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Dissertation

The double obstacle problem for functionals with linear growth

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Contents

1 Introduction

This thesis is concerned with the study of double obstacle problems for functionals with linear growth mainly focusing on the area functional. The underlying problem is the minimization of functionals over some domain $\Omega \subset \mathbb{R}^n$, i.e. can be written in general like

$$\min \int_\Omega f(\mathrm{D}u(x))\,\mathrm{d}x$$

for some integrand $f : \mathbb{R}^n \to \mathbb{R}$. Often the integrand is assumed to be convex. Those functionals, or more precisely the integrands, can be classified by their growths for large arguments. For example, they can have linear growth (1-growth), like the total variation $\mathrm{TV}[\,.\,]$ or the area functional $\mathrm{A}[\,.\,]$, quadratic (2-growth), like Dirichlet integral the $\mathrm{Dir}[\,.\,]$ or more generally of p-growth, like the p-energy for $1 \leq p < +\infty$. The total variation and the area functional are defined by

$$\mathrm{TV}[u] = \int_\Omega |\mathrm{D}u(x)|\,\mathrm{d}x \text{ and } \mathrm{A}[u] = \int_\Omega \sqrt{1 + |\mathrm{D}u(x)|^2}\,\mathrm{d}x$$

for suitable function u and it is easy to see, that the integrands $|\,.\,|$ and $\sqrt{1+|\,.\,|^2}$ behave like linear functions for large arguments. In contrast, the Dirichlet integral has an integrand which growths quadratic and the functional is given by

$$\mathrm{Dir}[u] = \int_\Omega |\mathrm{D}u(x)|^2\,\mathrm{d}x.$$

Similarly, the p-energy is given by

$$\mathcal{E}_p[u] = \int_\Omega |\mathrm{D}u|^p\,\mathrm{d}x$$

and thus coincides with the total variation for $p = 1$ and the Dirichlet integral for $p = 2$.
In addition to the minimization of those functionals, it is common to include further requirements, like the Dirichlet boundary values

$$u = u_0 \text{ on } \partial\Omega$$

for given boundary values u_0, or to constrain the admissible functions by obstacles or even both. There exists different versions of obstacle constraints, like the single lower obstacle constraint $u \geq \psi_1$, the single upper obstacle constraint $u \leq \psi_2$ or the combination of both, the double obstacle constraint, with

$$\psi_1 \leq u \leq \psi_2.$$

Here, ψ_1, ψ_2 are functions defined on $\mathbb{R}^n$ with values in $\mathbb{R} \cup \{\pm\infty\}$ and for the double obstacle problem those the lower obstacle is assumed to be less or equal to the upper obstacle, i.e. $\psi_1 \leq \psi_2$. The double obstacle problem contains the single ones by setting, for example, the upper obstacle $\psi_2 = +\infty$ and thus obtain the single lower obstacle case. There are many different obstacles which can be considered. To mention a few for the single lower obstacles, one can set the lower obstacle ψ_1 to be a continuous functions or a characteristic functions $\mathbb{1}_A$ of a sets $A \subset \mathbb{R}^n$, which are equal 1 on A and 0 else, but one could also implement constraints that are active only on a given set

$A \subset \mathbb{R}^n$ by considering the obstacle function with given real-valued function on a set A and $-\infty$ else. Further, it is possible to consider thin obstacles, like a function having the value 0 on an $(n-1)$-dimensional submanifold of Ω like the boundary of a (smooth) domain inside or intersecting Ω and $-\infty$, the characteristic functions of $(n-1)$-dimensional submanifolds and many more and more general ones. As it turns out, the consideration of functionals with linear growth under the thin single or double obstacles with or without Dirichlet boundary condition is challenging, since the usual theory regarding the typically considered function spaces and the thinness of the obstacles, which can be considered for those, reach a borderline case. This fact will be explained in little more detailed after more light is shed on the mathematical background. For functionals with p-growth, especially the Dirichlet integral and other functionals with quadratic growth, a broader literature than for the linear case exist, parts of which will be presented or referred to in Section 1.5. First results related to the study of functionals of linear growth with thin obstacles was done for the minimization of the area by a related approach to the here presented, namely by the parametric approach in contrast to the here presented non-parametric approach. This and the many other useful results connecting both theories are the reason, why this thesis is mainly concerned with the study of the area functional. As a side note, one readily speaks about the non-parametric Plateau problem and the non-parametric Plateau problem with obstacles if those are included instead of the minimization of the area functional with or without obstacles, respectively. Before those concepts are introduced mathematically, an example is presented to illustrate the obstacle problem and make the thin obstacle problem more tangible and depict difficulties which indeed arises mathematically. This example originates from the description of minimal surfaces by soap films or bubbles. Those minimal surfaces are, as it will be described in the following section, closely related to minimizers of the area functional $\mathrm{A}[\,.\,]$ and the mentioned parametric approach for minimal surfaces. This kind of analogy is often used, see for example [39, Introduction] or [52, Introduction], and can be described as follows:

Minimizing the area functional for some given boundary value means to search for a surface with least area under those fulfilling the boundary condition. If the boundary values are represented by a closed wire curve, a spanned soap film will assume a shape with minimal area spanned by the given wire, thus forming a minimal surface.

To depict the obstacle constraint, a simple configuration may be assumed: The boundary is given by a circular wire and the minimal surface will obviously be the enclosed circular disc. If an object, e.g. a book or a ball, is placed from below part-way through the circle, the soap film will dislocate by bending upwards and the object will represent an obstacle from below. Similarly, if the object is placed from above, an obstacle from above is given.

For the thin obstacle, a very thin sheet can be taken as an obstacle, which is placed perpendicular to the plane containing the wire. If the sheet is thin and sharp enough, a phenomenon can occur which also appears in the mathematical treatment of the given problem:

The soap film can slide downwards on the sides of the sheet without tearing and forms a soap film with a slit where the sheet is. The difficulty here is that the surface does not fulfill the obstacle constraint anymore, since the soap film does not lie above the sheet and additionally, because the slit is thin, the surface area is not changed in comparison to the starting configuration and thus the violation of the obstacle condition is not penalized.

The next question, which may arise, is what happens if an even lower dimensional object is placed

as an obstacle like a pencil lead or a needle. Unfortunately, there is very little theory available and there is no hope to provide results for this case with the approaches contained in this thesis.
The first ones to tackle double obstacle problem with thin obstacles in context of minimal surfaces were De Giorgi, Colombini and Piccinini in [13] using the parametric approach, which will be explained in the following paragraphs. Therefore, the next section introduces the parametric and non-parametric approach and gives a brief historical overview. Afterwards, the double obstacle problem is stated in the parametric and non-parametric setting with emphasis on the difficulties arising from enforcing thin obstacles and the first and second major result of this thesis is presented. Then, a different and again connected approach to obstacle problems via variational inequalities is summarized and the third major result of this thesis is stated. Next, a small overview over the existing theory and literature is provided before the introduction is closed by the structure of this thesis.

1.1 Parametric and non-parametric minimal surfaces

The study of minimal surfaces is a very interesting topic on its own. The problem was stated in various ways and in many different branches in mathematics like Analysis, Geometry, Partial Differential Equations, Geometric Measure Theory and Calculus of Variations. The general task is to minimize the (generalized) area of an n-dimensional surface under given constraints like a given $(n-1)$-dimensional boundary and in different ambient spaces like in the space $\mathbb{R}^{n+N}$, where $n, N \in \mathbb{N}$. Here, N is called the co-dimension of the underlying problem and $n \geq 2$ is often assumed.
In the research on minimal surfaces of arbitrary dimension n and co-dimension $N = 1$, the two main approaches besides the classical parametric theory, which is restricted to $n = 2$ and was solved independently by Douglas in [22] and Radó in [55, 56] around 1930 using parametrized disc-type surfaces, and the semi-classical theory, which uses only Lipschitz functions and is basically restricted to domains satisfying the boundary slope condition, are the parametric and non-parametric approach.
The parametric approach is due to articles by De Giorgi and by Caccioppoli in the early 1960s, who studied minimal surfaces as boundaries of $(n+1)$-dimensional sets. They introduced the perimeter $\mathrm{P}(E)$ of a set E as a generalized circumference of a set where only points with a generalized normal vector are considered. While De Giorgi introduced the concept, Caccioppoli gave a simpler equivalent definition which is used nowadays. For sets E with C^1 boundaries, the perimeter gives the same value as the $(n-1)$-dimensional Hausdorff measure of the boundary of the set:

$$\mathrm{P}(E) = \mathcal{H}^{n-1}(\partial E).$$

The definition of the perimeter can be found in Definition 2.20. Sets with finite perimeter are called Caccioppoli sets. One way to state the minimization problem is to only allow variations E of a given set Caccioppoli set E_0 in a given open set $\Omega \subset \mathbb{R}^{n+1}$ and define the boundary values by requiring $E = E_0$ outside of Ω:

$$\inf\{\mathrm{P}(E) \,:\, E \text{ has finite perimeter and } E \setminus \Omega = E_0 \setminus \Omega\}$$

A more localized version requires only that E_0 has finite perimeter in an open set A which contains Ω. Then it is possible to minimize the perimeter $\mathrm{P}(E, A)$ of E in A:

$$\inf\{\mathrm{P}(E, A) \,:\, E \text{ has finite perimeter in } A \text{ and } E \setminus \Omega = E_0 \setminus \Omega\}.$$

Using this version, it is possible to treat sets E_0 with unbounded perimeter.
Those results extend the classical parametric Plateau problem to higher dimensions than $n = 2$. This is also the reason, why it is still called parametric approach besides not really involving parametric surfaces like in the classic formulation.
The parametric approach leads to a well-established regularity theory, showing that minimal surfaces have no singularities in dimensions 1 to 7 and for higher dimensions can only have singularities up to Hausdorff dimension $n - 8$.
Returning to the initial example, the spanned soap film is considered a part of some object like a soap bubble and the wire defines the boundary outside of which the the bubble may not be varied anymore.
In contrast, the non-parametric approach deals with surfaces which are graphs of scalar functions defined on a given set $\Omega \subset \mathbb{R}^n$. The name originates from the fact that the parameterization cannot be chosen freely, since one is restricted only to graphs. The area functional for a function $u \in \mathrm{C}^1(\Omega, \mathbb{R})$ is defined by

$$\mathrm{A}[u] = \int_\Omega \sqrt{1 + |\nabla u|^2}\,\mathrm{d}x.$$

This definition of area can be generalized for Lipschitz and even Sobolev functions. A further generalization is the extension to BV functions. Those are functions u in $\mathrm{L}^1(\Omega)$ whose distributional derivative can be represented by a Radon measure $\mathrm{D}u$. In that sense, the area can be written with the help of the total variation of a vector-valued measure as $\mathrm{A}[u] = |(\mathcal{L}^n, \mathrm{D}u)|(\Omega)$.
In the first given example, the soap film is modeled by the graph of the function u.
Comparing both approaches, M. Miranda proved in [50], see also [39], that both lead to the same minimal surface in the following sense:

$$u \text{ is a minimizer of the area functional}$$
$$\Leftrightarrow$$
$$U = \{(x, t) \in \Omega \times \mathbb{R} \,:\, t < u(x)\} \text{ is of least perimeter in } \Omega \times \mathbb{R}.$$

Even a local statement of this type remains true as well as one for unbounded graphs. By connecting these theories, the regularity results for the parametric case are extended the non-parametric setting. In reverse, it is possible to use well known comparison principles established in the non-parametric setting to prove uniqueness of minimal surfaces for given boundary values.

1.2 Parametric obstacle problems and the De Giorgi measure

In the parametric setting, unlike in the introductory example, obstacles can be seen as *inner* obstacle, i.e. the objects are contained in the soap bubble and the final configuration has to contain the object, or *outer* obstacles, i.e. objects outside the bubble which have to stay outside of the configuration.

Thus the double obstacle may be stated like in [20]: For a given open set Ω and given obstacle sets $O_1, O_2 \subset \mathbb{R}^n$ with $O_1 \cap O_2 = \emptyset$, the goal is to investigate the following infimum

$$\inf\{\mathrm{P}(E,\Omega) \,:\, E \text{ has finite perimeter in } \Omega,\, O_1 \subset E \text{ and } E \cap O_2 = \emptyset\}.$$

Here, two major difficulties occur which have to be solved for the non-parametric problem in a similar way:

1. An additional term or a way to penalize the violation of the obstacles is needed which provides a lower semicontinuous functional and does not change the infimum.
2. To realize the first point a precise notion of the sets is needed, since thin parts should be taken into account but are ignored by the perimeter. For example, a line segment L in $\mathbb{R}^2$ has perimeter $\mathrm{P}(L, \mathbb{R}^2) = 0$.

The process described in the first point is called relaxation. The general idea is to replace a given functional, in this case the perimeter on those restricted sets, with a lower semicontinuous one which agrees on the domain of definition with the original one and has the same infimum, which is enforced by requiring a so-called recovery sequence. For more details, see Definition 2.48. The second point was solved by De Giorgi using the measure theoretic closure which gives a notion of the set up to $\mathcal{H}^{n-1}$ null sets.
In [20], the relaxed problem was proven to be

$$\inf\left\{\mathrm{P}(E,\Omega) + \varsigma((O_1 \setminus E) \cap \Omega) + \varsigma(O_2 \cap E \cap \Omega) \,:\, E \text{ has finite perimeter in } \Omega\right\} \tag{1.1}$$

with suitable assumptions. Here, ς denotes the De Giorgi measure, which in a broad sense measures the circumference of lower dimensional sets. As before, the goal is to minimize the perimeter, but violations of the obstacles are penalized by the two additional terms, each for the given obstacle separately. While the Hausdorff measure might seem sufficient for the measurement of the subsets violating the obstacles, since for sets contained in a countable union of C^1-hypersurfaces the measure coincides with two times the Hausdorff measure, Hutchinson proved in [41] that $\varsigma \neq 2\mathcal{H}^{n-1}$. The construction by Hutchinson uses the main difference between the two geometrical measures $\frac{1}{2}\varsigma$ and $\mathcal{H}^{n-1}$, namely that the first is measuring the half of the circumference of a set while the later is defined as the infimum over the sum of the diameter to the power $n-1$ of coverings by suitable sets, like balls, of the set and thus is sort of measuring the diameter for $n = 2$. The construction and a sketch of the related proofs will be presented in Section 3.1, since it justifies the interest in the De Giorgi measure.
Thus, the functional in (1.1), with the De Giorgi measure replaced by two times the Hausdorff measure, may be used as the relaxation with a priori knowledge of the obstacle like in the introductory example but not in general. For more details regarding the counterexample and the usage of the De Giorgi measure, see Section 3.

1.3 Non-parametric obstacle problems

May a domain $\Omega \subset \mathbb{R}^n$, a suitable function space X, boundary values $u_0 : \partial\Omega \to \mathbb{R}$ and two obstacle functions $\psi_1 \leq \psi_2$ be given. For a convex functional with linear growth, i.e. functionals where the integrand $f : \mathbb{R}^n \to [0, +\infty)$ is convex and fulfills the estimates

$$a|z| \leq f(z) \leq b(1 + |z|) \quad \forall z \in \mathbb{R}^n$$

for some positive constants a, b, the minimization problem can be stated as follows:

$$\inf \int_\Omega f(\mathrm{D}u)\,\mathrm{d}x \text{ in } X_{u_0,\psi_1,\psi_2}$$

where the admissible class X_{u_0,ψ_1,ψ_2} is defined by

$$X_{u_0,\psi_1,\psi_2} := \{u \in X \,:\, u = u_0 \text{ on } \partial\Omega \text{ and } \psi_1 \leq u \leq \psi_2\}.$$

Since the classical space C^1 for which such functionals are well-defined has bad analytic properties, the considered function space X is usually the Sobolev space $\mathrm{W}^{1,1}(\Omega)$ or the space of functions of bounded variations $\mathrm{BV}(\Omega)$. The Sobolev space $\mathrm{W}^{1,1}$ consists of L^1-functions with a weak/distributional derivative that can be represented by an L^1 function while for the space BV the derivative is a finite Radon measure. While the functional is well-defined on $\mathrm{W}^{1,1}$ without any further input, since the derivative is given by a function, it needs to be extended for BV functions to functionals on measures. The reason to consider this larger function space is that $\mathrm{W}^{1,1}$ is not closed under weak* convergence and its closure is the space of functions of bounded variation. Concerning those spaces a proper meaning has to be given to the boundary and obstacle constraint, since every function in $\mathrm{W}^{1,1}$ and BV has many different representatives. While the first task can be interpreted using trace theorems for the spaces, respectively, the obstacle condition yields different results assuming the condition to hold almost everywhere with respect to the Lebesgue measure $\mathcal{L}^n$ or almost everywhere with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$.
The $\mathcal{L}^n$-a.e. constraint does not add to the difficulty of proving the existence of solutions, since, for example, $\mathrm{BV}(\Omega) \cap \{\psi_1 \leq u \leq \psi_2 \ \mathcal{L}^n\text{-a.e.}\}$ is closed under weak* convergence in BV for a suitable domain Ω and suitable obstacles $\psi_1 \leq \psi_2$. The direct method then yields the existence of a minimizer under the usual convexity assumption for the integrand and coercivity assumption for the functional.
Meanwhile, the $\mathcal{H}^{n-1}$-a.e. constraint makes the problem harder even for the single obstacle case: With the introductory example in mind, the area functional for given boundary values $|(\mathcal{L}^n, \mathrm{D}u)|(\overline{\Omega})$ on $\mathrm{BV}(\Omega)$ is to be minimized on the open ball of radius 2 around the origin, i.e. on $\Omega = \mathrm{B}^2_2(0) \subset \mathbb{R}^2$. Furthermore, let the boundary values be $u_0 = 0$ on the sphere $\mathrm{S}^1_2(0)$ and the obstacle be $\psi = 1$ on the line $L := \{0\} \times [-1, 1]$. To define ψ on the whole of $\mathbb{R}^2$ set $\psi = -\infty$ on $\mathbb{R}^n \setminus L$. Since $\mathrm{W}^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ it is possible to consider the following sequence:

$$u_k(x) = \max\{1 - k \operatorname{dist}(x, L), 0\} \in \mathrm{W}^{1,1}(\Omega).$$

While for every k the sequence fulfills the constraint $u_k \geq \psi$ everywhere even in the classical sense, since the u_k are continuous, the weak* limit is $u_k \overset{*}{\rightharpoonup} u$ with $u \equiv 0$ on $\mathrm{B}_2(0)$, which obviously violates the obstacle constraint on L. In addition, $\mathcal{A}[0] < \mathcal{A}[w]$ on the set $\{w \in \mathrm{BV}(\Omega) \,:\, w = 0 \text{ on } \mathrm{S}^1_2(0) \text{ and } w \geq 1 \text{ enforced } \mathcal{H}^{n-1}\text{-a.e. on } L\}$. Even if another representative of u is chosen like $u = 0$ on $\Omega \setminus L$ and $u = 1$ on L, it does not change the value of $\mathcal{A}[u]$ and thus does not take the obstacle into account. The main problem here and as for the parametric problem is that the set of admissible functions, which in general is convex but not a subspace anymore, is not closed and thus the infimum over the functional values of the admissible set differs from the possible limits. In addition, the functions in the considered function space are only defined $\mathcal{L}^n$-a.e., but the function values has to be made sense of $\mathcal{H}^{n-1}$-a.e. Those problems leads to the same difficulties as in the parametric settings:

1. A precise representative of the involved functions is needed.
2. A penalizing term has to be added to account for violations of the obstacle constraints.

In [13], Carriero, Dal Maso, Leaci and Pascali considered the single obstacle problem for functionals with linear growth and proved that the relaxation $\overline{\mathrm{F}}^{\psi}_{u_0}$ in L^1 to the functional

$$\mathrm{F}^{\psi}_{u_0}[u] = \int\limits_{\Omega} f(\mathrm{D}u)\,\mathrm{d}x \text{ on } \mathrm{W}^{1,1}_{u_0,\psi}(\Omega)$$

extended by the value $+\infty$ to $\mathrm{L}^1(\Omega)\setminus \mathrm{W}^{1,1}_{u_0,\psi}(\Omega)$, with f like above together with some additional assumptions, Ω Lipschitz domain, $u_0 \in \mathrm{L}^1(\partial\Omega)$ and

$$\mathrm{W}^{1,1}_{u_0,\psi}(\Omega) := \{u \in \mathrm{W}^{1,1}(\Omega) \,:\, u = u_0 \text{ on } \partial\Omega \text{ and } u^* \geq \psi\, \mathcal{H}^{n-1}\text{-a.e. on } \Omega\}$$

is given by

$$\overline{\mathrm{F}}^{\psi}_{u_0}[u] = \int\limits_{\overline{\Omega}} f(\mathrm{D}u) + \int\limits_{\Omega} (\psi - u^+)_+ \,\mathrm{d}\varsigma_{f^\infty} \text{ on } \mathrm{BV}(\Omega).$$

In this formulation, ς_{f^∞} is the generalized De Giorgi measure, which is influenced by the integrand and is connected to the De Giorgi measure ς in the parametric problem and sometimes even coincides with it, for example, for the area functional and the total variation. The $\int_{\overline{\Omega}} f(\mathrm{D}u)$ is an extension of the function $\int_{\Omega} f(\mathrm{D}u)$ to $\mathrm{BV}(\Omega)$ where the boundary values and a possible difference to the prescribed boundary values is accounted for and thus is actually dependent of u_0. The superscripts '$*$' and '$+$' indicate the Lebesgue and the upper representative of the function, respectively, and the subscript '$+$' stands for the maximum function of the term and 0. A precise description is given in the first part of the Preliminaries 2.
Using that as a starting point, it seems reasonable to assume that the relaxation in L^1 of

$$\mathrm{F}^{\psi_1,\psi_2}_{u_0}[u] = \int\limits_{\Omega} f(\mathrm{D}u)\,\mathrm{d}x \text{ on } \mathrm{W}^{1,1}_{u_0,\psi_1,\psi_2}(\Omega)$$

with

$$\mathrm{W}^{1,1}_{u_0,\psi_1,\psi_2}(\Omega) := \{u \in \mathrm{W}^{1,1}(\Omega) \,:\, u = u_0 \text{ on } \partial\Omega \text{ and } \psi_1 \leq u \leq \psi_2 \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } \Omega\}$$

and the same extension to L^1 is given similarly by the following functional:

$$\overline{\mathrm{F}}^{\psi_1,\psi_2}_{u_0}[u] = \int\limits_{\overline{\Omega}} f(\mathrm{D}u) + \int\limits_{\Omega} (\psi_1 - u^+)_+ \,\mathrm{d}\varsigma_{f^\infty} + \int\limits_{\Omega} (u^- - \psi_2)_+ \,\mathrm{d}\varsigma_{\tilde{f}^\infty} \text{ on } \mathrm{BV}(\Omega)$$

with $u^- = -(-u)^+$ and a similar notation as above. The only significant change is a modification of the generalized De Giorgi measure to adjust for upper obstacles, which coincides with ς_{f^∞}, for example, for symmetric integrands f and thus is also equal to ς in the case of the area functional or total variation. Indeed, this is first major result proven in Section 4:

Result 1. *Let an open bounded set Ω with Lipschitz boundary, a convex function $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that $a, b > 0$ exists with $a|z| \leq f(z) \leq b(1 + |z|)$ and boundary values $u_0 \in \mathrm{L}^1(\partial\Omega)$ be given. If there exists at least a function v in $\mathrm{W}^{1,1}_{u_0,\psi_1,\psi_2}(\Omega)$ such that $\mathrm{F}^{\psi_1,\psi_2}_{u_0}[v] < +\infty$, then the relaxation is given by*

$$\overline{\mathrm{F}}^{\psi_1,\psi_2}_{u_0}[u] = \int_{\overline{\Omega}} f(\mathrm{D}u) + \int_{\Omega} (\psi_1 - u^+)_+ \, \mathrm{d}\varsigma_{f^\infty} + \int_{\Omega} (u^- - \psi_2)_+ \, \mathrm{d}\varsigma_{\tilde{f}^\infty} \ \textit{on } \mathrm{BV}(\Omega).$$

The additional requirement of the existence of the function v corresponds to the assumption, that the original problem has at least one finite competitor, i.e. $\mathrm{F}^{\psi_1,\psi_2}_{u_0}[u] < \infty$ at least for one function $u \in \mathrm{W}^{1,1}(\Omega)$.
Additionally, results without boundary condition and with the original functional having the domain of definition being the corresponding subspace of BV instead of $\mathrm{W}^{1,1}$ are presented.
Relying on the first major result, a similar connection as between the parametric and non-parametric minimal surfaces is proven for the double thin obstacle problem in Section 5:

Result 2. *The minimizers of the relaxed parametric and relaxed non-parametric double obstacle problem for minimal surfaces coincide in the graph-type setting, i.e. if the lower parametric obstacle O_1 and the upper parametric obstacle are given as the subgraph of the lower non-parametric obstacle function ψ_1 and the supergraph of the non-parametric obstacle functions ψ_2, respectively:*

$$\begin{gathered} u \textit{ is (locally) a minimizer of } \overline{\mathrm{A}}^{\psi_1,\psi_2}_{u_0} \\ \Leftrightarrow \\ U = \{(x,t) \in \Omega \times \mathbb{R} : t < u(x)\} \textit{ (locally) minimizes} \\ \mathrm{P}(E, \Omega \times \mathbb{R}) + \varsigma((O_1 \setminus E) \cap \Omega \times \mathbb{R}) + \varsigma(O_2 \cap E \cap \Omega \times \mathbb{R}). \end{gathered}$$

To close the paragraph, the main difference of obstacle problems for functionals with linear growth and functionals with p-growth for $p > 1$ is enlightened. For functionals with p-growth, the function space of choice is $\mathrm{W}^{1,p}$, i.e. functions in L^p with the weak derivative representable by an L^p function or a suitable subspace. To work with functions in $\mathrm{W}^{1,p}$ with $p > 1$ has some benefits such as the closedness under weak(*) convergence and the availability of representatives up to quasi every point, i.e. up to null sets of the p-capacity cap_p. This notion is finer than the $\mathcal{H}^{n-1}$-measure and thus allows to study problems with even thinner obstacles enforced, see for example [49]. In comparison, in the case $p = 1$ the cap_1 null sets agree with those having zero Hausdorff measure and thus the representatives available are defined on sets comparable to the sets where the obstacle constraints are enforced.

1.4 Variational inequalities with obstacles and free boundary problems

Another way to treat obstacle problems is to use variational inequalities. While a connection between variational problems and partial differential equations is made by the Euler-Lagrange-Equation, the theory changes as soon as there are obstacles involved. In a simple setting with a smooth obstacle and the restriction to $\mathrm{W}^{1,1}$-functions, i.e. without considering functions of bounded variations, a connection of the theories in calculus of variations, variational inequalities and partial differential equations is simple to deduce:

Assuming $u \in \mathrm{W}^{1,1}(\Omega)$ is the minimizer of $\mathrm{A}[\,.\,]$ in $\mathrm{W}^{1,1}_{u_0,\psi}(\Omega)$ for some given $\psi \in \mathrm{W}^{1,1}(\Omega)$ and a Lipschitz domain Ω to given boundary values $u_0 \in \mathrm{L}^1(\partial\Omega)$. To calculate the Euler-Lagrange-Equation one usually would compute the directional derivative at u in every direction which does not change the boundary values, i.e. $\mathrm{W}^{1,1}_0(\Omega)$, which consists of Sobolev functions having zero boundary values in some sense, or a dense subset of the function space like $\mathrm{C}^\infty_0(\Omega)$. Since u is assumed to be the minimizer, the derivative of the functional in each direction must be 0. By the obstacle condition there is a restriction for the admissible functions and thus not every direction is possible. For example, on the contact set $\{u = \psi\}$ the direction must equal 0 or the obstacle constraint is violated. It seems reasonable to allow positive variations on the contact set, which means to restrict the direction to one-sided ones. Allowing that, the derivative at the minimizer u does not need to be 0 anymore but instead greater or equal to 0, meaning that in every direction the value of the area functional increases:

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\mathrm{A}[u+t\varphi]\bigg|_{t=0} \geq 0 \qquad \forall \varphi \text{ admissible.} \tag{1.2}$$

The superscript '+' indicates that only variations in the positive direction of φ are permitted. To tighten the notation, the direction φ can be rewritten in the following way:

A function φ such that the function $v = u + t_0\varphi$ is a valid competitor which fulfills the obstacle constraint $v \geq \psi$ at least for a small $t_0 > 0$ will also fulfill those constraints for all $0 \leq t < t_0$. Rewriting leads to $\tilde{\varphi} = t_0\varphi = v - u$. Since this is possible for every direction and one gets a legal direction $\tilde{\varphi}$ for every function $v \geq \psi$ one can assume, without loss of generality, $\varphi = v - u \in \mathrm{W}^{1,1}_0(\Omega)$ with $v \in \mathrm{W}^{1,1}_{u_0,\psi}(\Omega)$. Returning to (1.2) leads to

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}t^+}\mathrm{A}[u+t\varphi]\bigg|_{t=0} = \int_\Omega \frac{\mathrm{D}u \cdot \mathrm{D}(v-u)}{\sqrt{1+|\mathrm{D}u|^2}}\,\mathrm{d}x \tag{1.3}$$

for all φ constructed in the above way and we thus obtain the variational inequality (1.3). In addition, if u is in $\mathrm{C}^2(\Omega)$, one can integrate by parts and, since each v can be written as the sum of the obstacle ψ and a non-negative function w, obtains

$$\begin{aligned}
0 &\leq \int_\Omega -\mathrm{div}\left(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\right)(v-u)\,\mathrm{d}x \\
&= \int_{\Omega\cap\{u=\psi\}} -\mathrm{div}\left(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\right) w\,\mathrm{d}x + \int_{\Omega\cap\{u>\psi\}} -\mathrm{div}\left(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\right)\varphi\,\mathrm{d}x
\end{aligned}$$

for all admissible φ and thus by construction w. Applying the fundamental theorem of the calculus of variations, with $w \geq 0$ in mind, yields the following obstacle problem/free boundary problem formulated in terms of partial differential equations

$$\begin{cases} \operatorname{div}\left(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\right) \leq 0 & \text{on } \{u = \psi\}, \\ \operatorname{div}\left(\frac{\mathrm{D}u}{\sqrt{1+|\mathrm{D}u|^2}}\right) = 0 & \text{on } \{u > \psi\}, \\ u \geq \psi & \text{on } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The interpretation of the partial differential equations is that, away from obstacles, the solution fulfills the minimal surface equation and, if the solution u is equal to the obstacle, with the comparison principle for minimal surfaces in mind, it must be a supersolution. Often the boundary $\partial\{u = \psi\}$ is called *free boundary* and the regularity of that is fascinating on its own.
Another interesting question is, under which assumptions on the minimizer and the integrand is the variational inequality sufficient to give a minimizer, i.e. when will u be a minimizer if it satisfies a variational inequality.
This connection and Euler-Lagrange-Equations become more challenging if the BV case is considered and, in addition, the thin double obstacle constraint is involved. With a glance at the paper by Ancelotti [4], it seems reasonable to prove a similar version for BV functions and is presented in section 6 as the third major result in this thesis:

Result 3. *Under usual assumptions on f, u_0, ψ_2 and ψ_2, the one-sided derivatives of $\overline{\mathrm{F}}^{\psi_1,\psi_2}_{(u_0)}$ exist, and thus an associated variational inequality can be computed. Further, a subset $\mathcal{D}_u$ of the admissible directions exists, such that the derivative for those directions $\varphi \in \mathcal{D}_u$ is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\overline{\mathrm{F}}^{\psi_1,\psi_2}_{(u_0)}[u+t\varphi]\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t^+}\mathcal{F}[u+t\varphi]\bigg|_{t=0} - \int\limits_{\{\psi - u^+ > 0\}} \varphi^+ \,\mathrm{d}\varsigma_{f^\infty} + \int\limits_{\{u^- - \psi_2 > 0\}} \varphi^- \,\mathrm{d}\varsigma_{\tilde{f}^\infty}.$$

Additionally, if a function u fulfills the variational inequality

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\overline{\mathrm{F}}^{\psi_1,\psi_2}_{(u_0)}[u+t\varphi]\bigg|_{t=0} \geq 0 \text{ for all } \varphi \in \mathcal{D}_u,$$

it is also a minimizer of $\overline{\mathrm{F}}^{\psi_1,\psi_2}_{(u_0)}$.

1.5 The De Giorgi measure and thin obstacles problems in the literature

The De Giorgi measure has different applications, the first being to consider lower dimensional obstacles and the second to give a sufficient condition for some Radon measure μ and function g such that the functional

$$\int\limits_\Omega f(x,u,\mathrm{D}u)\,\mathrm{d}x + \int\limits_\Omega g(x,u)\,\mathrm{d}\mu$$

is lower semicontinuous for suitable functions f. It was introduced in [20] to solve the *parametric* thin double obstacle problem. This result was partially presented by Piccinini at a summer school

with published lecture notes [54] and a similar version to the original was published by De Giorgi in [19]. Thereafter Colombini proved a new representation for the De Giorgi measure in [15]. Afterwards, de Acutis presented in [18] a first regularity result near the obstacle for the parametric minimal surface problem with a single thin obstacle being an $(n-1)$-dimensional manifold in $\mathbb{R}^n$. Next, Hutchinson revealed a counterexample to prove $\varsigma \neq 2\mathcal{H}^{n-1}$ in [41]. Later Carriero, Leaci and Pascali introduced in [11] and [12] a generalized version of the De Giorgi measure which can handle more general functionals than the area functional and in some sense is capable of dealing with integrands with p growth, $p \geq 1$, to suit and be able to treat the second application. Those results were again used by Pallara in [53]. In [13], Carriero, Dal Maso, Leaci and Pascali determined the relaxation for the non-parametric single obstacle problem, using the generalized versions of the De Giorgi measure with $p = 1$, and later considered limits of such relaxed functionals in [14]. The results by De Giorgi in [20] were extended to metric measures spaces by Kinnunen, Korte, Shanmugalingam and Tuominen in [43] and some regularity results for minimizers of the parametric Plateau problem with a single thin obstacle was established by Fernández-Real and Serra in [26] by approximating thin obstacles with wedge-like sets.
Next, a small overview of the literature dealing with the *non-parametric* minimal surface problem with a single thin obstacle and the corresponding variational inequality is given: Those were mainly studied by Giusti in [35, 36, 37, 38], Frehse in [31, 32] and Nitsche in [52]. A regularity result for $n = 2$ was proven by Kinderlehrer in [42]. Lately, Focardi and Spadaro proved an advanced regularity result at the thin obstacle in [29] based, amongst other sources, on [27, 28] and using a version of Almgren's frequency function. Other important contributions, including Frehse [32], were based on variational inequalities and many others dealing with thin obstacles problems for quadratic functionals, mainly the Dirichlet integral

$$\mathrm{Dir}[u] = \int_\Omega |\mathrm{D}u|^2 \,\mathrm{d}x.$$

A very rich and broad literature is available for studying this and similar quadratic functionals. Combined with a version of a thin obstacle, such problems are known as the *Signorini problem* with lots of contributions by various authors. Although some ideas and proofs can be used for the study of thin obstacles for functionals of linear growth, this problem is not this thesis' main concern and thus the presentation of some literature is skipped and the reader is referred to [29, Introduction] for a short and thus not complete overview on this topic.
At this stage, it should be mentioned, that almost all the presented sources for the non-parametric problem, besides those in [13], have in common that the thin obstacle is defined only on nice sets like (a countable union of) smooth manifolds or even segments of hyperplanes. Many restrict themselves to only consider the ball $\mathrm{B}_1^n(0)$ or half ball $\mathrm{B}_1^n(0) \cap \{x_n > 0\}$ as the domain and the thin obstacle to live on $\mathrm{B}_1^n(0) \cap \{x_n = 0\}$. Often the values of the obstacle function are assumed to be regular or even 0.
Another recent result is the dual formulation for the non-parametric minimal surface and total variation problem with a single obstacle due to Scheven and Schmidt in [62].
The study of thin obstacle problems appears in many branches of mathematics, in physics and other fields of application. Sometimes they are stated as free boundary problems or as a type of Signorini problem. Those keywords lead to a very rich literature with a varying degree of connection to the here presented topics.

1.6 Structure of the thesis

The goal of this thesis is to generalize the results achieved by Carriero, Dal Maso, Leaci and Pascali in [13], to prove the representation formula for the double obstacle problem and derive a comparison between the parametric and non-parametric solutions of the double obstacle problem for the area functional. Further, variational inequalities are studied for functionals with linear growth and involving thin obstacles. The thesis is divided into 5 further sections:

- In the first, some general notions for sets, basic function spaces and measures are fixed. Then the theory of BV functions, sets of finite perimeter and the extension of functionals of linear growth to functionals on measures are recalled as well as the idea of relaxation including some examples. For this kind of extension, a closer look on the recession function is provided. Main tools like Reshetnyak's theorem with some extended results are presented.
- Based on the results from De Giorgi, Colombini and Piccinini [20], Hutchinson [41], Piccinini [54], and Carriero, Dal Maso, Leaci and Pascali [13], a short overview on the De Giorgi measure, the mentioned counterexample by Hutchinson and the generalization of the De Giorgi measure, which is using an anisotropic perimeter in the construction, is provided. Some basic proofs for estimates and inequalities are presented to show the basic properties of those measures and to point out one of the differences between the proof in [13] and the generalized result for the single obstacle problem proven at the end of this section.
- The third section is devoted to the study of the non-parametric double obstacle problem. First, a convergence almost everywhere result is shown for the gradients of an area-strictly converging sequence. Next, a cut-off theorem is presented to overcome the difficulty of using only one-sided approximations like in the single obstacle case. Subsequently, a recovery sequence is constructed first for $\mathrm{W}^{1,1}$ and then for BV functions with finite energy without enforcement of the boundary condition. Afterwards, a Dirichlet boundary condition is implemented for the double obstacle problem.
- Since the relaxation of the area functional was determined in the previous section, a comparison between the non-parametric and the parametric case can be investigated and the equivalence result is proven like in the obstacle-free setting. This is mainly done by following the proof of M. Miranda, approximations of the sets and estimating the De Giorgi measure for general sets of the type $S \times (0,1)$. This section closes with some notes regarding the double obstacle problem for the area functional with a separation function $v \in \mathrm{BV}$ and some regularity observations for a certain double obstacle problem.
- In the final section, the theory of Euler equations for functionals with linear growth based on [4] is summarized and a version with only one-sided derivatives is proven leading to variational inequalities. Those results are then extended to the obstacle case and a simpler inequality is presented, which is sufficient for minimality.

2 Preliminaries

In this section, we first gather some general notation on sets, measures and basic function spaces before we define functions of bounded variations and sets of finite perimeter, note some fine properties and later give the needed definitions and main tools to treat functionals on measures and construct relaxations in our setting. Definitions and theorems concerned with Radon measures and convergence theorems by Reshetnyak are partially stated in a more extensive way to be able to provide some more general results for auxiliary statements, but are not needed for the main results in this generality.

The open ball with radius $r > 0$ and center $x \in \mathbb{R}^n$ is denoted by $\mathrm{B}^n_r(x) := \{y \in \mathbb{R}^n : |y-x| < r\}$, the sphere with the same properties $\mathrm{S}^{n-1}_r(x) = \partial\mathrm{B}^n_r(x)$ is the topological boundary of $\mathrm{B}^n_r(x)$. The superscript n is often omitted in both definitions if the dimension is clear through the setting or other information.

$A\Delta B$ stands for the symmetric difference $(A \setminus B) \cup (B \setminus A)$. A set A is compactly contained in Ω if $\overline{A} \subset \Omega^\circ$ where the superscript '$\circ$' denotes the topological interior of the set and the overlined set is the topological closure of the given set. $A \Subset \Omega$ stands for the closure of A being compact and contained in the open set Ω, i.e. $\overline{A} \subset \Omega$.

$\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure and $\mathcal{H}^k$ the k-dimensional Hausdorff measure with $k \geq 0$. For a $\mathcal{L}^n$-measurable set A, we write $|A|$ for the Lebesgue measure of A while for a function $|\,.\,|$ is the modulus and for a measure it is the total variation. In general, measurability refers to $\mathcal{L}^n$-measurability if not stated otherwise. The Lebesgue measure of the unit sphere is denoted by $\omega_n := |\mathrm{B}^n_1(0)|$ and, as an reminder, we have $\mathcal{H}^{n-1}(\mathrm{S}^{n-1}_1(0)) = n\omega_n$. For general definitions, lemmas and theorems like Fatou's Lemma, dominated convergence or those concerned with convergence in measure, almost everywhere convergence, convergence in the L^p spaces, we stick to the notion and versions provided in [23, Kapitel IV, §5] and [3, Theorem 1.19, 1.20 and 1.21].

As is customary, $\mathcal{B}(X)$ denotes the Borel σ-algebra of all Borel subsets, i.e. the smallest σ-algebra generated by open sets in X. For a (signed and/or vector-valued) measure ν and a positive measure μ defined on the same measure space, we say ν is absolutely continuous with respect to μ if $|\mu|(A) = 0$ implies $|\nu|(A) = 0$ and write $\nu \ll \mu$. For example, we have $\mathcal{L}^n \ll \mathcal{H}^{n-1}$. For two measures $\nu \ll \mu$ with μ being σ-finite, we define by $\frac{\nu}{\mu}$ the Radon-Nikodym density of ν with respect to μ. Two measures μ and ν defined on the same measure space are singular if a set A exists with $|\mu|(A) = 0 = |\nu|(X \setminus A)$ and we write $\mu \perp \nu$, see for example [24, Definition 1.22].

For a locally compact and separable metric space X, the space of all signed Radon measures is denoted $\mathrm{RM}_{\mathrm{loc}}(X, \mathbb{R}^m)$. If in addition the total variation of the signed Radon measure is finite, the space is denoted by $\mathrm{RM}(X, \mathbb{R}^m)$. Further, we always assume the completion of the σ-algebra with respect to a given Radon measure μ, like presented in [3, Definition 1.11 (c)] for the measure space. As usual, the target space is omitted for $m = 1$ or if the target space is clear from the context. We are usually going to deal with locally compact subsets $X \subset \mathbb{R}^n$ or even just open bounded sets Ω endowed with the euclidean norm $|\,.\,|$, but stick here to the general notation to be able to present some theorems and lemmas in the general framework. The weak* convergence on $\mathrm{RM}_{\mathrm{loc}}(X, \mathbb{R}^m)$ and $\mathrm{RM}(X, \mathbb{R}^m)$ is defined as the weak convergence of measures. The weak convergence of measures is given by the duality with the (completion of the) space of smooth functions with compact support $\mathrm{C}^0_{\mathrm{cpt}}(X, \mathbb{R}^m) := \{u : X \to \mathbb{R}^m \text{ is continuous on } X \text{ and } \operatorname{supp} u \Subset X\}$ with

respect to the supremum norm.
For the restriction of a measure μ to a given set A, we use Federer's notation $\mu \llcorner A$ which is defined by

$$\mu \llcorner A(B) := \mu(B \cap A).$$

If μ is a Borel measure or Radon measure, then $\mu \llcorner A$ is a Borel or Radon measure for a Borel set A or locally compact Borel set A, respectively.
Regarding typical function spaces on an open set Ω, we stick to the following notation: $\mathrm{C}^k(\Omega)$ denotes the k times continuously differentiable functions, a subscript 'cpt' indicates compact support and a subscript '0' the closure of $\mathrm{C}^k_{\mathrm{cpt}}(\Omega)$ under the supremum norm. Further, a subscript 'b' indicates the associated subspace of bounded functions.
The space of Lipschitz function on Ω is denoted by $\mathrm{C}^{0,1}(\Omega)$ and we have the following theorem by Rademacher, which ensures $\mathcal{L}^n$-a.e. differentiability of Lipschitz functions:

Theorem 2.1 (Rademacher, [24, Theorem 3.2.]).
A (locally) Lipschitz function $f : \mathbb{R}^k \to \mathbb{R}^m$ is differentiable $\mathcal{L}^k$-a.e.

This theorem is important, for instance, in connection with chain rules for Sobolev functions and functions of bounded variation.
Other important spaces are the Sobolev spaces. Since we are interested only in functionals with linear growth, we only give the definition of the Sobolev space $\mathrm{W}^{1,1}$:

Definition 2.2 (Sobolev space $\mathrm{W}^{1,1}$, see [3, Definition 2.3 and 2.4]).
A function $f \in \mathrm{L}^1(\Omega)$ belongs to the Sobolev space $\mathrm{W}^{1,1}(\Omega)$ if its distributional derivative is representable by a $\mathrm{L}^1(\Omega, \mathbb{R}^n)$ function v, i.e.

$$\int_\Omega u \nabla \varphi \, \mathrm{d}x = - \int_\Omega v \, \varphi \, \mathrm{d}x \quad \forall \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$$

holds and we write $v = \nabla u$.

Next, we take a closer look at the rectifiability of sets, which plays an important role for representatives of BV functions.

Definition 2.3 (Rectifiable sets, see [3, Definition 2.57] or [63, Definition 0.57]).
Let $E \subset \mathbb{R}^n$ be a $\mathcal{H}^k$ measurable set: Then E is

1. countably k-rectifiable if countably many Lipschitz maps $f_i : \mathbb{R}^k \to \mathbb{R}^n$ exist with

$$E \subset \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k).$$

2. countably $\mathcal{H}^k$-rectifiable if countably many Lipschitz maps $f_i : \mathbb{R}^k \to \mathbb{R}^n$ exist with

$$\mathcal{H}^k \left(E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k) \right) = 0.$$

3. $\mathcal{H}^k$-rectifiable if E is countably $\mathcal{H}^k$-rectifiable and $\mathcal{H}^k(E) < +\infty$.

Remark. Further, we say that a set $E \subset \mathbb{R}^n$ is locally $\mathcal{H}^k$-rectifiable if $E \cap K$ is $\mathcal{H}^k$-rectifiable for all compact sets $K \subset \mathbb{R}^n$.

An extension to this definition is provided in [3, Proposition 2.76] using so-called Lipschitz k-graphs, but is here slightly rephrased to better suite the statements:

Proposition 2.4 (Alternative characterization for rectifiable sets)**.**
A $\mathcal{H}^k$-measurable set is countably $\mathcal{H}^k$-rectifiable if countably many graphs of Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^{n-k}$ and rotations $R_i : \mathbb{R}^n \to \mathbb{R}^n$ exist such that

$$\mathcal{H}^k\left(E \setminus \bigcup_{i=0}^{\infty} R_i G_{f_i}\right) = 0$$

with $R_i G_{f_i} := \{R_i(x, f(x))^t \,:\, x \in \mathbb{R}^k\} \subset \mathbb{R}^n$ for $i \in \mathbb{N}$.

Using [3, Theorem 2.83] we obtain that each countably $\mathcal{H}^k$-rectifiable set admits an approximate tangent space $\mathcal{H}^k$-a.e. and vice versa, i.e if a set admits an approximate tangent space $\mathcal{H}^k$-a.e., it is countably $\mathcal{H}^k$-rectifiable. Thus a normal vector exists $\mathcal{H}^{n-1}$-a.e. if $k = n-1$ is chosen, where the normal vector is defined as a perpendicular (unit) vector to the approximate tangent space. Next, [3, Proposition 2.85] guarantees that on the intersection of two rectifiable sets the approximate tangent spaces of both sets coincide $\mathcal{H}^k$-a.e. on the intersection and thus the normal vector is inherited from the rectifiable sets at least $\mathcal{H}^{n-1}$-a.e. in the case $k = n-1$ and the normal vectors agree up to a sign. Further, this implies that the intersection of two rectifiable sets is again rectifiable. We collect this results in the following proposition:

Proposition 2.5 (Normal vectors and intersections of rectifiable sets)**.**
A $\mathcal{H}^{n-1}$-rectifiable set admits $\mathcal{H}^{n-1}$-a.e. a normal vector. The intersection of two $\mathcal{H}^{n-1}$-rectifiable sets S_1 and S_2 is again $\mathcal{H}^{n-1}$-rectifiable. The normal vector for $\mathcal{H}^{n-1}$-a.e. $x \in S_1 \cap S_2$ coincides with those obtained through S_1, S_2 and $S_1 \cap S_2$. Further, each measurable subset of an $\mathcal{H}^{n-1}$-rectifiable set is again $\mathcal{H}^{n-1}$-rectifiable and admits a normal vector.

Next, we define the measure theoretical closure of a measurable set (Borel set) A in $\mathbb{R}^n$ like in [13, Section 1] by:

$$A^+ = \{x \in \mathbb{R}^n \,:\, \limsup_{\rho \searrow 0} \rho^{-n} \mathcal{L}^n(\mathrm{B}_\rho(x) \cap A) > 0\}.$$

This coincides with the following definition for L^1 functions: We set $\{u > t\} := \{y \in \Omega \,:\, u(y) > t\}$ and denote by

$$u^+(x) = \inf_{t \in \mathbb{R}} \left\{ \limsup_{\rho \searrow 0} \rho^{-n} \mathcal{L}^n(\mathrm{B}_\rho(x) \cap \{u > t\}) = 0 \right\},$$

$$u^-(x) = \sup_{t \in \mathbb{R}} \left\{ \limsup_{\rho \searrow 0} \rho^{-n} \mathcal{L}^n(\mathrm{B}_\rho(x) \cap \{u < t\}) = 0 \right\}$$

two representatives of the equivalence class of a function $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$.
Further, we introduce the function $u^* := \frac{1}{2}(u^+ + u^-)$. Obviously we have $\mathbb{1}_A^{\pm} = \mathbb{1}_{A^{\pm}}$ with the

characteristic function of A, which is given by

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Referring to the indices '+' and '−' we distinguish between sub- and superscript. The superscript versions are given above and the subscript ones are defined by

$$u_+(x) := \max\{u(x), 0\} \text{ and } u_-(x) := \max\{-u(x), 0\}.$$

2.1 Functions of bounded variation and sets of finite perimeter

Functions of bounded variations

For the definitions regarding BV functions and sets of finite perimeter and their properties, we mainly follow [3, Chapter 3], [24, Chapter 5] and [63].
As mentioned in the introduction, the space $\mathrm{W}^{1,1}(\Omega)$ for some open $\Omega \subset \mathbb{R}^n$ is often not suitable to treat problems involving functionals of linear growth since the space is not closed under L^1-convergence or weak* convergence. For example, the sequence $u_k(x) := kx\mathbb{1}_{[0,\frac{1}{k}]}(x) + \mathbb{1}_{(\frac{1}{k},2)}(x)$ for $k \in \mathbb{N}$ is bounded in $\mathrm{W}^{1,1}((-2,2))$, but has no convergent subsequence. The natural extension is the space BV, since every bounded sequence in $\mathrm{W}^{1,1}$ has a converging subsequence with limit in BV and BV is closed under weak* convergence.

