Defects and orbifolds in 3-dimensional TQFTs

Dissertation with the aim of achieving a doctoral degree
at the Faculty of Mathematics, Informatics and Natural Sciences
Department of Mathematics
of Universität Hamburg

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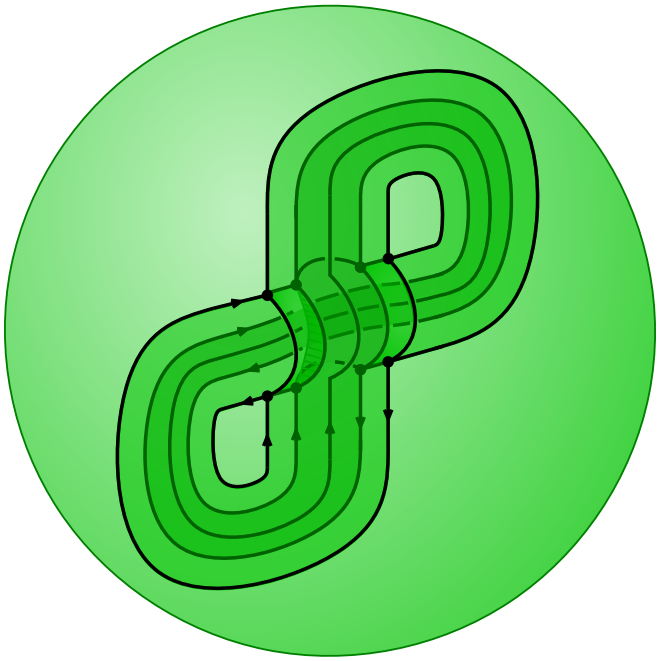
2021, Hamburg



Submitted: 30th of July 2021
Defended: 17th of September 2021

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Zusammenfassung

In dieser Arbeit betrachten wir Defekte und verallgemeinerte Orbifolds in 3-dimensionalen topologischen Quantenfeldtheorien (TQFTn) des Reshetikhin-Turaev-Typs (RT). Die verallgemeinerten Orbifolds erlauben es, neue TQFTn zu konstruieren, und zwar mittels einer Zustandssumme innerhalb einer fixierten TQFT mit Defekten (d.h., dass deren Quellkategorie die von dekorierten stratifizierten Bordismen ist). Den Input, den dieses Verfahren benötigt, nennt man ein Orbifold-Datum. Die wichtigsten Ergebnisse dieser Arbeit sind die Konstruktion einer modularen Fusionskategorie aus einem Orbifold-Datum in TQFTn vom RT-Typ, sowie der Beweis, dass die entsprechenden verallgemeinerten Orbifolds selbst wieder vom RT-Typ sind. Weiter untersuchen wir auch einige Beispiele von modularen Fusionskategorien, die sich auf diese Weise erhalten lassen, wodurch wir unseren Ansatz mit anderen Konstruktionen auf modularen Fusionskategorien in Verbindung setzen. Wir zeigen, wie sich das verwenden lässt, um einige Klassifikationsprobleme von modularen Fusionskategorien anzugehen. Schließlich wenden wir die obigen Ergebnisse an, um Flächendefekte zu untersuchen, die zwei verschiedene TQFTn des RT-Typs trennen.

Abstract

In this work we discuss defects and generalised orbifolds in 3-dimensional topological quantum field theories (TQFTs) of Reshetikhin-Turaev (RT) type. Generalised orbifolds provide one with a way to construct new TQFTs via a state-sum construction, internal to a fixed defect TQFT, i.e. having decorated stratified bordisms as the source category. The input needed for this procedure is called an orbifold datum. The main results of this work consist of constructing modular fusion categories out of orbifold data in the defect TQFTs of RT type and proving that the resulting generalised orbifolds are themselves TQFTs of RT type. We also explore several examples of modular fusion categories obtained this way, relating our approach to other constructions on modular fusion categories and demonstrating how it can be used to address some classification problems concerning them. Finally, we apply the above results to study surface defects separating two different theories of RT type.

Relevant publications

This thesis is based on the following preprints:

- V. Mulevičius, I. Runkel, *Constructing modular categories from orbifold data*, arXiv:2002.00663 [math.QA]
- V. Mulevičius, I. Runkel, *Fibonacci-type orbifold data in Ising modular categories*, arXiv:2010.00932 [math.QA]
- N. Carqueville, V. Mulevičius, I. Runkel, D. Scherl, G. Schaumann, *Orbifold graph TQFTs*, arXiv:2101.02482 [math.QA]
- V. Koppen, V. Mulevičius, I. Runkel, C. Schweigert, *Domain walls between 3d phases of Reshetikhin-Turaev TQFTs*, arXiv:2105.04613 [hep-th]
- N. Carqueville, V. Mulevičius, I. Runkel, D. Scherl, G. Schaumann, *Reshetikhin-Turaev TQFTs close under generalised orbifolds*, arXiv:2109.04754 [math.QA]

They are also listed in the bibliography at the end of the thesis and are cited respectively as [MR1, MR2, CMRSS1, KMRS, CMRSS2] throughout the text. The results in them were developed together with the coauthors, whose contributions I fully acknowledge.

Paper [MR1] is the basis of Chapters 5 and 7. The initial idea for the definition of the modular fusion category associated to an orbifold datum was suggested by Ingo Runkel. My contributions consist of formalising it (which resulted in Definition 5.3 below), as well as proving its properties (monoidal/ribbon structure in Sections 5.1 and 5.3, semisimplicity in Proposition 5.13 and modularity in Theorem 5.20). The proofs of Theorems 7.3 and 7.6 concerning the two examples in Chapter 7 were planned by both of us, the computations in them were performed by me.

The work [MR2] contains the material on which Chapter 8 is based. The main results in it (Proposition 8.1, Lemma 8.5, Proposition 8.8) are based on long computations, which both of us have done independently of each other and then compared the outcomes. I also came up with and executed the idea of analysing the fusion rules with the help of adjunctions as in Section 8.4.

The results of [CMRSS1] revolve around generalised orbifolds of arbitrary 3-dimensional defect TQFTs. They feature somewhat less prominently in this thesis, since here I focus solely on the specialisation to the Reshetikhin-Turaev defect TQFT, which are considered by me and the same coauthors in [CMRSS2]. My contribution to [CMRSS1] is the idea of using 2-skeleta of 3-manifolds instead of dual triangulations when defining generalised orbifolds, as well as the definition

of orbifold graph TQFTs, a version of which appears as Construction 6.7 in this work. The coauthors Nils Carqueville, Gregor Schaumann and Daniel Scherl have worked out a large amount of details regarding the proofs of the crucial Lemmas 6.2 and 6.4. In this work I sketch them in the Appendix B, which is a somewhat simplified version of a similar appendix in [CMRSS1]. My contribution to [CMRSS2] is both the idea and the details of the proof of the isomorphism between the Reshetikhin-Turaev orbifold graph TQFT and the graph TQFT obtained from the modular fusion category associated to an orbifold datum (here stated in Theorem 6.13). As explained below, the proof was inspired by a similar equivalence of the TQFTs of Turaev-Viro-Barrett-Westburry and Reshetikhin-Turaev types laid out by Turaev-Virelizier in the book [TV]. My idea of constructing 2-skeleta out of surgery presentations of 3-manifolds and using the pipe functors (introduced by me in [MR1] and reviewed in Section 5.4) is however characteristic to [CMRSS2] and by extension to this thesis.

The last chapter is based on the work [KMRS]. This project was initiated by Christoph Schweigert, who suggested to implement domain walls between two non-identical theories of Reshetikhin-Turaev type via an internal state sum construction and compare the outcome to the model independent analysis of [FSV]. My contributions to [KMRS] include the definition of the resulting defect TQFT worked out with the help of Ingo Runkel (see Section 9.1), as well as the definition of the algebraic datum which we used to label the domain walls (Definition 9.1). The details in Section 9.3 on module traces obtained from sphere defects and Section 9.4 on the Witt trivialisation needed for the comparison to [FSV] were largely worked out by Vincent Koppen and myself.

Eidesstattliche Versicherung/Declaration of oath

Hiermit versichere ich an Eides statt, dass ich die vorliegende Arbeit selbstständig und nur mit den angegebenen Quellen als Hilfsmittel erstellt habe. Übernommene Formulierungen und Ergebnisse sind als solche gekennzeichnet.

I hereby declare upon oath that I have written the present dissertation independently and have not used further resources and aids than those stated.

Hamburg _____

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Vincentas Mulevičius

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1. Introduction

The concept of a defect in a field theory is used to describe a variety of phenomena: phase transitions, domain walls, boundary conditions, etc. More abstractly, a defect can be understood as a spacetime region of positive codimension in which the field theory behaves somehow differently. A generic spacetime of a field theory which admits defects is therefore *stratified*. Defects usually carry labels (e.g. a boundary condition) describing the nature of them. One can also talk about codimension-1 defects between two different bulk theories. By abuse of terminology, we apply the term "defect" to the codimension-0 strata as well, which are then labelled by bulk theories. Line defects in theories of dimension 3 and higher (e.g. in Chern-Simons theory) are also commonly referred to as *Wilson lines*.

Mathematically, an *n-dimensional topological field theory (TQFT) with defects* can be formalised as a symmetric monoidal functor [CRS1]

$$Z^{\mathrm{def}}\colon\ \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}\,, \tag{1.1}$$

in an analogous way to that of the Atiyah-Segal approach. Here $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ is the category of n-dimensional oriented[1] stratified bordisms with strata carrying labels from a predetermined set $\mathbb{D}$. A very simple, yet illustrative example is that of 1-dimensional defect TQFTs in which the bulk theories (1-strata) are labelled by finite dimensional vector spaces and point defects (0-strata) by linear maps. It inspires the point of view that looking at implementations of defects in TQFTs is as natural as looking at morphisms in a category. Similarly, each n-dimensional defect TQFT can be thought of as giving rise to an n-category, which has bulk theories as objects and codimension-k defects as k-morphisms. Naturally, such algebraic structures are hardly tractable for higher values of n, but in low dimensions, namely $n = 2, 3$, they were addressed rigorously in [Ca, CMS, CR2, DKR, FSV].

Defects can also be used to describe symmetries of field theories. In particular, in the presence of a symmetry by a finite group G one can define a codimension-1 defect labelled by $g \in G$ such that the states on both sides are related by the action of g. Associated to such defects there is the *orbifold construction* which produces a new field theory by gauging the symmetry. Roughly it works as follows: the spacetime manifold is filled with a G-foam, i.e. a network of defects whose codimension-0 strata are contractible, codimension-1 strata are labelled by elements of G and the labels for higher codimension strata are produced by some appropriate construction. The evaluation then works by summing over all possible labels of all codimension-1 strata.

[1] One can adapt the definition of defect TQFTs to other tangential structures as well, but in this work we focus on oriented bordisms only.

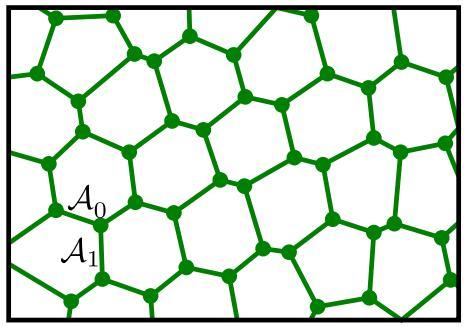


Figure 1.1: 2d foam labelled by an orbifold datum $\mathbb{A} = \{\mathcal{A}_0, \mathcal{A}_1\}$

Having a defect TQFT Z^{def} one can similarly define a *generalised orbifold construction* which produces a new ordinary TQFT (i.e. a symmetric monoidal functor $\mathrm{Bord}_n \to \mathrm{Vect}_{\Bbbk}$). In it, the role of the G-symmetry is played by a so called *orbifold datum*. An orbifold datum consists of a subset $\mathbb{A} \subseteq \mathbb{D}$ containing a single label for a stratum of each positive codimension, which are then used to label the strata of the foam (see Figure 1.1). $\mathbb{A}$ must satisfy certain requirements which ensure independence of the choice of the foam. Generalised orbifolds incorporate both the ones obtained from a G-symmetry, as well as some new instances. For example, the 3-dimensional Turaev-Viro-Barrett-Westbury theory can be obtained as a generalised orbifold of the trivial 3-dimensional TQFT [CRS3, Sec. 4].

The primary goal of this work is to explore 3-dimensional defect TQFTs and the associated generalised orbifold constructions. In particular we focus on the Reshetikhin-Turaev (RT) construction, which given a modular fusion category (MFC) $\mathcal{C}$ yields a 3-dimensional TQFT [RT, Tu]

$$Z^{\mathrm{RT}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C}) \to \mathrm{Vect}_{\Bbbk} \ . \tag{1.2}$$

Here $\widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C})$ denotes the category of 3-dimensional bordisms with embedded $\mathcal{C}$-coloured ribbon graphs (which are built into the construction to mimic Wilson lines and their junctions) and the hat indicates extra geometrical structure necessary to cancel a gluing anomaly. The construction can be extended to a defect TQFT [KSa, FSV, CRS2]

$$Z^{\mathrm{def}}_{\mathcal{C}} \colon \widehat{\mathrm{Bord}}^{\mathrm{def}}_3(\mathbb{D}^{\mathcal{C}}) \to \mathrm{Vect}_{\Bbbk} \, , \tag{1.3}$$

where the set $\mathbb{D}^{\mathcal{C}}$ of defect labels is as follows:

- all 3-strata have the same label $\mathcal{C}$;
- 2-strata are labelled by symmetric separable Frobenius algebra objects in $\mathcal{C}$;

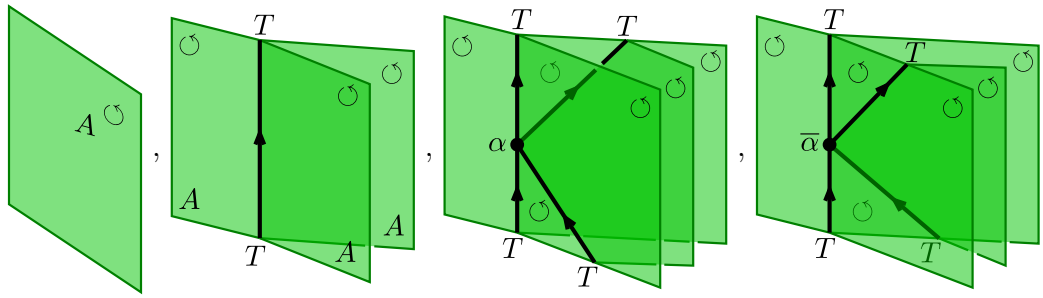


Figure 1.2: Orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$.

- 1-strata are labelled by multimodules of the Frobenius algebras which label the adjacent 2-strata (a multimodule being an object in $\mathcal{C}$ which is simultaneously a module over multiple algebra objects, such that the different actions commute);
- 0-strata are labelled with multimodule morphisms.

In this case an orbifold datum is a tuple $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$, where A is the Frobenius algebra labelling 2-strata, T is an A-$A \otimes A$-bimodule labelling trivalent edges, α and $\overline{\alpha}$ are labels for the 0-strata at the intersections of two T-lines (two labels for each orientation) and the morphism $\psi : \mathbb{1}_{\mathcal{C}} \to A$, as well as the scalar $\phi \in \Bbbk^\times$ constitute some extra data to account for certain normalisation factors [CRS3].

The main task we address in this work is determining, whether the generalised orbifold

$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}} \colon \widehat{\mathrm{Bord}}_3 \to \mathrm{Vect}_{\Bbbk} \tag{1.4}$$

of the defect RT TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ is itself a TQFT of RT type, i.e. equivalent to the functor (1.2) with $\mathcal{C}$ replaced by some other modular fusion category $\mathcal{C}_{\mathbb{A}}$, constructed out of the orbifold datum $\mathbb{A}$. The answer turns out to be affirmative and is shown in three steps:

i) given an orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ in the defect RT TQFT obtained from a modular fusion category $\mathcal{C}$, constructing a new modular fusion category $\mathcal{C}_{\mathbb{A}}$;

ii) extending the construction of generalised orbifolds in dimension 3 to include embedded ribbon graphs; for the TQFTs of RT type this extends the functor (1.4) to

$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}} \colon \widehat{\mathrm{Bord}}_3(\mathcal{C}_{\mathbb{A}}) \to \mathrm{Vect}_{\Bbbk}\,; \tag{1.5}$$

iii) showing that the two TQFTs $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ and $Z_{\mathcal{C}_{\mathbb{A}}}^{\mathrm{RT}}$ are isomorphic.

One can isolate two directions in which the above problem can be expanded. The first one is more algebraic in nature and revolves around understanding the construction yielding the category $\mathcal{C}_{\mathbb{A}}$ from step i) above. This also involves providing one with families of examples, of which we consider three in this work:

- when the orbifold datum $\mathbb{A}$ is given by a *condensable algebra* (i.e. haploid (or connected) commutative Δ-separable Frobenius algebra) A in a modular fusion category $\mathcal{C}$;
- when $\mathbb{A}$ is an orbifold datum in $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$, built from a spherical fusion category $\mathcal{S}$ with non-vanishing global dimension $\mathrm{Dim}\,\mathcal{S}$;
- when $\mathbb{A}$ is an orbifold datum in a multiplicity-less modular fusion category $\mathcal{C}$, which satisfies a number of simplifying assumptions, allowing one to reduce the conditions on $\mathbb{A}$ into a set of polynomial equations.

In the first example the category $\mathcal{C}_{\mathbb{A}}$ turns out to be equivalent to the category $\mathcal{C}_A^{\mathrm{loc}}$ of local A-modules and in the second one to the Drinfeld centre $\mathcal{Z}(\mathcal{S})$. The category $\mathcal{C}_{\mathbb{A}}$ thus unifies these two important constructions into a single algebraic setting. The third example was designed as an illustration how more complicated modular fusion categories can arise from simpler ones with the outcome not necessarily being a Drinfeld centre. In particular we solve the aforementioned polynomial equations for an ansatz of an orbifold datum in the categories of Ising type. We note that completely understanding the construction of $\mathcal{C}_{\mathbb{A}}$ is at the moment of writing still a work in progress.

The second direction in which we expand the analysis of generalised orbifolds of RT TQFTs is more to the side of mathematical physics and aims to exploit the isomorphism of TQFTs $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}} \cong Z_{\mathcal{C}_{\mathbb{A}}}^{\mathrm{RT}}$ from step iii) above to better understand some aspects of the theories of RT type. In particular, for two orbifold data $\mathbb{A}$ and $\mathbb{A}'$ in a modular fusion category $\mathcal{C}$ we look at domain walls between two theories of RT type, obtained from $\mathcal{C}_{\mathbb{A}}$ and $\mathcal{C}_{\mathbb{A}'}$. Domain walls between non-identical bulk theories could not be handled with the defect TQFT (1.3), but treating them as generalised orbifolds of a single theory $\mathcal{C}$ allows one to replace both of them with defect foams, the domain wall being a surface defect in $Z_{\mathcal{C}}^{\mathrm{def}}$ at which the two foams can end. In this work we formulate this generalisation for the orbifold data obtained from condensable algebras and compare it to a more model-independent way to analyse the domain walls proposed in [FSV].

Below we give a more detailed overview of the above results, along with some of their implications and possible future questions to address.

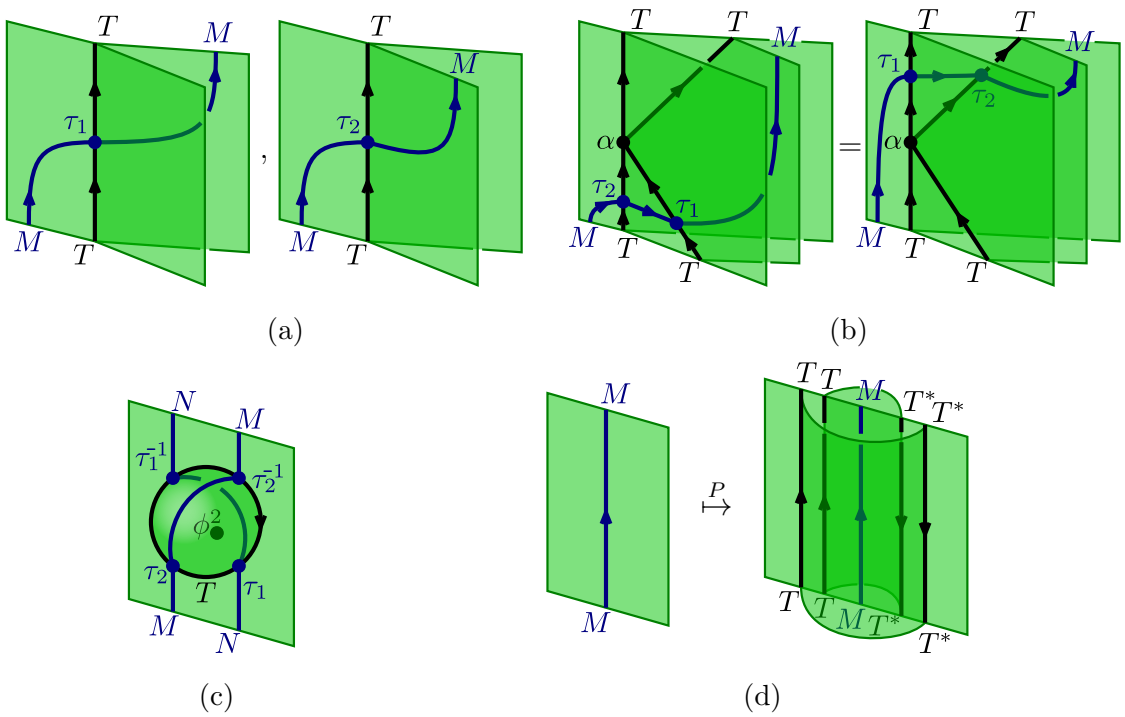


Figure 1.3: (a) The structure maps and (b) one of the conditions on an object (M, τ_1, τ_2) of $\mathcal{C}_{\mathbb{A}}$. (c) Braiding morphism of two objects $M, N \in \mathcal{C}_{\mathbb{A}}$. (d) Pipe functor $P\colon {}_A\mathcal{C}_A \to \mathcal{C}_{\mathbb{A}}$.

Modular categories from orbifold data

The construction of the category $\mathcal{C}_{\mathbb{A}}$ was done in [MR1] in collaboration with Ingo Runkel. The objects of $\mathcal{C}_{\mathbb{A}}$ turn out to be triples of the form (M, τ_1, τ_2) where M is an A-A-bimodule and τ_1, τ_2 are A-$A \otimes A$ module morphisms $M \otimes T \to T \otimes M$ subject to certain conditions. Equivalently, they describe line defects living on the 2-strata of the foam which can cross to adjacent 2-strata with the help of morphisms τ_1, τ_2 (see Figure 1.3a). The conditions that objects of $\mathcal{C}_{\mathbb{A}}$ must satisfy allow these strata to be moved across the 0-strata of the foam (see Figure 1.3b).

The interpretation of the objects of $\mathcal{C}_{\mathbb{A}}$ as defect labels enable one to perform some intricate constructions with them. For example, the braiding of two objects $M, N \in \mathcal{C}_{\mathbb{A}}$ is defined by creating a "bubble", along the two sides of which the M- and N-strands can pass over one another (see Figure 1.3c). One can also introduce objects of $\mathcal{C}_{\mathbb{A}}$ having an "internal structure", for example we define the "pipe functor" $P\colon {}_A\mathcal{C}_A \to \mathcal{C}_{\mathbb{A}}$ by surrounding a line, labelled by an A-A-bimodule M, with two A-labelled 2-strata and four T-labelled 1-strata (see Figure 1.3d). This functor can be thought of as the free construction of objects of $\mathcal{C}_{\mathbb{A}}$, as it is

adjoint to the forgetful functor $U\colon \mathcal{C}_{\mathbb{A}} \to {}_A\mathcal{C}_A$. It plays a great role both in showing that $\mathcal{C}_{\mathbb{A}}$ is in fact semisimple, and in showing that $\mathcal{C}_{\mathbb{A}}$ is modular.

Let us mention some notions, which seem to have close relations to the construction of the category $\mathcal{C}_{\mathbb{A}}$ that at the moment of writing are still yet to be explored. Firstly, the notion of an orbifold datum in RT theories is very similar to that of a monoidal category enriched over a braided category [MP], and, equivalently, to module tensor categories [MPP]. The latter also appear in the description of so-called anchored planar algebras [HPT]. A monoidal category enriched over a modular category $\mathcal{C}$ is shown in [MP] to be equivalently given by a pair $(\mathcal{S}, F)$, where $\mathcal{S}$ is a fusion category and $F\colon \mathcal{C} \to \mathcal{Z}(\mathcal{S})$ a braided functor. Most notably, the notion of an *enriched centre* [KZ3] seems to be closely related: a functor $F\colon \mathcal{C} \to \mathcal{Z}(\mathcal{S})$ yields a new braided category by taking the commutant of $F(\mathcal{C})$ in $\mathcal{Z}(\mathcal{S})$. The examples of condensations and Drinfeld centres mentioned above (which are also investigated in [MR1]) are both instances of enriched centres. We expect $\mathcal{C}_{\mathbb{A}}$ to be an enriched centre in general[2].

Other constructions on modular categories are given by taking modules of a *Hopf monad*. In [CZW] it was shown how it specialises to condensations. On the other hand, it also specialises to taking the Drinfeld centre which cannot be achieved from our construction, since $\mathcal{C}$ and $\mathcal{C}_{\mathbb{A}}$ have the same anomaly factor. Hence while the two constructions are non-equivalent, there is still a non-trivial overlap between the theory of Hopf monads and orbifold data, which would be interesting to explore.

Finally, we note that orbifold data are related to the so called *condensation algebras*, a notion which was introduced in [GJ] with the aim to formalise the relation between two TQFTs arising as topological orders of two gapped quantum systems due to a phase change. Indeed, the datum of a condensation algebra when applied to TQFTs of RT type seems very similar to an orbifold datum and their physical interpretations also seem to coincide.

RT theories are closed under generalised orbifolds

This is a two-part work [CMRSS1, CMRSS2] in collaboration with Nils Carqueville, Ingo Runkel, Gregor Schaumann and Daniel Scherl. The construction of the modular category from [MR1] can be adapted to any 3-dimensional defect TQFT together with an orbifold datum in it. Using it one can extend the orbifold TQFT to include also embedded ribbon graphs. It is known from [CMS] that such TQFTs yield tricategories with duals. An orbifold datum and the associated category of Wilson lines can be respectively seen as a datum in the corresponding tricategory

[2] We are grateful to David Penneys and David Reutter for bringing the enriched centre to our attention and for suggesting its relation to $\mathcal{C}_{\mathbb{A}}$.

and an algebraic invariant of it. We analyse this point of view in [CMRSS1] along with some questions on combinatorial presentations of 3-manifolds with embedded ribbon graphs.

In the follow-up paper [CMRSS2] we specialise the above outcome to the case of RT TQFT. The equivalence of the two TQFTs $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ and $Z_{\mathcal{C}_{\mathbb{A}}}^{\mathrm{RT}}$ is then shown by modifying the proof of a known equivalence between TQFTs of RT and Turaev-Viro-Barrett-Westbury (TVBW) types as explained in [TVire, TV]. Having the explicit description of the category $\mathcal{C}_{\mathbb{A}}$ from [MR1] is essential for the proof, as it relies greatly e.g. on the properties of the pipe functor $P\colon {}_A\mathcal{C}_A \to \mathcal{C}_{\mathbb{A}}$ mentioned above.

The isomorphism of the two TQFTs suggests an interesting parallel to the comparison of TQFTs of TVBW and RT types. Let us denote the former by $Z_{\mathcal{S}}^{\mathrm{TV}}$, where $\mathcal{S}$ denotes a spherical fusion category, which is needed as an input. The isomorphism $Z_{\mathcal{S}}^{\mathrm{TV}} \cong Z_{\mathcal{Z}(\mathcal{S})}^{\mathrm{RT}}$ (when treating both functors as having the source and target as in (1.2) with $\mathcal{C} = \mathcal{Z}(\mathcal{S})$) to the RT type theory obtained from the Drinfeld centre $\mathcal{Z}(\mathcal{S})$ is provided in [TVire, TV]. It enables higher flexibility when working with $Z_{\mathcal{Z}(\mathcal{S})}^{\mathrm{RT}}$, as the TVBW-type theory is defined by a state-sum construction and is therefore more "local" than a theory of RT type, which is obtained from surgery invariants of 3-manifolds, by definition being "global". Due to the definition of $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ via an *internal* state sum construction, the isomorphism $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}} \cong Z_{\mathcal{C}_{\mathbb{A}}}^{\mathrm{RT}}$ provides similar opportunities to an RT theory obtained from a MFC that is not necessarily a Drinfeld centre. The orbifold datum $\mathbb{A}$ is then an analogous input to that of the spherical fusion category $\mathcal{S}$ in the TVBW construction. As conjectured above, $\mathcal{C}_{\mathbb{A}}$ is an enriched centre, so one sees that just like state-sum constructions are equivalent to RT theories for Drinfeld centres, the internal state-sum constructions should be equivalent to RT theories for the enriched centres.

We note that an alternative proof of the isomorphism $Z_{\mathcal{S}}^{\mathrm{TV}} \cong Z_{\mathcal{Z}(\mathcal{S})}^{\mathrm{RT}}$ is laid out in [BalK1, BalK2, BalK3]. It differs from the one in [TVire, TV] in that it treats both TQFTs as *3-2-1 extended*, i.e. 2-functors with 3-dimensional bordisms with corners as the source category. By replacing $Z_{\mathcal{C}_{\mathbb{A}}}^{\mathrm{RT}}$ with the 3-2-1 extended TQFT introduced in [BDSV] (also requiring a MFC as an input[3], which conjecturally in this case would also be $\mathcal{C}_{\mathbb{A}}$), one could try to generalise the isomorphism $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}} \cong Z_{\mathcal{C}_{\mathbb{A}}}^{\mathrm{RT}}$ to this setting as well. At the moment, the obstacle is that the 3-2-1 extended TQFT in [BDSV] has not been yet generalised to a 3-2-1 extended *defect* TQFT, which one would need to perform the internal state-sum construction. Exploring the extended versions of defect TQFTs is a possible direction for future projects.

[3] It was not yet shown that the construction in [BDSV] when restricted to bordisms without corners yields an RT theory, but this claim is widely believed to be true.

Inversion of E_6-condensation

Let us discuss the last of the three examples of orbifold data and the associated MFCs mentioned above. It was explored in detail in the work [MR2], in collaboration with Ingo Runkel[4]. The idea of this project was to look at the modular category $\mathcal{C}(sl(2), 10)$ of integrable modules of the affine Lie algebra $\widehat{sl}(2)$ at level 10. Algebra objects in $\mathcal{C}(sl(2), 10)$ are classified by Dynkin diagrams of certain type, one of which, E_6, gives a condensable algebra. The category of its local modules is equivalent to $\mathcal{C}(sp(4), 1)$, having 3 simple objects (i.e. of rank 3) which exhibit Ising fusion rules. The goal was then to "invert" this, i.e. find an orbifold datum in the E_6 condensation which would give back $\mathcal{C}(sl(2), 10)$. This was achieved by making an ansatz for an orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ in a modular category $\mathcal{C}$ whose fusion rules are multiplicity-less. The ansatz in particular required that the Frobenius algebra A is a direct sum of copies of the tensor unit in $\mathcal{C}$. This allows one to write down the conditions for $\mathbb{A}$ as a system of polynomial equations which, in a concrete case like the categories of Ising type, can be solved using computer algebra. In the end this illustrated how finding an orbifold datum in a seemingly non-complicated category (i.e. the Ising one, of rank 3) can yield a rather complicated one (i.e. $\mathcal{C}(sl(2), 10)$, of rank 11) also when they are not equivalent to Drinfeld centres.

One of the motivations for this example stems from the classification of modular fusion categories. There is, up to equivalence, only a finite number of modular fusion categories with a given number of simple objects [BNRW1], a property called *rank-finiteness*. It therefore makes sense to classify such categories by number of simple objects (the rank). This has been done up to rank 5 [RSW, BNRW2], and rank 6 is in progress [Gr, Cr]. A systematic approach to produce more exotic examples of modular categories is to consider Drinfeld centres, see e.g. [HRW, EG, JMS, GM] for more details and references. The problem we would like to advocate with the simple example is:

> Try to construct new MFCs by systematically studying orbifold data $\mathbb{A}$ in a given MFC $\mathcal{C}$ of low rank e.g. by solving the simplified polynomial equations obtained from the conditions on $\mathbb{A}$ and then computing $\mathcal{C}_{\mathbb{A}}$.

Another motivation stems from the study of topological phases of matter. Namely, unitary modular categories $\mathcal{C}$ model anyons in two-dimensional topological phases of matter, see e.g. [RW], and MFCs, obtained by condensations, describe the process of anyon condensation [Ko] (hence our use of the terms *condensation, condensable algebra*). Since the Ising category does not support universal quantum computation, but $\mathcal{C}(sl(2), 10)$ does (see [NR, RW]), it would be interesting to see

[4]Both of us are grateful to Terry Gannon for suggesting this example

if the fact that a generalised orbifold can turn the former into (a close relative of) the latter has applications in topological quantum computation.

Domain walls between RT theories

The last result presented in this thesis is based on the work [KMRS], in collaboration with Vincent Koppen, Ingo Runkel and Christoph Schweigert, where we sought to provide a construction of a defect TQFT labelling the 3-strata with orbifold data in a fixed modular category $\mathcal{C}$. To achieve this one needs to look at surface defects between two foams labelled by different orbifold data $\mathbb{A}$ and $\mathbb{A}'$. In light of the equivalence of RT and orbifold TQFTs, this is the same as looking at surface defects between two RT theories labelled by modular categories $\mathcal{C}_{\mathbb{A}}$ and $\mathcal{C}_{\mathbb{A}'}$. It is argued in [FSV] that such defects are parametrised by a pair $(\mathcal{W}, F)$, where $\mathcal{W}$ is a fusion category and

$$F\colon \mathcal{C}_{\mathbb{A}} \boxtimes \widetilde{\mathcal{C}_{\mathbb{A}'}} \to \mathcal{Z}(\mathcal{W}) \tag{1.6}$$

is a braided equivalence (here for a braided category $\mathcal{B}$, $\widetilde{\mathcal{B}}$ denotes the category with the mirrored braiding). Two modular categories are called *Witt equivalent* if there exists an equivalence like the one in (1.6) for some fusion category $\mathcal{W}$ (to be referred to as a *Witt trivialisation*). We therefore can relate the analysis of 3-dimensional defect TQFTs of RT type with that of the Witt equivalence relation. In [KMRS] we have completed this project for orbifold data obtained from condensable algebras in a fixed modular category $\mathcal{C}$. We have found that a surface defect between two 3-strata, labelled by condensable algebras $A, B \in \mathcal{C}$ (or rather by the corresponding orbifold data) can be labelled by a symmetric separable Frobenius algebras in the category ${}_A\mathcal{C}_B$ of A-B-bimodules, which can be equipped with a specific monoidal structure due to A and B being commutative. The category ${}_A\mathcal{C}_B$ also is the one playing the above role of $\mathcal{W}$ in a choice of Witt trivialisation of the pair of condensations $\mathcal{C}_A^{\mathrm{loc}}$ and $\mathcal{C}_B^{\mathrm{loc}}$.

Structure of the thesis

Excluding the introduction, this work consists of 8 chapters numbered 2 to 9, which thematically can be further subdivided into prerequisite chapters 2, 3 and 4, main material chapters 5 and 6, and the chapters on examples and applications 7, 8 and 9. The appendix sections A-E contain supplementary material on various topics, some of it original (e.g. some more lengthy computations) and the other part being reviews of some secondary prerequisites.

The chapters 3, 4 and 6 are dedicated to defining the TQFT functors (1.2), (1.3) and (1.5) respectively. All of them are organised after the following pattern: i)

define the source category of decorated bordisms; ii) describe the aspects of its construction; iii) state the definition; iv) explore some of the properties which are used later. In the last step we use the "Property" environment, which should be interpreted as a mixture of the "Proposition", "Example" and "Remark" environments. In each of them we state a property of the respective TQFT, which in principle could be formulated (and in most cases has been formulated in one of the references) rigorously as a proposition, but doing so we deemed as too much of a digression and instead we either described it more informally as a remark, or illustrated it with an example, assuming that the generalisation is straightforward.

Let us briefly review the material presented in each of the chapters. A more detailed overview is positioned at the beginning of each chapter.

- Chapter 2 collects most of the general categorical notions that are used throughout the text, a lot of it based on the books [BakK, EGNO, TV]. In it we also introduce the separable Frobenius algebras, which are treated slightly differently than in the source literature [FRS1, FFRS], which is why we review them in more detail.

- Chapter 3 contains the definition of the Reshetikhin-Turaev TQFT $Z_{\mathcal{C}}^{\mathrm{RT}}$, obtained from a MFC $\mathcal{C}$. We mostly follow [Tu, Ch. IV], although the definition itself is stated differently using the universal construction of [BHMV], as it makes some properties, e.g. the functoriality of $Z_{\mathcal{C}}^{\mathrm{RT}}$, more transparent. The main prerequisite for it is understanding the material on MFCs laid out in section 2.3.

- Chapter 4 discusses stratified manifolds and defect TQFTs in general and states the definition and some properties of the Reshetikhin-Turaev defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$. It relies heavily on the works [CRS1, CMS, CRS2], with only very minor changes, the most noteworthy of which is our use of 1-skeleta, instead of the dual triangulations in defining the internal 2-dimensional state-sum construction. We review the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ quite thoroughly, since it is one of the most important tools used later. The prerequisites for this chapter include the sections 2.4 and 2.5 on separable Frobenius algebras, as well as the definition and the properties of $Z_{\mathcal{C}}^{\mathrm{RT}}$ in chapter 3.

