality that if Φ is a strong k-pseudoanemone, then it is a $\leq_A$-maximal strong k-pseudoanemone and in particular $\leq$-maximal. Let P_1 and P_2 be two elements of $\mathcal{P}$ which are contained in $\vec{s}$. As $\phi(\vec{s})$ points towards all elements of $V(\Phi)$, it also points towards all $\phi(\vec{t})$ where $\vec{t}$ is a union of petals of Φ which is also contained in S. In particular no union of petals of Φ distinguishes P_1 and P_2. Thus by Lemma 6.28 the profiles P_1 and P_2 are located at the same petal $\vec{p}$ of Φ. As $P_1 \in \phi(\vec{s}) \cap \phi(\vec{p})$, the fact that $\phi(\vec{s})$ points towards $\phi(\vec{p})$ implies that $\phi(\vec{s}) \leq \phi(\vec{p})$. □

Corollary 6.38. *Let Φ and Φ' be k-pseudoflowers which distinguish at least four profiles each and such that $V(\Phi) \neq V(\Phi')$. Then Φ and Φ' have petals P_i and P'_j respectively such that $\phi(P_i)^* \leq \phi(P'_j)$.*

[3] Recall that $\phi(\vec{s})$ is a subset of $\mathcal{P}$ and that, if it has at least two elements, then there are at least two profiles in $\mathcal{P}$ which contain $\overleftarrow{s}$.

PROOF. Let $S(I')$ be a separation displayed by Ψ such that $\phi(S(I')) \in V(\Psi)$. As $V(\Phi)$ and $V(\Psi)$ are distinct and thus disjoint, $\phi(S(I')) \notin V(\Phi)$. So there is by Lemma 4.3 an orientation of $\phi(S(I'))$ which points towards all elements of $V(\Phi)$. Assume, by replacing $S(I')$ with its inverse if necessary, that $\phi(S(I'))$ points towards the elements of $V(\Phi)$. By Lemma 6.37 there is a petal P_i of Φ such that $\phi(S(I')) \leq \phi(P_i)$. As $\phi(P_i)^*$ points towards $\phi(S(I'))$ and thus towards all elements of $V(\Psi)$, there is again by Lemma 6.37 a petal P'_j of Ψ such that $\phi(P_i)^* \leq \phi(P'_j)$. $\square$

CHAPTER 7

Algorithmic aspects of Chapter 4 in the context of separation systems of bipartitions

This section works with profiles of the universe of bipartitions of some finite ground set E, together with an order function λ. The goal to keep in mind is to compute, given a $k-1$-profile P_0, a tree set which distinguishes as many k-profiles with truncation P_0 as possible. Computing, given a tangle T_0, a tree set distinguishing all tangles whose truncation is T_0, is a common step in computing trees of tangles.

Lemma 3.1 of [**29**] describes an algorithm which computes, given a connectivity function κ on ground set E, $k \in \mathbb{N}$, a κ-tangle T_0 of order at most k and subsets $X_1, \ldots, X_n$ of the ground set with order at most k, whether there is a κ-tangle of order $k+1$ which has T_0 as truncation and contains the inverses of all the X_i. Note that κ is just a suitable order function in the context of this thesis, and thus T is a κ-tangle of order k for some $k \in \mathbb{N}$ if and only if $\{\overleftarrow{s} : \overrightarrow{s} \in T\}$ is a k-profile in the sense of this thesis for which some big side only has one element. The algorithm is used in [**29**] to determine all κ-tangles which have T_0 as a truncation.

The algorithm can easily be adapted to compute, given subsets $X_1, \ldots, X_n$ of E of order at most $k-1$, whether there is a k-profile which contains P_0 and all sets $X_1, \ldots, X_n$. Thus, if $\mathcal{P}$ is the set of all profiles whose truncation is P_0, it is possible for two separations $\overrightarrow{s}$ and $\overrightarrow{t}$ of order $k-1$ to check whether there is a k-profile with truncation P_0 which contains both $\overrightarrow{s}$ and $\overrightarrow{t}$, whether there is a k-profile with truncation P_0 which contains both $\overrightarrow{s}$ and $\overleftarrow{t}$ and so on. If all four profiles exist, then $\overrightarrow{s}$ and $\overrightarrow{t}$ cross properly and otherwise they do not cross properly. In the latter case, which orientations are contained in a common profile and which don't also shows whether $\overrightarrow{s}$ and $\overrightarrow{t}$ are relevant and how $\phi(\overrightarrow{s})$ and $\phi(\overrightarrow{t})$ are nested. Note that for this approach to determine nestedness, it is not necessary to know the elements of $\mathcal{P}$ explicitly. The algorithm is polynomial if k is considered as a constant, but it is exponential in k.

Part of what is so time-consuming, both in the algorithm itself and in the computation of the set of κ-tangles of order k whose truncation is T_0, is the need to find all separations $\overrightarrow{s}$ of order $k-1$ such that every k-profile containing T_0 as well as all X_i also needs to contain $\overrightarrow{s}$. In order to shorten the computation, one approach might be to not insist on distinguishing all k-profiles, but to be given some separations $\overrightarrow{s}_1, \ldots, \overrightarrow{s}_n$ of order $k-1$, and then determining a tree decomposition distinguishing all k-profiles which are distinguished by some $\overrightarrow{s_i}$ by using only corners of the separations $\overrightarrow{s_i}$. One possible such approach is described in Section 7.1 by using the theory developed in Chapter 4, using an oracle (possibly to be replaced with the above algorithm) which determines whether the images under ϕ of two separations of order $k-1$ are nested and if yes, how. A routine which determines whether the images under ϕ of two separations are nested is in particular a routine which decides whether two separations are contained in a common profile with the correct truncation. As such a routine is known but not yet a fast one, as described above, Section 7.2 contains a few statements about what might work when trying to work without such a routine. Those are ideas which have not been pursued in detail yet, that topic needs further research.

7.1. Assuming a routine to determine nestedness

This section describes an algorithm to obtain, given a set $\{\vec{s_1}, \dots, \vec{s_n}\}$ of relevant separations in S, a tree set with flowers at some nodes which are as close to the tree with flowers described in Section 4.1 as possible. Recall that ϕ maps every separation to the set of profiles not containing it, and that the image of ϕ without $\emptyset$ and $\mathcal{P}$ is denoted by $\mathcal{B}$. The algorithm uses an oracle which checks whether two separations $\vec{s}$ and $\vec{t}$ of order $k-1$ properly cross or if their images under ϕ are nested in some way.

The algorithm keeps track of a tree set and a set of flowers fitting together in the following way:

Definition 7.1. A *tree set with flowers* (T, Φ) consists of a tree set T of separations of order $k-1$ and a set of k-flowers Φ such that

- ϕ is injective on T.
- For every $\mathcal{F} \in \Phi$ there is a consistent orientation O of T with set of maximal elements σ such that every element of σ is contained in some petal of $\mathcal{F}$ and every petal of $\mathcal{F}$ is equivalent to an element of σ.
- For any two distinct elements $\mathcal{F}$ and $\mathcal{F}'$ of Φ there is $\vec{t} \in T$ such that $\vec{t}$ points towards all separations displayed by $\mathcal{F}$ and $\overleftarrow{t}$ points towards all separations displayed by $\mathcal{F}'$.