Definition 2.6 (BV functions, [3, Definition 3.1]).
A function $u \in \mathrm{L}^1(\Omega)$ is a function of bounded variation if the distributional derivative of u has a representation as a finite $\mathbb{R}^n$-valued Radon measure μ in Ω, i.e. if

$$\int_\Omega u \operatorname{div} \varphi \, \mathrm{d}x = -\int_\Omega \varphi \, \mathrm{d}\mu \quad \forall \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega, \mathbb{R}^n).$$

We write $\mathrm{D}u = \mu$. The vector space of all such functions is denoted by $\mathrm{BV}(\Omega)$. The space $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ contains all functions $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with distributional derivative in $\mathrm{RM}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.

Obviously the space $\mathrm{W}^{1,1}$ is contained in BV, since the distributional derivative of an $\mathrm{W}^{1,1}$-function can be written as the measure $\nabla u \mathcal{L}^n$ with the weak derivative ∇u. It is worth mentioning that $|\mathrm{D}u| \ll \mathcal{H}^{n-1}$ for every BV function u. This fact answers the question why thinner obstacles like the 1-dimensional needle for a 2-dimensional surface in $\mathbb{R}^3$ can not be considered with the approach relying on BV functions. Further, as the name implies, BV functions have a finite variation:

Definition 2.7 (Variation, [3, Definition 3.4]).
For $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, one defines the variation of u by

$$V(u, \Omega) := \sup \left\{ \int_\Omega u \operatorname{div} \varphi \, \mathrm{d}x \, : \, \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega, \mathbb{R}^n), \, \sup_\Omega |\varphi| \leq 1 \right\}.$$

A function $u \in \mathrm{L}^1(\Omega)$ belongs only to $\mathrm{BV}(\Omega)$ if and only if $V(u,\Omega) < \infty$. Then the total variation $|\mathrm{D}u|(\Omega)$ of the measure $\mu = \mathrm{D}u$ coincides with the variation $V(u,\Omega)$.
An extension to Borel sets B is given by approximation with open sets containing B:

$$V(u,B) = \inf\{V(u,A) \,:\, A \text{ open and } B \subset A\}.$$

In Definition 2.7, the function space of φ can be replaced with $\mathrm{C}_0(\Omega,\mathbb{R}^n)$ or $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega,\mathbb{R}^n)$, without loss of generality. For the total variation of u, we will also use the notation $\int_\Omega |\mathrm{D}u|$. An important property of the total variation is the lower semicontinuity with respect to L^1-convergence:

Theorem 2.8 (L.s.c. of the total variation, [3, Proposition 3.6])**.**
For a converging sequence $u_k \in \mathrm{L}^1(\Omega)$ for $k \in \mathbb{N}$, $u_k \to u$ for $k \to \infty$ in $\mathrm{L}^1(\Omega)$, the following estimate holds:

$$|\mathrm{D}u|(\Omega) \leq \liminf_{k\to\infty} |\mathrm{D}u_k|(\Omega).$$

The space $\mathrm{BV}(\Omega)$ is a Banach space with the norm

$$||u||_{\mathrm{BV}} := ||u||_1 + |\mathrm{D}u|(\Omega),$$

but the norm is too strong for most applications. Instead weak* convergence and strict convergence are used:

Definition 2.9 (Weak* and strict convergence, [3, Definition 3.11 and 3.14])**.**
For $u_k, u \in \mathrm{BV}(\Omega)$ for $k \in \mathbb{N}$, the sequence u_k converges

- weakly* to u in BV if $u_k\mathcal{L}^n \to u\mathcal{L}^n$ weakly* in $\mathrm{L}^1(\Omega)$ and $\mathrm{D}u_k \to \mathrm{D}u$ for $k \to \infty$ converges weakly* as Radon measures, i.e.

$$\lim_{k\to\infty} \int_\Omega \varphi \,\mathrm{dD}u_k = \int_\Omega \varphi \,\mathrm{dD}u \quad \forall \varphi \in \mathrm{C}_0(\Omega),$$

- strictly to u in BV if $u_k \to u$ in $\mathrm{L}^1(\Omega)$ and $|\mathrm{D}u_k|(\Omega) \to |\mathrm{D}u|(\Omega)$ for $k \to \infty$.

Unlike the weak* convergence, the strict convergence is induced by the metric

$$\mathrm{d}(u,v) = ||u - v||_1 + \big|\, |\mathrm{D}u|(\Omega) - |\mathrm{D}v|(\Omega) \,\big| \,.$$

The weak* convergence is often considered since the following compactness result holds:

Theorem 2.10 (Compactness in BV, [3, Theorem 3.23])**.**
Every bounded sequence $u_k \in \mathrm{BV}(\Omega)$ for $k \in \mathbb{N}$ has a weakly converging subsequence in $\mathrm{BV}(\Omega)$.*

The derivative of BV functions can be decomposed in different parts, which helps to understand the function space better and gives some insight on different representatives. First, we need to introduce some further sets involved in the study of BV functions:

Definition 2.11 (Approximate discontinuity, [3, Definition 3.63 and 3.67, Proposition 3.69])**.**
Let a function $u \in \mathrm{BV}(\Omega)$ be given. The set S_u denotes the set of approximate discontinuity, i.e.

all points where the function does not have an approximate limit $z \in \mathbb{R}$. The approximate limit is defined by

$$\lim_{r\searrow 0} \fint_{\mathrm{B}_r(x)} |u(x) - z| \,\mathrm{d}x = 0.$$

A subset of S_u is the so-called jump set J_u, where at least one-sided approximate limits exist, i.e. a normal vector ν_u and two values $u_{\text{ext}}, u_{\text{int}} \in \mathbb{R}$ exist with

$$\lim_{r\searrow 0} \fint_{\mathrm{B}_r^{\text{ext}}(x,\nu)} |u(x) - u_{\text{ext}}| \,\mathrm{d}x = 0 \text{ and } \lim_{r\searrow 0} \fint_{\mathrm{B}_r^{\text{int}}(x,\nu)} |u(x) - u_{\text{int}}| \,\mathrm{d}x = 0.$$

with

$$\begin{aligned} \mathrm{B}_r^{\text{ext}}(x,\nu) &= \{y \in \mathrm{B}_r(x) \,:\, (y-x)\cdot\nu > 0\} \\ \mathrm{B}_r^{\text{int}}(x,\nu) &= \{y \in \mathrm{B}_r(x) \,:\, (y-x)\cdot\nu < 0\}. \end{aligned}$$

Thus, for every point $x \in J_u$, a triple $(u_{\text{ext}}(x), u_{\text{int}}(x), \nu_u(x))$ exists which is unique up to simultaneous permutation of the first two components and the change of the sign of the normal vector ν_u to $-\nu_u$. This construction can be performed on general countably $\mathcal{H}^{n-1}$-rectifiable sets. Taking points outside of J_u into account resolves in the values u_{ext} and u_{int} being equal.

The following theorem states, how large the set $S_u \setminus J_u$ is and gives some property of J_u.

Theorem 2.12 (Federer-Vol'pert, [3, Theorem 3.78])**.**
Let $u \in \mathrm{BV}(\Omega)$ be given. Then the set S_u is countably $\mathcal{H}^{n-1}$-rectifiable and

$$\mathcal{H}^{n-1}(S_u \setminus J_u) = 0 \text{ and } \mathcal{L}^n(S_u) = 0$$

and thus J_u is $\mathcal{H}^{n-1}$-rectifiable. Moreover, $\mathrm{D}u \llcorner J_u = (u_{\text{ext}} - u_{\text{int}})\,\nu_u \mathcal{H}^{n-1} \llcorner J_u$ and

$$\mathrm{Tan}^{n-1}(J_u, x) = \nu_u^{\perp}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$.

A direct consequence and important statement is that for $u, v \in \mathrm{BV}$, we have for $\mathcal{H}^{n-1}$-a.e. $x \in J_u \cap J_v$ either $\nu_u = \nu_v$ or $\nu_u = -\nu_v$ since $J_u \cap J_v$ is rectifiable by Proposition 2.5 and inherits the normal vector from the larger sets.
Further, we can see u^+ as the function taking the highest value at jump points. Similar for the lower representative u^- taking the lowest and the Lebesgue representative u^* taking the average value, see for example [3, Section 3.6]. From this definition it is easy to see, that for $u \in \mathrm{W}^{1,1}(\Omega)$ we have $u^+ = u^- = u^*$.
Besides jump points, there exist the absolutely continuous part and another component of the derivative, which is for example known from the devil's staircase function, a continuous function from $[0,1] \to [0,1]$ with $\mathcal{L}^1$-a.e. vanishing derivative which attains the value 0 and 1 at $x = 0$ and $x = 1$, respectively. For the standard version of the devil's staircase function, the derivative measure is concentrated on the famous $\frac{1}{3}$-Cantor set, which has Hausdorff dimension between 0 and 1. In total, we can decompose the derivative measure in the following parts:

Definition 2.13 (Decomposition of the derivative, [3, Definition 3.91, Proposition 3.92]).
For a function $u \in \mathrm{BV}(\Omega)$, the derivative measure $\mathrm{D}u$ can be decomposed in the following parts:

$$\mathrm{D}u = \mathrm{D}^a u + \mathrm{D}^s u = \mathrm{D}^a u + \mathrm{D}^j u + \mathrm{D}^c u$$

$\mathrm{D}^a u$ denotes the absolutely continuous part of $\mathrm{D}u$ which is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$. It can be written as $\mathrm{D}u \llcorner (\Omega \setminus S)$ with

$$S := \left\{ x \in \Omega \,:\, \lim_{r \searrow 0} r^{-n} |\mathrm{D}u|(\mathrm{B}_r(x)) = \infty \right\}.$$

In contrast, $\mathrm{D}^s u = \mathrm{D}u \llcorner S$ is the singular part of the derivative with respect to the Lebesgue measure and the existence is provided by the Lebesgue decomposition theorem, see for example [23, Kapitel VII, 2.6]. The singular part can be further decomposed in the following two parts:

- The jump part $\mathrm{D}^j u = \mathrm{D}^s u \llcorner S_u = \mathrm{D}^s u \llcorner J_u$ with the use of Theorem 2.12.
- The Cantor part $\mathrm{D}^c u = \mathrm{D}^s u \llcorner (\Omega \setminus S_u)$.

Remark. As often used, we sometimes do not distinguish between the measure $\mathrm{D}^a u$ or the density of $\mathrm{D}^a u$ with respect to the Lebesgue measure. It will always be clear from the context, which is used.

Those parts have many useful properties, one of which is given in the following proposition:

Proposition 2.14 (Properties of the parts of the derivative, [3, Proposition 3.92]).
For given $u \in \mathrm{BV}(\Omega)$, the Cantor part $\mathrm{D}^c u$ vanishes on σ-finite sets with respect to $\mathcal{H}^{n-1}$.

This proposition ensures that the Cantor part does not interfere with the jump part.
With this knowledge obtained it is possible to define traces at least on rectifiable sets, for example, on interior rectifiable sets like in [3, Theorem 3.77] or on boundaries of Lipschitz domains. To treat boundary value problems this information is not enough since the trace is not conserved under weak* convergence. A way to identify boundary values on Lipschitz domains is to find a BV function defined on open set O containing Ω with the desired trace. For $u_0 \in \mathrm{L}^1(\partial\Omega) := \mathrm{L}^1(\partial\Omega, \mathcal{H}^{n-1})$, this can be done using Gagliardo's theorem found in [33] giving the existence even of an $\mathrm{W}^{1,1}$-function on $O \setminus \Omega$. Gluing two BV functions along the boundary $\partial\Omega$ gives a BV function on the superset maybe with a jump part on $\partial\Omega$. This construction sets the outer trace to the desired one, i.e. $u_{\text{ext}} = u_0$ with ν_Ω the outer normal vector to Ω, but does not imply anything for the values of u_{int}. This has to be taken into account when dealing with partial differential equations or minimizers of variational integrals. Summarized we have:

Theorem 2.15 (Boundary traces of BV functions, see [3, Theorem 3.87]).
On a Lipschitz domain Ω the trace of a function $u \in \mathrm{BV}(\Omega)$ is given for $\mathcal{H}^{n-1}$-almost every $x \in \partial\Omega$ as u_{int} of the extension of u on $\partial\Omega$.

To be able to solve problems on merely open and bounded sets Ω, a different approach is to use Dirichlet classes. The basic idea is to prescribe boundary values by a given function v and define the desired function space by adding only functions with 0 boundary values, which can be defined in more general cases than non-zero boundaries. A notion of those classes is provided, for example, in [46, Section 2.3]:

Definition 2.16 (Dirichlet classes)**.**
For an open bounded set Ω and prescribed boundary value function $v \in \mathrm{BV}(\Omega)$, we define $w(u) = (u-v)\mathbb{1}_\Omega$ on $\mathbb{R}^n$ and set the to v associated Sobolev and BV Dirichlet class

$$\mathrm{W}^{1,1}_v(\Omega) := \left\{u \in \mathrm{W}^{1,1}(\Omega) \,:\, w(u) \in \mathrm{BV}(\mathbb{R}^n) \text{ and } |\mathrm{D}w|(\partial\Omega) = 0\right\}$$

$$\mathrm{BV}_v(\Omega) := \{u \in \mathrm{BV}(\Omega) \,:\, w(u) \in \mathrm{BV}(\mathbb{R}^n) \text{ and } |\mathrm{D}w|(\partial\Omega) = 0\}\,.$$

Remark. For $v \in \mathrm{W}^{1,1}$ and domains Ω with Lipschitz boundary, the presented definitions are equal and (in even some more cases) equivalent to the commonly used notion $\mathrm{W}^{1,1}_v(\Omega) = v + \mathrm{W}^{1,1}_0(\Omega)$, where $\mathrm{W}^{1,1}_0$ is defined as the closure of $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with respect to the $\mathrm{W}^{1,1}$-Sobolev norm. Similarly for the BV case.

To be able to penalize wrong boundary values, a useful tool is to extend BV functions outside of a given set Ω and thus fix the exterior trace:

Definition 2.17 (BV extension, see [65, Introduction])**.**
For an open set Ω, given $u \in \mathrm{BV}(\Omega)$ and $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$, we set

$$\overline{u}(x) = \begin{cases} u(x) & \text{for } x \in \Omega \\ u_0(x) & \text{for } x \in \mathbb{R}^n \setminus \Omega \end{cases},$$

and define

$$\mathrm{BV}_{u_0}(\overline{\Omega}) := \{u \in \mathrm{BV}(\Omega) \,:\, \overline{u} \in \mathrm{BV}(\mathbb{R}^n)\}\,.$$

Remark. This is the way to interpret $\int_{\overline{\Omega}} |\mathrm{D}u|$ for a function $u \in \mathrm{BV}$ and given boundary values u_0 as mentioned in the introduction:
If $u \in \mathrm{BV}_{u_0}(\overline{\Omega})$, the boundary term can be interpreted as a the difference of u and u_0 at the boundary. If $u \in \mathrm{BV}(\Omega) \setminus \mathrm{BV}_{u_0}(\overline{\Omega})$ the value is $+\infty$.
Additionally, if Ω is a Lipschitz domain, we have $\mathrm{BV}_{u_0}(\overline{\Omega}) = \mathrm{BV}(\Omega)$ by Theorem 2.15.

Another useful result is the chain rule for BV functions:

Theorem 2.18 (Chain rule, see [3, Theorem 3.99, 3.101])**.**
Let $u \in \mathrm{BV}(\Omega)$ and a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ be given. Then $f \circ u$ is belongs to $\mathrm{BV}(\Omega)$ and we have

$$\mathrm{D}(f \circ u) = (f' \circ u^*)\mathrm{D}^a u \mathcal{L}^n + (f \circ u_{\mathrm{ext}} - f \circ u_{\mathrm{int}})\nu_u \mathcal{H}^{n-1} \llcorner J_u + f' \circ u^* \mathrm{D}^c u.$$

Similarly, the chain rule holds for vector-valued BV functions $v \in \mathrm{BV}(\Omega, \mathbb{R}^k)$ and Lipschitz-functions $f : \mathbb{R}^k \to \mathbb{R}^m$ with similar formulation. In addition, the following estimate holds:

$$|\mathrm{D}(f \circ u)| \leq \mathrm{Lip}_f |\mathrm{D}u|$$

with the Lipschitz constant Lip_f of the function f.

The last inequality enables us to estimate the derivative of the minimum and maximum of two different BV functions. A direct consequence is a product rule for bounded functions of bounded variation:

Corollary 2.19 (Product rule, see Remark to [3, Theorem 3.96])**.**
For two functions $u, v \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ *and* $f(x,y) = xy$*, we have*

$$uv \in \mathrm{BV}(\Omega) \text{ and } \mathrm{D}(uv) = u^*(\mathrm{D}^a v + \mathrm{D}^c v) + v^*(\mathrm{D}^a u + \mathrm{D}^c u) + (f(u_{\mathrm{ext}}) - f(u_{\mathrm{int}}))\, \nu_u \mathcal{H}^{n-1} \llcorner J_u.$$

Proof. The proof follows directly from the mentioned vector-valued case in 2.18 with $f(x,y) = xy$ which needs only to be defined on a bounded set, since u, v are assumed to be bounded. □

Next, we introduce sets of finite perimeter, some of their properties and a first version of the co-area formula.

Sets of finite perimeter

Sets of finite perimeter are very important in the study of minimal surfaces and are the base on which the parametric theory is build. Let us first define the perimeter:

Definition 2.20 (Perimeter, see [3, Definition 3.35])**.**
A $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ is a set of finite perimeter in an open set Ω if the perimeter $\mathrm{P}(E, \Omega)$ of E in Ω defined by

$$\mathrm{P}(E,\Omega) = |\mathrm{D}\mathbb{1}_E|(\Omega) = \sup \left\{ \int_E \operatorname{div} \varphi \, \mathrm{d}x \;:\; \varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\Omega, \mathbb{R}^n) \text{ with } \sup_\Omega |\varphi| \leq 1 \right\}$$

is finite. If $\Omega = \mathbb{R}^n$ is chosen, we will omit this in the notation and write $\mathrm{P}(E)$.

Remark. As mentioned in the introduction, sets E with C^1-boundary inside a set Ω have finite perimeter if $\mathcal{H}^{n-1}(\partial E \cap \Omega) < \infty$ and indeed the equality $\mathcal{H}^{n-1}(\partial E \cap \Omega) = \mathrm{P}(E, \Omega)$ holds.

The perimeter inherits some useful properties from the total variation formulation introduced for BV functions and has other helpful properties:

Proposition 2.21 (Properties of the perimeter, see [3, Proposition 3.38, Theorem 3.39])**.**
For open $\Omega' \subset \Omega$ *and measurable sets* E, F*, we have:*

1. *The perimeter is lower semicontinuous with respect to* L^1 *convergence of sets. The convergence of sets is defined by:* E_k*,* $k \in \mathbb{N}$ *converges to* E *in* L^1 *if* $\mathbb{1}_{E_k} \to \mathbb{1}_E$ *for* $k \to \infty$ *in* L^1.
2. *A sequence* E_k*,* $k \in \mathbb{N}$ *of bounded volume* $\sup |E_k| < \infty$ *and of bounded perimeter* $\sup \mathrm{P}(E_k) < \infty$ *admits a converging subsequence, i.e. there exist* k_l*,* $l \in \mathbb{N}$ *such that* $\mathrm{P}(E_{k_l}) \to \mathrm{P}(E)$.
3. $\mathrm{P}(E, \Omega') \leq \mathrm{P}(E, \Omega)$.

4. $\mathrm{P}(E,\Omega) = \mathrm{P}(\mathbb{R}^n \setminus E, \Omega)$ *and*

$$\mathrm{P}(E \cup F, \Omega) + \mathrm{P}(E \cap F, \Omega) \leq \mathrm{P}(E,\Omega) + \mathrm{P}(F,\Omega).$$

An important definition concerning sets of finite perimeter is the reduced boundary which quantifies the perimeter by giving insight on points considered for the calculation:

Definition 2.22 (Reduced boundary, see [3, Definition 3.54])**.**
For a $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ and the largest open subset Ω^* of $\mathbb{R}^n$ such that E has locally finite perimeter in Ω^*, the reduced boundary $\mathcal{F}E$ is defined as the set of all points $x \in \operatorname{supp}|\mathrm{D}\mathbb{1}_E| \cap \Omega^*$ such that the following limit exists in $\mathbb{R}^n$ and has modulus equal 1:

$$\nu_E := \lim_{r \searrow 0} \frac{\mathrm{D}\mathbb{1}_E(\mathrm{B}_r(x))}{|\mathrm{D}\mathbb{1}_E|(\mathrm{B}_r(x))}.$$

ν_E is a generalized inner normal to the set E.

Next, we can note some properties of the reduced boundary.

Proposition 2.23 (Properties of the reduced boundary, see [3, Chapter 3.5])**.**

1. *For boundary points where a normal vector exists in a classical sense, it agrees with the given generalization.*
2. $|\mathrm{D}\mathbb{1}_E|$ *is concentrated on* $\mathcal{F}E$ *and we have* $\mathrm{D}\mathbb{1}_E = \nu_E |\mathrm{D}\mathbb{1}_E|$.
3. *The reduced boundary is countably* $(n-1)$*-rectifiable and the perimeter can be represented by*

$$\mathrm{P}(E,\Omega) = \mathcal{H}^{n-1} \llcorner \mathcal{F}E(\Omega)$$

for E *and* Ω *as in Definition 2.20. This statement is part of De Giorgi's structure theorem, see [3, Theorem 3.59].*

4. *With the reduced boundary the following version of the Gauss-Green formula can be stated:*

$$\int\limits_E \operatorname{div} u \,\mathrm{d}x = \int\limits_{\mathcal{F}E} u \nu_E \,\mathrm{d}\mathcal{H}^{n-1}.$$

Another interesting boundary type is the essential boundary which is connected to the notion of densities.

Definition 2.24 (Density and essential boundary)**.**
For a $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ and a value $t \in [0,1]$, the set of points $E(t)$ with density t is given by

$$E(t) = \left\{ x \in \mathbb{R}^n : \lim_{r \searrow 0} \frac{|E \cap \mathrm{B}_r(x)|}{|\mathrm{B}_r(x)|} = t \right\},$$

where

$$\lim_{r \searrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|}$$

is called the density at a point x. There can exist points, where the density is not defined, i.e. the

limit in the definition does not exist. In those cases, one often uses the upper density, where the limes is changed to a limes superior or to the lower density with the limes inferior.
The essential boundary $\partial^* E$ is then given by $\mathbb{R}^n \setminus (E(0) \cup E(1)$.

This definition implies that the essential boundary consists of all points which are not fully in or outside of E. An important role play the points with density $\frac{1}{2}$, since those are closely connected to the reduced boundary as Federer's structure theorem implies:

Theorem 2.25 (Federer's structure theorem)**.**
For a set E with finite perimeter in Ω, we have the following inclusion and related estimate:

$$\mathcal{F}E \subset E(0.5) \subset \partial^* E$$

and

$$\mathcal{H}^{n-1}(\Omega \setminus (E(0) \cup \mathcal{F}E \cup E(1))) = 0.$$

This implies that a set of finite perimeter has $\mathcal{H}^{n-1}$-a.e. density 0, 0.5 or 1 and that $\mathcal{H}^{n-1}(E(0.5) \setminus \mathcal{F}E) = 0$. Comparing this definitions with the choices of representatives E^+ and E^-, we have that $E(t) \subset E^+$ for all $t > 0$ and E^+ also contains the $\mathcal{H}^{n-1}$-null set, where the limit does not exist but the limes superior is larger 0, i.e. $E^+ = E(1) \cup \partial^* E$, while E^- contains the density 1 points.

Co-area formula and an advanced trace theorem

An important formula to prove many useful results and applications in all kinds off branches in mathematics is the co-area formula. There exist many different versions in different settings for the co-area formula, the most common ones are for smooth and Lipschitz functions. Using the presented definitions, the co-area formula for BV functions is given by:

Theorem 2.26 (Co-area formula, [63, Theorem 1.42] or [3, Theorem 3.40])**.**
For $u \in \mathrm{BV}(\Omega)$, we have

$$V(u, \Omega) = \int_{-\infty}^{+\infty} \mathrm{P}(\{x \in \Omega : u(x) > t\}, \Omega)\, \mathrm{d}t$$

and for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, the set $\{u > t\} := \{x \in \Omega : u(x) > t\}$ has finite perimeter. Further, we have

$$\mathrm{D}u(\Omega) = \int_{-\infty}^{+\infty} \mathrm{D}\mathbb{1}_{\{u>t\}}(\Omega)\, \mathrm{d}t$$

and by rewriting the first equality for the variation

$$|\mathrm{D}u|(\Omega) = \int_{-\infty}^{+\infty} |\mathrm{D}\mathbb{1}_{\{u>t\}}|(\Omega)\, \mathrm{d}t.$$

Remark. More generalized versions of the co-area exist, where on the left-hand side another function appears as a product or outer composition part in the integrand which has to be implemented on the right-hand side. For this generalized version, we first need to understand how compositions behave at points or on sets with unbounded derivative. To achieve that, the recession function is introduced in the next subsection.

With the definition of the perimeter in hand, we can state another trace theorem:

Theorem 2.27 (Trace theorem on sets with $\mathrm{P}(\Omega) = \mathcal{H}^{n-1}(\partial\Omega) < +\infty$, [65, Proposition 4.1])**.**
For a function $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and on an open domain Ω with $\mathrm{P}(\Omega) = \mathcal{H}^{n-1}(\partial\Omega) < +\infty$ and every $\varepsilon > 0$, there exists an open set $\Omega_\varepsilon \subset \Omega$ with finite perimeter such that

$$\Omega_\varepsilon \Subset \Omega, \quad |\Omega \setminus \Omega_\varepsilon| < \varepsilon, \quad \int\limits_{\mathcal{F}\Omega_\varepsilon} |u_{\mathrm{int}}| \,\mathrm{d}\mathcal{H}^{n-1} \leq \int\limits_{\partial\Omega} |u_{\mathrm{int}}| \,\mathrm{d}\mathcal{H}^{n-1} + \varepsilon.$$

This suffices to explain boundary values on such domains using lower semicontinuity of the total variation and the sequence $u\mathbb{1}_{\Omega_\varepsilon} \in \mathrm{BV}(\mathbb{R}^n)$ for $\varepsilon \searrow 0$.

2.2 Recession function and functionals on measures

Recession function

The recession function approximates a given function at infinity to gain a grasp on how, for example, an integrand $f(\mathrm{D}u)$ behaves at points of unbounded derivative $\mathrm{D}u$. Also the recession function is necessary to extend functionals from $\mathrm{W}^{1,1}$ to the BV setting. There are many different approaches and many different settings with several nuances to the recession function, but we will stick to the more basic versions. We mostly follow the definition provided in [13] and compare it with the definition typical for Young measures provided for example in Kristensen's and Rindler's paper [46] or in the article by Alibert and Bouchitté [1] and thus achieve some further properties.

Definition 2.28 (Recession function part I)**.**
For $g : \Omega \times \mathbb{R}^n \to \mathbb{R}$, we define the strong recession function

$$g^\infty(x, z) = \lim_{\substack{t\to\infty \\ z'\to z \\ x'\to x}} \frac{g(x', tz')}{t},$$

if the limit exists for $x \in \overline{\Omega}$, $z \in \mathbb{R}^n$. This definition holds as well for g without dependency on x.

Remark. It is useful to assume $x \in \overline{\Omega}$ to be able to treat boundary conditions, as we will see later.

In the case, that is of interest here, the recession function f^∞ of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ with linear growth

$$a|z| \leq f(z) \leq b(1 + |z|)$$

turns out to be

$$f^\infty(z) = \lim_{t\to\infty} \frac{f(tz)}{t} = \sup_{v\in\mathbb{R}^n} \{f(z + v) - f(v)\} \tag{2.1}$$

and thus fulfills the following inequalities:

$$a|z| \leq f^{\infty}(z) \leq b|z|. \tag{2.2}$$

Additionally, we have the following estimate

$$f(z+w) \leq f(z) + f^{\infty}(w), \tag{2.3}$$

which is obtained from the second equality in (2.1).

Remark.

1. In the convex setting, it is not necessary to assume the existence of the limit, since by the linear boundedness the existence is assured.
2. In the case of the area functional $a(z) = \sqrt{1+|z|^2}$, we have $a^{\infty}(z) = |z|$ and for every positively 1-homogeneous function with or without continuous x-dependency the recession function coincides with the original function.
3. There are many other definitions of different recession functions. The keyword 'strong' indicates, that we consider the limit with respect to all variables, other definitions use the limes superior or inferior to suite their setting better or give the most general version.
4. Even though we assume linear growth in a very restricted way to be able to define a generalized De Giorgi measure later, it is possible to take other growth conditions into account. Useful are for example just the upper bound $f(z) \leq b(1+|z|)$ or for functions f which can be negative $|f(z)| \leq b(1+|z|)$. Further, general lower limits may be assumed like $f(z) \geq -a(1+|z|)$. If the function is x dependent, it is common to allow further alteration by allowing a function Ψ to appear in the estimates. The function is usually bounded, bounded on bounded sets for unbounded domains Ω or just in $\mathrm{L}^1_{\mathrm{loc}}$ or L^1. Thus, growth of the following forms may also be considered like $g(x,z) \leq \Psi + b|z|$, $|g(x,z)| \leq \Psi + b|z|$, $g(x,z) > \Psi - a|z|$ and many more. For the discussion of quite general Reshetnyak's theorems, we will see some of the mentioned modifications. In all conditions, $a, b > 0$ is assumed.

The last equality in (2.1) can be obtained by convex theory using properties of the supergraph and recession cones [60, Theorem 8.5.] or by a simple calculation as follows:
Part 1: First, we show that $f(z+w) - f(w) \leq f^{\infty}(z)$ for all $w \in \mathbb{R}^n$:
For fixed $w \in \mathbb{R}^n$, we use the convexity property of f to obtain:

$$\begin{aligned} f(z+w) - f(w) &= f\left(\frac{1}{2}(2z+w) + \frac{1}{2}w\right) - f(w) \\ &\leq \frac{1}{2} f\left(2z+w\right) + \frac{1}{2} f(w) - f(w). \end{aligned}$$

Iterating this step, we arrive at the following inequality for all $n \in \mathbb{N}$ and can further estimate:

$$
\begin{aligned}
f(z+w) - f(w) &\leq \frac{1}{2^n} f(2^n z + w) - f(w) + \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k f(w) \\
&= \frac{1}{2^n} f\left(\frac{1}{2} 2^{n+1} z + \frac{1}{2} 2w\right) - f(w) + \left(\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} - 1\right) f(w) \\
&\leq \frac{1}{2^{n+1}} f(2^{n+1} z) + \frac{1}{2^{n+1}} f(2w) - \left(\frac{1}{2}\right)^n f(w) \\
&\overset{n\to\infty}{\longrightarrow} f^\infty(z)
\end{aligned}
$$

The last limit is computed correctly, since $f(2w)$ and $f(w)$ are both bounded by $b(1 + |2w|) < \infty$ for fixed w. Thus, the supremum $\sup_{w \in \mathbb{R}^n} f(z+w) - f(w)$ is bounded from above by the value of the recession function $f^\infty(z)$.

Part 2: For the reversed inequality, we use

$$
f(tz) = f\left(\frac{1}{1+t} 0 + \frac{t}{1+t}(1+t)z\right) \leq \frac{1}{1+t} f(0) + \frac{t}{1+t} f((1+t)z),
$$

which implies

$$
\frac{t}{1+t} f((1+t)z) - f(tz) \geq -\frac{1}{1+t} f(0).
$$

Using the last inequality with $w = tz$ for $t > 0$ we obtain

$$
\begin{aligned}
f(z+tz) - f(tz) &= \frac{1}{1+t} f((1+t)z) + \frac{t}{1+t} f((1+t)z) - f(tz) \\
&\geq \frac{1}{1+t} f((1+t)z) - \frac{1}{1+t} f(0) \\
&\overset{t\to\infty}{\longrightarrow} f^\infty(z),
\end{aligned}
$$

where again the boundedness of $f(0) \leq b$ is used. Thus, the supremum $\sup_{w \in \mathbb{R}^n} f(z+w) - f(w)$ is bounded from below by the value of the recession function $f^\infty(z)$.

Part 1 and *Part 2* combined prove the desired equality. It is possible to use this results for x-dependent functionals $g(x, z)$ to obtain this pointwise estimate and therefore (2.3) holds for functions g as well.

A different view on recession functions, which comes from the theory involving generalized Young measure, is to define the recession function g^∞ of a continuous g as an extension of a transformed version of g. For that, the two maps T and T^{-1} are introduced:

$$
Tg(x, z) := (1 - |z|) g\left(x, \frac{z}{1 - |z|}\right) \quad \text{for } x \in \Omega,\ z \in \mathrm{B}_1(0)
$$

and

$$
T^{-1} p(x, z) := (1 + |z|) p\left(x, \frac{z}{1 + |z|}\right) \quad \text{for } x \in \Omega,\ z \in \mathbb{R}^n.
$$

Obviously, those are inverses of each other. To define the recession function, a bounded and continuous extension of Tf to $\mathrm{S}_1(0)$ is required.

Definition 2.29 (Recession function, alternative definition)**.**
Assume $g : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is continuous and Tg has a bounded continuous extension to $\overline{\Omega \times \mathrm{B}_1(0)}$, then the recession function is defined as the positively 1-homogeneous extension of the function $Tg : \overline{\mathrm{B}_1(0)} \times \mathrm{S}_1(0) \to \mathbb{R}$, i.e.

$$g^\infty(x, z) = |z| Tg\left(x, \frac{z}{|z|}\right) \text{ for } x \in \overline{\Omega}\, z \neq 0$$

and $g^\infty(x, 0) = 0$.

Remark.

1. This definition works also for functions $f : \mathbb{R}^n \to \mathbb{R}$ without x-dependency.
2. For such an extension to exist, it is sufficient that g and g^∞ are continuous on $\overline{\Omega} \times \mathbb{R}^n$, g has linear growth at infinity and is convex in the second argument. For the functions considered in this thesis, we have $a|z| \leq f(z) \leq b(1 + |z|)$ which suffices for the existence if f is convex, since f does not depend on x. In those cases, the definitions of the recession functions are equivalent. It is noteworthy that for $g \in \mathrm{C}^0(\overline{\Omega} \times \mathbb{R}^n)$, which is positively 1-homogeneous one has $g^\infty = g$.

The important impact of this definition in our setting is that the recession function can be approximated in dependence of the modulus of z at infinity:

Theorem 2.30 (Convergence to the recession function)**.**
For $f : \mathbb{R}^n \to \mathbb{R}$ convex and with linear growth $a|z| \leq f(z) \leq b(1 + |z|)$ for positive constants a, b, we have:

$$\forall \varepsilon > 0\ \exists \delta > 0 : |f(z) - f^\infty(z)| \leq \varepsilon |z| \text{ for } |z| \geq \frac{1}{\delta}. \tag{2.4}$$

Proof. Since Tf is extendable to $\mathrm{S}_1(0)$ and since we obtained a bounded continuous function $Tf : \overline{\mathrm{B}_1(0)} \to \mathbb{R}$ on the compact set $\overline{\mathrm{B}_1(0)}$, the function Tf is uniformly continuous and for any given $\tilde{\varepsilon} > 0$ there exists a $\tilde{\delta} > 0$ such

$$|Tf(y) - Tf(y_0)| = |Tf(y) - f^\infty(y_0)| < \tilde{\varepsilon} \text{ for all } y_0 \in \mathrm{S}_1(0) \text{ and } |y - y_0| < \tilde{\delta},$$

especially for $y = (1 - t)y_0$ or $y_0 = \frac{y}{1-t}$ with $t \in (0, \tilde{\delta})$ for which we have $|y| \geq 1 - \tilde{\delta}$. Since f^∞ is positively 1-homogeneous, we further have

$$|f^\infty(y_0) - f^\infty(y)| = |f^\infty(y_0) - f^\infty((1 - t)(y_0))| \leq t|f^\infty(y_0)| \leq \tilde{\delta} b(1 + |y_0|) = \tilde{\delta} 2b.$$

Without loss of generality, we may assume $\tilde{\delta} < \tilde{\varepsilon}$ and thus get

$$|f^\infty(y) - f(y)| \leq (1 + 2b)\tilde{\varepsilon}.$$

For $y = \frac{z}{1+|z|}$, we obtain

$$Tf(y) = \left(1 - \frac{|z|}{1+|z|}\right) f\left(\frac{\frac{z}{1+|z|}}{1-\frac{|z|}{1+|z|}}\right) = \frac{1}{1+|z|} f(z)$$
$$f^\infty(y) = \frac{1}{1+|z|} f^\infty(z)$$

and together with $|z| \geq 1$ we conclude

$$|f^\infty(z) - f(z)| \leq (1+2b)\tilde{\varepsilon}(1+|z|) \leq 2(1+2b)\tilde{\varepsilon}|z| \tag{2.5}$$

for $|y| = \frac{|z|}{1+|z|} \geq 1 - \tilde{\delta}$ which can be resolved to

$$|z| \geq \frac{1-\tilde{\delta}}{\tilde{\delta}}.$$

In conclusion, we achieve the desired estimate for example with the choice $\delta := \min\{\tilde{\delta}, 1\}$ and $\varepsilon := 2(1+2b)\tilde{\varepsilon}$. □

Remark. The outcome is the same if we demand uniform convergence of $\frac{f(tz)}{t}$ to $f^\infty(z)$ for $t \to \infty$ for all $z \in \mathrm{S}_1(0)$ (or any other $\mathrm{S}_r(0)$ or $\mathrm{B}_r(0)$), i.e. for all $z \in \mathrm{S}_1(0)$ and

$$\forall \varepsilon > 0\, \exists \delta > 0 : \left| f^\infty(z) - f\left(\frac{tz}{t}\right)\right| < \varepsilon \text{ for } t \geq \frac{1}{\delta}.$$

Using that property, we can estimate for $z \neq 0$ using the positive 1-homogeneity of the recession function

$$|f^\infty(z) - f(z)| = \left| |z| f^\infty\left(\frac{z}{|z|}\right) - |z| \frac{f\left(|z|\frac{z}{|z|}\right)}{|z|}\right| \leq |z|\varepsilon \text{ for } |z| \geq \frac{1}{\delta},$$

since $\frac{z}{|z|} \in \mathrm{S}_1(0)$. Again it is possible to include the x-dependency with further assumption on the continuity of g in x like mentioned in [6, Section 2.3].

With the definition of the recession function we can define linear functionals on measures and on BV functions.

Functionals on measures

The first to study functionals on measures were Goffman and Serrin [40] in the positively 1-homogeneous case. This result was eventually generalized to the non-homogeneous case using the perspective function or homogenized integrand, see for example [34], [63, Chapter 2.1] or [64, Chapter 2.1]. In the following, we provide the definition and derivation in several different cases, starting with the positively 1-homogeneous version on which the other build upon and follow the descriptions in [64, Chapter 2.1]:

Definition 2.31 (Functionals on measures, part I).
Let $\Omega \subset \mathbb{R}^n$ and $g : \Omega \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, which is positively 1-homogeneous in the second variable, i.e. $h(x, tz) = th(x, z)$ for all $t \geq 0$ and $(x, z) \in \Omega \times \mathbb{R}^n$, and which fulfills the lower bound constraint

$$h(x, z) \geq -\Gamma|z| \text{ for all } (x, z) \in \Omega \times \mathbb{R}^n$$

with some positive constant $\Gamma < \infty$. Then, for a $\nu \in \mathrm{RM}_{\mathrm{loc}}(X, \mathbb{R}^n)$ with X locally compact subset of Ω, we define the signed Borel measure $h(\,.\,, \nu)$ on X by choosing an arbitrary Radon measure μ with $|\nu| \ll \mu$ and setting

$$\int_E h(\,.\,, \nu) := \int_E h\left(\,.\,, \frac{\nu}{\mu}\right) \mathrm{d}\mu \tag{2.6}$$

for every $|\nu|$-finite Borel set $E \subset X$.

It is easy to see that the value does not depend on the choice of μ and one can always take $\mu = |\nu|$. Further, the dimension of Ω and the target space of the Radon measure do not need to agree.
To be able to treat more general functionals, we introduce the perspective function or homogenized integrand: For an open $\Omega \subset \mathbb{R}^n$ and a Borel function $g : \Omega \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, which is not necessarily positively 1-homogeneous, we define

$$\overline{g} : (\Omega \times (0, \infty) \times \mathbb{R}^n) \cup (\overline{\Omega} \times \{0\} \times \mathbb{R}^n) \to \mathbb{R} \cup \{\pm\infty\}$$

with

$$\overline{g}(x, t, z) = tg\left(x, \frac{z}{t}\right) \text{ for } (x, t, z) \in (\Omega \times (0, \infty) \times \mathbb{R}^n)$$

with the extension $\overline{g}(x, 0, 0) = 0$ for $x \in \overline{\Omega}$ and

$$g^\infty(x, z) := \overline{g}(x, 0, z) := \liminf_{\substack{t \to 0 \\ z' \to z \\ x' \to x}} \overline{g}(x', t, z') \text{ for } (x, z) \in \overline{\Omega} \times (\mathbb{R}^n \setminus \{0\}) \tag{2.7}$$

for $t = 0$. The constructed function $\overline{g}$ can reproduce the starting function g with $t = 1$ and a version of the recession function g^∞ for $t = 0$ and is positively 1-homogeneous with respect to (t, z) and g^∞ is positively 1-homogeneous in z. If the limit $t \searrow 0$ exists and is not only a limes inferior, we even get the strong recession function like in Definition 2.28. The additional assumption

$$f(x, z) \geq -\Psi(x) - \Gamma|z|$$

for some non-negative function Ψ and $\Gamma < +\infty$, which entails $\overline{g}(x, t, z) \geq -t\Psi(x) - \Gamma|z|$ for $t > 0$, paired with Ψ being either bounded on bounded subsets of Ω or being such that the limes inferior is indeed a limes, which ensures g^∞ to be the strong recession function, leads to

$$g^\infty(x, t) \geq -\Gamma|z| \text{ for all } (x, z) \in \overline{\Omega} \times \mathbb{R}^n.$$

This provides, that the functional can not attain the value $-\infty$ and that at every point of the set $\overline{\Omega} \times \{0\} \times \mathbb{R}^n$ the function $\overline{g}$ is lower semicontinuous in z as well as g^∞. As before, convexity of g implies convexity of $\overline{g}$ in (t, z) and convexity of g^∞ in z. Relying on the results from [40] and as shown in [64], we can state the following definition:

Definition 2.32 (Functionals on measures, part II)**.**
Let Ω be an open subset $\Omega \subset \mathbb{R}^n$ and $g : \Omega \times \mathbb{R}^n \to \mathbb{R}$ a Borel function fulfilling

$$g(x, z) \geq -\Psi(x) - \Gamma|z| \text{ and } g^\infty(x, z) \geq -\Gamma|z|$$

for $(x, z) \in \Omega \times \mathbb{R}^n$ and $(x, z) \in \overline{\Omega} \times \mathbb{R}^n$, respectively, and with $\Gamma < +\infty$. Then, for a Radon measure $\nu \in \mathrm{RM}_{\mathrm{loc}}(X, \mathbb{R}^n)$ on a locally compact subset $X \subset \overline{\Omega}$ with $|X \cap \partial\Omega| = 0$, $\Psi \in \mathrm{L}^1_{\mathrm{loc}}(X)$ and a Radon measure μ on X with $\mathcal{L}^n \llcorner X + |\nu| \ll \mu$, we define a signed Borel measure $g(\,.\,, \nu)$ on X by setting

$$\int_E g(\,.\,, \nu) := \int_E \overline{g}\left(\,.\,, \frac{\mathcal{L}^n \llcorner \Omega}{\mu}, \frac{\nu}{\mu}\right) \mathrm{d}\mu \text{ for every } (\Psi\mathcal{L}^n + |\nu|)\text{-finite Borel set } E \subset \Omega. \tag{2.8}$$

Remark.

1. The definition does not depend on the choice of μ, which is easy to check using the positive 1-homogeneity of $\overline{g}$ in (t, z). Further, the Definitions 2.31 and 2.32 coincide for lower semicontinuous and positively 1-homogeneous g.
2. With the particular choice $\mu = \mathcal{L}^n \llcorner X + |\nu|^s$ with the usual Lebesgue decomposition of $\nu = \nu^a + \nu^s$ we arrive at the usually considered functional on measures

$$\int_E g(\,.\,, \nu) = \int_{E \cap \Omega} g(\,.\,, \nu^a)\,\mathrm{d}x + \int_E g^\infty\left(\,.\,, \frac{\nu^s}{|\nu|^s}\right) \mathrm{d}|\nu|^s \tag{2.9}$$

   for all Borel sets $E \subset X$ with $|\Psi|\mathcal{L}^n(E) + |\nu|(E) < +\infty$. At this point we stress that $|X \cap \partial\Omega| = 0$ ensures, that the integral involving the absolute continuous parts with respect to the Lebesgue measure can be written without dependency of the boundary.
3. In the paper [46] by Kristensen and Rindler, such functionals are treated in the setting of generalized Young measures, which has added to the theory and provided some interesting features on the description of such convergences in the BV setting through generalized Young measures. This allows a better understanding in context of relaxation and by provides better convergence theorems, as presented in the next part.
4. As a side note, we mention that f may take negative values, and thus Γ is allowed to take arbitrary real values in contrast to the used constants $a, b > 0$. The additional constraint which we put upon our integrand for the double obstacle problem, comes from the interaction of the recession function and the generalized De Giorgi measure.
5. In general, we are interested in $E = \overline{\Omega}$ or $E = \Omega$. Those cases are considered, for example, in [4], where the difference between the closure and the open set is under considerations in connection with boundary concentration effects. For example, with the derivative of BV functions in mind, jumps can occur at the boundary which has to be tracked for prescribed boundary values.

Corollary 2.33 (Addendum to Definition 2.32)**.**

1. *The definition still holds under the assumption* $\Psi \in \mathrm{L}^1(\Omega)$, $\nu \in \mathrm{RM}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ *with* $X = \Omega$ *for some open set* Ω *and with an additional upper bound in the form*

$$|g(x,z)| \leq \Psi(x) + b|z|$$

for some finite $b > 0$*. In that case,* $g(\,.\,,\nu)$ *is a Radon measure. This formulation is useful to look at the local behavior of the functional and without considering boundary values.*

2. *With a given boundary condition the boundary values have to be tracked. In that case, one usually assumes* $\mathcal{L}^n(\partial\Omega) = 0$, $\Psi \in \mathrm{L}^1(\Omega)$ *and considers only finite Radon measure* $\nu \in \mathrm{RM}(\overline{\Omega}, \mathbb{R}^n)$ *with* $X = \overline{\Omega}$ *for some open set* Ω*. Here,* $g(\,.\,,\nu)$ *is even a finite Radon measure.*

Back to our BV setting for functionals with convex integrands of linear growth, we can plug in the derivative measure of a BV function and get dependent on the boundary constraint the following definitions:

Definition 2.34 (Functionals on BV)**.**
For an open bounded set Ω, $f : \mathbb{R}^n \to \mathbb{R}$ convex and with $a|z| \leq f(z) \leq b(1+|z|)$ for some positive constants a, b, we define for $u \in \mathrm{BV}(\Omega)$:

$$\mathcal{F}[u] := \int\limits_{\Omega} f(\mathrm{D}^a u)\,\mathrm{d}x + \int\limits_{\Omega} f^\infty\left(\frac{\mathrm{D}^s u}{|\mathrm{D}^s u|}\right) \mathrm{d}|\mathrm{D}^s u|. \tag{2.10}$$

If in addition boundary values are prescribed by a function $u_0 \in \mathrm{L}^1(\partial\Omega)$ on a suitable domain, i.e. at least $|\partial\Omega| = 0$, or by considering $\mathrm{BV}_{u_0}(\Omega)$ for some $u_0 \in \mathrm{BV}(\mathbb{R}^n)$, we define

$$\mathcal{F}_{u_0}[u] := \int\limits_{\Omega} f(\mathrm{D}^a u)\,\mathrm{d}x + \int\limits_{\Omega} f^\infty\left(\frac{\mathrm{D}^s u}{|\mathrm{D}^s u|}\right) \mathrm{d}|\mathrm{D}^s u| + \int\limits_{\partial\Omega} f^\infty(\nu_\Omega(u_0 - u_{\mathrm{int}}))\,\mathrm{d}\mathcal{H}^{n-1}. \tag{2.11}$$

for $u \in \mathrm{BV}(\Omega)$ and with the exterior normal vector ν_Ω to the domain Ω.

Remark.

1. The last term in (2.11) can be understood as a penalizing term for wrong boundary values. Since weak* convergence does not preserve boundary values, there is no way to enforce them in general settings but only to penalize the violation of the boundary condition. The term itself is the portion of the singular part of (2.9) contained in the boundary $\partial\Omega$.
2. These representations hold for integrands g with x-dependency as well as under the assumptions of Definition 2.32 with the versions explained in Corollary 2.33.

In fact, the functionals (2.10) and (2.11) are the relaxation of the associated $\mathrm{W}^{1,1}$-functionals. To be able to prove that and before we define and explain relaxation more detailed, we need to provide some convergence theorems for such functionals. Prior to this, we state a more generalized version of the co-area formula for BV functions, which can be found in [17, Lemma 2.4]:

Theorem 2.35 (Generalized co-area formula).
For an open set $\Omega \subset \mathbb{R}^n$, $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and a Borel function $g : \Omega \times \mathbb{R}^n \to [0, +\infty]$, which is convex and positively 1*-homogeneous as a function $z \mapsto g(x, y, z)$ for all $(x, y) \in \Omega \times \mathbb{R}$, we have*

$$\int_\Omega g(x, \mathrm{D}u) = \int_{\mathbb{R}} \left(\int_\Omega g\left(x, \frac{\mathrm{D}\mathbb{1}_{U_t}}{|\mathrm{D}\mathbb{1}_{U_t}|}\right) \mathrm{d}|\mathrm{D}\mathbb{1}_{U_t}| \right) \mathrm{d}t$$

with $U_t = \{x \in \Omega : u(x) \geq t\}$.

This co-area formula especially holds for integrands $f : \mathbb{R}^n \to [0, +\infty]$ which are convex and positively 1-homogeneous and, as shown in [17, Lemma 2.4], even holds, if the integrand is additionally dependent on the function u itself.