- Chapter 5 introduces the most important construction of this work: that of the category $\mathcal{C}_{\mathbb{A}}$. We discuss the various structures on it: tensor product, dualities, braiding, twists. We also prove that it is semisimple and, under a very natural assumption of $\mathbb{A}$ being *simple* (i.e. $\mathcal{C}_{\mathbb{A}}$ being fusion, instead of multifusion), that it is in fact a MFC. The material in this chapter first appeared in [MR1]. New to this thesis is the use of *separable* (instead of

stronger Δ-*separable*) Frobenius algebras, which results in a slight generalisation of the results in [MR1]. Naturally, sections 2.3 on MFCs and 2.4 and 2.5 on separable Frobenius algebras are prerequisites for this chapter. We also make use of the interpretation of $\mathcal{C}_{\mathbb{A}}$ in terms of defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$, so chapter 4 is a prerequisite as well.

- Chapter 6 both introduces the generalised orbifold TQFT $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ and proves the isomorphism of TQFTs $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}} \cong Z_{\mathcal{C}_{\mathbb{A}}}^{\mathrm{RT}}$. The definition of $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ is comparable to that of $Z_{\mathcal{C}}^{\mathrm{def}}$, being itself an internal 3-dimensional state-sum construction. The main prerequisites for this chapter therefore are the previous chapters 4 and 5. We base this chapter on the works [CMRSS1, CMRSS2], acknowledging the results of [CRS3], where the first simpler version of $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ was introduced, as an inspiration for them.

- Chapter 7 considers the first two examples of orbifold data mentioned above (i.e. obtained from condensible algebras and spherical fusion categories). The orbifold data in both cases were introduced in [CRS3], our focus is on the associated MFCs $\mathcal{C}_{\mathbb{A}}$. The results in this chapter appeared in the second part of [MR1], a lot of them consist of technicalities, needed for their proofs. As this chapter is more algebraic in nature, one just needs the material from chapter 5 as a prerequisite.

- Chapter 8 revolves around the third example of orbifold data mentioned above (i.e. orbifold data in multiplicity-less MFCs satisfying certain assumptions). It is based on the results in [MR2], which are again somewhat technical. The longest computations are moved to appendix D. Again, the chapter 5 is enough as a prerequisite, with the results of chapter 6 also being marginally used.

- Chapter 9 is based on the work [KMRS] and discusses the domain walls between Reshetikhin-Turaev theories. The results in it are twofold: a generalisation of the defect TQFT (1.3) to include different phases of 3-strata, as well as a comparison of two descriptions of domain walls: one as given by the new defect TQFT and the other as proposed in [FSV]. The latter reference was a great inspiration for the results in this chapter and is summarised in section 9.2. This is the only chapter that uses some higher-categorical notions more extensively; we review them very briefly in sections 2.6 and 2.7.

Finally, we note that the appendix B, in which the 1- and 2-skeleta of 2- and 3-manifolds is discussed, is based on a similar appendix to the work [CMRSS1]. The two of them are somewhat complementary in the material they present: the former focuses on 1-skeleta of surfaces and treats 2-skeleta for 3-manifolds by analogy, while the presentation in the latter is the opposite.

Acknowledgements

The main hero of this work is my supervisor Ingo Runkel. His guidance and the constant reminder of the wrath of the "gods of unfinished papers" is what brought all my humble contributions to their conclusions, for otherwise they all would have left in the limbo of "just a few more details". A similar thank you applies to all the other people with which I had a pleasure to work: Nils Carqueville, Vincent Koppen, Gregor Schaumann, Daniel Scherl, Christoph Schweigert.

I cannot thank enough to my family for their unconditional support: my parents Dalia Mulevičienė and Rimvydas Mulevičius, my grandmother Valerija Bumbulienė, my brother Adomas Mulevičius, his wife Jiang Pan and their son, my dear nephew, Aleksandras Mulevičius.

My very dear friends, which were my first "scientific community" back during the years of secondary school, deserve a credit in whatever I manage to achieve: Thanks to Mykolas Mikučiūnas and Alfonsas Juršėnas. Also, thanks to Eligijus Sakalauskas and Vidas Regelskis for supervising me during the years of Bachelor's studies and introducing me to the fields of algebra and mathematical physics.

A sincere thank you goes to my very first group of friends in Hamburg: Claudia Hoogenstraaten, Tom Weber, Hannes Malcha, Philipp Tontsch, Eren Kovanlıkaya, and especially to Tim Berberich, who has not missed a single week without asking about my state of mind and reminding me to take breaks while working.

For all kinds of help and support I am indebted to the Ingo's squad: Johannes Berger, Iordanis Romaidis and Daniel Scherl. Until the next beer seminar!

During my PhD years I experienced many wonderful moments, all of which I owe to the wonderful people with which I shared them: Yang Yang, Áron Szabó, Arpan Saha, Manasa Manjunatha, David Krusche, Vincent Koppen, Danu Thung, Manuel Araújo, Ehud Meir, Lóránt Szegedy, Mauro Mantegazza, and especially la mia amica geniale, Ilaria Flandoli.

The disadvantage of working from home is living at work. The endless quarantine (at the moment of writing, it would seem indeed, endless) would have been impossible to survive without the residents of the island of Wilhelmsburg, both native and honorary: Flavia Rossi, Thomas Bourton, Ilaria Flandoli, Max Siebert, and Alessandra Manganelli.

My respect goes to the creators of the software Inkscape, which was used to create all 864 pictures used in this work.

My position was supported by the DFG via the Research Training Group 1670 and the Fachbereich Mathematik of Universität Hamburg, to which I express my heartily gratitude. The contribution of our dear secretary Gerda Mierswa-Silva to

our well-being is impossible to overemphasise. Thanks also to Astrid Dörhöfer for constantly reminding us that we are contractually obligated to take holidays. I might just use them after finishing writing this sentence.

Addition after the defence: A big thank you for the comments, remarks, suggestions, questions and grammar corrections goes to the referees of this work: Nils Carqueville, David Penneys and Ingo Runkel.

2. Categorical preliminaries

This chapter contains the basic prerequisites on which we will rely in later sections. We mostly just review our preferences regarding terminology and notations. More details on the material in Sections 2.1-2.3 can be found, for example, in the books [BakK, EGNO, TV]. Sections 2.4, 2.5 introduce a slight generalisation of the notion of a symmetric Δ-separable Frobenius algebra and then adapt some of the results, found e.g. in [FRS1, FFRS]. The remaining Sections 2.6 and 2.7 serve as a very light exposition to some higher categorical notions, most of them emerging in applications to defect TQFTs (to be discussed in Section 4.1 below). Our reliance on them will be relatively marginal (with a possible exception in Chapter 9), and so they are discussed with less rigour, referring e.g. to [Ca, CMS, CR2, Schm] for more details.

2.1. Categories and functors

We assume some familiarity with monoidal (rigid, braided, etc.) categories, functors and natural transformations. In this section we review some aspects of these and related notions, in particular those that are more specific to this work.

- A *monoidal category* $\mathcal{C}$ is equipped with a monoidal (or tensor) product functor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ which has a unit $\mathbb{1}=\mathbb{1}_{\mathcal{C}}\in\mathcal{C}$ and the associator and unitor natural isomorphisms

$$a\colon (-\otimes-)\otimes-\Rightarrow-\otimes(-\otimes-)\,,\quad l\colon \mathbb{1}\otimes-\Rightarrow\mathrm{Id}_{\mathcal{C}}\,,\quad r\colon -\otimes\mathbb{1}\Rightarrow\mathrm{Id}_{\mathcal{C}} \tag{2.1}$$

  satisfying the pentagon and triangle identities. For notational simplicity we will sometimes omit the tensor product symbol $\otimes$.

- A *monoidal functor* between two monoidal categories $\mathcal{C}$, $\mathcal{D}$ is a functor $F\colon\mathcal{C}\to\mathcal{D}$ with an assigned *monoidal structure*, i.e. a natural isomorphism $F_2\colon F(-)\otimes F(-)\Rightarrow F(-\otimes-)$ and an isomorphism $F_0\colon F(\mathbb{1}_{\mathcal{C}})\xrightarrow{\sim}\mathbb{1}_{\mathcal{D}}$ satisfying the usual compatibility conditions.

- A *monoidal natural transformation* between monoidal functors $F,G\colon\mathcal{C}\to\mathcal{D}$ is a natural transformation $\varphi\colon F\Rightarrow G$ which commutes with the monoidal structures, i.e.

$$\varphi_{\mathbb{1}_{\mathcal{D}}}\circ F_0=G_0\,,\quad \varphi_{X\otimes Y}\circ F_2(X,Y)=G_2(X,Y)\circ(\varphi_X\otimes\varphi_Y)\,. \tag{2.2}$$

  One infers the notion of *monoidal equivalence* between monoidal categories.

- A *rigid category* is a monoidal category $\mathcal{C}$ whose each object $X \in \mathcal{C}$ has distinguished left and right duals, i.e. objects $X^*, {}^*X \in \mathcal{C}$ together with (co)evaluation morphisms

$$\begin{aligned} &\mathrm{ev}_X \colon X^* \otimes X \to \mathbb{1}\,, \quad \mathrm{coev}_X \colon \mathbb{1} \to X \otimes X^*\,, \\ &\widetilde{\mathrm{ev}}_X \colon X \otimes {}^*X \to \mathbb{1}\,, \quad \widetilde{\mathrm{coev}}_X \colon {}^*X \otimes X \to \mathbb{1} \end{aligned} \tag{2.3}$$

  satisfying the "snake" identities. A monoidal functor automatically preserves left/right duals of objects up to a canonical isomorphism.

- A *pivotal category* is a rigid category $\mathcal{C}$ endowed with a *pivotal structure*, i.e. a monoidal natural isomorphism $\delta \colon \mathrm{Id}_{\mathcal{C}} \Rightarrow (-)^{**}$. Since for rigid categories one has $X \cong ({}^*X)^*$ for all objects $X \in \mathcal{C}$, a pivotal structure provides one with a canonical isomorphism $X^* \cong ({}^*X)^{**} \cong {}^*X$. It is therefore enough to consider one of the two duals in pivotal categories, and we will use X^*. A *pivotal functor* between pivotal categories is a monoidal functor preserving the pivotal structure (up to the canonical isomorphism of the double-duals).

- A *braided category* is a monoidal category $\mathcal{C}$ with an assigned *braiding*, i.e. a natural isomorphism $\{c_{X,Y} \colon X \otimes Y \to Y \otimes X\}_{X,Y \in \mathcal{C}}$ satisfying the two hexagon identities. The *reverse* of $\mathcal{C}$ is the braided category $\widetilde{\mathcal{C}}$ having the same underlying monoidal category, but equipped with the reverse braiding $\{c^{-1}_{Y,X} \colon X \otimes Y \to Y \otimes X\}_{X,Y \in \mathcal{C}}$. A braided category $\mathcal{C}$ is called *symmetric* if one has $c_{X,Y} = c^{-1}_{Y,X}$ for all $X, Y \in \mathcal{C}$, i.e. if the braiding and its reverse coincide. A *braided (resp. symmetric) functor* between braided (resp. symmetric) categories is a monoidal functor preserving the braiding (up to a composition with the canonical isomorphisms due to the monoidal structure of the functor).

Any monoidal category $\mathcal{C}$ has an associated braided monoidal category, its *Drinfeld centre* $\mathcal{Z}(\mathcal{C})$. Recall that a *halfbraiding* on an object $X \in \mathcal{C}$ is a natural isomorphism $\gamma \colon X \otimes - \Rightarrow - \otimes X$ satisfying the first hexagon identity, i.e. for all $U, V \in \mathcal{C}$ the following identity holds:

$$a_{U,V,X} \circ \gamma_{U \otimes V} \circ a_{X,U,V} = (\mathrm{id}_U \otimes \gamma_V) \circ a_{U,X,V} \circ (\gamma_U \otimes \mathrm{id}_V)\,. \tag{2.4}$$

$\mathcal{Z}(\mathcal{C})$ is then defined to be the braided monoidal category having:

- *objects:* pairs (X, γ) where $X \in \mathcal{C}$ and $\gamma \colon X \otimes - \Rightarrow - \otimes X$ is a halfbraiding,

- *morphisms:* $f \colon (X, \gamma) \to (Y, \delta)$ is a morphism $f \colon X \to Y$ in $\mathcal{C}$, such that for all $U \in \mathcal{C}$ one has

$$\delta_U \circ (f \otimes \mathrm{id}_U) = (\mathrm{id}_U \otimes f) \circ \gamma_U\,; \tag{2.5}$$

- *monoidal product:* $(X,\gamma)\otimes(Y,\delta) := (X\otimes Y,\Gamma)$, where for all $U\in\mathcal{C}$

$$\Gamma_U := \left[\begin{array}{l} (XY)U \xrightarrow{a_{X,Y,U}} X(YU) \xrightarrow{\mathrm{id}_X\otimes\delta} X(UY) \xrightarrow{a^{-1}_{X,U,Y}} (XU)Y \\ \qquad\qquad \xrightarrow{\gamma_U\otimes\mathrm{id}_Y} (UX)Y \xrightarrow{a_{U,X,Y}} U(XY) \end{array}\right] . \tag{2.6}$$

- *monoidal unit:* $\mathbb{1}_{\mathcal{Z}(\mathcal{C})} := (\mathbb{1}_{\mathcal{C}}, r^{-1}\circ l)$,
- *associator and unitors:* as in $\mathcal{C}$,
- *braiding:* $c_{(X,\gamma),(Y,\delta)} := \gamma_Y$.

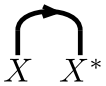
In case $\mathcal{C}$ is pivotal, so is $\mathcal{Z}(\mathcal{C})$.

A convenient tool when working with monoidal categories is graphical calculus. It depicts morphisms in a monoidal category $\mathcal{C}$ by so-called *string diagrams*, consisting of *strands* labelled by objects and *coupons* labelled by morphisms. A diagram is read from bottom to top. It is customary (although not required) to omit strands labelled with the unit object and coupons labelled with associator, unitor and identity morphisms.

When using graphical calculus for a pivotal category $\mathcal{C}$ one also adds directions to strands; a downwards direction corresponds to the dual of an object. The (co)evaluation morphisms of an object $X\in\mathcal{C}$ are depicted by bent lines:

$\mathrm{ev}_X =$

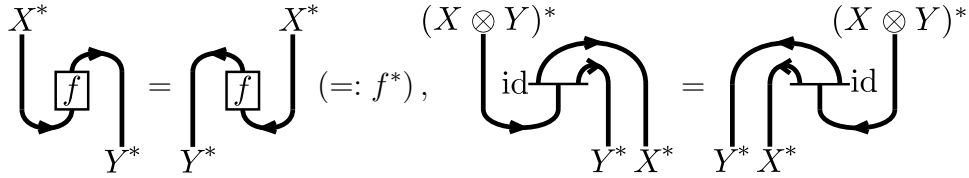
, $\mathrm{coev}_X =$

, $\widetilde{\mathrm{ev}}_X =$ 

, $\widetilde{\mathrm{coev}}_X =$

. (2.7)

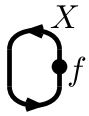
An equivalent way to define a pivotal structure on a monoidal category $\mathcal{C}$ is to fix for each object $X\in\mathcal{C}$ a (left and right) dual X^* such that the identities

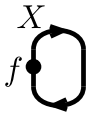


(2.8)

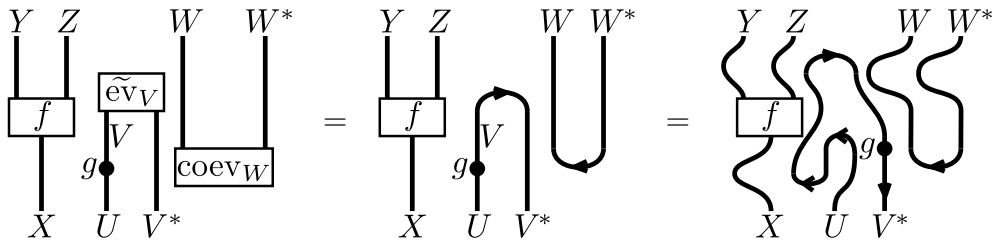
hold for all $[f\colon X\to Y]\in\mathcal{C}$ (see e.g. [CR1, Lem. 2.12]). These identities imply that string diagrams up to a plane isotopy with fixed ends of incoming and outgoing strands yield equal morphisms (see Figure 2.1a).

Let $\mathcal{C}$ be a pivotal category, $X\in\mathcal{C}$ and $f\in\mathrm{End}_{\mathcal{C}}\,X$. The *left trace* and the *right trace* of f are defined to be the following endomorphisms of the monoidal unit:

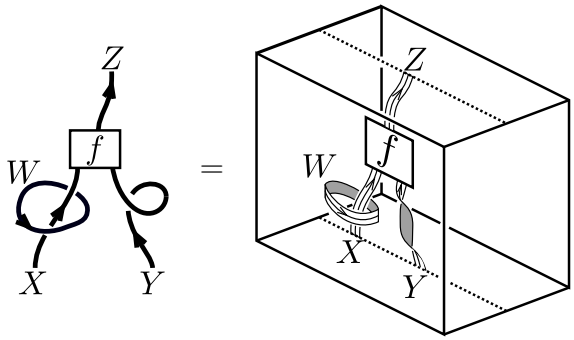
$\mathrm{tr}_l\, f =$ 

, $\mathrm{tr}_r\, f =$ 

. (2.9)



(a)



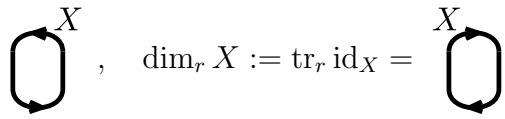
(b)

Figure 2.1: (a) Graphical calculus for a pivotal category $\mathcal{C}$. A morphism is depicted by a string diagram, consisting of *strands* labelled by objects and *coupons* labelled by morphisms. A diagram is read from bottom to top, in this example the diagram depicts a morphism $(\mathrm{id}_{Y\otimes Z} \otimes \widetilde{\mathrm{ev}}_V \otimes \mathrm{id}_{W\otimes W^*}) \circ (f \otimes g \otimes \mathrm{id}_{V^*} \otimes \mathrm{coev}_W)$ where $X, Y, Z, U, V, W \in \mathcal{C}$, $f\colon X \to Y \otimes Z$, etc. It is customary (although not required) to omit strands labelled with the tensor unit and coupons labelled with associators, unitors and identity objects, and to replace (co)evaluation morphisms with bent lines. In the latter case one also adds directions to strands; a downwards direction corresponds to the dual of an object. The axioms of a pivotal category imply that string diagrams up to a plane isotopy with fixed ends of incoming and outgoing strands yield equal morphisms. Coupons having one ingoing and one outgoing strand sometimes will also be replaced by points and referred to as *point insertions* on strands. If the domains are clear from the context, we will not relabel a point insertion g with the dual morphism g^* when it is read "upside down" (as is the case in the third equality in the picture).

(b) Graphical calculus for a ribbon category. In this case the strands can be seen as directed and *framed* lines with the framing given by the paper plane. The diagrams can be deformed up to isotopy as if embedded in $(0,1) \times (0,1) \times [0,1]$ with incoming strands starting at fixed points on the line $\{1/2\} \times (0,1) \times \{0\}$ and the outgoing ones ending at fixed points on $\{1/2\} \times (0,1) \times \{1\}$. The framing of strands allows one to depict them as ribbons. By a *($\mathcal{C}$-coloured) ribbon tangle* we mean an isotopy class of such embeddings.

One also defines the *left/right (categorical) dimensions* of X by

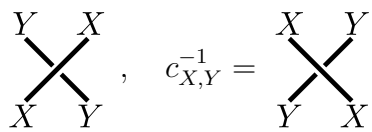
$$\dim_l X := \operatorname{tr}_l \operatorname{id}_X = \quad , \quad \dim_r X := \operatorname{tr}_r \operatorname{id}_X = \quad . \tag{2.10}$$



$\mathcal{C}$ is called *spherical* ([BW1]) if the left and the right traces coincide. In this case one speaks of the *trace* of f defined by $\operatorname{tr} f := \operatorname{tr}_l f = \operatorname{tr}_r f$ and the *(categorical) dimension* of X defined by $\dim X := \dim_l X = \dim_r X$. If the underlying category needs to be emphasised, we will add an index to the notation, e.g. $\operatorname{tr}_{\mathcal{C}} f$, $\dim_{\mathcal{C}} X$.

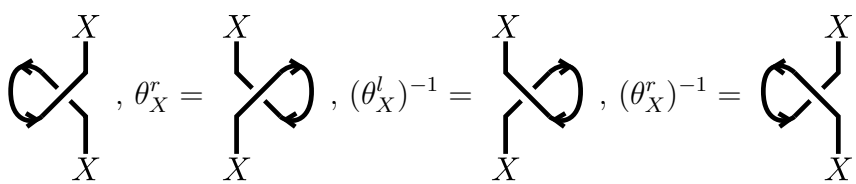
When using graphical calculus for a braided category $\mathcal{C}$ one uses overcrossings and undercrossings of strands labelled by $X, Y \in \mathcal{C}$ to depict the braiding morphisms and their inverses:

$$c_{X,Y} = \quad , \quad c_{X,Y}^{-1} = \quad . \tag{2.11}$$



Let $\mathcal{C}$ be a braided pivotal category, $X \in \mathcal{C}$ an object. The *left twist* and the *right twist* of X are defined to be the following invertible endomorphisms of X:

$$\theta_X^l = \quad , \theta_X^r = \quad , (\theta_X^l)^{-1} = \quad , (\theta_X^r)^{-1} = \quad . \tag{2.12}$$



$\mathcal{C}$ is called *ribbon* if the left and the right twists coincide. In this case one speaks of the *twist* of X defined as $\theta_X := \theta_X^l = \theta_X^r$. A *ribbon functor* between ribbon categories is a braided functor which preserves twists.

Graphical calculus allows one to represent a morphism in a ribbon category by a *$\mathcal{C}$-coloured ribbon tangle* embedded in $(0,1) \times (0,1) \times [0,1]$, see Figure 2.1b for an example. In fact, one can introduce the ribbon category $\mathrm{Rib}_{\mathcal{C}}$ having

- *objects:* lists $\big((X_1, \epsilon_1), \ldots, (X_n, \epsilon_n)\big)$, $X_i \in \mathcal{C}$, $\epsilon_i = \pm$;
- *monoidal product/unit:* concatenation/the empty list $\varnothing$;
- *morphisms/composition:* $\mathcal{C}$-coloured ribbon tangles/stacking.

The graphical calculus can then be formalised as the functor $F_{\mathcal{C}}\colon \mathrm{Rib}_{\mathcal{C}} \to \mathcal{C}$ sending a list $\big((X_1,\epsilon_1),\dots,(X_n,\epsilon_n)\big)$ to the tensor product $X_1^{\epsilon_1}\otimes\cdots\otimes X_n^{\epsilon_n}$ (where we use notation $X_i^+ := X_i$ and $X_i^- := X_i^*$) and a ribbon tangle to the corresponding morphism in $\mathcal{C}$.

Note that a ribbon tangle $[L\colon \varnothing \to \varnothing] \in \mathrm{Rib}_{\mathcal{C}}$ which does not have coupons can be seen as a directed framed link in $\mathbb{R}^3$ with components labelled by objects in $\mathcal{C}$. The element $F_{\mathcal{C}}(L) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is by definition an isotopy invariant of L.

2.2. Multifusion categories

Let $\Bbbk$ be an algebraically closed field. In this section we assume some familiarity with $\Bbbk$-linear and abelian categories. Functors between such categories are always assumed to be $\Bbbk$-linear, i.e. linear on the morphism spaces. They then automatically preserve direct sums of finite families of objects.

Let $\mathcal{A}$ be a $\Bbbk$-linear abelian category. Recall that an object $X \in \mathcal{A}$ is called *simple* if $\dim \mathrm{End}_{\mathcal{A}} X = 1$. The category $\mathcal{A}$ is called *semisimple* if all its objects are isomorphic to finite direct sums of simple objects (the empty direct sum is the zero object $0 \in \mathcal{A}$) and *finitely semisimple* if there is only a finite number of isomorphism classes of simple objects.

Definition 2.1. A *multifusion* category is a $\Bbbk$-linear, finitely semisimple, rigid monoidal category $\mathcal{A}$ such that the tensor product functor $\otimes : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ is bilinear on morphism spaces. If in addition the tensor unit $\mathbb{1}_{\mathcal{A}}$ is simple, it is called *fusion*.

A more general notion is that of *(multi-)tensor category* (see [EGNO, Def. 4.1.1]), in which one omits semisimplicity and adds the requirement of morphism spaces to be finite dimensional (which in the above definition is automatic). As most of the categories that we will encounter are semisimple, this notion, apart from a few exceptions, will not be used in the sections below.

For a simple object U of a $\Bbbk$-linear abelian category $\mathcal{A}$, it is customary to implicitly use the isomorphism $\Bbbk \xrightarrow{\sim} \mathrm{End}_{\mathcal{A}} U$, $\lambda \mapsto \lambda\,\mathrm{id}_U$. For an arbitrary object $X \in \mathcal{A}$ one can then identify the dual space $\mathcal{A}(X,U)^*$ with $\mathcal{A}(U,X)$ using the (non-degenerate) composition pairing:

$$\mathcal{A}(U,X)\otimes_{\Bbbk}\mathcal{A}(X,U) \to \Bbbk, \qquad f\otimes_{\Bbbk} g \mapsto g\circ f\,. \tag{2.13}$$

We will denote by $\mathrm{Irr}_{\mathcal{A}}$ a set of representatives of isomorphism classes of simple objects of $\mathcal{A}$. If $\mathcal{A}$ is fusion, we in addition assume that $\mathbb{1}_{\mathcal{A}} \in \mathrm{Irr}_{\mathcal{A}}$.

For two $\Bbbk$-linear categories $\mathcal{A}$, $\mathcal{B}$, their direct sum is defined as the $\Bbbk$-linear category $\mathcal{A}\oplus\mathcal{B}$ having pairs (X,Y), $X\in\mathcal{A}$, $Y\in\mathcal{B}$ as objects and $\mathcal{A}(X,X')\oplus$

$\mathcal{B}(Y, Y')$ as morphism spaces between $(X, Y), (X', Y') \in \mathcal{A} \oplus \mathcal{B}$. The direct sum is defined additively on each entry. Similarly one introduces the notion of a direct sum of functors. It is customary to identify e.g. $\mathcal{A}$ with the collection of objects $(X, 0) \in \mathcal{A} \oplus \mathcal{B}$, $X \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are abelian/(finitely) semisimple so is $\mathcal{A} \oplus \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are multifusion, the direct sum $\mathcal{A} \oplus \mathcal{B}$ too has a canonical structure of a multifusion category with $X \otimes Y = 0$ whenever $X \in \mathcal{A}$, $Y \in \mathcal{B}$ and the tensor unit $\mathbb{1}_{\mathcal{A}} \oplus \mathbb{1}_{\mathcal{B}}$. Naturally this can be generalised to define direct sums of finite families of $\Bbbk$-linear categories. A multifusion category is called *indecomposable* if it is not monoidally equivalent to a direct sum of two non-trivial (i.e. non-equivalent to $\{0\}$) multifusion categories [ENO1, Sec. 2.4]. We note that a multifusion category can have a non-simple tensor unit and still be indecomposable. The finite-dimensional bimodules of the semisimple $\mathbb{C}$-algebra $\mathbb{C} \oplus \mathbb{C}$ are an example of this.

Another construction on two finitely semisimple categories $\mathcal{A}$, $\mathcal{B}$ is their *Deligne product* $\mathcal{A} \boxtimes \mathcal{B}$, which is the semisimple category consisting of formal direct sums of objects of the form $X \boxtimes Y$, $X \in \mathcal{A}$, $Y \in \mathcal{B}$, such that direct sums in $\mathcal{A}$ and $\mathcal{B}$ distribute with respect to the symbol $\boxtimes$. Morphism spaces in $\mathcal{A} \boxtimes \mathcal{B}$ are tensor products of vector spaces of morphisms in $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are multifusion so is $\mathcal{A} \boxtimes \mathcal{B}$ with the tensor unit $\mathbb{1}_{\mathcal{A}} \boxtimes \mathbb{1}_{\mathcal{B}}$, see [EGNO, Cor. 4.6.2].

The simple summands of the tensor unit of a multifusion category can be used to decompose it (see [EGNO, Rem. 4.3.4]):

Proposition 2.2. Let $\mathcal{A}$ be a multifusion category, $\mathbb{1} \cong \bigoplus_{i \in I} \mathbb{1}_i$ be the decomposition of the tensor unit into simples and denote $\mathcal{A}_{ij} = \mathbb{1}_i \otimes \mathcal{A} \otimes \mathbb{1}_j$

(i) All $\mathbb{1}_i$ are mutually non-isomorphic and $\mathbb{1}_i \otimes \mathbb{1}_j \cong 0$ for $i \neq j$.

(ii) $\mathcal{A} \simeq \bigoplus_{i,j \in I} \mathcal{A}_{ij}$ as finitely semisimple $\Bbbk$-linear categories.

(iii) The tensor product is a direct sum of functors $\otimes : \mathcal{A}_{ij} \times \mathcal{A}_{kl} \to \mathcal{A}_{il}$ with the product being 0 if $j \neq k$.

The categories $\mathcal{A}_{ij}$ are called the *component categories* of $\mathcal{A}$. Note that each simple object of $\mathcal{A}$ must lie in one of the $\mathcal{A}_{ij}$. In particular, the diagonal component categories $\mathcal{A}_{ii}$ are fusion with tensor unit $\mathbb{1}_i$.

Recall that the Grothendieck ring $\mathrm{Gr}(\mathcal{A})$ of a multifusion category $\mathcal{A}$ is the commutative ring with $\mathbb{Z}$-basis given by the set $\mathrm{Irr}_{\mathcal{A}}$ together with the product

$$i \cdot j = \sum_{k \in \mathrm{Irr}_{\mathcal{A}}} N_{ij}^k \, k, \quad i, j \in \mathrm{Irr}_{\mathcal{A}} \ , \ N_{ij}^k := \dim \mathcal{A}(i \otimes j, k) \ . \tag{2.14}$$

If $\mathcal{A}$ is fusion, the Frobenius-Perron dimension is defined as the unique ring homomorphism $\mathrm{FPdim}\colon \mathrm{Gr}(\mathcal{A}) \to \mathbb{R}$, such that $\mathrm{FPdim}(i) > 0$ for all $i \in \mathrm{Irr}_{\mathcal{A}}$. The

Frobenius-Perron dimension of $\mathcal{A}$ is defined as

$$\mathrm{FPdim}(\mathcal{A}) := \sum_{i \in \mathrm{Irr}_{\mathcal{A}}} (\mathrm{FPdim}(i))^2 \,. \tag{2.15}$$

The following result then provides a convenient way to prove equivalence of two fusion categories, see [EGNO, Prop. 6.3.3].

Proposition 2.3. A functor $F\colon \mathcal{A} \to \mathcal{B}$ between fusion categories $\mathcal{A}$ and $\mathcal{B}$ is an equivalence iff it is fully faithful (i.e. isomorphism on morphism spaces) and $\mathrm{FPdim}(\mathcal{A}) = \mathrm{FPdim}(\mathcal{B})$.

Proposition 2.2(i) implies that for a multifusion category $\mathcal{A}$ one has $\mathrm{End}_{\mathcal{A}}(\mathbb{1}) \cong \Bbbk^{\oplus |I|}$. In particular, if $\mathcal{A}$ is in addition pivotal, the left/right traces of endomorphisms and the left/right categorical dimensions of objects are tuples of scalars in $\Bbbk$. The sphericality condition can be simplified as follows, see e.g. [TV, Lem. 4.4].[5]

Proposition 2.4. A pivotal multifusion category $\mathcal{A}$ is spherical iff $\dim_l i = \dim_r i$ for all $i \in \mathrm{Irr}_{\mathcal{A}}$.

For a spherical fusion category $\mathcal{S}$, the categorical dimensions are simply scalars in $\Bbbk$. One defines its *global dimension* to be the scalar

$$\mathrm{Dim}\,\mathcal{S} := \sum_{i \in \mathrm{Irr}_{\mathcal{S}}} (\dim i)^2 \,. \tag{2.16}$$

Note that the categorical and Frobenius-Perron dimensions are in general different notions: the former is an element of the field $\Bbbk$, whereas the latter is always real and non-negative.

2.3. Modular fusion categories

Let $\mathcal{B}$ be a braided multifusion category. The braiding provides equivalences $\mathcal{B}_{ij} \simeq \mathcal{B}_{ji}$ of the component categories, which together with Proposition 2.2(i) implies that the off-diagonal component categories are trivial. It follows that if $\mathcal{B}$ is indecomposable then it is fusion. We will focus on the case of braided fusion categories only.

We remark that it is not a priori clear, that the Drinfeld centre $\mathcal{Z}(\mathcal{A})$ of a multifusion category $\mathcal{A}$ is itself a multifusion category, since in general it need not be semisimple. Still, the decomposition in Proposition 2.2(ii) at least allows one to focus on the Drinfeld centres of fusion categories only (see e.g. [KZ1, Thm. 2.5.1]):

[5] In this reference the fusion case is considered, but the argument for the multifusion case is the same.

Proposition 2.5. If $\mathcal{A}$ is an indecomposable multifusion category, then $\mathcal{Z}(\mathcal{A})$ is a braided tensor category and is braided equivalent to $\mathcal{Z}(\mathcal{A}_{ii})$ for each i.

The following definition will be of central importance in the upcoming chapters:

Definition 2.6. Let $\mathcal{B}$ be a braided fusion category.

i) An object $T \in \mathcal{B}$ is called *transparent* if for all $X \in \mathcal{B}$ one has

$$c_{T,X} \circ c_{X,T} = \mathrm{id}_{X \otimes T} \ . \tag{2.17}$$

ii) $\mathcal{B}$ is called *non-degenerate* if each of its transparent objects is isomorphic to $\mathbb{1}^{\oplus n}$ for some $n \geq 0$.

iii) A *modular fusion category (MFC)* is a non-degenerate ribbon fusion category.

The most trivial example of a MFC is the category $\mathrm{Vect}_{\Bbbk}$ of finite dimensional vector spaces. Another class of examples is provided by the following (see [Mü2, Thm. 3.16], [ENO1, Thm. 2.3, Rem. 2.4]):

Proposition 2.7. Let $\mathcal{S}$ be a spherical fusion category. Then $\mathcal{Z}(\mathcal{S})$ is a MFC if and only if $\mathrm{Dim}\,\mathcal{S} \neq 0$. The condition $\mathrm{Dim}\,\mathcal{S} \neq 0$ always holds if $\mathrm{char}\,\Bbbk = 0$.

The following statement is convenient when looking for equivalences between non-degenerate (and therefore also modular) braided fusion categories (see [DMNO, Cor. 3.26]):

Proposition 2.8. Suppose $\mathrm{char}\,\Bbbk = 0$. Then a braided functor $F\colon \mathcal{C} \to \mathcal{D}$ between braided fusion categories $\mathcal{C}$ and $\mathcal{D}$, with $\mathcal{C}$ non-degenerate, is automatically fully-faithful.

For a MFC $\mathcal{C}$, let us define the following object/endomorphism pair:

$$C := \bigoplus_{i \in \mathrm{Irr}_{\mathcal{C}}} i \quad , \qquad d := \bigoplus_{i \in \mathrm{Irr}_{\mathcal{C}}} \dim i \cdot \mathrm{id}_i \quad (\in \mathrm{End}_{\mathcal{C}}\, C)\,, \tag{2.18}$$

along with the scalars:

$$p^+ := \sum_{i \in \mathrm{Irr}_{\mathcal{C}}} \theta_i \cdot (\dim i)^2 \quad , \qquad p^- := \sum_{i \in \mathrm{Irr}_{\mathcal{C}}} \theta_i^{-1} \cdot (\dim i)^2\,, \tag{2.19}$$

where here and below, for $i \in \mathrm{Irr}_{\mathcal{C}}$, $\theta_i \in \Bbbk^\times$ denotes the scalar which corresponds to the twist morphism of i under the usual identification $\mathrm{End}_{\mathcal{C}}\, i \cong \Bbbk$. When using graphical calculus for the morphisms of $\mathcal{C}$, we say that a closed strand carries the *Kirby colour*, in case it is coloured with the object C and has a single point insertion labelled by the morphism d.

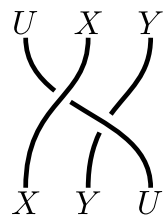
Proposition 2.9. Let $\mathcal{C}$ be a ribbon fusion category. Then the following are equivalent

i) $\mathcal{C}$ is a MFC;

ii) The ribbon functor

$$\mathcal{C} \boxtimes \widetilde{\mathcal{C}} \to \mathcal{Z}(\mathcal{C}), \qquad X \boxtimes Y \mapsto (X \otimes Y, \gamma^{\mathrm{dol}}_{X,Y}) \ , \tag{2.20}$$

where $\gamma^{\mathrm{dol}}_{X,Y}$ is the "dolphin" half-braiding defined for all $U \in \mathcal{C}$ by

$(\gamma^{\mathrm{dol}}_{X,Y})_U :=$ 

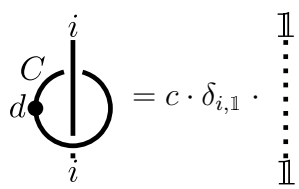
, (2.21)

is an equivalence;

iii) the s-matrix is invertible, where

$$s_{ij} := \mathrm{tr}_{\mathcal{C}} \left(\ c_{i,j} \circ c_{j,i} \ \right), \quad i,j \in \mathrm{Irr}_{\mathcal{C}}; \tag{2.22}$$

iv) for all $i \in \mathrm{Irr}_{\mathcal{C}}$ the following identity holds



$= c \cdot \delta_{i,\mathbb{1}} \cdot$

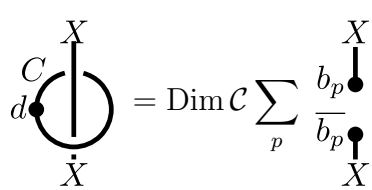
(2.23)

for some $c \neq 0$; moreover, in this case one necessarily has $c = \mathrm{Dim}\,\mathcal{C}$ (in particular, the global dimension of a MFC is automatically non-zero).