A separation of order $k-1$ is *displayed* by the tree set with flowers if it is either contained in T or a non-trivial union of petals of an element of Φ[1].

Remark 7.2. In more general separation systems $\vec{S}$, it might not be possible to keep track of flowers in S as is done here. In that case it should still be possible to keep track of a set Φ of abstract flowers such that for every element in Φ the set of petals is equal to the set of maximal elements of a consistent orientation of $\phi(T)$. Even if $\mathcal{P}$ is not known explicitly, elements of $\mathcal{B}$ can be determined by stating a separation $\vec{s} \in \vec{S}$ which is mapped to that particular element of $\mathcal{B}$ by ϕ.

The algorithm has at every step a tree set with flowers (T_i, Φ_i) such that T_i consists of relevant corners of the separations $\{\vec{s_1}, \dots, \vec{s_i}\}$ and their inverses, and every $\vec{s_j}$ with $j \le i$ is equivalent to a separation displayed by (T_i, Φ_i). Furthermore, if $i \ge 2$ then every separation displayed by (T_{i-1}, Φ_{i-1}) is equivalent to a separation displayed by (T_i, Φ_i) and no separation in $T_i \setminus T_{i-1}$ is equivalent to a separation displayed by (T_{i-1}, Φ_{i-1}). The rest of the section is devoted to a description of how to obtain T_{i+1} and Φ_{i+1} from T_i and Φ_i.

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[1]If some element $\mathcal{F}$ of Φ is a daisy and $\vec{s}$ is a union of petals of Φ, then that union of petals has order $k-1$ and is thus a union of an interval of petals. So for daisies the separations which are non-trivial unions of petals are exactly the separations displayed by the daisy.

If $\overrightarrow{s_i}$ is equivalent to a separation displayed by (T_i, Φ_i), then let T_{i+1} and Φ_{i+1} be T_i and Φ_i. So assume from here on that $\overrightarrow{s_i}$ is not equivalent to a separation displayed by (T_i, Φ_i).

Do the following recursively with all elements $\overrightarrow{t}$ of T_i: If $\phi(\overrightarrow{s_i}) \geq \phi(\overrightarrow{t})$, then replace $\overrightarrow{s_i}$ with $\overrightarrow{s_i} \vee \overrightarrow{t}$. If $\phi(\overrightarrow{s_i}) \leq \phi(\overrightarrow{t})$, then replace $\overrightarrow{s_i}$ with $\overrightarrow{s_i} \wedge \overrightarrow{t}$. Both separations have order $k-1$ and are equivalent to the former version of $\overrightarrow{s_i}$ by Lemmas 4.32 and 4.33. Also as T_i is nested, the new version of $\overrightarrow{s_i}$ is nested with all elements of T_i with which the old version of $\overrightarrow{s_i}$ was nested. So after finishing this process for all $\overrightarrow{t}$ in T_i, $\overrightarrow{s_i}$ is nested with all elements of T_i which it does not properly cross.

Let X_i be the set of separations in T_i which $\overrightarrow{s_i}$ properly crosses. If X_i is empty, then let T_{i+1} be $T_i + \overrightarrow{s_i} + \overleftarrow{s_i}$ and $\Phi_{i+1} = \Phi_i$. In this case the only non-obvious fact to show about T_{i+1} and Φ_{i+1} is that for all $\mathcal{F} \in \Phi$ there is a suitable consistent orientation of T_{i+1}. In order to show that, let O be the consistent orientation of T_i with set of maximal elements σ such that every element of σ is contained in some petal of $\mathcal{F}$ and every petal of $\mathcal{F}$ is equivalent to an element of σ. As $\overrightarrow{s_i}$ neither crosses nor is not equivalent to a separation displayed by $\mathcal{F}$, there is a petal $\overrightarrow{p}$ of $\mathcal{F}$ such that $\phi(\overrightarrow{s_i}) < \phi(\overrightarrow{p})$ or $\phi(\overrightarrow{s_i})^* < \phi(\overrightarrow{p})$. Assume that $\phi(\overrightarrow{s_i}) < \phi(\overrightarrow{p})$, the other case is symmetric. Let $\overrightarrow{s}$ be an element[2] of σ such that $\phi(\overrightarrow{s}) = \phi(\overrightarrow{p})$. Then $\phi(\overrightarrow{s_i}) < \phi(\overrightarrow{s})$, and as $\overrightarrow{s_i}$ and $\overrightarrow{s}$ are nested this implies $\overrightarrow{s_i} < \overrightarrow{s}$. Thus $O + \overrightarrow{s_i}$ is a consistent orientation of T_{i+1} whose set of maximal elements is σ.

Next consider the case that X_i contains at least one element. Let Y_i be the set of flowers in Φ_i which have a petal properly crossing $\overrightarrow{s}_i$. Let R be the union of $X_i + \overrightarrow{s}_i$ and all separations which are displayed by an element of Φ_i which is neither a petal nor the inverse of a petal. If $\overrightarrow{s}$ properly crosses a petal of a k-flower, then it also properly crosses a separation which is displayed by the k-flower but neither a petal nor the inverse of a petal of that k-flower. Thus $\phi(R)$ is connected under crossing. Let $\overrightarrow{t}$ be an element of X_i and define a k-flower with four petals as follows: Let $P_1 = \overrightarrow{s_i} \wedge \overrightarrow{t}$, $P_2 = \overrightarrow{s_i} \wedge \overleftarrow{t}$, $P_3 = \overleftarrow{s_i} \wedge \overleftarrow{t}$ and $P_4 = \overleftarrow{s_i} \wedge \overrightarrow{t}$. By repeated application of Lemma 6.24 to separations in R there is a k-flower $\mathcal{F}$ which displays $\overrightarrow{s_i}$ and $\overrightarrow{t}$ such that no element of R properly crosses a petal of $\mathcal{F}$. Because applications of Lemma 6.24 only create relevant petals, $\mathcal{F}$ only has relevant petals. Every element $\overrightarrow{t}$ of $T_i \backslash X_i$ is nested with both $\overrightarrow{s_i}$ and $\overrightarrow{t}$ and has thus a petal P_i such that $\overrightarrow{t} \leq P_i$ or $\overleftarrow{t} \leq P_i$. This property is maintained by applications of Lemma 6.24 as every element of $T_i \backslash X_i$ is nested with all elements of R. Because $\phi(R)$ is connected under crossing, every separation in R is equivalent to a separation displayed by $\mathcal{F}$. Define Φ_{i+1} to be $\Phi_i \backslash Y_i + \mathcal{F}$. Let T_{i+1} be the union of $T_i \backslash X_i$ and all petals and inverses of petals of $\mathcal{F}$ which are not equivalent to an element of T_i.

In order to show that T_{i+1} is nested, let $\overrightarrow{s}$ and $\overrightarrow{t}$ be elements of T_{i+1}. If the separations are both contained in T_i or both petals or inverses of $\mathcal{F}$ then they are nested. So assume that $\overrightarrow{s}$ is contained in T_i and $\overrightarrow{t}$ is a petal or the inverse of a petal of $\mathcal{F}$. Let $\overrightarrow{q}$ be a petal of $\mathcal{F}$ such that $\overrightarrow{s} \leq \overrightarrow{q}$ or $\overleftarrow{s} \leq \overrightarrow{q}$. By taking inverses of $\overrightarrow{s}$ and/or $\overrightarrow{t}$ if necessary it suffices to consider the case that $\overrightarrow{s} \leq \overrightarrow{q}$ and that $\overrightarrow{t}$ is a petal of $\mathcal{F}$. If $\overrightarrow{t} = \overrightarrow{q}$, then $\overrightarrow{s} \leq \overrightarrow{t}$ and if $\overrightarrow{t} \neq \overrightarrow{q}$ then $\overrightarrow{s} \leq \overrightarrow{q} \leq \overleftarrow{t}$, so in both cases $\overrightarrow{s}$ and $\overrightarrow{t}$ are nested.