2.3 Reshetnyak's theorems and area-strict convergence

The lower semicontinuity theorem and continuity theorem for functionals on measures were provided by Reshetnyak in his paper in [57] for positively 1-homogeneous functionals. With the construction in the last section the semicontinuity theorem can be transferred without much change, but for the continuity theorem the strict convergence used in the positively 1-homogeneous case must be adjusted.

Definition 2.36 (Area-strict convergence for Radon measures).
For a locally compact subset X of $\mathbb{R}^n$, we say that ν_k converges area-strict in $\mathrm{RM}(X, \mathbb{R}^n)$ to ν if ν_k converges weakly* to ν in $\mathrm{RM}(X, \mathbb{R}^n)$ and in addition

$$\lim_{k \to \infty} \int_X \sqrt{1 + |\nu_k|^2} = \int_X \sqrt{1 + |\nu|^2}. \tag{2.12}$$

Remark. Area-strict convergence is the strict convergence for the vector-valued measure of ν_k with $\mathcal{L}^n$, i.e. it can be written as $|(\mathcal{L}^n, \nu_k)|(X) \to |(\mathcal{L}^n, \nu)|(X)$ for $k \to \infty$. For BV functions, this is analogous to $\mathcal{A}[u_k] \to \mathcal{A}[u]$ with the addition of $u_k \to u$ in $\mathrm{L}^1(\Omega)$ for $k \to \infty$.

Indeed the area-strict convergence is stronger than strict convergence. To see that, the following example as presented in [59, in Section 2.2] may be used:
Consider on the interval $(0, 1)$ the sequence $u_k(x) = x + \frac{1}{2\pi k}\sin(2\pi k x)$ in $\mathrm{W}^{1,1}((0, 1))$. Then $\nabla u_k(x)$ is given by $1 + \cos(2\pi k x)$ and we can compute:

1. $u_k \to u$ in $\mathrm{L}^1((0, 1))$ with $u(x) = x$.
2. $u_k \to u$ weakly* in $\mathrm{W}^{1,1}((0, 1)) \subset \mathrm{BV}((0, 1))$.
3. $$\int_{(0,1)} |\nabla u_k| \,\mathrm{d}x = \int_{(0,1)} 1 + \cos(2\pi k x) \,\mathrm{d}x = 1 + \frac{1}{2\pi k}\left[\sin(2\pi k x)\right]_0^1 \overset{k\to\infty}{\longrightarrow} 1 = \int_{(0,1)} |\nabla u| \,\mathrm{d}x.$$

4. Using Jensen's inequality on the convex function $t \mapsto \sqrt{1+t^2}$, we obtain

$$\begin{aligned}\int\limits_{(0,1)} \sqrt{1+(1+\cos(2\pi k x))^2}\,\mathrm{d}x &= \frac{1}{2\pi k} \int\limits_{(0,2\pi k)} \sqrt{1+(1+\cos(x))^2} \\ &= \frac{1}{2\pi} \int\limits_{(0,2\pi)} \sqrt{1+(1+\cos(x))^2}\,\mathrm{d}x \\ &> \left(1+\left(\frac{1}{2\pi}\int\limits_{(0,2\pi)} 1+\cos(x)\,\mathrm{d}x\right)^2\right)^{\frac{1}{2}} \\ &= \sqrt{1+1} = \sqrt{2} = \int\limits_{(0,1)} \sqrt{1+1}\,\mathrm{d}x,\end{aligned}$$

where we have the strict inequality, since the requirement for the equality in the Jensen inequality is not fulfilled, here: $z \mapsto \sqrt{1+|z|^2}$ is strict convex and $1+\cos(x)$ is not constant.

As we will see later, area-strict convergence implies strict convergence and thus the statement is proven. One can interpret this fact the following way: In general, strict convergence in BV does not prohibit cancelling oscillation effects as presented in the example, while the area integral with a strict convex integrand accounts for such effects.
Next, we can state both Reshetnyak's theorems in different versions following [64, Section 2.2]. The first stated result is the lower semicontinuity theorem for the positively 1-homogeneous case from which the general case follows according to the generalization presented in the last section. For more details, see for example [3, Theorem 2.38].

Theorem 2.37 (Lower semicontinuity theorem, part I)**.**
For an open set $\Omega \subset \mathbb{R}^n$ and a lower semicontinuous function $h : \Omega \times \mathbb{R}^n \to [0, +\infty]$, which is convex and positively 1-homogeneous in the second argument, we have the following estimate: If ν_k converges locally weakly to ν in $\mathrm{RM}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, then*

$$\liminf_{k\to\infty} \int\limits_{\Omega} h(\,.\,,\nu_k) \geq \int\limits_{\Omega} h(\,.\,,\nu) \tag{2.13}$$

holds.

As mentioned, this result easily extends to the non-homogeneous version.

Corollary 2.38 (Lower semicontinuity theorem, part II)**.**
For an open set $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| = 0$ and a lower semicontinuous function $g : \Omega \times \mathbb{R}^n \to [0, \infty]$, which is convex in the second argument, we have the following estimate: If ν_k converges locally weakly to ν in $\mathrm{RM}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, then*

$$\liminf_{k\to\infty} \int\limits_{\overline{\Omega}} g(\,.\,,\nu_k) \geq \int\limits_{\overline{\Omega}} g(\,.\,,\nu). \tag{2.14}$$

Similarly, for the BV case, we have:

Corollary 2.39 (Lower semicontinuity theorem, part III).
For an open set $\Omega \subset \mathbb{R}^n$ *with* $|\partial\Omega| = 0$ *and a lower semicontinuous function* $g : \Omega \times \mathbb{R}^n \to [0, \infty]$*, which is convex in the second argument, we have the following estimate: If* u_k *converges weakly* to* u *in* $\mathrm{BV}(\Omega)$*, then*

$$\liminf_{k\to\infty} \int_{\overline{\Omega}} g(\,.\,, \mathrm{D}u_k) \geq \int_{\overline{\Omega}} g(\,.\,, \mathrm{D}u). \tag{2.15}$$

Remark. This statement coincides with what we defined and know about boundary value problems: If for a suitable domain Ω, the sequence $u_k \in \mathrm{BV}_{u_0}(\Omega)$ converges weakly* to some $u \in \mathrm{BV}(\Omega)$ which does not satisfy the boundary condition, in the term on the left-hand side the closure of Ω may be dropped, i.e. the functional over Ω and the functional over $\overline{\Omega}$ have the same value, but it can still appear on the right-hand side.
Further, the closure on both sides may be dropped as presented in [3, Theorem 2.38] for finite Radon measures even if the sequence does not have the right boundary values.

In contrast to the lower semicontinuity theorems, the continuity theorem provides an explicit value for strict convergent sequences of Radon measures in the homogeneous case, see for example [3, Theorem 2.39]:

Theorem 2.40 (Continuity theorem, part I).
For an open set $\Omega \subset \mathbb{R}^n$*, a continuous and in the second argument positively* 1*-homogeneous function* $h : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *satisfying*

$$|h(x, z)| \leq \Gamma|z| \text{ for } (x, z) \in \Omega \times \mathbb{R}^n$$

for some constant $\Gamma < +\infty$ *and a sequence* ν_k *converging strictly in* $\mathrm{RM}(\Omega, \mathbb{R}^n)$ *to* ν*, we have*

$$\lim_{k\to\infty} \int_{\Omega} h(\,.\,, \nu_k) = \int_{\Omega} h(\,.\,, \nu). \tag{2.16}$$

For the non-homogeneous version, the strict convergence has to hold for (t, z), implying that the sequence of Radon measures has to converge in area. Thus, the theorem can be written as:

Corollary 2.41 (Continuity theorem, part II).
For an open set $\Omega \subset \mathbb{R}^n$ *with* $|\Omega| < +\infty$*, a continuous function* $g : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *satisfying*

$$|g(x, z)| \leq \Gamma(1 + |z|) \text{ for } (x, z) \in \Omega \times \mathbb{R}^n$$

for some constant $\Gamma < +\infty$ *and such that the strong recession function* g^∞ *of* g *exists, we have for a sequence* ν_k *converging area-strictly in* $\mathrm{RM}(\Omega, \mathbb{R}^n)$ *to* ν *the following equality:*

$$\lim_{k\to\infty} \int_{\Omega} g(\,.\,, \nu_k) = \int_{\Omega} g(\,.\,, \nu). \tag{2.17}$$

Remark.

1. The additional continuity assumption on g is rather obstructive, since it is required in both arguments and not only the last one. This restriction was removed by Kristensen and Rindler in [46] and further generalized by Beck and Schmidt in [6, Theorem 3.10] to merely measurability in the first argument. The version is presented after the remarks.
2. With the first remark in mind, the prerequisite of the lower semicontinuity theorem and continuity theorem differ by the kind of convergence and by the convexity or continuity assumption in the second variable. In many cases considered, all requirements are fulfilled.
3. In many cases, the area-strict convergence for the continuity theorem can be replaced by Ψ-strict convergence, i.e. $\sqrt{\Psi + |\,.\,|^2}$ is used instead of $\sqrt{1 + |\,.\,|^2}$, with $\Psi \in \mathrm{L}^1(\Omega)$ bounded away from 0 on every bounded subset of Ω or similar requirements. This is very important on sets Ω with infinite Lebesgue measure, i.e. where $|\Omega| = \infty$, since for sets with finite Lebesgue measure one can choose $\Psi \equiv 1$. Indeed, the Ψ-strict and area strict convergence are equivalent on sets with finite Lebesgue measure and we will thus only consider the area-strict convergence on such sets and thus especially on bounded sets Ω.
4. All results regarding recession functions, functionals on measures and (semi-)continuity results for Radon measures from the last and this subsection as well as the following version of continuity theorem on Radon measures hold for $\mathbb{R}^m$-valued Radon measures as well with the basically the same definitions and proofs. We choose here to state only the $\mathbb{R}^n$-valued cases, since we are mostly interested in functionals on BV, where the dimensions of the domain and target space of the Radon measure coincide because of the gradient structure.

Theorem 2.42 (Continuity theorem, part III).
For an open subset $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| = 0$, a function $g : \Omega \times \mathbb{R}^n \to \mathbb{R}$ such that

- *$g(\,.\,, z)$ is measurable for all $z \in \mathbb{R}^n$ and such that $g(x, \,.\,)$ is continuous for all $x \in \Omega$ (i.e. the function is a Carathéodory function),*
- *fulfills*
$$|g(x, z)| \le \Psi(x) + \Gamma|z| \text{ for all } (x, z) \in \Omega \times \mathbb{R}^n,$$
with a constant $\Gamma < \infty$ and some $\Psi \in \mathrm{L}^1(\Omega)$, which is bounded away from 0 on every bounded subset of Ω, and
- *has a strong recession function g^∞,*

and some ν_k converging Ψ-strictly to ν in $\mathrm{RM}(\overline{\Omega}, \mathbb{R}^n)$, we have

$$\lim_{k\to\infty} \int_{\overline{\Omega}} g(\,.\,, \nu_k) = \int_{\overline{\Omega}} g(\,.\,, \nu). \tag{2.18}$$

Again we can use this theorem for the BV case and obtain:

Corollary 2.43 (Continuity theorem, part IV).
For an open subset $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| = 0$, a function $g : \Omega \times \mathbb{R}^n \to \mathbb{R}$ like in the theorem before

and some u_k converging Ψ-strictly to u in $\mathrm{BV}(\overline{\Omega})$*, we have*

$$\lim_{k\to\infty} \int_{\overline{\Omega}} g(\,.\,, \mathrm{D}u_k) = \int_{\overline{\Omega}} g(\,.\,, \mathrm{D}u). \tag{2.19}$$

Remark. This result also proves that area-strict convergence in BV implies strict convergence in BV and with the counterexample provided, that merely strict convergence does not suffice for the continuity theorems in the non-homogeneous case.

For more details and proofs on the advanced versions, consider reading [6, Theorem 3.10], [64, Section 2.2], [46]. Some simple proofs and some different versions are also presented in [66] and a further discussion in the homogeneous case can be found in [21].
To be able to generate area-strict convergent sequences for BV functions and to use the continuity theorems for BV functions, the following result was proven in [13] and is helpful for obstacle problems, since it allows to approximate BV functions in area with $\mathrm{W}^{1,1}$-functions:

Theorem 2.44 (Area-strict approximation from above, see [13, Theorem 3.3])**.**
Let Ω be a bounded open set of $\mathbb{R}^n$ and $u \in \mathrm{BV}(\Omega)$. Then there exists a sequence $u_k \in \mathrm{W}^{1,1}(\Omega)$ converging in area to u on Ω such that $u_k^* \geq u^+$ $\mathcal{H}^{n-1}$-a.e. on Ω.

This theorem implies the following density property, which also can be proven much easier without the last theorem:

Corollary 2.45 (Density with respect to area-strict convergence)**.**
For open bounded sets Ω, the space $\mathrm{W}^{1,1}(\Omega)$ is dense in $\mathrm{BV}(\Omega)$ with respect to convergence in area.

Another important convergence theorem is provided in [58, Lemma 11.1], see also [46, Lemma 1]:

Theorem 2.46 (Area-strict approximation)**.**
For each bounded open Ω and $u \in \mathrm{BV}(\Omega)$, there exists a sequence $u_k \in \mathrm{W}^{1,1}_u(\Omega) \cap \mathrm{C}^\infty(\Omega)$ with $u_k = u$ on $\partial\Omega$ and $u_k \to u$ area-strict in $\mathrm{BV}(\Omega)$. If u is additionally in $\mathrm{W}^{1,1}(\Omega)$, the sequence u_k can be chosen such that it converges in $\mathrm{W}^{1,1}(\Omega)$ to u.

For the proof, see [58, Lemma 11.1] or [46, Appendix A]. The proofs show, that for $u \in \mathrm{L}^\infty(\partial\Omega)$ the sequence can be chosen such that each element is in L^∞ in addition to the other constraints and for $u \in \mathrm{L}^\infty(\Omega)$ the sequence can be chosen bounded in $\mathrm{L}^\infty(\Omega)$ with $\sup ||u_k||_\infty \leq ||u||_\infty$, see for example [59, Proposition 2.3].
As mentioned, the space $\mathrm{BV}_{u_0}(\Omega)$ for suitable Ω is not closed, taking both traces into account. This plays an important role for boundary value problems and thus an approximation for any $u \in \mathrm{BV}(\Omega)$ in $\mathrm{BV}_{u_0}(\Omega)$ is needed. This is no problem for Lipschitz domains, but is also possible for domains with a weaker condition to the boundary, see [6, Lemma 3.12] and [65, Theorem 1.2]:

Theorem 2.47 (Area-strict approximation with fixed boundary)**.**
For a bounded domain $\Omega \subset \mathbb{R}^n$ with $\mathbb{1}_\Omega \in \mathrm{BV}(\mathbb{R}^n)$ and $|\mathrm{D}\mathbb{1}_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega$, a given $u_0 \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and a function $u \in \mathrm{BV}(\Omega)$ with $\mathbb{1}_\Omega u \in \mathrm{BV}(\mathbb{R}^n)$, there exists a sequence $u_k \in \mathrm{W}^{1,1}_{u_0}(\Omega)$ converging

in area to u on $\overline{\Omega}$, i.e.

$$u_k \longrightarrow u \textit{ in } \mathrm{L}^1,$$
$$\int\limits_{\overline{\Omega}} \sqrt{1+|\mathrm{D}u_k|^2}\,\mathrm{d}x \longrightarrow \int\limits_{\overline{\Omega}} \sqrt{1+|\mathrm{D}u|^2}.$$

In addition, the sequence may be chosen such that $\nabla u_k \to \mathrm{D}^a u$ $\mathcal{L}^n$-a.e.

Remark. The results in Theorem 2.47 may be considered on unbounded Ω. For that, the area-strict convergence must be switched to Ψ-strict convergence with a non-negative function $\Psi \in \mathrm{L}^1(\Omega)$. Further, in both cases the sequence can be chosen in $u_0 + \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega) \subset \mathrm{W}^{1,1}_{u_0}(\Omega)$.

Next, we state the definition of the process of relaxation.

2.4 Relaxation of Dirichlet and obstacle problems

A main tool to treat variational problems is the method of relaxation. The underlying idea is to replace a given functional with a lower semicontinuous one which attains the same infimum or minimum. For example, as seen in the introduction, lower dimensional obstacles are not preserved in general, but the values of the functional may be bounded from below on the class of functions considering the obstacle. Another example is minimization in Dirichlet classes or in general with boundary conditions. For example, in the space BV the boundary values are preserved for strictly convergent sequences, but a minimal sequence does not have to have a strictly convergent subsequence. As mentioned, the Sobolev space $\mathrm{W}^{1,1}$ is not closed under the typical convergences and thus functionals may not attain the infimum in $\mathrm{W}^{1,1}$ and thus the relaxation will involve BV functions. In all mentioned cases, a larger space has to be considered, but to do so, the initial functionals often have to be extended to a larger space. In case of $\mathrm{W}^{1,1}$, the relaxation is often constructed in the space L^1.
For a general functional on a metric space X, the relaxation can be defined in the following way, see for example [13, Remark 6.1] or [5, Chapter 11]:

Definition 2.48 (Relaxation)**.**
The relaxation $\overline{\mathrm{F}}$ of a given (proper) functional F on a metric space X with values in $\mathbb{R} \cup \{+\infty\}$ is uniquely determined by the following two conditions:

a) For every $u \in X$ and every sequence $(u_k)_{k\in\mathbb{N}}$ converging to u in X, we have

$$\overline{\mathrm{F}}[u] \leq \liminf_{k\to\infty} \mathrm{F}[u_k].$$

b) For every $u \in X$, there exists a sequence $(u_k)_{k\in\mathbb{N}}$ converging to u in X with

$$\overline{\mathrm{F}}[u] \geq \limsup_{k\to\infty} \mathrm{F}[u_k].$$

Equivalently, the relaxation of such a functional F can be defined by

$$\overline{\mathrm{F}} = \inf \left\{ \liminf_{k\to\infty} \mathrm{F}[u_k] \,:\, (u_k)_{k\in\mathbb{N}} \text{ converges to } u \text{ in } X \right\}.$$

Remark.

1. A function f is called proper, if $f > -\infty$ and there exists at least some $x \in X$ such that $f(x) < +\infty$.
2. The property noted in a) in Definition 2.48 ensures the lower semicontinuity of $\overline{\mathrm{F}}$. The sequence in b), which is required to exist, is often called recovery sequence and the existence enforces, that the infimum of the values of the functional and its relaxation are the same.
3. The relaxation can be seen as a special case of Γ-convergence with only constant sequences of functionals. For more details, see [5, Chapter 12], and on general notes to Γ-convergence [9] or [48] may be recommended.
4. In general, we are going to use the space L^1 with the strong topology, i.e. the topology induced by the L^1-norm, to define our functionals and compute their relaxation. Theoretically, the space BV could also be considered using the weak* convergence without much change.

Since we will work with convex functions and integrands, we skip convex and quasiconvex envelopes which are often presented in combination with relaxation and are (partially) contained, for example, in [5] and are well known aiming experts and instead study some examples.
In the first example, we investigate the relaxation of the area functional in L^1 without any additional constraints:
Example 1: Consider an open bounded set Ω and the area functional on $\mathrm{L}^1(\Omega)$ given by

$$\mathrm{A}[u] = \begin{cases} \int\limits_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x & \text{for } u \in \mathrm{W}^{1,1}(\Omega), \\ +\infty & \text{for } u \in \mathrm{L}^1(\Omega) \setminus \mathrm{W}^{1,1}(\Omega). \end{cases}$$

Then the relaxation is given by $\overline{\mathrm{A}} = \mathcal{A}$ with

$$\mathcal{A}[u] = \begin{cases} \int\limits_\Omega \sqrt{1+|\mathrm{D}u|^2} & \text{for } u \in \mathrm{BV}(\Omega), \\ +\infty & \text{for } u \in \mathrm{L}^1(\Omega) \setminus \mathrm{BV}(\Omega). \end{cases}$$

Proof. For a sequence $u_k \in \mathrm{L}^1(\Omega)$ converging with respect to the strong topology to $u \in \mathrm{L}^1$, we can encounter the case where the limit is a function in $\mathrm{L}^1(\Omega)\setminus\mathrm{BV}(\Omega)$. Here, the sequence consists either of $\mathrm{L}^1(\Omega)$ functions, where the functional A has value $+\infty$ and thus the limit in a) of Definition 2.48 is $+\infty$, an unbounded sequence in $\mathrm{W}^{1,1}(\Omega)$ with unbounded $||\nabla u_k||_1$, thus the limit is again $+\infty$, or a combination of both. In both cases, the limit is $+\infty$. Since $u_k \to u$ in L^1, $||u_k||_1$ is always bounded and thus the case with unbounded $||u_k||_1$ cannot occur. If the limit u is in $\mathrm{BV}(\Omega)$, we have either a sequence with $u_k \in \mathrm{L}^1(\Omega) \setminus \mathrm{W}^{1,1}(\Omega)$ and thus $\liminf_{k\to\infty} \mathrm{A}[u_k] = +\infty \geq \mathcal{A}[u]$, a sequence in $\mathrm{W}^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ for which we can use Theorem 2.39 to prove lower semicontinuity or we have the combination of both. In conclusion, the provided functional $\mathcal{A}$ fulfills the lower semicontinuity requirement for the relaxation. For the second part, we use Corollary 2.45 to obtain a sequence $u_k \in \mathrm{W}^{1,1}(\Omega)$ with $u_k \to u$ in area and thus especially in $\mathrm{L}^1(\Omega)$, see remark to Definition 2.36. For this sequence, we get, by applying Corollary 2.43, that it is the desired sequence, which fulfills b) in Definition 2.48. □

The second example deals with an additional boundary value constraint again for the area functional:

Example 2: Assume Ω is a bounded Lipschitz domain, which is why we can compute a trace on $\partial\Omega$ and let $u_0 \in \mathrm{W}^{1,1}(\Omega)$ define the boundary values via the Dirichlet class $\mathrm{W}^{1,1}_{u_0}(\Omega)$. The relaxation of

$$\mathrm{A}_{u_0}[u] = \begin{cases} \int\limits_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x & \text{for } u \in \mathrm{W}^{1,1}_{u_0}(\Omega), \\ +\infty & \text{for } u \in \mathrm{L}^1(\Omega) \setminus \mathrm{W}^{1,1}_{u_0}(\Omega) \end{cases}$$

is given by $\overline{\mathrm{A}}_{u_0} = \mathcal{A}_{u_0}$ with

$$\mathcal{A}_{u_0}[u] = \begin{cases} \int\limits_\Omega \sqrt{1+|\mathrm{D}u|^2} + \int\limits_{\partial\Omega} |u_0 - u_{\mathrm{int}}|\,\mathrm{d}\mathcal{H}^{n-1} & \text{for } u \in \mathrm{BV}(\Omega), \\ +\infty & \text{for } u \in \mathrm{L}^1(\Omega) \setminus \mathrm{BV}(\Omega). \end{cases}$$

Proof. The first part can be proven like in *Example 1.* Using Theorem 2.46 instead of Corollary 2.45, we obtain the desired functional for $u \in \mathrm{BV}_{u_0}(\Omega)$ and $u \in \mathrm{L}^1(\Omega) \setminus \mathrm{BV}(\Omega)$. For this, note that since Ω is a Lipschitz domain and by Definition 2.16 for the Dirichlet classes and $u \in \mathrm{BV}_{u_0}(\Omega)$, we have

$$\int\limits_{\Omega} \sqrt{1+|\mathrm{D}u|^2} = \int\limits_{\Omega} \sqrt{1+|\mathrm{D}u|^2} + \int\limits_{\partial\Omega} |u_0 - u_{\mathrm{int}}|\,\mathrm{d}\mathcal{H}^{n-1}.$$

For $u \in \mathrm{BV}(\Omega) \setminus \mathrm{BV}_{u_0}(\Omega)$, we note that the weak* closure of $\mathrm{BV}_{u_0}(\Omega)$ is $\mathrm{BV}(\Omega)$ and explicit approximations can be construction like in [7, Lemma B.2]. □

On more general open domains a similar result may be achieved:

Example 2': The result in *Example 2* remains true for domains Ω satisfying $\mathbb{1}_\Omega \in \mathrm{BV}(\mathbb{R}^n)$ and $|\mathrm{D}\mathbb{1}_\Omega| = \mathcal{H}^{n-1} \llcorner \partial\Omega$. Here, instead of the construction in [7, Lemma B.2] the last step is provided by Theorem 2.47.

The last example is concerned with obstacles to hold $\mathcal{L}^n$-a.e.

Example 3: If in the setting of *Example 1* or *Example 2* the additional lower obstacle constraint with a Borel obstacle function ψ is added, we obtain that the relaxation of

$$\mathrm{A}^\psi[u] = \begin{cases} \int\limits_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x & \text{for } u \in \{w \in \mathrm{W}^{1,1}(\Omega) \,:\, w \geq \psi \text{ holds } \mathcal{L}^n\text{-a.e.}\}, \\ +\infty & \text{for } u \in \mathrm{L}^1(\Omega) \setminus \{w \in \mathrm{W}^{1,1}(\Omega) \,:\, w \geq \psi \text{ holds } \mathcal{L}^n\text{-a.e.}\} \end{cases}$$

is given by $\overline{\mathrm{A}}^\psi = \mathcal{A}^\psi$ with

$$\mathcal{A}^\psi[u] = \begin{cases} \int\limits_\Omega \sqrt{1+|\mathrm{D}u|^2} & \text{for } u \in \{w \in \mathrm{BV}(\Omega) \,:\, w \geq \psi \text{ holds } \mathcal{L}^n\text{-a.e.}\}, \\ +\infty & \text{for } u \in \mathrm{L}^1(\Omega) \setminus \{w \in \mathrm{BV}(\Omega) \,:\, w \geq \psi \text{ holds } \mathcal{L}^n\text{-a.e.}\} \end{cases}$$

and the relaxation of

$$\mathrm{A}^\psi_{u_0}[u] = \begin{cases} \int\limits_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x & \text{for } u \in \{w \in \mathrm{W}^{1,1}_{u_0}(\Omega) \,:\, w \geq \psi \text{ holds } \mathcal{L}^n\text{-a.e.}\}, \\ +\infty & \text{for } u \in \mathrm{L}^1(\Omega) \setminus \{w \in \mathrm{W}^{1,1}_{u_0}(\Omega) \,:\, w \geq \psi \text{ holds } \mathcal{L}^n\text{-a.e.}\} \end{cases}$$

is given by $\overline{\mathrm{A}}_{u_0}^{\psi} = \mathcal{A}_{u_0}^{\psi}$ with

$$\mathcal{A}_{u_0}^{\psi}[u] = \begin{cases} \mathcal{A}^{\psi}[u] + \int\limits_{\partial\Omega} |u_0 - u_{\mathrm{int}}| \, \mathrm{d}\mathcal{H}^{n-1} & \text{for } u \in \{w \in \mathrm{BV}(\Omega) \, : \, w \geq \psi \text{ holds } \mathcal{L}^n\text{-a.e.}\}, \\ +\infty & \text{for } u \in \mathrm{L}^1(\Omega) \setminus \{w \in \mathrm{BV}(\Omega) \, : \, w \geq \psi \text{ holds } \mathcal{L}^n\text{-a.e.}\}. \end{cases}$$

The proof is identical to the ones of *Example 1 and 2* with the specification that L^1 convergence preserves the obstacle inequality for the limit and finer arguments involving the restriction of using area-strict approximation from above from Theorem 2.44 instead of approximation preserving the trace from Theorem 2.46. This can be handled using a similar approach like in the last part of the proof for *Example 2*.
The case with the double obstacle problem to hold $\mathcal{L}^n$-a.e. for two Borel obstacles $\psi_1 \leq \psi_2$ can be treated similarly using an additional truncation argument, which keeps the area-strict sequence in between the obstacles. This result can be also established by using the argument presented in Section 4.

Remark. For the readability, we write for the relaxation of functionals involving obstacle term $\overline{\mathrm{F}}^{\psi}$ instead of $\overline{\mathrm{F}^{\psi}}$ and so on.

For more general notes on relaxation for the BV setting, the articles [2], [8], [17], [30] and [45] and the book [10] may be considered. They are adding to the theory among many other sources.

3 The De Giorgi measure and the single obstacle problem

3.1 The De Giorgi measure and a counterexample by Hutchinson

In this section, we take a closer look at the De Giorgi measure, which is essential for the relaxation of parametric and non-parametric obstacle problems. We start with the definitions and variations provided in [20, Chapter 4] and [15].

Definition 3.1 (De Giorgi measure)**.**
For a set $E \subset \mathbb{R}^n$ and $\delta > 0$, let

$$\varsigma_\delta(E) = \inf\left\{\mathrm{P}(B) + \frac{|B|}{\delta} \,:\, E \subset B,\, B \text{ open}\right\} \tag{3.1}$$

be the δ-De Giorgi measure, where P is the perimeter, and, as for the Hausdorff measure, define the De Giorgi measure ς by

$$\varsigma(E) = \lim_{\delta \searrow 0} \varsigma_\delta(E) = \sup_{\delta > 0} \varsigma_\delta(E). \tag{3.2}$$

In the original article, the set B had to be in the class $\mathcal{G}_n$, which fixes the representative of a given set sufficiently for their use of it. More precisely, a set B was in $\mathcal{G}_n$ iff the following properties were fulfilled:

$$\begin{aligned} &\lim_{r \searrow 0} r^{-n}|B \cap \mathrm{B}_r(x)| = 0 \quad &&\Rightarrow x \notin B, \\ &\lim_{r \searrow 0} r^{-n}|\mathrm{B}_r(x) \setminus B| = 0 \quad &&\Rightarrow x \in B. \end{aligned}$$

Colombini then proved, that the definition provided with open sets is equivalent to the one with the class $\mathcal{G}_n$. There exist different approaches to De Giorgi's measure, which we will see later when a generalized version is presented.
In [20], many extended results were already proven like the fact that ς is a Borel regular measure and a first estimate of the type

$$c(n)\mathcal{H}^{n-1} \leq \varsigma \leq C(n)\mathcal{H}^{n-1} \tag{3.3}$$

with two constants c, C depending only on the dimension n. At this stage, it was not clear, whether the introduced measure is equal to $2\mathcal{H}^{n-1}$ or not. Also in [20] was proven that for a set E contained in a countable union of C^1-hypersurfaces, which can easily be generalized to E being $\mathcal{H}^{n-1}$-rectifiable, the equality

$$\varsigma(E) = 2\mathcal{H}^{n-1}(E) \tag{3.4}$$

holds. Still it seemed not possible to prove a formula for the relaxation of the parametric double obstacle problem with the Hausdorff measure instead of the De Giorgi measure. It turned out that a set could be found in Federer's book [25, 2.10.28 and 3.3.20] (with $m = \frac{1}{2}$) and was proven to be a counterexample for the equality (3.4) to hold for general Borel sets by Hutchinson in [41]. The constructed set is a Cantor dust, i.e. a 1-dimensional Cantor set in $\mathbb{R}^2$. The main idea behind

this construction is to provide a sequence of sets, where each set has a circumference of 4, but can be rearranged such that the set converges to the diagonal of the unit square with length $\sqrt{2}$ and therefore indicates that the Hausdorff measure is equal to $\sqrt{2}$. In the following, we introduce the constructed set and revisit the (sketch of the) proof given by Hutchinson in [41].
First, we define $I_0 = [0, 1]$ and set

$$I_k = \left(\frac{1}{4}I_{k-1}\right) \cup \left(\frac{3}{4} + \frac{1}{4}I_{k-1}\right) \quad \forall k \in \mathbb{N}.$$

For $k \in \mathbb{N}_0$, we further set

$$C_k = I_k \times I_k,$$
$$C = \bigcap_{k=0}^{\infty} C_k$$

and thus obtain a decreasing sequence of sets with $C \subset \ldots \subset C_{k+1} \subset C_k \subset C_{k-1} \subset \ldots \subset C_0$.

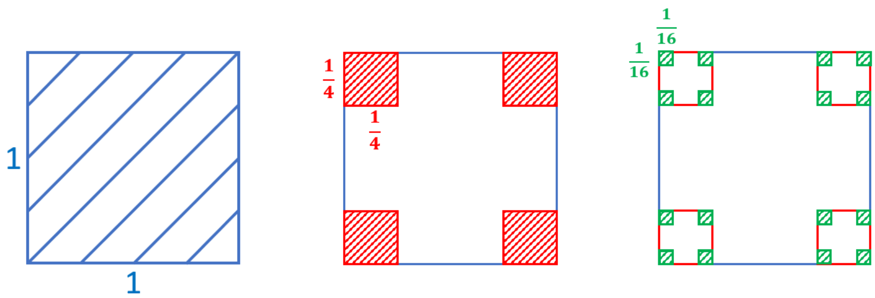


Figure 1: construction steps k=0, 1, 2 (from left to right)

For this purely $\mathcal{H}^1$-unrectifiable set, we are able to prove the following first estimate:

Theorem 3.2 (Hausdorff measure of C)**.**
The Hausdorff measure of the set C is bounded from above by $\sqrt{2}$.

Proof. Since in each construction step the set C_k consists of 4^k squares with side length 4^{-k} and diagonal length $\sqrt{2} \cdot 4^{-k}$, we can estimate the Hausdorff premeasure $\mathcal{H}^1_\delta(C)$:
By the choice of k_0 such that $\sqrt{2} \cdot 4^{-k_0} < \delta$ and by covering each square with a ball of diameter $\sqrt{2} \cdot 4^{-k_0}$, we obtain:

$$\mathcal{H}^1_\delta(C) \leq \mathcal{H}^1_\delta(C_{k_0}) \leq 4^{k_0} 4^{-k_0} \sqrt{2} = \sqrt{2}.$$

With $\mathcal{H}^1(C) = \lim_{\delta \searrow 0} \mathcal{H}^1_\delta(C) \leq \sqrt{2}$, the result is proven. □

Indeed, in [25, 3.3.20] it is shown, that $\mathcal{H}^1(C) = \sqrt{2}$, but because of the length of the proof and because it is not necessary for our primary goal to disprove $\varsigma = 2\mathcal{H}^1$, this is not proven here. Next, we calculate the value of $\varsigma(C)$:

Theorem 3.3 (De Giorgi measure of C)**.**
For C as given above, the De Giorgi measure of C is equal to 4.

The proof is divided into two parts, the first provides the inequality $\varsigma(C) \leq 4$, which is not necessarily needed but easy to obtain:

Proof. Part I: We first observe that $\mathrm{P}(C_k) = 4$ and $|C_k| = 4^{-k}$ for all $k \in \mathbb{N}_0$. For a given $\varepsilon > 0$ and arbitrary $k \in \mathbb{N}$, we thus can find a suitable open set B_k with $C \subset C_k \subset B_k$, $\mathrm{P}(B_k) \leq 4 + \frac{\varepsilon}{2}$ and $|B_k| \leq 4^{-k} + \frac{\varepsilon\delta}{2}$. With that we obtain by monotonicity:

$$\varsigma_\delta(C) \leq 4 + \frac{1}{4^k \delta} + \varepsilon \quad \forall k \in \mathbb{N} \text{ and } \varepsilon > 0,$$

which implies $\varsigma_\delta(C) \leq 4$ ($k \to \infty$ and $\varepsilon \searrow 0$) and hence $\varsigma(C) \leq 4$.

Part II: For the second part, we need some auxiliary statements and definitions. The main idea is to prove that every open set containing C contains the set C_k for some k and estimate the perimeter using an approximation of the normal vector field to each square contained in C_k. This is accomplished in the following steps:

1. For any open set A with $C \subset A$, there exists $k \in \mathbb{N}$ such that $C_k \subset A$.

   Proof. Assume that $C_k \setminus A \neq \emptyset$ for all k. Then there exists a sequence $x_k \in C_k \setminus A$ with a converging subsequence $x_k \to x$, since all C_k are compact. Because x is in $C \subset A$ and A is open, we reach a contradiction. □

2. Next, we look at the construction of the set and fix a notation for each square and the 4 squares obtained through performing one construction step $k \mapsto k + 1$. For a center point $a = (a_1, a_2) \in \mathbb{R}^2$ and some radius $r > 0$, we define the square $S = S(a, r) \subset \mathbb{R}^2$ given by

   $$S(a, r) = \left\{x = (x_1, x_2) \in \mathbb{R}^2 \, : \| \, x - a \, \|_\infty \leq r\right\}.$$

   Further, starting at the top left corner and going left to right and top to down, we define the corner squares S_1, S_2, S_3 and S_4, see Figure 2:

   $$S_1 = S\left(\left(a_1 - \frac{3}{4}r, a_2 + \frac{3}{4}r\right), \frac{r}{4}\right), \quad S_2 = S\left(\left(a_1 + \frac{3}{4}r, a_2 + \frac{3}{4}r\right), \frac{r}{4}\right),$$
   $$S_3 = S\left(\left(a_1 - \frac{3}{4}r, a_2 - \frac{3}{4}r\right), \frac{r}{4}\right), \quad S_4 = S\left(\left(a_1 + \frac{3}{4}r, a_2 - \frac{3}{4}r\right), \frac{r}{4}\right).$$

   The corresponding corners of the larger square S are denoted by s_1, s_2, s_3 and s_4.

3. Next, we define a property which states an approximating quality to the normal vectorfield of each square. A vectorfield $\varphi \in \mathrm{C}^1(S, \mathbb{R}^2)$ is (S, ε)-normal iff

   - $|\varphi(x)| \leq 1$ for all $x \in S$,
   - $|\varphi(x)| = 0$ if $x \in \partial S$ and $|x - s_i| \leq \varepsilon$ for any $i = 1, 2, 3, 4$,
   - $\frac{\varphi(x)}{|\varphi(x)|} = \nu(x)$ for all $x \in \partial S \cap \{|\varphi(x)| \neq 0\}$ (ν outward normal to S),
   - $\mathrm{D}\varphi(x) \cdot \nu(x) = 0$ for all $x \in \partial S$.

   Here, '$\cdot$' denotes the matrix-vector product and we note, that the function $\varphi \equiv 0$ is (S, ε)-normal.

4. For any S, it is easy to construct a (S,ε)-normal function φ, for example, by using mollification: One possibility would be to take rectangular 'neighborhoods' of parts of the vertices, which have for some sufficiently small $0 < \delta \ll \frac{r}{4}$ a distance of $\varepsilon + \delta$ from the corners of the square S and expands by δ in the perpendicular direction to the contained vertices. On those 'neighborhoods' define φ as the constant outward normal vector of the only vertex contained in the neighborhood and mollify with the usual symmetric mollifier scaled accordingly with $\frac{\delta}{2}$. It is easy to verify that the such constructed field φ is (S,ε)-normal. By Stokes formula, we have
$$\int_S \operatorname{div}\varphi \,\mathrm{d}x = \int_{\partial S} \varphi \cdot \nu \,\mathrm{d}\mathcal{H}^1 \leq 8r - 8\varepsilon$$
providing a lower bound. Further, it is possible to construct a (S,ε)-normal function φ such that $\int_S \operatorname{div}\varphi\,\mathrm{d}x$ is arbitrary close to $8r - 8\varepsilon$ or, for example, equal to $8r - 9\varepsilon$.
5. Given (S_i,ε)-normal φ_i on S_i for $i = 1,2,3,4$, we can extend them to a (S,ε)-normal function φ on S with $\varphi|_{S_i} = \varphi_i$ and $\operatorname{div}\varphi = 0$ on $S \setminus \bigcup_{i=1}^4 S_i$.

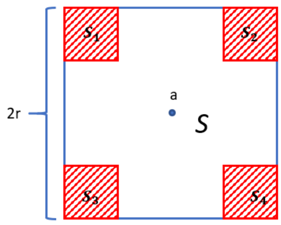
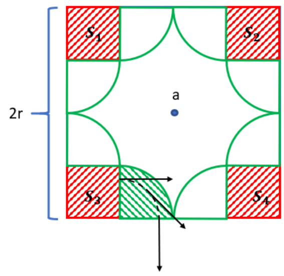


Figure 2: Square S and subsquares S_i



Extension φ_i to φ with sample vectors

Proof. The main idea is to 'rotate' the approximating normal vector provided by each φ from inner boundary parts of S_i in S to boundary parts of S. For example, for the right side of the lower left square, we can define the extension using polar coordinates in the following way:
For $x = (x_1, x_2) = (a_1 - \frac{r}{2} + \rho\cos(\vartheta), a_2 - r + \rho\sin(\vartheta))$ with $0 \leq \rho \leq \frac{r}{2}$ and $0 \leq \vartheta \leq \frac{\pi}{2}$, we define
$$\varphi(x_1, x_2) = -\left|\varphi_3\left(a_1 - \frac{r}{2}, a_2 - r + \rho\right)\right| (-\sin(\vartheta), \cos(\vartheta))^T .$$
Applying this to every side and by setting $\varphi = 0$ on the remaining part of S, the desired extension is found. Since all vertices of S are vertices of some S_i and the normal vectors are 'rotated' smoothly in a way to match the normal vector field of S or be equal to 0, we obtain that the defined extension is C^1 by the last property of (S,ε)-normal fields and the other properties are passed on accordingly. Thus the extension is indeed (S,ε)-normal for the same ε as the φ_i before.
It remains to check, if the extension is divergence free outside of the S_i, $i = 1,\ldots,4$. This is clear by definition or can be checked using the divergence in polar coordinates. □

Finally, we are able to implement the proof. For an open set A containing C, we can assume A to be bounded: For A with $|A| = +\infty$, we have $\varsigma(A) = +\infty$, since $\varsigma_\delta(A)$ is

$+\infty$ for each $\delta > 0$. For A with $|A| < +\infty$, we can use a suitable projection p on the ball $\mathrm{B}_2(0)$, which is Lipschitz continuous with Lipschitz constant 1, decreases the Lebesgue measure of A and by $\mathrm{P}(p(A)) \leq \mathrm{Lip}_p \mathrm{P}(A)$ the perimeter, too, using the well known estimate $\mathcal{H}^{n-1}(f(E)) \leq \mathrm{Lip}_f \mathcal{H}^{n-1}(E)$ for Lipschitz functions f and measurable sets E.
By the first point, we find some k_0 such that $C_{k_0} \subset A$. For given $\varepsilon > 0$, we can find for all 4^k squares S_i for $i = 1, \ldots, 4^k$ in C_k a (S_i, ε)-normal function φ_i with $\int_{S_i} \operatorname{div} \varphi_i \, \mathrm{d}x \geq 4 \cdot 4^{-k} - 9\varepsilon$, where the radius is provided by $2r = 4^{-k}$. Next, we apply the extension iteratively until we get a (C_0, ε)-normal function φ on $C_0 = [0,1]^2$, which is further extended in the vertical and horizontal direction by the values of the nearest border point and 0 everywhere else from S to $\mathbb{R}^2$. Next, we can find a function $\eta \in \mathrm{C}^1_{\mathrm{cpt}}(\mathbb{R}^2)$ with $\eta(x) = 1$ for $x \in A$ and can conclude:

$$\begin{aligned} \int_A \operatorname{div} (\eta\varphi) \, \mathrm{d}x &= \int_A \operatorname{div} \varphi \, \mathrm{d}x = \int_{C_0} \operatorname{div} \varphi \, \mathrm{d}x \\ &= \int_{C_k} \operatorname{div} \varphi \, \mathrm{d}x \geq 4 - 4^k \cdot 9\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary and $\eta\varphi \in \mathrm{C}^1_{\mathrm{cpt}}(\mathbb{R}^2, \mathbb{R}^2)$ with $|\eta\varphi| \leq 1$ is a suitable candidate in the Definition 2.20 for the perimeter, $\mathrm{P}(A) \geq 4$ follows for each considered open set containing C and, together with the first part, we obtain $\varsigma(C) = 4$.

□

In addition to this counterexample, Hutchinson improved the upper bound constant in (3.3) and proved that

$$C(n) = \frac{n\omega_n}{\omega_{n-1}}$$

is optimal. The constant seems reasonable, since it gives the ratio between the circumference of an sphere and its cross-section area, which is interpreted as a kind of diameter for $n = 2$. Thus, the difference between the De Giorgi measure and the Hausdorff measure is reflected in this constant.

3.2 The generalized De Giorgi measure

In this section, we will investigate the De Giorgi measure in a slightly more general version as presented in [13, Chapter 4] with some aspects influenced by the articles [11], [12] and [53]. In the following, we are interested in a possibly non-symmetric version of the (generalized) De Giorgi measure in comparison to the introduced version in [13, Definition 4.1 and 4.2]. Let $q : \mathbb{R}^n \to \mathbb{R}^+_0$ with

1. $q(z + w) \leq q(z) + q(w)$ for all $z, w \in \mathbb{R}^n$,
2. $q(tz) = tq(z)$ for every $t > 0$ and all $z \in \mathbb{R}^n$ and
3. $a|z| \leq q(z) \leq b|z|$ for some positive constants $a, b > 0$

be given.
Then we define the generalized δ-De Giorgi measure by:

Definition 3.4 (δ-De Giorgi measure)**.**
For $\delta > 0$ and for every $E \subset \mathbb{R}^n$, we define

$$\varsigma_q^\delta(E) = \inf \left\{ \int\limits_{\mathbb{R}^n} q(\mathrm{D}u) + \frac{|u|}{\delta} \,\mathrm{d}x \,:\, u \in \mathrm{W}^{1,1}(\mathbb{R}^n), u^* \geq 1 \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } E \right\}. \tag{3.5}$$

The corresponding generalized De Giorgi measure is then given by the same construction as the original De Giorgi measure:

Definition 3.5 (Generalized De Giorgi measure)**.**
For E as above, we define the generalized De Giorgi measure ς_q of the set E by

$$\varsigma_q(E) = \lim_{\delta \searrow 0} \varsigma_q^\delta(E) = \sup_{\delta > 0} \varsigma_q^\delta(E). \tag{3.6}$$

The role of the function q will be played by the recession function f^∞ of f. The required properties of q are automatically fulfilled for convex f with linear growth in the form $a|z| \leq f(z) \leq b(1+|z|)$ for some $a, b > 0$: With the convexity of f and Definition 2.28 with remarks we have

1. $$\begin{aligned} f^\infty(z+w) &= \lim_{t\to\infty} \frac{f(tz+tw)}{t} \leq \lim_{t\to\infty} \left(\frac{f(2tz)}{2t} + \frac{f(2tw)}{2t} \right) \\ &= f^\infty(z) + f^\infty(w). \end{aligned}$$

2. Clear from Definition 2.29.
3. See Equation (2.1).

Using an anisotropic perimeter, i.e. we use $q(\mathrm{D}\mathbb{1}_E)$ instead of $|\mathrm{D}\mathbb{1}_E|$ for sets with finite perimeter, we can rewrite Definition 3.4 like in [13, Proposition 4.1]:

Proposition 3.6 (Another characterization of the δ-De Giorgi measure)**.**
For $E \subset \mathbb{R}^n$ and $\delta > 0$, we have

$$\varsigma_q^\delta(E) = \inf \left\{ \int\limits_{\mathcal{F}B} q(v) \,\mathrm{d}\mathcal{H}^{n-1} + \frac{|B|}{\delta} \,:\, B \textit{ is } \mathcal{L}^n\textit{-measureable and } \mathcal{H}^{n-1}(E \setminus B^+) = 0 \right\}. \tag{3.7}$$

The proof is similar to the one presented in the original source, but for the sake of completeness we state the adapted version:

Proof. To distinguish both definitions of the generalized δ-De Giorgi measure we set $\tilde{\varsigma}_q^\delta$ as the δ-De Giorgi measure of Proposition 3.6. We prove the statement by showing the inequalities $\varsigma_q^\delta \geq \tilde{\varsigma}_q^\delta$ and $\varsigma_q^\delta \leq \tilde{\varsigma}_q^\delta$.

Step 1: With the Definition 3.4 of ς_q^δ and for all $\varepsilon > 0$ we find a function $u \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ such that $u^* \geq 1$ $\mathcal{H}^{n-1}$-a.e. on E and

$$\varsigma_q^\delta(E) + \varepsilon > \int_{\mathbb{R}^n} q(\mathrm{D}u) + \frac{|u|}{\delta}\,\mathrm{d}x.$$

For $t \in \mathbb{R}$, we can set $S^t = \{u > t\}$ and denote by ν_{S^t} the outward unit normal to S^t defined on $\mathcal{F}S^t$. Using Theorem 2.35, we have

$$\int_{\mathbb{R}^n} u_+\,\mathrm{d}x = \int_0^{+\infty} |S^t|\,\mathrm{d}t$$

and

$$\int_{\mathbb{R}^n} q(\mathrm{D}u)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \int_{\mathcal{F}S^t} q(\nu_{S^t})\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}t.$$

Thus, we obtain

$$\varsigma_q^\delta(E) + \varepsilon \geq \int_0^1 \int_{\mathcal{F}S^t} q(\nu_{S^t})\,\mathrm{d}\mathcal{H}^{n-1} + \frac{|S^t|}{\delta}\,\mathrm{d}t.$$

Since $u^* \geq 1$ $\mathcal{H}^{n-1}$-a.e. on E, we have $\mathcal{H}^{n-1}\left(E \setminus (S^t)^+\right) = 0$ for all $t \in (0,1)$ and arrive at $\varsigma_q^\delta(E) + \varepsilon \geq \tilde{\varsigma}_q^\delta(E)$.