The equivalence of the statements i), ii) and iii) in the above proposition are shown e.g. in [EGNO, Prop. 8.20.12]. The two directions of the equivalence to the statement iv) can be found in [BakK, Cor. 3.1.11] and [KO, Lem. 4.6]

Proposition 2.9 implies the following properties of a MFC $\mathcal{C}$:

- the *scissors identity*: for an object $X \in \mathcal{C}$ one has



$= \mathrm{Dim}\,\mathcal{C} \sum_p$

(2.24)

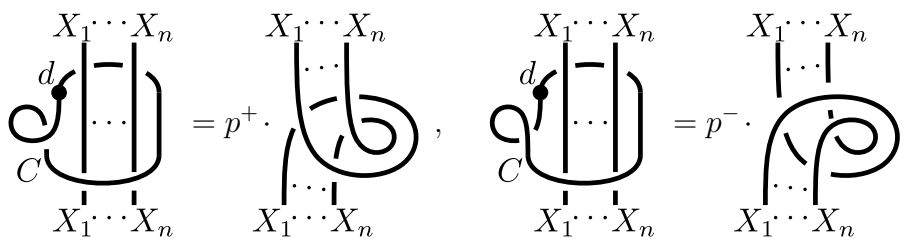
where b_p run through a basis of $\mathcal{C}(\mathbb{1}, X)$ and $\overline{b_p}$ is the dual basis of $\mathcal{C}(X, \mathbb{1})$ with respect to the composition pairing, i.e. $\overline{b_q} \circ b_p = \delta_{pq}\, \mathrm{id}_{\mathbb{1}} \in \mathrm{End}_{\mathcal{C}}\, \mathbb{1} \cong \Bbbk$.;

- one has the following relation (see [BakK, Cor. 3.1.11]):

$$p^+ p^- = \mathrm{Dim}\, \mathcal{C}\,; \tag{2.25}$$

  in particular, the scalars p^+, p^- are both non-zero;

- the following identities hold for any collection of objects $X_1, \dots, X_n \in \mathcal{C}$ (see [BakK, Lem. 3.1.5])



(2.26)

In the upcoming applications to TQFTs, a MFC $\mathcal{C}$ will always have an assigned choice for the square root

$$\mathcal{D} := \sqrt{\mathrm{Dim}\, \mathcal{C}}\,. \tag{2.27}$$

The scalar

$$\delta := \frac{p^+}{\mathcal{D}} = \frac{\mathcal{D}}{p^-} \tag{2.28}$$

is then called the *anomaly* of $\mathcal{C}$. In case we need to differentiate between several MFCs, we will add an index to the notation, e.g. $C_{\mathcal{C}}$, $d_{\mathcal{C}}$, $p^{\pm}_{\mathcal{C}}$, $\mathcal{D}_{\mathcal{C}}$, $\delta_{\mathcal{C}}$.

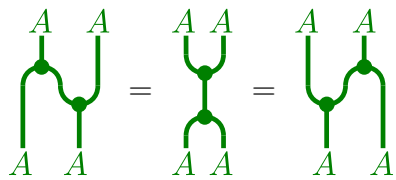
2.4. Algebras and modules

We assume some familiarity with algebras in a (multi)fusion category $\mathcal{C}$ and refer to [FRS1, FFRS] for more details. For an algebra $A \in \mathcal{C}$, we denote the categories of left and right modules by ${}_A\mathcal{C}$ and $\mathcal{C}_A$ respectively. For two algebras $A, B \in \mathcal{C}$, the category of A-B-bimodules is denoted by ${}_A\mathcal{C}_B$. We call an algebra $A \in \mathcal{C}$ *semisimple*, if the category of its left modules ${}_A\mathcal{C}$ (equivalently, that of the right modules $\mathcal{C}_A$ or bimodules ${}_A\mathcal{C}_A$, see [DMNO, Prop. 2.7]) is semisimple. An algebra A is called *haploid* (or *connected*) if one has $\dim \mathcal{C}(\mathbb{1}, A) = 1$. A haploid algebra A is necessarily a simple object in ${}_A\mathcal{C}$, $\mathcal{C}_A$ and ${}_A\mathcal{C}_A$, but the converse is not true (take for example the matrix algebra in $\mathrm{Vect}_{\Bbbk}$, which is simple as a bimodule over itself, but not haploid).

A *Frobenius algebra* in a multifusion category $\mathcal{C}$ is a tuple

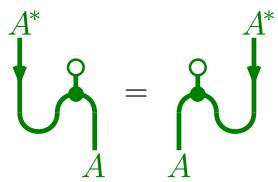
$$A \in \mathcal{C}\,, \quad \mu\colon A \otimes A \to A\,, \quad \eta\colon \mathbb{1} \to A\,, \quad \Delta\colon A \to A \otimes A\,, \quad \varepsilon\colon A \to \mathbb{1} \tag{2.29}$$

where (A, μ, η) is an associative unital algebra and (A, Δ, ε) is a coassociative counital coalgebra, such that with , , , denoting the multiplication, comultiplication, unit and counit morphisms, one has



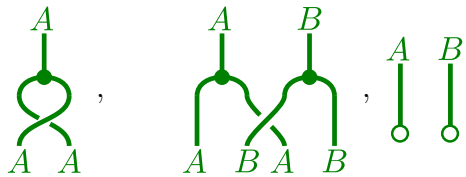
(2.30)

If $\mathcal{C}$ is pivotal, a Frobenius algebra A is in addition called *symmetric* if



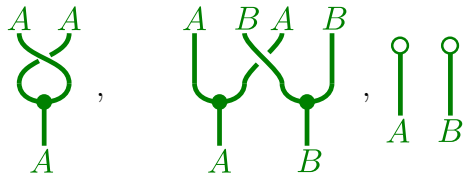
(2.31)

If $\mathcal{C}$ is braided, we define the *opposite* of an algebra A (to be denoted by A^{op}) and the *tensor product* of two algebras $A, B \in \mathcal{C}$ to be the algebras obtained from A and $A \otimes B$ by equipping them with the following multiplication (and, in the latter, case unit) morphisms:



(2.32)

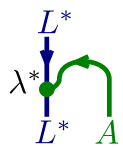
If A and B are Frobenius algebras, so are A^{op} and $A \otimes B$ upon equipping them with the comultiplications (and, in the latter case, counit)



(2.33)

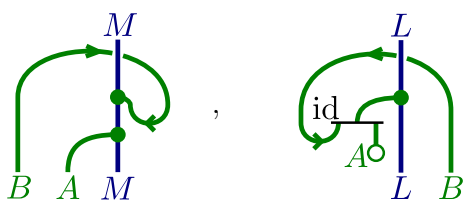
If A, B are in addition symmetric, so are A^{op} and $A \otimes B$.

For any algebra $A \in \mathcal{C}$, the dual L^* of a left module $L = (L, \lambda\colon A \otimes L \to L) \in {}_A\mathcal{C}$ is a right A-module with the action



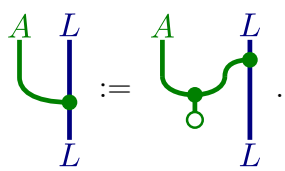
(2.34)

Similarly, the dual of a right module is a left module and consequently the bimodules are closed under taking duals. For two algebras $A, B \in \mathcal{C}$, the objects of ${}_A\mathcal{C}_B$ and ${}_{B^{\mathrm{op}} \otimes A}\mathcal{C}$ are in bijection with the left action of $B^{\mathrm{op}} \otimes A$ on $M \in {}_A\mathcal{C}_B$ and the right action B on $L \in {}_{B^{\mathrm{op}} \otimes A}\mathcal{C}$ given by:



(2.35)

A (left) module over a Frobenius algebra $A \in \mathcal{C}$ is a module L of the underlying algebra. If A is symmetric, it has a canonical (left-)comodule structure with the coaction

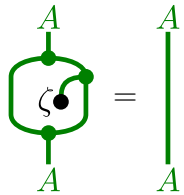


(2.36)

One easily generalises this to right modules and bimodules of Frobenius algebras.

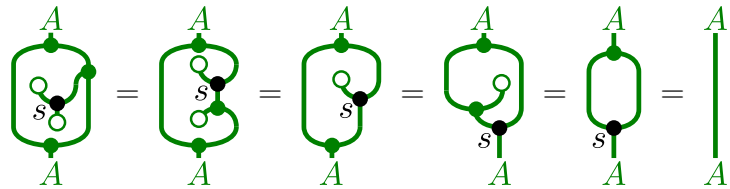
Recall that an algebra $A \in \mathcal{C}$ is called *separable* if the multiplication $\mu\colon A \otimes A \to A$ has a section in ${}_A\mathcal{C}_A$, i.e. an A-A-bimodule morphism $s\colon A \to A \otimes A$ such that $\mu \circ s = \mathrm{id}_A$. One has:

Proposition 2.10. A Frobenius algebra $A \in \mathcal{C}$ is separable if and only if there is a morphism $\zeta\colon \mathbb{1} \to A$ such that



(2.37)

Proof. A is obviously separable if such ζ exists. On the other hand, if A is separable with $s\colon A \to A \otimes A$ such that $\mu \circ s = \mathrm{id}_A$, take $\zeta := (\varepsilon \otimes \mathrm{id}_A) \circ s \circ \eta$. Then



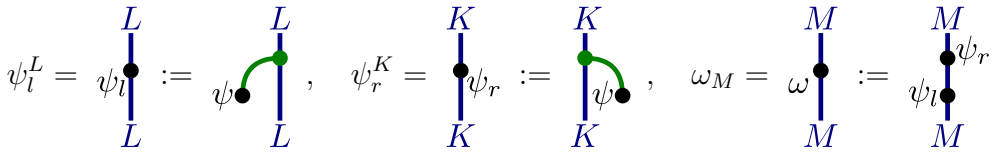
(2.38)

where one uses that s is an A-A-bimodule morphism and therefore commutes with (co)multiplication morphism of A. □

Convention 2.11. For later convenience, we will assume that a morphism $\zeta\colon \mathbb{1} \to A$ as in (2.37) is obtained as a multiplicative square of another morphism $\psi\colon \mathbb{1} \to A$, i.e. $\zeta = \mu \circ (\psi \otimes \psi)$. Moreover, we will always make an additional assumption that such ψ has a multiplicative inverse, i.e. a morphism $\psi^{-1}\colon \mathbb{1} \to A$ such that

$$\mu \circ (\psi \otimes \psi^{-1}) = \mu \circ (\psi^{-1} \otimes \psi) = \eta\,. \tag{2.39}$$

Slightly abusing the terminology, by *separable Frobenius algebra* we will mean a pair (A, ψ), i.e. the morphism ψ (and therefore also the section of μ) is fixed as a structure. If $\psi = \eta$, the section is given by the comultiplication $\Delta\colon A \to A \otimes A$. In this case we say that A is Δ-*separable*.
For a left A-module L, right A-module K and an A-A-bimodule M we introduce the following invertible endomorphisms



(2.40)

Note that ψ_l^L, ψ_r^K, ω_M are not in general (left-, right-, bi-)module morphisms, as ψ is not in general central. However they do commute with (left-, right- bi-)module morphisms under composition.

We will mostly work with symmetric separable Frobenius algebras $A = (A, \psi)$. In graphical calculus, morphisms, consisting of compositions of (co)unit and (co)multiplication as well as (co)evaluation maps of A can be interpreted as surfaces, consisting of ribbon strands which branch out at the vertices. Together, (co)unitality, (co)associativity, the Frobenius property (2.30) and the symmetry property (2.31) allow one to freely deform such a surface. The strands need not even be directed, as the symmetry property provides one with a canonical isomorphism $A \cong A^*$.

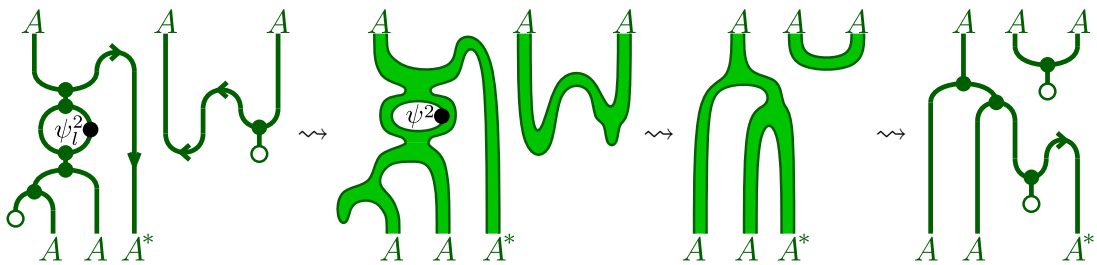
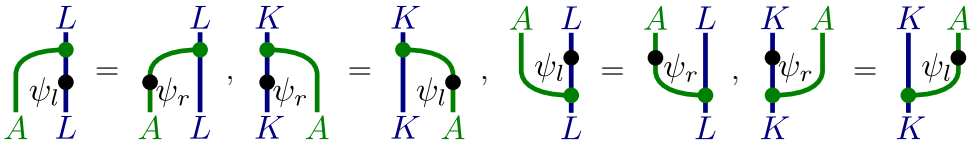


Figure 2.2: In graphical calculus, morphisms composed of (co)unit, (co)multiplication and (co)evaluation morphisms of a symmetric separable Frobenius algebra A can be changed by replacing the corresponding graph by a thin surface and deforming it. The separability property (2.37) allows one to omit holes in the surface as indicated by the second step in the figure. The symmetry property is used to canonically identify A with its dual A^*.

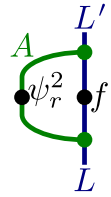
The holes in such surfaces can also be omitted if there is a ψ^2-insertion on the boundary of it, see Figure 2.2. In graphical calculus, the modules of A serve as boundary lines for such surfaces. The ψ-insertions are compatible with action and coaction on modules $L \in {}_A\mathcal{C}$ and $K \in \mathcal{C}_A$, in the sense that the following identities hold



. (2.41)

Working with separable Frobenius algebras is not much different from working with Δ-separable ones. For example, having a separable Frobenius algebra $A = (A, \psi)$ and two left modules $L, L' \in {}_A\mathcal{C}$, the map

$$\mathcal{C}(L, L') \to {}_A\mathcal{C}(L, L')\,, \quad f \mapsto$$



(2.42)

is an idempotent projecting onto the subspace ${}_A\mathcal{C}(L, L') \subseteq \mathcal{C}(L, L')$. Analogous maps can also be defined to project onto the spaces of right- and bimodule morphisms.

Separable algebras are known to be semisimple ([DMNO, Prop. 2.7]). For a symmetric separable Frobenius algebra $A = (A, \psi)$ this can be conveniently shown

using adjunctions: As explained in Appendix A, for any pair of $\Bbbk$-linear categories $\mathcal{A}$, $\mathcal{B}$, a biadjunction[6] between functors $X : \mathcal{A} \to \mathcal{B}$, $Y : \mathcal{B} \to \mathcal{A}$ is called *separable*, if the natural transformation $\varepsilon \circ \widetilde{\eta} : \mathrm{Id}_{\mathcal{B}} \Rightarrow \mathrm{Id}_{\mathcal{B}}$ is invertible (here $\widetilde{\eta} : \mathrm{Id}_{\mathcal{B}} \Rightarrow XY$ is the unit of the adjunction $Y \dashv X$ and $\varepsilon : XY \Rightarrow \mathrm{Id}_{\mathcal{B}}$ is the counit of the adjunction $X \dashv Y$). Suppose now that the category $\mathcal{A}$ is finitely semisimple, $\mathcal{B}$ is idempotent complete, and that there exists a separable biadjunction between $\mathcal{A}$ and $\mathcal{B}$. Then it is shown in Proposition A.3 that $\mathcal{B}$ is finitely semisimple as well. We apply this for $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = {}_A\mathcal{C}$, $\mathcal{C}_A$ or ${}_A\mathcal{C}_A$. To this end, let

$$\begin{array}{rcl} \mathrm{Ind}^l_A : \mathcal{C} & \to & {}_A\mathcal{C} \\ X & \mapsto & A \otimes X \end{array}, \quad \begin{array}{rcl} \mathrm{Ind}^r_A : \mathcal{C} & \to & \mathcal{C}_A \\ X & \mapsto & X \otimes A \end{array}, \quad \begin{array}{rcl} \mathrm{Ind}_{AA} : \mathcal{C} & \to & {}_A\mathcal{C}_A \\ X & \mapsto & A \otimes X \otimes A \end{array} \tag{2.43}$$

be the induced module functors and denote by U^l_A, U^r_A, U_{AA} the corresponding forgetful functors. One has

Proposition 2.12. $(\mathrm{Ind}^l_A, U^l_A)$, $(\mathrm{Ind}^r_A, U^r_A)$, $(\mathrm{Ind}_{AA}, U_{AA})$ are pairs of biadjoint functors and in each case the biadjunction is separable.

Proof. We only show the argument for the pair $(\mathrm{Ind}^l_A, U^l_A)$. Recall that a datum for the two adjunctions $\mathrm{Ind}^l_A \dashv U^l_A$ and $U^l_A \dashv \mathrm{Ind}^l_A$ can be given by the two pairs of unit/counit natural transformations

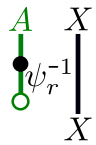
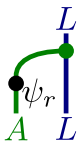
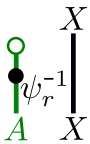
$$\begin{aligned} &\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow U^l_A \circ \mathrm{Ind}^l_A\,, \quad && \varepsilon\colon \mathrm{Ind}^l_A \circ U^A_l \Rightarrow \mathrm{Id}_{{}_A\mathcal{C}}\,, \\ &\widetilde{\eta}\colon \mathrm{Id}_{{}_A\mathcal{C}} \Rightarrow \mathrm{Ind}^l_A \circ U^l_A\,, \quad && \widetilde{\varepsilon}\colon U^l_A \circ \mathrm{Ind}^l_A \Rightarrow \mathrm{Id}_{\mathcal{C}}\,, \end{aligned} \tag{2.44}$$

such that for all $X \in \mathcal{C}$ and $L \in {}_A\mathcal{C}$ one has

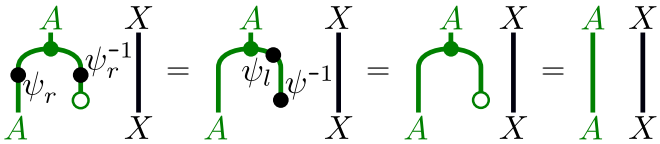
$$\varepsilon_{\mathrm{Ind}^l_A X} \circ \mathrm{Ind}^l_A(\eta_X) = \mathrm{id}_{\mathrm{Ind}^l_A X}\,, \qquad U(\varepsilon_L) \circ \eta_{UL} = \mathrm{id}_{UL}\,, \tag{2.45}$$

$$\widetilde{\varepsilon}_{UL} \circ U(\widetilde{\eta}_L) = \mathrm{id}_{UL}\,, \qquad \mathrm{Ind}^l_A(\widetilde{\varepsilon}_X) \circ \widetilde{\eta}_{\mathrm{Ind}^l_A X} = \mathrm{id}_{\mathrm{Ind}^l_A X}\,. \tag{2.46}$$

We set

$\eta_X =$  , $\varepsilon_L :=$  , $\widetilde{\eta}_L =$, $\widetilde{\varepsilon}_X :=$  . (2.47)

The conditions (2.45) and (2.46) are then clear, for example one has

$\varepsilon_{\mathrm{Ind}^l_A X} \circ \mathrm{Ind}^l_A(\eta_X) =$  $= \mathrm{id}_{\mathrm{Ind}^l_A X}$. (2.48)

[6] In this work, the term *biadjunction* means a pair of functors, which are both left and right adjoints of each other, not to be confused with a notion of adjunction on bicategories.

Finally, the separability condition $\varepsilon_L \circ \widetilde{\eta}_L = \mathrm{id}_L$ follows from the identity (2.37) (with $\zeta = \psi^2$, see Convention 2.11) and the definition of the coaction (2.36). $\square$

It is easy to check that the categories ${}_A\mathcal{C}$, $\mathcal{C}_A$ and ${}_A\mathcal{C}_A$ are idempotent complete, which yields:

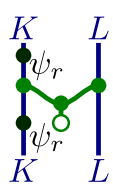
Corollary 2.13. The categories ${}_A\mathcal{C}$, $\mathcal{C}_A$ and ${}_A\mathcal{C}_A$ are finitely semisimple.

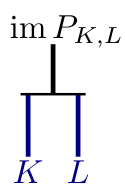
2.5. Relative tensor products

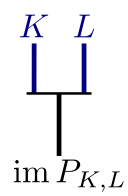
For an algebra A in a (multi)fusion category $\mathcal{C}$, the relative tensor product of a right A-module $K = (K, \rho)$ and a left A-module $L = (L, \lambda)$ is defined as the difference cokernel [EGNO, Def. 7.8.21]

$$K \otimes_A L := \operatorname{coker}[K \otimes A \otimes L \xrightarrow{\rho \otimes \mathrm{id}_L - \mathrm{id}_K \otimes \lambda} K \otimes L]\,. \tag{2.49}$$

In case $A = (A, \psi)$ is a separable Frobenius algebra in a multifusion category $\mathcal{C}$, the relative tensor product can be computed as follows: Recall that the image of an idempotent $p \in \mathrm{End}_{\mathcal{C}} X$, $p \circ p = p$, is an object $\operatorname{im} p \in \mathcal{C}$ together with projection and inclusion morphisms $\pi \colon X \leftrightarrow \operatorname{im} p : \imath$ such that $\imath \circ \pi = p$ and $\pi \circ \imath = \mathrm{id}_{\operatorname{im} p}$. For K, L as above, consider the following idempotent and the corresponding projection/inclusion:

$P_{K,L} :=$ 

, $\pi =$ 

, $\imath =$ 

. (2.50)

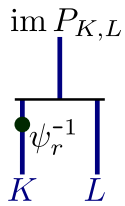
Proposition 2.14. Let $A = (A, \psi)$ be a separable Frobenius algebra, $K \in \mathcal{C}_A$ and $L \in {}_A\mathcal{C}$. Then $K \otimes_A L \cong \operatorname{im} P_{K,L}$.

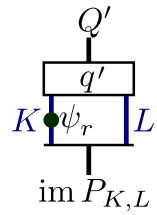
Proof. The diagram

$$\begin{tikzcd} K \otimes A \otimes L \arrow[r, "\rho \otimes \mathrm{id}_L - \mathrm{id}_K \otimes \lambda"] \arrow[dr, "0"'] & K \otimes L \arrow[r, "q"] \arrow[d, "q'"] & \operatorname{im} P_{K,L} \arrow[dl, "u"] \\ & Q' & \end{tikzcd} \tag{2.51}$$

commutes for an arbitrary object $Q' \in \mathcal{C}$ and a morphism $q' \colon K \otimes L \to Q'$ such

that $q' \circ (\rho \otimes \mathrm{id}_L - \mathrm{id}_K \otimes \lambda) = 0$, where one takes

$q =$ 

, $u =$ 

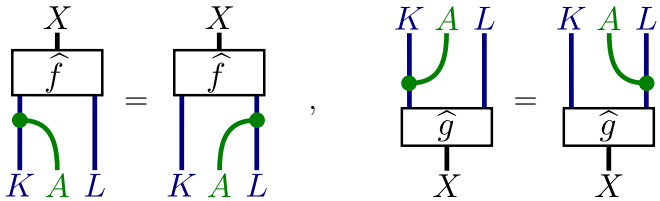
. (2.52)

Using the properties of π, $\imath$, q' and the separability of A, one can also check that u is unique. The pair $(\mathrm{im}\, P_{K,L}, q)$ is therefore a cokernel. □

We will use the proof of Proposition 2.14 to define morphisms involving the relative tensor product $K \otimes_A L$, as morphisms involving the regular tensor product $K \otimes L$ in $\mathcal{C}$ which satisfy some conditions. More explicitly, for arbitrary objects $X \in \mathcal{C}$, $K \in \mathcal{C}_A$, $L \in {}_A\mathcal{C}$ define the subspaces of morphisms

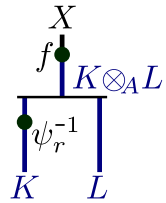
$$\mathcal{C}^X_{K,A,L} \subseteq \mathcal{C}(K \otimes L, X) \quad , \qquad \mathcal{C}^{K,A,L}_X \subseteq \mathcal{C}(X, K \otimes L) \tag{2.53}$$

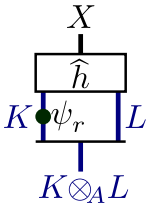
where $\widehat{f} \in \mathcal{C}^X_{K,A,L}$ and $\widehat{g} \in \mathcal{C}^{K,A,L}_X$ satisfy



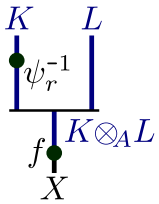
. (2.54)

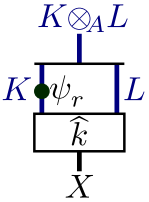
We say that the elements of $\mathcal{C}^X_{K,A,L}$ and $\mathcal{C}^{K,A,L}_X$ commute with the A-actions. One has the bijections

$\mathcal{C}(K \otimes_A L, X) \leftrightarrow \mathcal{C}^X_{K,A,L}$, $f \mapsto \widehat{f} :=$ 

, $\widehat{h} \mapsto h :=$ 

,

$\mathcal{C}(X, K \otimes_A L) \leftrightarrow \mathcal{C}^{K,A,L}_X$, $g \mapsto \widehat{g} :=$ 

, $\widehat{k} \mapsto k :=$ 

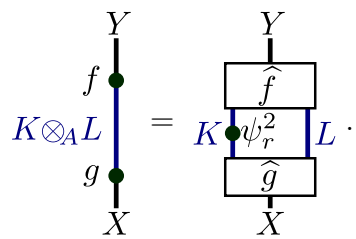
. (2.55)

Note that the ψ-insertions in (2.55) cannot be eliminated by choosing other possible conventions for the idempotent P_{KL} in (2.50), for example altering its definition so that both of the ψ_r-insertions are at the bottom (resp. top) would change the expressions of $\widehat{f}$ (resp. $\widehat{g}$) to have two ψ_r^{-1}-insertions.

Remark 2.15. For objects $X, Y \in \mathcal{C}$, the composition

$$\mathcal{C}(X, K \otimes_A L) \times \mathcal{C}(K \otimes_A L, Y) \to \mathcal{C}(X, Y) \tag{2.56}$$

is preserved by the bijection (2.55) only up to additional ψ-insertions:

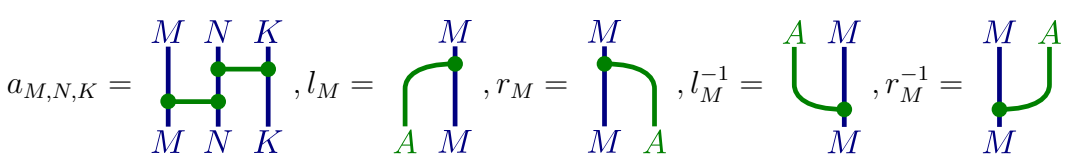


(2.57)

The composition is preserved properly if A is Δ-separable.

Convention 2.16. Later we will often encounter objects X defined as images of idempotents similar to the one in (2.50) (possibly involving several actions of different algebras). The corresponding projection/inclusion morphisms will also be denoted by horizontal lines. We will assume it to be clear how to generalise the bijections (2.55) and we will occasionally use them to define morphisms having a tensor factor X in the domain or codomain. By abuse of notation, we will drop the overhats as in (2.53) and will refer to the Remark 2.15 to account for the possible appearance of ψ-insertions upon composing.

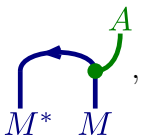
For any algebra $A \in \mathcal{C}$, the tensor product (2.49) provides the category of bimodules ${}_A\mathcal{C}_A$ with a monoidal structure with unit $\mathbb{1}_{{}_A\mathcal{C}_A} := A$. For a symmetric separable Frobenius algebra $A = (A, \psi)$ we can instead use Proposition 2.14, as the morphisms in (2.50) for $M, N \in {}_A\mathcal{C}_A$ are A-A-bimodule morphisms. The associator for $M, N, K \in {}_A\mathcal{C}_A$, as well as the left/right unitors of $M \in {}_A\mathcal{C}_A$ and their inverses are given by

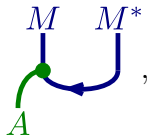


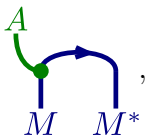
(2.58)

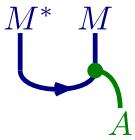
(or rather obtained by mapping them to morphisms $(M \otimes_A N) \otimes_A K \to M \otimes_A (N \otimes_A K)$, $A \otimes_A M \to M$, etc. adapting the bijection (2.55)).

If $\mathcal{C}$ is pivotal and A symmetric, ${}_A\mathcal{C}_A$ inherits from $\mathcal{C}$ a natural pivotal structure where the dual of $M \in {}_A\mathcal{C}_A$ is M^* (with the two A-actions given similarly as in (2.34)) and the (co)evaluation morphisms given by

$\mathrm{ev}_M =$ 

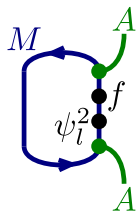
, $\mathrm{coev}_M =$ 

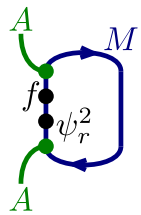
, $\widetilde{\mathrm{ev}}_M =$ 

, $\widetilde{\mathrm{coev}}_M =$ 

. (2.59)

Note that because of Remark 2.15, the left and right traces of $f \in \mathrm{End}_{{}_A\mathcal{C}_A} M$ have additional ψ^2-insertions, for example

$\mathrm{tr}_l\, f =$ 

, $\mathrm{tr}_r\, f =$ 

. (2.60)

Note also that ${}_A\mathcal{C}_A$ need not be spherical even if $\mathcal{C}$ is (however if A is in addition haploid, ${}_A\mathcal{C}_A$ is automatically spherical [Mü1, Thm. 5.12]).

Proposition 2.17. Let $\mu \in {}_A\mathcal{C}_A$ be a simple A-A-bimodule. Then the morphisms $\dim_l \mu, \dim_r \mu \in \mathrm{End}_{{}_A\mathcal{C}_A}(A)$ are non-zero.

Proof. Since A is the monoidal unit in the pivotal category ${}_A\mathcal{C}_A$ and μ is simple, one has for example $\dim {}_A\mathcal{C}_A(A, \mu^* \otimes_A \mu) = \dim {}_A\mathcal{C}_A(\mu, \mu) = \dim {}_A\mathcal{C}_A(\mu^* \otimes_A \mu, A) = 1$. Since the morphisms $\widetilde{\mathrm{coev}}_\mu$, ev_μ are non-zero, so is their composition $\dim_l \mu$. The argument for $\dim_l \mu$ is then the same. □

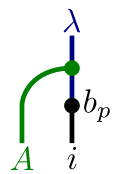
Applying Proposition 2.17 for $A = \mathbb{1}_\mathcal{C}$, one gets the following ([BakK, Lem. 2.4.1])

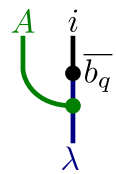
Corollary 2.18. Let $\mathcal{C}$ be a pivotal fusion category. Then for a simple object $i \in \mathrm{Irr}_\mathcal{C}$ the left/right categorical dimensions $\dim_l i / \dim_r i$ are non-zero.

Proposition 2.19. Let (A, ψ) be a symmetric separable Frobenius algebra in a pivotal fusion category $\mathcal{C}$ and $\lambda \in {}_A\mathcal{C}$, $\kappa \in \mathcal{C}_A$, $\mu \in {}_A\mathcal{C}_A$ simple objects. Then $\mathrm{tr}_\mathcal{C}(\psi_l^\lambda)^2$, $\mathrm{tr}_\mathcal{C}(\psi_r^\kappa)^2$, $\mathrm{tr}_\mathcal{C}\, \omega_\mu^2$ are non-zero.

Proof. We show this for the left module λ only, the proofs for simple right and bimodules are similar.

For a simple object $i \in \mathrm{Irr}_{\mathcal{C}}$, fix a basis $\{b_p\}$ of the space $\mathcal{C}(i,\lambda)$ along with the dual basis $\{\overline{b_p}\}$ of $\mathcal{C}(\lambda,i)$ with respect to the composition pairing, i.e. $\overline{b_q} \circ b_p = \delta_{pq} \cdot \mathrm{id}_i$. Define the following bimodule morphisms

$\beta_p :=$ 

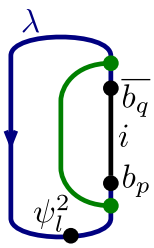
, $\overline{\beta_q} :=$ 

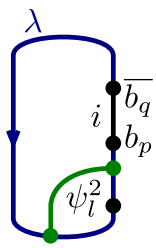
(2.61)

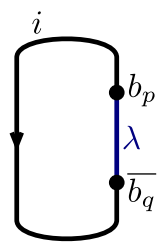
Since λ is simple, there are scalars $X_{pq}^{i,\lambda} \in \Bbbk$, such that

$$\beta_p \circ \overline{\beta_q} = X_{pq}^{i,\lambda} \cdot \mathrm{id}_\lambda \ . \tag{2.62}$$

Precomposing both sides of (2.62) with $(\psi_l^\lambda)^2$ and taking the trace in $\mathcal{C}$ yields:

$X_{pq}^{i,\lambda} \cdot \mathrm{tr}_{\mathcal{C}}(\psi_l^\lambda)^2 =$ 

$=$ 

$=$ 

$= \dim_{\mathcal{C}} i \cdot \delta_{pq}$. (2.63)

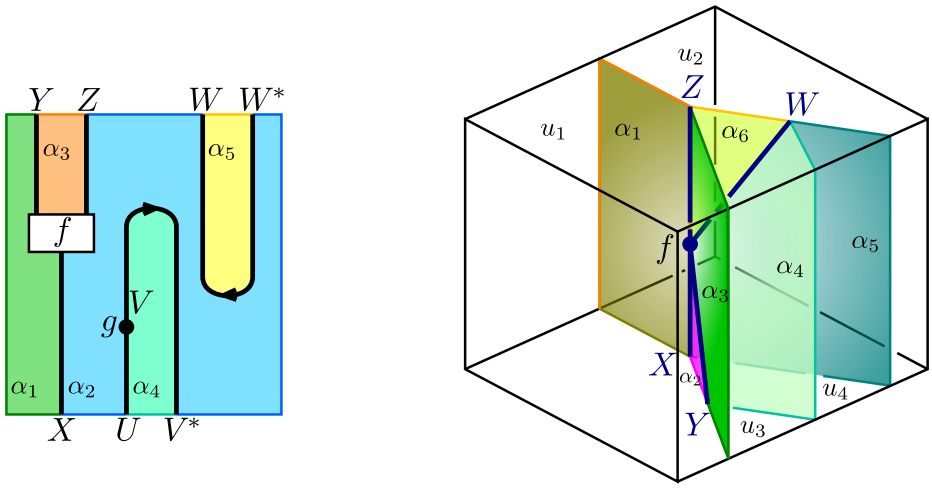
Since for $p = q$ and i such that $\mathcal{C}(i,\lambda) \neq \{0\}$ the right hand side is non-zero, one gets $\mathrm{tr}_{\mathcal{C}}(\psi_l^\lambda)^2 \neq 0$. □

2.6. Bi- and tricategories

Recall that a *bicategory* $\mathcal{B}$ consists of

- a collection of *objects*, which we will also call $\mathcal{B}$ by abuse of notation,
- categories $\mathcal{B}(\alpha,\beta)$ for each pair of objects $\alpha,\beta \in \mathcal{B}$, whose objects are called 1-*morphisms* morphisms are in turn called 2-*morphisms* of $\mathcal{B}$,
- *composition functors* $\otimes\colon \mathcal{B}(\beta,\gamma) \times \mathcal{B}(\alpha,\beta) \to \mathcal{B}(\alpha,\gamma)$ for all $\alpha,\beta,\gamma \in \mathcal{B}$, together with associativity isomorphisms,
- a *unit* for each $\mathcal{B}(\alpha,\alpha)$, together with unit isomorphisms for the composition.

For more details and for the axioms these data have to satisfy, see e.g. [Le]. Bicategories have a version of graphical calculus, in which strands and coupons are labelled with 1- and 2-morphisms while the 2-dimensional patches between them



(a) (b)

Figure 2.3: Examples of graphical calculus for (a) a pivotal bicategory $\mathcal{B}$ (the picture depicts a 2-morphism involving objects $\alpha_1, \ldots, \alpha_5 \in \mathcal{B}$, 1-morphisms X, Y, Z, U, V, W and 2-morphisms f, and g between them) (b) a Gray category $\mathcal{T}$ (the picture depicts a 3-morphism involving objects $u_1, \ldots, u_4 \in \mathcal{T}$, 2-morphisms $\alpha_1, \ldots, \alpha_6$, 1-morphisms X, Y, Z, W and a 3-morphism f between them).

are labelled with the corresponding objects. In particular, the datum of a bicategory with one object is equivalent to a monoidal category. One can also define structures on bicategories which allow one to make their graphical calculus richer, for example in a *pivotal bicategory* each 1-morphism $X \in \mathcal{B}(\alpha, \beta)$ has a two-sided dual (or biadjoint) $X^* \in \mathcal{B}(\beta, \alpha)$ together with (co)evaluation 2-morphisms satisfying analogous conditions to those in (2.8), see [KSt, §2], [Ca, Sec. 2.2]. The graphical calculus then allows one to introduce orientations to the lines, see Figure 2.3a.