[2]It is true but not needed here that there is exactly one such element.

In order to show that ϕ is injective on T_{i+1}, let $\vec{s}$ and $\vec{t}$ be distinct elements of T_{i+1}. If $\vec{s}$ and $\vec{t}$ are both contained in T_i, then ϕ maps them to different elements of $\mathcal{B}$ as ϕ is injective on T_i. If one of $\vec{s}$ and $\vec{t}$ is contained in T_i and the other is not, then ϕ maps them to different elements of $\mathcal{B}$ as $T_{i+1} \backslash T_i$ only contains separations which are not equivalent to separations contained in T_i. So assume that both $\vec{s}$ and $\vec{t}$ are contained in $T_{i+1} \backslash T_i$. Then they have orientations which are petals of $\mathcal{F}$, so as every petal of $\mathcal{F}$ is relevant ϕ maps $\vec{s}$ and $\vec{t}$ to distinct elements of $\mathcal{B}$.

By definition of T_{i+1}, every petal of $\mathcal{F}$ is equivalent to an element of T_{i+1}, and as ϕ is injective on T_{i+1} that element is unique. Denote the set of elements of T_{i+1} which are equivalent to petals of $\mathcal{F}$ by σ. In order to show that every element of σ is contained in a petal of $\mathcal{F}$, let $\vec{s}$ be an element of σ and $\vec{t}$ the petal of $\mathcal{F}$ which is equivalent to σ. If $\vec{s} \in T_{i+1} \backslash T_i$, then $\vec{s} = \vec{t}$ and in particular $\vec{s} \leq \vec{t}$. Otherwise $\vec{s} \in T_i \backslash X_i$ and thus there is a petal $\vec{p}$ of $\mathcal{F}$ such that $\vec{s} \leq \vec{p}$ or $\overleftarrow{s} \leq \vec{p}$. Then $\phi(\vec{s}) = \phi(\vec{t})$ implies that $\vec{t} = \vec{p}$ and that $\vec{s} \leq \vec{p}$, so also in this case $\vec{s} \leq \vec{t}$. In order to show that σ is the set of maximal elements of a consistent orientation O of T_{i+1} it suffices to show that for all separations $\vec{t}$ in T_{i+1} there is an element $\vec{s}$ of σ such that $\vec{t} \leq \vec{s}$ or $\overleftarrow{t} \leq \vec{s}$. This property is clear if $\vec{t} \in T_{i+1} \backslash T_i$, so assume otherwise. Then $\vec{t} \in T_i \backslash X_i$ and thus there is a petal $\vec{p}$ of $\mathcal{F}$ such that $\vec{t} \leq \vec{p}$ or $\overleftarrow{t} \leq \vec{p}$. Let $\vec{s}$ be the element of σ which is equivalent to $\vec{p}$. Then $\phi(\vec{t}) < \phi(\vec{s})$ or $\phi(\overleftarrow{t}) < \phi(\vec{s})$ because ϕ is injective on T_{i+1}, so the fact that T_{i+1} is nested implies $\vec{t} < \vec{s}$ or $\overleftarrow{t} < \vec{s}$.

If it exists, let $\mathcal{F}'$ be a flower in $\Phi_i \backslash Y_i$. Let $O_{\mathcal{F}'}$ be the consistent orientation of T_i which is associated with $\mathcal{F}'$ and let σ' be the set of maximal elements of the orientation. As $\vec{s_i}$ does not properly cross any petal of $\mathcal{F}'$ and the elements of σ' are equivalent to petals of $\mathcal{F}'$, σ' is a subset of T_{i+1}. Assume for a contradiction that σ' is a subset of the consistent orientation O of T_{i+1} which is associated with $\mathcal{F}$. Then every element of σ' is contained in a petal of $\mathcal{F}$ and thus points towards $\vec{s_i}$. But then for all petals $\vec{p}$ of $\mathcal{F}'$ the set $\phi(\vec{p})$ points towards $\phi(\vec{s_i})$. So $\vec{s_i}$ is equivalent to a union of petals of $\mathcal{F}'$, contradicting the fact that $\vec{s_i}$ is not equivalent to a separation displayed by (T_i, Φ_i). Hence there is a separation $\vec{s} \in \sigma'$ such that $\vec{s} \in O_{\mathcal{F}'}$ but $\overleftarrow{s} \in O$. In particular $\overleftarrow{s}$ is contained in a petal $\vec{s'}$ of $\mathcal{F}'$ and thus points towards all separations displayed by $\mathcal{F}'$ and similarly $\vec{s}$ points towards all separations displayed by $\mathcal{F}$. In order to show that σ' is the set of maximal elements of a consistent orientation of T_{i+1}, let $\vec{t}$ be an element of $T_{i+1} \backslash T_i$. Then $\vec{t}$ is a petal of $\mathcal{F}$. If $\vec{t} = \vec{s'}$, then $\overleftarrow{t} \leq \vec{s}$ and otherwise $\vec{t} \leq \overleftarrow{s'} \leq \vec{s}$, so in both cases $\vec{t}$ is less than some element of σ'.

Furthermore every separation in $T_i \backslash T_{i+1}$ is contained in X_i and thus equivalent to a union of petals of $\mathcal{F}$, so (T_{i+1}, Φ_{i+1}) has all the desired properties.

Remark 7.3. Let (T, Φ) be a tree set with flowers. Given a k-flower $\mathcal{F} \in \Phi$, the consistent orientation $O_{\mathcal{F}}$ of T with set of maximal elements σ such that every petal of $\mathcal{F}$ is equivalent to an element of σ and every element of σ is contained in a petal of $\mathcal{F}$ is unique. Also the function which maps every element of σ to the petal of $\mathcal{F}$ in which it is contained is a bijection. Furthermore, just as in Section 4.3, if a profile P in $\mathcal{P}$ induces a consistent orientation O on T, then O differs from all orientations $O_{\mathcal{F}}$ where $\mathcal{F} \in \Phi$.

7.2. Assuming a membership oracle for the common truncation

The previous section describes an algorithm creating a tree set with flowers for a set of profiles with a common truncation P_0 in the context where there is an oracle determining for two separations of order $k-1$ whether they properly cross and if not, how their images under ϕ are nested. This section presents ideas, not finished research but speculation on what might be a suitable approach, how to proceed in a setting where the above oracle is not available but instead an oracle to test whether separations of order at most $k-2$ are contained in P_0. Under these circumstances, it is not feasible any more to frequently check whether two separations $\vec{s}$ and $\vec{t}$ properly cross. But it is still possible to check the necessary condition that all four corners of $\vec{s}$ and $\vec{t}$ have the correct order, namely $k-1$. If some corner has a different order than $k-1$, then by submodularity of the order function there is a corner, $\vec{s} \vee \vec{t}$ say, which has order less than $k-1$. Then either $\vec{s} \vee \vec{t}$ is contained in P_0, and thus $\phi(\overleftarrow{s}) = \phi(\overleftarrow{t}) = \emptyset$, or $(\vec{s} \vee \vec{t})^*$ is contained in P_0, and thus $\phi(\overleftarrow{s}) \leq \phi(\vec{t})$.