Step 2: We may assume that $\tilde{\varsigma}_q^\delta(E)$ is finite, since otherwise the inequality is trivial. Then, for every $\varepsilon > 0$, we find some $\mathcal{L}^n$-measurable set $B \subset \mathbb{R}^n$ with $\mathcal{H}^{n-1}(E \setminus B^+)$ and

$$\tilde{\varsigma}_q^\delta(E) + \varepsilon > q(\mathrm{D}\mathbb{1}_B)(\mathbb{R}^n) + \frac{|B|}{\delta}.$$

Using [13, Lemma 3.1], we find a sequence $w_k \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ for $k \in \mathbb{N}$ such that $w_k^* \geq 1$ $\mathcal{H}^{n-1}$-a.e. on B^+ and $w_k \to \mathbb{1}_B$ strictly in $\mathrm{BV}(\mathbb{R}^n)$ and thus especially $w_k \to \mathbb{1}_B$ in $\mathrm{L}^1(\mathbb{R}^n)$ for $k \to \infty$. Using Theorem 2.40, we obtain

$$q(\mathrm{D}\mathbb{1}_B)(\mathbb{R}^n) = \lim_{k\to\infty} \int_{\mathbb{R}^n} q(w_k)\,\mathrm{d}x.$$

Thus, we have

$$\tilde{\varsigma}_q^\delta(E) + \varepsilon > \lim_{k\to\infty} \int_{\mathbb{R}^n} q(\mathrm{D}w_k) + \frac{|w_k|}{\delta}\,\mathrm{d}x \geq \varsigma_q^\delta(E)$$

and, since $\varepsilon > 0$ was arbitrary, we get the second inequality. □

The statements of Proposition 4.2 as well as Proposition 4.3 in [13] hold as well:

Proposition 3.7 (Properties of ς_q^δ and ς_q)**.**
The generalized δ-De Giorgi measure ς_q^δ has for all $\delta > 0$ the following properties:

1. *ς_q^δ is non-negative, non-decreasing and countably subadditive.*

2. ς_q^δ *is strongly subadditive, i.e. for every* $E, F \subset \mathbb{R}^n$ *the following inequality holds:*

$$\varsigma_q^\delta(E \cup F) + \varsigma_q^\delta(E \cap F) \leq \varsigma_q^\delta(E) + \varsigma_q^\delta(F).$$

3. For two sets $E, F \subset \mathbb{R}^n$ *with* $\operatorname{dist}(E, F) \geq \varepsilon$*, we have*

$$\varsigma_q^\delta(E) + \varsigma_q^\delta(F) \leq \left(1 + \frac{3b\delta}{\varepsilon}\right) \varsigma_q^\delta(E \cup F).$$

As a conclusion for the measure ς_q*, we have the following properties:*

1'. ς_q *is non-negative, non-decreasing and countably subadditive.*

2'. ς_q *is strongly subadditive:*

$$\varsigma_q(E \cup F) + \varsigma_q(E \cap F) \leq \varsigma_q(E) + \varsigma_q(F).$$

3'. For two sets $E, F \subset \mathbb{R}^n$ *with* $\operatorname{dist}(E, F) > 0$*, we have*

$$\varsigma_q(E) + \varsigma_q(F) = \varsigma_q(E \cup F)$$

and thus the De Giorgi measure is a Borel regular measure on the Borel σ*-algebra by the Carathéodory criterion [3, Theorem 1.49].*

Further, we have a similar estimate like in (3.3)*:*

4'. With the same constants as in (3.3)*, we have*

$$ac(n)\mathcal{H}^{n-1} \leq \varsigma_q \leq bC(n)\mathcal{H}^{n-1}.$$

Proof. The facts that ς_q^δ and ς_q are non-negative and non-decreasing follow easily from the definitions. The countable subadditivity follows from [11, Proposition 3.1] for the δ-De Giorgi measure, and thus for the generalized De Giorgi measure as well, in the same fashion as stated there. Thus point 1. and 1'. hold.

The strong subadditivity in 2. and 2'. follows from the strong subadditivity of the Lebesgue measure and of the anisotropic perimeter:

For the sets E and F, we find sequences $e_k \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ and $f_k \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ converging strictly to $\mathbb{1}_E$ and $\mathbb{1}_F$, respectively. Thus $e_k f_k$ converges in $\mathrm{L}^1(\mathbb{R}^n)$ to $\mathbb{1}_{E \cap F}$ and $e_k + f_k - e_k f_k$ to $\mathbb{1}_{E \cup F}$. Since characteristic functions are bounded by 1 and non-negative, the sequences e_k and f_k can be chosen bounded by 1 and non-negative as well, like in Theorem 2.44. With that, the product rule holds and we obtain the following estimate:

$$\begin{aligned}\int\limits_{\mathbb{R}^n} q(\mathrm{D}(e_k f_k))\,\mathrm{d}x &+ \int\limits_{\mathbb{R}^n} q(\mathrm{D}e_k + \mathrm{D}f_k - \mathrm{D}(e_k f_k))\,\mathrm{d}x \\ &= \int\limits_{\mathbb{R}^n} q(e_k \mathrm{D}f_k + f_k \mathrm{D}e_k)\,\mathrm{d}x + \int\limits_{\mathbb{R}^n} q(\mathrm{D}e_k + \mathrm{D}f_k - e_k \mathrm{D}f_k - f_k \mathrm{D}e_k)\,\mathrm{d}x\end{aligned}$$

The first integral on the right-hand side can be estimated using the convexity as well as the positive

1-homogeneity of q and the non-negativity of e_k and f_k:

$$\int_{\mathbb{R}^n} q(e_k \mathrm{D} f_k + f_k \mathrm{D} e_k)\,\mathrm{d}x \leq \int_{\mathbb{R}^n} q(e_k \mathrm{D} f_k) + q(f_k \mathrm{D} e_k)\,\mathrm{d}x$$
$$\leq \int_{\mathbb{R}^n} e_k q(\mathrm{D} f_k) + f_k q(\mathrm{D} e_k)\,\mathrm{d}x.$$

Similarly, we get for the second integral on the right-hand side with the upper bound 1 for e_k and f_k:

$$\int_{\mathbb{R}^n} q(\mathrm{D}e_k + \mathrm{D}f_k - e_k \mathrm{D} f_k - f_k \mathrm{D} e_k)\,\mathrm{d}x = \int_{\mathbb{R}^n} q\left((1-f_k)\mathrm{D}e_k + (1-e_k)\mathrm{D}f_k\right)\,\mathrm{d}x$$
$$\leq \int_{\mathbb{R}^n} (1-e_k) q(\mathrm{D}f_k) + (1-f_k) q(\mathrm{D}e_k)\,\mathrm{d}x.$$

Combining both estimates leads to

$$\int_{\mathbb{R}^n} q(\mathrm{D}(e_k f_k))\,\mathrm{d}x + \int_{\mathbb{R}^n} q(\mathrm{D}e_k + \mathrm{D}f_k - \mathrm{D}(e_k f_k))\,\mathrm{d}x \leq \int_{\mathbb{R}^n} q(\mathrm{D}e_k) + q(\mathrm{D}f_k)\,\mathrm{d}x$$

and, with the strict convergence of e_k and f_k and Theorem 2.38 used on the left-hand side and Theorem 2.40 used on the right-hand side of the last equation, we obtain

$$q(\mathrm{D}\mathbb{1}_{E\cap F})(\mathbb{R}^n) + q(\mathrm{D}\mathbb{1}_{E\cup F})(\mathbb{R}^n) \leq q(\mathrm{D}\mathbb{1}_E)(\mathbb{R}^n) + q(\mathrm{D}\mathbb{1}_F)(\mathbb{R}^n),$$

which proves 2. and thus 2'.
For 3., we slightly modify the proof in [13, Proposition 4.2]:
For each $\lambda > 0$, there exists by the definition of ς_q^δ a function $u \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ with $u^* \geq 1$ $\mathcal{H}^{n-1}$-a.e. on $E \cup F$ and

$$\varsigma_q^\delta(E \cup F) + \lambda > \int_{\mathbb{R}^n} q(\mathrm{D}u) + \frac{|u|}{\delta}\,\mathrm{d}x.$$

Next, we choose cut-off functions $\eta_i \in \mathrm{C}^\infty(\mathbb{R}^n)$, $i = 1, 2$, with $0 \leq \eta_i \leq 1$, $|\mathrm{D}\eta_i| \leq \frac{3}{\varepsilon}$ and $\eta_1 = 1$ on E as well as $\eta_2 = 1$ on F such that the support is disjoint, i.e. $\operatorname{supp}\eta_1 \cap \operatorname{supp}\eta_2 = \emptyset$. The product $\eta_i u$ is in $\mathrm{W}^{1,1}(\mathbb{R}^n)$ and fulfills $(\eta_i u)^* \geq 1$ on E and F, respectively, for $i = 1, 2$. Thus, we can estimate

$$\varsigma_q^\delta(E) \leq \int_{\mathbb{R}^n} q(\mathrm{D}(\eta_1 u)) + \frac{|\eta_1 u|}{\delta}\,\mathrm{d}x,$$
$$\varsigma_q^\delta(F) \leq \int_{\mathbb{R}^n} q(\mathrm{D}(\eta_2 u)) + \frac{|\eta_2 u|}{\delta}\,\mathrm{d}x.$$

Combining both inequalities, using the disjoint supports of the cut-off functions and the properties of q, we obtain:

$$\begin{aligned}
\varsigma_q^\delta(E) + \varsigma_q^\delta(F) &\leq \int\limits_{\mathbb{R}^n} q\left((\eta_1+\eta_2)\mathrm{D}u + u\mathrm{D}(\eta_1+\eta_2)\right) + \frac{|(\eta_1+\eta_2)u|}{\delta}\,\mathrm{d}x \\
&\leq \int\limits_{\mathbb{R}^n} q(\mathrm{D}u) + q(u\mathrm{D}(\eta_1+\eta_2)) + \frac{|u|}{\delta}\,\mathrm{d}x \\
&\leq \int\limits_{R^n} q(\mathrm{D}u) + b|u|\frac{3}{\varepsilon} + \frac{|u|}{\delta}\,\mathrm{d}x \\
&\leq \left(1+\frac{3b\delta}{\varepsilon}\right)(\varsigma_q^\delta(E\cup F)+\lambda).
\end{aligned}$$

Since λ was arbitrary, 3. follows and proves 3'. by the Carathéodory criterion.
Point 4'. is a direct consequence of the estimate $a|z| \leq q(z) \leq b|z|$ and the inequality (3.3). □

Next, an auxiliary lemma estimating the δ-De Giorgi measure of superlevel sets is presented.

Lemma 3.8 (Estimate of superlevel sets)**.**
For $E \subset \mathbb{R}^n$ and an arbitrary function $\psi : E \to \mathbb{R} \cup \{\pm\infty\}$, we have for each $\delta > 0$ the following relation:

$$\int\limits_0^{+\infty} \varsigma_q^\delta(\{x \in E \,:\, \psi(x) > t\})\,\mathrm{d}t$$

$$= \inf\left\{\int\limits_{\mathbb{R}^n} q(\mathrm{D}u) + \frac{|u|}{\delta}\,\mathrm{d}x \,:\, u \in \mathrm{W}^{1,1}(\mathbb{R}^n),\ u^* \geq \psi \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } E\right\}.$$

The statement can be found in [12, Proposition 4.3], using the here described setting, and is similar to [13, Lemma 4.5].
This lemma can be used to prove [13, Proposition 4.4], which is key to prove lower semicontinuity and to construct recovery sequences for the relaxation of the discussed obstacle problem:

Proposition 3.9 (Lower semicontinuity and a continuity statement for the De Giorgi measure)**.**
For a Borel set $E \subset \mathbb{R}^n$ and a Borel function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ with $\psi_+\varsigma_q(E) < +\infty$, we obtain the following two results:

1. *For every sequence $(u_k) \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ such that $u_k \to 0$ in $\mathrm{L}^1(\mathbb{R}^n)$ and $u_k^* \geq \psi$ $\mathcal{H}^{n-1}$-a.e. on E, we have*
$$\int\limits_E \psi_+\,\mathrm{d}\varsigma_q \leq \liminf_{k\to\infty} \int\limits_{\mathbb{R}^n} q(\mathrm{D}u_k)\,\mathrm{d}x.$$
2. *There exists a sequence $u_k \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ with $u_k \to 0$ in $\mathrm{L}^1(\Omega)$ and $u_k^* \geq \psi$ $\mathcal{H}^{n-1}$-a.e. on E with*
$$\int\limits_E \psi_+\,\mathrm{d}\varsigma_q = \lim_{k\to\infty} \int\limits_{\mathbb{R}^n} q(\mathrm{D}u_k)\,\mathrm{d}x.$$

Remark. In [62, Lemma 2.10], a slightly different version was established, where for 2. in Proposition 3.9 the sequence could be chosen in $\mathrm{W}_0^{1,1}(\Omega)$ for an open set Ω as long as ψ has compact support in Ω.

Proof. The proof is identical to the proof of Proposition 4.4. in [13]. □

To be able to treat upper obstacle problems, we obtain as a direct consequence of Proposition 3.9:

Proposition 3.10 (Another lower semicontinuity and continuity statement)**.**
For a Borel set $E \subset \mathbb{R}^n$ and a Borel function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ with $\psi_+ \varsigma_q(E) < +\infty$ and $\tilde{q}(z) = q(-z)$, we obtain the following two results:

1. *For every sequence $(u_k) \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ such that $u_k \to 0$ in $\mathrm{L}^1(\mathbb{R}^n)$ and $u_k^* \leq \psi$ $\mathcal{H}^{n-1}$-a.e. on E, we have*
$$\int_E (-\psi)_+ \,\mathrm{d}\varsigma_{\tilde{q}} \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} \tilde{q}(-\mathrm{D}u_k)\,\mathrm{d}x = \liminf_{k\to\infty} \int_{\mathbb{R}^n} q(\mathrm{D}u_k)\,\mathrm{d}x.$$

2. *There exists a sequence $u_k \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ with $u_k \to 0$ in $\mathrm{L}^1(\Omega)$ and $u_k^* \leq \psi$ $\mathcal{H}^{n-1}$-a.e. on E with*
$$\int_E (-\psi)_+ \,\mathrm{d}\varsigma_{\tilde{q}} = \lim_{k\to\infty} \int_{\mathbb{R}^n} \tilde{q}(-\mathrm{D}u_k)\,\mathrm{d}x = \lim_{k\to\infty} \int_{\mathbb{R}^n} q(\mathrm{D}u_k)\,\mathrm{d}x.$$

Proof. Apply for 1. the Proposition 3.9 with $\tilde{q}$ on $-u_k^* \geq -\psi$ and note that $-u_k^* = (-u_k)^*$. Similar for 2. □

Remark.

1. It is convenient to use the definition $\tilde{q}(z) = q(-z)$, since usually only $q(\mathrm{D}u)$ appears in the most calculations. Since we will plug in f^∞ instead of q, we also write $\tilde{f}^\infty(z) = f^\infty(-z)$ for the adjusted recession function. For a symmetric function f^∞ like the absolute value function, we have $\tilde{f}^\infty = f^\infty$ and thus in this case nothing needs to be adjusted. This implies that the same recession function can be used for the lower and upper obstacle problem for the area functional and the total variation functional.
2. Since $\tilde{q}$ fulfills the properties required on the function q stated at the beginning of this section, all results and properties hold accordingly for $\varsigma_{\tilde{q}}$ instead of ς_q.
3. On the first glance it might seem strange that for the upper obstacle a 'different' recession function has to be used. This fact becomes clearer, if one considers Definition 3.5 and notices, that a possibly preferred direction for the function u to 'rise' to or above 1 on E (since we have no symmetry), needs to be inverted if the function has to 'decrease' down and lie below -1 on a given set E.

Next, we state a useful result on how to calculate the generalized De Giorgi measure of a Lipschitz graph. This is similar to Proposition 4.6 in [13]:

Proposition 3.11 (De Giorgi measure of Lipschitz graphs)**.**
Let $G \subset \mathbb{R}^n$ be the graph of a Lipschitz function $g : \mathbb{R}^{n-1} \to \mathbb{R}$. For a Borel set $E \subset G$, we can compute
$$\varsigma_q(E) = \int_E q(\nu_G) + q(-\nu_G)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.8}$$

Proof. Again, the proof is similar to that of Proposition 4.6 in [13] with the main difference, that we do not have $q(\nu) = q(-\nu)$. Knowing that, the proof can easily be obtained. □

A consequence of this result is that we can rewrite the De Giorgi measure on rectifiable sets:

Proposition 3.12 (De Giorgi measure on rectifiable sets)**.**
Let $R \subset \mathbb{R}^n$ be a countably $\mathcal{H}^{n-1}$-rectifiable set. For a Borel subset E of R, we can compute

$$\varsigma_q(E) = \int\limits_E q(\nu_R) + q(-\nu_R)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.9}$$

Proof. With the alternative definition of rectifiability provided by Proposition 2.4, we may use Proposition 3.11 on components of the rectifiable sets contained in the Lipschitz graphs. Applying this on components of E yields the result. □

Remark. For the last two propositions, it does not matter if q or $\tilde{q}$ is used in the sense that

$$q(\nu_R) + q(-\nu_R) = q(\nu_R) + \tilde{q}(\nu_R) = \tilde{q}(-\nu_R) + \tilde{q}(\nu_R).$$

Thus we will not distinguish on rectifiable sets between the two De Giorgi measures ς_q and $\varsigma_{\tilde{q}}$.

3.3 The single obstacle problem revisited

In this section, we derive a representation formula for the relaxation of the functional, related to the single obstacle problem, at first without boundary constraint:
For some open bounded set Ω and a Borel function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, we set

$$\mathrm{F}^{\psi}[u] = \int\limits_\Omega f(u)\,\mathrm{d}x \text{ on } \mathrm{W}^{1,1}_{\circ,\psi}(\Omega)$$

with F^{ψ} set $+\infty$ on $\mathrm{L}^1(\Omega) \setminus \mathrm{W}^{1,1}_{\circ,\psi}(\Omega)$ and with convex integrand $f : \mathbb{R}^n \to [0, \infty)$, which fulfills $a|z| \leq f(z) \leq b(1 + |z|)$ for some constants $a, b > 0$. The function space $\mathrm{W}^{1,1}_{\circ,\psi}(\Omega)$ is given by

$$\mathrm{W}^{1,1}_{\circ,\psi}(\Omega) := \{u \in \mathrm{W}^{1,1}(\Omega) \,:\, u^* \geq \psi \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } \Omega\}.$$

The relaxation is proven to be

$$\overline{\mathrm{F}}^{\psi}[u] = \int\limits_\Omega f(\mathrm{D}u) + \int\limits_\Omega (\psi - u^+)_+ \,\mathrm{d}\varsigma_{f^\infty} \text{ on } \mathrm{BV}(\Omega)$$

and $+\infty$ on $\mathrm{L}^1 \setminus \mathrm{BV}(\Omega)$. In contrast to [13, Chapter 5], we manage to prove the result with the convexity and growth condition and leaving out the symmetry and a weaker form of the triangle inequality $f(z+w) \leq f(z) + f(w) + c$ for some $c \geq 1$. The last inequality ensures, that functionals with integrands like $\sqrt{1 + |\,.\,|} - 1$ can be treated without further issues and gives control over the recession function in the form $f^\infty(z) \leq f(z) + c$. With the results achieved in Theorem 2.30, we

are able to drop such extra assumptions.
We begin by proving the lower semicontinuity of

$$\int_\Omega f(\mathrm{D}u)\,\mathrm{d}x + \int_\Omega (\psi - u^*)_+ \,\mathrm{d}\varsigma_{f^\infty}.$$

For the proof, we need the auxiliary statement proven in [12, Lemma 5.3] for symmetric q, see also [13, Lemma 5.3] for not necessary symmetric q. The difference lies in the usage of $\tilde{q}$ instead of q in one of the terms or in other words in the sign of the evaluation of one integrand.

Lemma 3.13 (Estimate for the obstacle term)**.**
For an open bounded $\Omega \subset \mathbb{R}^n$ and a Borel function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ and a sequence $u_k \in \mathrm{W}^{1,1}(\Omega)$ converging in $\mathrm{L}^1(\Omega)$ to $u \in \mathrm{W}^{1,1}(\Omega)$, we have for every open set $A \subset \Omega$ the following estimate:

$$\begin{aligned}\int_A (\psi - u^*)_+ \,\mathrm{d}\varsigma_q &\leq \int_A \tilde{q}(\mathrm{D}u)\,\mathrm{d}x + \liminf_{k\to\infty}\left(\int_A q(\mathrm{D}u_k)\,\mathrm{d}x + \int_A (\psi - u^*)_+ \,\mathrm{d}\varsigma_q\right)\\ &= \int_A q(-\mathrm{D}u)\,\mathrm{d}x + \liminf_{k\to\infty}\left(\int_A q(\mathrm{D}u_k)\,\mathrm{d}x + \int_A (\psi - u^*)_+ \,\mathrm{d}\varsigma_q\right),\end{aligned}$$

where q is like in the previous section.

With this result in hand, we can now prove the following lower semicontinuity result:

Theorem 3.14 (Lower semicontinuity on $\mathrm{W}^{1,1}$)**.**
For an open bounded $\Omega \subset \mathbb{R}^n$ and Borel function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, the functional

$$\int_\Omega f(\mathrm{D}u)\,\mathrm{d}x + \int_\Omega (\psi - u^*)_+ \,\mathrm{d}x$$

is lower semicontinuous on $\mathrm{W}^{1,1}(\Omega)$ with respect to the convergence in $\mathrm{L}^1(\Omega)$, i.e. the functional is lower semicontinuous along sequences $u_k \in \mathrm{W}^{1,1}(\Omega)$ with $u_k \to u$ in $\mathrm{L}^1(\Omega)$ with $u \in \mathrm{W}^{1,1}(\Omega)$.

Proof. We first follow the lines of the proof of [13, Theorem 5.1]:
Let $u_k \in \mathrm{W}^{1,1}(\Omega)$ be a sequence converging in L^1 to u. The convergence in L^1 implies, that if $|\{\psi - u^* > 0\}| > 0$, then this holds also true for u_k for k large enough and thus $|\{\psi - u_k^* > \lambda\}| > 0$ for some λ small enough. With that, we have $\int_\Omega (\psi - u_k^*)_+ \,\mathrm{d}\varsigma_{f^\infty} = +\infty$ which is clear from Proposition 3.6, see also Remark 4.2 in [13].
If $|\{\psi - u^* > 0\}| = 0$, we find for arbitrary $\lambda > 0$ an open set A such that $\{\psi - u^* > 0\} \subset A \subset \Omega$, $|A| < \lambda$ and

$$\int_A b + b|\mathrm{D}u|\,\mathrm{d}x \leq \lambda. \tag{3.10}$$

We set $I = \int_\Omega (\psi - u^*)_+ \,\mathrm{d}\varsigma_{f^\infty} = \int_A (\psi - u^*)_+ \,\mathrm{d}\varsigma_{f^\infty}$ and can find for each value $t < I$ some compact set $K \subset A$ such that $t < \int_K (\psi - u^*)_+ \,\mathrm{d}\varsigma_{f^\infty}$. Further, we can find two disjoint open sets Ω_1 and Ω_2

such that $\Omega \setminus A \subset \Omega_1$ and $K \subset \Omega_2 \subset A$. By Theorem 2.39, the functional $\int_{\Omega_1} f(\mathrm{D}u)\,\mathrm{d}x$ is lower semicontinuous and thus we can estimate, using (3.10) and Lemma 3.13:

$$
\begin{aligned}
\int_{\Omega} f(\mathrm{D}u)\,\mathrm{d}x + t <& \lambda + \int_{\Omega_1} f(\mathrm{D}u)\,\mathrm{d}x + \int_{\Omega_2} (\psi - u^*)_+ \,\mathrm{d}\varsigma_{f^\infty} \\
\leq& \lambda + \liminf_{k\to\infty} \int_{\Omega_1} f(\mathrm{D}u_k)\,\mathrm{d}x + \int_{\Omega_2} f^\infty(-\mathrm{D}u)\,\mathrm{d}x \\
&+ \liminf_{k\to\infty} \left(\int_{\Omega_2} f^\infty(\mathrm{D}u_k)\,\mathrm{d}x + \int_{\Omega_2} (\psi - u_k^*)_+ \,\mathrm{d}\varsigma_{f^\infty} \right)
\end{aligned}
$$

Since $\Omega_2 \subset A$, we have

$$
\int_{\Omega_2} f^\infty(-\mathrm{D}u)\,\mathrm{d}x \leq \int_{\Omega_2} b|\mathrm{D}u|\,\mathrm{d}x \leq \lambda.
$$

For $\int_{\Omega_2} f^\infty(\mathrm{D}u_k)\,\mathrm{d}x$, we can estimate in the following way:
For some fixed $\varepsilon > 0$, we use Theorem 2.30 to get a $\delta > 0$ and obtain for any fixed $k \in \mathbb{N}$:

- On $\Omega_2 \cap \{|\mathrm{D}u_k| \leq \frac{1}{\delta}\}$:

$$
\int_{\Omega_2 \cap \{|\mathrm{D}u_k| \leq \frac{1}{\delta}\}} f^\infty(\mathrm{D}u_k)\,\mathrm{d}x \leq \int_A \frac{b}{\delta}\,\mathrm{d}x \leq \frac{\lambda}{\delta}.
$$

- On $\Omega_2 \cap \{|\mathrm{D}u_k| > \frac{1}{\delta}\}$:

$$
\int_{\Omega_2 \cap \{|\mathrm{D}u_k| > \frac{1}{\delta}\}} f^\infty(\mathrm{D}u_k)\,\mathrm{d}x \leq \int_{\Omega_2} f(\mathrm{D}u_k)\,\mathrm{d}x + \varepsilon \int_{\Omega_2} |\mathrm{D}u_k|\,\mathrm{d}x.
$$

Plugging those estimates in, we obtain

$$
\int_{\Omega} f(\mathrm{D}u)\,\mathrm{d}x + t < 2\lambda + \frac{\lambda}{\delta} + \liminf_{k\to\infty} \left(\int_{\Omega} f(\mathrm{D}u_k)\,\mathrm{d}x + \int_{\Omega} (\psi - u_k^*)_+ \,\mathrm{d}\varsigma_{f^\infty} + \varepsilon \int_{\Omega} |\mathrm{D}u_k|\,\mathrm{d}x \right).
$$

With $\lambda \searrow 0$, $\varepsilon \searrow 0$ and $t \nearrow I$ we arrive at

$$
\int_{\Omega} f(\mathrm{D}u)\,\mathrm{d}x + \int_{\Omega} (\psi - u^*)_+ \,\mathrm{d}\varsigma_{f^\infty} \leq \liminf_{k\to\infty} \left(\int_{\Omega} f(\mathrm{D}u_k)\,\mathrm{d}x + \int_{\Omega} (\psi - u_k^*)_+ \,\mathrm{d}\varsigma_{f^\infty} \right).
$$

□

Next, we show that for $u \in \mathrm{W}^{1,1}(\Omega)$ the relaxation of the functional agrees with $\mathcal{F}^\psi$, which is given by

$$
\mathcal{F}[u] = \int_{\Omega} f(\mathrm{D}u) + \int_{\Omega} (\psi - u^+)_+ \,\mathrm{d}\varsigma_{f^\infty} \text{ for } u \in \mathrm{BV}(\Omega)
$$

and $\mathcal{F} = +\infty$ on $\mathrm{L}^1(\Omega) \setminus \mathrm{BV}(\Omega)$.

Theorem 3.15 (Equality on $\mathrm{W}^{1,1}$).
For ψ, F^ψ and Ω as above, we have

$$\overline{\mathrm{F}}^\psi[u] = \mathcal{F}^\psi[u]$$

for $u \in \mathrm{W}^{1,1}(\Omega)$.

Proof. The proof is identical to the proof for Proposition 6.2 in [13]. By Theorem 3.14 we obtain $\overline{F}^\psi \geq \mathcal{F}^\psi[u]$ on $\mathrm{W}^{1,1}(\Omega)$ since the relaxation is the largest lower semicontinuous functional, which is smaller than the initial functional. The other inequality follows easily with 2. from Proposition 3.9: For fixed $u \in \mathrm{W}^{1,1}(\Omega)$ with $\mathcal{F}^\psi[u] < +\infty$, we find a sequence $w_k \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ with $w_k^* \geq (\psi - u^+)_+$ $\mathcal{H}^{n-1}$-a.e. on Ω and with

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f^\infty(\nabla w_k)\,\mathrm{d}x = \int_\Omega (\psi - u^+)_+ \,\mathrm{d}\varsigma_{f^\infty}.$$

With this and with (2.3), we can estimate

$$\begin{aligned}
\limsup_{k\to\infty} \int_\Omega f(\nabla(u + w_k))\,\mathrm{d}x &\leq \limsup_{k\to\infty} \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty(\nabla w_k)\,\mathrm{d}x \\
&\leq \limsup_{k\to\infty} \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_{R^n} f^\infty(\nabla w_k)\,\mathrm{d}x \\
&= \int_\Omega f(\nabla u)\,\mathrm{d}x + \int_\Omega (\psi - u^+)_+ \,\mathrm{d}\varsigma_{f^\infty}.
\end{aligned}$$

Functions $u \in \mathrm{W}^{1,1}(\Omega)$ such that $\mathcal{F}^\psi[u] = +\infty$ violate the obstacle constraint and thus a recovery sequence in this case is given by $u_k \equiv u$. □

Finally, we state the theorem which extends the relaxation to BV functions.

Theorem 3.16 (Extension of the relaxation to $\mathrm{BV}(\Omega)$).
For ψ, F^ψ and Ω as above, we have

$$\overline{\mathrm{F}}^\psi = \mathcal{F}^\psi.$$

The proof is identical to the proof of Theorem 6.1 presented in [13, Chapter 6] and relies on a representation lemma for monotone non-increasing functionals on BV, which is used on the obstacle term. This strategy does not extend to the double obstacle case, which is our main concern in the subsequent chapters, since we cannot obtain monotonicity for the sum of both obstacle terms. Therefore, we abstain from going into the details of the proof of Theorem 3.16. Additionally, another proof is obtained through the proof for the double obstacle problem by setting the upper obstacle $\psi_2 \equiv +\infty$.

Remark. If one wants to add a Dirichlet boundary constraint with a function $u_0 \in \mathrm{L}(\partial\Omega)$ as in [13, Chapter 7], Ω is assumed to be a Lipschitz domain and the construction is more involved, since boundary data has to be tracked. Using the results on the double obstacle problem, this result is easier obtained, as we will see.

4 The double obstacle problem

The main goal of this section is to prove the representation formula for the double obstacle problem for the functional

$$\mathrm{F}_{u_0}^{\psi_1,\psi_2}[u] = \int_\Omega f(\nabla u)\,\mathrm{d}x \text{ for } u \in \mathrm{W}^{1,1}_{u_0,\psi_1,\psi_2}(\Omega),$$

which is extended to $\mathrm{L}^1(\Omega)$ by $+\infty$ on $\mathrm{L}^1 \setminus \mathrm{W}^{1,1}_{u_0,\psi_1,\psi_2}(\Omega)$. As usual, Ω is an open bounded set, $\psi_1 \le \psi_2$ are Borel functions, where the inequality holds $\mathcal{H}^{n-1}$-a.e., the integrand $f : \mathbb{R}^n \to [0,+\infty)$ is convex and of linear growth with $a|z| \le f(z) \le b(1+|z|)$ for some constants $a, b > 0$. As in [13] we will first start with the problem without the boundary constraint and thus it suffices, that Ω is just an open bounded set. When we later add the boundary constraint in a distinct section, further requirements on the boundary of Ω will have to be assumed. Before we can start, we need some auxiliary statements. To be able to adapt parts of proof from the single obstacle problem, a truncation argument is needed: The main issue is that in the single obstacle case we are free to approximate from the other side of the obstacle, i.e. if we deal with a single lower obstacle, we can approximate from above or add non-negative functions like in the proof of Theorem 3.15. For the double obstacle problem, such constructed functions may disobey the other obstacle and thus we cannot use one-sided approximations anymore without further care. However, assuming the existence of a separation function $v \in \mathrm{W}^{1,1}_{\circ,\psi_1,\psi_2}(\Omega)$, with $\mathrm{W}^{1,1}_{\circ,\psi_1,\psi_2}(\Omega)$ defined similar to $\mathrm{W}^{1,1}_{\circ,\psi}$ as $\mathrm{W}^{1,1}_{u_0,\psi_1,\psi_2}$ without the boundary value constraint, we can still use the space in between u and v to construct a recovery sequence. To be able to do that, a truncation argument is required and to implement this, we need to find a subsequence for any sequence converging in area such that the absolutely continuous part of the gradients converge a.e. with respect to the Lebesgue measure. The later result is presented in the more general setting with arbitrary Radon measures.

4.1 Almost everywhere convergence of the gradients

We want to show that if a sequence of measures converges area-strictly, then it already converges almost everywhere. For the proof, we need some shifted versions of both Reshetnyak theorems. For that, we consider a lower semicontinuous function $g : \Omega \times \mathbb{R}^n \to [0,\infty)$, measurable in the first argument, convex in the second and with strong recession function g^∞ and linear growth, i.e.

$$g(\,.\,, z) \le b(1+|z|)$$

for some $b > 0$. We remember that we have

$$g(x, z+w) \le g(x,z) + g^\infty(x,w) \tag{4.1}$$

and are able to prove the following versions of convergence theorems with that inequality:

Corollary 4.1 (Shifted lower semicontinuity theorem).
For an open set $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| = 0$ and g as above, we have: If ν_k converges weakly to ν in*

$\mathrm{RM}(\overline{\Omega}, \mathbb{R}^n)$*, then for* $v \in \mathrm{L}^1(\overline{\Omega}, \mathbb{R}^n)$ *we have*

$$\liminf_{k\to\infty} \int_{\overline{\Omega}} g(\,.\,, \nu_k + v) \geq \int_{\overline{\Omega}} g(\,.\,, \nu + v). \tag{4.2}$$

Proof. Using mollification, we find for a given $v \in \mathrm{L}^1(\overline{\Omega}, \mathbb{R}^n)$ and for each $\varepsilon > 0$ a function $v_\varepsilon \in \mathrm{C}^0_\mathrm{b}(\Omega)$ such that

$$\int_{\overline{\Omega}} |v_\varepsilon - v| \,\mathrm{d}x < \varepsilon.$$

Then we investigate $\bar{g}_\varepsilon(x, z) = g(x, z + v_\varepsilon(x))$ and find by boundedness of v_ε that the strong recession function of the shifted function agrees with the original recession function g^∞, since we have using (4.1):

$$\bar{g}_\varepsilon^\infty(x,z) = \lim_{\substack{t\to\infty\\ z'\to z\\ x'\to x}} \frac{g(x', tz' + v_\varepsilon)}{t} \leq \lim_{\substack{t\to\infty\\ z'\to z\\ x'\to x}} \frac{g(x', tz') + g^\infty(x', v_\varepsilon)}{t} \leq g^\infty(x,z) + \lim_{t\to\infty} \frac{b||v_\varepsilon||_\infty}{t} = g^\infty(x,z),$$

$$\bar{g}_\varepsilon^\infty(x,z) = \lim_{\substack{t\to\infty\\ z'\to z\\ x'\to x}} \frac{g(x', tz' + v_\varepsilon)}{t} \geq \lim_{\substack{t\to\infty\\ z'\to z\\ x'\to x}} \frac{g(x', tz') - g^\infty(x', v_\varepsilon)}{t} \geq g^\infty(x,z) - \lim_{t\to\infty} \frac{b||v_\varepsilon||_\infty}{t} = g^\infty(x,z)$$

and thus the claim. Further, $g(\,.\,, z + v_\varepsilon)$ is still lower semicontinuous, convex in the second argument and there exists a constant $\tilde{b} = b + b||v_\varepsilon||_\infty > 0$ such that $\bar{g}_\varepsilon(\,.\,, z) \leq \tilde{b}(1 + |z|)$. Then we use (4.1) and Corollary 2.38 on $g(\,.\,, z + v_\varepsilon)$ and obtain:

$$\begin{aligned}
\liminf_{k\to\infty} \int_{\overline{\Omega}} g(x, \nu_k + v) &\geq \liminf_{k\to\infty} \int_{\overline{\Omega}} g(x, \nu_k + v_\varepsilon) - \int_{\overline{\Omega}} g^\infty(x, v_\varepsilon - v) \,\mathrm{d}x \\
&\geq \int_{\overline{\Omega}} g(x, \nu + v_\varepsilon) - \int_{\overline{\Omega}} b|v - v_\varepsilon| \,\mathrm{d}x \\
&\geq \int_{\overline{\Omega}} g(x, \nu + v) - 2b \int_{\overline{\Omega}} |v - v_\varepsilon| \,\mathrm{d}x \\
&\geq \int_{\overline{\Omega}} g(x, \nu + v) - 2b\varepsilon.
\end{aligned}$$

With $\varepsilon \searrow 0$ the proof is complete. □

Remark. This theorem obviously also holds with the open set Ω instead of the closure and similarly we can use the theorem for BV functions, i.e.

$$\liminf_{k\to\infty} \int_{\Omega} g(\,.\,, \mathrm{D}u_k + v) \geq \int_{\Omega} g(\,.\,, \mathrm{D}u + v) \tag{4.3}$$

for $u_k, u \in \mathrm{BV}(\Omega)$, $u_k \to u$ weakly* in $\mathrm{BV}(\Omega)$ and some $v \in \mathrm{L}^1(\Omega, \mathbb{R}^n)$.

For the continuity theorem version, we get:

Corollary 4.2 (Shifted continuity theorem).
For an open bounded set $\Omega \subset \mathbb{R}^n$ with $|\partial\Omega| = 0$ and g as above, we have: If ν_k converges area-strictly to ν in $\mathrm{RM}(\overline{\Omega}, \mathbb{R}^n)$, then for $v \in \mathrm{L}^1(\overline{\Omega}, \mathbb{R}^n)$ we have

$$\lim_{k\to\infty} \int_{\overline{\Omega}} g(\,.\,, \nu_k + v) = \int_{\overline{\Omega}} g(\,.\,, \nu + v). \tag{4.4}$$

Proof. We argue similarly as for the lower semicontinuity theorem and obtain with Theorem 2.42:

$$\begin{aligned}
\limsup_{k\to\infty} \int_{\overline{\Omega}} g(x, \nu_k + v) &\leq \limsup_{k\to\infty} \int_{\overline{\Omega}} g(x, \nu_k + v_\varepsilon) + \int_{\overline{\Omega}} g^\infty(x, v - v_\varepsilon)\,\mathrm{d}x \\
&\leq \int_{\overline{\Omega}} g(x, \nu + v_\varepsilon) + \int_{\Omega} b|v - v_\varepsilon|\,\mathrm{d}x \\
&\leq \int_{\overline{\Omega}} g(x, \nu + v) + 2b \int_{\overline{\Omega}} |v - v_\varepsilon|\,\mathrm{d}x \\
&\leq \int_{\overline{\Omega}} g(x, \nu + v) + 2b\varepsilon.
\end{aligned}$$

With $\varepsilon \searrow 0$ and Corollary 4.1 the result is proven. □

Remark. This theorem holds obviously for non-convex but still continuous integrands as well as on open sets Ω and we obtain a version for BV functions, too, namely

$$\limsup_{k\to\infty} \int_{\Omega} g(\,.\,, \mathrm{D}u_k + v) \leq \int_{\Omega} g(\,.\,, \mathrm{D}u + v). \tag{4.5}$$

for $u_k, u \in \mathrm{BV}(\Omega)$, $u_k \to u$ weakly* in $\mathrm{BV}(\Omega)$ and some $v \in \mathrm{L}^1(\Omega, \mathbb{R}^n)$.
A similar result was established in [46, Section 5] for gradient Young measures.

Now we are able to prove the following convergence a.e. result:

Theorem 4.3 (Convergence almost everywhere of the absolutely continuous parts).
Let Ω be an open bounded set and $\nu_k \in \mathrm{RM}(\Omega, \mathbb{R}^n)$ converge area-strictly to $\nu \in \mathrm{RM}(\Omega, \mathbb{R}^n)$ for $k \to \infty$, then there exists a subsequence such that $\nu_k^a \to \nu^a$ a.e., where ν_k^a and ν^a are the absolute continuous parts (or densities) with respect to the Lebesgue measure.

Proof. Since ν_k converges area-strictly to ν, it also converges weakly*. Next, we set S as the union of all singular sets of ν and ν_k with respect to the Lebesgue measure and thus have $|S| = 0$ and $\nu^a(S) = 0 = \nu^s(\Omega \setminus S)$ with the singular part with respect to the Lebesgue measure ν^s. Similarly, the equations hold for all ν_k, $k \in \mathbb{N}$ as well. By regularity we can find open sets O_l for $l \in \mathbb{N}$ such that $S \subset O_l$ and $|O_l| \leq \frac{1}{l}$ and have, using Corollary 4.1 for open sets (see remark)

$$\liminf_{k\to\infty} \int_{O_l} |\nu_k - \nu^a| \geq |\nu^s|(O_l) = |\nu^s|(S). \tag{4.6}$$

Using Corollary 4.2 for open sets (see remark), we also obtain

$$\limsup_{k\to\infty} \int_{\Omega} |\nu_k - \nu^a| \leq |\nu^s|(\Omega) = |\nu^s|(O_l) = |\nu^s|(S). \tag{4.7}$$

Combining both estimates (4.6) and (4.7), we obtain

$$0 \leq \limsup_{k\to\infty} \int_{\Omega\setminus O_l} |\nu_k^a - \nu^a| \leq \limsup_{k\to\infty} \int_{\Omega\setminus O_l} |\nu_k - \nu^a| \leq 0.$$

By a standard argument we can choose for each l an almost everywhere convergent subsequence on $\Omega \setminus O_l$ and by a simple diagonal argument with $l \to \infty$ we get an a.e. convergent subsequence on $\Omega \setminus S$ and thus on Ω. Since $\nu_k^a \mathcal{L}^n = \nu_k$ on $\Omega \setminus S$, the proof is complete. □

Remark. This construction is obviously also possible for more general locally compact metric measure spaces with some base measure μ. The concept of area-strict convergence on such space (X, μ) has to be adjusted to strict convergence of the measure $|(\mu, \nu_k)|(X)$ to $|(\mu, \nu)|(X)$ for $k \to \infty$ and we obtain an μ-a.e. convergent subsequence $\nu_k^a \to \nu^a$ with ν_k^a and ν^a now being the absolute continuous parts with respect to the measure μ.

We can apply this theorem for example on sequences in $\mathrm{W}^{1,1}(\Omega)$ converging in area to a limit in $\mathrm{BV}(\Omega)$:

Corollary 4.4 (A.e. convergence of gradients)**.**
Let $(u_k)_{k\in\mathbb{N}} \in \mathrm{W}^{1,1}$ (or BV) converge in area to $u \in \mathrm{BV}$. Then there exists a subsequence with $\nabla u_k \to \nabla u$ a.e.

4.2 A truncation argument

To construct a recovery sequence in the way of Theorem 3.15 which obeys both obstacles, we approximate $\min\{u, v\}$ and $\max\{u, v\}$ separately using only the space provided in $\{u < v\}$ and $\{u > v\}$, respectively. More precisely, we want use the one-sided approximation on $\min\{u, v\}$ from above and truncate the sequence by v and similarly for $\max\{u, v\}$. Thus a truncation argument for sequence converging from one side, here above, is sufficient. As usual, $\Omega \subset \mathbb{R}^n$ is an open and bounded set.

Theorem 4.5 (Truncation argument)**.**
Let f and Ω be as in the beginning of this section and f^∞ the corresponding recession function. Let $u \in \mathrm{BV}(\Omega)$ with $u \leq h$ $\mathcal{L}^n$-almost everywhere for a function $h \in \mathrm{W}^{1,1}(\Omega)$. If $u_k \in \mathrm{W}^{1,1}(\Omega)$ converge area-strictly from above to u and $w_k \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ converge in L^1 to 0 from above, we have

$$\begin{aligned}\limsup_{k\to\infty} \int_{\Omega} f(\mathrm{D}\tilde{u}_k)\,\mathrm{d}x &= \lim_{k\to\infty} \int_{\Omega} f(\mathrm{D}u_k)\,\mathrm{d}x + \limsup_{k\to\infty} \int_{\mathbb{R}^n} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x \\ &= \int_{\Omega} f(\mathrm{D}u) + \limsup_{k\to\infty} \int_{\mathbb{R}^n} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x\end{aligned}$$

for a not relabeled sub-sequence of $\tilde{u}_k = \min\{u_k + w_k, h\}$.

Proof. The proof relies on the fact that $\nabla h\mathcal{L}^n$ is absolute continuous with respect to $\mathcal{L}^n$ and convergence in measure as well as the Lemma of Fatou. For a given $\varepsilon > 0$, we divide Ω into three subsets, $\Omega_0 := \{u = h\}$, $\Omega_\varepsilon := \{h - u > \varepsilon\}$ and $\tilde{\Omega}_\varepsilon = \Omega \setminus (\Omega_0 \cup \Omega_\varepsilon)$. For reason of simplicity, we drop the notion of the precise representative in the following proof. We can now estimate on each of these sets using that, for two BV functions v_1 and v_2 and thus especially for $\mathrm{W}^{1,1}(\Omega)$ functions, we have $\mathrm{D}^a v_1 = \mathrm{D}^a v_2$ almost everywhere on $\{v_1 = v_2\}$, see [3, Proposition 3.92(a) and Remark 3.93]:

1. On Ω_0 we have

$$\begin{aligned}\int_{\Omega_0} f(\mathrm{D}\tilde{u}_k)\,\mathrm{d}x &= \int_{\Omega_0} f(\nabla u)\,\mathrm{d}x \\ &= \int_{\Omega_0} f(\mathrm{D}u_k)\,\mathrm{d}x + \int_{\Omega_0} f(\nabla u)\,\mathrm{d}x - \int_{\Omega_0} f(\nabla u_k)\,\mathrm{d}x.\end{aligned}$$

Restricting to a subsequence, where ∇u_k converges to ∇u almost everywhere provided by Corollary 4.4, we can use Fatou's Lemma and conclude, since f is non-negative, with

$$\limsup_{k\to\infty}\left(\int_{\Omega_0} f(\nabla u)\,\mathrm{d}x - \int_{\Omega_0} f(\mathrm{D}u_k)\,\mathrm{d}x\right) \leq 0$$

that

$$\limsup_{k\to\infty}\int_{\Omega_0} f(\mathrm{D}\tilde{u}_k)\,\mathrm{d}x \leq \limsup_{k\to\infty}\int_{\Omega_0} f(\mathrm{D}u_k)\,\mathrm{d}x.$$

2. On Ω_ε we subdivide further for arbitrary but fixed $k \in \mathbb{N}$:
If $\tilde{u}_k - u < \varepsilon$ the truncation has no effect on $\tilde{u}_k$ and we can estimate with (2.3):

$$\int_{\Omega_\varepsilon\cap\{\tilde{u}_k-u<\varepsilon\}} f(\mathrm{D}\tilde{u}_k)\,\mathrm{d}x \leq \int_{\Omega_\varepsilon\cap\{\tilde{u}_k-u<\varepsilon\}} f(\mathrm{D}u_k)\,\mathrm{d}x + \int_{\Omega_\varepsilon\cap\{\tilde{u}_k-u<\varepsilon\}} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x.$$

Else, the truncation may have an effect and we obtain, using (2.3) and an inequality of the type $f(\mathrm{D}\min\{a,b\}) \leq f(\nabla a) + f(\nabla b)$ for $a, b \in \mathrm{W}^{1,1}$ which holds at least $\mathcal{L}^n$-a.e.:

$$\begin{aligned}\int_{\Omega_\varepsilon\setminus\{\tilde{u}_k-u<\varepsilon\}} f(\mathrm{D}\tilde{u}_k)\,\mathrm{d}x \leq &\int_{\Omega_\varepsilon\setminus\{\tilde{u}_k-u<\varepsilon\}} f(\mathrm{D}u_k)\,\mathrm{d}x + \int_{\Omega_\varepsilon\setminus\{\tilde{u}_k-u<\varepsilon\}} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x \\ &+ \int_{\Omega_\varepsilon\setminus\{\tilde{u}_k-u<\varepsilon\}} f(\mathrm{D}h)\,\mathrm{d}x.\end{aligned}$$

These estimates together imply

$$\int_{\Omega_\varepsilon} f(\mathrm{D}\tilde{u}_k)\,\mathrm{d}x \leq \int_{\Omega_\varepsilon} f(\mathrm{D}u_k)\,\mathrm{d}x + \int_{\Omega_\varepsilon} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x + \int_{\Omega_\varepsilon\cap\{\tilde{u}_k-u\geq\varepsilon\}} f(\mathrm{D}h)\,\mathrm{d}x.$$

At this stage we notice that for $k \to \infty$ and by the convergence of w_k to 0 and $u_k + w_k$ to u

in measure and $\mathrm{D}h$ being in $\mathrm{L}^1(\Omega)$ we can estimate:

$$\int\limits_{\Omega_\varepsilon \cap \{\tilde{u}_k - u \geq \varepsilon\}} f(\mathrm{D}h)\,\mathrm{d}x \to 0.$$

Thus the last term may be omitted in the limit.

3. For $\tilde{\Omega}_\varepsilon$, we have $\mathcal{L}^n(\tilde{\Omega}_\varepsilon) \to 0$ for $\varepsilon \searrow 0$ and can simply estimate

$$\int\limits_{\tilde{\Omega}_\varepsilon} f(\mathrm{D}\tilde{u}_k)\,\mathrm{d}x \leq \int\limits_{\tilde{\Omega}_\varepsilon} f(\mathrm{D}u_k)\,\mathrm{d}x + \int\limits_{\tilde{\Omega}_\varepsilon} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x + \int\limits_{\tilde{\Omega}_\varepsilon} f(\mathrm{D}h)\,\mathrm{d}x.$$

Further, we have

$$\lim_{\varepsilon \searrow 0} \int\limits_{\tilde{\Omega}_\varepsilon} f(\mathrm{D}h)\,\mathrm{d}x = 0$$

and again this term may be omitted in the final calculation.

Combining all results, we arrive at

$$\limsup_{k\to\infty} \int\limits_{\Omega} f(\mathrm{D}\tilde{u}_k)\,\mathrm{d}x \leq \limsup_{k\to\infty} \int\limits_{\Omega} f(\mathrm{D}u_k)\,\mathrm{d}x + \int\limits_{\mathbb{R}^n} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x.$$

Since u_k converges in area to u, the rest of theorem follows easily, where the equality instead of $=$ is obtained through the lower semicontinuity of $\int\limits_{\Omega} f(\mathrm{D}u)\,\mathrm{d}x$ and the non-negativity of the other terms. □

This truncation theorem has some nice implications on the manipulation of the sequences occurring in this thesis. The first ensures the area-strict convergence for a truncated sequence.

Corollary 4.6 (Truncation of an area-strictly convergent sequence)**.**
If $u_k \in \mathrm{W}^{1,1}(\Omega)$ converge area-strictly to $u \in \mathrm{BV}(\Omega)$ from above with $u \leq h$ $\mathcal{L}^n$-a.e. for some $h \in \mathrm{W}^{1,1}(\Omega)$, then there exists a subsequence such that $\min\{u_k, h\}$ converges in area to u.

Proof. Apply Theorem 4.5 with $w_k = 0$ and $f(z) = \sqrt{(1 + |z|^2)}$. With the lower semicontinuity of the functional we obtain the desired result. □

Next, we show that the from above in area converging sequence can be chosen monotonic.