A functor between bicategories $\mathcal{B}$, $\mathcal{B}'$ is a mapping between objects $F\colon \mathcal{B} \to \mathcal{B}'$ together with a collection of functors $\{\mathcal{B}(A, B) \to \mathcal{B}'(F\alpha, F\beta)\}_{\alpha,\beta\in\mathcal{B}}$ equipped with natural transformations, analogous to the monoidal structure on a functor between two monoidal categories. As with 1-categories, we say that a functor $F\colon \mathcal{B} \to \mathcal{B}'$ is an *equivalence* if there is a functor in the opposite direction such that the two compositions are isomorphic to identity functors. Again analogous to 1-categories, F is an equivalence iff the corresponding functors on the categories of 1-morphisms are equivalences (i.e. F is fully faithful) and each object of $\mathcal{B}'$ has an invertible 1-morphism to an object in the image of F (i.e. F is essentially surjective), cf. [Le].

A *tricategory* is a "3-dimensional" analogue of a bicategory. A *Gray category*

is a strictified notion of a tricategory, i.e. with some coherence data eliminated (a full strictification of tricategories is however not possible in general, see [GPS]). A Gray category consists of a collection of objects $\mathcal{T}$, for each pair of objects $u, v \in \mathcal{T}$ a (strict) bicategory $\mathcal{T}(u, v)$, as well as more data and requirements related to the composition of 1- and 2-morphisms. For more details, see e.g. [CMS, Sec. 3.1.2]. Gray categories have a version of graphical calculus, in which 3-morphisms are depicted as cubes, subdivided by planes, lines and points, labelled by 1-, 2- and 3-morphisms respectively, see Figure 2.3b for an example. In a *Gray category with duals*, one adds orientations to these components, as well as structure allowing one to bend the planes and lines in the graphical calculus, much like one can bend the lines in pivotal categories and bicategories, see [CMS, Sec. 3.2.2].

2.7. Module categories

Let $\mathcal{A}$ be a multifusion category. By a (left) $\mathcal{A}$-module category we will mean a finitely semisimple $\Bbbk$-linear category $\mathcal{M}$ together with a $\Bbbk$-bilinear action $\triangleright\colon \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ and natural isomorphisms $(- \otimes -) \triangleright - \Rightarrow - \triangleright (- \triangleright -)$ and $\mathbb{1}_{\mathcal{C}} \triangleright - \Rightarrow \mathrm{Id}_{\mathcal{M}}$ satisfying the usual pentagon and triangle identities. An $\mathcal{A}$-module functor between $\mathcal{A}$-module categories $\mathcal{M}$ and $\mathcal{N}$ is a $\Bbbk$-linear functor $F\colon \mathcal{M} \to \mathcal{N}$ equipped with natural isomorphism $F(- \triangleright -) \Rightarrow - \triangleright F(-)$ satisfying compatibility conditions. Natural transformations of $\mathcal{A}$-module functors $F, G\colon \mathcal{M} \to \mathcal{N}$ are required to commute with the structure morphisms of F and G. A module category is called indecomposable if it is not equivalent (as a module category) to a direct sum of two non-trivial module categories. Detailed definitions can be found in [EGNO].[7]

Let $A \in \mathcal{A}$ be a semisimple algebra. Then the category $\mathcal{A}_A$ of *right* A-modules is a *left* $\mathcal{A}$-module category with the action $X \triangleright (M, \rho) := (X \otimes M, \widetilde{\rho})$, where the right A-action is given by $\widetilde{\rho} = \big[(X \otimes M) \otimes A \xrightarrow{\sim} X \otimes (M \otimes A) \xrightarrow{\mathrm{id}_X \otimes \rho} X \otimes M\big]$. Two algebras are said to be *Morita equivalent* if their respective categories of right modules are equivalent as $\mathcal{A}$-module categories. One has by [Os], and e.g. [EGNO, Cor. 7.10.5.(i)]:

Proposition 2.20. Let $\mathcal{M}$ be an $\mathcal{A}$-module category. Then there exists a semisimple algebra $A \in \mathcal{A}$ such that $\mathcal{M} \simeq \mathcal{A}_A$ as module categories.

Let $\mathcal{A}$-Mod be the bicategory of $\mathcal{A}$-module categories, functors and natural transformations and $\mathrm{Alg}_{\mathcal{A}}$ be the bicategory of semisimple algebras in $\mathcal{A}$, their bimodules and bimodule morphisms. Building on Proposition 2.20 one gets (cf. [EGNO, Prop. 7.11.1, Thm. 7.10.1]):

[7] We stress again that we assume all module categories to be *semisimple*. This is only a special case of the exact module categories considered in [EGNO] but sufficient for our purposes.

Proposition 2.21. Let $A, B \in \mathcal{A}$ be semisimple algebras, $M, N \in \mathcal{A}$ be A-B-bimodules and $[f\colon M \to N]$ a bimodule morphism. The functor $\mathrm{Alg}_{\mathcal{A}} \to \mathcal{A}\text{-Mod}$ defined on

$$\begin{array}{lcl} \text{objects:} & A & \mapsto & \mathcal{A}_A \\ \text{1-morphisms:} & M & \mapsto & [- \otimes_A M\colon \mathcal{A}_A \to \mathcal{A}_B] \\ \text{2-morphisms:} & f & \mapsto & \{\mathrm{id}_L \otimes_A f\}_{L \in \mathcal{A}_A} \end{array} \tag{2.64}$$

is an equivalence of bicategories.

The following is a convenient criterion to compare the module categories of two fusion categories $\mathcal{A}$ and $\mathcal{B}$, see [Mü1] and [ENO2, Thm. 3.1]:

Proposition 2.22. The bicategories $\mathcal{A}$-Mod and $\mathcal{B}$-Mod are equivalent if and only if $\mathcal{Z}(\mathcal{A}) \simeq \mathcal{Z}(\mathcal{B})$ as braided tensor categories.

Let $\mathcal{C}$ be a pivotal fusion category and let $A \in \mathcal{C}$ be a symmetric separable Frobenius algebra. The question of what extra structure the $\mathcal{C}$-module category $\mathcal{C}_A$ has in this situation was addressed in [Schm], where the following notion was introduced:

Definition 2.23. A *module trace* on a module category $\mathcal{M}$ of a pivotal fusion category $\mathcal{C}$ is a collection of linear maps $\Theta_M\colon \mathrm{End}_{\mathcal{M}} M \to \Bbbk$, $M \in \mathcal{M}$, such that for all $X \in \mathcal{C}$ and $M, N \in \mathcal{M}$:

i) One has
$$\Theta_M(g \circ f) = \Theta_N(f \circ g) \tag{2.65}$$
for all $f \in \mathcal{M}(M, N)$ and $g \in \mathcal{M}(N, M)$.

ii) The following pairing is non-degenerate:
$$\omega_{M,N}\colon \mathcal{M}(M, N) \otimes_{\Bbbk} \mathcal{M}(N, M) \to \Bbbk, \quad f \otimes_{\Bbbk} g \mapsto \Theta_M(g \circ f)\,. \tag{2.66}$$

iii) For $f \in \mathrm{End}_{\mathcal{M}}(X \triangleright M)$ let $\overline{f}\colon M \to M$ be given by
$$\overline{f} := (\mathrm{ev}_X \triangleright \mathrm{id}_M) \circ (\mathrm{id}_{X^*} \triangleright f) \circ (\widetilde{\mathrm{coev}}_X \triangleright \mathrm{id}_M)$$
(we have omitted coherence isomorphisms for readability). Then
$$\Theta_{X \triangleright M}(f) = \Theta_M(\overline{f}). \tag{2.67}$$

Module traces satisfy the following uniqueness property [Schm, Prop. 4.4]:

Proposition 2.24. Suppose $\mathrm{char}\,\Bbbk = 0$. Then if an indecomposable module category $\mathcal{M}$ has a module trace Θ, then any other module trace Θ' will be proportional to Θ, i.e. $\Theta' = z \cdot \Theta$, $z \in \Bbbk^\times$.

For a symmetric separable Frobenius algebra $A = (A, \psi)$ in a spherical fusion category $\mathcal{C}$, the module category $\mathcal{C}_A$ has a module trace

$$\Theta_M(f) := \mathrm{tr}_r(f \circ \psi_r^2)\,, \qquad M \in \mathcal{C}_A\,, \quad f \in \mathrm{End}_{\mathcal{C}_A}(M)\,. \tag{2.68}$$

Indeed, the cyclicity condition (2.65) is implied since ψ-insertions commute with module morphisms because of the definitions (2.40), the non-degeneracy of the pairing (2.66) is implied by Proposition 2.19 and the partial trace condition (2.67) follows since $\mathcal{C}$ is spherical. Combining Proposition 2.20 with [Schm, Sec. 6], one has:

Proposition 2.25. Let $\mathcal{C}$ be a spherical fusion category and $\mathcal{M}$ a $\mathcal{C}$-module category with module trace. Then there exists a symmetric separable Frobenius algebra $A \in \mathcal{C}$, such that $\mathcal{M} \simeq \mathcal{C}_A$ as module categories.

Definition 2.26. We let $\mathcal{C}\text{-Mod}^{\mathrm{tr}}$ be the bicategory of $\mathcal{C}$-module categories with module trace, module functors and natural transformations and $\mathrm{FrobAlg}_{\mathcal{C}}^{\mathrm{ssep}}$ be the bicategory of symmetric separable Frobenius algebras in $\mathcal{C}$, their bimodules and bimodule morphisms.

Note that in the definition of $\mathcal{C}\text{-Mod}^{\mathrm{tr}}$ we do not require the module functors to be compatible with module traces (so-called *isometric functors*, see [Schm, Def. 3.10]). By Propositions 2.21 and 2.25 we get

Proposition 2.27. The functor $\mathrm{FrobAlg}_{\mathcal{C}}^{\mathrm{ssep}} \to \mathcal{C}\text{-Mod}^{\mathrm{tr}}$ defined analogously as in (2.64) is an equivalence of bicategories.

3. Reshetikhin-Turaev graph TQFT

A modular fusion category $\mathcal{C}$ yields a *3-dimensional graph TQFT*

$$Z_{\mathcal{C}}^{\mathrm{RT}} \colon \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}) \to \mathrm{Vect}_{\Bbbk} \tag{3.1}$$

via the Reshetikhin-Turaev (RT) construction. Its source category is the (centrally extended) category of 3-dimensional bordisms with embedded $\mathcal{C}$-coloured ribbon graphs (hence the term *graph TQFT*). The purpose of the ribbon graphs is to mimic networks of Wilson line operators; an object of $\mathcal{C}$ is interpreted as the datum assigned to a Wilson line (by analogy with Chern-Simons theory, where Wilson lines are labelled by representations of the gauge group).

In this chapter we review the construction and some properties of the Reshetikhin-Turaev graph TQFT. It is one of the most important prerequisites to understand the material presented afterwards.

3.1. Bordisms with embedded ribbon graphs

In this section we review the source category of the Reshetikhin-Turaev TQFT. For our purposes it is more convenient to work with smooth rather than topological manifolds, even though originally the latter were used for this construction, see [Tu, Sec. IV]. We assume it to be clear how to make this transition rigorously and accordingly skip most of the technicalities (the smooth setting is also used in [BakK, Ch. 4]).

Recall that a smooth *3-dimensional bordism* $M \colon \Sigma_- \to \Sigma_+$ is a tuple $M = (M, \Sigma_-, \Sigma_+, \varphi_-, \varphi_+)$ where M is a compact oriented 3-dimensional manifold (possibly with boundary), $\Sigma_\pm$ are closed compact oriented *surfaces* (i.e. 2-dimensional manifolds) and

$$\varphi_- \colon \Sigma_- \times [0,1) \to M\,, \quad \varphi_+ \colon \Sigma_+ \times (-1,0] \to M \tag{3.2}$$

are orientation preserving embeddings such that

$$\partial M = \partial_- M \sqcup \partial_+ M\,, \quad \text{where} \quad \partial_\pm M := \varphi_\pm(\Sigma_\pm \times \{0\})\,. \tag{3.3}$$

The surfaces Σ_- and Σ_+ are then called the *incoming* and *outgoing* boundaries of M. The maps $\varphi_\pm$ are called the *(germs of) parametrisations* of $\Sigma_\pm$. An *equivalence* of two bordisms $M, M' \colon \Sigma_- \to \Sigma_+$ is an orientation preserving diffeomorphism $f \colon M \to M'$ such that $f \circ \varphi_\pm = \varphi'_\pm$. The category of 3-dimensional bordisms Bord_3 is as usual defined to have

- *objects:* closed compact oriented surfaces (including the empty surface $\varnothing$);
- *morphisms:* equivalence classes of bordisms;
- *composition:* the class obtained by gluing two representing bordisms, i.e. for two bordisms $[M_1 \colon \Sigma \to \Sigma']$, $[M_2 \colon \Sigma' \to \Sigma'']$ one takes the composition to be

$$[M_2] \circ [M_1] = [M_1 \cup_{\Sigma'} M_2]; \tag{3.4}$$

- *identities:* for an object Σ, the identity morphism is the class of the cylinder $C_\Sigma := \Sigma \times [0,1]$ (with identity parametrisations).

The operation of the disjoint union makes Bord_3 into a symmetric monoidal category with the empty surface $\varnothing$ as the monoidal unit.

Remark 3.1. In what follows we will encounter several categories, which are defined by equipping the objects and morphisms of Bord_3 with extra structure. In doing so one must be careful in defining the analogous notions of parametrisations of surfaces, equivalences of bordisms, gluing, etc. (in [Tu, Ch. III] this was formalised with the notion of *cobordism theory*). We will avoid diving into such technicalities too deep and will instead provide references where these categories are explored in more detail. Deviations from the literature, if any, will be kept marginal.

Fix a ribbon category $\mathcal{C}$.

- Let Σ be an oriented 2-dimensional smooth manifold. A *(framed) puncture* is a triple $p = (p, n, v)$ where $p \in \Sigma$ is a point and (n, v) is a basis of $T_p\Sigma$. We call the puncture *positively oriented* if (n, v) is a positive basis and *negatively oriented* otherwise. A *$\mathcal{C}$-coloured* puncture is a puncture with an assigned object $X \in \mathcal{C}$ (its *label*).
- A *($\mathcal{C}$-coloured) punctured surface* is a pair $\Sigma = (\Sigma, P)$ where Σ is a compact oriented 2-dimensional smooth manifold and P is a finite set of ($\mathcal{C}$-coloured) punctures. The *orientation reversal* of a punctured surface $\Sigma = (\Sigma, P)$ is defined to be $-\Sigma := (-\Sigma, P)$. Note that this means that the orientations of punctures are automatically changed as well.
- A *($\mathcal{C}$-coloured) ribbon bordism* is a pair $M = (M, R)$ where M is a compact oriented smooth 3-bordism and $R \subseteq M$ is an embedded ($\mathcal{C}$-coloured) ribbon graph (i.e. a subspace consisting of smooth oriented strands and coupons which locally look like ($\mathcal{C}$-coloured) ribbon tangles). Note that the boundary ∂M is a ($\mathcal{C}$-coloured) punctured surface with the set of punctures $\partial M \cap R$, which inherit the framing from the adjacent strands. If $\partial M = \varnothing$, we call M a closed ($\mathcal{C}$-coloured) ribbon 3-manifold.

Definition 3.2. The *category of ribbon bordisms* $\mathrm{Bord}_3^{\mathrm{rib}}$ is defined to have

- *objects:* punctured surfaces;
- *morphisms:* equivalence classes of ribbon bordisms;
- *identities:* classes of cylinders $[0,1] \times \Sigma$ for each object Σ.

Similarly one defines the category of $\mathcal{C}$*-coloured ribbon bordisms* $\mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C})$. Both $\mathrm{Bord}_3^{\mathrm{rib}}$ and $\mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C})$ are symmetric monoidal categories with disjoint union as the monoidal product.

Remark 3.3. Equivalent ribbon bordisms are related by a smooth orientation preserving diffeomorphism of the underlying 3-bordisms. This allows one to deform the embedded ribbon graphs up to a smooth isotopy of their tubular neighbourhoods (see e.g. [Hir, Thm. 8.1.8]). When working with morphisms of $\mathrm{Bord}_3^{\mathrm{rib}}$ and $\mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C})$ we will mostly ignore the difference between the equivalence class of a ribbon bordism and its representative. When necessary, we will also depict them by pictures, see Figure 3.1 for an example.

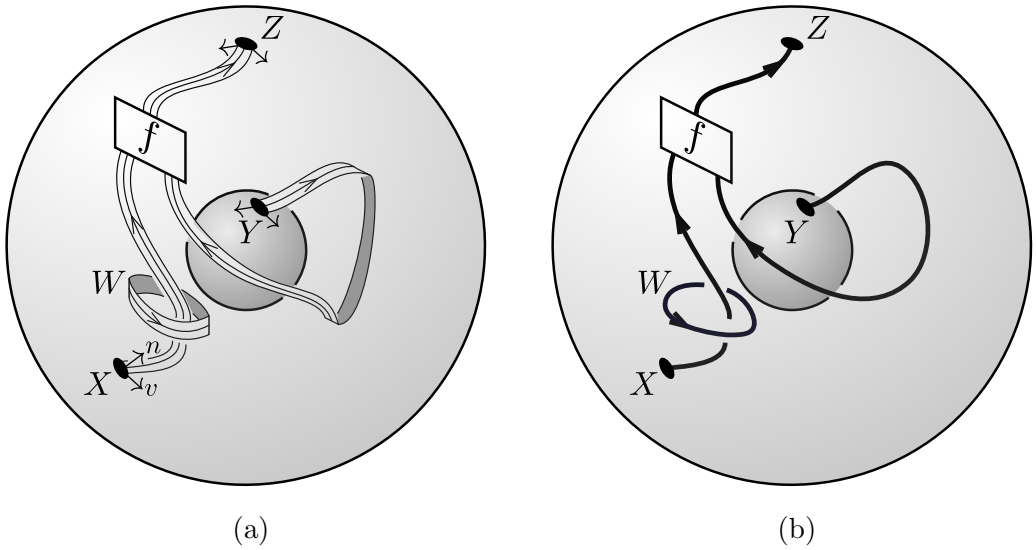
As we will see later, the invariant that the Reshetikhin-Turaev TQFT assigns to a closed ribbon 3-manifold $M = (M, R)$ depends on an auxiliary datum resulting in a choice of a bounding 4-manifold, i.e. a compact oriented 4-manifold W such that $\partial W = M$. The dependence is rather weak, in fact only the signature $\sigma(W)$ of the intersection pairing $H_2(W;\mathbb{R}) \times H_2(W;\mathbb{R}) \to \mathbb{R}$ is important. Still, this causes problems when generalising the invariants to a TQFT, since the signatures of bounding manifolds are not additive upon gluing, which causes a *gluing anomaly*. To eliminate it one equips the objects and morphisms of $\mathrm{Bord}_3^{\mathrm{rib}}$ and $\mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C})$ with extra structure which we now describe.

Recall the following notions (see [Tu, Sec. IV.3] for more details):

- A *Lagrangian subspace* of a symplectic vector space (H, ω) is a maximal isotropic subspace $\lambda \subseteq H$, i.e. such that $\omega|_{\lambda\times\lambda} = 0$ and if λ' is another subspace with this property such that $\lambda \subseteq \lambda'$ then $\lambda = \lambda'$.
- Given three Lagrangian subspaces $\lambda_1, \lambda_2, \lambda_3 \subseteq H$, the *Maslov index* $\mu(\lambda_1, \lambda_2, \lambda_3)$ is defined as follows: define the (not necessarily non-degenerate) pairing on the subspace $(\lambda_1 + \lambda_2) \cap \lambda_3$ by

$$\langle a, b\rangle := \omega(a_2, b), \quad \text{where } a = a_1 + a_2, \quad a_1 \in \lambda_1,\, a_2 \in \lambda_2\,. \tag{3.5}$$

  The Maslov index $\mu(\lambda_1, \lambda_2, \lambda_3)$ is then defined to be the signature of $\langle -, -\rangle$. By definition μ is antisymmetric in its arguments.



(a) (b)

Figure 3.1: (a) A morphism in $\mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C})$ with underlying manifold $S^2 \times [0,1]$, depicted here by a closed solid ball with an open ball removed from the interior. The strands intersect the boundary transversally at the punctures, which are framed by the pair of tangent vectors (n, v), so that (n, v, s), with s being the orientation of the strand, is positive with respect to the orientation of the underlying manifold (assumed right-handed in the picture). The strands can be though of as ribbons with n being the normal vector.

(b) A simplified notation for the same bordism: the framing of the strands is assumed to be that of the paper plane, the framings of the punctures are omitted.

Let Σ be a compact oriented 2-manifold. Then the intersection pairing makes $H_1(\Sigma;\mathbb{R})$ into a symplectic vector space. Let us denote by $\Lambda(\Sigma)$ the set of Lagrangian subspaces of $H_1(\Sigma;\mathbb{R})$. By a Lagrangian space of Σ we mean an element $\lambda \in \Lambda(\Sigma)$. A bordism $[M\colon \Sigma \to \Sigma'] \in \mathrm{Bord}_3$ yields two maps, $N_*(M) : \Lambda(\Sigma) \to \Lambda(\Sigma')$ and $N^*(M)\colon \Lambda(\Sigma') \to \Lambda(\Sigma)$, defined as follows: for $\lambda \in \Lambda(\Sigma)$ one has $\gamma' \in N_*(M)(\lambda)$ if and only if there is $\gamma \in \lambda$ such that $\gamma - \gamma' = 0$ in $H_1(M;\mathbb{R})$ ($N^*(M)$ is defined similarly).

Definition 3.4. The category $\widehat{\mathrm{Bord}}_3^{\mathrm{rib}}$ is defined to have

- *objects:* pairs (Σ, λ) where $\Sigma \in \mathrm{Bord}_3^{\mathrm{rib}}$ is a punctured surface and $\lambda \in \Lambda(\Sigma)$ is a Lagrangian subspace;
- *morphisms:* a morphism $(\Sigma, \lambda) \to (\Sigma', \lambda')$ is a pair $M = (M, n)$ where $[M\colon \Sigma \to \Sigma'] \in \mathrm{Bord}_3^{\mathrm{rib}}$ is a ribbon bordism and $n \in \mathbb{Z}$ is an integer (called

the *signature* of M);

- *composition:* for two morphisms $(M,n)\colon (\Sigma,\lambda)\to(\Sigma',\lambda')$ and $(M',n')\colon (\Sigma',\lambda')\to(\Sigma'',\lambda'')$ the composition is defined by

$$(M',n')\circ(M,n) = \big(\, M'\circ M,\; n+n'-\mu\big(M_*(\lambda),\,\lambda',\,M'^*(\lambda'')\big)\big)\,; \qquad (3.6)$$

- *identities:* $(\Sigma\times[0,1],0)$ for each object (Σ,λ).

Similarly one defines the category of $\widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C})$. Both of them are symmetric monoidal categories with the monoidal product $(\Sigma,\lambda)\sqcup(\Sigma',\lambda')=(\Sigma\sqcup\Sigma',\lambda\oplus\lambda')$ on objects and $(M,n)\sqcup(M',n')=(M\sqcup M',n+n')$ on morphisms. We will refer to them as *signature extensions* of $\mathrm{Bord}_3^{\mathrm{rib}}$ and $\mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C})$.

Remark 3.5. In the later sections we will also encounter the signature extensions of other categories based on Bord_3, where we assume it to be clear how to adapt Definition 3.4. In the computational examples below we will never need to handle e.g. the Lagrangian subspaces of a surface $\Sigma=(\Sigma,\lambda)$ explicitly. Still, the hat notation will be kept to emphasise when the signature extension is in principle necessary.

3.2. Invariants of closed manifolds

We now turn to defining the Reshetikhin-Turaev invariants for closed ribbon 3-manifolds. Roughly, if the underlying 3-manifold is the 3-sphere S^3, the invariant is defined (up to a predetermined factor) to be the number obtained by projecting the embedded ribbon graph R on a plane and using graphical calculus to read it as a morphism in $\mathrm{End}_{\mathcal{C}}(\mathbb{1})\cong\Bbbk$, where $\mathcal{C}$ is a modular fusion category (abbr. MFC, see Section 2.3), needed for the construction as an input. Other underlying 3-manifolds can be achieved by representing the corresponding 3-manifold as a framed link L in S^3 using *surgery* and treating the components of L also as strands of the embedded ribbon graph, but labelled in a specific way (in particular by the Kirby colour).

We now recall the surgery presentation of closed compact oriented 3-manifolds on which the Reshetikhin-Turaev invariants are based. More details can be found e.g. in [PS].

Let $S^3\simeq\mathbb{R}^3\cup\{\infty\}$ be the 3-sphere with the right-hand orientation and let $L=(L,n,v)$ be a closed unlabelled ribbon strand (i.e. a framed knot) in S^3. Consider a tubular neighbourhood of L given by an embedding $\imath\colon\mathbb{R}^2\times L\hookrightarrow M$ such that for all $x\in L$ one has $\imath(0,0,x)=x$, $d\imath_{(0,0,x)}(e_1)=n(x)$, $d\imath_{(0,0,x)}(e_2)=v(x)$ and denote $U=\mathrm{im}\,\imath$. The normal vector field n of L induces an oriented curve $\gamma\colon L\to\partial U$ on the boundary $\partial U=\overline{U}\setminus U$ defined as $\gamma(x)=\lim_{t\to\infty}\imath(t,0,x)$.

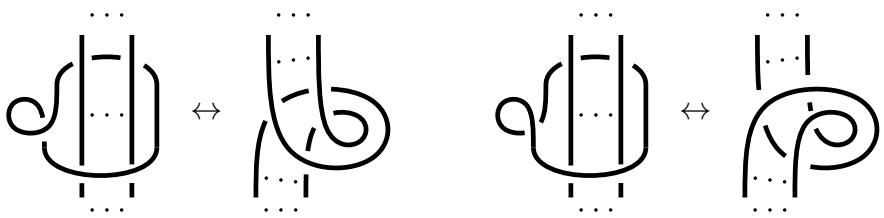


Figure 3.2: Kirby-Fenn-Rourke moves relating two surgery presentations.

We let ∂U have the orientation induced from $S^3 \setminus U$. Pick a diffeomorphism $\varphi\colon S^1 \times S^1 \to -\partial U$ such that $\varphi(S^1 \times \{1\}) = \operatorname{im}\gamma$. We say that the manifold

$$M = (S^3 \setminus U) \cup_\varphi (B^2 \times S^1) \tag{3.7}$$

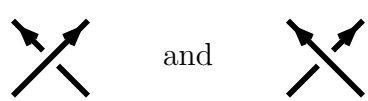
is obtained by surgery on S^3 along L. In other words, M is obtained from S^3 by cutting out a solid torus and glueing it back via a diffeomorphism of its boundary. By construction, changing L by an isotopy does not change the diffeomorphism class of M. Moreover, the orientation of L does not matter, since surgery along $(-L, n, -v)$ gives the same result. One can however generalise this construction by performing surgery along a link of ribbon strands in S^3 with multiple components. One has (see [PS, Thm. 19.3, Thm. 19.5])

Theorem 3.6. Every oriented compact connected 3-manifold is diffeomorphic to a one obtained by surgery on S^3 along a framed link. Two links yield diffeomorphic manifolds if and only if they are related by a finite sequence of *Kirby-Fenn-Rourke moves* depicted in Figure 3.2.

For a 3-manifold M, pick a surgery link L with components $L_1, \dots, L_l$. It also provides M with a choice of a bounding 4-manifold W_L (see e.g. [Tu, Sec. 2.1]). The signature of W_L can then be computed directly from L by taking the signature $\sigma(L)$ of its *linking number matrix* $\big(\operatorname{lk}(L_i, L_j)\big)_{ij}$ defined as follows: Take a planar projection of L, so that only two strands are allowed to intersect at one point, with the intersection being necessarily transversal. For two distinct components L_i, L_j let

$$\operatorname{lk}(L_i, L_j) = \frac{1}{2}\,(\#\text{overcrossings} - \#\text{undercrossings}) \tag{3.8}$$

where by overcrossings and undercrossings we mean the crossings like



(3.9)

between the components L_i and L_j (meaning that the crossings of e.g. of L_i with itself are not counted). For the diagonal entries one sets $\operatorname{lk}(L_i, L_i) := \operatorname{lk}(L_i, L_i')$,

where L_i' is a curve obtained from L_i by shifting it to the v-direction of its framing (n, v). The linking number matrix is by definition a symmetric matrix with integer entries. The signature $\sigma(L)$ is then computed as usual - the number of positive eigenvalues minus the number of negative ones - and is an isotopy invariant of the framed link L.

We are now ready to introduce the Reshetikhin-Turaev invariants of closed 3-manifolds. The definition needs a modular fusion category $\mathcal{C}$ and a choice $\mathcal{D}$ of the square root of the global dimension of $\operatorname{Dim}\mathcal{C}$ as an input. Recall from Section 2.3 also the definitions of the scalar δ, the Kirby colour object $C = \bigoplus_i i \in \mathcal{C}$ and the morphism $d = \bigoplus_i \dim i \cdot \mathrm{id}_i \in \operatorname{End}_{\mathcal{C}} C$.

For an oriented compact connected 3-manifold M, let $L \subseteq S^3$ be a choice of the surgery link. Let $L(C, d)$ be the coloured version of L whose components are in addition assigned the object C and contain a single insertion labelled by d. Recall from Section 2.1, that the Reshetikhin-Turaev functor $F_{\mathcal{C}}\colon \operatorname{Rib}_{\mathcal{C}} \to \mathcal{C}$ assigns to $L(C, d)$ an isotopy invariant $F_{\mathcal{C}}\big(L(C, d)\big) \in \operatorname{End}_{\mathcal{C}} \mathbb{1} \cong \Bbbk$.

Definition 3.7. Let $\mathcal{C}$ be a MFC together with a choice of a square root $\mathcal{D} = (\operatorname{Dim}\mathcal{C})^{1/2}$. The *Reshetikhin-Turaev invariant* of an oriented closed connected 3-manifold M represented by a surgery link L with $|L|$ components is the scalar

$$\tau(M) := \delta^{-\sigma(L)}\, \mathcal{D}^{-|L|-1}\, F_{\mathcal{C}}\big(L(C, d)\big)\,. \tag{3.10}$$

If M is not connected, $\tau(M)$ is defined to be the product of the invariants of the connected components.

We refer to [Tu, Thm.II.2.2.2] for the proof that $\tau(M)$ is indeed a topological invariant of the manifold M. Note however that the identities (2.26) already imply the Kirby-Fenn-Rourke moves up to a factor.

The invariant $\tau(M)$ is readily lifted to ribbon 3-manifolds with signatures, or more precisely, bordisms of the form

$$[(M, R, n)\colon \varnothing \to \varnothing] \in \widehat{\operatorname{Bord}}_3^{\mathrm{rib}}(\mathcal{C})\,. \tag{3.11}$$

Indeed, upon representing M by a surgery link $L \subseteq S^3$ one can always deform R so that it does not intersect the regions of M corresponding to the solid tori that are glued in place of L. The $\mathcal{C}$-coloured ribbon 3-manifold (M, R) can then be represented by the tangle $L \sqcup R$ in S^3 (where the components of L and R may be non-trivially entangled). One then defines the invariant by (see [Tu, Thm.II.2.3.2])

$$\tau(M, R, n) := \delta^{n-\sigma(L)}\, \mathcal{D}^{-|L|-1}\, F_{\mathcal{C}}\big(\, L(C, d) \sqcup R\,\big)\,. \tag{3.12}$$

3.3. Definition of the graph TQFT

We are now ready to define the Reshetikhin-Turaev graph TQFT

$$Z_{\mathcal{C}}^{\mathrm{RT}} \colon \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}) \to \mathrm{Vect}_{\Bbbk} \ . \tag{3.13}$$

The definition we provide uses the universal construction of TQFTs as introduced in [BHMV]. This approach is slightly different from the one in [Tu], but yields the same result (see Remark 3.10 below).

For a given object $\Sigma \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C})$, consider the vector spaces

$$\begin{aligned} \mathcal{V}(\Sigma) &= \mathrm{span}_{\Bbbk} \left\{ [M] \;\middle|\; [M \colon \varnothing \to \Sigma] \in \mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C}) \right\}, \\ \mathcal{V}'(\Sigma) &= \mathrm{span}_{\Bbbk} \left\{ [M] \;\middle|\; [M \colon \Sigma \to \varnothing] \in \mathrm{Bord}_3^{\mathrm{rib}}(\mathcal{C}) \right\}, \end{aligned} \tag{3.14}$$

i.e. the infinite dimensional vector spaces with bases given by the sets of equivalence classes of extended $\mathcal{C}$-coloured ribbon bordisms of types $\varnothing \to \Sigma$ and $\Sigma \to \varnothing$. One utilises the Reshetikhin-Turaev invariant to define the pairing

$$\begin{array}{rccc} \beta_\Sigma \colon & \mathcal{V}'(\Sigma) \times \mathcal{V}(\Sigma) & \to & \Bbbk \\ & ([M'], [M]) & \mapsto & \tau(M' \circ M) \end{array} \ . \tag{3.15}$$

Let

$$\mathrm{rad}_{\mathrm{r}}\, \beta_\Sigma = \left\{ v \in \mathcal{V}(\Sigma) \;\middle|\; \beta_\Sigma(v', v) = 0 \text{ for all } v' \in \mathcal{V}'(\Sigma) \right\} \tag{3.16}$$

be the right radical of the pairing β_Σ.

Definition 3.8. One defines the functor $Z_{\mathcal{C}}^{\mathrm{RT}} \colon \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}) \to \mathrm{Vect}_{\Bbbk}$

- on objects $\Sigma \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C})$:

$$Z_{\mathcal{C}}^{\mathrm{RT}}(\Sigma) := \mathcal{V}(\Sigma) / \mathrm{rad}_{\mathrm{r}}\, \beta_\Sigma \, ; \tag{3.17}$$

- on morphisms $[M : \Sigma \to \Sigma'] \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C})$:

$$\begin{array}{rccc} Z_{\mathcal{C}}^{\mathrm{RT}}(M) \colon & Z_{\mathcal{C}}^{\mathrm{RT}}(\Sigma) & \to & Z_{\mathcal{C}}^{\mathrm{RT}}(\Sigma') \\ & [\varnothing \xrightarrow{H} \Sigma] & \mapsto & [\varnothing \xrightarrow{H} \Sigma \xrightarrow{M} \Sigma'] \end{array} \ . \tag{3.18}$$

To obtain a well defined graph TQFT it remains to equip $Z_{\mathcal{C}}^{\mathrm{RT}}$ with a monoidal structure, for which there is a natural candidate:

Proposition 3.9. $Z_{\mathcal{C}}^{\mathrm{RT}} = (Z_{\mathcal{C}}^{\mathrm{RT}}, Z_{\mathcal{C},2}^{\mathrm{RT}}, Z_{\mathcal{C},0}^{\mathrm{RT}})$, where

$$\begin{array}{rccc} Z_{\mathcal{C},2}^{\mathrm{RT}}(\Sigma, \Sigma') \colon Z_{\mathcal{C}}^{\mathrm{RT}}(\Sigma) \otimes_{\Bbbk} Z_{\mathcal{C}}^{\mathrm{RT}}(\Sigma') & \to & Z_{\mathcal{C}}^{\mathrm{RT}}(\Sigma \sqcup \Sigma') \\ [M] \otimes_{\Bbbk} [M'] & \mapsto & [M \sqcup M'] \end{array} , \quad \begin{array}{rccc} Z_{\mathcal{C},0}^{\mathrm{RT}} \colon Z_{\mathcal{C}}^{\mathrm{RT}}(\varnothing) & \to & \Bbbk \\ [M] & \mapsto & \tau(M) \end{array}$$

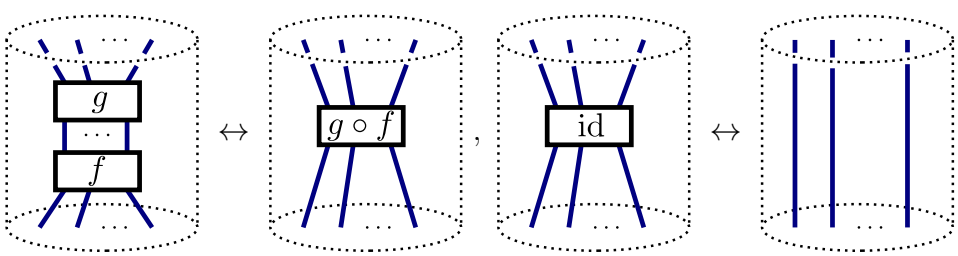
is a symmetric monoidal functor.

Remark 3.10. 1. The proof of Proposition 3.9 is not trivial, as the universal construction of [BHMV] does not guarantee that the maps $Z^{\mathrm{RT}}_{\mathcal{C},2}$ are indeed isomorphisms. This however was proven in [DGGPR] for a generalisation of the Reshetikhin-Turaev TQFT obtained from a not necessarily semisimple modular tensor category. This generalisation yields the above construction in the semisimple case. The proof of monoidality fails if the objects of $\widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C})$ are not equipped with Lagrangian subspaces, so the use of the extended category $\widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C})$ is necessary (see [GW] for more details on this).

2. The original definition in [Tu, Ch. IV] does not use the universal construction, so a priori it is not clear that the TQFT of [Tu] and $Z^{\mathrm{RT}}_{\mathcal{C}}$ as defined above are isomorphic. However this follows from Lemma 6.14 below, as the two TQFTs give the same invariants on closed manifolds (cf. [Tu, Thm. II.2.3.2]) and have isomorphic state spaces (compare [Tu, (IV. 1.4.a)] and [DGGPR, Prop. 4.16]).