There is still research to do but there is an idea to design an algorithm for these circumstances as follows: Very similarly to the algorithm of Section 7.1, in each step there should be a current tree set T_i of elements of S and a current set Φ_i of k-flowers. For every k-flower $\mathcal{F}$ there should be a consistent orientation of T_i with set of maximal elements $\sigma_{\mathcal{F}}$ such that every petal of $\mathcal{F}$ contains exactly one element of $\sigma_{\mathcal{F}}$ and every element of $\sigma_{\mathcal{F}}$ is contained in a petal of $\mathcal{F}$. Furthermore, any two k-flowers in the set of k-flowers should have different such orientations.

Now, the algorithm should try to follow the algorithm of Section 7.1 but in places where that other algorithm checks whether $\phi(\vec{s})$ and $\phi(\vec{t})$ are nested and how, this algorithm should check whether some corner has too small order and otherwise assume that $\vec{s}$ and $\vec{t}$ are properly crossing. Of course, as T_i may contain irrelevant elements or distinct elements which are not yet known to be equivalent and k-flowers in Φ_i might contain k-flowers with irrelevant petals, while inserting $\vec{s_i}$ into (T_i, Φ_i), something might go wrong. But something going wrong means in that case that there is a corner which has too low order. If a separation in T_i or a union of petals of some k-flower in Φ_i happens to be shown to be contained in all profiles in $\mathcal{P}$, then all other separations displayed by (T_i, Φ_i) which are less than that separation are also contained in all profiles in $\mathcal{P}$ and thus it is possible to delete all these separations from (T_i, Φ_i).

CHAPTER 8

Matroids

Most statements about flowers in matroids are already true in separation systems of bipartitions whose order function is limit-closed and can thus be found in Chapter 6. One fact which does need extra properties of matroids is the fact that there are no infinite daisies. That statement is proved in this chapter.

Remark 8.1. Let M be an infinite matroid and $\{P_1, \ldots, P_n\}$ a finite k-flower of M. Then by applying Lemma 1.42 several times, once to each petal, a finite minor N of M is obtained in which $\{P_1 \cap E(N), \ldots, P_n \cap E(N)\}$ is a finite k-flower such that $\sqcap_N(P_i \cap E(N), P_j \cap E(N)) = \sqcap_M(P_i, P_j)$ for all distinct indices i and j. As a finite k-flower in a finite matroid is a k-flower in the sense of [**5**], Lemma 1.88 can be applied to the k-flower of N. Thus Lemma 1.88 holds for $\{P_1, \ldots, P_n\}$. In other words, Lemma 1.88 can be applied to finite flowers in infinite matroids.

Lemma 8.2. *Let Φ be a finite k-flower of M with at least five petals and parameters $\sqcap_M(P_1, P_2) = c$ and $\sqcap_M(P_1, P_3) = d$. Denote $\sqcap_{M^*}(P_1, P_2)$ by c^* and $\sqcap_{M^*}(P_1, P_3)$ by d^*. Then $c + c^* = k - 1$ and $c^* - d^* = c - d$.*

Proof. Let i and j be distinct indices. Then

$$\begin{aligned}\sqcap_{M^*}(P_i, P_j) &= \lambda_{M^* \restriction (P_i \cup P_j)}(P_j) = \lambda_{M.(P_i \cup P_j)}(P_j) \\ &= \lambda_M(E \setminus P_i) - \lambda_{M \setminus P_j}(E \setminus (P_i \cup P_j)) \\ &= k - 1 - \sqcap_M(P_i, E \setminus (P_i \cup P_j))\end{aligned}$$

where the third equality holds by Lemma 1.39. So

$$c^* = \sqcap_{M^*}(P_1, P_2) = k - 1 - \sqcap_M(P_1, E \setminus (P_1 \cup P_2)) = k - 1 - c$$

where the last equation holds by Lemma 1.88. Furthermore,

$$\begin{aligned}d^* &= \sqcap_{M^*}(P_1, P_3) = k - 1 - \sqcap_M(P_1, E \setminus (P_1 \cup P_3)) \\ &= k - 1 - (2c - d) = k - 1 - c - (c - d) = c^* - (c - d),\end{aligned}$$

where the third equality also holds by Lemma 1.88. □

Lemma 8.3. *Let $(P_1, \ldots, P_n)$ be a k-flower with at least five petals and denote $\sqcap_M(P_1, P_2)$ by c and $\sqcap_M(P_1, P_3)$ by d. Then $(P_n \cup P_2, P_3, \ldots, P_{n-1})$ is a $2c-d+1$-flower of $M \setminus P_1$ in which adjacent petals have local connectivity c and non-adjacent petals have local connectivity d.*

Proof. Deleting P_1 does not change the local connectivity of sets which are disjoint from P_1. So it suffices to show that $(P_n \cup P_2, P_3, P_4, \ldots, P_{n-1})$ is a $2c-d+1$-flower in $M \setminus P_1$. In order to show that let I' be a non-empty interval of the set $\{3, 4, \ldots, n-1\}$. Then by Lemma 1.88

$$\lambda_{M \setminus P_1}(P_{I'}) = \sqcap_M(P_{I'}, E \setminus (P_1 \cup P_{I'})) = 2c - d. \qquad \square$$

Corollary 8.4. *The partition $(P_n \cup P_2, P_3, \dots, P_{n-1})$ is a $k-d$-flower of M/P_1 in which adjacent petals have local connectivity $c-d$ and non-adjacent petals have local connectivity* 0.

Proof. By Lemma 8.2 the partition $(P_1, \dots, P_n)$ is a k-flower of M^* with local connectivities $k-1-c$ for adjacent petals and $(k-1-c)-(c-d)$ for non-adjacent petals. So by Lemma 8.3 $(P_n \cup P_2, P_3, \dots, P_{n-1})$ is a $k-d$-flower of $M^* \backslash P_1$ with local connectivities $k-1-c$ for adjacent petals and $(k-1-c)-(c-d)$ for non-adjacent petals. Applying Lemma 8.2 again yields that $(P_n \cup P_2, P_3, \dots, P_{n-1})$ is a $k-d$-flower of M/P_1 with local connectivities $c-d$ for adjacent petals and 0 for non-adjacent petals. $\square$

Lemma 8.5. *For every $k \in \mathbb{N}$: There are no infinite k-daisies.*

Proof. Assume for a contradiction that there is an infinite k-daisy. Then there also is a k-daisy of the form $(P_i)_{i \in \mathbb{N}}$. Denote the local connectivity of adjacent petals by c and the local connectivity of non-adjacent petals by d. For all $n \geq 5$, apply Lemma 8.3 to the k-flowers $(P_0, \dots, P_n, P_{[n,\infty[})$. So the partitions

$$(P_0 \cup P_2, P_3, , \dots, P_n, P_{[n,\infty[})$$

are $2c-d+1$-flowers of $M \backslash P_1$ with local connectivities c for adjacent petals and d for non-adjacent petals. Thus the partition $(P_0 \cup P_2, P_3, \dots)$ is a $2c-d+1$-daisy for $M \backslash P_1$ with the same local connectivities. Similarly, by applying Corollary 8.4 to finite flowers, $(P_0 \cup P_2 \cup P_4, P_5, \dots)$ is a $2(c-d)+1$-flower of $M \backslash P_1/P_3$ with local connectivities $c-d$ for adjacent petals and 0 for non-adjacent petals. Thus there is a matroid N with an infinite daisy of the form $(Q_i)_{i \in \mathbb{N}}$ and $c \in \mathbb{N}$ such that the connectivity of intervals is $2c$ and the local connectivity between adjacent and non-adjacent petals is c and 0 respectively. By Lemma 8.2, $(Q_i)_{i \in \mathbb{N}}$ is also a $2c+1$-flower in N^* with local connectivities c for adjacent petals and 0 for non-adjacent petals.