Corollary 4.7 (Monotonic area-strict approximation from above)**.**
For $u \in \mathrm{BV}(\Omega)$, there exists a non-increasing sequence $u_k \in \mathrm{W}^{1,1}(\Omega)$ converging in area from above to u.

Proof. By Theorem 2.44 we obtain a sequence u_k, converging in area from above to u. Using Corollary 4.6 iteratively with $h_n = u_n$, $n \in \mathbb{N}$ on u_k with $k \geq n$ we obtain the desired sequence. □

Further, it follows that a sequence provided by 2. from Proposition 3.9 can also be truncated by a function $h \geq \psi$, $h \in \mathrm{W}^{1,1}$ without changing the provided estimate:

Corollary 4.8 (Truncation of sequence converging to the obstacle term).
If $w_k \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ *converge in* $\mathrm{L}^1(\mathbb{R}^n)$ *to* 0 *with the estimate from 2. in Proposition 3.9, then for* $h > \psi$, $h \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ *the sequence* $\min\{w_k, h\}$ *fulfills the estimate as well.*

Proof. Use Theorem 4.5 with $u \equiv 0 \equiv u_k$. □

Remark. Such sequence w_k can also be chosen monotone decreasing with the same argument like in the proof of Corollary 4.7.

Last and most important, the statement of Theorem 4.5 remains true if instead of f a shifted integrand is used:

Corollary 4.9 (Truncation argument for shifted integrand).
Assume the same assumptions as in Theorem 4.5. For a given $v \in \mathrm{L}^1(\Omega;\mathbb{R}^n)$ *and with the integrand* $f(.+v)$*, we have*

$$\begin{aligned}\limsup_{k\to\infty}\int_\Omega f(\mathrm{D}\tilde{u}_k + v)\,\mathrm{d}x &\le \limsup_{k\to\infty}\int_\Omega f(\mathrm{D}u_k + v)\,\mathrm{d}x + \limsup_{k\to\infty}\int_{\mathbb{R}^n} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x\\ &\le \int_\Omega f(\mathrm{D}u + v) + \limsup_{k\to\infty}\int_{\mathbb{R}^n} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x.\end{aligned}$$

Proof. The proof is identical to the one provided for Theorem 4.5 with the use of the shifted version of Reshetnyak's continuity theorem provided in Corollary 4.2, see also the associated remark. □

Remark. All results in this section hold as well if one considers sequences area-strict approximations from below and $w_k \le 0$, since the truncation argument can be applied for such sequences in a similar fashion. For $w_k \le 0$ it is convenient to use $f^\infty(\mathrm{D}w_k) = \tilde{f}^\infty(-\mathrm{D}w_k)$ to be able to apply the continuity part of Proposition 3.10 without further changes.

4.3 Relaxation of the double obstacle problem

We first begin with proving that the functional

$$\mathcal{F}^{\psi_1,\psi_2}[u] = \int_\Omega f(\mathrm{D}u) + \int_\Omega (\psi_1 - u^+)_+ \,\mathrm{d}\varsigma_{f^\infty} + \int_\Omega (u_- - \psi_2)_+ \,\mathrm{d}\varsigma_{\tilde{f}^\infty} \text{ on } \mathrm{BV}(\Omega)$$

and $\mathcal{F}^{\psi_1,\psi_2}[u] = +\infty$ on $\mathrm{L}^1(\Omega)\setminus\mathrm{BV}(\Omega)$ is lower semicontinuous for open bounded Ω, f and ψ_1,ψ_2 like in the beginning of this section. Under the assumption that a function $v \in \mathrm{W}^{1,1}(\Omega)$ exists such that for the two Borel functions (obstacles) $\psi_1,\psi_2 : \mathbb{R}^n \to \mathbb{R}\cup\{\pm\infty\}$ holds $\psi_1 \le v^* \le \psi_2$ $\mathcal{H}^{n-1}$-a.e., we provide afterwards a recovery sequences to prove that this functional is indeed the relaxation of $\mathrm{F}^{\psi_1,\psi_2}$ and thus the first major result announced in the introduction, yet first without boundary conditions.

Theorem 4.10 (Lower semicontinuity of $\mathcal{F}^{\psi_1,\psi_2}$).
The functional $\mathcal{F}^{\psi_1,\psi_2}$ *is lower semicontinuous with respect to convergence in* L^1.

Proof. We are only looking at limit functions $u \in \mathrm{BV}(\Omega)$ with $\mathcal{F}^{\psi_1,\psi_2}[u] < \infty$. In the other cases, either the function u

a) is in $\mathrm{L}^1(\Omega) \setminus \mathrm{BV}(\Omega)$ and we trivially get the lower semicontinuity,

b) violates the obstacle condition even in the $\mathcal{L}^n$-a.e. sense and thus u_k for all k large enough as well, similar to the proof in Theorem 3.14, and one obtains that at least one of the integrals with respect to the generalized De Giorgi measure is infinite, or

c) at least one of the integrals with respect to the generalized De Giorgi measure is $+\infty$, where the following estimate can be used:
If, without loss of generality, $\int_\Omega (\psi_1 - u^+)_+ \,\mathrm{d}\varsigma_{f^\infty} = +\infty$ and we have $u_k \to u$ in L^1, we can restrict ourselves to the case $u_k \in \mathrm{BV}(\Omega)$, since otherwise the functional takes the value $+\infty$ constantly. For this sequence, we can estimate, using the lower semicontinuity of $\mathcal{F}^{\psi_1}$ provided by Theorem 3.16:

$$\liminf_{k\to\infty} \mathcal{F}^{\psi_1,\psi_2}[u_k] \geq \liminf_{k\to\infty} \mathcal{F}^{\psi_1}[u_k] \geq \mathcal{F}^{\psi_1}[u] = +\infty.$$

For the case $\mathcal{F}^{\psi_1,\psi_2}[u] < \infty$ and given $u_k \in \mathrm{L}^1(\Omega)$ with $u_k \to u$ in $\mathrm{L}^1(\Omega)$, we again can restrict ourselves to $u_k \in \mathrm{BV}(\Omega)$. Then we divide Ω in the two Borel sets $\Omega_1 := \{u^- \leq \psi_2\}$ and $\Omega_2 := \{u^- > \psi_2\}$. We have $(u^- - \psi_2)_+ = 0$ on Ω_1 and since $u^+ \geq u^- > \psi_2 \geq \psi_1$, we get $(\psi_1 - u^+)_+ = 0$ on Ω_2. Since $f(\mathrm{D}u) + (\psi_1 - u^+)_+ \varsigma_{f^\infty}$ and $f(\mathrm{D}u) + (u^- - \psi_2)_+ \varsigma_{\tilde{f}^\infty}$ are regular measures, we can find for given $\varepsilon > 0$ disjoint compact sets $K_1^\varepsilon \subset \Omega_1$ and $K_2^\varepsilon \subset \Omega_2$ such that

$$\begin{aligned}
[f(\mathrm{D}u) + (\psi_1 - u^+)_+ \varsigma_{f^\infty}](\Omega_1) - \varepsilon &\leq [f(\mathrm{D}u) + (\psi_1 - u^+)_+ \varsigma](K_1^\varepsilon),\\
[f(\mathrm{D}u) + (u^- - \psi_2)_+ \varsigma](\Omega_2) - \varepsilon &\leq [f(\mathrm{D}u) + (u^- - \psi_2)_+ \varsigma_{\tilde{f}^\infty}](K_2^\varepsilon).
\end{aligned}$$

Since K_1^ε and K_2^ε are disjoint, we can find disjoint open sets $K_1^\varepsilon \subset O_1^\varepsilon$ and $K_2^\varepsilon \subset O_2^\varepsilon$. Using Theorem 3.16 on O_1^ε and O_2^ε, we obtain

$$\begin{aligned}
[f(\mathrm{D}u) + (\psi_1 - u^+)_+ \varsigma_{f^\infty}](\Omega_1) - \varepsilon &\leq [f(\mathrm{D}u) + (\psi_1 - u^+)_+ \varsigma_{f^\infty}](K_1^\varepsilon)\\
&\leq [f(\mathrm{D}u) + (\psi_1 - u^+)_+ \varsigma_{f^\infty}](O_1^\varepsilon)\\
&\leq \liminf_{k\to\infty} \int_{O_1^\varepsilon} f(\mathrm{D}u_k)\,\mathrm{d}x + \int_{O_1^\varepsilon} (\psi_1 - u_k^+)_+ \,\mathrm{d}\varsigma_{f^\infty},\\
[f(\mathrm{D}u) + (u^- - \psi_2)_+ \varsigma_{\tilde{f}^\infty}](\Omega_2) - \varepsilon &\leq [f(\mathrm{D}u) + (u^- - \psi_2)_+ \varsigma_{\tilde{f}^\infty}](K_2^\varepsilon)\\
&\leq [f(\mathrm{D}u) + (u^- - \psi_2)_+ \varsigma_{\tilde{f}^\infty}](O_2^\varepsilon)\\
&\leq \liminf_{k\to\infty} \int_{O_2^\varepsilon} f(\mathrm{D}u_k)\,\mathrm{d}x + \int_{O_2^\varepsilon} (u_k^- - \psi_2)_+ \,\mathrm{d}\varsigma_{\tilde{f}^\infty}
\end{aligned}$$

for any given $u_k \in \mathrm{L}^1(\Omega)$, $k \in \mathbb{N}$ with $u_k \to u$ in L^1. Both estimates combined and with $\varepsilon \searrow 0$ the proof is complete. □

Next, we provide the proof for the representation of the relaxation, first for functions $u \in \mathrm{W}^{1,1}(\Omega)$ and then the general case $u \in \mathrm{BV}(\Omega)$.

Theorem 4.11 (Relaxation for functions in $\mathrm{W}^{1,1}(\Omega)$)**.**
For $u \in \mathrm{W}^{1,1}(\Omega)$, we have with f, f^∞ and Ω as before:

$$\overline{\mathrm{F}}^{\psi_1,\psi_2}[u] = \mathcal{F}^{\psi_1,\psi_2}[u].$$

Proof. By lower semicontinuity provided by Theorem 4.10 we have $\overline{\mathrm{F}}^{\psi_1,\psi_2} \geq \mathcal{F}^{\psi_1,\psi_2}$. It remains to construct a recovery sequence, to prove $\overline{\mathrm{F}}^{\psi_1,\psi_2} \leq \mathcal{F}^{\psi_1,\psi_2}$:
First of all we only consider $u \in \mathrm{W}^{1,1}(\Omega)$ with $\mathcal{F}^{\psi_1,\psi_2}[u] < \infty$ since otherwise the constant sequence $u_k = u$ suffices. Using Proposition 3.9 and 3.10, we obtain two sequences w_k and $w^k \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ with $w_k^* \geq (\psi_1 - u^*)_+$, $(w^k)^* \leq -(u^* - \psi_2)_+ = -(\psi_2 - u^*)_-$ and with

$$\limsup_{k\to\infty} \int_{\mathbb{R}^n} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x \leq \int_\Omega (\psi_1 - u^*)_+\,\mathrm{d}\varsigma_{f^\infty}$$

as well as

$$\limsup_{k\to\infty} \int_{\mathbb{R}^n} \tilde{f}^\infty(-\mathrm{D}w^k)\,\mathrm{d}x \leq \int_\Omega (u^* - \psi_2)_+\,\mathrm{d}\varsigma_{\tilde{f}^\infty}.$$

Using Corollary 4.8 the sequences can be chosen such that $0 \leq w_k \leq (v-u)_+$ and $0 \geq w^k \geq -(v-u)_-$, where $v \in \mathrm{W}^{1,1}(\Omega)$ is the given function, which lies in between the obstacles, i.e. $\psi_1 \leq v^* \leq \psi_2$. This leads to $\psi_1 \leq u + w_k + w^k \leq \psi_2$ for the Lebesgue representatives $\mathcal{H}^{n-1}$-a.e., since we can estimate on $\{u \leq v\}$ with $w^k = 0$ and $(v-u)_+ = v - u$:

$$\psi_1 \leq u + w_k = u + w_k + w^k = u + w_k \leq u + v - u = v \leq \psi_2$$

and on $\{u \geq v\}$ with $w_k = 0$ and $-(v-u)_- = v - u$ we obtain:

$$\psi_1 \leq v = u + v - u = u + w^k = u + w_k + w^k = u + w^k \leq \psi_2.$$

Plugging the sequence in, we can estimate, using (2.3):

$$\begin{aligned}
\limsup_{k\to\infty} \mathrm{F}^{\psi_1,\psi_2}[u + w_k + w^k] &\leq \mathrm{F}[u] + \limsup_{k\to\infty}\left[\int_{\mathbb{R}^n} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x + \int_{\mathbb{R}^n} f^\infty(\mathrm{D}w^k)\,\mathrm{d}x\right] \\
&= \mathrm{F}[u] + \limsup_{k\to\infty}\left[\int_{\mathbb{R}^n} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x + \int_{\mathbb{R}^n} \tilde{f}^\infty(-\mathrm{D}w^k)\,\mathrm{d}x\right] \\
&\leq \mathcal{F}^{\psi_1,\psi_2}[u].
\end{aligned}$$

And thus we have $\overline{\mathrm{F}}^{\psi_1,\psi_2} = \mathcal{F}^{\psi_1,\psi_2}$ on $\mathrm{W}^{1,1}$. □

Next, we need a way to adapt this method for $u \in \mathrm{BV}(\Omega)$.

Theorem 4.12 (Relaxation for functions in $\mathrm{BV}(\Omega)$)**.**
For an open bounded set Ω and the functional $\mathrm{F}^{\psi_1,\psi_2}$ with two Borel functions $\psi_1 \leq \psi_2$ $\mathcal{H}^{n-1}$-a.e., a convex integrand $f : \mathbb{R}^n \to [0, +\infty)$ of linear growth with $a|z| \leq f(z) \leq b(1+|z|)$, the relaxation

of $\mathrm{F}^{\psi_1,\psi_2}$ *is given by*

$$\overline{\mathrm{F}}^{\psi_1,\psi_2} = \mathcal{F}^{\psi_1,\psi_2}.$$

Proof. Since we know, that $\mathcal{F}^{\psi_1,\psi_2}$ is lower semicontinuous and agrees with $\overline{\mathrm{F}}^{\psi_1,\psi_2}$ on $\mathrm{W}^{1,1}(\Omega)$, we only need to provide a recovery sequence for $u \in \mathrm{BV}(\Omega)$ and can limit ourselves to provide this exclusively for $u \in \mathrm{BV}(\Omega)$ with $\mathcal{F}^{\psi_1,\psi_2}[u] < +\infty$ since otherwise $u_k \equiv u$ for all $k \in \mathbb{N}$ may be used. Next, we use Theorem 2.44 to obtain a sequence $u_k \in \mathrm{W}^{1,1}(\Omega)$ converging in area to $u_v = \min\{u, v\}$ from above. Further, we obtain a sequence $w_k \in \mathrm{W}^{1,1}(\Omega)$ from Proposition 3.9 with $0 \le w_k \to 0$ in L^1, $(w_k)^* \ge (\psi_1 - u^+)_+$ and with

$$\limsup_{k\to\infty} \int_{\mathbb{R}^n} f^\infty(\mathrm{D}w_k)\,\mathrm{d}x \le \int_\Omega (\psi_1 - u^+)_+\,\mathrm{d}\varsigma_{f^\infty}$$

and define $\tilde{u}_k := \min\{u_k + w_k, v\}$. Similarly, we get a sequence $u^k \in \mathrm{W}^{1,1}(\Omega)$ converging in area from below to $u^v := \max\{u, v\}$ and by Proposition 3.10 a sequence $w^k \in \mathrm{W}^{1,1}(\Omega)$ with $0 \ge w^k \to 0$ in L^1, $(w_k)^* \le -(u^- - \psi_2)_+$ and with

$$\limsup_{k\to\infty} \int_{\mathbb{R}^n} \tilde{f}^\infty(-\mathrm{D}w^k)\,\mathrm{d}x \le \int_\Omega (u^- - \psi_2)_+\,\mathrm{d}\varsigma_{\tilde{f}^\infty}$$

and define $\tilde{u}^k := \max\{u^k + w^k, v\}$.
Then we define $\overline{u}_k = \tilde{u}_k + \tilde{u}^k - v$ with $\psi_1 \le \overline{u}_k \le \psi_2$, since on $\{u \le v\}$, we have $\tilde{u}^k = v$ and $\tilde{u}_k$ fulfills the constraint by construction like in the $\mathrm{W}^{1,1}$-case and on $\{u \ge v\}$, we have similarly $\tilde{u}_k = v$ and $\tilde{u}^k$ fulfills the obstacle constraint. Further, $\overline{u}_k$ converges to u in $\mathrm{L}^1(\Omega)$. Now we can use Corollary 4.9 with the integrands $f^\infty(\,.\, - \mathrm{D}^a u_v)$, $f^\infty(\,.\, - \mathrm{D}^a u^v)$ and with the associated recession function which is equal to f^∞, and the fact $\mathrm{D}^a u + \mathrm{D}v = \mathrm{D}^a u_v + \mathrm{D}^a u^v$ to conclude

$$\begin{aligned}
\limsup_{k\to\infty} \int_\Omega f(\mathrm{D}\overline{u}_k)\,\mathrm{d}x &\le \limsup_{k\to\infty} \int_\Omega f(\mathrm{D}\overline{u}_k - \mathrm{D}^a u + \mathrm{D}^a u)\,\mathrm{d}x \\
&\le \limsup_{k\to\infty} \left(\int_\Omega f(\mathrm{D}^a u)\,\mathrm{d}x \right. \\
&\quad \left. + \int_\Omega f^\infty(\mathrm{D}\tilde{u}_k - \mathrm{D}^a u_v)\,\mathrm{d}x + \int_\Omega f^\infty(\mathrm{D}\tilde{u}^k - \mathrm{D}^a u^v)\,\mathrm{d}x \right) \\
&\le \int_\Omega f(\mathrm{D}^a u)\,\mathrm{d}x + \int_\Omega f^\infty(\mathrm{D}u_v - \mathrm{D}^a u_v) + \int_\Omega f^\infty(\mathrm{D}u^v - \mathrm{D}^a u^v) \\
&\quad + \int_\Omega (\psi_1 - u^+)_+\,\mathrm{d}\varsigma_{f^\infty} + \int_\Omega (u^- - \psi_2)_+\,\mathrm{d}\varsigma_{\tilde{f}^\infty} \\
&= \int_\Omega f(\mathrm{D}u) + \int_\Omega (\psi_1 - u^+)_+\,\mathrm{d}\varsigma_{f^\infty} + \int_\Omega (u^- - \psi_2)_+\,\mathrm{d}\varsigma_{\tilde{f}^\infty}.
\end{aligned}$$

In the last step, we used that $f^\infty(\mathrm{D}^s u_v) + f^\infty(\mathrm{D}^s u^v)$ sum up to $f^\infty(\mathrm{D}^s u)$, since $\mathrm{D}^c u_v$ and $\mathrm{D}^c u^v$ are mutually singular and possible split jump parts, i.e. if $u^- < v^* < u^+$, add up correctly, since they have the same direction ν_u and $f^\infty(\nu_u)(v^* - u^-) + f^\infty(\nu_u)(u^+ - v^*) = f^\infty(\nu_u)(u^+ - u^-)$. □

Since the major difficulty in the proofs is the construction of the recovery sequence with the help of the truncation argument which is applicable for limits in BV, we obtain the following trivial corollary:

Corollary 4.13 (Double obstacle problem in a BV-setting).
Let Ω be an open set, $f, f^\infty, \psi_1, \psi_2$ as before. Further define the functional

$$\int_\Omega f(\mathrm{D}u) \text{ for } u \in \mathrm{BV}_{\circ,\psi_1,\psi_2}(\Omega)$$

extended by $+\infty$ to $\mathrm{L}^1(\Omega) \setminus \mathrm{BV}_{\circ,\psi_1,\psi_2}(\Omega)$, where

$$\mathrm{BV}_{\circ,\psi_1,\psi_2}(\Omega) = \{u \in \mathrm{BV}(\Omega) : \psi_1 \le u^+ \text{ and } u^- \le \psi_2 \text{ holds } \mathcal{H}^{n-1}\text{-a.e. on } \Omega\}.$$

If there exists a function $v \in \mathrm{W}^{1,1}(\Omega)$ with $\psi_1 \le v^* \le \psi_2$ $\mathcal{H}^{n-1}$-a.e., then the relaxation of the functional is given by $\mathcal{F}^{\psi_1,\psi_2}$.

It is unsatisfactory that the separation function v has to be in $\mathrm{W}^{1,1}$, but the proofs heavily rely on the absolute continuity of the separation functions and thus concentration effects are a problem. The relaxation with only a BV separation function v, i.e. $\psi_1 \le v^+$ and $v^- \le \psi_2$ is a point of interest for further research in the general setting, while for the area functional a result is obtained using the parametric theory, see Section 5.4.

4.4 Relaxation of the Dirichlet double obstacle problem

In this section, we additionally assume that the open bounded set Ω has either Lipschitz boundary or that Ω fulfills $\mathrm{P}(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$ and there exists a constant $M > 0$ such that we have the bounds $-M \le \psi_1 \le \psi_2 \le M$ and thus are able to define a trace on the boundary, see Theorem 2.27 and [61, Lemma 2.3]. The boundedness of the obstacles enforces the boundedness of all $u \in \mathrm{BV}(\Omega)$ with finite functional value $\mathcal{F}_{u_0}^{\psi_1,\psi_2} < +\infty$, i.e. provides $u \in \mathrm{L}^\infty(\Omega)$. Further, we assume that the function $v \in \mathrm{W}^{1,1}(\Omega)$ with $\psi_1 \le v^* \le \psi_2$ on $\overline{\Omega}$ can be extended to a $\mathrm{W}^{1,1}$ function on a ball B_r such that $\Omega \Subset \mathrm{B}_r$ and in addition fulfills $v* = u_0$ on $\partial\Omega$ for given boundary values $u_0 \in \mathrm{L}^1(\partial\Omega, \mathcal{H}^{n-1})$ or $u_0 \in \mathrm{W}^{1,1}(\mathrm{B}_r)$. Under those assumptions, we have:

Theorem 4.14 (Relaxation of the Dirichlet double obstacle problem).
For Ω and v as described above and for integrands as defined at the beginning of Section 4, we have

$$\overline{\mathrm{F}}_{u_0}^{\psi_1,\psi_2} = \mathcal{F}_{u_0}^{\psi_1,\psi_2},$$

with

$$\mathcal{F}_{u_0}^{\psi_1,\psi_2}[u] = \int_{\overline{\Omega}} f(\mathrm{D}u) + \int_\Omega (\psi_1 - u^+)_+ \,\mathrm{d}\varsigma_{f^\infty} + \int_\Omega (u^- - \psi_2)_+ \,\mathrm{d}\varsigma_{\tilde{f}^\infty}$$

for $u \in \mathrm{BV}(\Omega)$ and $\mathcal{F}_{u_0}^{\psi_1,\psi_2} = +\infty$ on $\mathrm{L}^1(\Omega) \setminus \mathrm{BV}(\Omega)$.

Remark. Although $\mathcal{F}_{u_0}^{\psi_1,\psi_2}$ seems to not depend on u_0, the boundary penalization is again encoded in $\int_{\overline{\Omega}} f(\mathrm{D}u)$, since a portion of the integral on the singular set u is given by

$$\int_{\partial\Omega} f^\infty(\nu_\Omega(u_0 - u_{\mathrm{int}}))\,\mathrm{d}\mathcal{H}^{n-1},$$

where ν_Ω is the exterior normal to Ω and u_{int} the inner trace of u on $\partial\Omega$.

Proof. First we extend $v \in \mathrm{W}^{1,1}(\Omega)$ to $v \in \mathrm{W}^{1,1}(\mathrm{B}_r)$ with $\Omega \Subset \mathrm{B}_r$. Further, we set $\psi_1 = \psi_2 = v$ on $\mathrm{B}_r \setminus \overline{\Omega}$ and consider the relaxation of $\mathrm{F}^{\psi_1,\psi_2}$ on B_r. Theorem 4.12 yields that the relaxation is equal to $\mathcal{F}^{\psi_1,\psi_2}$ on B_r. Next, we notice that all $u \in \mathrm{BV}$, where $\mathcal{F}^{\psi_1,\psi_2}[u]$ is finite, are equal on $\mathrm{B}_r \setminus \overline{\Omega}$ and, since $\psi_1 \le v^* = u_{\mathrm{ext}} \le \psi_2$ on $\partial\Omega$, we have $(\psi_1 - u^+)_+ = (u^- - \psi_2)_+ = 0$ on $\partial\Omega$ and thus the integral terms involving the obstacles are 0 on $\partial\Omega$. In reverse, every function $u \in \mathrm{BV}(\Omega)$ such that $\tilde{u} = \mathbb{1}_\Omega u + \mathbb{1}_{\mathrm{B}_r\setminus\Omega} v \in \mathrm{BV}(\mathrm{B}_r)$ can be extended in this way and yields finite values of $\mathcal{F}^{\psi_1,\psi_2}[u]$ on B_r. Thus

$$\mathcal{F}^{\psi_1,\psi_2}_{u_0}[u,\Omega] + \int_{\mathrm{B}_r\setminus\overline{\Omega}} f(\mathrm{D}v) = \mathcal{F}^{\psi_1,\psi_2}[\tilde{u},\mathrm{B}_r],$$

which proves the claim. □

Remark. In the same way, the result of Corollary 4.13 extends to the addition of boundary values.

Another application of these results is to allow inner boundary parts like for slit domains, i.e. domains with interior cuts, enforcing the values by the obstacles. If, for example, the domain $\mathrm{B}^2_1(0) \setminus ([0,1] \times \{0\})$ is considered with some boundary values u_0 on $\mathrm{S}^1_1(0) \cup ([0,1] \times \{0\})$, we can consider the obstacles $\psi_1 = \psi_2 = u_0$ on $\mathrm{S}^1_1(0) \cup ([0,1] \times \{0\})$ and $\psi_1 = -\infty = -\psi_2$ else, the relaxed functional for integrands f like before is given for $u \in \mathrm{BV}(\mathrm{B}^2_1(0))$ by

$$\int_{\mathrm{B}_1(0)\setminus([0,1]\times\{0\})} f(\mathrm{D}u) + \int_{\mathrm{S}^1_1(0)} f^\infty(\nu_{\mathrm{B}_1(0)}(u_0 - u_{\mathrm{int}}))\,\mathrm{d}\mathcal{H}^1$$
$$+ \int_{[0,1]\times\{0\}} f^\infty\left(\begin{pmatrix}0\\1\end{pmatrix}(u_0 - u_{\mathrm{int},1})\right) + f^\infty\left(\begin{pmatrix}0\\-1\end{pmatrix}(u_0 - u_{\mathrm{int},-1})\right)\mathrm{d}\mathcal{H}^1$$

where Proposition 3.11 was used to rewrite the obstacle term involving the slit. The representatives $u_{\mathrm{int},\pm1}$ are the corresponding (inner) traces with respect to the given normal vector $(0,\pm1)^t$.
In general cases, where the slit is the graph of an Lipschitz function, a similar result can be established. If the slit is unrectifiable, wrong boundary values are penalized in the sense of the obstacle constraint, i.e with an integral with respect to the generalized De Giorgi measures ς_{f^∞} and $\varsigma_{\tilde{f}^\infty}$. Obviously one can consider domains with more slits and theoretically also domains, where slits touch each other and so on.

5 Connection between the parametric and non-parametric double obstacle problem for the area functional

A connection between the parametric and non-parametric minimal surface problem is their coincidence if dealing with a graph-type setting. As stated in the introduction, u is a minimizer of the non-parametric problem iff the subgraph is of least perimeter. Since for the single and double obstacle problems the additional obstacle terms in both settings look quite similar, it seems that even in this setting, they may have the same or a similar connection. Since we have no general uniqueness result yet, we can only hope to show that in a non-parametric setting, both problems yields the same infimum and thus the subgraph of a non-parametric minimizer is a parametric minimizer as well. In the parametric and non-parametric case we therefore have to deal with De Giorgi-measures acting on different spaces, i.e. on $\mathbb{R}^{n+1}$ for the parametric and on $\mathbb{R}^n$ for the non-parametric and we thus switch and adapt our notation to $\varsigma^n := \varsigma_{|\,.\,|}$ for the De Giorgi measure with n-dimensional domain of definition and similarly ς^{n+1} and thus indicate the dimension using a superscript if needed.
In this section we will only consider single and double obstacle problems of the non-parametric type on a bounded open set Ω, i.e. the parametric obstacles O_1 and O_2 are given as the sub- and supergraphs of Borel functions $\psi_1, \psi_2 : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ such that $\psi_1 \leq \psi_2$, $\psi_1 \leq v^+$ and $v^- \leq \psi_2$ holds $\mathcal{H}^{n-1}$-a.e. on Ω for some separation function v which we first assume to be in $\mathrm{W}^{1,1}$. Thus the $\mathcal{H}^{n-1}$-a.e. required inequalities can be stated as $\psi_1 \leq v^* \leq \psi_2$. For the single obstacle problem we will without loss of generality only consider the lower single obstacle problem and in that case set $\psi_2 = +\infty$. Under the given assumptions, the relaxation from Section 4 is defined and the corresponding non-parametric functional is given by

$$\mathcal{A}^{\psi_1,\psi_2}[u] = \int_\Omega \sqrt{1+|\mathrm{D}u|^2} + \int_\Omega (\psi_1 - u^+)_+ \,\mathrm{d}\varsigma^n + \int_\Omega (u^- - \psi_2)_+ \,\mathrm{d}\varsigma^n \text{ for } u \in \mathrm{BV}(\Omega).$$

The parametric area functional can be written as

$$\mathrm{P}^{\psi_1,\psi_2}(E) = \mathrm{P}(E, \Omega \times \mathbb{R}) + \varsigma^{n+1}((O_1 \setminus E^+) \cap (\Omega \times \mathbb{R})) + \varsigma^{n+1}(O_2 \cap E^- \cap (\Omega \times \mathbb{R})).$$

We start by stating the important results for the equality of parametric and non-parametric problems for the area functional presented in [39] which is based on [50] and introduces a construction of a subgraph from a suitable set of finite perimeter, which decreases the perimeter. Further, we state some approximation results, which are helpful so avoid reduction effects and be able to treat the obstacle problem and discuss the right translation between parametric and non-parametric problems. After the preparatory section, we prove a first inequality, which shows that the presented construction does not only decrease the perimeter but also correctly takes into account the obstacle functionals. In the next step, we show that the obstacle terms in the parametric and non-parametric setting of the right representative of a subgraph and function, respectively, are equal and thus prove that the subgraph of a non-parametric minimizer is a parametric minimizer of the corresponding problem. In the last two sections, we improve the non-parametric double obstacle problem for the area-functional by allowing BV separation functions and provide a continuity result at the obstacle for minimizers of the double obstacle problem for the area functional.

5.1 Graphification and approximation of Caccioppoli sets

We begin with a connection between the subgraph and the area integral as presented in [39, Theorem 14.6]:

Theorem 5.1 (Connection between the area integral and perimeter of the subgraph)**.**
For $u \in \mathrm{BV}$, we have

$$\int_\Omega \sqrt{1+|\mathrm{D}u|^2} = \mathrm{P}(S_u, \Omega \times \mathbb{R})$$

with $S_u = \{(x,t) \in \Omega \times \mathbb{R} \,:\, t < u(x)\}$.

Remark. Considering obstacle problems, one may be inclined to use a notation, where the graph is included, i.e. define S_u with the defining inequality $t \leq u(x)$. In this case, the theorem remains true but will not yield any advantage regarding the coverage of obstacle parts as we will discuss towards the end of this subsection. We therefore use the definition in the theorem.

Next, we present [39, Theorem 14.8], which allows us to pass from arbitrary sets to subgraph-type sets with smaller perimeter:

Theorem 5.2 (Graphification)**.**
Let $E \subset \Omega \times \mathbb{R}$ be measurable. Suppose that

(i) for almost every $x \in \Omega$ we have

$$\lim_{t\to+\infty} \mathbb{1}_E(x,t) = 0,$$

$$\lim_{t\to-\infty} \mathbb{1}_E(x,t) = 1,$$

(ii) the symmetric difference $E_0 = E\Delta(\Omega \times (-\infty, 0))$ has finite measure.

Then the function

$$w_E(x) = \lim_{k\to\infty} \left(\int_{-k}^{k} \mathbb{1}_E(x,t)\,\mathrm{d}t - k \right)$$

is in $\mathrm{L}^1(\Omega)$ and

$$\int_\Omega \sqrt{1+|\mathrm{D}w_E|^2} \leq \mathrm{P}(E, \Omega \times \mathbb{R}).$$

This is already enough to prove the equality for the obstacle-free case.

Remark. By Fubini's theorem, we have that $|E_0| = ||w_E||_1$ and that the $w_{k,E} = \int_{-k}^{k} \mathbb{1}_E(x,t)\,\mathrm{d}t - k$ converge in L^1 to w_E. Further, if we consider a set E of finite perimeter in $\Omega \times \mathbb{R}$, an application of the isoperimetric inequality will often guarantee the part (ii) of the assumptions and from (ii) and again with the isoperimetric inequality one can show the assumption (i). More precisely, if we consider suitable boundary values, which, for example, are given as the subgraph of a $\mathrm{W}^{1,1}$ function outside of $\Omega \times \mathbb{R}$, the prerequisites are fulfilled. For the double obstacle problem it is often possible to consider the problem without boundary conditions where the assumptions can be obtained through the obstacle condition. In contrast, it is not possible to do so for the single obstacle problem, since $\Omega \times \mathbb{R}$ obviously will contain the obstacle and have zero perimeter in $\Omega \times \mathbb{R}$ and, for example, in the case with a single lower obstacle it is easy to construct an from above unbounded obstacle such that no non-parametric minimizer can exist.

Next we revisit the definitions of the De Giorgi measure to be able to state a first approximation result, which is interesting on its own:

Theorem 5.3 (Equivalence of definitions of the De Giorgi measure, see also [15, Theorema 4])**.** *Let Ω be open and E be a Borel set. It is then equivalent to define the δ-De Giorgi measure in Ω with open sets*

$$\varsigma^\delta(E,\Omega) = \inf\left\{\mathrm{P}(B,\Omega) + \frac{|B\cap\Omega|}{\delta} \,:\, B \text{ open and } E\cap\Omega \subset B\right\}$$

or measurable sets

$$\tilde{\varsigma}^\delta(E,\Omega) = \inf\left\{\mathrm{P}(B,\Omega) + \frac{|B\cap\Omega|}{\delta} \,:\, B \text{ is } \mathcal{L}^n-\text{measurable and } \mathcal{H}^{n-1}((E\setminus B^+)\cap\Omega) = 0\right\}$$

and we have

$$\varsigma^\delta(E,\Omega) = \tilde{\varsigma}^\delta(E,\Omega).$$

Thus, both generate the same measure for $\delta \searrow 0$ which agrees with the in Section 3 defined one for $\Omega = \mathbb{R}^n$, i.e.

$$\varsigma(E,\Omega) = \lim_{\delta\searrow 0}\varsigma^\delta(E,\Omega) = \lim_{\delta\searrow 0}\tilde{\varsigma}^\delta(E,\Omega).$$

Similar as before, the limits can be changed to suprema.

The first approximation results allows us to cover the reduced/essential boundary in an open set with arbitrary small change in the perimeter and the Lebesgue measure:

Proposition 5.4 (Approximation of the perimeter covering the essential/reduced boundary)**.** *Let Ω be an open set and E a Borel set with finite perimeter in Ω. Then for each $\varepsilon > 0$ there exist open sets B_ε such that $\mathcal{F}E \subset \partial^* E \subset B_\varepsilon$, $|B_\varepsilon \cap \Omega| < \varepsilon$ and*

$$\mathrm{P}(E\cup B_\varepsilon,\Omega) \le \mathrm{P}(E,\Omega) + \varepsilon.$$

Similarly, we can find for an arbitrary subset A of E^+ with finite De Giorgi measure open covers B_ε with the same properties, i.e. $|B_\varepsilon \cap \Omega| < \varepsilon$ and $\mathrm{P}(E\cup B_\varepsilon,\Omega) \le \mathrm{P}(E,\Omega) + \varepsilon$.

Proof. First we note, that the essential boundary of E in Ω has finite Hausdorff measure, i.e. $\mathcal{H}^{n-1}(\partial^* E\cap\Omega) < +\infty$, and thus has also finite De Giorgi measure $\varsigma(\partial^* E\cap\Omega) < +\infty$ by (3.3). With the definition of the De Giorgi measure with open sets, we can find for each $\varepsilon > 0$ some $\delta_1 > 0$ such that $\varsigma^\delta(\partial^* E) > \varsigma(\partial^* E) - \varepsilon$ for $0 < \delta \le \delta_1$. Further we can find by the definition of the δ-De Giorgi measure open sets $B_{\delta,\varepsilon}$ containing $\partial^* E$ such that $\mathrm{P}(B_{\delta,\varepsilon},\Omega) + \frac{|B_{\delta,\varepsilon}\cap\Omega|}{\delta} \le \varsigma^\delta(\partial^* E,\Omega) + \varepsilon$. Since $\varsigma^\delta(\partial^* E,\Omega) + \varepsilon \le \varsigma(\partial^* E,\Omega) + \varepsilon < +\infty$, we can estimate

$$|B_{\delta,\varepsilon}\cap\Omega| \le \delta\left(\varsigma(\partial^* E,\Omega) + \varepsilon\right)$$

and thus can find δ_2 such that $|B_{\delta,\varepsilon} \cap \Omega| < \varepsilon$ for $0 < \delta \leq \delta_2$. Therefore we can find for each $\varepsilon > 0$ some $0 < \delta_\varepsilon \leq \min\{\delta_1, \delta_2, \varepsilon\}$, which can be chosen monotonically increasing, i.e. $\delta_\varepsilon \leq \delta_{\varepsilon'}$ for $0 < \varepsilon < \varepsilon'$, and such that $\varsigma^{\delta_\varepsilon}(\partial^* E, \Omega) \geq \varsigma(\partial^* E, \Omega) - \varepsilon$ as well as corresponding sets $B_\varepsilon = B_{\delta_\varepsilon,\varepsilon}$ which fulfill $\mathrm{P}(B_\varepsilon, \Omega) + \frac{|B_\varepsilon \cap \Omega|}{\delta_\varepsilon} \leq \varsigma^{\delta_\varepsilon}(\partial^* E, \Omega) + \varepsilon$ and $|B_\varepsilon \cap \Omega| < \varepsilon$. Since the intersection of finitely many open sets is again open, we may replace B_ε by $B_\varepsilon \cap \Omega$ without losing any properties. It now remains to estimate the perimeter. We have, using Proposition 2.21:

$$\begin{aligned}
\mathrm{P}(E \cup B_\varepsilon) &\leq \mathrm{P}(E, \Omega) + \mathrm{P}(B_\varepsilon, \Omega) - \mathrm{P}(E \cap B_\varepsilon, \Omega) \\
&\leq \mathrm{P}(E, \Omega) + \mathrm{P}(B_\varepsilon, \Omega) + \frac{|B_\varepsilon|}{\delta_\varepsilon} - \mathrm{P}(E \cap B_\varepsilon, \Omega) - \frac{|B_\varepsilon \cap E|}{\delta_\varepsilon} \\
&\leq \mathrm{P}(E, \Omega) + \varsigma^{\delta_\varepsilon}(\partial^* E, \Omega) + \varepsilon - \varsigma^{\delta_\varepsilon}(\partial^* E, \Omega) \\
&\leq \mathrm{P}(E, \Omega) + \varepsilon,
\end{aligned}$$

where we used that $\mathrm{P}(E \cap B_\varepsilon, \Omega) + \frac{|B_\varepsilon \cap E|}{\delta_\varepsilon} \geq \varsigma^{\delta_\varepsilon}(\partial^* E, \Omega)$, since by definition of the open set B_ε we have $\partial^* E \subset E^+ \cap B_\varepsilon \subset (E \cap B_\varepsilon)^+$ and thus the claimed is proven using the alternate definition of the δ-De Giorgi measures and Theorem 5.3.
The proof for arbitrary subsets of E^+ with finite De Giorgi measure is identical. □

Since we will have to construct parametric recovery sequences for the double obstacle problem, we need approximations which do not interfere with the obstacle constraint. Thus, we state the following proposition:

Proposition 5.5 (Truncated approximation of obstacle parts).
Consider some set E with finite perimeter in $\Omega \times \mathbb{R}$ and a lower obstacle O_1 as the subgraph of a Borel function ψ_1 such that $\varsigma^{n+1}(O_1 \setminus E^+, \Omega \times \mathbb{R})$ is finite. Then there exists a sequence C_ε of open sets of finite perimeter with $O_1 \setminus E^+ \subset C_\varepsilon$, $|C_\varepsilon| < \varepsilon$ and

$$\mathrm{P}(E \cup C_\varepsilon, \Omega \times \mathbb{R}) \to \mathrm{P}(E, \Omega \times R) + \varsigma^{n+1}(O_1 \setminus E^+, \Omega \times \mathbb{R})$$

for $\varepsilon \to 0$. In addition, if a separation function $v \in \mathrm{W}^{1,1}(\Omega)$ (or $v \in \mathrm{BV}(\Omega)$) is given and we have $\psi_1 \leq v^+$ and thus $O_1 \subset S_v^+$, we instead may use the sets $C_\varepsilon \cap S_v$ instead of C_ε.

Proof. With a similar argument as in Theorem 5.4, we obtain open sets C_ε containing $O_1 \setminus E^+$ and with $\mathrm{P}(C_\varepsilon, \Omega) + \frac{|C_\varepsilon \cap \Omega|}{\delta_\varepsilon} \leq \varsigma^{\delta_\varepsilon}(O_1 \setminus E^+, \Omega) + \varepsilon$, where δ_ε is chosen like above. We then can use the lower semicontinuity of P^{ψ_1} $(= \mathrm{P}^{\psi_1, +\infty})$ to obtain

$$\mathrm{P}^{\psi_1}(E) \leq \liminf_{\varepsilon \searrow 0} \mathrm{P}(E \cup C_\varepsilon, \Omega \times \mathbb{R})$$

and use Proposition 2.21 as well as that $\varsigma^\delta \leq \varsigma$ holds for all $\delta > 0$ and all sets from the definition of the De Giorgi measure to get the upper bound

$$\begin{aligned}
\mathrm{P}(E \cup C_\varepsilon, \Omega \times \mathbb{R}) &\leq \mathrm{P}(E, \Omega \times \mathbb{R}) + \mathrm{P}(C_\varepsilon, \Omega \times \mathbb{R}) \\
&\leq \mathrm{P}(E, \Omega \times \mathbb{R}) + \mathrm{P}(C_\varepsilon, \Omega \times \mathbb{R}) + \frac{|C_\varepsilon|}{\delta_\varepsilon} \\
&\leq \mathrm{P}(E, \Omega \times \mathbb{R}) + \varsigma^{\delta_\varepsilon, n+1}(O_1 \setminus E^+, \Omega \times \mathbb{R}) + \varepsilon \\
&\leq \mathrm{P}(E, \Omega \times \mathbb{R}) + \varsigma^{n+1}(O_1 \setminus E^+, \Omega \times \mathbb{R}) + \varepsilon
\end{aligned}$$

and thus the first part of the claim. For the second, we replace C_ε with $C_\varepsilon \cap S_v$ and note that $C_\varepsilon \cup S_v$ converges in L^1 to S_v. With the inequality

$$\mathrm{P}(C_\varepsilon \cap S_v, \Omega \times \mathbb{R}) \leq \mathrm{P}(C_\varepsilon, \Omega) + \mathrm{P}(S_v, \Omega \times \mathbb{R}) - \mathrm{P}(C_\varepsilon \cup S_v, \Omega \times \mathbb{R}),$$

together with the lower semicontinuity of the perimeter and thus

$$\limsup_{\varepsilon \searrow 0} \left(\mathrm{P}(S_v, \Omega \times \mathbb{R}) - \mathrm{P}(C_\varepsilon \cup S_v, \Omega \times \mathbb{R}) \right) \leq \mathrm{P}(S_v, \Omega \times \mathbb{R}) - \mathrm{P}(S_v, \Omega \times \mathbb{R}) = 0,$$

we obtain the second part of the claim. □

Remark. We can obviously combine Proposition 5.4 and 5.5 and replace E with $E \cup \mathrm{B}_\varepsilon$ in the statement of Proposition 5.5. Another direct consequence of this approximation is stated in the following corollary:

Corollary 5.6 (Truncated approximation of the reduced boundary).
In the setting of Proposition 5.5, we can extend Proposition 5.4 as follows: We can find open sets B_ε containing (a subset of) $\partial^* E$ such that

$$\mathrm{P}(E \cup (B_\varepsilon \cap S_v), \Omega \times \mathbb{R}) \to \mathrm{P}(E, \Omega),$$

where S_v is the subgraph of the separation function $v \in \mathrm{W}^{1,1}(\Omega)$ (or $\mathrm{BV}(\Omega)$). Further, with C_ε from Proposition 5.5, we obtain the convergence

$$\mathrm{P}^{\psi_1}(E) = \lim_{\varepsilon \searrow 0} \mathrm{P}(E \cup ((B_\varepsilon \cup C_\varepsilon) \cap S_v)).$$

Although, the sets $B_\varepsilon \cap S_v$ and $C_\varepsilon \cap S_v$ are possibly not open anymore, we still have that $\partial^* E \cap S_v^+ \subset B_\varepsilon \cap S_v^+$ and $O_1 \setminus E^+ \subset C_\varepsilon \cap S_v^+$.

Proof. Follows directly from the proof of Proposition 5.4 and 5.5 together with Proposition 2.21. □

Next we state an approximation stemming from the study of the minimizers of constant mean curvature functionals: Consider some nonempty set Ω with finite positive volume and perimeter, i.e. $0 < \mathrm{P}(\Omega) + |\Omega| < +\infty$ and consider the minimizers $E_\lambda \subset \Omega$ of the functional

$$\mathrm{P}(E) + \lambda |\Omega \setminus E| \tag{5.1}$$

which is defined on sets E contained in Ω. Then one obtains the following result, see for example [68, Theorem 2.3] and for the proof [16] and [67].

Theorem 5.7 (Approximation of Caccioppoli sets and $\lambda \nearrow \infty$)**.**
The (possibly non-unique) minimizers E_λ *of* (5.1) *fulfill*

(i) $\partial E_\lambda \cap E_\lambda(0) = \emptyset$ *for all* $\lambda > 0$*, i.e. the topological boundary contains no points of density zero for the set* E_λ*.*

Furthermore, as $\lambda \nearrow +\infty$*, we have*

(ii) $E_\lambda \nearrow \Omega$ *in the sense that* $E_{\lambda_1} \subset E_{\lambda_2}$ *if* $\lambda_1 < \lambda_2$ *and* $|\Omega \setminus E_\lambda| \to 0$*.*

(iii) $\mathrm{P}(E_\lambda) \nearrow \mathrm{P}(\Omega)$

Additionally, one can show that ∂E_λ *converges to* $\partial\Omega$ *in the Hausdorff distance if* Ω *is bounded.*

Remark.

1. We will use this approximation to reduce certain parts of considered sets to tackle the upper obstacle constraint and on the complement of considered sets to enlarge other parts of the set in a suitable sense to tackle the lower obstacle constraint. The main goal is to get rid of points of density 1 and 0 in the topological boundary, respectively.

2. The problem of the mentioned density 1 (and 0) points in the topological boundary also appears in the regularity theory for parametric minimal surfaces where one usually wants to show that the topological boundary is contained in the reduced boundary/essential boundary and use this information to provide regularity.

Before we begin with the proofs of the desired results, we need to answer the question on how to relate the parametric and non-parametric obstacle constraint. Starting from the non-parametric constraint with a separation function $v \in \mathrm{W}^{1,1}$ which fulfills $\psi_1 \leq v^* \leq \psi_2$, we need to find corresponding sets for the parametric problem. For the results like lower semicontinuity of the parametric double obstacle functional proved in [20] and [43] to hold, we need disjoint obstacle sets. This implies, that the choice $U_1 := \{(x,t) \in \Omega \times \mathbb{R} : t \leq \psi_1(x)\}$ and $U_2 := \{(x,t) \in \Omega \times \mathbb{R} : t \geq \psi_2(x)\}$ is not suitable, since they intersect on the coincidence set $\{\psi_1 = \psi_2\}$. Another point against adding the graphs of ψ_1 and ψ_2 to the obstacles is a problem regarding density 0 and 1 points of the sub-/supergraph which are in the graph of the considered function. At this stage it is not clear whether this problem is of solely technical nature but if, for example, the graph of a function $u \in \mathrm{BV}$ has so many points $(x, u^*(x))$ which are both density 0 points for the subgraph of u and contact points with the lower obstacle in the sense of $u * (x) = \psi_1(x)$, that they together have positive De Giorgi measure, then those points are not seen in the non-parametric setting but are seen in the parametric one and thus would lead to a discrepancy between both theories. One can easily show that at least for minimizers in the single obstacle case such a configuration can not occur and similarly can be excluded in the double obstacle case if enough space between the two obstacles is left. But if a BV-function u exists such that the mentioned density 0 or 1 points have positive De Giorgi measure, a minimizer could be forced to contain those points in its graph using suitable obstacles, for example $\psi_1 = u = \psi_2$ on a suitable subset of Ω.
If we instead consider the subgraph of ψ_1 as the lower obstacle, i.e. $O_1 = S_{\psi_1}$, and the supergraph

of ψ_2 as the upper obstacle, i.e. $O_2 = S^{\psi_2} = \{(x,t) \in \Omega \times \mathbb{R} : t > \psi_2(x)\}$, such problems do not occur: The sets are disjoint (or at least there exist disjoint representatives, since $\psi_1 \leq \psi_2$ holds $\mathcal{H}^{n-1}$-a.e.) and we still have that if a set S with finite perimeter and of subgraph type contains O_1, that the function corresponding function fulfills $\psi_1 \leq w_S^+$. As mentioned in the definition of the subgraph of a BV function, the perimeter of the subgraph and the perimeter of the union of graph and subgraph have the same perimeter and thus the area-part of the considered problem is not touched.