3.4. Properties of the graph TQFT

Property 3.11. From the definition it follows that the graph TQFT $Z^{\mathrm{RT}}_{\mathcal{C}}$ is *linear* with respect to addition and scalar multiplication of coupons of embedded ribbon graphs. Moreover, upon evaluation one can compose the coupons as well as remove a coupon labelled with an identity morphism, i.e. the following exchanges are allowed:



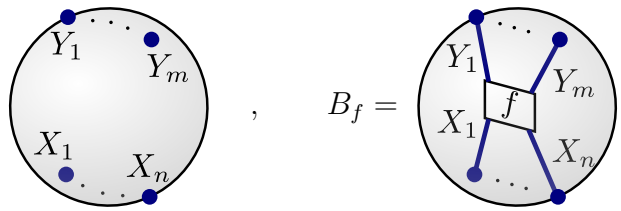
(3.19)

This means in particular that one can perform graphical calculus on the embedded ribbon graphs. For example, if a ribbon graph R in a 3-bordism M has no free ends and has a contractible open neighbourhood in M, it can be exchanged for a single coupon labelled with $F_{\mathcal{C}}(R) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$. By the terminology of [TV, Sec. 15.2.3], graph TQFTs with this property are called *regular*.

Property 3.12. Let $S^2_P \in \widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C})$ be the unit 2-sphere in $\mathbb{R}^3$ with $(n+m)$ punctures, n of which are located at the bottom of the sphere at the intersection with yz plane and are labelled by objects $X_1, \ldots, X_n \in \mathcal{C}$ (in this order from left to right), while the rest m are located at the top and labelled by $Y_1, \ldots, Y_m \in \mathcal{C}$.

Furthermore, for a morphism $f \in \mathcal{C}(X_1 \cdots X_n, Y_1 \cdots Y_m)$ let $[B_f \colon \varnothing \to S_P] \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C})$ be the unit ball with an embedded ribbon graph consisting of an f-labelled coupon and strands connecting it to the punctures of S_P, all of which lie in the yz-plane, i.e.

$$S_P = \quad , \qquad B_f = \quad . \tag{3.20}$$

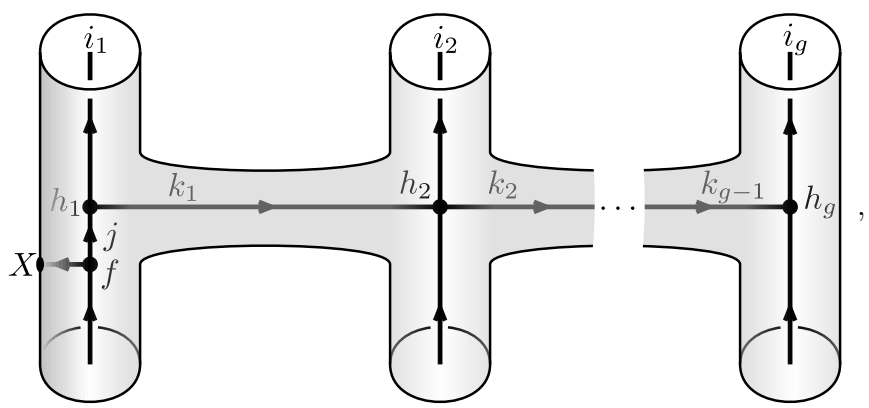


Then one has the following isomorphism of vector spaces:

$$\mathcal{C}(X_1 \cdots X_n, Y_1 \cdots Y_m) \cong Z_{\mathcal{C}}^{\mathrm{RT}}(S_P^2)\,, \quad f \mapsto [B_f \colon \varnothing \to S_P^2]\,. \tag{3.21}$$

This follows from the fact that any bordism $\varnothing \to S_P^2$ can be obtained by performing surgery on the interior of B_f and the resulting components of the embedded ribbon graph and the Kirby-coloured surgery link can be absorbed into a single coupon.

Property 3.13. An analogous argument as in Property 3.12 can be made to determine the vector space assigned by $Z_{\mathcal{C}}^{\mathrm{RT}}$ to a punctured surface $\Sigma \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C})$ of genus $g > 0$. In particular, $Z_{\mathcal{C}}^{\mathrm{RT}}(\Sigma)$ is spanned by bordisms of the form $[(H_g, R) \colon \varnothing \to \Sigma]$, where H_g is a solid handlebody with the boundary Σ. The spanning set can then be further reduced by only taking the ribbon graphs R lying at the core of H_g and having strands labelled by the simple objects of $\mathcal{C}$. For such handlebodies we will use the graphical presentation



, (3.22)

where each of the vertical solid cylinder represents one of the handles of H_g, by taking its top and bottom ends as identified. For simplicity, the above depiction

is only for the case of Σ having a single $X \in \mathcal{C}$ labelled puncture, the other labels being

$$\begin{aligned} &i_1, i_2, \dots, i_g, \quad j, \quad k_1, \dots, k_{g-1} \quad \in \quad \mathrm{Irr}_{\mathcal{C}} \\ &f \in \mathcal{C}(i_1, Xj), \quad h_1 \in \mathcal{C}(j, i_1 k_1), \quad h_2 \in \mathcal{C}(k_1 i_2, i_2 k_2), \quad \dots \quad h_g \in \mathcal{C}(k_{g-1} i_g, i_g)\,. \end{aligned}$$

If one lets $f, h_1, \dots h_g$ run through the bases of their respective spaces, one can check that the handlebodies (3.22) actually form a basis of the space $Z^{\mathrm{RT}}_{\mathcal{C}}(\Sigma)$, which yields the following isomorphism:

$$\begin{aligned} Z^{\mathrm{RT}}_{\mathcal{C}}(\Sigma) &\cong \bigoplus_{i_1,\dots,k_{g-1} \in \mathrm{Irr}_{\mathcal{C}}} \mathcal{C}(i_1, Xj) \otimes_{\Bbbk} \mathcal{C}(j, i_1 k_1) \otimes_{\Bbbk} \mathcal{C}(k_1 i_2, i_2 k_2) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathcal{C}(k_{g-1} i_g, i_g) \\ &\cong \mathcal{C}(\mathbb{1}, XL^{\otimes g}), \ , \qquad L := \bigoplus_{i \in \mathrm{Irr}_{\mathcal{C}}} i \otimes i^*\,, \end{aligned} \tag{3.23}$$

where one uses the duals and the braiding of $\mathcal{C}$ to obtain the isomorphism in the second line.

Property 3.14. Some simple invariants of 3-manifolds are:

$$Z^{\mathrm{RT}}(S^3, 0) = \mathcal{D}^{-1}, \qquad Z^{\mathrm{RT}}(S^2 \times S^1, 0) = 1, \tag{3.24}$$

$$Z^{\mathrm{RT}}(S^1 \times S^1 \times S^1, 0) = |\,\mathrm{Irr}_{\mathcal{C}}\,| \bmod \mathrm{char}\,\Bbbk\,. \tag{3.25}$$

The S^3-invariant allows one to rewrite the formula (3.12) for the invariant of a ribbon 3-manifold (M, R, n) represented by the surgery link $L \sqcup R$ in S^3 as

$$Z^{\mathrm{RT}}_{\mathcal{C}}(M, R, n) = \delta^{n-\sigma(L)} \cdot \mathcal{D}^{-|L|} \cdot Z^{\mathrm{RT}}(S^3, L_C \sqcup R, 0)\,. \tag{3.26}$$

The other two invariants are simply instances of a more general observation: Due to the functoriality of Z^{RT} (or in fact any graph TQFT), for any surface $\Sigma \in \widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C})$ the invariant of $\Sigma \times S^1$ is the trace of the identity map on $Z^{\mathrm{RT}}(\Sigma)$. If $\mathrm{char}\,\Bbbk = 0$, it is equal to the dimension of the state space of Σ, otherwise to the dimension modulo $\mathrm{char}\,\Bbbk$, i. e.

$$Z^{\mathrm{RT}}(\Sigma \times S^1, 0) = \dim Z^{\mathrm{RT}}(\Sigma) \bmod \mathrm{char}\,\Bbbk\,. \tag{3.27}$$

We have already seen explicitly in (3.12) that $\dim Z^{\mathrm{RT}}(S^2) = 1$, and the state space of a punctureless torus $S^1 \times S^1$ has a basis consisting of solid tori with non-contractible loops labelled by $i \in \mathrm{Irr}_{\mathcal{C}}$ (see Property 3.13).

4. Reshetikhin-Turaev defect TQFT

In this chapter we review the construction and some properties of the Reshetikhin-Turaev defect TQFT

$$Z_{\mathcal{C}}^{\mathrm{def}} : \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) \to \mathrm{Vect}_{\Bbbk} \tag{4.1}$$

obtained from a modular fusion category $\mathcal{C}$. In a nutshell, $Z_{\mathcal{C}}^{\mathrm{def}}$ extends the graph TQFT $Z_{\mathcal{C}}^{\mathrm{RT}}$ so that it can evaluate stratified bordisms, i.e. with both embedded ribbon graphs, embedded surfaces and even networks of intersecting surfaces. The evaluation works by reducing stratifications into networks of ribbon graphs using an internal state-sum procedure which exploits the properties of symmetric separable Frobenius algebras in $\mathcal{C}$. This idea was already used in [FRS1] for computing correlators in rational conformal field theory and was adapted for surface defects in TQFTs in [KSa]. It was then explained in a more model independent analysis in [FSV] and brought to the functorial form (4.1) in [CRS2].

4.1. Defect bordisms and defect TQFTs

In the following two sections we explain the source category in (4.1). More details can be found in [CMS, Sec. 2] on which this review is based.

- An *n-dimensional stratified manifold* is a pair $M = (M, \mathrm{T})$ where M is an oriented n-dimensional topological manifold and T is a filtration of M into topological spaces

  $$\varnothing = \mathrm{T}^{(-1)} \subseteq \mathrm{T}^{(0)} \subseteq \mathrm{T}^{(1)} \subseteq \cdots \subseteq \mathrm{T}^{(n)} = M \tag{4.2}$$

  to be called the *stratification* of M, such that

  - for each $j = 0, \dots, n$, $\mathrm{T}^j := \mathrm{T}^{(j)} \setminus \mathrm{T}^{(j-1)}$ has a structure of an oriented smooth j-dimensional manifold (for $j = n$ the orientation is taken to be the same as that of M); the connected components of T^j are called the *strata* (or *j-strata* if the dimension needs to be emphasised) of M;
  - if s and t are two strata and $s \cap \bar{t} \neq \varnothing$, then $s \subseteq \bar{t}$ (in this case we say that s and t are *adjacent*);
  - the total number of strata of M is finite.

  In case M has boundary one requires in addition:

  - the interior of M is a stratified manifold with stratification $\mathrm{T} \cap (M \setminus \partial M)$;
  - ∂M is a stratified manifold with stratification $\partial\,\mathrm{T}$ where $\partial\,\mathrm{T}^{(j-1)} = \mathrm{T}^{(j)} \cap \partial M$, $j = 0, \dots, n$ (in particular all 0-strata of M lie in the interior);

– for all strata s of M, either $\partial s = \varnothing$ or $\partial s \subseteq \partial M$ and s, ∂M intersect transversally.

The orientation reversal $-M$ of a stratified manifold is defined by reversing the orientation of all strata.

- A morphism between two stratified manifolds (M, T), (M', T') is a continuous map $f\colon M \to M'$ so that one has $f(\mathrm{T}^j) \subseteq \mathrm{T}^{j'}$ and for each stratum s of M there is a stratum s' of M' such that $f(s) \subseteq s'$ and the restriction $f\big|_s\colon s \to s'$ is smooth and orientation preserving. In case M and M' have boundaries, one in addition requires $f\big|_{\partial M}\colon \partial M \to \partial M'$ to be a morphism of stratified manifolds.

- We will work with the so-called *defect manifolds*, which are stratified manifolds with an additional regularity condition, imposed by requiring each point to have a neighbourhood which, as a stratified manifold, is isomorphic to a certain local model. For dimensions 2 and 3 they are depicted in Figures 4.1 and 4.2. Compact closed defect 2-manifolds will also be called *defect surfaces*.

- A 3-*dimensional defect bordism* is defined by analogy to a usual 3-dimensional oriented bordism: it is a tuple $M = (M, \Sigma_-, \Sigma_+, \varphi_-, \varphi_+)$ where M is a 3-dimensional compact defect manifold, $\Sigma_\pm$ are defect surfaces, $\varphi_\pm$ are germs of embeddings of defect manifolds

$$\varphi_-\colon \Sigma_- \times (-1, 0] \hookrightarrow M\,, \quad \varphi_+\colon \Sigma_+ \times [0, 1) \hookrightarrow M \tag{4.3}$$

  such that $\partial M = \partial_- M \sqcup \partial_+ M$, where $M_\pm := \varphi_\pm(\Sigma_\pm \times \{0\})$. An equivalence between defect bordisms M and M' is an isomorphism $f\colon M \to M'$ of the underlying stratified manifolds such that $f \circ \varphi_\pm = \varphi'_\pm$.

Like for strands and coupons of ribbon graphs in Section 3.1, we consider labels for the strata of a defect bordism. The labelling scheme is formalised by a so-called *(3-dimensional) defect datum* $\mathbb{D} = (D_3, D_2, D_1, D_0)$ where D_j, $j = 0, \ldots, 3$ are the sets of labels for the j-strata (technically, the possible labels of a stratum also depend on the labels of the adjacent strata; we will avoid making this too general since the adjacency rules will be clear for each example of a defect datum that we will encounter; see [CRS3, Def. 2.4] for more details). We use the terms $\mathbb{D}$-*coloured defect surface* Σ and $\mathbb{D}$-*coloured defect bordism* M to emphasise that the strata of Σ and M are labelled according to the defect datum $\mathbb{D}$. The isomorphisms and equivalences between $\mathbb{D}$-coloured defect surfaces and bordisms are always assumed to preserve the labels.

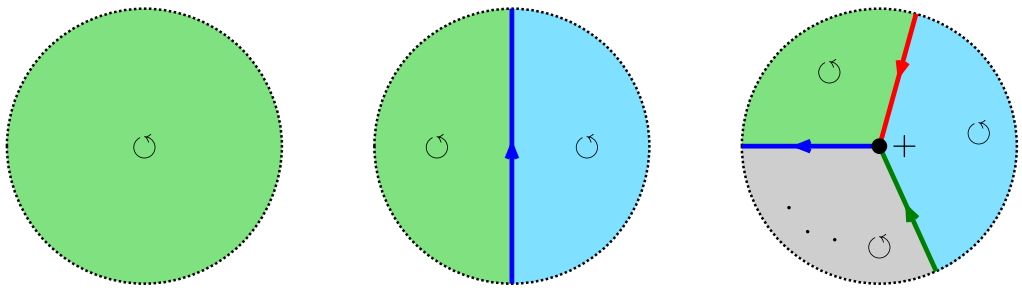


Figure 4.1: Local models for a defect surface Σ: each point has a neighbourhood, isomorphic to one of the above stratified open discs. There are infinitely many models of the third type (with arbitrary many (or none) 1-strata adjacent to the 0-stratum), and any choice of orientation for the 0-stratum and the 1-strata is allowed.

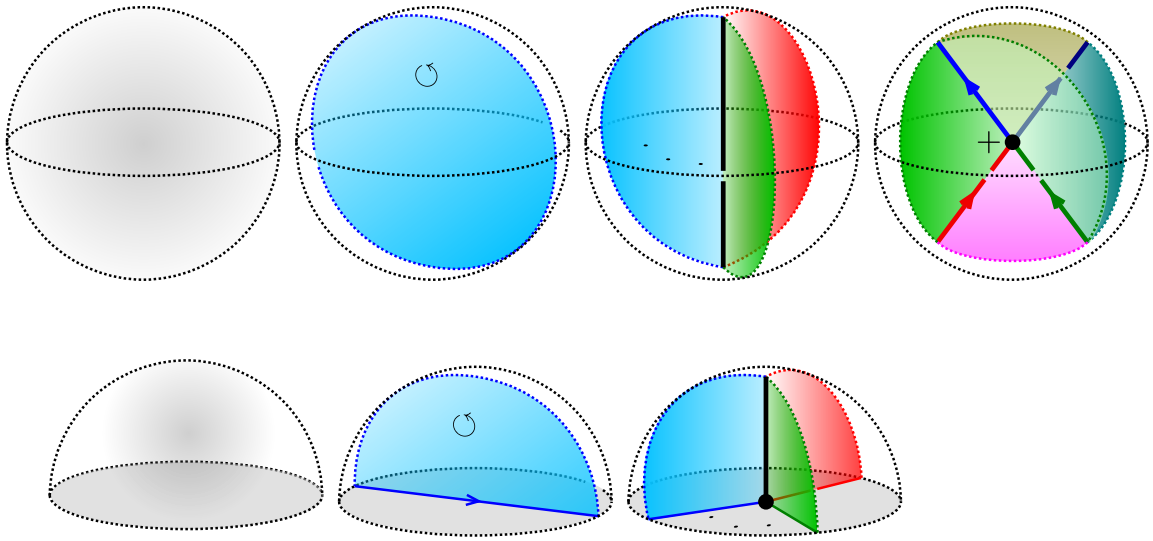


Figure 4.2: Local models for a 3-dimensional defect bordism: each point has a neighbourhood, isomorphic to one of the above stratified open balls (or open half-balls if the point lies in the boundary). There are infinitely many models of the third and the fourth type, and any choice of orientations is allowed. Note that the model of the fourth type is a cone of a defect 2-sphere with a 0-stratum at the tip. In the four pictures in the first line the purpose of the horizontal equator circle is solely to emphasise that the pictures are to be thought of as 3-dimensional.

Definition 4.1. Let $\mathbb{D}$ be a 3-dimensional defect datum.

- We denote by $\mathrm{Bord}_3^{\mathrm{def}}$ and $\widehat{\mathrm{Bord}}_3^{\mathrm{def}}$ correspondingly the category of defect bordisms and its signature extension (by analogy to Definition 3.4). Similarly one introduces the categories $\mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D})$ and $\widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D})$ of $\mathbb{D}$-coloured defect bordisms. Disjoint union equips them with a structure of symmetric monoidal categories.

- A *(3-dimensional) defect TQFT* is a symmetric monoidal functor

$$Z\colon \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}\ . \tag{4.4}$$

Let us review some generalities and terminology related to defect TQFTs

- Formally, by *defect* we will mean a stratum which has a label assigned. We will also use the words *point*, *line*, *surface* and *bulk* as synonyms of 0-, 1-, 2- and 3-stratum. Point defects are also called *point insertions* while the labelled 3-strata will also be referred to as *phases* or *bulk theories*. We will call a surface defect a *domain wall* if we want to stress that it can separate different bulk phases.

- For any defect TQFT Z there is a natural choice of the set D_0 of labels of point defects: For a 0-stratum p of a ($\mathbb{D}$-coloured) defect bordism, take the vector space $Z(S_p^2)$ as the new label set, where S_p^2 is a defect 2-sphere obtained as a boundary component after removing a small open ball B_p° surrounding p. The evaluation procedure is then defined as follows: Let $[M\colon \Sigma \to \Sigma'] \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D})$ be a defect bordism with point defects $p_1, \dots, p_n$ labelled by $v_i \in Z(S_{p_i}^2)$, $i = 1, \dots, n$. Denoting $M_\circ := M \setminus (B_{p_1}^\circ \sqcup \cdots \sqcup B_{p_n}^\circ)$, one gets a linear map

$$Z(M_\circ)\colon Z(S_{p_1}^2) \otimes \cdots \otimes Z(S_{p_n}^2) \otimes Z(\Sigma) \to Z(\Sigma')\,. \tag{4.5}$$

  The invariant of M is then defined by

$$Z(M)(-) := Z(M_\circ)(v_1 \otimes \cdots \otimes v_n \otimes -)\,. \tag{4.6}$$

  Note that effectively this procedure can only extend the set of labels for a point p, as for a predefined label set D_0 and a suitable label $f \in D_0$, the stratified ball $B_p(f) := [\overline{B_p^\circ}(f)\colon \varnothing \to S_p^2]$, obtained as the closure of the stratified open ball B_p° with p labelled by f, provides one with the canonical element in the space $Z(S_p^2)$, obtained as the image of $1 \in \Bbbk$ under the map $Z(B_p(f))\colon \Bbbk \to Z(S_p^2)$. If the map $f \mapsto Z(B_p(f))(1)$ is surjective for all point defects, the defect TQFT Z is called D_0-*complete*. If two labels in D_0 have the same image under this map, Z does not distinguish them.

- Point defects in a D_0-complete defect TQFT can be *fused* (or *composed*): two point defects p_1, p_2 next to each other can be replaced by a single point defect p, labelled by the vector in $Z(S^2_p)$, obtained from the closed ball $B_{p_1,p_2}\colon \varnothing \to S^2_p$ containing both p_1 and p_2. Removing a small open ball B° from a 1-, 2- or 3-stratum (not surrounding any other point defects) provides one with the *trivial* (or *identity*) label for point defects in this configuration. In this case, two point defects are said to be inverses of each other if their composition is the trivial point defect.

- For a D_0-complete defect TQFT $Z\colon \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ one can extend the defect datum $\mathbb{D}$ to a defect datum $\mathbb{D}^\odot$, whose labels for an i-stratum s, $i = 1,2,3$, consist of pairs (L,γ), where $L \in D_i$ is a suitable label for s and $\gamma \in D_0$ is a label for an invertible point defect on s. This yields a new defect TQFT $Z^\odot\colon \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^\odot) \to \mathrm{Vect}_{\Bbbk}$, the *Euler completion* of Z, defined as follows: Introduce the map[8] $W\colon \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^\odot) \to \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D})$, acting as the forgetful functor on objects and on a $\mathbb{D}^\odot$-coloured bordism M replacing the label (L,γ) for an i-stratum s with the label L while adding an additional point insertion on s labelled with $\gamma^{\chi_{\mathrm{sym}}(s)}$, where

$$\chi_{\mathrm{sym}}(s) := 2\chi(s) - \chi(s \cap \partial M) \tag{4.7}$$

  is the symmetric Euler characteristic (which ensures that the number of γ-insertions is compatible upon gluing and does not make a distinction between the incoming and outgoing boundaries of a bordism). One then defines $Z^\odot := Z \circ W$. We call a defect TQFT Z *Euler complete* if there exist relabelling maps $\rho_i\colon D_i^\odot \to D_i$, $i = 1,2,3$, extending to a functor $\rho\colon \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^\odot) \to \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D})$, such that $Z^\odot = Z \circ \rho$. Similarly one can define the Euler completeness with respect to i-strata for a fixed value of $i = 1,2,3$ only.

- The defect datum can encode a variety of information and/or restrict the possible stratifications. For example, the Reshetikhin-Turaev graph TQFT $Z^{\mathrm{RT}}_{\mathcal{C}}$ obtained from a modular fusion category $\mathcal{C}$ as defined in Section 3.3 can be seen as a defect TQFT with no surface defects, lines labelled by both framings and the objects of $\mathcal{C}$, and points labelled by the morphisms of $\mathcal{C}$. Note that $Z^{\mathrm{RT}}_{\mathcal{C}}$ is D_0-complete (see Property 3.12).

- Let $Z\colon \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_{\Bbbk}$ be a defect TQFT. To each pair of labels $u, v \in D_3$ of 3-strata one can assign a pivotal bicategory $\mathcal{B}_{u,v}$ which has (see [CMS, Sec. 3.3], [DKR, Sec. 2.4]):

[8] W is not a functor since upon composing the stratified bordisms in its image one gets more than one point insertion on the strata, see [CRS1, Sec. 2.5].

– *objects:* labels for 2-strata between two 3-strata labelled by u, v, oriented so that the normal vector points towards u;

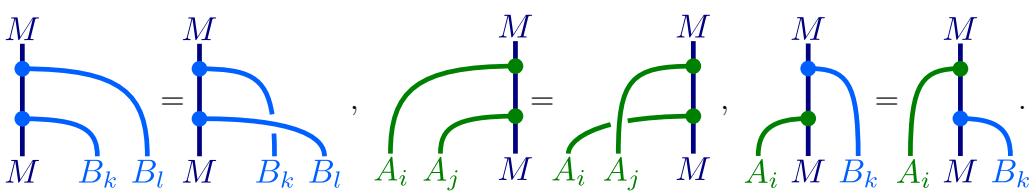
– 1-*morphisms:* (lists of)[9] labels for 1-strata having two adjacent 2-strata labelled with objects in $\mathcal{B}_{u,v}$;

– 2-*morphisms:* labels for 0-strata whose adjacent 1-strata are 1-morphisms of $\mathcal{B}_{u,v}$.

The *bicategory of surface defects* as sketched above belongs to a more general construction assigning a tricategory with duals to a 3-dimensional defect TQFT. This is explained in detail in [CMS].

4.2. Defect labels for RT TQFT

Let $A_1, \dots, A_n$ and $B_1, \dots, B_m$ be symmetric separable Frobenius algebras. By an $A_1 \cdots A_n$-$B_1 \cdots B_m$-*multimodule* we will mean an object $M \in \mathcal{C}$ with structures of a left A_i-module and a right B_k-module for all $i = 1, \dots, n$ and $k = 1, \dots, m$ such that



(4.8)

with the first two identities holding for all $1 \leq i < j \leq n$ and $1 \leq k < l \leq m$ and the third one for all $i = 1, \dots, n$ and $k = 1, \dots, m$. For the cases $n = 0$ (resp. $m = 0$) the algebra acting from the left (right) is not present. Alternatively, a multimodule M can be defined as a $A_1 \otimes \cdots \otimes A_n$-$B_1 \otimes \cdots \otimes B_m$-bimodule. One can also talk about relative tensor products of multimodules. A morphism between (relative tensor products of) multimodules can be seen either as a morphism of the corresponding bimodules, or a morphism between (regular tensor products of) the underlying objects in $\mathcal{C}$ which commutes with actions of all the algebras (cf. Convention 2.16).

For a modular fusion category $\mathcal{C}$, there is a set of defect data $\mathbb{D}^{\mathcal{C}}$, with

- $D_3^{\mathcal{C}}$: 3-strata have no labels, or equivalently they all carry the same label $\mathcal{C}$.
- $D_2^{\mathcal{C}}$: 2-strata are labelled by symmetric separable Frobenius algebras in $\mathcal{C}$.

[9] Using lists of 1-strata-labels as 1-morphisms allows one to define composition as concatenation. We will ignore this technicality in what follows.

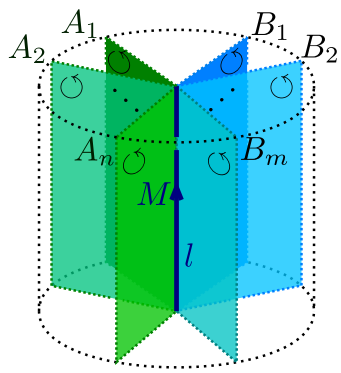


Figure 4.3: Neighbourhood of a line defect l with several adjacent 2-strata. The framing of l as well as the orientations of the 2-strata are paper plane.

- $D_1^{\mathcal{C}}$: in our setting all 1-strata will be framed. A framed 1-stratum that has no adjacent 2-strata is labelled by an object of $\mathcal{C}$ (alternatively, a $\mathbb{1}_{\mathcal{C}}$-$\mathbb{1}_{\mathcal{C}}$-multimodule).

  Suppose a 1-stratum l has $n + m > 0$ adjacent 2-strata. We require l to have a neighbourhood isomorphic to the one shown in Figure 4.3 with $A_1, \ldots, A_n, B_1, \ldots, B_m \in D_2^{\mathcal{C}}$ labelling the adjacent 2-strata. We then label l with an $A_1 \cdots A_n$-$B_1 \cdots B_m$-multimodule $M \in \mathcal{C}$.

- $D_0^{\mathcal{C}}$: point insertions on an $A \in D_2^{\mathcal{C}}$ labelled 2-stratum are labelled by A-A-bimodule morphisms. 0-strata that have adjacent 1-strata are labelled by the morphisms of (relative tensor products of) multimodules as well as data to make them into coupons: a plane orientation, compatible with the framings of the adjacent 1-strata and an order on them. We will later see that the resulting defect TQFT will turn out to be D_0-complete.

The use of the local model in Figure 4.3 in principle restricts the possible orientations of the 2-strata adjacent to a 1-stratum. Although in later sections this will not be used, we argue that in Property 4.12 of the defect TQFT $Z_{\mathcal{C}}^{\text{def}}$ defined below, that the construction we provide can be generalised to also include arbitrary orientations, as for given $A \in D_2^{\mathcal{C}}$, labelling a 2-stratum s with the opposite algebra A^{op} instead of A will yield the same result upon evaluation with $Z_{\mathcal{C}}^{\text{def}}$ as flipping the orientation of s. A further generalisation also allows one to eliminate the framings of the 1-strata by equipping the multimodules with a so-called *cyclic structure*. These technicalities are discussed in detail in [CRS2].

Remark 4.2. For an algebra $A \in D_2^{\mathcal{C}}$, an A-A-bimodule M is the same as an A-A-multimodule and can be used to label 1-strata having two adjacent A-labelled

2-strata. Diverting from the rigorous definition of a stratum, in this case we will sometimes say "an M-labelled line on an A-labelled 2-stratum", as if these two adjacent 2-strata are the same.

Example 4.3. Let $M = (M, \mathrm{T}) \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$ be a $\mathbb{D}^{\mathcal{C}}$-coloured defect bordism which has no 2-strata, i.e. $\mathrm{T}^{(1)} = \mathrm{T}^{(2)}$. Then M can be seen in a natural way as a ribbon bordism $(M, R) \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C})$ where the embedded ribbon graph is $R := \mathrm{T}^{(1)}$, i.e. it has the 1-strata of M as strands (already framed and labelled by the objects of $\mathcal{C}$) and the 0-strata as coupons (or more precisely, as junctions labelled by the morphisms of $\mathcal{C}$, where the parametrisations are encoded in the parts of the defect datum which we do not discuss here). For such defect bordisms, the defect TQFT (4.1) will restrict to the Reshetikhin-Turaev TQFT, i.e. one will have $Z_{\mathcal{C}}^{\mathrm{def}}(M, \mathrm{T}) = Z_{\mathcal{C}}^{\mathrm{RT}}(M, R)$.

4.3. Ribbonisation

We will define the Reshetikhin-Turaev defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ in terms of the "ribbonisation" map

$$R\colon \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) \to \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C})\,, \tag{4.9}$$

which, as will momentarily become apparent, is not a functor since it does not preserve identity morphisms and depends on auxiliary data. It converts a $\mathbb{D}^{\mathcal{C}}$-coloured ribbon bordism M into a $\mathcal{C}$-coloured ribbon bordism by exchanging each surface defect with a ribbon graph. Intuitively, this exchange can be thought of as punching a number of holes in the surface defect and taking the deformation retract, see Figure 4.4 (in [FSV, Sec. 6], exactly this idea was introduced to classify surface defects satisfying certain conditions). The graphical calculus of symmetric separable Frobenius algebras (see Figure 2.2) allows one to label such ribbon graphs in a very natural way. In the rest of the section we formalise this idea, following the treatment of [CRS2].

For $M = (M, \mathrm{T}) \in \mathrm{Bord}_3^{\mathrm{def}}$ and $\epsilon > 0$, let $U_\epsilon\,\mathrm{T}^{(1)}$ denote a shrinking family of tubular neighbourhoods of $\mathrm{T}^{(1)}$. For a 2-stratum s of M we define the *external closure* of s to be the topological space

$$\overline{\overline{s}} := \lim_{\epsilon\to 0} \overline{s \setminus U_\epsilon\,\mathrm{T}^{(1)}}\,, \tag{4.10}$$

i.e. obtained as a limit (in the category of topological spaces) by removing from s smaller and smaller tubular neighbourhoods of 0- and 1-strata from s. In practice, $\overline{\overline{s}}$ is similar to the closure $\overline{s}$ of the stratum s in M, but keeps the edges from being identified e.g. if s is adjacent to the same line twice. The external closure $\overline{\overline{s}}$ is an oriented 2-manifold with boundary and has an obvious projection map $\pi\colon \overline{\overline{s}} \to \overline{s}$.

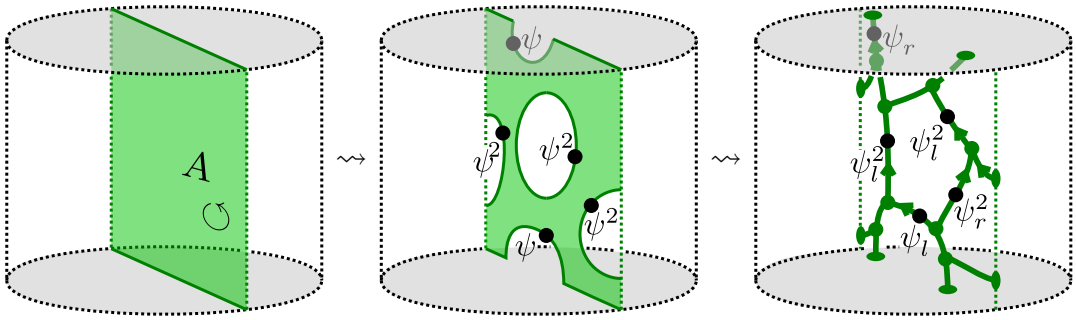


Figure 4.4: Ribbonisation of an $A = (A, \psi) \in D_2^{\mathcal{C}}$ labelled surface by "punching holes". The boundary of a hole receives a ψ-insertion, the strands can be oriented arbitrarily, as long as the vertices can be labelled by the (co)multiplication morphisms of A.

Definition 4.4. An *admissible 1-skeleton* of a 2-stratum s of a defect bordism $M = (M, \mathrm{T}) \in \mathrm{Bord}_3^{\mathrm{def}}$ is a stratification t of $\overline{\overline{s}}$ such that

- each 2-stratum is diffeomorphic to $\mathbb{R}^2$ or $\mathbb{R} \times [0, 1)$ (in particular contractible and does not intersect ∂M in more than one segment);
- each point of $t^{(1)}$ has a neighbourhood isomorphic to one of the local models in Figure 4.5;
- $\pi(\partial t^{(1)}) \cap \mathrm{T}^{(0)} = t \cap \partial T^{(1)} = \varnothing$.

A choice of an admissible 1-skeleton on s induces a stratification on the boundary $\partial s = s \cap \partial M$ consisting of a finite set of oriented points dividing it into a collection of open intervals; we will call such a stratification a *0-skeleton* of ∂s. When talking of admissible 1-skeleta for the 2-strata of M in plural we mean a collection $t = \{t(s)\}$ of admissible 2-skeleta such that $\pi(\partial t^{(1)}(s)) \cap \pi(\partial t^{(1)}(s')) = \varnothing$ for distinct 2-strata s and s' of M (so that the points as in Figure 4.5e-4.5h belonging to s and s' do not coincide). Similarly we will refer to 0-skeleta for 1-strata of a defect surface. □

Remark 4.5. The 1-strata of an admissible 1-skeleton of s have a canonical framing which is induced from the orientation of s. Similarly, the 0-strata of a 0-skeleton carry framings, which makes these points into framed punctures (see Section 3.1).

Lemma 4.6. Let $M \in \mathrm{Bord}_3^{\mathrm{def}}$.

i) Two admissible 1-skeleta t_1, t_2 for the 2-strata of M which restrict to the same 0-skeleta on the boundaries are related by a finite number of local moves shown in Figure 4.6.

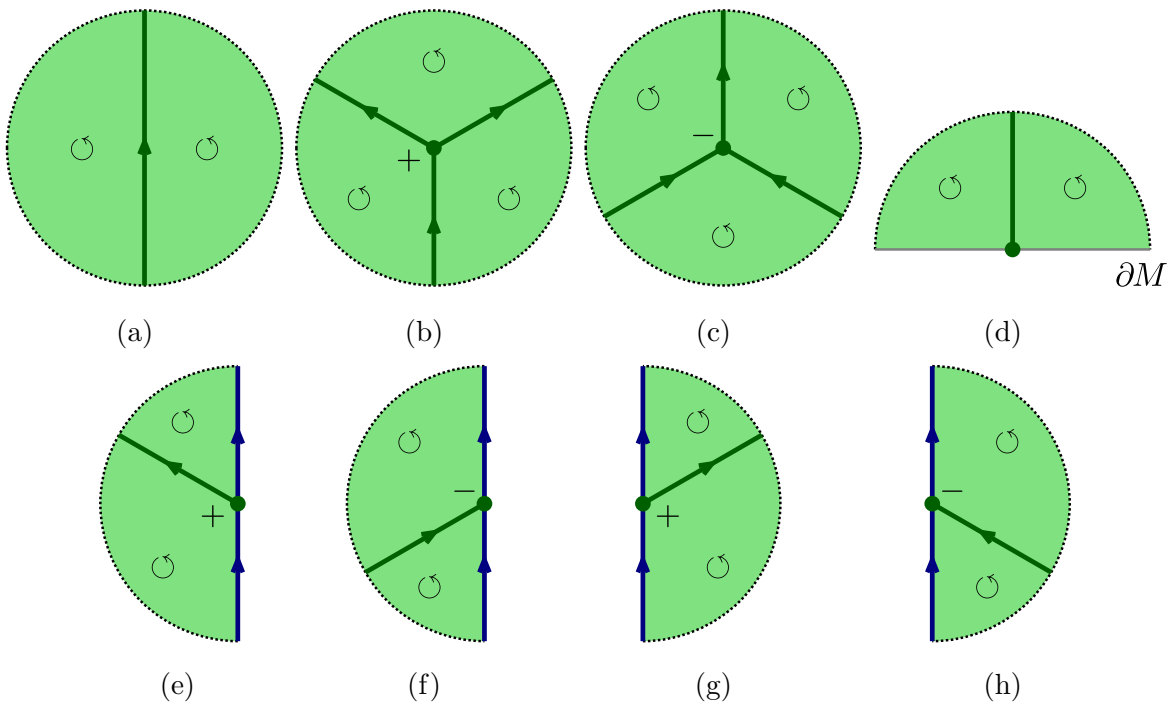


Figure 4.5: Local models of a point on a 0- or 1-stratum of an admissible 1-skeleton of a 2-stratum.

ii) Any set of 0-skeleta for the 1-strata of ∂M can be extended to a set of admissible 1-skeleta for the 2-strata of M.