For each $i \in \mathbb{N}$ let B_i be a base of $N \restriction Q_i$ and B'_i a base of $N.Q_i$ such that $B'_i \subseteq B_i$. As the local connectivity of Q_i and Q_{i+2} is 0, $B_i \cup B_{i+2}$ is a base of $N \restriction (Q_i \cup Q_{i+2})$. Thus $B_i \cup B'_{i+1} \cup B_{i+2}$ is independent in $N \restriction Q_{[i,i+2]}$. Dually, $B'_i \cup B_{i+1} \cup B'_{i+2}$ is spanning in $N.Q_{[i,i+2]}$. Let X be a base of $N \restriction Q_{[i,i+2]}$ such that $B_i \cup B'_{i+1} \cup B_{i+2} \subseteq X \subseteq B_i \cup B_{i+1} \cup B_{i+2}$. Let Y be a base of $N.Q_{[i,i+2]}$ such that $B'_i \cup B'_{i+1} \cup B'_{i+2} \subseteq Y \subseteq B'_i \cup B_{i+1} \cup B'_{i+2}$. Now $X \backslash Y$ is finite, so by Lemma 1.44

$$\begin{aligned}
\lambda_N(Q_{[i,i+2]}) &= |X \backslash Y| - |Y \backslash X| \\
&= |B_i \backslash B'_i| + \left|B_{i+2} \backslash B'_{i+2}\right| + |(X \backslash Y) \cap Q_{i+1}| - |Y \backslash X| \\
&\geq 4c + 0 - 2c.
\end{aligned}$$

As the connectivity of $Q_{[i,i+2]}$ in N is $2c$, the inequality has to be an equality, so $X = B_i \cup B'_{i+1} \cup B_{i+2}$ and $Y = B'_i \cup B_{i+1} \cup B'_{i+2}$. In particular, $B_i \cup B'_{i+1} \cup B_{i+2}$ is a base of $N \restriction Q_{[i,i+2]}$ and $B'_i \cup B_{i+1} \cup B'_{i+2}$ is a base of $N.Q_{[i,i+2]}$.

Denote the set $\bigcup_{i \text{ even}} B_i \cup \bigcup_{i \text{ odd}} B'_i$ by B. Then B spans all sets of the form $Q_{[i,i+2]}$ where i is even, so B is spanning in N. Assume for a contradiction that B contains a circuit C. Then for every odd $i \in \mathbb{N}$, the set $C \cap Q_{[i,i+2]}$ is a scrawl of $N.Q_{[i,i+2]}$. Also $C \cap Q_{[i,i+2]}$ is contained in $B'_i \cup B_{i+1} \cup B'_{i+2}$ which was already shown to be a base of $N.Q_{[i,i+2]}$. So $C \cap Q_{[i,i+2]}$ is empty for all odd $i \in \mathbb{N}$. Thus C is a subset of Q_0, so $C \subseteq B_0$, which is a contradiction to the fact that B_0 is independent in N. So B is a base of N. For every $i \in \mathbb{N}$, the set $E \backslash B'_i$ is a base of

$N^*\restriction Q_i$ and the set $E\backslash B_i$ is a base of $N^*.Q_i$. Thus, just as B is a base of N, the set $\bigcup_{i \text{ even}} E\backslash B_i' \cup \bigcup_{i \text{ odd}} E\backslash B_i$ is a base of N^*, and thus $B' := \bigcup_{i \text{ even}} B_i' \cup \bigcup_{i \text{ odd}} B_i$ is a base of N.

As $B_0' \cup B_1$ is independent in $N\restriction(Q_0 \cup Q_1)$ and $B_0 \cup B_1$ is spanning in $N\restriction(Q_0 \cup Q_1)$, there is a base R of $N\restriction(Q_0 \cup Q_1)$ such that $B_0' \cup B_1 \subseteq R \subseteq B_0 \cup B_1$. As the partition $(Q_0 \cup Q_1, Q_2, Q_3, \ldots)$ with the induced cyclic order is also a $2c+1$-daisy of N with local connectivities c for adjacent petals and 0 for non-adjacent petals, also

$$R \cup \bigcup_{i \text{ even},\, i \neq 0} B_i' \cup \bigcup_{i \text{ odd},\, i \neq 1} B_i$$

is a base of N. As this base contains B' as a subset, and a base cannot be properly contained in another one, $R = B_0' \cup B_1$. So $B_0' \cup B_1$ is a base of $N\restriction(Q_0 \cup Q_1)$, implying that B_0' is a base of $N\backslash Q_{[2,\infty[}/Q_1$. As B_0' is also a base of $N/Q_{[2,\infty[}/Q_1$, the connectivity of Q_0 in N/Q_1 is 0. But by Lemma 1.39

$$\begin{aligned}\lambda_{N/Q_1}(Q_0) &= \lambda_N(Q_0 \cup Q_1) - \lambda_{N\backslash Q_0}(Q_1) \\ &= \lambda_N(Q_0 \cup Q_1) - \sqcap_N(Q_1, Q_{[2,\infty[}) \\ &= 2c - c = c.\end{aligned}$$

So $c = 0$, contradicting the fact that $(Q_0, Q_1, \ldots)$ is a $2c+1$-daisy of N. □

Corollary 8.6. *Let Φ be a k-daisy which distinguishes at least two profiles with the same truncation to $k-1$-profiles. Then the set of k-flowers extending Φ has no infinite increasing chain.*

Proof. Assume for a contradiction that there is an infinite increasing chain $(\Phi_j)_{j \in J}$ of k-flowers extending Φ. As a concatenation of k-anemone cannot be a k-daisy, all Φ_j are k-daisies. Given a k-pseudoflower $(Q_i)_{i \in \mathbb{N}}$, with cyclic order induced by the linear order of $\mathbb{N}$, denote for $i \geq 5$ the concatenation to $(Q_1, Q_2, \ldots, Q_{i-1}, Q_0 \cup Q_i \cup Q_{i+1} \cup \cdots)$ by Ψ_i. Then there is a partition $(Q_i)_{i \in \mathbb{N}}$ such that all Ψ_i are concatenations of some Φ_j and such that Φ equals some Ψ_i. As the order function is limit-closed, $(Q_i)_{i \in \mathbb{N}}$ is a k-pseudoflower. Consider the k-pseudoflower Ψ which arises from $(Q_i)_{i \in \mathbb{N}}$ by concatenating Q_0 and Q_1 into one petal. As $(Q_2, Q_3, Q_4, Q_0 \cup Q_1 \cup Q_5 \cup \cdots)$ is a concatenation of some Φ_j and thus is a k-daisy, Ψ is not a k-anemone. By Lemma 8.5 Ψ is also not a k-daisy, so it is not a k-flower. As all the Ψ_i are k-flowers, there is some $i \geq 1$ such that $Q_0 \cup \cdots \cup Q_i$ has order less than $k-1$. Then for all $i' \geq i$