Since we have to deal with representatives of functions and sets, we need to provide some connections and note some properties which we will use without further mention during the proofs: The definition $u^+(x) = \inf_{t\in\mathbb{R}} \left\{ \limsup_{\rho\searrow 0} \rho^{-n}\mathcal{L}^n(\mathrm{B}_\rho(x) \cap \{u > t\}) = 0 \right\}$ of the upper representative of a function $u \in \mathrm{BV}$ (or $\mathrm{L}^1_{\mathrm{loc}}$) function implies some useful properties and a connection to the representatives of the sub- and supergraph. We notice that $\mathcal{L}^n(\mathrm{B}_\rho(x) \cap \{u > t\})$ is monotonic in t, i.e. for $t_1 \leq t_2$ we always have

$$\mathcal{L}^n(\mathrm{B}_\rho(x) \cap \{u > t_2\}) \leq \mathcal{L}^n(\mathrm{B}_\rho(x) \cap \{u > t_1\}).$$

This implies that

- if for a given t_0 we have $\limsup_{\rho\searrow 0} \rho^{-n}\mathcal{L}^n(\mathrm{B}_\rho(x) \cap \{u > t_0\}) > 0$ we already have $u^+ \geq t_0$ and
- if for a given t_0 we have $\lim_{\rho\searrow 0} \rho^{-n}\mathcal{L}^n(\mathrm{B}_\rho(x) \cap \{u > t_0\}) = 0$ we already have $u^+ \leq t_0$.

Similarly for the lower representative $u^-(x) = \sup_{t\in\mathbb{R}} \left\{ \limsup_{\rho\searrow 0} \rho^{-n}\mathcal{L}^n(\mathrm{B}_\rho(x) \cap \{u < t\}) = 0 \right\}$.

If we now consider the relation to the measure theoretical concept on sets, we quickly find by the graph-type structure and Fubini's theorem that $S_{u^+} \subset S_u^+$ and the difference lies basically in the set $G_{u^+} \cap S_u^+$, where $G_u := \{(x, u(x)) \in \Omega \times \overline{\mathbb{R}}\}$. Similarly one has $S^{(u^-)} \subset (S^u)^-$ and the difference lies in the points contained in $S_u^- \cap G_{u^-}$. Since the graph of the function does not contribute to the obstacle terms by the choice of the obstacles, we can freely interchange those sets.

To close this subsection, we state some notes regarding representatives:

1. If we consider a separation function $v \in \mathrm{BV}$, we may choose some representative of the function which fulfills both obstacle constraints simultaneously and use that without further notice. For the certain representatives of the sets, the choice will not matter. Further, we may always assume $\psi_1 \leq \psi_2$ on Ω by changing the functions on a $\mathcal{H}^{n-1}$ null set.

2. As the obstacles are disjoint, it is possible to find for each set E with $E^- \subset E \subset E^+$ a suitable representative $\tilde{E}$ with

$$\mathrm{P}^{\psi_1,\psi_2}(E) = \mathrm{P}(\tilde{E}, \Omega \times \mathbb{R}) + \varsigma^{n+1}((O_1 \setminus \tilde{E}) \cap (\Omega \times \mathbb{R})) + \varsigma^{n+1}(O_2 \cap \tilde{E} \cap (\Omega \times \mathbb{R})).$$

A particular choice, like introduced in [20] and [43], is given by $\tilde{E} = (E \cup (E^+ \cap O_1)) \setminus (O_2 \setminus E^-)$, which obviously fulfills $\varsigma^{n+1}(O_1 \setminus E^+) = \varsigma^{n+1}(O_1 \setminus \tilde{E})$ and $\varsigma^{n+1}(O_2 \cap E^-) = \varsigma^{n+1}(O_2 \cap \tilde{E})$. With the discussion before, we may thus change the choice of obstacles and consider $O_1^+ \cup O_1$ or $S_{\psi_1^+} \cup S_\psi$ instead of O_1 and $O_2^- \cup O_2$ or $S^{(\psi_2^-)} \cup S^{\psi_2}$ instead of O_2 without really changing the functional.

3. This discussion gives us a hint on how to divide the given obstacles in thin and thick parts. The thick parts of the lower obstacle can be described by ψ_1^+ and the thin parts are given as ψ_1 on the set where $\psi_1 > \psi_1^+$. Similarly for the upper obstacle with ψ_2^- and the set $\psi_2 < \psi_2^-$. Obviously this decomposition is not unique.

5.2 Approximation of obstacle terms and the first inequality regarding the parametric and non-parametric double-obstacle problem

To tackle the general double obstacle problem, we first study the single obstacle problem and consider an example to enlighten some difficulties to tackle and explain, why the approximations introduced in the previous section are needed:
Consider the domain $\Omega = (-1, 1)$ and the obstacle function $\psi_1 = 2\mathbb{1}_{\{1\}}$ and the corresponding parametric obstacle $O_1 = ((-1, 1) \times (-\infty, 0)) \cup (\{0\} \times [0, 2))$. Then $E = (-1, 1) \times (-\infty, 0] \cup ([0, 1) \times (0, 1]) \cup ((-1, 0] \times [1, 2])$ is a set with finite perimeter and eligible for the single obstacle problem. We have $O_1 \setminus E^+ = \emptyset$, but the measure theoretical closure of the graphification is given by $(-1, 1) \times (-\infty, 1]$ and thus the set $\{0\} \times (1, 2) \subset O_1$ with positive ς^{n+1}-measure is not covered anymore. This implies, that the graphification may uncover obstacle parts which were covered before. In this particular case it is easy to see that the single obstacle functional is reduced by the graphification, but in more general cases this needs to be proven. A first step to counter such cancellation is to enlarge the set with the help of Proposition 5.4 as an approximation. To handle obstacle parts, which are not covered by the original set, Proposition 5.5 is helpful. In the double obstacle case one needs to be even more careful, since for the upper obstacle one needs to shrink the set during such approximation, we will apply Corollary 5.6 one time on a certain subset of the reduced boundary and the not covered part of the lower obstacle to extend the given set and a second time on the rest of reduced boundary and the covered part of the upper obstacle to reduce the set, thus forming a 'dual' approach.
The second approximation introduced in Theorem 5.7 will help to ensure that the graphification of the newly obtained approximation will indeed cover the obstacle. It unfortunately works only on Caccioppoli sets with finite volume, which in our case are subsets in $\mathbb{R}^{n+1}$. To circumvent this restriction, we will use truncations of the type $E \cap (\Omega \times (-k, k))$ for a given set E. Since this set will in general not have finite perimeter as Ω need not to have finite perimeter, we will later need to approximate general domains Ω. To start the proof, we start first with an easier case, namely with only a bounded single lower obstacle:

Lemma 5.8 (Inequality for the bounded single obstacle problem)**.**
Let Ω be a bounded open set in $\mathbb{R}^n$ with finite perimeter and let the parametric single lower obstacle $O_1 = \{(x, t) \in \Omega \times \mathbb{R} : t < \psi_1(x)\}$ for some bounded Borel function ψ_1 with $|\psi_1| \leq M$ be given. Then we have that the graphification w_E of a set E, which fulfills the assumptions of Theorem 5.2 and has finite parametric single obstacle functional value

$$\mathrm{P}^{\psi_1}(E) := \mathrm{P}(E, \Omega \times \mathbb{R}) + \varsigma^{n+1}(O_1 \setminus E^+, \Omega \times \mathbb{R}) < \infty,$$

has finite $\mathcal{A}^{\psi_1}$ value and the following inequality holds:

$$\mathcal{A}^{\psi_1}[w_E] \leq \mathrm{P}^{\psi_1}(E).$$

Proof. Let E be a set such that $\mathrm{P}^{\psi_1}(E) < +\infty$. This especially implies that the perimeter of E is finite. First we assume that E is bounded from above, i.e. there exists some k, such that $E \subset \Omega \times (-\infty, k)$ and later use the result to obtain the general case. Without loss of generality, we may choose $k > M$.
For given $\varepsilon > 0$ we find by Proposition 5.4 and 5.5 sets B_ε containing $\partial^* E$ and sets C_ε containing $O_1 \setminus E^+$ such that $E \cup B_\varepsilon \cup C_\varepsilon$ has finite perimeter in $\Omega \times \mathbb{R}$, converges for $\varepsilon \searrow 0$ to E and

$$\mathrm{P}(E \cup B_\varepsilon \cup C_\varepsilon, \Omega \times \mathbb{R}) \to \mathrm{P}^{\psi_1}(E).$$

Since $\psi_1(x) \leq M$, we may also truncate the sets B_ε and C_ε, and instead consider the approximation by $E \cup ((B_\varepsilon \cup C_\varepsilon) \cap \Omega \times (-\infty, k+1))$ and by Corollary 5.6, we are still left with

$$\mathrm{P}(E \cup ((B_\varepsilon \cup C_\varepsilon) \cap \Omega \times (-\infty, k+1)), \Omega \times \mathbb{R}) \to \mathrm{P}^{\psi_1}(E).$$

For easier notation we set $A_\varepsilon = E \cup ((B_\varepsilon \cup C_\varepsilon) \cap \Omega \times (-\infty, k+1))$ and note that the sets $\partial^* E \cap (\Omega \times (-\infty, M]$ and $O_1 \setminus E^+$ are still openly covered and contained in the set A_ε, since the truncations of $B_\varepsilon \cap (\Omega \times (-\infty, k+1))$ and $C_\varepsilon \cap (\Omega \times (-\infty, k+1))$ are still open and contain the mentioned sets, respectively.
If we want to verify that the graphification of A_ε fulfills the obstacle condition, i.e. $w^+_{A_\varepsilon} \geq \psi_1$, we quickly find that this can not be verified in general. A major problem are the points of density 1 which lie in the topological boundary of E. Thus we will use the second approximation given in Theorem 5.7 to circumvent this. To be able to apply this Theorem, we need to rewrite our terms, since it, at least without further proof, works only for bounded sets:
Since ψ_1 is bounded, we have that E^+ contains the set $\Omega \times (-\infty, -M]$ and thus A^+_ε as well. Next consider the set ${}^k A_\varepsilon = A_\varepsilon \cap (\Omega \times (-k-1, k+1))$. Since $\Omega_k := \Omega \times (-k-1, k+2)$ has finite perimeter in $\mathbb{R}^{n+1}$, and A_ε has finite perimeter in $\Omega \times \mathbb{R}$, we obtain that ${}^k A_\varepsilon$ has finite perimeter in $\mathbb{R}^{n+1}$:

$$\mathrm{P}({}^k A_\varepsilon) \leq \mathrm{P}(A_\varepsilon, \Omega \times \mathbb{R}) + |\Omega| + 2(k+2)\mathrm{P}(\Omega).$$

The inequality can easily be obtained using the Hausdorff measure of the reduced boundary which obviously is contained in $(\mathcal{F} A_\varepsilon) \cap (\Omega \times \mathbb{R}) \cup \partial\Omega_k$, since $-k-1 < -M$ holds and $A_\varepsilon \subset \Omega \times (-\infty, k+1)$.
Next, consider a ball B_R such that $\overline{\Omega} \subset \mathrm{B}_R$ and the set $Q_k = \mathrm{B}_R \times (-k-1, k+2)$ and define the set ${}^k Z_\varepsilon = Q_k \setminus {}^k A_\varepsilon$ as the complement of ${}^k A_\varepsilon$ in Q_k. It is easy to see, compare for example [3, Remark 4.2], that

$$\mathrm{P}({}^k A_\varepsilon, Q_k) = \mathrm{P}({}^k Z_\varepsilon, Q_k) = \frac{1}{2}\left(\mathrm{P}({}^k A_\varepsilon) + \mathrm{P}({}^k Z_\varepsilon) - \mathrm{P}(Q_k)\right) \tag{5.2}$$

Now we apply Theorem 5.7 on ${}^k Z_\varepsilon$ and obtain for $\lambda \gg 1$ sets ${}^k Z^\lambda_\varepsilon \subset {}^k Z_\varepsilon$ which converge in L^1 and in the perimeter to ${}^k Z_\varepsilon$ for $\lambda \to \infty$ and such that $\partial {}^k Z^\lambda_\varepsilon \cap {}^k Z^\lambda_\varepsilon(0) = \emptyset$. We further define the sets ${}^k A^\lambda_\varepsilon := Q_k \setminus ({}^k Z^\lambda_\varepsilon \cup (Q_k \setminus (\Omega \times (-k-1, k+1)))$ and observe that ${}^k A_\varepsilon \subset {}^k A^\lambda_\varepsilon$, ${}^k A^\lambda_\varepsilon$ converges to ${}^kA_\varepsilon$ for $\lambda \to \infty$ in L^1 and in the perimeter, i.e. $|{}^kA_\varepsilon^\lambda \setminus {}^kA_\varepsilon| \to 0$ and $\mathrm{P}({}^kA_\varepsilon^\lambda) \to \mathrm{P}({}^kA_\varepsilon)$ for $\lambda \to \infty$. We then can easily verify that $\mathcal{F}({}^kA_\varepsilon^\lambda) \cap \mathcal{F}Q_k = \mathcal{F}({}^kA_\varepsilon) \cap \mathcal{F}Q_k$ for all $\lambda > 0$, and that $\mathcal{F}({}^kA_\varepsilon) \cap \mathcal{F}\Omega \times (-k-1, k+1) \subset \mathcal{F}({}^kA_\varepsilon^\lambda) \cap \mathcal{F}\Omega \times (-k-1, k+1)$. Using those estimates and the principle used in equation (5.2), we obtain

$$\begin{aligned} \mathrm{P}({}^kA_\varepsilon^\lambda, \Omega_k) &\leq \mathrm{P}({}^kA_\varepsilon^\lambda, Q_k) - \mathcal{H}^n(\mathcal{F}({}^kA_\varepsilon^\lambda) \cap (\mathcal{F}\Omega) \times (-k-1, k+1)) \\ &\leq \mathrm{P}({}^kA_\varepsilon^\lambda, Q_k) - \mathcal{H}^n(\mathcal{F}^kA_\varepsilon \cap (\mathcal{F}\Omega) \times (-k-1, k+1)) \end{aligned}$$

and thus

$$\mathrm{P}({}^kA_\varepsilon, \Omega_k) = \lim_{\lambda \to \infty} \mathrm{P}({}^kA_\varepsilon^\lambda, \Omega_k).$$

Further, we can choose for each $\varepsilon > 0$ some λ_ε, such that $|\mathrm{P}({}^kA_\varepsilon^{\lambda_\varepsilon}, \Omega_k) - \mathrm{P}({}^kA_\varepsilon, \Omega_k)| < \varepsilon$ and $|{}^kA_\varepsilon^{\lambda_\varepsilon} \setminus {}^kA_\varepsilon| < \varepsilon$. Next we extend the constructed sets to $A_{k,\varepsilon} := {}^kA_\varepsilon^{\lambda_\varepsilon} \cup \Omega \times (-\infty, -M]$ and observe that $E \subset A_{k,\varepsilon}$, $A_{k,\varepsilon} \to E$ in L^1 and $\mathrm{P}(A_{k,\varepsilon}, \Omega \times \mathbb{R}) \to \mathrm{P}(E, \Omega \times \mathbb{R})$ for $\varepsilon \searrow 0$.
Next we consider the graphification of $A_{k,\varepsilon}$ and take a closer look at the obstacle O_1:
By construction, we have that $O_1 \subset A_{k,\varepsilon}$. Further, we have that for each point $(x,t) \in O_1 \subset \Omega \times \mathbb{R}$ there exists a radius $\rho > 0$ such that $\mathrm{B}_\rho(x,t) \subset A_{k,\varepsilon}$: To verify this, we notice that either $(x,t) \notin E^+$ and thus is covered by the open set $C_\varepsilon \subset A_{k,\varepsilon}$ or $(x,t) \in E^+$. Now there are again two possibilities: Either $(x,t) \in E^+ \setminus E(1)$ and is thus contained in the open set B_ε or $(x,t) \in E(1)$ in which case we will show that there exists either such ball or we arrive at a contradiction:
If no such ball exist, we find for each $\rho > 0$ a point in $\mathrm{B}_\rho((x,t))$ which is outside $A_{k,\varepsilon}$ and thus get that $(x,t) \in \partial A_{k,\varepsilon} \cap \Omega_k$, noting that $k > M$ and thus $t < k$. This implies that the point $(x,t) \in E(1) \cap \partial A_{k,\varepsilon} \subset A_{k,\varepsilon}(1) \cap \partial A_{k,\varepsilon}$ and thus a contradiction to (i) of Theorem 5.7, since this implies that, by construction of ${}^kA_\varepsilon^\lambda$, $\partial^k Z_\varepsilon^{\lambda_\varepsilon} \cap {}^kZ_\varepsilon^{\lambda_\varepsilon}(0) \neq \emptyset$.
Next we fix $x \in \Omega$ such that $w^\pm_{A_{k,\varepsilon}}$ are defined and consider the line $L_x = \{x\} \times [-M, \psi_1(x)) \subset \{x\} \times [-M, M]$. For any given $\eta > 0$, the set $L_x^\eta = \{x\} \times [-M, \psi_1(x) - \eta]$ is compact. Thus at each point of L_x^η there exists an open ball contained in $A_{k,\varepsilon}$ and we can find a finite open cover of L_x^η with finitely many of such balls and an $r_\eta > 0$ such that a cylinder of the form $\{z \in \Omega : |z - x| < r_\eta\} \times [-M, \psi_1(x) - \eta]$ is contained in $A_{k,\varepsilon}$. The radius r_η can be chosen monotonically increasing in η, which we will assume from now on. Since $\mathrm{B}^n_{r_\eta}(x) \times [-\infty, -M]$ is automatically contained in $A_{k,\varepsilon}$, we obtain that $\mathrm{B}^n_{r_\eta}(x) \times [-\infty, \psi_1(x) - \eta] \subset A_{k,\varepsilon}$ and for the graphification $w_{A_{k,\varepsilon}}$ holds $\mathcal{L}^n(\{w_{A_{k,\varepsilon}}(x) < \psi_1(x) - \eta\} \cap \mathrm{B}^n_\rho(x)) = 0$ for $\rho < r_{\frac{\eta}{2}}$. Since $\eta > 0$ was arbitrary, we even have by the definition of the lower representative $w^-_{A_{k,\varepsilon}}(x) \geq \psi_1(x)$. Thus the lower and upper representative of $w_{A_{k,\varepsilon}}$ fulfill $w^\pm_{A_{k,\varepsilon}}(x) \geq \psi_1(x)$ and $w_{A_{k,\varepsilon}}$ satisfies the obstacle constraint. Further and similar to the proof of Theorem 5.2 and the preceding auxiliary lemma in [39] together with the estimates obtained through Proposition 5.4 and 5.5, we can estimate

$$\begin{aligned} \int_\Omega \sqrt{1 + |\mathrm{D}w_{A_{k,\varepsilon}}|^2} &\leq \mathrm{P}(A_{k,\varepsilon}, \Omega \times \mathbb{R}) \\ &\leq \mathrm{P}^{\psi_1}(E) + 3\varepsilon. \end{aligned}$$

The term 3ε stems from the approximation of ${}^kA_\varepsilon$ with $A_{k,\varepsilon}$ and the approximation of $\mathrm{P}^{\psi_1}({}^kE)$ by $\mathrm{P}({}^kA_\varepsilon)$ using the truncated versions of B_ε and C_ε. Now we can let $\varepsilon \searrow 0$ and obtain that $A_{k,\varepsilon} \to E$ in L^1 which implies that $w_{A_{k,\varepsilon}} \to w_E$ in L^1, see, for example, remark to Theorem 5.2.

By lower semicontinuity of $\mathcal{A}^{\psi_1}$, we arrive at

$$\mathcal{A}^{\psi_1}[w_E] \le \mathrm{P}^{\psi_1}(E)$$

and thus the result is proven in the case where E is bounded from above in the x_{n+1}-variable. To obtain the general case, use truncations $E_l := E \cap (\Omega \times (-\infty, l)$ of E with $l > M$ together with the estimate

$$\mathrm{P}(E_l, \Omega \times \mathbb{R}) \le \mathrm{P}(E, \Omega \times \mathbb{R}) + \int\limits_\Omega \mathbb{1}_E(x,l)\,\mathrm{d}\mathcal{H}^n(x),$$

like in the proof of Theorem 5.2 in [39]. Applying the bounded from above case with $k = l$, we observe that

$$\mathcal{A}^{\psi_1}[w_{E_l}] \le \mathrm{P}^{\psi_1}(E_l) \le \mathrm{P}^{\psi_1}(E) + \int\limits_\Omega \mathbb{1}_E(x,l)\,\mathrm{d}\mathcal{H}^n(x),$$

since l was chosen larger then M. Noticing that we have

$$\int\limits_\Omega \mathbb{1}_E(x,l)\,\mathrm{d}\mathcal{H}^n(x) \to 0 \text{ for } l \to \infty$$

and $E_l \to E$ in L^1, which can easily be obtained like in the remark to Theorem 5.2, we arrive at

$$\mathcal{A}^{\psi_1}[w_E] \le \liminf_{l\to\infty} \mathcal{A}^{\psi_1}[w_{E_l}] \le \lim_{l\to\infty} \left(\mathrm{P}^{\psi_1}(E) + \int\limits_\Omega \mathbb{1}_E(x,l)\,\mathrm{d}\mathcal{H}^n(x) \right) = \mathrm{P}^{\psi_1}(E)$$

and the general claim is proven. □

Next, we consider the first double obstacle case. A noteworthy problem which arises in the double obstacle case is that the approximation by enlargement like in the single obstacle case may stand in conflict with the upper obstacle. Similarly, reducing the set may interfere with the lower obstacle. The idea is to truncate such approximations by the subgraph and supergraph of a separation function.

Lemma 5.9 (Inequality for the bounded double obstacle problem).
Let Ω be a bounded open set in $\mathbb{R}^n$ with finite perimeter and let $O_1 = S_{\psi_1}$ and $O_2 = S^{\psi_2}$ for some bounded Borel functions ψ_1, ψ_2 with $|\psi_1|, |\psi_2| \le M$ and such that a function $v \in \mathrm{W}^{1,1}(\Omega)$ exists with $\psi_1 \le v^* \le \psi_2$. Then, we have that the graphification w_E of a set E with finite parametric double obstacle functional value

$$\mathrm{P}^{\psi_1,\psi_2}(E) := \mathrm{P}(E, \Omega \times \mathbb{R}) + \varsigma^{n+1}(O_1 \setminus E^+, \Omega \times \mathbb{R}) + \varsigma^{n+1}(E^- \cap O_2, \Omega \times \mathbb{R}) < \infty$$

has finite $\mathcal{A}^{\psi_1,\psi_2}$ value and the following inequality holds:

$$\mathcal{A}^{\psi_1,\psi_2}[w_E] \le \mathrm{P}^{\psi_1,\psi_2}(E).$$

Remark. In the following proof, one has to be more careful regarding the representatives. For the single obstacle case, we could state that $\Omega \times (-\infty, -M] \subset E^+$ and had to some extent not to worry

about the exact representatives. In the double obstacle case, we again have that $\Omega \times (-\infty, -M] \subset E^+ \subset \Omega \times (-\infty, M]$ and the same holds for E^-, but not necessary for the set E itself. Since the choice of the representative does not change the perimeter and the right representative is automatically chosen for the De-Giorgi obstacle terms, we will assume, without loss of generality, that $\Omega \times (-\infty, -M] \subset E \subset \Omega \times (-\infty, M]$ and we will deliberately switch representatives when dealing with the perimeter of a set.
Further, it is clear, that the sets considered in this Lemma fulfill the requirements of Theorem 5.2.

Proof. We divide the proof into four steps. In the first, we approximate the set by enlarging/reducing the set to cover O_1 and uncover O_2 as well as the corresponding parts of the essential boundary. In the second step, we approximate the set to get rid of certain density 1 and 0 points as in the single obstacle case. In the third step, we check the graphification of the obtained sets for their interaction with the obstacles and, finally, consider the limit of our approximation in the last step of the proof.
Step 1. First, we have that E is bounded and thus no truncation of the set will be needed. We then start similar to the proof of Lemma 5.8 and by Corollary 5.6 find for given $\varepsilon > 0$ sets B_ε containing this time $(\partial^* E) \cap S_v$ and the portion $\{x \in \mathcal{F}E \cap \mathcal{F}S_v : \nu_E = \nu_{S_v}\}$ of $\mathcal{F}E \cap \mathcal{F}S_v$ where the normal vectors align as well as the null sets $(\partial^* E \cap \partial^* S_v) \setminus (\mathcal{F}E \cap \mathcal{F}S_v)$ and where the normal vector of $\mathcal{F}E \cap \mathcal{F}S_v$ does not exist, compare Proposition 2.5. Then, we define $S^v := \{(x,t) \in \Omega \times \mathbb{R} : t > v(x)\}$ and find similarly sets $\tilde{B}_\varepsilon$ which cover $\partial^* E \cap S^v$ and the portion of $\mathcal{F}E \cap \mathcal{F}S^v$ where the normal vectors align as well as the null set $(\partial^* E \cap \partial^* S_v) \setminus (\mathcal{F}E \cap \mathcal{F}S_v)$. We can now consider the set $(E \cup (B_\varepsilon \cap S_v)) \setminus (\tilde{B}_\varepsilon \cap S^v)$. By Corollary 5.6 we can ensure

$$\mathrm{P}((E \cup (B_\varepsilon \cap S_v)) \setminus (\tilde{B}_\varepsilon \cap S^v), \Omega \times \mathbb{R}) \leq \mathrm{P}(E, \Omega \times \mathbb{R}) + 2\varepsilon$$

and by construction find that

$$\varsigma^{n+1}(O_1 \setminus ((E \cup (B_\varepsilon \cap S_v)) \setminus (\tilde{B}_\varepsilon \cap S^v))^+, \Omega \times \mathbb{R}) \leq \varsigma^{n+1}(O_1 \setminus E^+, \Omega \times \mathbb{R}),$$
$$\varsigma^{n+1}(O_2 \cap ((E \cup (B_\varepsilon \cap S_v)) \setminus (\tilde{B}_\varepsilon \cap S^v))^-, \Omega \times \mathbb{R}) \leq \varsigma^{n+1}(O_2 \cap E^-, \Omega \times \mathbb{R}).$$

To verify those estimates, assume for example that $x \in O_2 \setminus E^-$ is in $((E \cup (B_\varepsilon \cap S_v)) \setminus (\tilde{B}_\varepsilon \cap S^v))^-$. We first note that $v^- \leq \psi_2$ and thus $O_2 \cap S^{v^-}$ and $O_2 \cap (S^v)^-$ is a $\mathcal{H}^n$ null set. Thus we may assume $x \notin E^-$ and $x \notin S_v^-$, which implies that $x \in \partial^* E \cap \partial^* S_v$. Since $(\partial^* E \cap \partial^* S_v) \setminus (\mathcal{F}E \cap \mathcal{F}S_v)$ is (besides being a null set) only reached by $E \cup (B_\varepsilon \cap S_v)$, because $\tilde{B}_\varepsilon \cap S^v$ was subtracted, we have that $x \notin (\partial^* E \cap \partial^* S_v) \setminus (\mathcal{F}E \cap \mathcal{F}S_v)$ and by construction also not in the subset of $\mathcal{F}E \cap \mathcal{F}S_v$ where no normal vector exists. As a side note, we can argue here with S_v instead of $B_\varepsilon \cap S_v$ since we obviously have $(B_\varepsilon \cap S_v)^+ \subset S_v^+$. For $x \in \mathcal{F}E \cap \mathcal{F}S_v$, we would obtain that x is in addition a density 1 point of $E \cup S_v$, which implies that the normal vectors of E and S_v show in opposite directions. The considerations leads to $x \in \tilde{B}_\varepsilon$ implying $x \notin ((E \cup (B_\varepsilon \cap S_v)) \setminus (\tilde{B}_\varepsilon \cap S^v))^-$ but merely $x \in \mathcal{F}((E \cup (B_\varepsilon \cap S_v)) \setminus (\tilde{B}_\varepsilon \cap S^v))$. Arguing similar for $x \in O_1 \cap E^+$, the claim is verified. Next, we find by Corollary 5.6 some sets C_ε containing $O_1 \setminus E^+$ and $\tilde{C}_\varepsilon$ containing $O_2 \cap E^-$ and observe that

$$\mathrm{P}^{\psi_1,\psi_2}((E \cup ((B_\varepsilon \cup C_\varepsilon) \cap S_v)) \setminus (\tilde{B}_\varepsilon \cup \tilde{C}_\varepsilon \cap S^v)), \Omega \times \mathbb{R}) = \mathrm{P}((E \cup ((B_\varepsilon \cup C_\varepsilon) \cap S_v)) \setminus (\tilde{B}_\varepsilon \cup \tilde{C}_\varepsilon \cap S^v)), \Omega \times \mathbb{R})$$

and

$$\mathrm{P}((E \cup ((B_\varepsilon \cup C_\varepsilon) \cap S_v)) \setminus (\tilde{B}_\varepsilon \cup \tilde{C}_\varepsilon \cap S^v)), \Omega \times \mathbb{R}) \leq \mathrm{P}^{\psi_1,\psi_2}(E) + 4\varepsilon.$$

For easier notation, we set $A_\varepsilon := (E \cup ((B_\varepsilon \cup C_\varepsilon) \cap S_v)) \setminus (\tilde{B}_\varepsilon \cup \tilde{C}_\varepsilon \cap S^v))$ and continue with the second step.
Step 2. Using Theorem 5.7 and arguing as in the proof of Lemma 5.8, we can again obtain a sequence A_ε^λ converging in L^1 and in the perimeter in $\Omega \times \mathbb{R}$ to A_ε: We have that $A_\varepsilon \subset A_\varepsilon^\lambda$, A_ε^λ converges for $\lambda \to \infty$ to A_ε in L^1 and in the perimeter and additionally fulfills that $\partial A_\varepsilon^\lambda \cap \tilde{A}_\varepsilon^\lambda(1) = \emptyset$. Relying on the truncation like in the proof of Proposition 5.5 and as will be shown below, we may adjust the sequence by truncating the parts added to A_ε by A_ε^λ and thus obtain $\hat{A}_\varepsilon^\lambda := A_\varepsilon \cup (A_\varepsilon^\lambda \cap S_v)$, which still converges in L^1 and in the perimeter to A_ε, and fulfills $A_\varepsilon \subset \hat{A}_\varepsilon^\lambda$ but maybe loses the property regarding density 1 points in the topological boundary if the considered point is contained in the reduced/essential boundary of S_v. The approximation properties regarding the convergence in L^1 are trivial to show as well as the fact that $A_\varepsilon \subset \hat{A}_\varepsilon^\lambda$. The stated convergence in the perimeter can be verified using $A_\varepsilon \subset A_\varepsilon^\lambda$ and Proposition 2.21, to obtain

$$\begin{aligned}\mathrm{P}(A_\varepsilon \cup (A_\varepsilon^\lambda \cap S_v), \Omega \times \mathbb{R}) &= \mathrm{P}(A_\varepsilon^\lambda \cap (A_\varepsilon \cup S_v), \Omega \times \mathbb{R}) \\ &\leq \mathrm{P}(A_\varepsilon^\lambda, \Omega \times \mathbb{R}) + \mathrm{P}(A_\varepsilon \cup S_v, \Omega \times \mathbb{R}) - \mathrm{P}(\tilde{A}_\varepsilon^\lambda \cup S_v, \Omega \times \mathbb{R})\end{aligned}$$

and thus the convergence in the perimeter follows with the lower semicontinuity of the perimeter. We may further choose suitable λ_ε as in the proof of Lemma 5.8, i.e. that A_ε and $A_\varepsilon^{\lambda_\varepsilon}$ differ in the perimeter and in L^1 only by an ε.
In a similar fashion and now using interior approximation of A_ε, we can construct $\tilde{A}_\varepsilon^\lambda \subset A_\varepsilon$ which converge in L^1 and in the perimeter to A_ε and fulfill that $\partial \tilde{A}_\varepsilon^\lambda \cap \tilde{A}_\varepsilon^\lambda(1) = \emptyset$. More precisely, we use Theorem 5.7 on $A_\varepsilon \cap (\Omega \times (-M-1, M))$ and add $\Omega \times (-\infty, -M]$ to the newly obtained approximation. Arguing like in the single obstacle cases ensures, that the stated properties hold. Now we can consider the set $E_\varepsilon = (A_\varepsilon \cup (A_\varepsilon^{\lambda_\varepsilon} \cap S_v)) \setminus (\tilde{A}_\varepsilon^{\lambda_\varepsilon} \cap S^v)$ after adjusting λ_ε, such that

$$\mathrm{P}(E_\varepsilon, \Omega \times \mathbb{R}) \leq \mathrm{P}(A_\varepsilon, \Omega \times \mathbb{R}) + 2\varepsilon$$

holds. This can be done arguing like in the first part of this step and by monotonicity of A_ε^λ and $\tilde{A}_\varepsilon^\lambda$ in λ.
Altogether, we constructed a sequence of sets E_ε which, considering each time the right representative, contain O_1 but do not contain O_2, converge to E in L^1 for $\varepsilon \searrow 0$ as well as in the perimeter $\mathrm{P}(E_{k,\varepsilon}, \Omega \times \mathbb{R}) \to \mathrm{P}^{\psi_1,\psi_2}(E)$. Further, the sets E_ε have some boundary properties regarding density 1 points in the topological boundary on parts of the set and contain by S_v truncated cylinders around $[-M, \psi_1(x)-\eta]$ and by S^v truncated cylinders around $[\psi_2(x)+\eta, M]$ for all $\eta > 0$, which we will explain and use in the third step to conclude properties of the graphification of E_ε regarding the obstacles.
Step 3. We can now consider the (subgraph of the) graphification $S_{w_{E_\varepsilon}}$ and observe, as in the proof in the single obstacle case, that E_ε contains for each $\eta > 0$ a truncated cylinder around each line contained in $O_1 - \eta$, i.e. for all x we have that $L_x^\eta := \{x\} \times [-M, \psi_1(x) - \eta]$ is contained in a cylinder $U_x^\eta = \mathrm{B}_{r_\eta}^n(x) \times (-\infty, \psi_1(x) - \eta] \subset A_\varepsilon^{\lambda_\varepsilon}$ and thus $U_x^\eta \cap S_v$ is contained in $A_\varepsilon^{\lambda_\varepsilon}$, since the adjustments with $\tilde{A}_\varepsilon \cap S^v$ in step 2 do not take away any points in S_v and hence do not

interfere. This implies that $U_x^\eta \cap S_v$ is still a subset of E_ε. Remembering that by our setting we have that $\Omega \times (-\infty, M] \subset E_\varepsilon^+$, we can draw the conclusion that $w_{E_\varepsilon}^+(x) \geq \psi_1(x)$ for all $x \in \Omega$ by the definition of the upper representative and $U_x^\eta \cap S_v$ implying $w_{E_\varepsilon}^+(x) \geq \psi_1(x) - \eta$ for all $\eta > 0$ since S_{v^+} contains O_1. Arguing in a similar fashion for the upper obstacle, i.e. finding a cylinder in the complement of E_ε^- around $\{x\} \times [\psi_2(x) + \eta, M] \subset \{x\} \times [-M, M]$ which get truncated by S^v, we have that $w_{E_\varepsilon}^-(x) \leq \psi_2(x)$ for all $x \in \Omega$.
Step 4. By estimating all approximation steps, we get in the same fashion as for the single obstacle case that

$$|E_\varepsilon \Delta E| \leq 6\varepsilon \text{ and } \mathrm{P}(E_\varepsilon, \Omega \times \mathbb{R}) \leq \mathrm{P}^{\psi_1,\psi_2}(E) + 6\varepsilon$$

and thus $E_\varepsilon \to E$ in L^1 and $\mathrm{P}(E_\varepsilon, \Omega \times \mathbb{R}) \to \mathrm{P}^{\psi_1,\psi_2}(E)$ by lower semicontinuity for $\varepsilon \searrow 0$. This implies as described for the single obstacle case that $w_{E_\varepsilon} \to w_E$ in L^1, which leads to

$$\mathcal{A}^{\psi_1,\psi_2}[w_E] \leq \liminf_{\varepsilon \searrow 0} \mathcal{A}^{\psi_1,\psi_2}[w_{E_\varepsilon}] \leq \mathrm{P}^{\psi_1,\psi_2}(E)$$

and the desired inequality is proven. □

With Lemma 5.9 in hand, we can prove the general statement:

Theorem 5.10 (Inequality for the general double obstacle case)**.**
Let Ω be a bounded open set in $\mathbb{R}^n$ and let $O_1 = S_{\psi_1}$ and $O_2 = S^{\psi_2}$ for some bounded Borel functions ψ_1, ψ_2 be given such that a function $v \in \mathrm{W}^{1,1}(\Omega)$ exists with $\psi_1 \leq v^* \leq \psi_2$. Then, we have that the graphification w_E of set E, which fulfills the assumptions of Theorem 5.2 and has finite parametric double obstacle functional value

$$\mathrm{P}^{\psi_1,\psi_2}(E) := \mathrm{P}(E, \Omega \times \mathbb{R}) + \varsigma^{n+1}(O_1 \setminus E^+, \Omega \times \mathbb{R}) + \varsigma^{n+1}(E^- \cap O_2, \Omega \times \mathbb{R}),$$

has finite $\mathcal{A}^{\psi_1,\psi_2}$ value and the following inequality holds:

$$\mathcal{A}^{\psi_1,\psi_2}[w_E] \leq \mathrm{P}^{\psi_1,\psi_2}(E).$$

Proof. We first assume that Ω has finite perimeter. For given $k > 0$, we observe that the set $E_k = E \cap (-k, k) \cup \Omega \times (-\infty, k]$ has finite perimeter in $\Omega \times \mathbb{R}$, more precisely, we have

$$\mathrm{P}(E_k, \Omega \times \mathbb{R}) \leq \mathrm{P}(E, \Omega \times \mathbb{R}) + \int_\Omega 1 + \mathbb{1}_E(x, k) - \mathbb{1}_E(x, -k) \, \mathrm{d}\mathcal{H}^n(x) \leq \mathrm{P}(E, \Omega \times \mathbb{R}) + 2|\Omega|,$$

compare for example the proof of Theorem 5.2 in [39]. Further, we set $\psi_{1,m} = \min\{\psi_1, m\}$ and $\psi_{2,m} = \max\{\psi, -m\}$ and observe that for $k \geq m$ we have

$$\mathrm{P}^{\psi_{1,m},\psi_{2,m}}(E_k) \leq \mathrm{P}^{\psi_1,\psi_2}(E) + \int_\Omega 1 + \mathbb{1}_E(x, k) - \mathbb{1}_E(x, -k) \, \mathrm{d}\mathcal{H}^n(x).$$

For this, it is important to note that we have $\psi_{1,m} \leq \psi_1$ and $\psi_{2,m} \geq \psi_2$ and thus the estimate follows trivially for $k \geq m$. We notice further, that E_k contains $\Omega \times (-\infty, -k-1)$ and does not intersect $\Omega \times (k+1, +\infty)$. Thus we may replace the function $\psi_{1,m}$ by $\psi_{1,m,k} = \max\{\psi_{1,m}, -k-1\}$

and the function $\psi_{2,m}$ by $\psi_{2,m,k} = \min\{\psi_{2,m}, k+1\}$ without changing the value of the functional, i.e. $\mathrm{P}^{\psi_{1,m},\psi_{2,m}}(E_k) = \mathrm{P}^{\psi_{1,m,k},\psi_{2,m,k}}(E_k)$. We now can apply Lemma 5.9 with the separation function $v_m = \min\{\max\{v, -m\}, m\}$ and obtain for each $k \geq m$ that

$$\mathcal{A}^{\psi_{1,m},\psi_{2,m}}[w_{E_k}] \leq \mathrm{P}^{\psi_1,\psi_2}(E) + \int\limits_\Omega 1 + \mathbb{1}_E(x,k) - \mathbb{1}_E(x,-k)\,\mathrm{d}\mathcal{H}^n(x).$$

Taking the limit $k \to \infty$, we have $w_{E_k} \to w_E$ in L^1 and observe that

$$\mathcal{A}^{\psi_{1,m},\psi_{2,m}}[w_E] \leq \mathrm{P}^{\psi_1,\psi_2}(E),$$

for all $m \in \mathbb{N}$, where we used that by the remark to Theorem 5.2, the integral term $\int\limits_\Omega 1 + \mathbb{1}_E(x,k) - \mathbb{1}_E(x,-k)\,\mathrm{d}\mathcal{H}^n(x)$ vanishes for $k \to \infty$. Since this holds for all $m \in N$, we may send $m \to \infty$ and obtain with the monotone convergence theorem that

$$\mathcal{A}^{\psi_1,\psi_2}[w_E] \leq \mathrm{P}^{\psi_1,\psi_2}(E).$$

It remains to prove that this result holds for arbitrary open bounded sets Ω and not only for those of finite perimeter:
For fixed E consider the Radon measure $\mu(A) = \int\limits_A \sqrt{1 + |\mathrm{D}w_E|^2} + \int\limits_A (\psi_1 - w_E^+)_+ + (w_E^- - \psi_2)_+ \,\mathrm{d}\varsigma^n$.
First we prove that $\mu(\Omega) < \infty$:
Assume $\mu(\Omega) = +\infty$. Then we find for each $l > 0$ a compact set $K_l \subset \Omega$ such that $\mu(K_l) \geq l$. Since Ω is open and K_l compact, we can find for each point $x \in \partial K_l$ a ball $\mathrm{B}_{r_{x,l}}(x) \subset \Omega$. Covering the topological boundary of K_l in such a way and noting that the boundary is again compact, we find a finite subcover of the topological boundary with $\mathrm{B}_{r_{x_i,l}}(x_i)$, $i = 1, \dots, n$. Consider now the set $\Omega_l := K_l \cup \bigcup_{i=1}^n \mathrm{B}_{r_{x_i,l}}(x_i)$, which has finite perimeter and fulfills $\mu(\Omega_l) \geq \mu(K_l) \geq l$ by the non-negativity of μ. Choosing $l > \mathrm{P}^{\psi_1,\psi_2}(E)$ and applying the result for sets of finite perimeter on the set Ω_l, we reach a contradiction and thus $\mu(\Omega) < +\infty$.
Since $\mu(\Omega) < +\infty$, we find for given ε by regularity some compact set K_ε such that $|\Omega \setminus K_\varepsilon| < \varepsilon$ and $\mu(K_\varepsilon) \geq \mu(\Omega) - \varepsilon$. We can again cover ∂K_ε with balls $\mathrm{B}_{r_{x,l}}(x) \subset \Omega$ for $x \in \partial K_\varepsilon$, choose a finite subcover $\mathrm{B}_{r_{x_i,l}}(x_i)$, $i = 1, \dots, n$ and define the set $\Omega_\varepsilon = K_\varepsilon \cup \bigcup_{i=1}^n \mathrm{B}_{r_{x_i,l}}(x_i)$. Now we arrive again at a set Ω_ε of finite perimeter with $|\Omega \setminus \Omega_\varepsilon| \leq |\Omega \setminus K_\varepsilon| \leq \varepsilon$ and $\mu(\Omega_\varepsilon) \geq \mu(K_\varepsilon) \geq \mu(\Omega) - \varepsilon$. Applying again the result obtained for sets of finite perimeter, we arrive at

$$\mathcal{A}^{\psi_1,\psi_2}[w_E] - \varepsilon \leq \mathrm{P}^{\psi_1,\psi_2}(E)$$

for all $\varepsilon > 0$ and the claimed result follows for $\varepsilon \searrow 0$. □

Remark. This proof can also be applied locally on suitable subsets of the domain Ω and on sufficiently regular domains even with boundary values given by a $\mathrm{W}^{1,1}$ separation function, which is defined on a suitable larger set like a ball containing Ω for the non-parametric problem and as a subgraph of the given function for the parametric problem, similar to Theorem 14.9 in [39].

To obtain that the subgraph of w_E is indeed a minimizer of the parametric double obstacle problem if E is already a minimizer in the cases, where the assumptions of Theorem 5.2 automatically hold, we need to prove that $\mathcal{A}^{\psi_1,\psi_2}[w_E] \geq \mathrm{P}^{\psi_1,\psi_2}(S_{w_E})$. While the part of the area functional and the perimeter is already obtained by Theorem 5.1, a connection for the obstacle terms is missing and its acquirement is the goal of the next section.

5.3 Relationship between the n- and (n+1)-dimensional De Giorgi measure and the second inequality

To prove the desired result, we need to show that

$$\varsigma^{n+1}(S_u^+ \setminus O_1 \cap (\Omega \times \mathbb{R})) = \int\limits_{\Omega} (\psi_1 - u^+)_+ \,\mathrm{d}\varsigma^n$$

or at least that the left hand side is smaller then the right hand side together with an analogous result for the upper obstacle. To achieve that, we first prove a connection between the n- and the $(n+1)$-dimensional De Giorgi measure for sets of the type $E \subset \mathbb{R}^n$ and $E \times I \subset \mathbb{R}^n \times \mathbb{R}$ with an interval I, respectively. From this case, the desired (in-)equality will follow easily by applying a Cavalieri-type principle. Before we state and prove the theorem involving this equality, we shall recall some helpful tools:
First we state a version of the co-area formula for rectifiable sets from [47, Theorem 18.8], which will allow us to estimate the perimeter of an $(n+1)$-dimensional object by its slices:

Theorem 5.11 (Co-area formula for (locally) $\mathcal{H}^{n-1}$-rectifiable sets)**.**
If E is a locally $\mathcal{H}^{n-1}$-rectifiable set in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function, then

$$\int\limits_{\mathbb{R}} \mathcal{H}^{n-2}(E \cap \{f = t\}) \,\mathrm{d}t = \int\limits_{E} |\nabla_E f| \,\mathrm{d}\mathcal{H}^{n-1} \tag{5.3}$$

with tangential gradient

$$\nabla_E f = \nabla f - (\nabla f \cdot \nu_E)\nu_E$$

for a normal vector ν_E of E which exists by the remark to Definition 2.3.
In addition, for any Borel function $g \geq 0$ or $g \in \mathrm{L}^1(\mathbb{R}^n, \mathcal{H}^{n-1} \llcorner E)$, we have

$$\int\limits_{\mathbb{R}} \int\limits_{E \cap \{f=t\}} g \,\mathrm{d}\mathcal{H}^{n-2} \,\mathrm{d}t = \int\limits_{E} g |\nabla_E f| \,\mathrm{d}\mathcal{H}^{n-1}. \tag{5.4}$$

Remark. At this stage we have to mention, that a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is not only differentiable $\mathcal{L}^n$-a.e. as stated in Theorem 2.1 but tangentially differentiable at $\mathcal{H}^{n-1}$-a.e. point of a (locally) $\mathcal{H}^{n-1}$-rectifiable set, see [47, Theorem 11.4], and thus the original statement is for merely Lipschitz functions.