The proof reduces an arbitrary 1-skeleton to one obtained as the Poincaré dual of a triangulation. The bl-moves in Figure 4.6 are known to imply the (dual) Pachner moves, which are used to transform one triangulation into another. The only complication arises due to admissibility constraints on the orientations. We sketch these details in Appendix B.

We now discuss the labelling conventions for an admissible 2-skeleton t for a 2-stratum s of $M = (M, \mathrm{T}) \in \mathrm{Bord}_3^{\mathrm{def}}$. Let us introduce some terminology:

- We call the points in $\pi(t^{(1)}) \cap \mathrm{T}^{(1)}$ the *intersection points* of t and T. The orientation of an intersection point is positive if the adjacent 1-stratum of t is directed away from it and negative otherwise (as depicted on the second line in Figure 4.5).

- Let σ be a 2-stratum of $\overline{\overline{s}}$. By a *1-stratum adjacent to* σ we will mean either an adjacent 1-stratum of $\overline{\overline{s}}$ or a non-empty open segment $(\pi(\sigma) \cap l) \setminus (\pi(t^{(1)}) \cap \mathrm{T}^{(1)})$ of a 1-stratum l of M (oriented the same way as l).

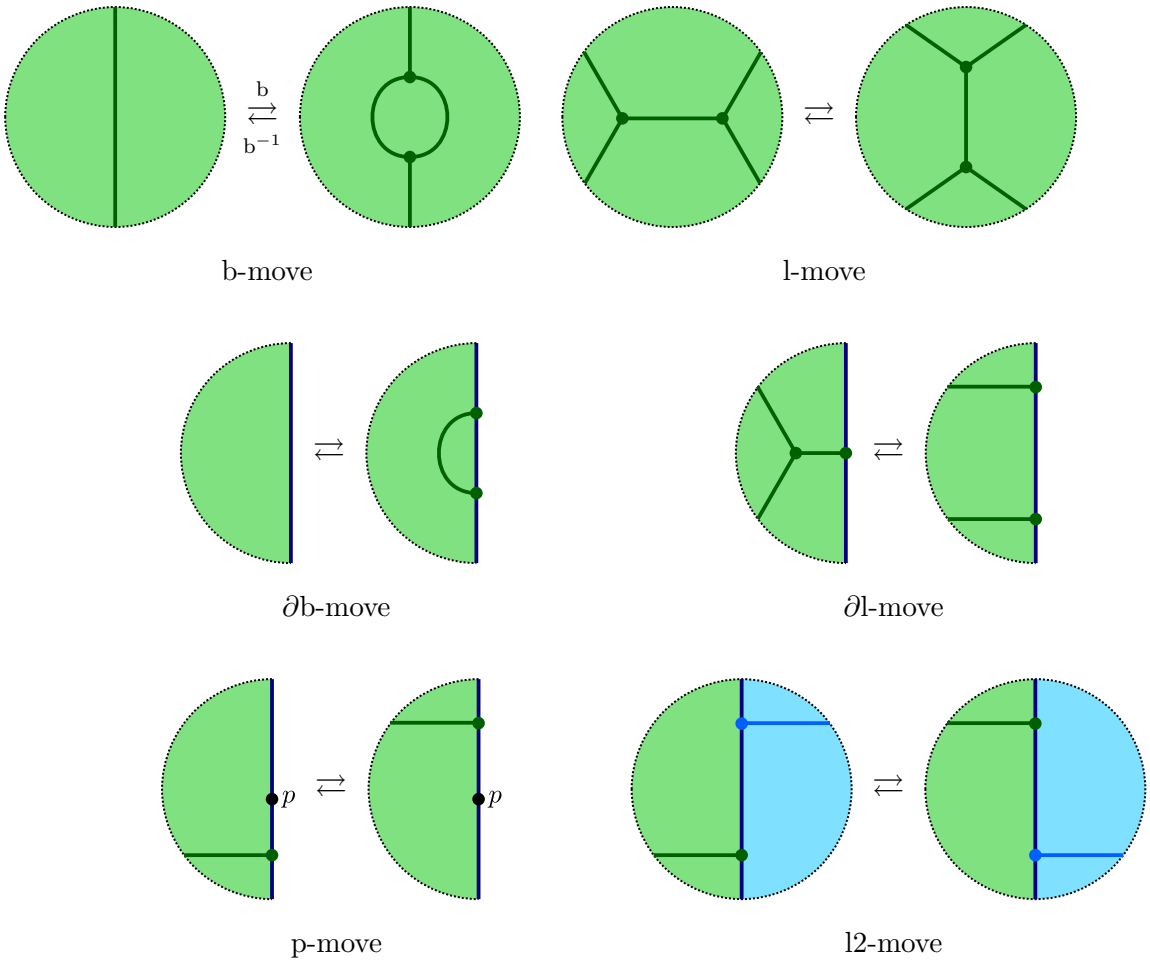


Figure 4.6: Moves on a set of admissible 1-skeleta for the 2-strata of a defect bordism M. The b- and l-moves are applied in the interior of a 2-stratum s, while the rest of them are applied in the neighbourhood of a point of $\partial\overline{s}$ which is projected on a 1-stratum of M, rather than on ∂M. The l2-move is the only one involving 1-skeleta from two different (germs of) 2-strata, adjacent to the same 1-stratum; the other adjacent 2-strata are not depicted. The label p in the depiction of the p-move refers to a 0-stratum of M, it too can have other adjacent 1- and 2-strata. There are several versions of each move, as the orientations are not listed; each move can be applied if the orientations on both sides are admissible (i.e. compatible with the local models in Figure 4.5) and agree at the boundary. This can sometimes prevent an inverse b-move, while the rest of the moves are always possible if the orientation of a newly created 1-stratum is chosen admissibly.

Definition 4.7. Let $M \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$ and s be an $A = (A, \psi) \in D_2^{\mathcal{C}}$ labelled 2-stratum of M. An *admissible A-coloured 1-skeleton for s* consists of

- an admissible 1-skeleton t for s;
- for each 2-stratum σ of s (where the stratification of s is given by t) a *special point* $p_\sigma \in l$ where l is a 1-stratum adjacent to σ; in addition, for distinct 2-strata σ, σ' of s one requires $p_\sigma \neq p_{\sigma'}$.

These data are assigned the following labels:

- the 1-strata of s are labelled by A, the positively (resp. negatively) oriented 0-strata of s as depicted in the local model in Figure 4.5b (resp. Figure 4.5c) are labelled by the coproduct (resp. product) morphisms of A;
- each positively (resp. negatively) oriented intersection point as depicted in the local models in Figures 4.5e and 4.5g (resp. 4.5f and 4.5h) is labelled by the coaction (resp. action) morphisms of A on K, where $K \in D_1^{\mathcal{C}}$ is the multimodule labelling the 1-stratum of M containing the intersection point;
- for each 2-stratum σ of s, if the adjacent 1-stratum l such that $p_\sigma \in l$ is labelled by $K \in D_1^{\mathcal{C}}$ (in particular labelled by $A \in D_1^{\mathcal{C}}$ if l lies in the interior of s), then p_σ is labelled by $\psi_{l/r}^K$ if $\sigma \cap \partial M \neq \varnothing$ and $(\psi_{l/r}^K)^2$ otherwise. Here $\psi_{l/r}^K$ means the choice between the morphisms ψ_l^K and ψ_r^K determined as follows: If l is a 1-stratum of t bounding σ from the right (resp. left) in the local model in Figure 4.5a, one chooses ψ_l^A (resp. ψ_r^A); if l is a 1-stratum of M bounding σ from the right (resp. left) like in the local models in Figures 4.5e and 4.5f (resp. in Figures 4.5g and 4.5h) one chooses ψ_l^K (resp. ψ_r^K);

A set of *admissible $\mathbb{D}^{\mathcal{C}}$-coloured 1-skeleta* for the 2-strata of M is a set $\{t(s)\}$ of admissible A_s-coloured 1-skeleta for each 2-stratum s of M labelled by $A_s \in D_2^{\mathcal{C}}$ such that all special points are distinct. □

Let $\Sigma = (\sigma, \mathrm{t}) \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$ be a $\mathbb{D}^{\mathcal{C}}$-coloured defect surface with the stratification t and let τ be a set of 0-skeleta for its 1-strata. We define

$$R(\Sigma, \tau) \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}) \tag{4.11}$$

to be the underlying oriented surface Σ with the set of punctures $\mathrm{t}^{(0)} \cup \bigcup_l \tau^{(0)}(l)$ (see Remark 4.5), where l runs over the 1-strata of Σ. A point $p \in \mathrm{t}^{(0)}$ is already labelled by a multimodule $K \in D_1^{\mathcal{C}}$, whereas a point $p \in \tau^{(0)}(l)$ is labelled by the algebra object $A \in D_1^{\mathcal{C}}$ labelling l.

For a defect bordism $[M = (M, \mathrm{T})\colon \Sigma \to \Sigma'] \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$, let t be a set of $\mathbb{D}^{\mathcal{C}}$-coloured 1-skeleta for the 2-strata of M which restricts to sets of 0-skeleta τ and τ' for the 1-strata of Σ and Σ'. Denote by t_I and t_S the sets of intersection points of t and T, and of special points of t. We define

$$\big[R(M,t)\colon R(\Sigma,\tau) \to R(\Sigma',\tau')\big] \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}) \tag{4.12}$$

to be the underlying 3-manifold M with the embedded ribbon graph

$$R = \mathrm{T}^{(1)} \cup \bigcup_s t^{(1)}(s) \cup t_I \cup t_S\,, \tag{4.13}$$

i.e. its strands are the 1-strata of both M (labelled by the multimodules in $D_1^{\mathcal{C}}$) and t (labelled by the algebra objects in $D_2^{\mathcal{C}}$) and the coupons are the 0-strata of both M (labelled by multimodule morphisms) and t (labelled by (co)multiplication morphisms), as well as the intersection points (labelled by (co)actions) and the special points of t (labelled by the ψ-morphisms of the corresponding algebra).

4.4. Definition of the defect TQFT

The penultimate step in defining the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ is the following

Lemma 4.8. Let $[M\colon \Sigma \to \Sigma'] \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$ and let t_1, t_2 be two sets of $\mathbb{D}^{\mathcal{C}}$-coloured 1-skeleta for the 2-strata of M which restrict to the same 0-skeleta τ, τ' on Σ and Σ'. Then one has

$$Z_{\mathcal{C}}^{\mathrm{RT}}\big(\,R(M,t_1)\,\big) = Z_{\mathcal{C}}^{\mathrm{RT}}\big(\,R(M,t_2)\,\big)\,. \tag{4.14}$$

Proof. We need to show that upon the evaluation with $Z_{\mathcal{C}}^{\mathrm{RT}}$ one can apply the moves in Figure 4.6 on t_1, t_2. This is however guaranteed by how the algebraic structures constituting the label sets $\mathbb{D}^{\mathcal{C}}$ were chosen. In particular, the b-move follows from the separability (2.37) and the symmetry (2.31) properties of algebras in $D_2^{\mathcal{C}}$ (the special points labelled by ψ-insertions in Definition 4.7 were introduced solely for this purpose). The l-move follows from the symmetry and Frobenius property (2.30). Similarly, the moves $\partial\,$b and $\partial\,$l follow from the symmetry property and the fact that the adjacent line is labelled by a module of the corresponding algebra. The p-move follows from labelling the 0-strata with multimodule morphisms, which commute with the algebra actions. The l2-move follows from the identities (4.8) defining multimodules. □

In light of Lemma 4.8, one only needs to eliminate the dependence on the 0-skeleta on the boundary, which is achieved by a standard limit construction which

we now describe. Let Σ be an object in $\widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$ and let τ, τ' be two sets of 0-skeleta for its 1-strata. Consider the defect cylinder $C_\Sigma := \Sigma \times [0,1]$ and the linear map

$$\Phi_\tau^{\tau'} := \Big[Z_{\mathcal{C}}^{\mathrm{RT}}\big(R(C_\Sigma, t) \big) \colon Z_{\mathcal{C}}^{\mathrm{RT}}\big(R(\Sigma, \tau) \big) \to Z_{\mathcal{C}}^{\mathrm{RT}}\big(R(\Sigma, \tau') \big) \Big] , \tag{4.15}$$

where t is an arbitrary set of 1-skeleta for the 2-strata of C_Σ restricting to τ, τ' on $\Sigma \times \{0\}$, $\Sigma \times \{1\}$. For three sets τ, τ', τ'' of 0-skeleta of Σ one has

$$\Phi_{\tau'}^{\tau''} \circ \Phi_\tau^{\tau'} = \Phi_\tau^{\tau''} . \tag{4.16}$$

In particular, each map Φ_τ^τ is an idempotent.

With this preparation, we can now reformulate the construction in [CRS2] as follows:

Construction 4.9. Let $\mathcal{C}$ be a modular fusion category. The Reshetikhin-Turaev defect TQFT

$$Z_{\mathcal{C}}^{\mathrm{def}} \colon \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) \to \mathrm{Vect}_{\Bbbk} \tag{4.17}$$

is defined as follows:

1. For an object $\Sigma \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$, we set

$$Z_{\mathcal{C}}^{\mathrm{def}}(\Sigma) = \mathrm{colim}\big\{ \Phi_\tau^{\tau'} \big\} , \tag{4.18}$$

   where τ, τ' range over all sets of 1-skeleta for the 1-strata of Σ.

2. For a morphism $[M \colon \Sigma \to \Sigma'] \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$, we set $Z_{\mathcal{C}}^{\mathrm{def}}(M)$ to be

$$Z_{\mathcal{C}}^{\mathrm{def}}(\Sigma) \hookrightarrow Z_{\mathcal{C}}^{\mathrm{RT}}\big(R(\Sigma, \tau)\big) \xrightarrow{Z^{\mathrm{RT}}(R(M,t))} Z_{\mathcal{C}}^{\mathrm{RT}}\big(R(\Sigma', \tau')\big) \twoheadrightarrow Z_{\mathcal{C}}^{\mathrm{def}}(\Sigma') , \tag{4.19}$$

   where t is an arbitrary set of 1-skeleta for the 2-strata of M that restricts to sets τ and τ' of 0-skeleta for the 1-strata of Σ and Σ' respectively. The inclusion is part of the data of the colimit, and the projection is obtained from the universal property.

Remark 4.10. In practice, for an object $\Sigma \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$, the state space can be conveniently computed as the image of the idempotent Φ_τ^τ, i.e.

$$Z_{\mathcal{C}}^{\mathrm{def}}(\Sigma) \cong \mathrm{im}\, \Phi_\tau^\tau . \tag{4.20}$$

Remark 4.11. Labelling a 2-stratum with a symmetric separable Frobenius algebra $(A, \psi) \in D_2^{\mathcal{C}}$ and introducing the ψ-insertions in the ribbonisation procedure is similar to the 2-dimensional state-sum construction in [LP]. There one requires the morphism $\mu \circ \Delta \circ \eta \colon \mathbb{1} \to A$ (the *window element*) to have a multiplicative inverse, which is then used to cancel the extra factors appearing at each 2-stratum of a 1-skeleton. Our approach is however slightly more general: take for example the case $\operatorname{char} \Bbbk = 0$ and the algebra $A = X \otimes X^*$ for a non-zero object $X \in \mathcal{C}$ with $\dim X = 0$. The window element is then zero hence non-invertible, but one can nonetheless find a suitable ψ-insertion as follows: Let $\bigoplus_\alpha i_\alpha$ be a decomposition of X into simple objects with π_α, $\imath_\alpha$ denoting the corresponding projection/inclusion. Define a morphism $\xi \in \operatorname{End}_{\mathcal{C}}(X)$ by $\xi := \sum_\alpha (\dim i_\alpha)^{-1/2} \imath_\alpha \circ \pi_\alpha$. Then one has $\operatorname{tr} \xi^2 = \dim \operatorname{End}_{\mathcal{C}}(X) \neq 0$, which can be used to define the ψ-insertion

$$\psi := (\dim \operatorname{End}_{\mathcal{C}}(X))^{-1/2} \cdot [\mathbb{1} \xrightarrow{\operatorname{coev}_X} X \otimes X^* \xrightarrow{\xi \otimes \operatorname{id}_{X^*}} X \otimes X^*] \,, \tag{4.21}$$

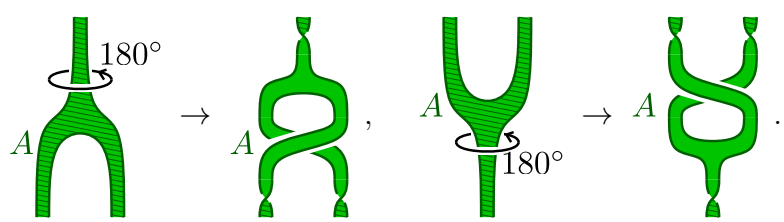
so that the assumptions in Convention 2.11 hold. Note also, that as long as the condition (2.37) holds, the ψ-insertions in the ribbonisation procedure need not even be invertible. Such generalisation is however not necessary for our purposes.

4.5. Properties of the defect TQFT

Property 4.12. Let $M_{A,\epsilon}$ be a $\mathbb{D}^{\mathcal{C}}$-coloured defect bordism, where $A \in D_2^{\mathcal{C}}$ and $\epsilon \in \{\pm\}$ parametrise the label and the orientation and of a fixed closed 2-stratum s of M. Then one has

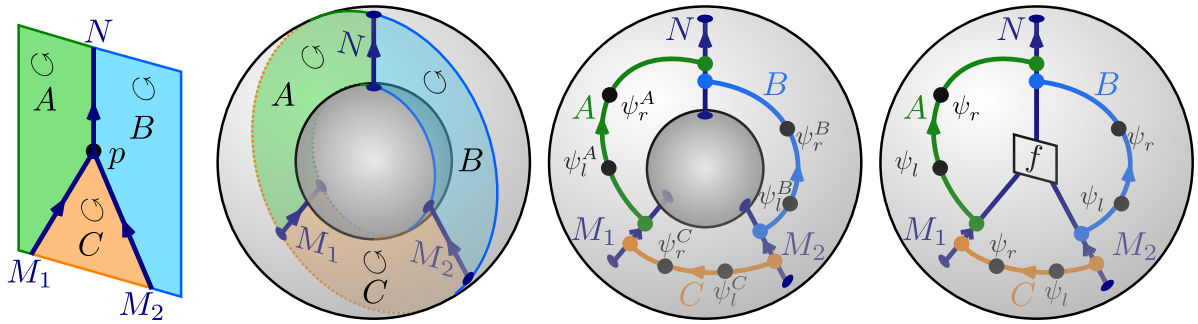
$$Z_{\mathcal{C}}^{\mathrm{def}}(M_{A,-}) = Z_{\mathcal{C}}^{\mathrm{def}}(M_{A^{\mathrm{op}},+}) \,, \tag{4.22}$$

i.e. the orientation of s can be flipped by relabelling it with the opposite algebra (see Section 2.4). This is because after the ribbonisation procedure one can rotate the strands coming from the skeleta of the 2-strata so that their framing faces the opposite direction. Doing this to the coupons labelled by the (co)multiplication morphisms results in crossings as in the definition of the opposite algebra (cf. (2.32) and (2.33)):



(4.23)

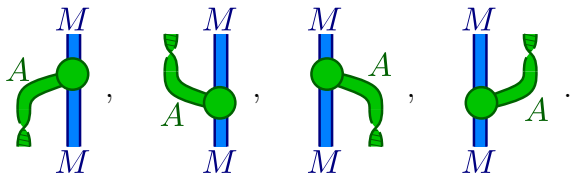
One can also use this to introduce orientations for the 2-strata adjacent to a multi-module M which are different from those imposed by the local model in Figure 4.3:



(a) (b) (c) (d)

Figure 4.7: (a) A 0-stratum p . (b) The defect cylinder $C = S^2_p \times [0,1]$ presented as a closed ball with an open ball removed. (c) A choice $C_R\colon S^2_R \to S^2_R$ of ribbonisation of C. (d) The closed ball $\varnothing \to S^2_R$ corresponding to the image $\Phi(f)$.

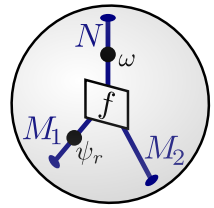
If an A-labelled 2-stratum in the local model does not have the paper plane orientation, the multimodule M has instead an A^{op}-action and the ribbonisation procedure is changed so that in this case the intersection points are labelled with (co)action morphisms with a half-twist of the adjacent A-strand as follows:



(4.24)

Property 4.13. The defect TQFT $Z^{\mathrm{def}}_{\mathcal{C}}$ is D_0-complete (see Section 4.1). Let us demonstrate this with a simple example of a point defect p adjacent to three 2-strata, labelled by algebras $A = (A, \psi^A)$, $B = (B, \psi^B)$, $C = (C, \psi^C)$ in $D^{\mathcal{C}}_2$ and three 1-strata, labelled by A-C-bimodule M_1, C-B-bimodule M_2 and A-B-bimodule N as shown in Figure 4.7a. We need to compute the vector space assigned by $Z^{\mathrm{def}}_{\mathcal{C}}$ to the stratified 2-sphere S^2_p, obtained as a new boundary component after removing a small open ball B°_p surrounding p, which can be done using the formula (4.20). To this end, let us use the presentation of the cylinder $C = S^2_p \times [0,1]$ as well as a choice $[C_R\colon S^2_R \to S^2_R] \in \widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C})$ for its ribbonisation as depicted in Figures 4.7b and 4.7c and set $\Phi := Z^{\mathrm{RT}}_{\mathcal{C}}(C_R)\colon Z^{\mathrm{RT}}_{\mathcal{C}}(S^2_R) \to Z^{\mathrm{RT}}_{\mathcal{C}}(S^2_R)$. From Property 3.12 of the Reshetikhin-Turaev TQFT we obtain the explicit isomorphism $Z^{\mathrm{RT}}_{\mathcal{C}}(S^2_R) \cong \mathcal{C}(M_1M_2, N)$. For $f \in \mathcal{C}(M_1M_2, N)$, the image $\Phi(f)$ corresponds to a bordism $\varnothing \to S^2_R$ depicted in Figure 4.7d. One recognises the pair of A- and B-lines as the projector onto the space ${}_A\mathcal{C}_B(M_1M_2, N)$ of A-B-bimodule morphisms,

and the C-line as the projector onto the relative tensor product $M_1 \otimes_C M_2$. One therefore obtains the isomorphism

$$ {}_A\mathcal{C}_B(M_1 \otimes_C M_2, N) \cong \operatorname{im}\Phi\,, \quad f \mapsto $$  $$. \tag{4.25}$$

The space on the left-hand side is precisely the subset of $D_0^{\mathcal{C}}$ of the suitable labels for the point p. The ribbon bordism on the right is a choice of ribbonisation for the defect bordism $B_p(f)\colon \varnothing \to S_p^2$.

This example can be generalised for point defects p with an arbitrary configuration of adjacent 1- and 2-strata. We note in particular that the fusion of point insertions on 1- and 2-strata correspond to compositions of the respective multimodule and bimodule morphisms, with the trivial label being given by the identity morphism.

Property 4.14. The defect TQFT $Z_{\mathcal{C}}^{\text{def}}$ is Euler complete with respect to surfaces. Indeed, let $(A, \psi) \in D_2^{\mathcal{C}}$ be the symmetric separable Frobenius algebra having the (co)multiplication and (co)unit morphisms μ, η (Δ, ε) which labels a 2-stratum s. Then for an invertible A-A-bimodule morphism $\gamma\colon A \to A$, adding a $\gamma^{\chi_{\text{sym}}(s)}$-insertion on s corresponds to relabelling s with a symmetric separable Frobenius algebra (A_γ, ψ_γ), having the same underlying object $A \in \mathcal{C}$ and the rescaled morphism $\psi_\gamma := \gamma \circ \psi$, as well as the following rescaled structure morphisms:

$$\begin{aligned} &\text{multiplication:} \quad \gamma^{-1} \circ \mu\,, \qquad &&\text{unit:} \quad \gamma \circ \eta\,, \\ &\text{comultiplication:} \quad \Delta \circ \gamma^{-1}\,, \qquad &&\text{counit:} \quad \varepsilon \circ \gamma\,. \end{aligned} \tag{4.26}$$

To see this one can argue as follows: If an admissible 1-skeleton t of a 2-stratum s (assumed for simplicity to have no adjacent 1-strata) is obtained as the Poincaré dual of a triangulation of s, one can employ the usual formula $\chi(s) = V - E + F$ to compute its Euler characteristic. We assign to each triangle a γ^2-insertion to its face (i.e. a 0-stratum of t) and a γ^{-1}-insertion to each of its three edges (so that a 1-stratum of t gets a γ^{-1}-insertion if it touches the boundary and a γ^{-2}-insertion otherwise), while the vertices (i.e. 2-strata of t) already receive the γ^2-insertion from the ψ_γ-labelled special points of t. Since γ is an A-A-bimodule morphism, for each triangle one can fuse the insertions assigned to it into a single γ^{-1}-insertion, which is why the (co)multiplication of A_γ are as in (4.26). Collecting all γ-insertions on s, the number of them adds exactly to $\chi_{\text{sym}}(s)$ (see (4.7)).

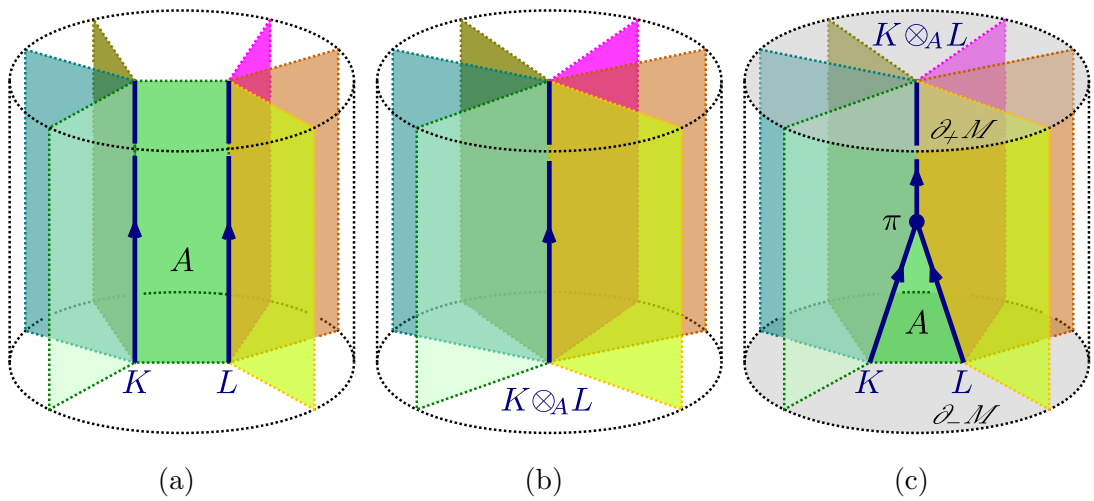


Figure 4.8: Upon evaluation with $Z_{\mathcal{C}}^{\mathrm{def}}$, two parallel lines labelled by K, L (a) can be fused into a single line labelled by $K \otimes_A L$ (b). Evaluating the cylinder containing the projection morphism $\pi\colon K \otimes L \to K \otimes_A L$ (c) provides an isomorphism between the two state spaces. Its inverse is obtained from a similar cylinder containing the inclusion $\imath\colon K \otimes_A L \to K \otimes L$.

The modules of the algebras A and A_γ are in a bijection: one makes a left A-module (L, λ) and a right A-module (K, ρ) into the corresponding A_γ-modules by taking the rescaled action morphisms λ_γ and ρ_γ defined by

$$\lambda_\gamma := \lambda \circ (\gamma^{-1} \otimes \mathrm{id}_L) \quad , \qquad \rho_\gamma := \rho \circ (\mathrm{id}_K \otimes \gamma^{-1}) \,. \tag{4.27}$$

We remark that the Euler completeness for surfaces would not have been achieved if the label set $D_2^{\mathcal{C}}$ was taken to be the symmetric Δ-separable Frobenius algebras only, as was the case in [CRS2]. Our approach allows one to hide the point insertions inside the ribbonisation procedure, which makes e.g. the figures depicting $\mathbb{D}^{\mathcal{C}}$-coloured defect bordisms easier to understand since the point insertions do not clutter them.

Property 4.15. Let M be a defect bordism and let l_1, l_2 be two parallel 1-strata labelled by multimodules K and L, connected by a 2-stratum s labelled by $A \in D_2^{\mathcal{C}}$ (see Figure 4.8a). Upon ribbonisation, s gets replaced by a ribbon graph connecting l_1 and l_2, which upon evaluation with $Z_{\mathcal{C}}^{\mathrm{RT}}$ can be replaced by the idempotent (2.50), projecting $K \otimes L$ onto $K \otimes_A L$. The same result can be obtained by replacing l_1, l_2 and s with a single 1-stratum labelled with $K \otimes_A L$ (see Figure 4.8b). If l_1, l_2 end on the boundary or on 0-strata in M, the endpoints also need to be relabelled (and in the case of endpoints on the boundary, a cylinder as in Figure 4.8c provides an isomorphism between the surfaces with different decorations).

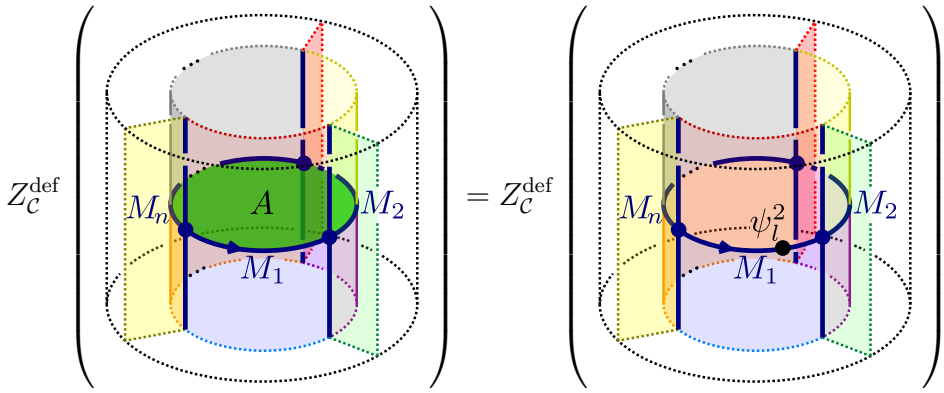


Figure 4.9: Upon evaluation with $Z_{\mathcal{C}}^{\mathrm{def}}$, a contractible 2-stratum s labelled by $A \in D_2^{\mathcal{C}}$ can be removed by forgetting all A-actions on the multimodules M_i which decorate adjacent 1-strata.

Property 4.16. Let s be a 2-stratum labelled with an algebra $(A, \psi) \in D_2^{\mathcal{C}}$, which is bounded by 1-strata labelled by multimodules $M_1, M_2, \ldots, M_n$ (and by some 0-strata between them). Assume that s is contractible, as illustrated on the left-hand side of Figure 4.9. Then, upon evaluation with $Z_{\mathcal{C}}^{\mathrm{def}}$, s can be removed by adding a single ψ^2-insertion on one of the 1-strata, adjacent to s, with the actions of A on $M_1, M_2, \ldots, M_n$ otherwise forgotten to make them into valid new labels. This is summarised in Figure 4.9. To show this identity, one can argue that during the ribbonisation procedure the 1-skeleton of s can be taken to have a single 2-stratum.

Property 4.17. The vector spaces assigned by $Z_{\mathcal{C}}^{\mathrm{def}}$ to a $\mathbb{D}^{\mathcal{C}}$-coloured defect surface Σ can be constructed similarly as the vector spaces assigned by $Z_{\mathcal{C}}^{\mathrm{RT}}$ to the punctured surfaces (see Property 3.13). In particular, they are spanned by defect handlebodies H_g, obtained by retracting the stratification on the boundary Σ to the core of H_g (much like for $Z_{\mathcal{C}}^{\mathrm{RT}}$, the ribbon graphs in the handlebodies (3.22) were positioned at the core with extra strands connecting them to the punctures). To see this, let us investigate an example of a 2-torus T_l^2 with a single non-contractible $A = (A, \psi) \in D_2^{\mathcal{C}}$ labelled loop. We use the presentation of the cylinder $C = T_l^2 \times [0, 1]$ as in Figure 4.10a, where it is depicted by a cylinder of an annulus with the top and bottom ends identified. A choice $C_R : T_R^2 \to T_R^2$ for the ribbonisation of C is shown in Figure 4.10b. We compute $Z_{\mathcal{C}}^{\mathrm{def}}(T_l^2)$ as the image of the map $\Phi := [Z_{\mathcal{C}}^{\mathrm{RT}}(C_R) \colon Z_{\mathcal{C}}^{\mathrm{RT}}(T_R^2) \to Z_{\mathcal{C}}^{\mathrm{RT}}(T_R^2)]$ using the formula (4.20). From Property 3.13 we know that the vector space $Z_{\mathcal{C}}^{\mathrm{RT}}(T_R^2)$ is spanned by the solid tori $H(i, f)$ with an $i \in \mathrm{Irr}_{\mathcal{C}}$ labelled strand at the core, and an $f \in \mathcal{C}(i, Ai)$ insertion, connected to the A-labelled puncture of T_R^2 by an A-labelled strand. The image

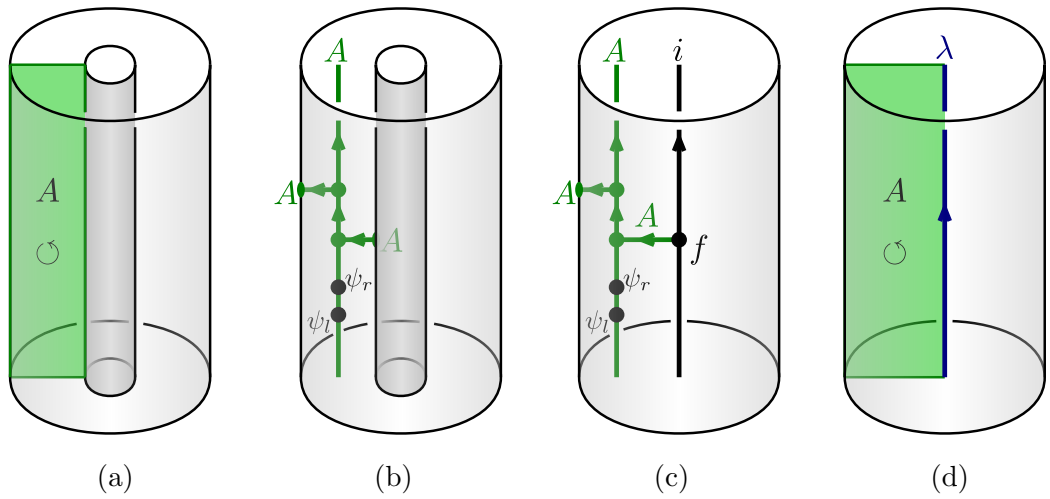


Figure 4.10: Computation of the vector space $Z_{\mathcal{C}}^{\mathrm{def}}(T_l^2)$

over Φ then corresponds to the composition $C_R \circ H(i,f)\colon \varnothing \to T_R^2$ shown in Figure 4.10c. Upon evaluating with $Z_{\mathcal{C}}^{\mathrm{RT}}$, the parallel A- and i-lines can be exchanged for a single line, labelled by the induced module $A \otimes i$, which has point insertion labelled by the left module morphism, obtained from composing the ψ_r- and f-insertions. By the semisimplicity of the category ${}_A\mathcal{C}$ of left modules, $A \otimes i$ can be decomposed into simple left modules, with the point insertions replaced by scalar factors. The space $\operatorname{im}\Phi$ is therefore spanned by the ribbonisations of the stratified solid tori $H(\lambda)$ as in Figure 4.10d, where λ is a simple left A-module, labelling the line at the core. In fact, the set $\{H(\lambda)\}_\lambda$ where $\lambda \in \mathrm{Irr}_{{}_A\mathcal{C}}$ constitutes a basis of the space $Z_{\mathcal{C}}^{\mathrm{def}}(T_l^2)$, see [FRS1, Thm. 5.18].

Property 4.18. As was explained in [FSV, Sec. 6], the dependence of the invariants, produced by the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$, on the label $(A,\psi) \in D_2^{\mathcal{C}}$ of a 2-stratum is only up to the Morita class (see Section 2.7) of the algebra A. This is because in the "hole punching" interpretation (Figure 4.4) one had a freedom to choose the label for the boundary of the hole, effectively exchanging the algebra A for the algebra $K \otimes_A K^*$ for some right A-module K, which is known to be Morita equivalent to A. Our approach with separable algebras, which yields the ψ-insertions, also allows one to control the various factors, which would otherwise appear after such an exchange. For example, when exchanging $A = \mathbb{1}$ with the Morita equivalent algebra $X \otimes X^*$ for an object X^*, the factor due to adding a hole is $\dim X$, which in the setting of Remark 4.11 can be eliminated if $X \otimes X^*$ is equipped with the ψ-insertion (4.21).