$$\lambda(Q_0 \cup \cdots \cup Q_{i'}) \leq \lambda(Q_0 \cup \cdots \cup Q_i) + \lambda(Q_i \cup \cdots \cup Q_{i'}) - \lambda(Q_i) < k-1$$

and for all $i' < i$

$$\begin{aligned}\lambda(Q_0 \cup \cdots \cup Q_{i'}) &\leq \lambda(Q_0 \cup \cdots \cup Q_i) + \lambda(Q_0 \cup \cdots \cup Q_{i'} \cup Q_{i+2} \cup \cdots) \\ &\quad - \lambda(Q_0 \cup \cdots \cup Q_i \cup Q_{i+2} \cup \cdots) \\ &= \lambda(Q_0 \cup \cdots \cup Q_i) < k-1.\end{aligned}$$

As Φ distinguishes two profiles P_1 and P_2 with the same truncation to $k-1$-profiles and is a concatenation of some Ψ_i, all Ψ_i with sufficiently large index i distinguish P_1 and P_2. Then also there is some i' such that $Q_0 \cup Q_1 \cup \cdots \cup Q_i$ distinguishes P_1 and P_2. That is a contradiction to $Q_0 \cup \cdots \cup Q_i$ having order less than $k-1$. □

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Appendix

A. Short summary of results

This thesis has two parts. The topic of the first part is the question whether nearly finitary matroids need be k-nearly finitary for some $k \in \mathbb{N}$. Section 2.2 contains a reduction of the original problem to matroids M with the property that for all edge sets P of finite connectivity one of $M \backslash P$ and $M \restriction P$ is finitary (Corollary 2.13). It then follows that every nearly finitary Ψ-matroid with finite parts is k-nearly finitary for some $k \in \mathbb{N}$ (Lemma 2.14).

Furthermore every nearly finitary matroid M that arises from a graph-like space in which some finite vertex set V_f meets all infinite pseudo-cycles is k-nearly finitary for some $k \in \mathbb{N}$ (Theorem 3.24). In order to show that the class of such matroids is not too small to be relevant, graph-like spaces are investigated in which any two distinct vertices are, after deletion of all edges between those two vertices and after deletion of a finite vertex set, contained in distinct topological components of the graph-like space. Every matroid M that is induced by graph-like space with this property and is not k-nearly finitary for any $k \in \mathbb{N}$ is not nearly finitary or has a minor that is not k-nearly finitary for any $k \in \mathbb{N}$ and that is induced by a graph-like space in which a vertex meets every infinite pseudo-cycle (Lemma 3.12 and Corollary 3.15).

The aim of the second part is to transfer the structure theory of [**17**], and in particular of flowers, to infinite separation systems. For this, in Section 4.3 a separation system $\mathcal{S}$, a number $k \in \mathbb{N}$ and a set of k-profiles that have all the same truncation is fixed. The map that maps every separation $\vec{s}$ in $\mathcal{S}$ to the set of profiles in $\mathcal{P}$ that do not contain $\vec{s}$ is a homomorphism of separation systems from $\mathcal{S}$ to the separation system of bipartitions on ground set $\mathcal{P}$ (Lemma 4.32). The image of that map is a separation system that is a subsystem of the universe of bipartitions on ground set $\mathcal{P}$, and it is closed under pairwise unions and intersections of non-nested elements (Corollary 4.35).

Such separation systems $\mathcal{B}$ are under investigation in Sections 4.1 and 4.2. The separations in $\mathcal{B}$ are organised into the set $\mathcal{E}$ of separations that are nested with all other separations in $\mathcal{B}$, and the finest partition $\mathcal{V}$ of the remaining separations in $\mathcal{B}$ such that any two non-nested separations are in the same partition class. For an element V of $\mathcal{V}$, the coarsest partition class of the ground set such that every element of V is a union of partition classes is denoted by $\partial(V)$, and the set of unions of partition classes that are also contained in $\mathcal{B}$ is denoted by $\overline{V}$. If V is finite, then either $\overline{V}$ is the set of all unions of partition classes, apart from the empty set and the whole ground set (anemone-like), or there is a cyclic order (unique up to mirroring) of the partition classes such that, apart from the empty set and the whole ground set, the elements of $\overline{V}$ are exactly the unions of intervals (daisy-like)(Corollary 4.15). If V is infinite, then there are finite subsets of V that behave very much like finite elements of $\mathcal{V}$ and for which in particular the characterisation of $\overline{\mathcal{F}}$ still applies. For a fixed $V \in \mathcal{V}$, those finite subsets are either all anemone-like or all daisy-like (Lemma 4.25). If V is infinite and all suitable finite subsets are daisy-like, then there is a cyclic order (unique up to mirroring) of $\partial(V)$ such that all elements of $\overline{V}$ are unions of intervals of partition classes (Lemma 4.27). If $\mathcal{B}$ is finite, then a structure theorem of a tree with flowers, very much like the original structure theorem from [**17**] can be recovered (Theorem 4.20).

The next setting, which is analysed in Chapter 5, is that of subsystems of universes of vertex separations that are limit-closed in the following sense: For

$k \in \mathbb{N}$ and for a chain $(A_i, B_i)_{i\in I}$ of separations of order at most k, the supremum of the chain in the subsystem exists and has order at most k and is of the form $(\bigcup_{i\in I} A_i \cup X, \bigcap_{i\in I} B_i)$. In this setting, a definition of k-pseudoflower is given and discussed such that every k-pseudoflower is inherently either anemone-like or daisy-like. There is a relation on the set of k-pseudoflowers such that $\Phi \leq \Psi$ if all separations displayed by Φ are also displayed by Ψ. For most k-pseudoflowers Φ, there is a $\leq$-maximal k-pseudoflower Ψ such that $\Phi \leq \Psi$ (Theorem 5.44). If $k \in \mathbb{N}$ and $\mathcal{P}$ is a set of k-profiles that all have the same truncation and are limit-closed, then most daisy-like $\leq$-maximal k-pseudoflowers are also maximal in the pre-order "distinguishing more profiles in $\mathcal{P}$" (Lemma 5.27).