Next, we define slices for a set $E \subset \mathbb{R}^{n+1}$ by

$$E_t := \{x \in E \,:\, x_{n+1} = t\} \tag{5.5}$$

and set x' as the first n components of $x \in \mathbb{R}^n$, i.e. $x = (x', x_{n+1})$. With that, we can connect points in the reduced boundary of a measurable set $E \subset \mathbb{R}^{n+1}$ and its slices E_t starting at (5.4): We use (5.4) on the set $\mathcal{F}E \subset \mathbb{R}^{n+1}$ and the function $f(x) = x_{n+1}$ and obtain

$$\int\limits_{\mathcal{F}E} g|\nu'_E| \,\mathrm{d}\mathcal{H}^n = \int\limits_{\mathbb{R}} \int\limits_{(\mathcal{F}E)_t} g \,\mathrm{d}\mathcal{H}^{n-1} \,\mathrm{d}t \tag{5.6}$$

for a Borel function g which is either non-negative or in $\mathrm{L}^1(\mathbb{R}^n, \mathcal{H}^n \llcorner \mathcal{F}E)$. Using $g \equiv 1$ and $|\nu'_E| \le |\nu_E| = 1$, we obtain

$$\int\limits_{\mathcal{F}E} \mathrm{d}\mathcal{H}^n \ge \int\limits_{\mathcal{F}E} |\nu'_E| \,\mathrm{d}\mathcal{H}^n = \int\limits_{\mathbb{R}} \int\limits_{(\mathcal{F}E)_t} \mathrm{d}\mathcal{H}^{n-1} \,\mathrm{d}t. \tag{5.7}$$

Further, one can use (5.6) to prove an adaptation of [47, Theorem 18.11]:

Theorem 5.12 (Slicing of boundaries).
If E is a set of locally finite perimeter in $\mathbb{R}^{n+1}$, then for a.e. $t \in \mathbb{R}$ the horizontal section E_t of E is a set of locally finite perimeter in $\mathbb{R}^n \times \{t\}$, with

$$\mathcal{H}^{n-1}\left(\mathcal{F}E_t \Delta \left(\mathcal{F}E\right)_t\right) = 0, \tag{5.8}$$

$$\nu'_E(x,t) \neq 0 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \left(\mathcal{F}E\right)_t \tag{5.9}$$

and

$$\nu_{E_t} = \frac{\nu'_E(\,.\,,t)}{|\nu'_E(\,.\,,t)|} \quad \mathcal{H}^{n-1}\text{-a.e. on } (\mathcal{F}E)_t. \tag{5.10}$$

At this point, it is important to distinguish and emphasize the difference between $\mathcal{F}E_t = \mathcal{F}(E_t)$ as the reduced boundary of the slice in $\mathbb{R}^n \times \{t\}$ and $(\mathcal{F}E)_t$ which is the slice of the reduced boundary of E.
Moreover, if E has finite Lebesgue measure and $\mathcal{H}^n(\{x \in \mathcal{F}E : \nu_E(x) = \pm e_{n+1}\}) = 0$, then $v_E(t) = \mathcal{H}^n(E_t)$ for $(t \in \mathbb{R})$ is such that $v_E \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\mathbb{R})$, with

$$\frac{\mathrm{d}}{\mathrm{d}t} v_E(t) = -\int\limits_{(\mathcal{F}E)_t} \frac{\left(\nu_E\right)_{n+1}(x,t)}{|\nu'_E(x,t)|} \,\mathrm{d}\mathcal{H}^{n-1}(x) \text{ for a.e. } t \in \mathbb{R}.$$

Using (5.8) in (5.7) we obtain:

$$\mathcal{H}^n(\mathcal{F}E) \ge \int\limits_{\mathbb{R}} \mathcal{H}^{n-1}(\mathcal{F}E_t) \,\mathrm{d}t. \tag{5.11}$$

With this preparation we are able to prove the following connection between the $(n+1)$- and the n-dimensional De Giorgi measure:

Theorem 5.13 (De Giorgi measure on specific sets).
For $E \subset \mathbb{R}^n$, we have

$$\varsigma^n(E) = \varsigma^{n+1}(E \times (0,1)). \tag{5.12}$$

We split the proof into different parts:

Lemma 5.14 (First inequality between De Giorgi measures of different dimensions)**.**
For $E \subset \mathbb{R}^n$, we have

$$\varsigma^{n+1}(E \times (0,1)) \leq \varsigma^n(E).$$

Proof. We can assume $\varsigma^{\delta,n}(E) < \infty$, since otherwise the inequality is trivial by monotonicity of $\varsigma^{\delta,n}$. By the definition of $\varsigma^{\delta,n}$ (see 3.1) we find for each $\varepsilon > 0$ an open set $E_{\delta,\varepsilon} \subset \mathbb{R}^n$ with $E \subset E_{\delta,\varepsilon}$ and

$$\mathrm{P}(E_{\delta,\varepsilon}) + \frac{|E_{\delta,\varepsilon}|}{\delta} \leq \varsigma^{\delta,n}(E) + \varepsilon \tag{5.13}$$

and thus

$$\begin{aligned} \mathrm{P}(E_{\delta,\varepsilon}) &\leq \varsigma^{\delta,n}(E) + \varepsilon \text{ and} \\ |E_{\delta,\varepsilon}| &\leq \delta(\varsigma^{\delta,n}(E) + \varepsilon). \end{aligned} \tag{5.14}$$

Next, we use the inequality $\mathrm{P}(A \times (0,1)) \leq \mathrm{P}(A) + 2\mathcal{L}^n(A)$, which is still to be proven, and obtain for the open set $E_{\delta,\varepsilon} \times (0,1) \subset \mathbb{R}^{n+1}$ with $(E \times (0,1)) \subset (E_{\delta,\varepsilon} \times (0,1))$ that

$$\begin{aligned} \varsigma^{\delta,n+1}(E_{\delta,\varepsilon} \times (0,1)) &\leq \mathrm{P}(E_{\delta,\varepsilon} \times (0,1)) + \frac{|E_{\delta,\varepsilon} \times (0,1)|}{\delta} \\ &\leq \mathrm{P}(E_{\delta,\varepsilon}) + 2|E_{\delta,\varepsilon}| + \frac{|E_{\delta,\varepsilon}|}{\delta} \\ &\leq \varsigma^{\delta,n}(E) + \varepsilon + 2\delta(\varsigma^{\delta,n}(E) + \varepsilon), \end{aligned}$$

where (5.13) and (5.14) were used. Sending $\delta \searrow 0$ and $\varepsilon \searrow 0$ proves the desired result.
It remains to prove

$$\mathrm{P}(E \times (0,1)) \leq \mathrm{P}(E) + 2\mathcal{L}^n(E).$$

First we note that $(\mathcal{F}(E \times (0,1)))_t = \mathcal{F}E \times \{t\}$ up to $\mathcal{H}^{n-1}$-null sets for a.e. $t \in (0,1)$ by Theorem 5.12, especially (5.8) and the fact that every slice is a shifted version of E. This results in combination with (5.7) in

$$\mathcal{H}^n\left(\mathcal{F}(E \times (0,1)) \cap (\mathbb{R}^n \times (0,1))\right) = \mathrm{P}(E).$$

Further, since $\mathcal{F}(E \times (0,1)) \subset ((\mathcal{F}E) \times (0,1)) \cup (E \times (\{0\} \cup \{1\}))$, it suffices to estimate

$$\mathcal{H}^n(\mathcal{F}(E \times (0,1)) \cap (E \times (\{0\} \cup \{1\}))) \leq 2\mathcal{L}^n(E).$$

Combining both estimates, we obtain the desired inequality. □

Next, we want to prove the second inequality:

Lemma 5.15 (Second inequality for De Giorgi measures of different dimensions)**.**
For $E \subset \mathbb{R}^n$, we have

$$\varsigma^{n+1}(E \times (0,1)) \geq \varsigma^n(E).$$

For this proof, we need a little more preparation, since we want to construct a set $Z \subset \mathbb{R}^n$ for each open set $S \subset \mathbb{R}^{n+1}$ with $E \times [0,1] \subset S$ and for which $\mathrm{P}(S) + \frac{|S|}{\delta} \leq \varsigma^{\delta,n+1}(E \times (0,1)) + \varepsilon$ holds, such that Z fulfills the corresponding inequality in $\mathbb{R}^n$ for the set E. To do so, we need to adjust the sets S:

Lemma 5.16 (Truncation with half-spaces reduces the perimeter)**.**
Given a set $E \subset \mathbb{R}^n$ with finite perimeter and finite volume, then the set $E \cap H$ with a half-space H has a smaller perimeter, i.e.

$$\mathrm{P}(E \cap H) \leq \mathrm{P}(E).$$

Proof. Step 1: For a function $\eta \in \mathrm{C}^1_{\mathrm{cpt}}(\mathbb{R}^n)$ with $0 \leq \eta$ and, without loss of generality, $H = \mathbb{R}^{n-1} \times (-\infty, 0)$, we have $\eta \mathbb{1}_H \in \mathrm{BV}(\mathbb{R}^n)$ with

$$|\mathrm{D}(\eta \mathbb{1}_H)| \leq |\nabla \eta| \mathcal{L}^n \llcorner H + \eta \mathcal{H}^{n-1} \llcorner \partial H.$$

Then we can estimate the following by using integration by part on the half-space $\mathbb{R}^n \setminus H$

$$\begin{aligned} |\mathrm{D}(\eta \mathbb{1}_H)|(\mathbb{R}^n) &\leq \int\limits_H |\nabla \eta| \,\mathrm{d}x + \int\limits_{\partial H} \eta \,\mathrm{d}\mathcal{H}^{n-1} \\ &= \int\limits_H |\nabla \eta| \,\mathrm{d}x - \int\limits_{\mathbb{R}^n \setminus H} \partial_n \eta \,\mathrm{d}x \\ &\leq \int\limits_{\mathbb{R}^n} |\nabla \eta| \,\mathrm{d}x. \end{aligned}$$

Step 2: Let E be a set with finite perimeter and volume. Then $\mathbb{1}_E$ is in BV and we can find a sequence $0 \leq \eta_k \leq 1$ with $\eta_k \in \mathrm{C}^1_{\mathrm{cpt}}(\mathbb{R}^n)$ and $\eta_k \to \mathbb{1}_E$ strict in BV. We obviously have $\eta_k \mathbb{1}_H \to \mathbb{1}_H \mathbb{1}_E = \mathbb{1}_{H \cap E}$ in $\mathrm{L}^1(\mathbb{R}^n)$. By the above argument the sequence is bounded in BV and there exists a weakly* converging subsequence $\eta_{k_l} \mathbb{1}_H \xrightarrow{*} \mathbb{1}_{H \cap E}$ in BV. Using the lower semi-continuity of the perimeter and step 1., we obtain

$$\begin{aligned} |\mathrm{D}\mathbb{1}_{H \cap E}|(\mathbb{R}^n) &\leq \liminf_{l \to \infty} |\mathrm{D}(\eta_{k_l} \mathbb{1}_H)|(\mathbb{R}^n) \\ &\leq \lim_{l \to \infty} \int\limits_{\mathbb{R}^n} |\nabla \eta_{k_l}| \,\mathrm{d}x = \mathrm{P}(E). \end{aligned}$$

□

Now we can proceed with the proof of Lemma 5.15:

Proof. W.l.o.g we can assume $\varsigma^{n+1} = (E \times (0,1)) < \infty$. To show the desired inequality we will show that for each open set $S \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E \times (0,1) \setminus S^+) = 0$ we can find a set $Z \subset \mathbb{R}^n$ with $\mathcal{H}^{n-1}(E \setminus Z^+) = 0$ and $\varsigma^{\delta,n}(Z) \leq \varsigma^{\delta,n+1}(S)$. First of all by truncating S with Lemma 5.16 with the hyper-spaces $H_1 := \mathbb{R}^n \times (-\infty, 1)$ and $H_2 := \mathbb{R}^n \times (0, +\infty)$, we only reduce the perimeter and Lebesgue-measure of the set S and do not change any properties regarding $E \times (0,1) \subset S^+$. Because of that it is enough to consider only sets $S \subset \mathbb{R}^n \times (0,1)$. Since $\mathcal{H}^n(E \times (0,1) \setminus S^+) = 0$ it is clear by using Fubini's theorem that for $\mathcal{H}^1$-a.e t the $(S^+)_t$ contains $E \times \{t\}$.

For choosing the correct slice for our inequality, we define $S_t^n := \{x \in \mathbb{R}^n : (x,t) \in (S^+)_t\}$ and can find for any $\lambda > 0$ a $t_0 \in (0,1)$ with

$$\mathrm{P}(S_{t_0}^n) + \frac{|S_{t_0}^n|}{\delta} \leq \operatorname*{ess\,inf}_{t\in(0,1)} \left(\mathrm{P}(S_t) + \frac{|S_t|}{\delta}\right) + \lambda \tag{5.15}$$

using Theorem 5.12 with (5.8) and (5.11). Further, we can assume $\left(S_{t_0}^n\right)^+$ contains E.
Now, since $\mathcal{F}S_{t_0}^n \times (0,1) \subset \mathcal{F}(S_{t_0}^n \times (0,1))$, we can estimate using (5.15) and (5.7) for $t \in (0,1)$ to obtain

$$\begin{aligned}
\mathrm{P}(S_{t_0}^n, \mathbb{R}^n) + \frac{|S_{t_0}^n|}{\delta} &\leq \mathcal{H}^n(\mathcal{F}S_{t_0}^n \times (0,1)) + \frac{|S_{t_0}^n \times (0,1)|}{\delta} = \int_0^1 \mathcal{H}^{n-1}(\mathcal{F}S_{t_0}^n) + \frac{|S_{t_0}^n|}{\delta}\,\mathrm{d}s \\
&\leq \int_0^1 \mathcal{H}^{n-1}(\mathcal{F}S_s) + \frac{|S_s|}{\delta} + \lambda\,\mathrm{d}s \\
&= \int_0^1 \int_{(\mathcal{F}S_s)} |v_S'|\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}s + \frac{|S \times (0,1)|}{\delta} + \lambda \\
&\leq \mathrm{P}(S) + \frac{|S|}{\delta} - \mathcal{H}^n(\mathcal{F}(S \times (0,1)) \setminus (\mathcal{F}S \times (0,1))) + \lambda.
\end{aligned}$$

In the last estimate $|v_S'| \leq 1$ was used and all parts where $|v_S'| = 0$ were added but only those at $t = 0$ and $t = 1$ subtracted. With λ small enough and $\delta \searrow 0$, the result follows. □

Proof. (of Theorem 5.13) Combining the estimates of Lemma 5.14 and 5.15 yields the result. □

Now we have $\varsigma^n(E) = \varsigma^{n+1}(E \times (0,1))$. By a simple scaling argument we get $\varsigma^n(E) = \frac{1}{\lambda}\varsigma^{n+1}(E \times (0,\lambda))$ and, using the translation invariance of ς, we obtain $\varsigma^{n+1}(E \times M) = \varsigma^n(E)\mathcal{L}^1(M)$ for each $M \in \mathcal{B}(\mathbb{R})$, since open intervals generate $\mathcal{B}(\mathbb{R})$. Since $\mathbb{R}$ is σ-finite with respect to $\mathcal{L}^1$, we have by [23, Kapitel V, 1.3 Satz] that $\varsigma^n \otimes \mathcal{L}^1$ is a measure equal to ς^{n+1} on sets in $\mathcal{B}(E) \times \mathcal{B}(\mathbb{R})$ and by [23, Kapitel V, 1.5 Satz und Definition] the product measure is unique as long as ς^n is defined on an σ-finite set with a corresponding (completed) σ-algebra, i.e. E is a σ-finite set. We thus obtain $\varsigma^{n+1} = \varsigma^n \otimes \mathcal{L}^1$ on such sets. Indeed we will consider the sets $\{\psi_1 - u^+ > 0\}$ and $\{u^- - \psi_2 > 0\}$ which, if $\int_{\mathbb{R}^n} (\psi_1 - u^+)_+ + (u^- - \psi_2)_+ \,\mathrm{d}\varsigma^n$ is finite, is also σ-finite with respect to the De Giorgi measure. Using the product measure on such sets we arrive at

$$\int_E f_+ \,\mathrm{d}\varsigma^n = \varsigma^{n+1}(F) \text{ with } F = \{(x,t) \in \mathbb{R}^{n+1} : x \in A \text{ and } 0 \leq t \leq f(x)\} \tag{5.16}$$

as long as $E \cap \{f > 0\}$ is σ-finite with respect to the De Giorgi measure ς^n.
Applying this result, we arrive at:

Theorem 5.17 (Equality of the obstacle term).
Let Ω, ψ_1, ψ_2 and O_1, O_2 be like in Theorem 5.10 and $u \in \mathrm{BV}$ with $\mathcal{A}^{\psi_1,\psi_2}[u] < \infty$. Then

$$\varsigma^{n+1}(O_1 \setminus S_u^+, \Omega \times \mathbb{R}) = \int_\Omega (\psi_1 - u^+)_+ \,\mathrm{d}\varsigma^n$$

and similar

$$\varsigma^{n+1}(S_u^- \cap O_2, \Omega \times \mathbb{R}) = \int_\Omega (u^- - \psi_2)_+ \, \mathrm{d}\varsigma^n.$$

Proof. Since $\mathcal{A}^{\psi_1,\psi_2}[u]$ is finite, we have that the set $T_1 = \{x \in \Omega : (\psi_1(x) - u^+(x)) > 0\}$ is σ-finite. Using the definition of S_u^+ and u^+, we find that $\mathcal{L}^1((S_u^+ \setminus O_1) \cap \{x\} \times \mathbb{R}) = (\psi_1(x) - u^+(x))_+$. Applying Cavalieri's principle for the product measure $\varsigma^{n+1} = \varsigma^n \times \mathcal{L}^1$ on $T_1 \times \mathbb{R}$, we arrive at the first claim. The second follows analogously. □

Combining the result of Theorem 5.10 and 5.17, we arrive at:

Theorem 5.18 (Non-parametric minimizer of the parametric double obstacle problem)**.** *Let Ω, ψ_1, ψ_2 and O_1 and O_2 be like in Theorem 5.10. Further, let E be a minimizer of* $\mathrm{P}^{\psi_1,\psi_2}$ *such that the assumptions of Theorem 5.2 are met. Then, S_{w_E} is also a minimizer of* $\mathrm{P}^{\psi_1,\psi_2}$.

Proof. Applying Theorem 5.1, 5.10 and 5.17, we have

$$\mathrm{P}^{\psi_1,\psi_2}(S_{w_E}) = \mathcal{A}^{\psi_1,\psi_2}[w_E] \leq \mathrm{P}^{\psi_1,\psi_2}(E) = \min_A \mathrm{P}^{\psi_1,\psi_2}(A)$$

and thus S_{w_E} is again a minimizer or $\mathrm{P}^{\psi_1,\psi_2}$. □

With Theorem 5.18 the second major result of this thesis follows:

Theorem 5.19 (Relation between parametric and non-parametric double obstacle problem)**.** *Let Ω be an open bounded set, $\psi_1, \psi_2 : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be two Borel functions such that a separation function $v \in \mathrm{W}^{1,1}$ exists with $\psi_1 \leq v^* \leq \psi_2$ holds $\mathcal{H}^{n-1}$-a.e. Further, let $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ be a local minimizer of $\mathcal{A}^{\psi_1,\psi_2}$, i.e. u minimizes $\mathcal{A}^{\psi_1,\psi_2}$ on each open set $A \Subset \Omega$. Then the set $U = \{(x,t) \in \Omega \times \mathbb{R} : t < u(x)\}$ minimizes the parametric double obstacle problem $\mathrm{P}^{\psi_1,\psi_2}$ locally (on $A \times \mathbb{R}$).*

Proof. Let $A \Subset \Omega \subset \mathbb{R}^n$ be an open set and B be a Caccioppoli set in $\Omega \times \mathbb{R}$ which coincides with U outside of some compact set $K \subset A \times \mathbb{R}$. Then $U \cap (A \times \mathbb{R})$ and $B \cap (A \times \mathbb{R})$ satisfy the conditions (i) and (ii) of Theorem 5.2 and the graphification w_B of the competitor B coincides with u outside of A.
Assume now that B has smaller parametric double obstacle functional value than U. Then we obtain by Theorem 5.1, 5.17 and 5.10 that $\mathcal{A}^{\psi_1,\psi_2}[w_B, A] < \mathcal{A}^{\psi_1,\psi_2}[u, A]$ and thus a contradiction. Therefore U is a local minimizer of $\mathrm{P}^{\psi_1,\psi_2}$. □

Remark.

1. The reverse implication of the main result, i.e. that a set of subgraph type that minimizes the parametric double obstacle problem with obstacles O_1, O_2 like in Theorem 5.19 follows trivially by 5.17 and that for each function u the subgraph S_u is eligible for the parametric double obstacle problem.

2. The general approach of Theorem 5.19 can be exploited for the functional $\mathcal{A}_{u_0}^{\psi_1,\psi_2}$ if we have a minimizer $u \in \mathrm{BV}(\Omega)$ and Ω is a bounded domain such that u_0 can be extended by a function $u_0 \in \mathrm{W}^{1,1}(\mathrm{B}_r)$ to a ball $\mathrm{B}_r \supset \Omega$ where $\psi_1 \le u_0^* \le \psi_2$ holds $\mathcal{H}^{n-1}$-a.e. in Ω. Then this equivalence holds on $\overline{\Omega} \times \mathbb{R}$ by the same argument, where the obstacles are set to $\psi_1 \equiv \psi_2 \equiv u_0$ on $\mathrm{B}_r \setminus \Omega$.
3. Obviously, this result also extends to the single obstacle problem using $\psi_1 \equiv -\infty$ or $\psi_2 \equiv +\infty$ for the upper or lower obstacle problem.

5.4 The non-parametric double obstacle problem revisited

In Section 4, we have seen that in the non-parametric setting a $\mathrm{W}^{1,1}$ separation function between the obstacles is needed to be able to use the truncation theorem in a proper way. Applying the truncation theorem with a BV-separation function failed and could not be fixed. The usage of the parametric theory can circumvent this difficulty. Consider the proof of Theorem 5.10. If it is possible to obtain the same result without using truncated obstacles, the in the proof constructed sequence w_{E_ε} could be used as a recovery sequence for arbitrary function $u \in \mathrm{BV}$ with finite functional value $\mathcal{A}^{\psi_1,\psi_2}[u] < \infty$ by setting $E = S_u$. The change from a $\mathrm{W}^{1,1}$ to a BV separating function is not an issue, since one can still apply the truncation results from Proposition 5.5 with the subgraph of a BV function v which satisfies $\psi_1 \le v^+$ and $v^- \le \psi_2$. The main part to notice is that S_v^+ still contains the lower obstacle O_1 and similarly $O_2 \subset \Omega \setminus S_v^{\cdot}$ and thus the proof works identically. Now we can state the full version of the double obstacle problem for the area functional and thus extend the result of 4.13 in that specific case:

Theorem 5.20 (The general non-parametric double obstacle problem for the area functional)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$ and let $O_1 = S_{\psi_1}$ and $O_2 = S^{\psi_2}$ for some bounded Borel functions ψ_1, ψ_2 be given such that $\psi_1 \le \psi_2$ holds $\mathcal{H}^{n-1}$-a.e. and a function $v \in \mathrm{BV}(\Omega)$ exists with $\mathcal{H}^{n-1}$-a.e. inequalities $\psi_1 \le v^+$ and $v^- \le \psi_2$. Then, the relaxation of*

$$\int_\Omega \sqrt{1+|\mathrm{D}u|^2} \text{ for } u \in \mathrm{BV}_{\circ,\psi_1,\psi_2}(\Omega)$$

extended by $+\infty$ to $\mathrm{L}^1(\Omega) \setminus \mathrm{BV}_{\circ,\psi_1,\psi_2}(\Omega)$ is given by $\mathcal{A}^{\psi_1,\psi_2}$.

Proof. We first introduce an improvement of the proof of Theorem 5.10 to apply it later on each subgraph of eligible function $u \in \mathrm{BV}$ with $\mathcal{A}^{\psi_1,\psi_2}[u]$: Given is a set E which satisfies the prerequisite of Theorem 5.2 with $\mathrm{P}^{\psi_1,\psi_2}(E) < \infty$. The main idea is to not simply cut off a given set at $\pm k$, but instead consider the set $E_k = E \cap (\Omega \times (-k,k)) \cup (S_v \setminus (\Omega \times (-k,k))$ and notice that

$$\mathrm{P}^{\psi_1,\psi_2}(E_k) \le \mathrm{P}^{\psi_1,\psi_2}(E, \Omega \times \mathbb{R}) + 2|\Omega| + \mathrm{P}(S_v, \Omega \times \mathbb{R})$$

and $E_k \to E$ in L^1 as well as

$$\begin{aligned}\mathrm{P}(E_k, \Omega \times \mathbb{R}) \leq &\mathcal{H}^n(\mathcal{F}E \cap (\Omega \times (-k,k)) + \mathcal{H}^n(\mathcal{F}S_v \setminus (\Omega \times (-k,k))) \\ &+ \int_\Omega \mathbb{1}_E(x,-k) + \mathbb{1}_{S_v}(x,k) - 2\mathbb{1}_E(x,-k)\mathbb{1}_{S_v}(x,k)\,\mathrm{d}\mathcal{H}^n(x) \\ &+ \int_\Omega \mathbb{1}_E(x,k) + \mathbb{1}_{S_v}(x,k) - 2\mathbb{1}_E(x,k)\mathbb{1}_{S_v}(x,k)\,\mathrm{d}\mathcal{H}^n(x)\end{aligned}$$

for almost all $k > 0$, where the inequality follows similarly as in the proof of Theorem 5.2 in [39]. Without loss of generality consider an Ω of finite perimeter in $\mathbb{R}^n$ or apply the approximation like in the end of proof of Theorem 5.10. Using the approximations like in the proof of Lemma 5.8 and 5.9 on $(E_k \cap \Omega \times (-k-1,k+1))$ with $Q_k = \mathrm{B}_R \times (-k-1,k+1)$ and the usual truncation of the set, which gets added to E_k, by S_v and the truncation of the set, which gets subtracted from E_k, by S^v, we obtain for almost all $k > 0$ sets $\tilde{E}_{k,\varepsilon} \subset \Omega \times (-k-1,k+1)$ which are close to $E_k \cap (\Omega \times (-k-1,k-1)$ in L^1 and the perimeter can be bounded by

$$\begin{aligned}\mathrm{P}(\tilde{E}_{k,\varepsilon}, \Omega \times (-k-1,k+1)) \leq &\mathrm{P}(E_k, \Omega \times (-k-1,k+1)) \\ &+ \varsigma^{n+1}(O_1 \setminus E^+, \Omega \times \mathbb{R}) + \varsigma^{n+1}(O_2 \cap E^-, \Omega \times \mathbb{R}) + \varepsilon,\end{aligned}$$

since the truncation enforces that E_k agrees with S_v in $\Omega \times (-k-1,-k)$ and $\Omega \times (k,k+1)$. This also yields that $E_{k,\varepsilon} := \tilde{E}_{k,\varepsilon} \cup S_v \setminus (\Omega \times (-k,k))$ is close to E_k in L^1 and in the perimeter to the parametric double obstacle functional, more precisely, we have

$$\mathrm{P}(E_{k,\varepsilon}, \Omega \times \mathbb{R}) \leq \mathrm{P}^{\psi_1,\psi_2}(E_k) + \varepsilon$$

for almost all $k > 0$. Considering the obstacle O_1, we easily can verify that $O_1 \subset E^+_{k,\varepsilon}$ and moreover that $O_1 \subset S_{w^+_{E_{k,\varepsilon}}}$ at least $\mathcal{H}^n$-a.e., since for almost every x with

1. $\psi_1(x) \leq -k$, we have $O_1 \cap \{x\} \times \mathbb{R} \subset S_{v^+} \cap \{x\} \times \mathbb{R} \subset S^+_v \cap \{x\} \times \mathbb{R}$,
2. $-k < \psi_1(x) \leq k$ we have that $E_{k,\varepsilon} \cap (\Omega \times [-k,k])$ contains for each $\eta > 0$ a by S_v truncated cylinder of the form $\mathrm{B}_{r_\eta} \times [-M, \psi_1(x) - \eta] \cap S_v$ around $x \times [-k, \psi_1(x) - \eta]$. This implies that by the definition of the upper representative v^+, S_{v^+} and S^+_v that $O_1 \cap (\mathrm{B}_{r_\eta}(x) \times \mathbb{R}) \subset S^+_v \cap (\mathrm{B}_r(x) \times \mathbb{R}) \subset S^+_{w_{E_{k,\varepsilon}}} \cap (\mathrm{B}_r(x) \times \mathbb{R})$, since $S_v \cap (\mathrm{B}_r(x) \times (-\infty, -k)$ is also contained in $E_{k,\varepsilon}$ by construction. Hence, $O_1 \cap \{x\} \times \mathbb{R} \subset S_{w^+_{E_{\varepsilon,k}}}$.
3. Finally and arguing like in 2. together with $v^+ \geq \psi_1$, the same result follows for $\psi_1(x) > k$.

Arguing similarly for the upper obstacle, we have that $w_{E_{k,\varepsilon}}$ fulfills the double obstacle constraint and converges for $\varepsilon \searrow 0$ to w_{E_k} together with

$$\mathcal{A}^{\psi_1,\psi_2}[w_{E_k}] \leq \mathrm{P}^{\psi_1,\psi_2}(E_k).$$

Letting $k \to \infty$, we obtain that $\mathrm{P}(E_k, \Omega \times \mathbb{R}) \to \mathrm{P}^{\psi_1,\psi_2}(E)$ and $w_{E_k} \to w_E$ in L^1 and thus

$$\mathcal{A}^{\psi_1,\psi_2}[w_E] \leq \mathrm{P}^{\psi_1,\psi_2}(E).$$

Now we may apply this result on each subgraph $E = S_u$ of functions u with finite $\mathcal{A}^{\psi_1,\psi_2}[u]$ and a suitable diagonal sequence with $\varepsilon \searrow 0$ and suitable $k \to \infty$ to obtain a recovery sequence. Together with Theorem 5.18, the proof is finished, since this gives us

$$\mathcal{A}^{\psi_1,\psi_2}[u] = \mathrm{P}^{\psi_1,\psi_2}(S_u).$$

□

Remark. In retrospective, the relaxation of the double obstacle problem for the area functional could be proven using Theorem 5.18 and the results for the parametric double obstacle problem for the area functional. Similarly, if for a non-parametric functional a parametric version with the same approximation properties exists, the results of Corollary 4.13 extends to those functionals.

5.5 Some continuity result at the obstacles for minimizers of the double obstacle problem

In this section, we briefly discuss the regularity of solutions of the double obstacle problem for the area functional in the non-parametric setting. In general, on the contact sets $\{u = \psi_i\}$ for $i = 1, 2$ we expect the regularity of the obstacle ψ_i and away from obstacles we have the usual local regularity of minimal surfaces. It is interesting to study regularity up to the obstacles, which in general may be as bad as possible depending on the obstacles. Even if only thin obstacles are allowed, they can define values on a dense subset of Ω and thus can restrict the minimizer to only one option. Because of that, it is reasonable to consider nice settings, like for example in [26] for the parametric or in [29] for the non-parametric case. Most of the time the domain $\Omega \subset \mathbb{R}^n$ is set to be a unit ball and smooth obstacles contained in one diameter of the ball are considered. The prescribed boundary values on the unit sphere should be compatible with the obstacle where, in the case $n \geq 2$, the diameter touches the boundary. Here, we consider $\Omega = \mathrm{B}_1(0)$ and thin obstacles whose support is contained in $\mathrm{B}_1'(0) = \mathrm{B}_1(0) \cap (\mathbb{R}^{n-1} \times \{0\})$. Even if the thin obstacle is smooth or constant, one cannot expect a global regularity, since even in the case $n = 1$ it is clear that for a domain $\Omega = (-1, 1)$, boundary values $u_0 = 0$ for $x = \pm 1$ and a single obstacle $\psi = 1$ at $x = 0$ and $\psi = -\infty$ on $[-1, 1] \setminus \{0\}$ the minimizer is not smooth on the whole domain. In this case, the minimizer is given by $u(x) = 1 - |x|$ which is only Lipschitz but not differentiable on the whole domain. Similar results are established in [26, Theorem 1.7] showing that for flat obstacles, at least on $\mathrm{B}_{\frac{1}{2}}(0)$, the minimizers are reasonably smooth on $\mathrm{B}^+_{\frac{1}{2}}(0) \cup \mathrm{B}'_{\frac{1}{2}}(0)$ with $\mathrm{B}^+_{\frac{1}{2}}(0) = \mathrm{B}_{\frac{1}{2}}(0) \cap \big(\mathbb{R}^{n-1} \times (0, +\infty)\big)$ and on $\mathrm{B}^-_{\frac{1}{2}}(0) \cup \mathrm{B}'_{\frac{1}{2}}(0)$ with $\mathrm{B}^-_{\frac{1}{2}}(0) = \mathrm{B}_{\frac{1}{2}}(0) \cap \big(\mathbb{R}^{n-1} \times (-\infty, 0)\big)$. Similarly, in the non-parametric setting, regularity on $\mathrm{B}^+_1(0) \cup \mathrm{B}'_1(0)$ and $\mathrm{B}^-_1(0) \cup \mathrm{B}'_1(0)$ is established in [29, Theorem 1.1] under a symmetry assumption with respect to $\mathbb{R}^{n-1} \times \{0\}$.
For the double obstacle problem, we note that if $\psi_1 = \psi_2$ on $\mathrm{B}'_1(0)$ and $\psi_1 = -\psi_2 = -\infty$ otherwise, the regularity/continuity at the obstacle switches to those at the boundary for the domains $\mathrm{B}^+_1(0)$ and $\mathrm{B}^-_1(0)$. For example, if u_0 is continuous on $\mathrm{B}'_1(0)$ and $\psi_i = u_0$ for $i = 1, 2$ on $\mathrm{B}'_1(0)$, we can use [39, Theorem 15.9] following the results of [51] to establish continuity at the active obstacle on $\mathrm{B}'_1(0)$:

Theorem 5.21 (Continuity at the boundary)**.**
For an open bounded set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and $u_0 \in \mathrm{L}^1(\partial\Omega, \mathcal{H}^{n-1})$, let v be the minimizer of

$$\mathcal{A}_{u_0}[u,\Omega] = \int\limits_{\Omega} \sqrt{1+|\mathrm{D}u|^2}\,\mathrm{d}x + \int\limits_{\partial\Omega} |u_0 - u_{\mathrm{int}}|\,\mathrm{d}\mathcal{H}^{n-1}.$$

Then, if u_0 is continuous at a point x_0 and $\partial\Omega$ has non-negative mean-curvature in a generalized sense near x_0, v attains continuously the value of u_0 at x_0, i.e.

$$\lim_{\Omega \ni x \to x_0} v(x) = u_0(x_0).$$

Now we are able to prove the following result:

Proposition 5.22 (Continuity at the obstacle/boundary)**.**
Let $\psi_0 \in \mathrm{C}^0(\mathrm{B}_1'(0))$ and $u_0 \in \mathrm{L}^1(\partial\mathrm{B}_1(0), \mathcal{H}^{n-1})$ be given. For an easier notation we set $u_0 = \psi_0$ on $\mathrm{B}_1'(0)$ and further define the obstacles $\psi_1 = \psi_2 = u_0 (= \psi_0)$ on $\mathrm{B}_1'(0)$ and $\psi_1 = -\infty = -\psi_2$ otherwise. Then, the minimizer v of $\mathcal{A}_{u_0}^{\psi_1,\psi_2}$ on $\mathrm{B}_1(0)$ coincides on $\mathrm{B}_1^+(0)$ with the minimizers of $\mathcal{A}_{u_0}[u, \mathrm{B}_1^+(0)]$ and on $\mathrm{B}_1^-(0)$ with the minimizers of $\mathcal{A}_{u_0}[u, \mathrm{B}_1^-(0)]$ and is continuous in $\mathrm{B}_1(0)$. Further, if u_0 is even continuous on $\mathrm{B}_1'(0) \cup \partial\mathrm{B}_1(0)$, the minimizer is continuous on the closed ball and thus at the obstacle and up to the boundary.

Remark. Even if u_0 is defined on $\partial\mathrm{B}_1(0) \cup \mathrm{B}_1'(0)$, the boundary values constraint in the functional $\mathcal{A}_{u_0}^{\psi_1,\psi_2}[u, E]$ is only considered on ∂E for $E \in \{\mathrm{B}_1(0), \mathrm{B}_1^{\pm}(0)\}$.

Proof. By classical results, see for example [39, Theorem 14.3], we have that the minimizers of $\mathcal{A}_{u_0}^{\psi_1,\psi_2}[u, \mathrm{B}_1(0)]$, $\mathcal{A}_{u_0}[u, \mathrm{B}_1^+(0)]$ and $\mathcal{A}_{u_0}[u, \mathrm{B}_1^-(0)]$ are continuous (smooth) on $\mathrm{B}_1(0) \setminus \mathrm{B}_1'(0)$, $\mathrm{B}_1^+(0)$ and $\mathrm{B}_1^-(0)$, respectively. Since the boundary of $\mathrm{B}_1^+(0)$ and $\mathrm{B}_1^-(0)$ is Lipschitz and has non-negative mean curvature, we obtain with the use of Theorem 5.21 two minimizers v_1 of $\mathcal{A}_{u_0}[u, \mathrm{B}_1^+(0)]$ and v_2 of $\mathcal{A}_{u_0}[u, \mathrm{B}_1^-(0)]$ are continuous on $\mathrm{B}_1^+(0) \cup \mathrm{B}_1'(0)$ and $\mathrm{B}_1^-(0) \cup \mathrm{B}_1'(0)$, respectively, and thus can be combined to a continuous function v over the whole domain $\mathrm{B}_1(0)$. It remains to prove that v is indeed a minimizer of $\mathcal{A}_{u_0}^{\psi_1,\psi_2}$:
For that, we use Proposition 3.11 and rewrite the functional, remembering $f^\infty(z) = |z|$ for the area functional for some arbitrary $u \in \mathrm{BV}(\mathrm{B}_1(0))$:

$$\begin{aligned}
\int\limits_{\mathrm{B}_1(0)} \sqrt{1+|\mathrm{D}u|^2} + \int\limits_{\partial\mathrm{B}_1(0)} |u_0 - u_{\mathrm{int}}|\,\mathrm{d}\mathcal{H}^{n-1} + \int\limits_{\mathrm{B}_1(0)} (\psi_1 - u^+)_+ + (u^- - \psi_2)_+ \,\mathrm{d}\varsigma \\
= \int\limits_{\mathrm{B}_1^+(0)} \sqrt{1+|\mathrm{D}u|^2} + \int\limits_{\partial\mathrm{B}_1(0)\cap\overline{\mathrm{B}_1^+(0)}} |u_0 - u_{\mathrm{int}}|\,\mathrm{d}\mathcal{H}^{n-1} \\
+ \int\limits_{\mathrm{B}_1^-(0)} \sqrt{1+|\mathrm{D}u|^2} + \int\limits_{\partial\mathrm{B}_1(0)\cap\overline{\mathrm{B}_1^-(0)}} |u_0 - u_{\mathrm{int}}|\,\mathrm{d}\mathcal{H}^{n-1} \\
+ 2 \int\limits_{\mathrm{B}_1'(0)} (u_0 - u^+)_+ + (u^- - u_0)_+ \,\mathrm{d}\mathcal{H}^{n-1} + \int\limits_{\mathrm{B}_1'(0)} u^+ - u^- \,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}$$

Now we need to investigate the last line. On the set, where u does not jump, we have

$$2 \int\limits_{B_1'(0)\setminus J_u} (u_0 - u^+)_+ + (u^- - u_0)_+ \, d\mathcal{H}^{n-1} + \int\limits_{B_1'(0)\setminus J_u} u^+ - u^- \, d\mathcal{H}^{n-1}$$
$$= 2 \int\limits_{B_1'(0)\setminus J_u} (u_0 - u)_+ + (u_0 - u)_- \, d\mathcal{H}^{n-1}$$
$$= 2 \int\limits_{B_1'(0)\setminus J_u} |u_0 - u| \, d\mathcal{H}^{n-1}.$$

On $B_1'(0) \cap J_u$ we have the triple $(u^+, u^-, \pm e_n)$ $\mathcal{H}^{n-1}$-a.e. In those cases, the trace is with respect to $B_1^+(0)$ and $B_1^-(0)$ has to be considered more carefully. Since $u^- < u^+$ we only have three cases: The first case is $u^- < u^+ \leq u_0$ and we obtain pointwise for the integrands:

$$2\left((u_0 - u^+)_+ + (u^- - u_0)_+\right) + u^+ - u^- = 2\left(u_0 - u^+\right) + u^+ - u^- = u_0 - u^+ + u_0 - u^-.$$

In the second case, we have $u_0 \leq u^- < u^+$ and we similarly obtain

$$2\left((u_0 - u^+)_+ + (u^- - u_0)_+\right) + u^+ - u^- = 2\left(u^- - u_0\right) + u^+ - u^- = u^+ - u_0 + u^- - u_0.$$

In the last case, we have $u^- < u_0 < u^+$ and thus

$$2\left((u_0 - u^+)_+ + (u^- - u_0)_+\right) + u^+ - u^- = u^+ - u^- = u^+ - u_0 + u_0 - u^-.$$

Combining this results, we arrive in each case at the integrand $|u_0 - u^+| + |u_0 - u^-|$ and thus obtain

$$\mathcal{A}_{u_0}^{\psi_1,\psi_2}[u, B_1(0)] = \mathcal{A}_{u_0}[u, B_1^+(0)] + \mathcal{A}_{u_0}[u, B_1^-(0)].$$

From this it is clear, that v is a minimizer of $\mathcal{A}_{u_0}^{\psi_1,\psi_2}$ since otherwise a minimizer w would satisfy either

$$\mathcal{A}_{u_0}[w, B_1^+(0)] < \mathcal{A}_{u_0}[v, B_1^+(0)] = \mathcal{A}_{u_0}[v_1, B_1^+(0)],$$
$$\mathcal{A}_{u_0}[w, B_1^-(0)] < \mathcal{A}_{u_0}[v, B_1^-(0)] = \mathcal{A}_{u_0}[v_2, B_1^-(0)]$$

or both. This contradicts that v_1 and v_2 are minimzers of $\mathcal{A}_{u_0}[u, B_1^+(0)]$ and $\mathcal{A}_{u_0}[u, B_1^-(0)]$, respectively. If u_0 is in addition continuous on $\partial B_1(0) \cup B_1'(0)$, the continuity of v on $\overline{B}_1(0)$ follows from Theorem 5.21. □

Similarly, some specific cases may be traced back to existing regularity theory for single obstacle problems, but we still miss a theory for the double obstacle problem even in the mentioned case with $\psi_1 \neq \psi_2$ on $B_1'(0)$. This could be an interesting point for further research, which may be enriched by the equivalence of the parametric and non-parametric problem from Theorem 5.19.

6 Variational inequalities via one-sided directional derivatives

In the introduction, the relationship between obstacle problems, variational inequalities and partial differential equations with additional constraints have been mentioned at least in the $W^{1,1}$-case. We have seen that to handle double obstacle problems for functionals with linear growth we need the space of functions of bounded variation for which Anzellotti introduced a notion of Euler equations in [4]. If dealing with obstacle problems, the choice for directional derivatives is restricted by the obstacle. This leads to a restriction on the directions which on the other side allows for more integrals to to be covered. For example, if we consider the total variation, we cannot compute the directional derivative at a point where the original function is 0, since the absolute value function is not differentiable at 0. If instead only one-sided derivatives are considered, this difficulty can be overcome.
To introduce this theory and compute the derivatives, we present in the first subsection the main results by Anzellotti, before we provide a framework for one-sided derivatives in the setting. Last, we tackle the derivatives and variational inequalities obtained through obstacle terms and provide a theorem which proves at least in some cases the equivalence between the solutions of variational inequalities and minimizers of associated functionals.
Before we begin, we have to mention another approach to Euler equations, which is also applicable to obstacle problems, and compare it with the approach given in this section: It is (often) possible and useful to derive Euler equations from dual formulations of a given problem, like done in [44], [7, Chapter 2] or [62]. The later also involves Euler equation for the one-sided (thin) obstacle problem for the total variation and the area functional and one of their main results, see [62, Introduction, Theorem 3.6 and 3.9], is the dual formulation of the minimization of the one-sided (thin) obstacle problem, which uses a duality between BV functions and S^∞_- functions. S^∞_- is defined as the space of sub-unit vector fields in L^∞ whose distributional divergence exists as a non-positive Radon measure. Further one can see that in the obstacle free case, the duality still holds with sub-unit vector-fields with vanishing distributional divergence.
If one tries to implement such result for the double obstacle problems many difficulties arise. A major one is that one loses the conditions on the sign of the divergence and thus no representation theorem of Riesz type can be applied for the required vector-fields and domain splitting arguments to get such condition back only work in trivial cases. Thus, for the double obstacle problem this directions seems to be not promising.

6.1 Differentiability results by Anzellotti

For the general setting, we consider on an open bounded set $\Omega \subset \mathbb{R}^n$, a non-negative Borel function $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$ such that the strong recession function f^∞ exists. For the following statements, like in [4], it is enough to demand the existence of a weaker form of the recession function of the form

$$f^\infty(x, z) = \lim_{t\to\infty} \frac{f(x, tz)}{t},$$

and thus dropping the additional regularity with regards to the x-variable. This is enough to explain a functional of the form

$$\mathrm{I}[\mu] := \int\limits_{\Omega} f(x, \mu)$$

for a Radon measure $\mu \in \mathrm{RM}(\Omega, \mathbb{R}^n)$ like in (2.9). Further, the non-negativity may be dropped if $|f(.,z)| \leq b(1+|z|)$ is assumed. For simplicity, we assume f to be non-negative nevertheless. In the following, if we consider boundary conditions or penalizing terms on the boundary appear, we also require Ω to have either Lipschitz boundary and the recession function to be continuously extendable to $\overline{\Omega}$ like in (2.7). Next, we state the main results from [4] with regards to functionals on Radon measures which can be found in Theorem 2.4 and 2.5

Theorem 6.1 (Differentiability of functionals on measures)**.**
Assume that for all $x \in \Omega$ the function $f(x,z)$ is differentiable in z for all $z \in \mathbb{R}^n$ and $f^\infty(x,z)$ is differentiable for all $z \in \mathbb{R}^n \setminus \{0\}$. Assume also that the derivatives with regard to the z-variable are bounded, i.e.

$$|f_z(x,z)| \leq M \text{ and } |f_z^\infty(x,z)| \leq M.$$

Then the functional $\mathrm{I}[\mu] := \int\limits_{\Omega} f(\,.\,,\mu)$ is differentiable at the point μ in direction β iff $|\beta|^s \ll |\mu|^s$ and in this case one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{I}[\mu + t\beta]\bigg|_{t=0} = \int\limits_{\Omega} f_z(\,.\,,\mu^a)\cdot\beta^a \,\mathrm{d}x + \int\limits_{\Omega} f_z^\infty\left(\,.\,,\frac{\mu^s}{|\mu|^s}\right)\cdot\frac{\beta^s}{|\beta|^s}\,\mathrm{d}|\beta|^s$$

with the common notation of the absolute continuous parts μ^a, β^a and the singular parts μ^s, β^s w.r.t. the Lebesgue measure.
If $f(x,z)$ is not differentiable in z for some $x \in \Omega$ at $z = 0$, the statement remains true under the additional assumptions that for such x, where f is not differentiable in z at $z = 0$ we have $f(x,0) = 0$ and that $\beta^a = 0$ almost everywhere on the set $T = \{x \in \Omega \,:\, \mu^a(x) = 0\}$.

Remark. Obviously, Theorem 6.1 can be stated with respect to some other measure than the Lebesgue measure and remains true if the x-dependency of f is dropped.

Next, we specifically turn to the BV case and sum up the results from Theorem 3.6 and parts of Theorem 3.7 and 3.9 in [4]:

Theorem 6.2 (Differentiability of functionals on BV)**.**
Assume that for all $x \in \Omega$ the function $f(x,z)$ is differentiable in z for all $z \in \mathbb{R}^n$ and $f^\infty(x,z)$ is differentiable for all $z \in \mathbb{R}^n \setminus \{0\}$. Assume also that

$$|f_z(x,z)| \leq M \text{ and } |f_z^\infty(x,z)| \leq M.$$

Then the functional $\mathcal{F}[u] := \int\limits_{\Omega} f(\,.\,,\mathrm{D}u)$ is differentiable at $u \in \mathrm{BV}(\Omega)$ in direction φ iff $|\mathrm{D}^s\varphi| \ll |\mathrm{D}^s u|$ and in such case one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[u + t\varphi]\bigg|_{t=0} = \int\limits_{\Omega} f_z(\,.\,,\mathrm{D}^a u)\cdot\mathrm{D}^a\varphi \,\mathrm{d}x + \int\limits_{\Omega} f_z^\infty\left(\,.\,,\frac{\mathrm{D}^s u}{|\mathrm{D}^s u|}\right)\cdot\frac{\mathrm{D}^s\varphi}{|\mathrm{D}^s\varphi|}\,\mathrm{d}|\mathrm{D}^s\varphi|$$

with the common notation of the absolutely continuous parts $\mathrm{D}^a u$, $\mathrm{D}^a \varphi$ *and the singular parts* $\mathrm{D}^s u$, $\mathrm{D}^s \varphi$ *w.r.t. the Lebesgue-measure.*
If f *is not differentiable at* $z = 0$ *for some* $x \in \Omega$*, the statement remains true under the additional assumptions that for such* $x \in \Omega$ *we have* $f(x, 0) = 0$ *and* $\mathrm{D}^a \varphi = 0$ *almost everywhere on the set* $T = \{x \in \Omega \,:\, \mathrm{D}^a u(x) = 0\}$*.*
Additionally, if boundary values are fixed by a given function u_0*, we have an extra penalizing term to consider. Thus we are looking at* $\int_\Omega f(x, \mathrm{D}u) + \int_{\partial\Omega} f^\infty(x, \nu_\Omega[u_0 - u_{\mathrm{int}}]) \,\mathrm{d}\mathcal{H}^{n-1}$ *with the outward normal* ν_Ω*. The additional term is again differentiable at* $u \in \mathrm{BV}$ *if* $\varphi = 0$ $\mathcal{H}^{n-1}$*-a.e. where* $u = u_0$ *in* $\partial\Omega$ *and the derivative can be computed by*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega} f^\infty(\,.\,, \nu_\Omega[u_0 - u - t\varphi]) \,\mathrm{d}\mathcal{H}^{n-1} \Bigg|_{t=0} = -\int_{\partial\Omega} f_z^\infty\left(\,.\,, \nu_\Omega \frac{u_0 - u}{|u_0 - u_{\mathrm{int}}|}\right) \nu_\Omega \varphi \,\mathrm{d}\mathcal{H}^{n-1}.$$

The theorems treat the two general cases of the total variation with $f(z) = |z|$, which is not differentiable in $\{0\}$, and the area functional with $f(z) = \sqrt{1 + |z|^2}$. For the total variation case, the directions are even further restricted because of the non-differentiability at $\{0\}$ and the additional condition has to be satisfied. For the boundary term, the problem is essentially the same.
Closing this part, we state that if u is a minimizer of such a given functional, the derivatives obtained through Theorem 6.2 have to vanish, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[u + t\varphi] \Bigg|_{t=0} = 0 \text{ for all } \varphi \text{ allowed.}$$

These equations are generalized Euler equations.
The remaining question is if, in reverse, solutions of those Euler equations are again minimizers. The result is given by Theorem 3.10 in [4], which we will compare later in context of the same question regarding variational inequalities for obstacle problems.

6.2 Variational inequalities for functionals of linear growth

Since we are interested in the one-sided directional derivative to be able to treat obstacle problems, we assume $t > 0$ and can state the following two versions connected to the Euler equations presented in Theorem 6.1 and Theorem 6.2 without distinguishing the two cases, since the absolute value function is one-sided differentiable in $\{0\}$. With the definition

$$\bar{f}_z(x, z, y) := \lim_{t \searrow 0} \frac{f(x, z + ty) - f(x, z)}{t},$$

i.e. the one-sided direction derivative in the z variable in the direction y, we obtain:

Theorem 6.3 (One-sided differentiability of functionals on measures)**.**
Assume that for all $x \in \Omega$ *the function* $f(x, z)$ *is one-sided directionally differentiable in* z *for all* $z \in \mathbb{R}^n$ *and* $f^\infty(x, z)$ *is one-sided directionally differentiable for all* $z \in \mathbb{R}^n$*. Assume also that all*

one-sided derivatives at all $z \in \mathbb{R}^n$ are bounded by M, i.e. for all $z \in \mathbb{R}^n$ we require

$$|\bar{f}_z(x,z,y)| \leq M \text{ and } |\bar{f}_z^\infty(x,z,y)| \leq M$$

to hold for each $y \in \mathrm{S}_1^{n-1}(0)$. Then the functional $\mathrm{I}[\mu] := \int_\Omega f(\,.\,,\mu)$ is one-sided directional differentiable at the point μ in direction β with

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\mathrm{I}[\mu+t\beta]\bigg|_{t=0} = \int_\Omega \bar{f}_z(\,.\,,\mu^a,\beta^a)\,\mathrm{d}x + \int_\Omega \bar{f}_z^\infty\left(\,.\,,\frac{\mu^s}{|\mu|^s},\frac{\beta^{as}}{|\beta|^{as}}\right)\mathrm{d}|\beta|^{as} + \int_\Omega f^\infty(\,.\,,\beta^{ss})$$

with the notation of the absolute continuous part β^{as} and the singular part β^{ss} of β^s w.r.t. μ^s.

Remark. If f is differentiable at some point then $f_z(x,z)\cdot y$ and $\bar{f}_z(x,z,y)$ coincide. In the case of the absolute value function $a(z) = |z|$ on $\mathbb{R}^n$, we can calculate $\bar{a}_z$ at 0 in direction $\varphi \in \mathbb{R}^n$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t^+}|(0+t\varphi)|\bigg|_{t=0} = |\varphi|.$$

In that case, $\bar{a}_z(0,\varphi) = |\varphi|$ and especially $\bar{a}_z(0,0) = 0$.