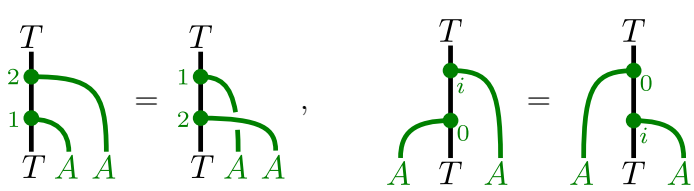
5. Orbifold data and the associated MFCs

A notion of an orbifold datum for an n-dimensional defect TQFT Z was introduced in [CRS1] as an input needed to perform an internal state sum construction, which in turn yields a new n-dimensional TQFT, a *generalised orbifold* of Z. The main goal of this work is to explore generalised orbifolds of the 3-dimensional Reshetikhin-Turaev defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$, obtained from a modular fusion category $\mathcal{C}$. In particular, we address the question whether they themselves are TQFTs of Reshetikhin-Turaev type. The answer, as will be shown in Chapter 6, turns out to be affirmative. Therefore, in this case an orbifold datum can be alternatively seen as a certain algebraic input $\mathbb{A}$ in the modular fusion category $\mathcal{C}$, which yields a new modular fusion category $\mathcal{C}_{\mathbb{A}}$. In this chapter we define the category $\mathcal{C}_{\mathbb{A}}$ and prove some of its properties. The exposition is mostly algebraic, we present some of the interpretation of orbifold data and the associated modular fusion categories in terms of defect TQFTs in Section 5.2, but it will mostly be postponed until Chapter 6.

Orbifold data for modular fusion categories were first introduced in [CRS3]. The material below is based on [MR1]. For the rest of the chapter we fix a modular fusion category $\mathcal{C}$.

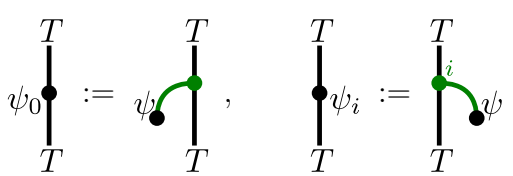
5.1. Algebraic definitions

Let $A = (A, \psi)$ be a symmetric separable Frobenius algebra in $\mathcal{C}$. In what follows, we will often encounter A-$A \otimes A$-bimodules and various relative tensor products involving them. Recall from Section 4.2, that an A-$A \otimes A$ bimodule T is the same as A-A, A-multimodule, i.e. an object $T \in \mathcal{C}$ together with a left and two right A-actions, such that



$$i \in \{1, 2\}\,, \tag{5.1}$$

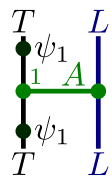
where the right actions of the first and the second A factor are distinguished by the indices 1, 2. Similarly, the left A-action will be indicated by 0 whenever we find it necessary to avoid ambiguity. The notation (2.40) for the ψ-insertions is adapted in this case as

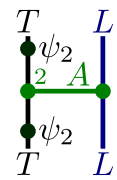


$$i \in \{1, 2\}\,. \tag{5.2}$$

Note that because of (5.1), the morphisms ψ_0, ψ_1, ψ_2 commute under composition.

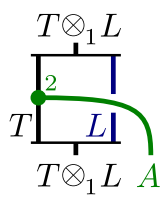
For a left A-module L, let us denote by $T \otimes_1 L$ and $T \otimes_2 L$ the relative tensor products with respect to the corresponding right A-action on T, i.e.

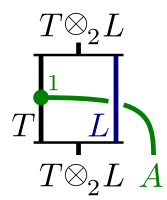
$T \otimes_1 L \cong \mathrm{im}$ 

, $T \otimes_2 L \cong \mathrm{im}$ 

. (5.3)

Note that $T \otimes_1 L$ and $T \otimes_2 L$ are A-A-bimodules, with the right A-actions given by



, 

(5.4)

and the left A-action obtained from the one on T. If M is an A-A-bimodule, $T \otimes_1 M$ and $T \otimes_2 M$ are themselves A-$A \otimes A$-bimodules. Indeed, for $T \otimes_1 M$ (resp. $T \otimes_2 M$) the first right A-action is obtained from the one on M (resp. as in (5.4)), while the second A-action is as in (5.4) (resp. obtained from the one on M). This can be easily generalised further, e.g. for an A-$A \otimes A$-bimodule T, the dual T^* is an $A \otimes A$-A-bimodule, $T \otimes_1 T$, $T \otimes_2 T$ are A-$A \otimes A \otimes A$ bimodules, $T^* \otimes_0 T$ is an $A \otimes A$-$A \otimes A$-bimodule, etc. When defining any explicit morphisms between objects obtained by taking such relative tensor products we will use the Convention 2.16 to present them as morphisms between the regular tensor products (between the underlying objects in $\mathcal{C}$), commuting with the corresponding A-actions.

With this preparation we are ready to state the following (cf. [CRS3, Sec. 3.2])

Definition 5.1. An *orbifold datum* in $\mathcal{C}$ is a tuple $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ where

- (A, ψ) is a symmetric separable Frobenius algebra in $\mathcal{C}$;
- T is an A-$A \otimes A$-bimodule in $\mathcal{C}$;
- $\alpha \colon T \otimes_2 T \to T \otimes_1 T$, is an A-$A \otimes A \otimes A$-bimodule morphism with the inverse $\overline{\alpha} \colon T \otimes_1 T \to T \otimes_2 T$;
- $\phi \in \Bbbk^\times$;

which satisfies the conditions (O1)-(O8) in Figure 5.1.

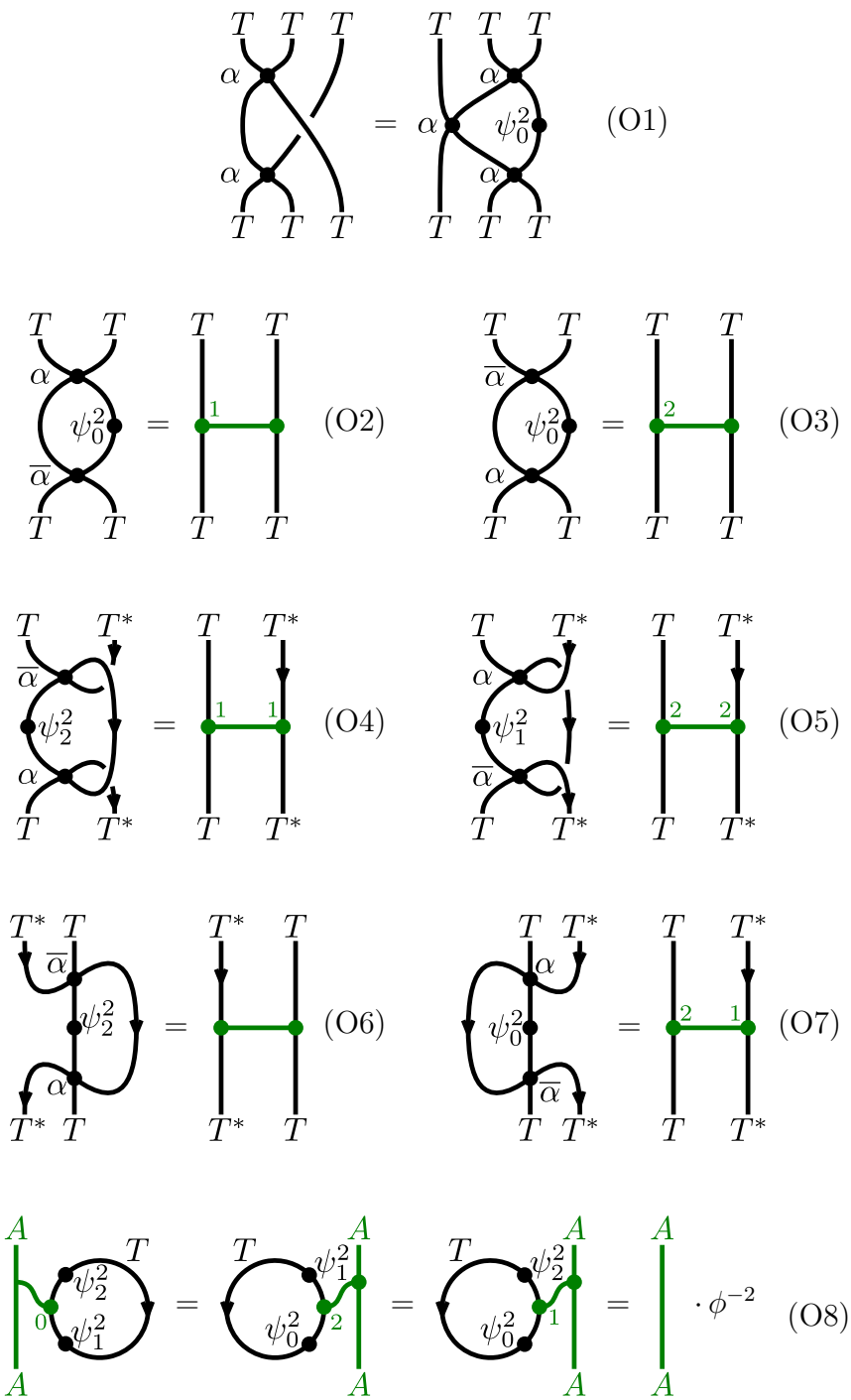


Figure 5.1: Conditions on an orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$.

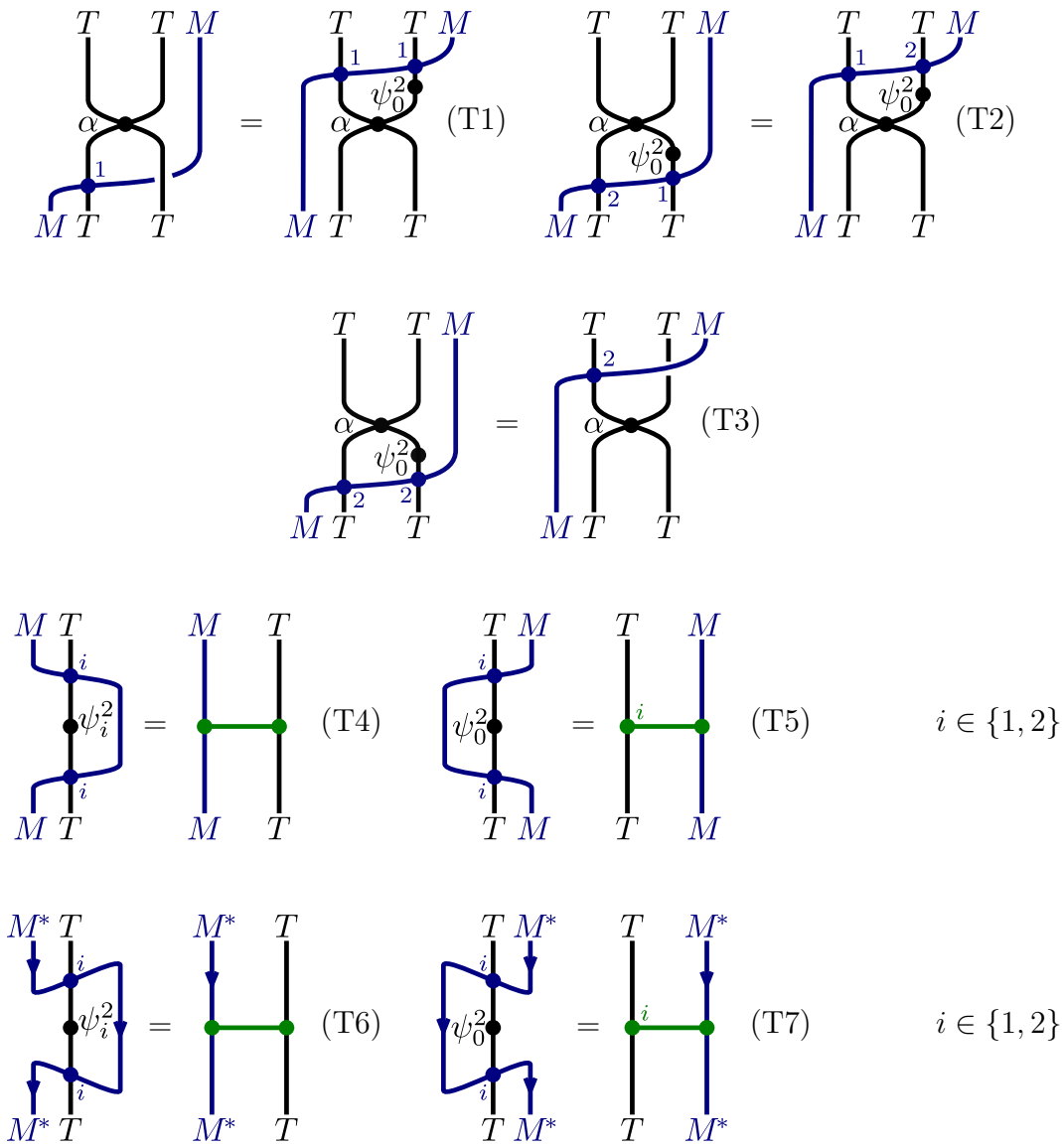
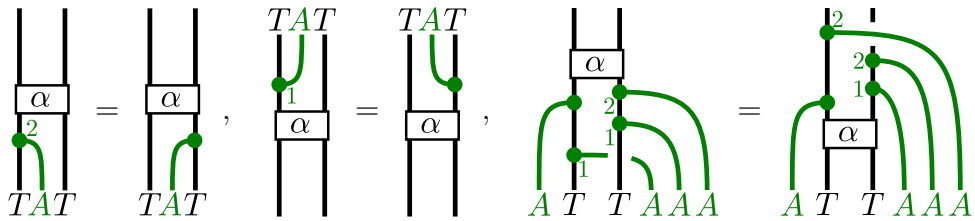
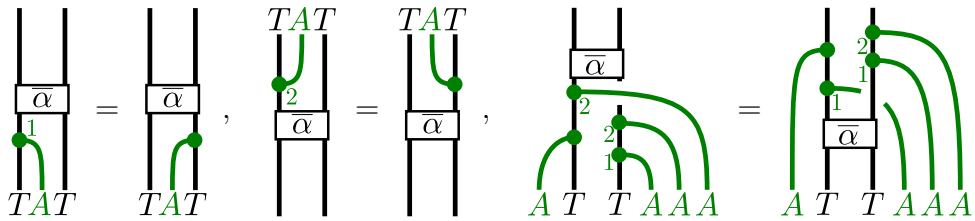


Figure 5.2: Conditions on an object $(M, \tau_1, \tau_2) \in \mathcal{C}_{\mathbb{A}}$.

Remark 5.2. Let $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ be an orbifold datum in $\mathcal{C}$. In the definition above we used the Convention 2.16 applied to the A-$A\otimes A\otimes A$-bimodule morphisms $\alpha\colon T\otimes_2 T \leftrightarrow T\otimes_1 T : \overline{\alpha}$ which allows one to define it as morphisms $T\otimes T \to T\otimes T$ in $\mathcal{C}$ such that

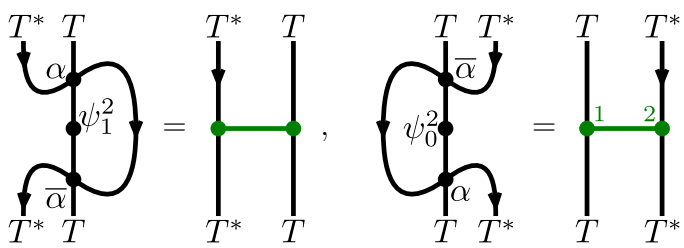


(5.5)



(5.6)

The conditions (O2), (O3) are just spelled-out requirements for α and $\overline{\alpha}$ to be inverses of each other (with the additional appearance of a ψ^2-insertion as explained in Remark 2.15). The identities (O1)-(O8) imply other similar identities, e.g. the following are the dual versions of (O6) and (O7)



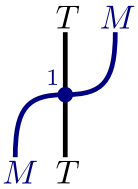
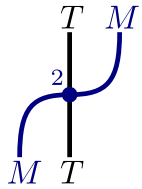
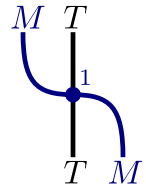
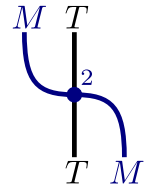
(5.7)

We are now ready to introduce the main construction of this work.

Definition 5.3. Let $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ be a defect datum in a modular fusion category $\mathcal{C}$. Define the category $\mathcal{C}_{\mathbb{A}}$ to have:

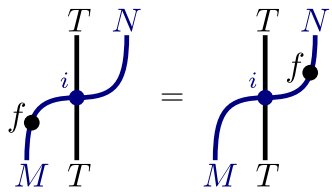
- *objects:* triples (M, τ_1, τ_2), where
  - M is an A-A-bimodule;
  - $\tau_1\colon M\otimes_0 T \to T\otimes_1 M, \quad \tau_2\colon M\otimes_0 T \to T\otimes_2 M$ are A-AAA-bimodule

isomorphisms, with inverses $\overline{\tau_1}$, $\overline{\tau_2}$, denoted by

$\tau_1 =$  , $\tau_2 =$  , $\overline{\tau_1} =$  , $\overline{\tau_2} =$  , (5.8)

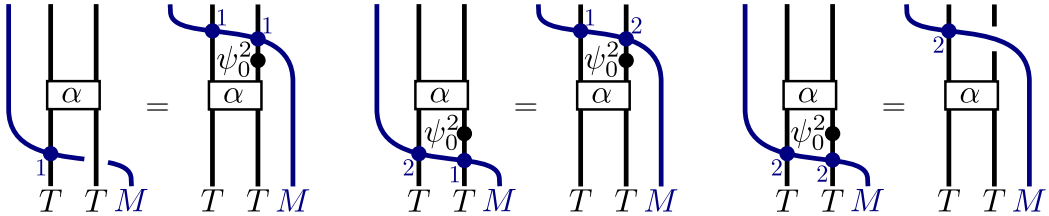
such that the identities (T1)-(T7) in Figure 5.2 are satisfied (note that the identities (T4) and (T5) are just spelled out requirements for τ_i and $\overline{\tau_i}$ to be the inverses of each other);

- *morphisms:* a morphism $f\colon (M, \tau_1^M, \tau_2^M) \to (N, \tau_1^N, \tau_2^N)$ is an A-A-bimodule morphism $f\colon M \to N$ such that $\tau_i^N \circ (f \otimes_0 \mathrm{id}_T) = (\mathrm{id}_T \otimes_i f) \circ \tau_i^M$, $i = 1, 2$, or graphically

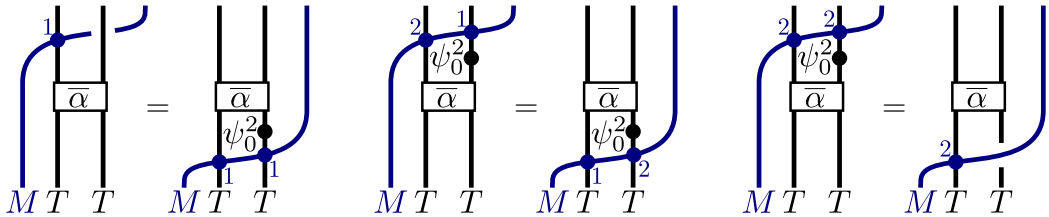
 , $i = 1, 2\,.$ (M)

For an object $(M, \tau_1, \tau_2) \in \mathcal{C}_{\mathbb{A}}$ we will refer to the morphisms τ_1, τ_2, $\overline{\tau_1}$, $\overline{\tau_2}$ as its *T-crossings.* □

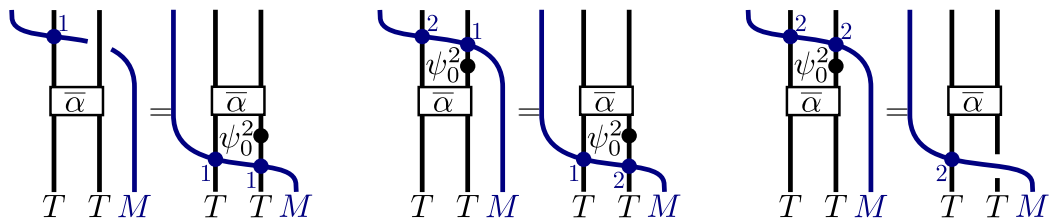
The conditions (T1)-(T3) imply other similar identities, for example:



(T8′) (T9′) (T10′)



(T11′) (T12′) (T13′)



(T14′) (T15′) (T16′)

An orbifold datum $\mathbb{A}$ also provides one with two further categories $\mathcal{C}_1^{\mathbb{A}}$, $\mathcal{C}_2^{\mathbb{A}}$ which we will find useful later:

Definition 5.4. Define the categories $\mathcal{C}_{\mathbb{A}}^i$, $i = 1, 2$ as follows:

- *objects* of $\mathcal{C}_{\mathbb{A}}^i$ are pairs (M, τ_i) where $M \in {}_A\mathcal{C}_A$ and $\tau_i \colon M \otimes_0 T \leftrightarrow T \otimes_i M : \overline{\tau_i}$ are a T-crossing and its inverse, i.e. they satisfy the identities (T1) and (T4)-(T7) (for $i = 1$) and (T3)-(T7) (for $i = 2$);
- *morphisms* of $\mathcal{C}_{\mathbb{A}}^i$ are bimodule morphisms satisfying the identity (M) (for the given value of i only).

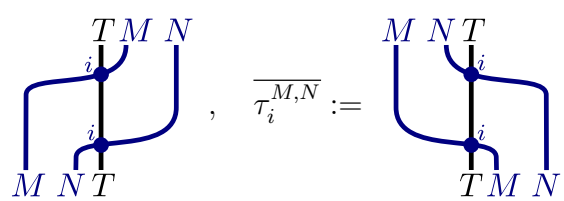
We equip $\mathcal{C}_{\mathbb{A}}$ with the following monoidal structure:

- *product:*

$$(M, \tau_1^M, \tau_2^M) \otimes (N, \tau_1^N, \tau_2^N) := (M \otimes_A N, \tau_1^{M,N}, \tau_2^{M,N}), \tag{5.9}$$

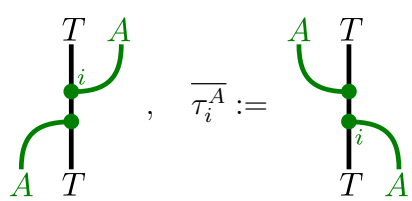
  where the T-crossings are

$$\tau_i^{M,N} := \quad , \quad \overline{\tau_i^{M,N}} := \quad , \quad i = 1, 2\,; \tag{5.10}$$



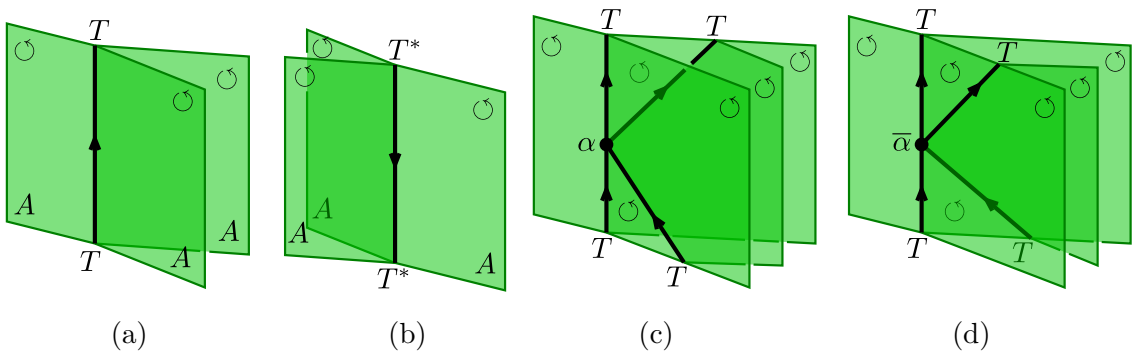
- *unit:* $\mathbb{1}_{\mathcal{C}_{\mathbb{A}}} := (A, \tau_1^A, \tau_2^A)$ where the T-crossings are

$$\tau_i^A := \quad , \quad \overline{\tau_i^A} := \quad , \quad i = 1, 2\,; \tag{5.11}$$



- *associators and unitors:* as in ${}_A\mathcal{C}_A$.

Similarly one also defines monoidal structures on $\mathcal{C}_{\mathbb{A}}^1$ and $\mathcal{C}_{\mathbb{A}}^2$.

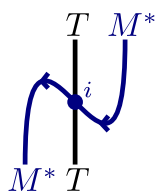
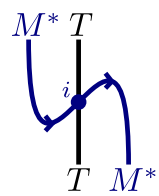


(a) (b) (c) (d)

Figure 5.3: Orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ as defect labels.

Definition 5.5. We call an orbifold datum $\mathbb{A}$ in $\mathcal{C}$ *simple* if $\dim \mathrm{End}_{\mathcal{C}_{\mathbb{A}}}(\mathbb{1}_{\mathcal{C}_{\mathbb{A}}}) = 1$.

If $M = (M, \tau_1, \tau_2)$ is an object of $\mathcal{C}_{\mathbb{A}}$, so is the dual bimodule M^*, where the T-crossings are

$\tau_i :=$  , $\overline{\tau_i} :=$  , $i = 1, 2\,.$ (5.12)

One can use the identities (T1)-(T7) to show that the duality morphisms (2.59) of ${}_A\mathcal{C}_A$ are also morphisms in $\mathcal{C}_{\mathbb{A}}$, i.e. they satisfy (M). The category $\mathcal{C}_{\mathbb{A}}$ (and similarly also $\mathcal{C}^1_{\mathbb{A}}$ and $\mathcal{C}^2_{\mathbb{A}}$) therefore has a natural pivotal structure.

5.2. Interpretation as defects

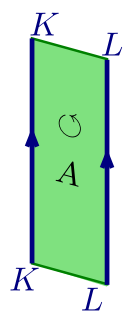
An orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ in a MFC $\mathcal{C}$ can also be interpreted as a collection of defect labels for the defect TQFT $Z^{\mathrm{def}}_{\mathcal{C}}$ (see Chapter 4):

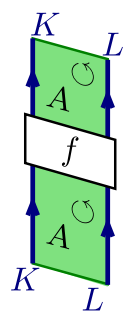
- (A, ψ), being a symmetric separable Frobenius algebra, belongs in the set $D^{\mathcal{C}}_2$ of labels for 2-strata (see Section 4.2).
- T is an A-AA-multimodule, i.e. belongs in the label set $D^{\mathcal{C}}_1$ and can therefore be used to label a 1-stratum with three adjacent A-labelled 2-strata whose orientations are as depicted in Figure 5.3a. Similarly, the dual object T^* is an AA-A-multimodule and can label a 1-stratum with the configuration of adjacent 2-strata as shown in Figure 5.3b.
- α and $\overline{\alpha}$ are morphisms between the relative tensor products $T \otimes_1 T$, $T \otimes_2 T$ of the respective multimodules, i.e. belong in the label set $D^{\mathcal{C}}_0$, and are

valid labels for 0-strata having four adjacent T-labelled 1-strata (two of them incoming and two outgoing), as well as six adjacent A-labelled 2-strata in configurations shown in Figures 5.3c and 5.3d. For conventional reasons only, we assign to a point which can be labelled by α (resp. $\overline{\alpha}$) the positive (resp. negative) orientation.

- ϕ is a scalar and can be used to label point insertions on 3-strata.

As was done in Figure 5.3, it is arguably clearer to present a morphism f between relative tensor products of multimodules by a 3-dimensional figure indicating a neighbourhood of a 0-stratum labelled by f. This way the commutation of the various algebra actions is transparent from the adjacent 2-strata, for example the identities (5.5) and (5.6) are evident in Figures 5.3c and 5.3d by how the 2-strata overlap. If two ingoing or outgoing lines share an adjacent 2-stratum labelled by $A \in D_2^{\mathcal{C}}$, a relative tensor product with respect to the A-action is implied between their respective multimodules. Like in string diagrams in $\mathcal{C}$, the identity morphisms need not be depicted and we sometimes use coupons, instead of points, to emphasise which strands are incoming/outgoing. For example, for a pair of right/left A-modules $K \in \mathcal{C}_A$, $L \in {}_A\mathcal{C}$ and a morphism $f \in \mathrm{End}_{\mathcal{C}}(K \otimes_A L)$ one denotes

$\mathrm{id}_{K\otimes_A L} =$ 

, $f =$ 

. (5.13)

We will sometimes omit some of the labels and/or orientations, if they are clear from the context. Composition of morphisms presented in this way corresponds to stacking the 3-dimensional pictures.

To obtain a string diagram in $\mathcal{C}$ from a 3-dimensional picture one first applies the ribbonisation procedure in the definition of $Z_{\mathcal{C}}^{\mathrm{def}}$ (as if the picture depicted a stratification of a closed 3-ball with the outer edges ending on the boundary) and then cancels the ψ-insertions at the strands which intersect the boundary (to ensure that the corresponding morphism between multimodules is given according to Convention 2.16). This is illustrated in Figure 5.4, the computation in Property 4.13 of $Z_{\mathcal{C}}^{\mathrm{def}}$ also serves as an example.

With the above convention in mind, the conditions (O1)-(O8) on an orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ can be depicted as in Figure 5.6.

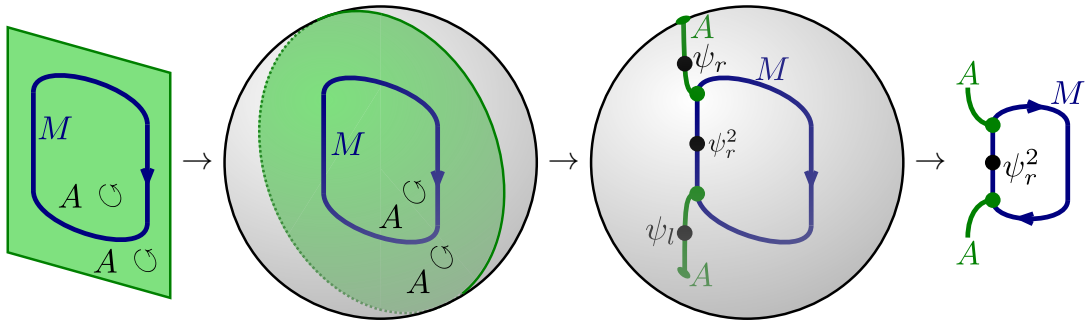


Figure 5.4: Using the ribbonisation procedure to convert a 3-dimensional picture into a string diagram in $\mathcal{C}$.

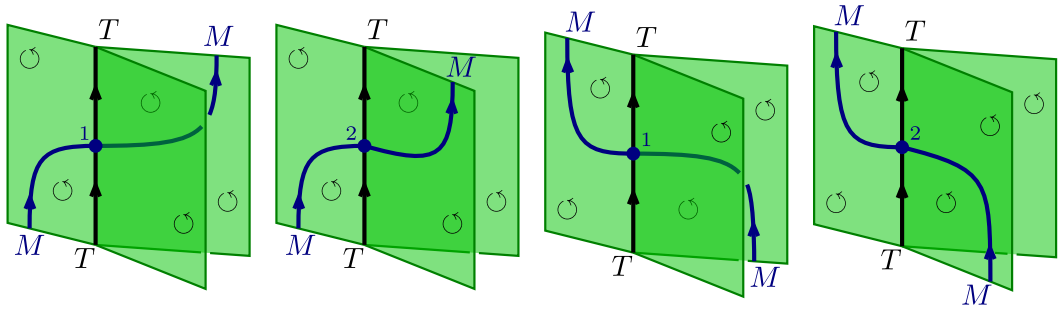


Figure 5.5: T-crossings τ_1, τ_2, $\overline{\tau_1}$, $\overline{\tau_2}$ of an object $M \in \mathcal{C}_{\mathbb{A}}$ (in this order) as defect labels.

Remark 5.6. More concisely, the 3-dimensional pictures we use can be interpreted as 3-morphisms in the tricategory associated to the defect TQFT $Z_{\mathcal{C}}^{\text{def}}$ (see Section 4.1), presented using the 3-dimensional graphical calculus as briefly reviewed in Section 2.6. We refer to [CMS] for more details on this.

As was the case with an orbifold datum, an object $(M, \tau_1, \tau_2) \in \mathcal{C}_{\mathbb{A}}$ can be interpreted as a collection of defect labels: M for 1-strata with two adjacent A-labelled 2-strata and τ_1, τ_2, $\overline{\tau_1}$, $\overline{\tau_2}$ for 0-strata at the intersection of two 1-strata, one labelled with M and the other with T (see Figure 5.5). In this notation, the conditions (T1)-(T7) correspond to the identities in Figure 5.7. For two objects $M, N \in \mathcal{C}_{\mathbb{A}}$, the T-crossings of of $M \otimes N$ as defined in (5.10) become

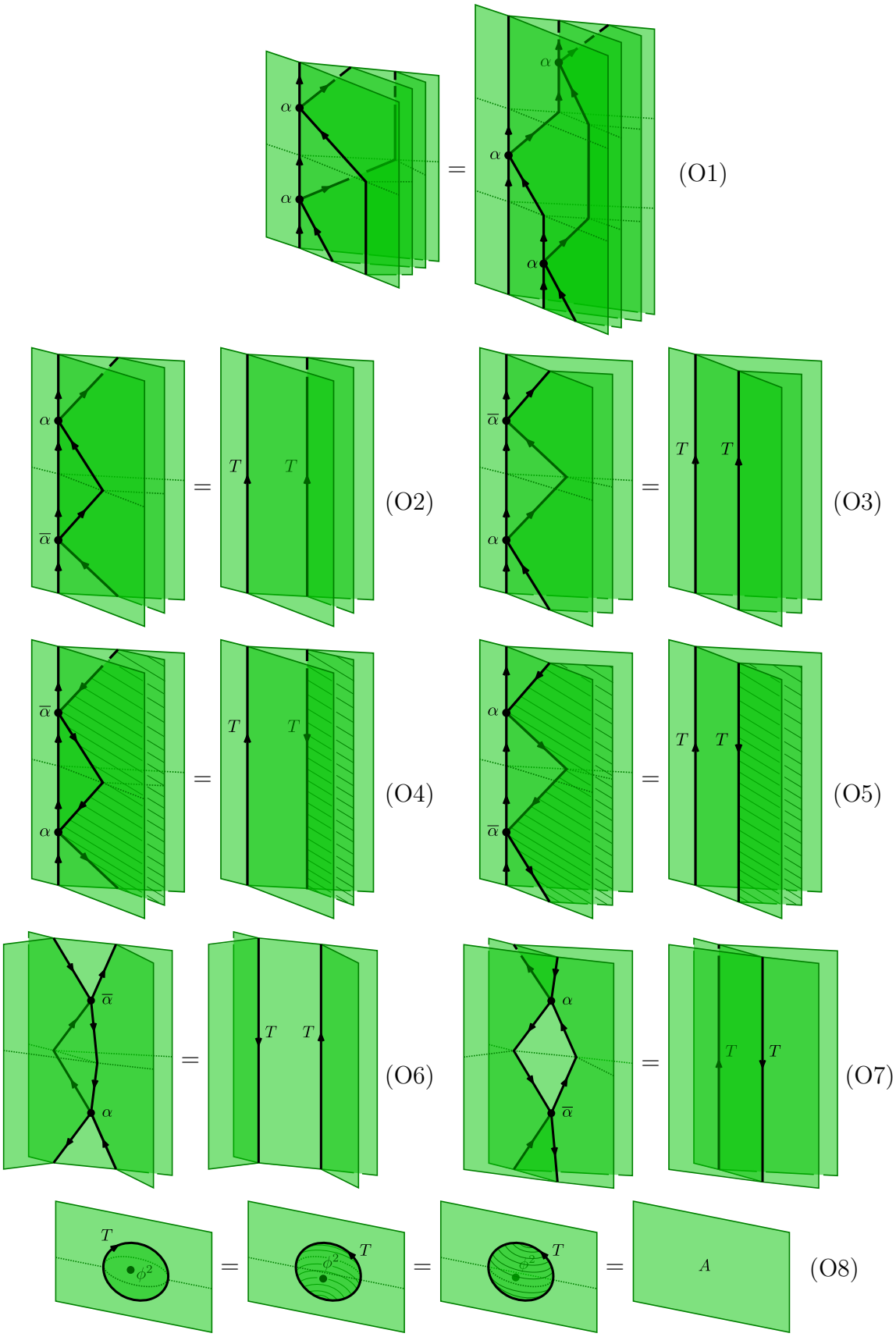


Figure 5.6: Conditions on an orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ in 3d-form.