In Chapter 6, the separation system under consideration is a subsystem of a universe with bipartitions together with a limit-closed order function, and again a set $\mathcal{P}$ of k-profiles with the same truncation is fixed. In this context, a k-pseudoflower is a partition together with a cyclic order of the partition classes such that every union of an interval of partition classes has order at most $k-1$. And the order $\leq$ on the set of k-pseudoflowers is the one where $\Phi \leq \Psi$ if the partition of Ψ is finer than that of Φ and the map between the partitions that is the inclusion is a homomorphism of cyclic orders. Then every k-pseudoflower can be extended to a $\leq$-maximal one (Corollary 6.15). Also, for most $\leq$-maximal k-pseudoflowers it is true that if there are two k-profiles with the same truncation that are not distinguished, then the k-pseudoflower can be concatenated into an infinite k-anemone (Lemmas 6.24 and 6.28). Therefore most $\leq$-maximal k-pseudoflowers that do not extend a k-anemone are also maximal in the pre-order where $\Phi \preccurlyeq \Psi$ if all profiles in $\mathcal{P}$ that are distinguished by the union of some interval of Φ are also distinguished by the union of some interval of Ψ (Lemma 6.29). For every k-anemone Φ with at least k many partition classes, all k-pseudoflowers Ψ with $\Phi \leq \Psi$ have the property that all their unions of partition classes have order at most $k-1$ (Corollary 6.12), k-pseudoflowers with this property are called strong k-pseudoanemones. Also, for every k-anemone Φ with at least k many partition classes, the set of all partitions of k-pseudoflowers Ψ with $\Phi \leq \Psi$ has a finest element (Theorem 6.22). And most $\leq$-maximal strong k-pseudoanemones are also maximal in the pre-order $\preccurlyeq_A$ where $\Phi \preccurlyeq_A \Psi$ if all k-profiles that are distinguished by some union of partition classes of Φ are also distinguished by some union of partition classes of Ψ (Lemma 6.34). In Chapter 8 it is shown that if the order function is the order function of a matroid, then there are no infinite k-daisies (Lemma 8.5).

Finally, Chapter 7 contains an algorithm to determine, from a small set of separations of a finite separation system of bipartitions, as much as possible from the tree with flowers given in the structure theorem Theorem 4.20.

B. Kurze Zusammenfassung der Ergebnisse

Diese Dissertation besteht aus zwei Teilen. Der erste Teil behandelt die Frage ob es zu jedem Matroiden M, der fast finitär ist, auch eine natürliche Zahl k gibt sodass M fast finitär mit Parameter k ist. In Abschnitt 2.2 wird gezeigt, dass es reicht, Matroide mit der Eigenschaft zu betrachten, dass für jede Kantenmenge P mit endlichem Zusammenhang einer der Matroide $M\backslash P$ und $M\restriction P$ finitär ist (Korollar 2.13). Daraus folgt, dass jeder fast finitäre Ψ-Matroid mit endlichen Teilen auch fast finitär mit mit einem Parameter k ist (Lemma 2.14).

Auch jeder fast finitäre Matroid M, der von einem graphenartigen Raum induziert wird in dem eine endliche Eckenmenge jeden unendlichen Pseudokreis trifft, ist fast finitär mit einem Parameter k ist (Satz 3.24). Um zu zeigen, dass die Klasse der Matroide mit diesen Eigenschaften gro genug ist um bedeutsam zu sein, werden graphenartige Räume mit der Zusatzeigenschaft betrachtet, dass je zwei unterschiedliche Ecken nach dem Löschen aller Kanten zwischen ihnen und dem Löschen einer geeigneten endlichen Eckenmenge in unterschiedlichen topologischen Komponenten des graphenartigen Raumes enthalten sind. Jeder Matroid, der von einem graphenartigen Raum mit dieser Eigenschaft induziert wird und für keine natürliche Zahl k fast finitär mit Parameter k ist, ist entweder nicht fast finitär oder hat einen Minor, der auch ffür keine natürliche Zahl k fast finitär mit Parameter k ist und von einem graphenartigen Raum induziert wird, in dem eine Ecke in allen unendlichen Pseudokreisen enthalten ist (Lemmas 3.12 und Korollar 3.15).

Der zweite Teil der Arbeit zielt darauf ab, den Struktursatz von [**17**], und insbesondere die Blumen, auf unendliche Teilungssysteme zu übertragen. Dazu werden in Abschnitt 4.3 ein Teilungssystem $\mathcal{S}$ und eine Menge $\mathcal{P}$ von k-Profilen, die alle zu demselben $k-1$-Profil eingeschränkt werden können, betrachtet. Die Funktion, die jede Teilung auf die Menge derjenigen Elemente von $\mathcal{P}$ abbildet, welche die Teilung nicht enhalten, ist dann ein Teilungssystem-Homomorphismus, dessen Bild im Teilungssystem von Bipartitionen der Menge $\mathcal{P}$ enthalten ist (Lemma 4.32). Das Bild ist ein Teilsystem, in dem zu je zwei nicht geschachtelten Elementen auch deren Schnitt und Vereinigung enthalten ist (Korollar 4.35).

Solche Teilungssysteme $\mathcal{B}$ werden in den Abschnitten 4.1 und 4.2 untersucht. Die Menge aller Teilungen in $\mathcal{B}$, die mit allen anderen Teilungen in $\mathcal{B}$ geschachtelt sind, wird mit $\mathcal{E}$ bezeichnet, und $\mathcal{V}$ ist die feinste Partition von $\mathcal{B}\backslash\mathcal{E}$ in der je zwei nicht geschachtelte Teilungen in derselben Partitionsklasse enthalten sind. Zu einem Element $V \in \mathcal{V}$ wird mit $\partial(V)$ die gröbste Partition der Grundmenge bezeichnet, in der jedes Element von V eine Vereinigung von Partitionsklassen ist. Die Menge von Elementen von $\mathcal{B}$, die Vereinigungen von Partitionsklassen in $\partial(V)$ sind, wird mit $\overline{V}$ bezeichnet. Falls V endlich ist, so ist $\overline{V}$ entweder gleich der Menge aller Vereinigungen von Partitionsklassen von $\partial(V)$ mit Ausnahme von $\emptyset$ und der Grundmenge (anemonenartig), oder es gibt eine zyklische Ordnung (die eindeutig bis auf Spiegelung ist) sodass die Elemente von $\overline{V}$ gerade die Vereinigungen von Intervallen sind, die weder $\emptyset$ noch $\partial(V)$ sind (gänseblümchenartig) (Korollar 4.15). Falls V dagegen unendlich ist, dann gibt es endliche Teilmengen von V, die sich genauso wie endliche Elemente von $\mathcal{V}$ verhalten, und für die daher die Charakterisierung für endliche V gilt. Für ein festes unendliches $V \in \mathcal{V}$ sind diese endlichen Teilmengen entweder alle anemonenartig oder alle gänseblümchenartig (Lemma 4.25). Falls die Teilmengen alle gänseblümchenartig sind, dann gibt es auch auf $\partial(V)$ eine zyklische Ordnung, die eindeutig bis auf Spiegelung ist, sodass alle Elemente von

$\overline{V}$ Vereinigungen von Intervallen sind (Lemma 4.27). Wenn $\mathcal{B}$ endlich ist, dann gilt ein Struktursatz, der eng verwandt ist mit dem Struktursatz von [**17**] (Satz 4.20).

Kapitel 5 handelt von dem Spezialfall, in dem das Teilungssystem ein Teilungssytem aus Eckentrennern ist, das auf folgende Weise abgeschlossen ist unter Grenzwertbildung: Für jedes $k \in \mathbb{N}$ hat jede Kette $(A_i, B_i)_{i\in I}$ von Teilungen, die höchstens Ordnung k haben, ein Supremum, welches Ordnung höchstens k hat und von der Form $(\bigcup_{i\in I} A_i \cup X, \bigcap_{i\in I} B_i)$ ist. Es wird eine für diese Situation passende Definition von k-Pseudoblumen gegeben, in der jede k-Pseudoblume schon eindeutig anemonenartig oder gänseblümchenartig ist. Auf der Menge dieser k-Pseudoblumen gibt es eine Ordnung, in der $\Phi \leq \Psi$ gilt falls jede Teilung, die von Φ dargestellt wird, auch von Ψ dargestellt wird. Die meisten k-Pseudoblumen lassen sich zu einer $\leq$-maximalen k-Pseudoblume Ψ erweitern (Satz 5.44). Falls $\mathcal{P}$, für eine natürliche Zahl k, eine Menge von k-Profilen ist sodass alle k-Profile aus $\mathcal{P}$ dieselbe Einschränkung auf ein $k-1$-Profil haben und abgeschlossen unter Grenzwertbildung sind, dann sind die meisten $\leq$-maximalen k-Pseudoblumen auch maximal in der Präordnung "mehr Profile aus $\mathcal{P}$ unterscheiden" (Lemma 5.27).