Proof. Following the proof of Anzellotti to Theorem 2.4 and Theorem 2.5 the result follows easily using $t > 0$: We have:

$$I[\mu+t\beta] = \int_\Omega f(\,.\,,\mu^a + t\beta^a)\,\mathrm{d}x + \int_\Omega f^\infty\left(\,.\,,\frac{\mu^s + t\beta^{as}}{|\mu^s|}\right)\mathrm{d}|\mu^s| + \int_\Omega f^\infty(\,.\,,t\beta^{ss}).$$

With the definition of $\bar{f}_z$, the first term is clear. The second follows through simple calculation like in [4]. For the last, we have by the positive homogeneity of f^∞ in the second variable

$$\int_\Omega f^\infty(\,.\,,t\beta^{ss}) = t\int_\Omega f^\infty(\,.\,,\beta^{ss}),$$

since $t > 0$ and we thus do not have to deal with the non-differentiability of $|z|$ at 0 and do not have to enforce β^{ss} to vanish like in the proof to Theorem 2.4. □

To be more precise, in the setting in [4] we would have to be able to differentiate $\int_\Omega f^\infty(\,.\,,t\beta^{ss})$. If $t > 0$ we simply have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f^\infty(\,.\,,t\beta^{ss}) = \frac{\mathrm{d}}{\mathrm{d}t}t\int_\Omega f^\infty(\,.\,,\beta^{ss}) = \int_\Omega f^\infty(\,.\,,\beta^{ss}).$$

For $t < 0$, we similarly have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f^\infty(\,.\,,t\beta^{ss}) = \frac{\mathrm{d}}{\mathrm{d}t} - t\int_\Omega f^\infty(\,.\,,-\beta^{ss}) = -\int_\Omega f^\infty(\,.\,,-\beta^{ss}).$$

For both derivatives to agree at $t = 0$ for all allowed β^{ss}, we either need β^{ss} to vanish or f^∞ to be linear, which, besides from the trivial case $f^\infty = 0$, does not occur since we assumed f to be non-negative.

Applying Theorem 6.3 to the BV case we obtain:

Corollary 6.4 (One-sided differentiability for functionals on BV)**.**
Assume that for all $x \in \Omega$ the function $f(x,z)$ is one-sided directional differentiable in z for all $z \in \mathbb{R}^n$ and $f^\infty(x,z)$ is one-sided directional differentiable for all $z \in \mathbb{R}^n$. Assume also that all one-sided derivatives at all $z \in \mathbb{R}^n$ are bounded by M as in Theorem 6.3. Then the functional $\mathcal{F}[u] := \int\limits_\Omega f(x, \mathrm{D}u)$ is one-sided directionally differentiable at the point u in direction φ with

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\mathcal{F}[u+t\varphi]\bigg|_{t=0} = \int\limits_\Omega \bar{f}_z(\,.\,,\mathrm{D}^a u, \mathrm{D}^a\varphi)\,\mathrm{d}x + \int\limits_\Omega \bar{f}_z^\infty\left(\,.\,,\frac{\mathrm{D}^s u}{|\mathrm{D}^s u|},\frac{\mathrm{D}^{as}\varphi}{|\mathrm{D}^{as}\varphi|}\right)\mathrm{d}|\mathrm{D}^{as}\varphi|$$
$$+\int\limits_\Omega f^\infty\left(\,.\,,\frac{\mathrm{D}^s\varphi}{|\mathrm{D}^{ss}\varphi|}\right)\mathrm{d}|\mathrm{D}^{ss}\varphi|$$

with the notation as above.

Proof. This is a direct consequence of Theorem 6.3. □

Remark. In the case, that the requirements of Theorem 6.1 or Theorem 6.2 are met, the results can be reproduced from Theorem 6.3 and Theorem 6.4, respectively, combining the results from the directions given by the measures β and $-\beta$ or the directions given by the functions φ and $-\varphi$. If obstacles are involved, it will in general be not possible anymore to choose both measures or both functions, since for example φ may be admissible but $-\varphi$ may not be. Such phenomenon occurs, for example, on the contact set of the function u and the obstacle ψ. If a lower obstacle is assumed, φ could still be chosen positive at such point, since $u + t\varphi \geq 0$, but $-\varphi$ would lead to $u - t\varphi < \psi$, which is not admissible. For the relaxations obtained in Section 4, the problem remains on *thick* obstacle parts. In addition, one has to deal with the additional obstacle terms.

Before we treat the obstacle terms, we take a closer look on the boundary terms for which we obtain the following theorem:

Theorem 6.5 (Differentiability of boundary terms)**.**
Under the assumptions of Theorem 6.4, we have

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\int\limits_{\partial\Omega} f^\infty(\,.\,,\nu_\Omega(u_0-u-t\varphi))\,\mathrm{d}\mathcal{H}^{n-1}\Bigg|_{t=0} = \int\limits_{\partial\Omega} f^\infty(\,.\,,\mathrm{sgn}_\varphi(u_0-u)\nu_\Omega)\tilde{a}_z(u_0-u,\varphi)\,\mathrm{d}\mathcal{H}^{n-1}$$

with $a(z) = |z|$.

Proof. The proof follows from direct calculation using

$$f^\infty(x,\nu_\Omega(u_0-u-t\varphi)) = f^\infty\left(x,\nu_\Omega\frac{u_0-u-t\varphi}{|u_0-u-t\varphi|}\right)|u_0-u-t\varphi|$$

if $|(u_0 - u - +t\varphi)| \neq 0$.
First, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ where $u_0 - u \neq 0$, we have that $\frac{u_0-u-t\varphi}{|u_0-u-t\varphi|} = \frac{u_0-u}{|u_0-u|}$ for $t \searrow 0$.

In the second case, we have $u_0 - u = 0$ and $\varphi \neq 0$ and thus

$$f^\infty(x, \nu_\Omega(u_0 - u - t\varphi)) = f^\infty\left(x, \nu_\Omega \frac{\varphi}{|\varphi|}\right) t|\varphi|.$$

In the last case, we have $\varphi = 0$ and thus the derivative vanishes.
Combining those results, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t^+} \int\limits_{\partial\Omega} f^\infty(\,.\,, \nu_\Omega(u_0 - u - t\varphi))\,\mathrm{d}\mathcal{H}^{n-1}\Bigg|_{t=0} = \int\limits_{\partial\Omega} f^\infty(\,.\,, \mathrm{sgn}_\varphi(u_0 - u)\nu_\Omega)\tilde{a}_z(u_0 - u, \varphi)\,\mathrm{d}\mathcal{H}^{n-1}$$

with

$$\mathrm{sgn}_\varphi(u_0 - u) = \begin{cases} \mathrm{sgn}(u_0 - u) & \text{for } |u_0 - u| \neq 0, \\ \mathrm{sgn}(\varphi) & \text{else.} \end{cases}$$

□

Similar to the Euler equations, we now get inequalities if we consider the derivatives at minimizers u. Thus, we get

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\mathcal{F}[u + t\varphi] \geq 0 \text{ for all } \varphi \text{ allowed.}$$

This type of inequalities we call variational inequality and here the same question arises as by the Euler equation about the connection between solutions of the variational inequality and minimizers of the related functional. Similarly, we will answer this in the relevant case, where the directions are restricted. In the other case, where all directions are allowed, the result is equivalent to the case of the Euler equation.

6.3 Differentiability of the obstacle terms

In this part, we assume that the function $f : \mathbb{R}^n \to [0, +\infty)$ is convex and of linear growth as in Section 4. We now take a closer look at the obstacle term $O[u + t\varphi] = \int\limits_\Omega (\psi - (u + t\varphi)^+)_+ \,\mathrm{d}\varsigma_{f^\infty}$, where $\psi : \Omega \to [-\infty, +\infty]$ is a Borel function and $u, \varphi \in \mathrm{BV}(\Omega)$. Further, for fixed u, we restrict ourselves to consider only functions φ such that $O[u + t\varphi]$ is finite for $0 \leq t \leq t_0$ for some $t_0 > 0$. As we will see in the next subsection it is enough to consider only such directions φ where $O[u+t\varphi]$ is finite for some $t = t_0 > 0$ and $t = 0$. This implies that the functional is finite for all $0 \leq t \leq t_0$. To compute the derivative we start by investigating the integrand $(\psi - (u + t\varphi)^+)_+$ and discuss it with respect to the different representatives, which may occur for $\varphi \in \mathrm{BV}(\Omega)$. We can distinguish the following three different cases with the reminder that on $\Omega \setminus J_u$ the representatives u^+ and u^* are equal:

i) On $\Omega \setminus S_\varphi$ the function φ^* is approximately continuous and thus we have

$$\psi - (u + t\varphi)^+ = \psi - u^+ - t\varphi^*.$$

ii) On the set, where only φ jumps, i.e. on $J_\varphi \setminus J_u$, and on the set where the functions u and φ jump and where the sides of u^+ and φ^* coincide, i.e. on $J_\varphi \cap J_u \cap \{\nu_u = \nu_\varphi\}$, where ν_u and

ν_φ are given by the triples (u^+, u^-, ν_u) and $(\varphi^+, \varphi^-, \nu_u)$, we have

$$\psi - (u + t\varphi)^+ = \psi - u^+ - t\varphi^+.$$

iii) On the set, where u and φ jump, but the sides of u^+ and φ^- coincide, i.e. on the set $J_\varphi \cap J_u \cap \{\nu_u = -\nu_\varphi\}$, where ν_u and ν_φ are given by the triples (u^+, u^-, ν_u) and $(\varphi^+, \varphi^-, \nu_\varphi)$, we have

$$\psi - (u + t\varphi)^+ = \psi - \max\{u^+ + t\varphi^-, u^- + t\varphi^+\}.$$

For the points ii) and iii), we use that J_u, J_φ and $J_u \cap J_\varphi$ are rectifiable by 2.5 and the normal vector on the intersections ν_u and ν_φ agrees $\mathcal{H}^{n-1}$-a.e. up to the sign.
In this situations, the derivative with respect to t can be computed and we obtain:

i)
$$\frac{\mathrm{d}}{\mathrm{d}t^+} \left(\psi - u^+ - t\varphi^*\right) = -\varphi^*.$$

ii)
$$\frac{\mathrm{d}}{\mathrm{d}t^+} \left(\psi - u^+ - t\varphi^+\right) = -\varphi^+.$$

iii) If $u^+ + t\varphi^- > u^- + t\varphi^+$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t^+} \left(\psi - u^+ - t\varphi^-\right) = -\varphi^-,$$

if $u^+ + t\varphi^- < u^- + t\varphi^+$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t^+} \left(\psi - u^- - t\varphi^+\right) = -\varphi^+$$

and, for completeness, if $u^+ + t\varphi^- = u^- + t\varphi^+$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t^+} \left(\psi - \max\{u^+ + t\varphi^-, u^- + t\varphi^+\}\right) = -\varphi^+.$$

Roughly speaking, this implies that the obstacle functional is monotonically decreasing and thus variations in negative direction increase the functional and in positive decrease it or keep it the same. Further, for $\varphi \geq 0$, we have $0 \leq O[u + t\varphi] \leq O[u] < +\infty$ for $t > 0$, but for $\varphi \leq 0$ we need to assure the finiteness to be able to compute the derivative. To do so, we have to determine the sets, where the variations play a role, since the set, where the integrand of $O[u + t\varphi]$ is greater than zero, is not necessarily a subset or superset of $\{(\psi - u^+)_+ > 0\}$, i.e. the set where u violates the obstacle. More precisely, we may have derivative terms on some subset of $A_1 = \{\psi - u^+ < 0\}$ or $A_2 = \{\psi - u^+ = 0\}$ for some $t > 0$ if the suitable representative of φ is negative and such that $\psi - (u + t\varphi)^+ > 0$, i.e. the variation with $t\varphi$ leads to a violation of the obstacle. In contrast, we may have no such term in a subset of $A_3 = \{\psi - u^+ > 0\}$ for fixed $t > 0$ if $(u + t\varphi)^+ \geq \psi$, i.e. the variation with $t\varphi$ leads the function to fulfill the obstacle constraint. Besides that, we need to distinguish which derivative matters in case iii).
The following lemmas show that the case of a subset of A_1 may be ignored for $t \searrow 0$ and the set A_3 is fully accounted for and also will allow us to observe, what happens on A_2. For the first

results, we consider only φ like in the cases i) and ii) from above, i.e. that either u does not jump or the jumps align like in case ii). Thus we have $(u + t\varphi)^+ = u^+ + t\varphi^+$ $\mathcal{H}^{n-1}$-a.e. and obtain:

Lemma 6.6 (Some convergence result)**.**
For u, φ, ψ as above and such that $O[u + t\varphi]$ is finite for all $0 \le t \le t_0$ for some $t_0 > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t^+} \int\limits_{\{u^+ > \psi\}} (\psi - u^+ - t\varphi^+)_+ \,\mathrm{d}\varsigma_{f^\infty} \Bigg|_{t=0} = 0.$$

Proof. Since $u^+ > \psi$ and thus u fulfills the obstacle condition, we have $\varphi^+ < 0$ on the set, where $u^+ + t\varphi^+ < \psi$. We now define the set $B(t) = \{\psi - u^+ - t\varphi^+ > 0\} \cap \{u^+ > \psi\}$ for $0 < t \le t_0$ and notice that $B(t_1) \subset B(t_2)$ for $0 < t_1 < t_2$ by monotonicity. Since $O[u + t\varphi]$ is finite, the sets $B(t)$ have at most Hausdorff dimensions $n - 1$ and are clearly contained in the thin part of the obstacle up to a $\mathcal{H}^{n-1}$-null set. To be able to compute the derivative of $O[u + t\varphi]$ on $B(t)$ with respect to (positive) t at 0, we will prove the finiteness of $\int\limits_{B(t)} -\varphi^+ \,\mathrm{d}t$ for (small) $t \ge 0$ and estimate the derivative by this term: We have

$$\int\limits_{\{u^+ > \psi\}} (\psi - u^+ - t\varphi^+)_+ \,\mathrm{d}\varsigma_{f^\infty} < +\infty$$

for $0 < t \le t_0$ and thus

$$\int\limits_{\{u^+ > \psi\}} (\psi - u^+ - t_0\varphi^+)_+ \,\mathrm{d}\varsigma_{f^\infty} = \int\limits_{B(t_0)} \psi - u^+ - t_0\varphi^+ \,\mathrm{d}\varsigma_{f^\infty} < +\infty$$

and

$$\int\limits_{\{u^+ > \psi\}} \left(\psi - u^+ - t_1\varphi^+\right)_+ \,\mathrm{d}\varsigma_{f^\infty} = \int\limits_{B(t_1)} \psi - u^+ - t_1\varphi^+ \,\mathrm{d}\varsigma_{f^\infty} < +\infty$$

for all $0 < t_1 < t_0$. Combining both estimates and by monotonicity, positivity of the integrand and the fact that

$$0 \le \int\limits_{B(t_1)} \psi - u^+ - t_0\varphi^+ \,\mathrm{d}\varsigma_{f^\infty} \le \int\limits_{B(t_0)} \psi - u^+ - t_0\varphi^+ \,\mathrm{d}\varsigma_{f^\infty} < +\infty,$$

we arrive at the finiteness of

$$\left| \int\limits_{B(t_1)} \varphi^+ \,\mathrm{d}\varsigma_{f^\infty} \right| < +\infty.$$

By monotonicity of $B(t)$ and since $\varphi^+ < 0$ we have the finiteness for each $0 \le t < t_1 < t_0$ and since t_1 was arbitrary but smaller than t_0 we have the finiteness for all $0 \le t < t_0$. Now we can estimate the difference quotient on those sets:

$$\frac{1}{t}\left(O[u + t\varphi, B(t)] - O[u, B(0)]\right) = \int\limits_{B(t)} \frac{\psi - u^+}{t} - \varphi^+ \,\mathrm{d}\varsigma_{f^\infty}.$$

We can further estimate by the definition of $B(t)$ that $0 \geq \frac{\psi - u^+}{t} \geq \varphi^+$ and we thus obtain

$$0 \leq \frac{1}{t}\left(O[u + t\varphi, B(t)] - O[u, B(0)]\right) \leq \int\limits_{B(t)} -\varphi^+ \,\mathrm{d}\varsigma_{f^\infty}$$

and it only remains to show that the integral on the right-hand side is vanishing for $t \searrow 0$. To do so, we reduce to the case where $B(t)$ has finite Hausdorff measure: Since $O[u + t\varphi]$ is finite for $0 \leq t \leq t_0$, we have that $|B(t)| = 0$, but $\mathcal{H}^{n-1}(B(t))$ may still be infinite.
Since $\int\limits_{B(t)} |\varphi^+| \,\mathrm{d}\varsigma_{f^\infty} < +\infty$, we obtain that $B(t)$ is σ-finite with respect to $\mathcal{H}^{n-1}$ and thus also with respect to ς_{f^∞}. Using this, we find for each $0 < t \leq t_0$ a subset $\tilde{B}_\varepsilon(t) \subset B(t)$ with $\mathcal{H}^n(\tilde{B}_\varepsilon(t)) < +\infty$ and

$$\int\limits_{B(t)\setminus\tilde{B}_\varepsilon(t)} |\varphi^+| \,\mathrm{d}\varsigma_{f^\infty} < \varepsilon$$

for any given $\varepsilon > 0$. A possible choice would be a set, where $|\varphi^+| > \delta$ for a suitable small δ. Further, by monotonicity, we can choose $\tilde{B}_\varepsilon(t_1) = B(t_1) \cap \tilde{B}_\varepsilon(t_2)$ for $0 < t_1 < t_2 = \frac{t_0}{2}$. Sending $\varepsilon \searrow 0$ allows us to restrict ourselves to the case where $B(t)$ has finite $\mathcal{H}^{n-1}$-measure.
Next, we consider the set $B_0 := \bigcap_{t>0} B(t)$. Since $B(t)$ has finite Hausdorff measure and the sequence is monotone, we have $\mathcal{H}^{n-1}(B_0) = \mathcal{H}^{n-1}(B(0) = \lim\limits_{t\searrow 0} \mathcal{H}^{n-1}(B(t))$, see for example [24, Theorem 1.2]. Since $B(0) = \emptyset$ up to $\mathcal{H}^{n-1}$-null sets, we are finished. □

This proof implies the following corollary:

Corollary 6.7 (Vanishing derivative).
For given $u \in \mathrm{BV}(\Omega)$, let u, φ be in $\mathrm{L}^1(B(t), \mathcal{H}^{n-1})$ with $B(t) = \{u < 0\} \cap \{u + t\varphi \geq 0\}$. Then, if $\int\limits_{B(t)} u + t\varphi \,\mathrm{d}\varsigma_{f^\infty}$ is finite for all $0 \leq t \leq t_0$, $\int\limits_{B(t)} \varphi \,\mathrm{d}\varsigma_{f^\infty}$ is finite for $0 < t < t_0$ and we have

$$\int\limits_{B(t)} \varphi \,\mathrm{d}\varsigma_{f^\infty} \to 0 \text{ for } t \searrow 0$$

and thus the limit set $B_0 = \lim\limits_{t\searrow 0} B(t)$ does not contribute to the derivative with respect to t of $\int\limits_\Omega (u + t\varphi)_+ \,\mathrm{d}\varsigma_{f^\infty}$.

Relying on this corollary and with the equivalent version where $B(t)$ is replaced by
$\{u > 0\} \cap \{t\varphi \geq u\}$, we can compute the derivative on A_3 if $(u + t\varphi)^+ = u^+ + t\varphi^+$:

Lemma 6.8 (Derivative on a set).
Under the finiteness assumption on $O[u + t\varphi]$ for $0 \leq t \leq t_0$ for some $t_0 > 0$ and for $u, \varphi \in \mathrm{BV}$ with $(u + t\varphi)^+ = u^+ + t\varphi^+$, we have on the set $\{\psi - u^+ > 0\}$ the following derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t^+} \int\limits_{\{\psi - u^+ > 0\}} (\psi - (u + t\varphi)^+ \,\mathrm{d}\varsigma_{f^\infty} \Bigg|_{t=0} = \int\limits_{\{\psi - u^+ > 0\}} -\varphi^+ \,\mathrm{d}\varsigma_{f^\infty},$$

where the integral on the right-hand side can take values in $[-\infty, +\infty)$.

Proof. For $\varphi^+ \leq 0$, we can compute the derivative directly, using the finiteness of $O[u + t\varphi]$:

$$\frac{1}{t}\left(O[u+t\varphi] - O[u]\right) = -\int\limits_{\{u^+<\psi\}} \varphi^+ \, \mathrm{d}\varsigma_{f^\infty}.$$

For $\varphi^+ > 0$, we have to be a bit more careful: Applying Corollary 6.7 and by a similar argumentation as for Lemma 6.6, we obtain that the set where $u^+ + t\varphi^+ > \psi$ is vanishing if $(\varphi^+)_+$ is summable over A_3. If it is not summable, we have by finiteness of $O[u + t\varphi]$ for the derivative:

$$\frac{1}{t}\left(O[u+t\varphi] - O[u]\right) = -\int\limits_{\{u^+<\psi\}\cap\{u^++t\varphi^+<\psi\}} \varphi^+ \, \mathrm{d}\varsigma_{f^\infty} - \frac{1}{t}\int\limits_{\{u^+<\psi\}\cap\{u^++t\varphi^+\geq\psi\}} \psi - u^+ \, \mathrm{d}\varsigma_{f^\infty}. \quad (6.1)$$

If the set $\{u^+ + t\varphi \geq \psi\}$ does not vanish with respect to $\mathcal{H}^{n-1}$, the derivative at $t = 0$ is $-\infty$, since we have for the last appearing integral

$$0 \geq -\frac{1}{t}\int\limits_{\{u^+<\psi\}\cap\{u^++t\varphi^+\geq\psi\}} \psi - u^+ \, \mathrm{d}\varsigma_{f^\infty} \to -\infty.$$

If the Hausdorff measure of the set vanishes, then the difference quotient behaves for $t > 0$ like

$$-\int\limits_{\{u^+<\psi\}\cap\{u^++t\varphi^+<\psi\}} \varphi^+ \, \mathrm{d}\varsigma_{f^\infty},$$

and since φ^+ is not summable by assumption and $\{u^+ < \psi\} \cap \{u^+ + t\varphi^+ < \psi\} \to \{u^+ < \psi\}$ for $t \searrow 0$, the derivative at $t = 0$ is again $-\infty$. ☐

Now it only remains to clarify, what happens on the set, where u and φ jump and u^+ and φ^- are aligned on one side. Before we do that, we fix some notation. As explained in Definition 2.11, we can identify the function values for some $x \in J_u$ for a BV function u by a triple $(u_{\mathrm{ext}}, u_{\mathrm{int}}, \nu)$ which is unique up to permutation of the first two components and a simultaneous change of the sign of the normal vector. In this Section, we fix the order by setting the triple to be (u^+, u^-, ν_u) and define the hereby given jump set to be $J_u^{\nu_u}$. For two BV functions u, φ and for $\mathcal{H}^{n-1}$-a.e. x in the intersection $J_u^{\nu_u} \cap J_\varphi^{\nu_\varphi}$, we have by Proposition 2.5 either $\nu_u = \nu_\varphi$ or $\nu_u = -\nu_\varphi$. To distinguish these two cases, we write $J_{u,\varphi}^+$ for the subset of $J_u^{\nu_u} \cap J_\varphi^{\nu_\varphi}$ if $\nu_u = \nu_\varphi$ and $J_{u,\varphi}^-$ else. Now we can state we following lemma:

Lemma 6.9 (Estimate on the intersection of jumpsets, the missing case)**.**
Let some function $u, \varphi \in \mathrm{BV}(\Omega)$ be given such that $O[u + t\varphi]$ is finite for $0 \leq t \leq t_0$ for some $t_0 > 0$. On the set $C(0) = J_{u,\varphi}^- \cap \{\psi - u^+ > 0\}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\int\limits_{C(t)} (\psi - (u+t\varphi)^+)_+ \, \mathrm{d}\varsigma_{f^\infty}\Bigg|_{t=0} = \int\limits_{C(0)} -\varphi^- \, \mathrm{d}\varsigma_{f^\infty}$$

with $C(t) = J_{u,\varphi}^- \cap \{\psi - (u + t\varphi)^+ > 0\}$.

Proof. We distinguish into three cases depending on the sign of the representatives of φ:

1. In the first case, we assume $\varphi^+ \leq 0$. Here, we have the finiteness on $C_1(t) = C(t) \cap \{\varphi^+ \leq 0\}$ of
$$\int\limits_{C_1(t)} (\psi - (u + t\varphi)^+)_+ \,\mathrm{d}\varsigma_{f^\infty} < +\infty \text{ for } 0 \leq t \leq t_0.$$
We can divide $C_1(t)$ further in the part $C_1^+(t) = C_1(t) \cap \{u^+ + t\varphi^- \geq u^- + t\varphi^+\}$ and $C_1^-(t) = C_1(t) \setminus C_1^+(t)$. On $C_1^+(t)$ we can now estimate, using Proposition 3.12, 4'. in Proposition 3.7 and the finiteness of $|\mathrm{D}^j\varphi|(\Omega)$ to obtain
$$\int\limits_{C_1^+(t)} \psi - u^+ - t\varphi^+ \,\mathrm{d}\varsigma_{f^\infty} \geq \int\limits_{C_1^+(t)} \psi - u^+ - t\varphi^- \,\mathrm{d}\varsigma_{f^\infty} - t_0 b C_n |\mathrm{D}^j\varphi| > -\infty$$
and we already have by monotony
$$\int\limits_{C_1^+(t)} \psi - u^+ - t\varphi^+ \,\mathrm{d}\varsigma_{f^\infty} \leq \int\limits_{C_1^+(t)} \psi - u^+ - t\varphi^- \,\mathrm{d}\varsigma_{f^\infty}.$$
By adding the integral over $C_1^-(t)$ and similarly interchanging u^+ and u^-, we obtain the finiteness of
$$\left| \int\limits_{C_1(t)} \psi - u^+ - t\varphi^+ \,\mathrm{d}\varsigma_{f^\infty} \right| < +\infty \text{ for all } 0 \leq t \leq t_0$$
and thus, similarly to the proof of finiteness in Lemma 6.6, the finiteness of
$$-\infty < \int\limits_{C_1(t)} \varphi^- \,\mathrm{d}\varsigma_{f^\infty} < \int\limits_{C_1(t)} \varphi^+ \,\mathrm{d}\varsigma_{f^\infty} \leq 0 \text{ for } 0 \leq t < t_0.$$
Now we can use Corollary 6.7 using $u^+ - u^-$ and $-t\varphi^+ \geq 0$ to obtain that the part of the integral, where $u^+ + t\varphi^- < u^-$ does not add to the derivative for $t \searrow 0$ and thus the integral part over $\{u^+ + t\varphi^- < u^- + t\varphi^+\} \subset \{u^+ + t\varphi^- < u^-\}$ does neither. This implies that only the function $-\varphi^-$ appears as the integrand of the derivative over $C_1(0)$.

2. In the second case, we have $\varphi^+ > 0 \geq \varphi^-$ and define $C_2(t) = C(t) \cap \{\varphi^+ > 0 \geq \varphi^-\}$. Here, we obtain the needed finiteness by estimating
$$0 \leq \int\limits_{C_2(t)} \varphi^+ \,\mathrm{d}\varsigma_{f^\infty} \leq \int\limits_{C_2(t)} \varphi^+ - \varphi^- \,\mathrm{d}\varsigma_{f^\infty} \leq bC_n|\mathrm{D}^j\varphi|(\Omega) < +\infty$$
and similarly
$$0 \geq \int\limits_{C_2(t)} \varphi^- \,\mathrm{d}\varsigma_{f^\infty} \geq \int\limits_{C_2(t)} \varphi^- - \varphi^+ \,\mathrm{d}\varsigma_{f^\infty} \geq -bC_n|\mathrm{D}^j\varphi|(\Omega) > -\infty.$$
Thus, we can use again similar arguments like in Lemma 6.6 and Corollary 6.7 with $u^+ - u^*$, where $u^* = \frac{u^+ + u^-}{2} < u^+$, and $-t\varphi^-$ and with $u^* - u^-$ and $t\varphi^+$ to obtain that the part of the

integral where $u^+ + t\varphi^- < u^*$ does not add to the integral in the limit $t \searrow 0$ and neither does the part where $u^- + t\varphi^+ > u^*$. Together this implies that only the function $-\varphi^-$ appears as the integrand of the derivative over $C_2(0)$.

3. In the last case, we have $\varphi^- \geq 0$. Here, we have to differ into two subcases:
   In the first, we assume that φ^+ and thus φ^- as well are summable on $C_3(0)$ with respect to the De Giorgi measure ς_{f^∞}, where $C_3(t) = C(t) \cap \{\varphi^- \geq 0\}$. By monotonicity, i.e. since $C_3(t_1) \supset C_3(t_2)$ for $0 \leq t_1 \leq t_2 \leq t_0$, we also have the summability on $C_3(t)$ for $0 \leq t \leq t_0$. Now can use Corollary 6.7 to obtain, that the integral over the set where $u^- + t\varphi^+ > u^+$ does not add to the derivative for $t \searrow 0$ and neither does $\{u^- + t\varphi^+ > u^+ + t\varphi^-\} \subset \{u^- + t\varphi^+ > u^+\}$. This implies that in this case the function $-\varphi^-$ appears as the integrand of the derivative over $C_3(0)$.
   In the second case, φ^+ and thus φ^- as well are not summable over $C_3(0)$. Here, we can argue like for 6.1 and note, that the representative chosen does not matter, since all combinations for the integrand of φ^+ and φ^- deliver the same value, namely $-\infty$.

Combining all cases we arrive at the stated claim. $\square$

Remark. The non-summability in the very last case is no trouble, since $O[u + t\varphi] < +\infty$ for $0 \leq t \leq t_0$. Thus the derivative can still be computed, but tends to $-\infty$.

By Lemma 6.6 together with Corollary 6.7 we achieve that the set A_1 does not contribute to the derivative, where the missing case of the derivative on $J^-_{u,\varphi} \cap A_1$ is added by using the concepts established in the proofs in Lemma 6.9 by the 1. and 2. point. For example, for $\varphi^+ \leq 0$, we have that $\mathcal{H}^{n-1}$-a.e. only the integrand $-\varphi^-$ matters and we obtain the finiteness of the derivative like in 1. for the adapted sets $B(t)$ and can then argue like in the proof of Lemma 6.6 with φ^- instead of φ^+. Similarly for the second case.
Further, we obtained by the calculations after Corollary 6.7 and by Lemma 6.9 point 3. that the set A_3 counts fully for the computation of the derivative and that the integrand to appear in the derivative at $t = 0$ in case iii) is given by $-\varphi^+$ on subset of $J^+_{u,\varphi}$ and $-\varphi^-$ on $J^-_{u,\varphi}$.
Now we are able to state the full derivative of such obstacle terms, remembering that we still may have some derivative on A_2:

Theorem 6.10 (Derivative of the obstacle term)**.**
Assume $u, \varphi \in \mathrm{BV}(\Omega)$ such that $O[u + t\varphi]$ is finite for all $0 \leq t \leq t_0$ for some $t_0 > 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t^+} O[u + t\varphi]\Big|_{t=0} = -\int\limits_{\{\psi - u^+ > 0\} \setminus J^-_{u,\varphi}} \varphi^+ \,\mathrm{d}\varsigma_{f^\infty} - \int\limits_{\{\psi - u^+ > 0\} \cap J^-_{u,\varphi}} \varphi^- \,\mathrm{d}\varsigma_{f^\infty} - \int\limits_{\{\psi - u^+ = 0,\, \varphi^+ < 0\} \setminus J^-_{u,\varphi}} \varphi^+ \,\mathrm{d}\varsigma_{f^\infty} - \int\limits_{\{\psi - u^+ = 0,\, \varphi^- < 0\} \cap J^-_{u,\varphi}} \varphi^- \,\mathrm{d}\varsigma_{f^\infty}, \tag{6.2}$$

where we used that $\varphi^+ = \varphi^*$ on $\Omega \setminus J_\varphi$.

Proof. The first two integrals on the right-hand side are known. The third term is clear and the finiteness is easily given by the finiteness of $O[u + t\varphi]$. The fourth integral term is obtained very similarly to the proof of point 1. and 2. in Lemma 6.9. $\square$

Remark. The important thing to notice is that the assumption $O[u+t\varphi] < +\infty$ for $0 \le t \le t_0$ for some $t_0 > 0$ is very powerful and necessary but, as we will see in the next section, exactly the condition, which we want. Here, we note that on the sets where $\varphi^+ < 0$ or $\varphi^- < 0$ we obtain summability of those parts by finiteness of $O[u+t\varphi]$. Next, we mention that, by the lower semicontinuity obtained in Section 4, the computation of a derivative is nonsensical if we have $O[u] = +\infty$ and thus $\lim_{t \searrow 0} O[u+t\varphi] = +\infty$.

In this section, we computed the derivative of the obstacle integral term for the lower obstacle. Similarly, this is possible for the upper obstacle term, but we skip the precise formulation because of the lengthy formula.

Combining the results of the previous subsection and this, we acquire the derivative of the relaxation of the double obstacle functional under the given conditions. The expression of the derivative is lengthy and hard to verify whether a function u satisfies the corresponding variational inequality. Therefore, we prove in the next section some useful results for the allowed directions φ and give an easier formula, which a given function under certain restriction has to verify to be an minimizer of the double obstacle functional.

6.4 Directions and minimizers

A main argument needed in the previous subsection is the finiteness of $O[u+t\varphi]$ for $0 \le t \le t_0$ for some constant $t_0 > 0$. To achieve that not only for the obstacle functional, but on the whole functionals $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}$ and $\mathcal{F}^{\psi}_{(u_0)} = \mathcal{F}^{\psi,+\infty}_{(u_0)}$, where boundary values u_0 may or may not be prescribed, we prove the following lemma:

Lemma 6.11 (Finiteness of the functionals)**.**
For functions $u, \varphi, v \in \mathrm{BV}$, a suitable domain Ω and f as in Subsection 4.4, we have:

1. $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}$ *is convex and thus*

$$\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[\lambda u + (1-\lambda)v] < +\infty \text{ for } \lambda \in [0,1]$$

   if $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u] < +\infty$ *and* $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[v] < +\infty$.

2. *If* $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u] < +\infty$ *and* $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u+t_0\varphi] < +\infty$ *for a* $t_0 > 0$*, then* $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u+t\varphi] < +\infty$ *for all* $0 \le t \le t_0$.

Proof. We only need to prove 1., since 2. easily follows from 1.
Since f and f^∞ are linear, as well as the boundary term, we have only to be careful at the obstacle terms, since the map $u+v \mapsto (u+v)^+$ is not always linear. More precisely we may have trouble on a subset of $J^-_{u,v}$. Here, we have to take the jump part into account. Since the J_u and J_φ are rectifiable, and we are on a subset of the intersection of those, this subset is rectifiable by Proposition 2.5, we have

$$f^\infty(D^j u) \llcorner J^-_{u,v} = f^\infty(\nu_u)(u^+ - u^-)\mathcal{H}^{n-1} \llcorner J^-_{u,v},$$
$$f^\infty(D^j v) \llcorner J^-_{u,v} = f^\infty(-\nu_u)(v^+ - v^-)\mathcal{H}^{n-1} \llcorner J^-_{u,v}.$$

On the set $D = \{(\lambda u + (1-\lambda)v)^+ = \lambda u^+ + (1-\lambda)v^-\} \cap \{\psi - (\lambda u + (1-\lambda)v)^+ > 0\}$ we can estimate:

$$\int_D (\psi - \lambda u^+ - (1-\lambda)v^-) f^\infty(\nu_u) + (\psi - \lambda u^+ - (1-\lambda)v^-) f^\infty(-\nu_u)\, \mathrm{d}\mathcal{H}^{n-1}$$
$$= \lambda \int_D \psi - u^+ \,\mathrm{d}\varsigma_{f^\infty} + (1-\lambda) \int_D \psi - v^- \,\mathrm{d}\varsigma_{f^\infty} \tag{6.3}$$

and

$$\begin{aligned} f^\infty(D^j(\lambda u + (1-\lambda)v))(D) &= \int_D ((\lambda u^+ + (1-\lambda)v^-) - (\lambda u^- + (1-\lambda)v^+)) f^\infty(\nu_u)\, \mathrm{d}\mathcal{H}^{n-1} \\ &= \int_D \lambda(u^+ - u^-) f^\infty(\nu_u)\, \mathrm{d}\mathcal{H}^{n-1} + \int_D (1-\lambda)(v^- - v^+) f^\infty(\nu_u)\, \mathrm{d}\mathcal{H}^{n-1} \\ &= \lambda f^\infty(\mathrm{D}^j u)(D) + (1-\lambda) f^\infty(\mathrm{D}^j v)(D) \\ &\quad + \int_D (1-\lambda)(v^- - v^+) f^\infty(\nu_u) + (1-\lambda)(v^- - v^+) f^\infty(-\nu_u)\, \mathrm{d}\mathcal{H}^{n-1} \\ &= \lambda f^\infty(\mathrm{D}^j u)(D) + (1-\lambda) f^\infty(\mathrm{D}^j v)(D) + \int_D (1-\lambda)(v^- - v^+)\, \mathrm{d}\varsigma_{f^\infty}, \end{aligned} \tag{6.4}$$

where we used Proposition 3.12 and its remark in the last step. Adding (6.3) and (6.4) we have

$$\begin{aligned} & f^\infty(\mathrm{D}^j(\lambda u + (1-\lambda)v)(D) + \int_D \psi - (\lambda u + (1-\lambda)v)^+ \,\mathrm{d}\varsigma_{f^\infty} \\ &\quad = \lambda f^\infty(\mathrm{D}^j u)(D) + (1-\lambda) f^\infty(\mathrm{D}^j v)(D) + \lambda \int_D \psi - u^+ \,\mathrm{d}\varsigma_{f^\infty} + (1-\lambda) \int_D \psi - v^+ \,\mathrm{d}\varsigma_{f^\infty} \\ &\quad \le \lambda f^\infty(\mathrm{D}^j u)(D) + (1-\lambda) f^\infty(\mathrm{D}^j v)(D) + \lambda \int_D (\psi - u^+)_+ \,\mathrm{d}\varsigma_{f^\infty} + (1-\lambda) \int_D (\psi - v^+)_+ \,\mathrm{d}\varsigma_{f^\infty}. \end{aligned}$$

A similar computation on $\{(\lambda u + (1-\lambda)v)^+ = \lambda u^- + (1-\lambda)v^+\} \cap \{\psi - (\lambda u + (1-\lambda)v)^+ > 0\}$ and for the upper obstacle terms gives us the convexity of $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}$ and thus the lemma is proven. For this, we note that by $\psi_1 \le \psi_2$ only one obstacle may be violated at a certain point and thus the calculation for the upper and lower obstacle terms do not intervene. □

Remark. Without such detailed insight on the calculation with the generalized De Giorgi measure, the convexity follows from the convexity of $\mathrm{F}^{\psi_1,\psi_2}$, too. One simply can chose the recovery sequence for u, v, namely sequences $u_k, v_k \in \mathrm{W}^{1,1}(\Omega)$, $k \in \mathbb{N}$, with

$$\lim_{k\to+\infty} \mathrm{F}[u_k] = \mathcal{F}^{\psi_1,\psi_2}[u] \text{ and } \lim_{k\to+\infty} \mathrm{F}[v_k] = \mathcal{F}^{\psi_1,\psi_2}[v].$$

Using the lower semicontinuity and $\lambda u_k + (1-\lambda)v_k \to \lambda u + (1-\lambda)v$ in $\mathrm{L}^1(\Omega)$, we obtain by the convexity of $\mathrm{F}^{\psi_1,\psi_2}$ the convexity of $\mathcal{F}^{\psi_1,\psi_2}$:

$$\begin{aligned}\mathcal{F}^{\psi_1,\psi_2}[\lambda u+(1-\lambda)v] &\leq \liminf_{k\to+\infty} \mathrm{F}^{\psi_1,\psi_2}[\lambda u_k+(1-\lambda)v_k] \\ &\leq \liminf_{k\to+\infty} \lambda\mathrm{F}^{\psi_1,\psi_2}[u_k]+(1-\lambda)\mathrm{F}^{\psi_1,\psi_2}[v_k] \\ &= \lambda\mathcal{F}^{\psi_1,\psi_2}[u]+(1-\lambda)\mathcal{F}^{\psi_1,\psi_2}[v].\end{aligned}$$

Lemma 6.11 implies, that for the double obstacle problem we are allowed to vary in direction of any v with finite functional value $\mathcal{F}^{\psi_1,\psi_2}_{u_0}[v]<+\infty$, which seems reasonable.
It remains to check, whether functions satisfying the variational inequality are also minimizers for the functional. Since it is very hard to check if a functional satisfies the general variational inequality for the double obstacle, we will limit the directions and will still be able to prove from this reduced version that functions satisfying the new variational inequality are minimizers of the double obstacle functional:

Definition 6.12 (Restricted directions)**.**
Given some function $u\in\mathrm{BV}(\Omega)$ such that $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u]<+\infty$, we define the set of restricted direction by

$$\mathcal{D}_u=\left\{\varphi=v-u\in\mathrm{BV}(\Omega)\,:\,v\in\mathrm{W}^{1,1}(\Omega)\text{ and }\mathrm{F}^{\psi_1,\psi_2}_{(u_0)}[v]<+\infty\right\},$$

i.e. directions $v-u$, where v satisfies the obstacle constraint and has the right boundary values.

This restriction seems reasonable, since the functions v in the Definition suffice to approximate $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[w]$ for all $w\in\mathrm{BV}(\Omega)$, such that the functional is finite. This implies a kind of density of the directions. Next, we state some obvious properties involving the directions $\mathcal{D}_u$.

Proposition 6.13 (Some properties of $u+t\varphi$ for $\varphi\in\mathcal{D}_u$)**.**
For u with $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u]<+\infty$ and $\mathcal{D}_u$ like in Definition 6.12, we have:

1. $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u+t\varphi]<+\infty$ *for* $t\in[0,1]$,
2. $\mathrm{D}^s(u+t\varphi)=(1-t)\mathrm{D}^s u$ *and thus* $|\mathrm{D}^s\varphi|\ll|\mathrm{D}^s u|$ *and* $|\mathrm{D}^s(u+t\varphi)|\ll|\mathrm{D}^s u|$ *for* $t\in[0,1]$,
3. $\{\psi_1-(u+t\varphi)^+>0\}\subset\{\psi_1-u^+>0\}$ *and* $\{(u+t\varphi)^- -\psi_2>0\}\subset\{u^- -\psi_2>0\}$ *for* $t\in[0,1]$.
4. *If a Dirichlet boundary condition is involved, we additionally have* $\{u_{\mathrm{int}}+t\varphi\neq u_0\}\cap\partial\Omega\subset\{u_{\mathrm{int}}\neq u_0\}\cap\partial\Omega$ *for* $t\in[0,1]$.

Proof.

1. Clear by Lemma 6.11.
2. By the definition of $\mathcal{D}_u$ we can write $\varphi=v-u$ for some $v\in\mathrm{W}^{1,1}(\Omega)$. Thus, we have $\mathrm{D}^s\varphi=-\mathrm{D}^s u$ and $\mathrm{D}^s(u+t\varphi)=(1-t)\mathrm{D}^s u$ and the statement follows.
3. Clear, since v satisfies the obstacle constraint in the definition of $\mathcal{D}_u$.
4. Follows easily from $u+t\varphi=(1-t)u+tv$ and v satisfying the boundary constraint in the trace sense.

□

Now we are able to state the reduced derivative of $\mathcal{F}^{\psi_1,\psi_2}$ at $u \in \mathrm{BV}(\Omega)$ in directions $\mathcal{D}_u$:

Theorem 6.14 (Reduced directions and variational inequality).
For u as described and $\varphi \in \mathcal{D}_u$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\mathcal{F}^{\psi_1,\psi_2}[u+t\varphi]\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t^+}\mathcal{F}[u+t\varphi]\bigg|_{t=0} - \int\limits_{\{\psi_1-u^+>0\}} \varphi^- \,\mathrm{d}\varsigma_{f^\infty} + \int\limits_{\{u^- - \psi_2>0\}} \varphi^+ \,\mathrm{d}\varsigma_{\tilde{f}^\infty}.$$

Thus, a minimizer u of $\mathcal{F}^{\psi_1,\psi_2}$ fulfills the reduced variational inequality

$$\frac{\mathrm{d}}{\mathrm{d}t^+}\mathcal{F}[u+t\varphi]\bigg|_{t=0} - \int\limits_{\{\psi_1-u^+>0\}} \varphi^- \,\mathrm{d}\varsigma_{f^\infty} + \int\limits_{\{u^- - \psi_2>0\}} \varphi^+ \,\mathrm{d}\varsigma_{\tilde{f}^\infty} \geq 0 \qquad \forall \varphi \in \mathcal{D}_u. \tag{6.5}$$

If boundary values u_0 are involved, we need to replace $\mathcal{F}$ by $\mathcal{F}_{u_0}$.

Proof. The proof follows trivially from Theorem 6.10 and the properties given in Proposition 6.13. More precisely, by 2. from Proposition 6.13, we have that $J^+_{u,\varphi}$ is empty for all $\varphi \in \mathcal{D}_u$ and by 3. we have that $u + t\varphi$ violates the obstacle constraint at most one points where u violates the obstacle constraint and thus we have no terms on the set $\{\psi_1 = u^+\}$ and $\{\psi_2 = u^-\}$. □

It remains to show that functions satisfying the reduced variational inequality are also minimizers of the functional.

Theorem 6.15 (Variational inequality and minimizers).
Assume $u \in \mathrm{BV}$ with $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u] < +\infty$ satisfies the variational inequality (6.5)*, then u is also a minimizer of $\mathcal{F}^{\psi_1,\psi_2}$.*

Proof. Consider any competitor $v \in \mathrm{BV}$. Without loss of generality, we assume $\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[v] < +\infty$. For this v, we find a recovery sequence $v_k \in \mathrm{W}^{1,1}(\Omega)$ with

$$\lim_{k\to+\infty} \mathrm{F}^{\psi_1,\psi_2}_{(u_0)}[v_k] = \mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[v].$$

Then, for each k, we have $\varphi_k := v_k - u \in \mathcal{D}_u$. Using the variational inequality and convexity, we obtain

$$\mathrm{F}^{\psi_1,\psi_2}_{(u_0)}[v_k] = \mathrm{F}^{\psi_1,\psi_2}_{(u_0)}[u+1\cdot\varphi_k] \geq \mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u+0\cdot\varphi_k] + \frac{\mathrm{d}}{\mathrm{d}t^+}\bigg|_{t=0} \mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u+t\varphi_k] \geq \mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u].$$

With $k \to \infty$ we have

$$\mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[v] \geq \mathcal{F}^{\psi_1,\psi_2}_{(u_0)}[u]$$

and the proof is complete. □

The proof is very similar to the proof of Theorem 3.10 in [4]. In the obstacle-free case, i.e. if we set $\psi_1 = -\infty = -\psi_2$, we obtain a more general result, since we can treat integrands $f(\,.\,,z)$ which are only one-sided differentiable in z like the total variation integrand $f(z) = |z|$. To achieve that, we allow more directions to be considered and only obtain a variational inequality instead of an equality, which is a natural trade-off. If the function f is differentiable in z, both theorems lead to the same result. Generally, there is no chance that one can still use variational equalities if obstacles are involved and one has to stick to the here presented (reduced) variational inequality.

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Abstract

Obstacle problems appear in many different fields of mathematics. In this thesis, we improve the known results of [13] by removing two of the stated prerequisites on the integrand for the relaxation of single obstacle problems for non-parametric functionals of linear growth by using insights on the recession function and modifications of the De Giorgi measure. Further, we identify the relaxation of the double obstacle problem and obtain a full counterpart to the parametric double obstacle problem for the area functional treated by De Giorgi in [20] by using a truncation argument which leads to new approximation results. For the proof of that truncation, a convergence almost everywhere result is established for gradients of an in area converging sequence of BV functions. Additionally, we are able to prescribe boundary values in a broader sense to the double obstacle problem, which allows us to treat usual Dirichlet problems with or without obstacles as well as 'inner' boundary parts on a slit domain. With the relaxation formula for the double obstacle problem to the area functional, we prove the equivalence of the non-parametric and parametric obstacle problem in the graph setting using tools from the obstacle-free case and approximations. Further, we use a similar approach to slightly generalize the relaxation result for the double obstacle problem for the area functional. Finally, we develop variational inequalities for functionals of linear growth and especially for relaxations of obstacle problems relying on one-sided directional derivatives. For the obstacle case, a reduced variational inequality is stated, which is sufficient to be a minimizer of the relaxation of the corresponding double obstacle problem.

Zusammenfassung

Hindernisprobleme treten in vielen verschiedenen Bereichen der Mathematik auf. In dieser Arbeit werden die bekannten Ergebnisse aus [13] verbessert und es wird gezeigt, dass zwei der dort angegebenen Anforderungen an den Integranden für die Relaxierung von Hindernisproblemen für nichtparametrische Funktionale mit linearem Wachstum weggelassen werden können. Dies wird durch eine genauere Betrachtung der Rezessionsfunktion und Modifikation des De-Giorgi-Maßes erreicht. Weiterhin wird eine Darstellungsformel für die Relaxierung des Doppelhindernisproblems bewiesen und somit ein Äquivalent zum parametrischen Doppelhindernisproblem für das Flächenfunktional, welches von De Giorgi in [20] behandelt wurde, auch für das nichtparametrische Problem gezeigt. Diese basiert auf der Verwendung eines Abschneidearguments, welches zusätzlich zu neuen Approximationsresultaten führt. In den Beweis dieses Abschneidearguments geht die fast-überall Konvergenz für die Gradienten einer in Fläche konvergierenden Folge von BV-Funktionen ein, welche ebenfalls bewiesen wird. Zusätzlich werden Randwertaufgaben im Zusammenhang mit Doppelhindernisproblemen behandelt, welche in einem verallgemeinerten Sinne normale Randwerteprobleme mit oder ohne Hindernisse als auch Gebiete mit inneren Randstücken berücksichtigen können. Mit der Relaxierung des Doppelhindernisproblems für das Flächenfunktional wird die Äquivalenz des nichtparametrischen und parametrischen Hindernisproblems mit der Herangehensweise für den Graphenfall ohne Hindernis und mit Hilfe von Approximationen bewiesen. Des Weiteren wird ein ähnlicher Ansatz genutzt um den Satz über die Relaxierung des Doppelhindernisproblems für das Flächenfunktional etwas zu verallgemeinern. Schließlich werden Variationsungleichungen für Funktionale mit linearem Wachstum und insbesondere für die Relaxierungen von Hindernisproblemen entwickelt. Dafür werden hauptsächlich einseitige Richtungsableitungen verwendet. Für den Hindernisfall wird eine reduzierte Variationsungleichung angegeben, die hinreichend ist, um Minimierer der Relaxierung des zugehörigen Doppelhindernisproblems zu sein.

Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den