In Kapitel 6 werden Teilungssyteme betrachtet, die in einem Teilungssytem von Bipartitionen enthalten sind und eine Zusammenhangsfunktion haben, die abgeschlossen ist unter Grenzwertbildung von Teilungen derselben Ordnung. Außerdem wird wieder eine Menge $\mathcal{P}$ von k-Profilen festgehalten. Unter diesen Umständen wird eine k-Pseudoblume als eine Partition der Grundmenge mit einer zyklischen Ordnung definiert, sodass jede Vereinigung eines Intervalls von Partitionsklassen höchstens Zusammenhang $k-1$ hat. Außerdem wird eine Ordnung $\leq$ auf den k-Pseudoblumen definiert, für die $\Phi \leq \Psi$ gilt falls die Partition von Ψ eine Verfeinerung der Partition von Φ ist und die Inklusionsabbildung zwischen den Partitionen ein Homomorphismus zyklischer Ordnungen ist. Dann kann jede k-Pseudoblume zu einer $\leq$-maximalen k-Pseudoblume erweitert werden (Korollar 6.15). Außerdem sind die meisten $\leq$-maximalen k-Pseudoblumen, in denen zwei Elemente von $\mathcal{P}$ nicht unterschieden werden, Erweiterungen einer unendlichen k-Anemone (Lemmas 6.24 und 6.28). Daher sind fast alle $\leq$-maximalen k-Pseudoblumen, die keine Erweiterung einer k-Anemone sind, auch maximal in der Präordnung $\preccurlyeq$ in der $\Phi \preccurlyeq \Psi$ wenn all Elemente von $\mathcal{P}$, die von Φ unterschieden werden, auch von Ψ unterschieden werden (Lemma 6.29). Für jede k-Anemone Φ mit mindestens k Partitionsklassen und jede k-Pseudoblume Ψ mit $\Phi \leq \Psi$ gilt, dass der Zusammenhang jeder Vereinigung von Partitionsklassen von Ψ höchstens $k-1$ ist (Korollar 6.12). k-Pseudoblumen mit dieser Eigenschaft werden starke k-Pseudoanemonen genannt. Außerdem hat die Menge aller Partitionen, die durch Wahl einer geeigneten zyklischen Ordnung zu einer starken k-Pseudoanemone Ψ mit $\Phi \leq \Psi$ werden, ein feinstes Element (Satz 6.22). Die meisten $\leq$-maximalen starken k-Pseudoanemonen sind auch $\preccurlyeq_A$-maximal in der Präordnung für starke k-Pseudoanemonen, in der $\Phi \preccurlyeq \Psi$ falls alle Elemente von $\mathcal{P}$, die von einer Vereinigung von Partitionsklassen von Φ unterschieden werden, auch von einer Vereinigung von Partitionsklassen von Ψ unterschieden werden (Lemma 6.34). Für den Spezialfall, dass die Zusammenhangsfunktion von einem Matroiden stammt, wird in Chapter 8 gezeigt, dass es keine unendlichen k-Gänseblümchen gibt (Lemma 8.5).

Schließlich wird in Kapitel 7 ein Algorithmus beschrieben, der für wenige Teilungen, die aus einem Teilungssytem von Bipartitionen gegeben sind, einen möglichst großen Anteil der in Satz 4.20 beschriebenen Baumstruktur bestimmt.

C. Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe. Darüber hinaus versichere ich, dass diese Dissertation nicht in einem früheren Promotionsverfahren eingereicht wurde.

Ort, Datum, Unterschrift

D. Abgrenzung gemeinsam erarbeiteten Inhalts

Anteile des zweiten Teils dieser Dissertation sind in Zusammenarbeit mit Hendrik Heine entstanden. Insbesondere entspricht Kapitel 5 in weiten Teilen einem bislang noch fertiggestellten gemeinsamen Artikel. Dabei ist die derzeitige Version von Abschnitt 5.1 überwiegend von mir geschrieben worden, wobei wir natürlich die schließlich verwendete Definition von k-Pseudoblumen gemeinsam erarbeitet haben. Das nächste Kapitel wurde überwiegend von Hendrik erstellt, wobei ich Lemma 5.16 bis Lemma 5.22 geschrieben habe. In Abschnitt 5.3 hatte Hendrik die Idee, inverse limits zu benutzen, und ich habe die Details ausgearbeitet. Die Erläuterungen vor dem ersten Abschnitt lassen sich keinem von uns zuordnen und sind zu gleichen Teilen von uns beiden erstellt worden.

Ansonsten stammt die Idee in Abschnitt 4.3, die Äquivalenzklassen (zwei Teilungen sind äquivalent, wenn sie in denselben Profilen enthalten sind) mit einer Ordnung auszustatten und so in ein Teilungssystem zu verwandeln, aus einer gemeinsamen ebenfalls nicht fertig gestellten Arbeit mit Hendrik. Der Teil der nicht fertig gestellten Arbeit, aus dem diese Idee stammt, wurde von Hendrik geschrieben. Die hier vorliegende Umformulierung mit der Funktion ϕ habe ich geschrieben. Die Erkenntnis, dass das Einschränken der Profile auf abgeschlossene Profile dazu führt, dass in der neu definierten Ordnung auf den Äquivalenzklassen Suprema von Ketten gebildet werden können, stammt aus derselben Arbeit und von Hendrik.

E. Liste der aus dieser Arbeit hervorgegangenen Artikel

Es gibt keine fertig gestellten Artikel, die aus dieser Arbeit hervorgegangen sind. Von den Artikeln, die in Vorbereitung sind, sind die folgenden beiden am weitesten fortgeschritten:

- *Flowers in graph-like spaces*, Ann-Kathrin Elm und Hendrik Heine, in Vorbereitung, entspricht in etwa Kapitel 5
- *Nearly finitary graph-like matroids and Psi-matroids*, Ann-Kathrin Elm, in Vorbereitung, entspricht in etwa dem ersten Teil

F. Danksagung

Vielen Dank an die vielen Menschen, die mich unterstützt haben. Die mir mit Rat und Tat zur Seite gestanden haben und dafür gesorgt haben, dass die Jahre meiner Promotion eine gute Zeit waren, auch wenn das letzte Jahr etwas schwierig wurde. Vielen Dank auch für die gute Arbeitsatmosphäre und das gemeinsame Rätsel lösen. Und zwar ganz besonders an meinen Betreuer Nathan Bowler, aber auch an die gesamte Arbeitsgruppe und alle anderen, mit denen ich an der Uni und im Rahmen meines Mathematikstudiums zu tun hatte. Auch vielen Dank an meine Familie, die mir das Studium und die Promotion ermöglicht haben.