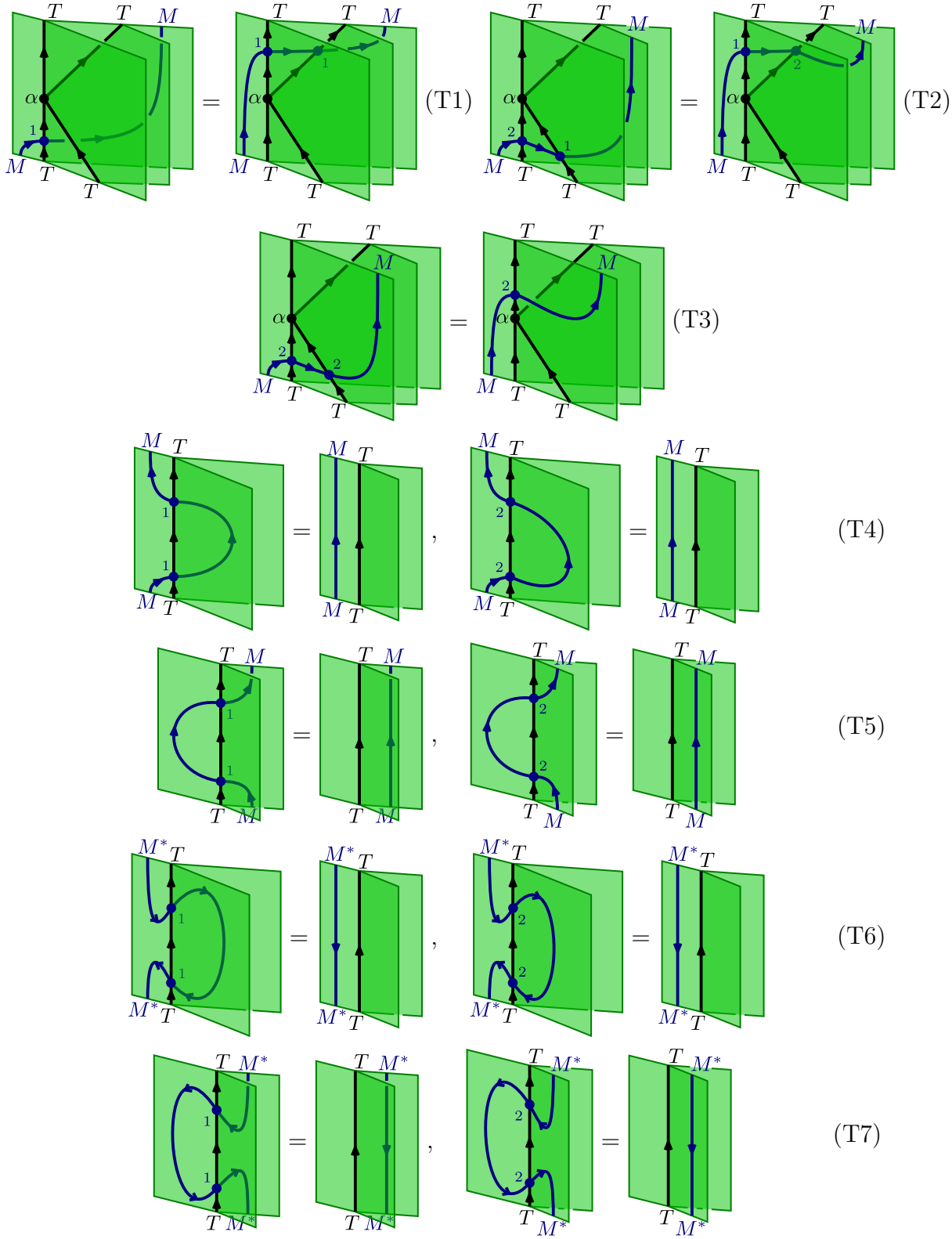
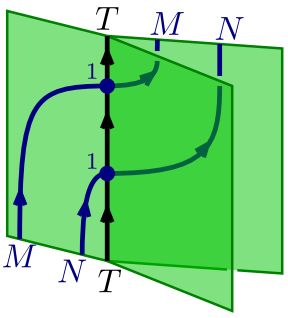
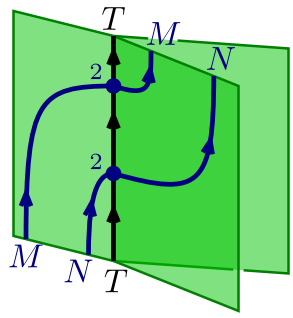


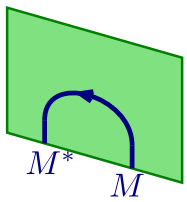
Figure 5.7: Conditions on an object $(M, \tau_1, \tau_2) \in \mathcal{C}_{\mathbb{A}}$ in 3d-form.

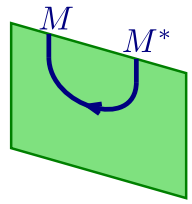
$$\tau_1^{M,N} =$$ 

, $$\tau_2^{M,N} =$$ 

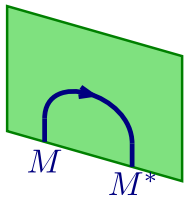
, (5.14)

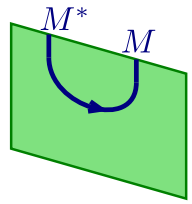
(similarly for the inverses $\overline{\tau_1^{M,N}}$, $\overline{\tau_2^{M,N}}$). The duality morphisms of $M \in \mathcal{C}_{\mathbb{A}}$, by definition being inherited from the category ${}_A\mathcal{C}_A$ of A-A-bimodules and hence given by (2.59), have the 3-dimensional presentations

$$\mathrm{ev}_M =$$ 

, $$\mathrm{coev}_M =$$ 

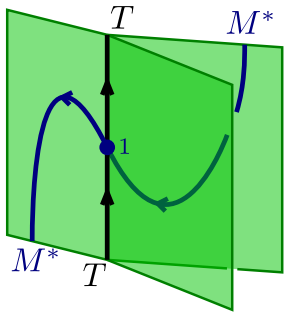
, (5.15)

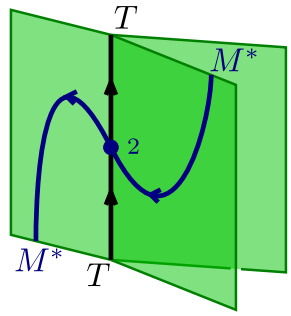
$$\widetilde{\mathrm{ev}}_M =$$ 

, $$\widetilde{\mathrm{coev}}_M =$$ 

. (5.16)

The T-crossings of the dual $M^* \in \mathcal{C}_{\mathbb{A}}$, as defined in (5.12), correspond to

$$\tau_1^{M^*} =$$ 

, $$\tau_2^{M^*} =$$ 

, (5.17)

(similarly for the inverses $\overline{\tau_1^{M^*}}$, $\overline{\tau_2^{M^*}}$).

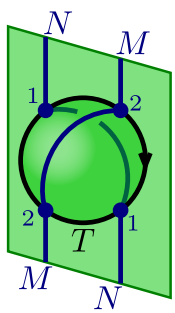
Remark 5.7. In the references [CRS3, MR1], on which this exposition is based, only the symmetric Δ-separable Frobenius algebras were used to label 2-strata. Consequently, the counterpart defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ was not Euler complete with

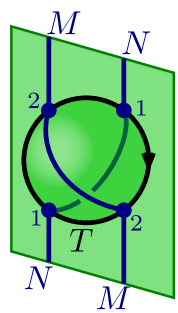
respect to surfaces (see Section 4.1 and Property 4.14), and an orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ was defined to take into account the completion, as this allowed one to consider more interesting examples (like the one producing Drinfeld centres from $\mathrm{Vect}_{\Bbbk}$, see Section 7.2 below). In particular, the entry ψ in $\mathbb{A}$ was taken to be the label for an invertible point insertion on A-labelled 2-strata, or more algebraically, an invertible A-A-bimodule morphism $A \to A$. Such ψ-labelled point insertions would then also appear in the 3-dimensional presentation of the conditions (T1)-(T7), etc.

Our approach with symmetric separable Frobenius algebras allows one to hide the ψ-insertions in the ribbonisation step and therefore makes the 3-dimensional presentations clearer. Still, when exploring the examples in later sections, we prefer to switch back to the setting of Δ-separable algebras. We review the definitions of orbifold data and the category $\mathcal{C}_{\mathbb{A}}$ in this setting in Appendix C.

5.3. Ribbon structure

For a pair of objects $M, N \in \mathcal{C}_{\mathbb{A}}$, define the morphisms $c_{M,N} \colon M \otimes_A N \to N \otimes_A M$, $c^{-1}_{M,N} \colon N \otimes_A M \to M \otimes_A N$ as follows:

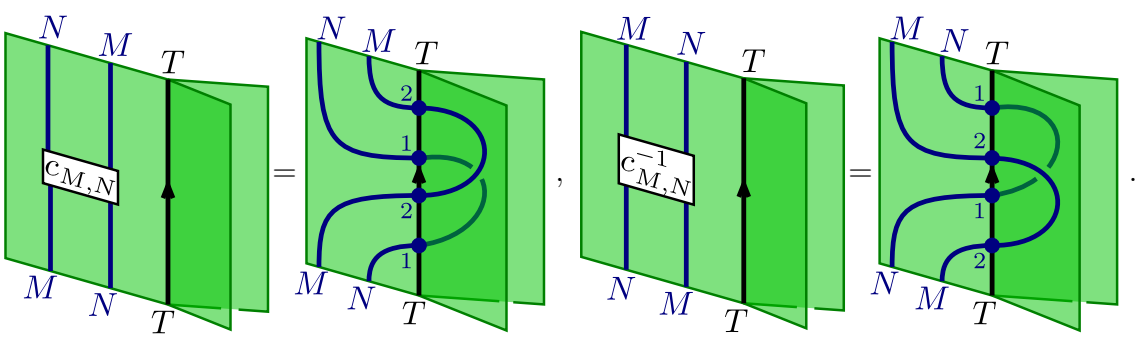
$c_{M,N} := \phi^2 \cdot$ 

, $c^{-1}_{M,N} := \phi^2 \cdot$ 

. (5.18)

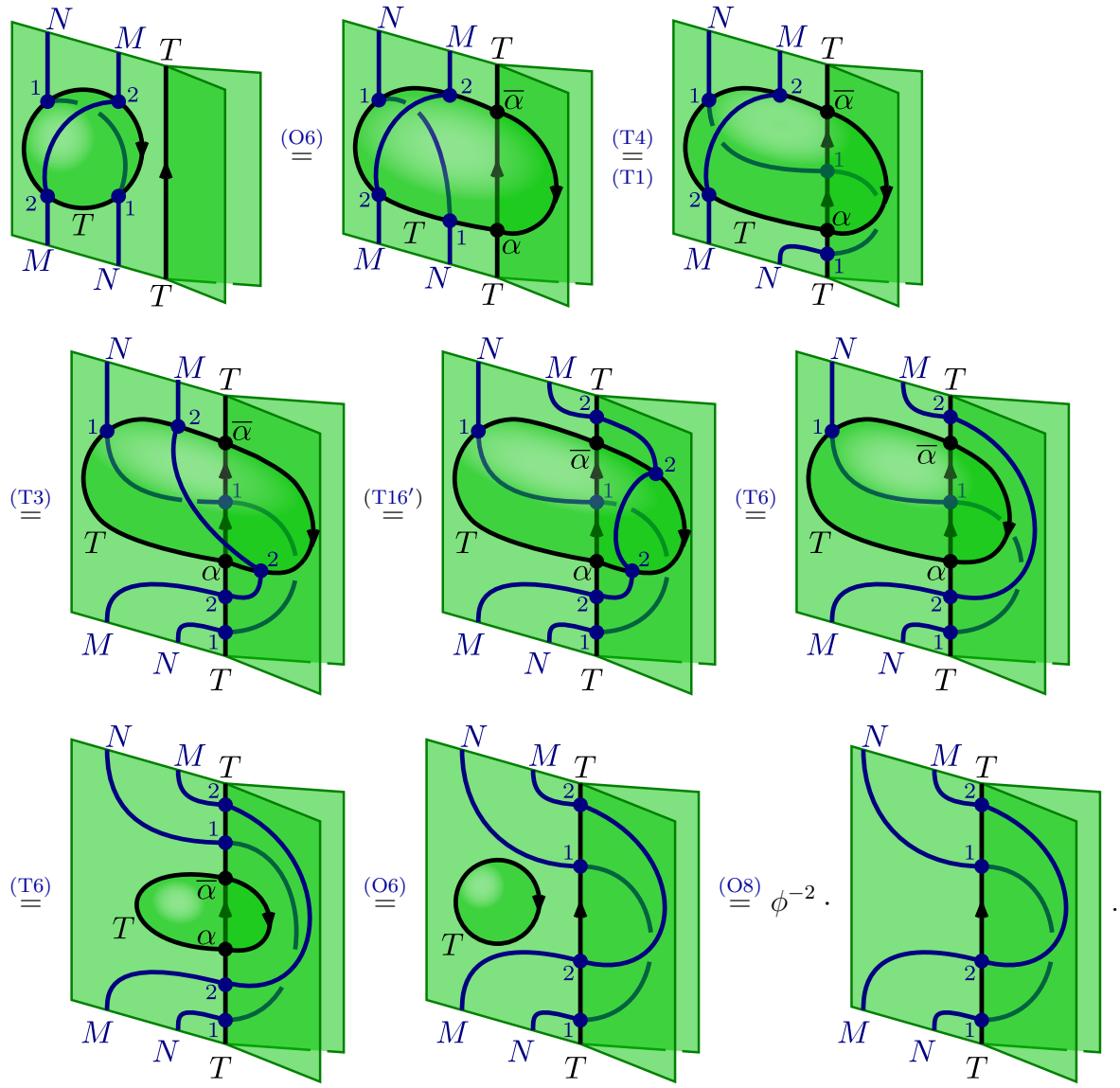
That the notation $c^{-1}_{M,N}$ is indeed justified is part of the claim in Proposition 5.9 below.

Lemma 5.8. For all $M, N \in \mathcal{C}_{\mathbb{A}}$ the following identities hold:



(5.19)

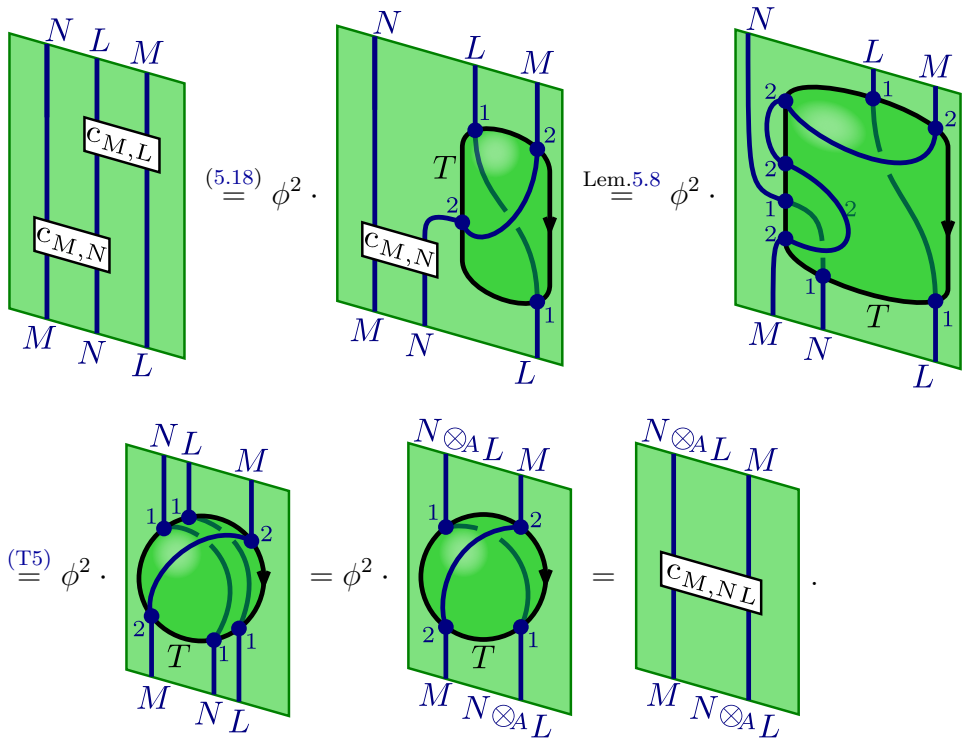
Proof. Repeatedly using the identities (O1)-(O8) and (T1)-(T7), for the first identity one has



Similarly one can show the second identity. □

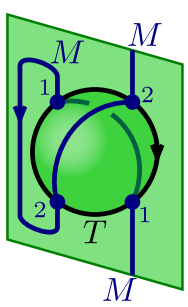
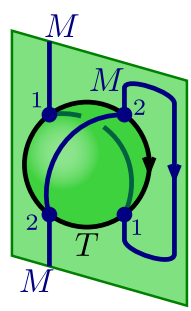
Proposition 5.9. $\{c_{M,N}\}_{M,N\in\mathcal{C}_{\mathbb{A}}}$ defines a braiding on $\mathcal{C}_{\mathbb{A}}$.

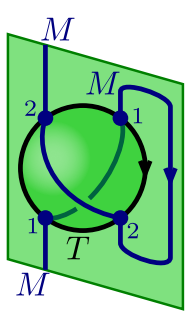
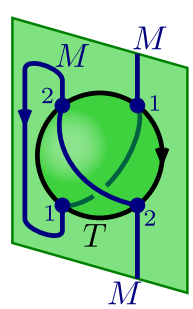
Proof. One must check that for $M, N \in \mathcal{C}_{\mathbb{A}}$, the morphisms $c_{M,N}$ and $c_{M,N}^{-1}$ are natural in M, N, satisfy the hexagon identities, are inverses of each other and that the identity (M) holds. All of this can be done by repeatedly applying (O1)-(O8), (T1)-(T7) and (M); we only show one of the hexagon identities for $c_{M,N}$. Using Lemma 5.8 one gets:



□

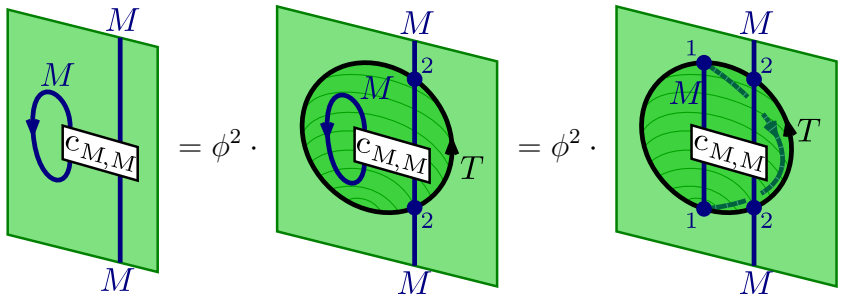
Since at this point the category $\mathcal{C}_{\mathbb{A}}$ is both braided and pivotal, one can introduce the left/right twist morphisms and their inverses for an object $M \in \mathcal{C}_{\mathbb{A}}$ in the usual way (2.12) as partial traces of the braiding morphisms $c_{M,M}$, $c^{-1}_{M,M}$, or explicitly

$\theta^l_M := \phi^2 \cdot$  , $\theta^r_M := \phi^2 \cdot$  , (5.20)

$(\theta^l_M)^{-1} := \phi^2 \cdot$  , $(\theta^r_M)^{-1} := \phi^2 \cdot$  . (5.21)

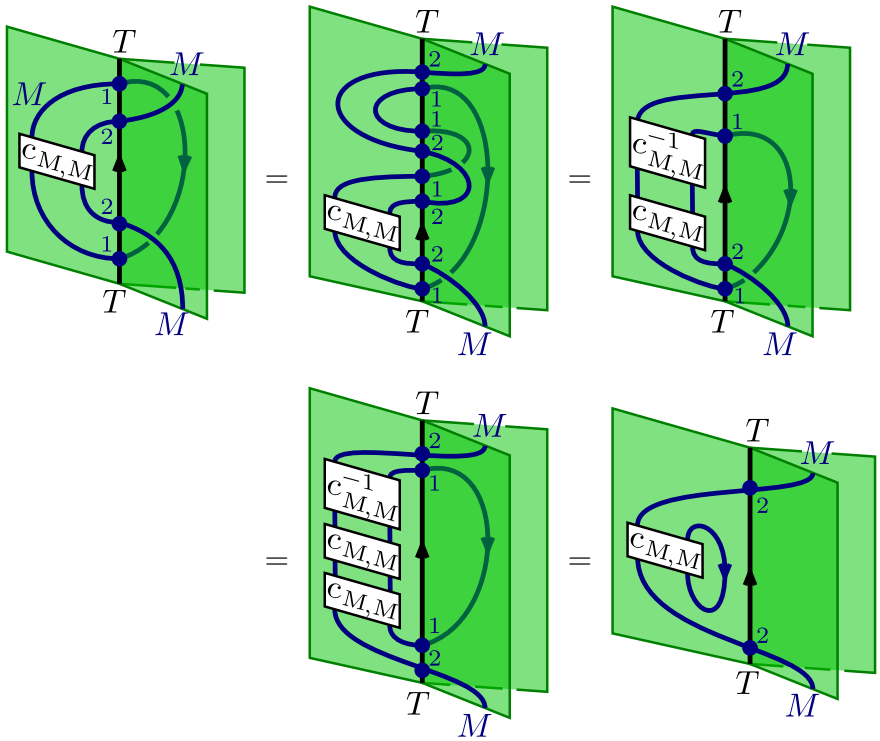
Proposition 5.10. $\mathcal{C}_{\mathbb{A}}$ is ribbon.

Proof. We need to show that for an object $M \in \mathcal{C}_{\mathbb{A}}$ one has $\theta^l_M = \theta^r_M$. One has:



(5.22)

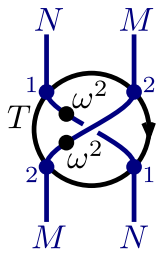
where in the first equality we used (O8), (T5) and (M) to create a bubble and move the coupon on it, and in the second equality we used (T4) to move the M-strand onto the 2-stratum to the back (note that both this 2-stratum and the strand lying in it have opposite to paper/screen plane orientation, hence the stripy pattern). Next, using the auxiliary identities in Lemma 5.8 together with (T5) and (T6) one gets:

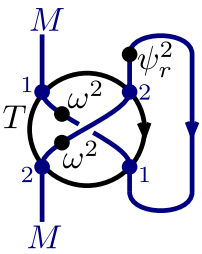


,

which upon substituting back to (5.22) and applying (O8), (T5) and (M) once more, yields the desired result. □

Remark 5.11. Applying the ribbonisation procedure as explained in Figure 5.4 to (5.18) and (5.20) one obtaines the following string diagrams for $M, N \in \mathcal{C}_{\mathbb{A}}$

$$c_{M,N} = \phi^2 \cdot$$ 

, $$\theta_M = \phi^2 \cdot$$ 

. (5.23)

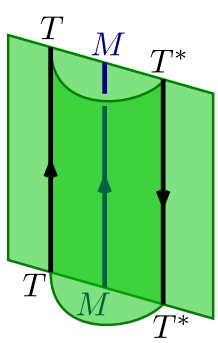
5.4. Pipe functors and semisimplicity

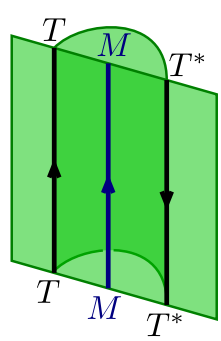
In this section we show that the categories $\mathcal{C}_{\mathbb{A}}$, $\mathcal{C}^1_{\mathbb{A}}$, $\mathcal{C}^2_{\mathbb{A}}$ are semisimple. This is done by constructing in each case a separable biadjunction to the semisimple category ${}_A\mathcal{C}_A$ and utilising Proposition A.3 from Appendix A.

Let us define four functors

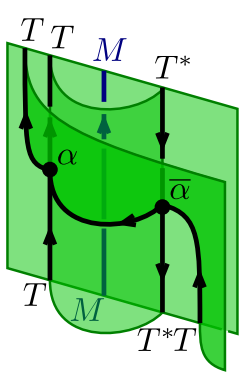
$$\begin{tikzcd} {}_A\mathcal{C}_A \arrow[r, "H_1"] \arrow[d, "H_2"] & \mathcal{C}^1_{\mathbb{A}} \arrow[d, "H_{21}"] \\ \mathcal{C}^2_{\mathbb{A}} \arrow[r, "H_{12}"] & \mathcal{C}_{\mathbb{A}} \end{tikzcd} . \tag{5.24}$$

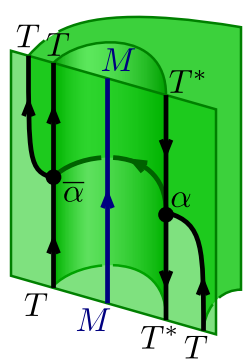
Namely, for any bimodule $M \in {}_A\mathcal{C}_A$, define two bimodules $H_1(M)$, $H_2(M)$ together with T-crossings $\tau_1^{H_1(M)}$, $\tau_2^{H_2(M)}$ and their inverses by

$$H_1(M) :=$$ 

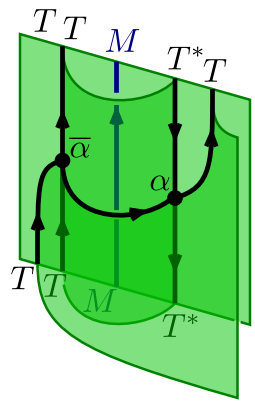
$$H_2(M) :=$$ 

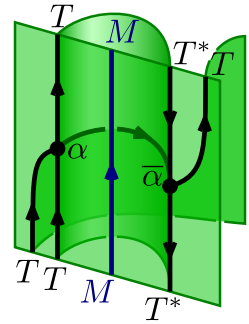
(5.25)

$$\tau_1^{H_1(M)} :=$$ 

$$\tau_2^{H_2(M)} :=$$ 

(5.26)

$$\overline{\tau_1^{H_1(M)}} :=$$ 

$$\overline{\tau_2^{H_2(M)}} :=$$ 

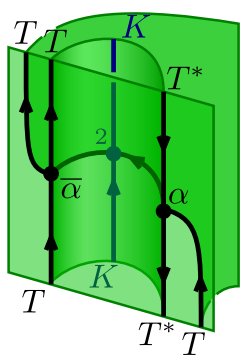
(5.27)

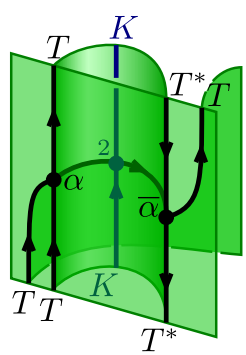
The definitions of $H_1(M)$ and $H_2(M)$ here mean the images of the idempotents like the one in Remark 5.16 below.

Next, for $K \in \mathcal{C}^2_{\mathbb{A}}$ define $H_{12}(K)$ to have the same underlying bimodule as $H_1(K)$, and set

$$\tau_1^{H_{12}(K)} := \tau_1^{H_1(K)}, \qquad \overline{\tau_1^{H_{12}(K)}} := \overline{\tau_1^{H_1(K)}}. \tag{5.28}$$

The remaining T-crossings are defined as follows:

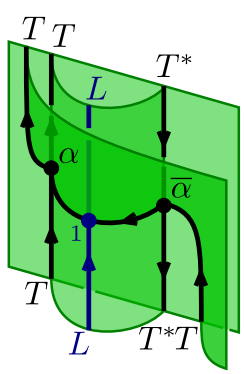
$$\tau_2^{H_{12}(K)} :=$$ 

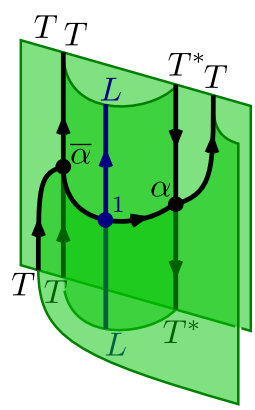
$$,\qquad \overline{\tau_2^{H_{12}(K)}} :=$$ 

$$.$$

(5.29)

For H_{21} one proceeds analogously. Given $L \in \mathcal{C}^1_{\mathbb{A}}$, $H_{21}(L)$ has the same bimodule and the T-crossing τ_2 as $H_2(L)$ with the remaining T-crossings given by

$$\tau_1^{H_{21}(L)} :=$$ 

$$,\qquad \overline{\tau_1^{H_{21}(L)}} :=$$ 

$$.$$

(5.30)

Consider now the following commuting square of forgetful functors:

$$\begin{tikzcd} {}_A\mathcal{C}_A & \mathcal{C}^1_{\mathbb{A}} \arrow[l, "U_1"'] \\ \mathcal{C}^2_{\mathbb{A}} \arrow[u, "U_2"] & \mathcal{C}_{\mathbb{A}} \arrow[l, "U_{12}"'] \arrow[u, "U_{21}"'] \end{tikzcd} \,. \tag{5.31}$$

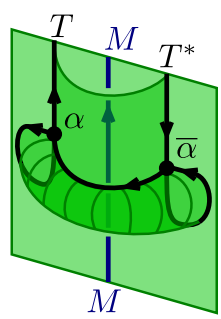
We have:

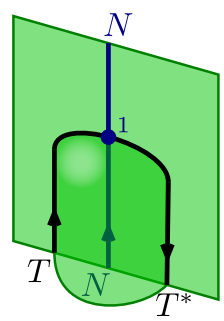
Proposition 5.12. (H_1, U_1), (H_{12}, U_{12}), (H_2, U_2), (H_{21}, U_{21}) are pairs of biadjoint functors and in each case the biadjunction is separable.

Proof. We only show the argument for the pair (H_1, U_1), the other cases can be handled similarly. One defines the unit/counit natural transformations

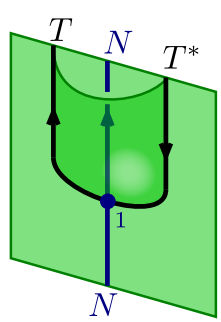
$$\begin{aligned} &\eta\colon \mathrm{Id}_{\mathcal{C}} \Rightarrow U_1 \circ H_1 \,, \quad && \varepsilon\colon H_1 \circ U_1 \Rightarrow \mathrm{Id}_{{}_A\mathcal{C}} \,, \\ &\widetilde{\eta}\colon \mathrm{Id}_{{}_A\mathcal{C}} \Rightarrow H_1 \circ U_1 \,, \quad && \widetilde{\varepsilon}\colon U_1 \circ H_1 \Rightarrow \mathrm{Id}_{\mathcal{C}} \,, \end{aligned} \tag{5.32}$$

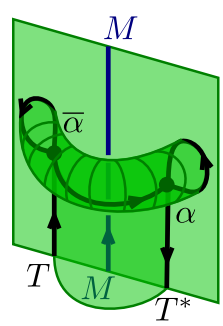
for the two adjunctions $H_1 \dashv U_1$ and $U_1 \dashv H_1$ for each object $M \in {}_A\mathcal{C}_A$, $N \in \mathcal{C}^1_{\mathbb{A}}$ by

$\eta_M = \phi^2 \cdot$ 

$\varepsilon_N =$ 

(5.33)

$\widetilde{\eta}_N =$ 

$\widetilde{\varepsilon}_M = \phi^2 \cdot$ 

(5.34)

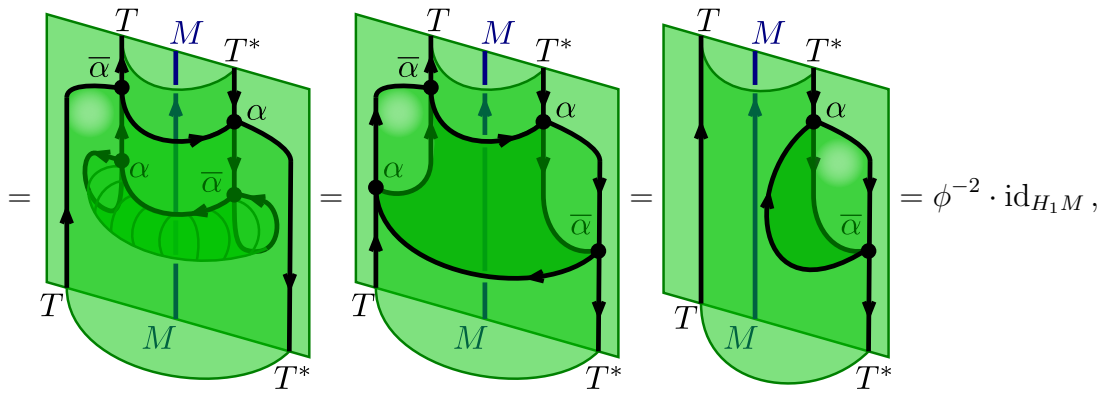
The identities

$$\varepsilon_{H_1 M} \circ H_1 \eta_M = \mathrm{id}_{H_1 M} \,, \qquad U_1 \varepsilon_N \circ \eta_{U_1 N} = \mathrm{id}_{U_1 N} \,, \tag{5.35}$$

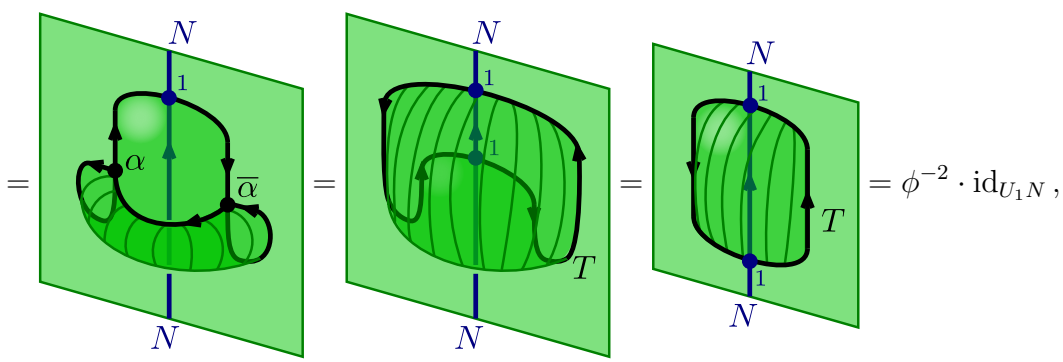
$$\widetilde{\varepsilon}_{U_1 N} \circ U_1 \widetilde{\eta}_N = \mathrm{id}_{U_1 N} \,, \qquad H_1 \widetilde{\varepsilon}_M \circ \widetilde{\eta}_{H_1 M} = \mathrm{id}_{H_1 M} \,. \tag{5.36}$$

are shown by repeatedly applying the identities (O1)-(O8), e.g. for (5.35) one has

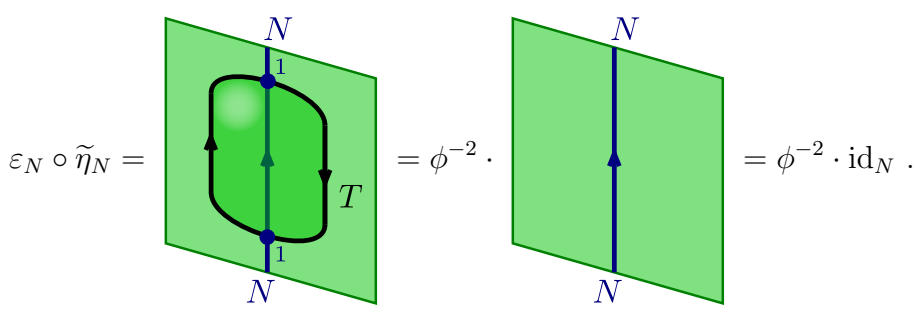
$\phi^{-2} \cdot \varepsilon_{H_1 M} \circ H_1 \eta_M$



$\phi^{-2} \cdot H_1 \widetilde{\varepsilon}_M \circ \widetilde{\eta}_{H_1 M}$



and the identities (5.36) are shown similarly. That the biadjunction is separable, follows from



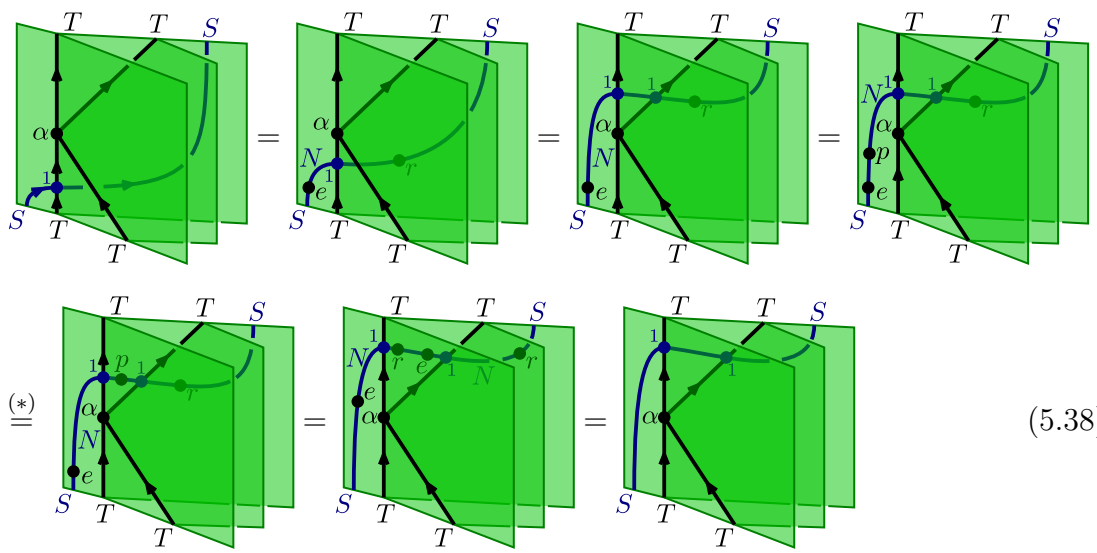
□

Proposition 5.13. The categories $\mathcal{C}^1_{\mathbb{A}}$, $\mathcal{C}^2_{\mathbb{A}}$ and $\mathcal{C}_{\mathbb{A}}$ are finitely semisimple.

Proof. We use the results of Appendix A. From Corollary 2.13 we already know that ${}_A\mathcal{C}_A$ is finitely semisimple. By Proposition 5.12, it is enough to show that $\mathcal{C}^1_{\mathbb{A}}$, $\mathcal{C}^2_{\mathbb{A}}$ and $\mathcal{C}_{\mathbb{A}}$ are idempotent complete. We show this for $\mathcal{C}^1_{\mathbb{A}}$ only, since other cases are analogous. Let $p\colon N \to N$ be an idempotent in $\mathcal{C}^1_{\mathbb{A}}$. Then it is also an idempotent in ${}_A\mathcal{C}_A$ and hence has a retract (S, e, r) in ${}_A\mathcal{C}_A$. We equip S with the T-crossings

$$\begin{aligned} \tau_1^S &:= [S \otimes_0 T \xrightarrow{e\otimes_0 \mathrm{id}_T} N \otimes_0 T \xrightarrow{\tau_1^N} T \otimes_1 N \xrightarrow{\mathrm{id}_T \otimes_1 r} T \otimes_1 S]\,, \\ \overline{\tau_1^S} &:= [T \otimes_1 S \xrightarrow{\mathrm{id}_T \otimes_1 e} T \otimes_1 N \xrightarrow{\overline{\tau_1^N}} N \otimes_0 T \xrightarrow{r\otimes_0 \mathrm{id}_T} S \otimes_0 T]\,. \end{aligned} \tag{5.37}$$

It is easy to check that they indeed satisfy the axioms of a T-crossing, e.g.



(5.38)

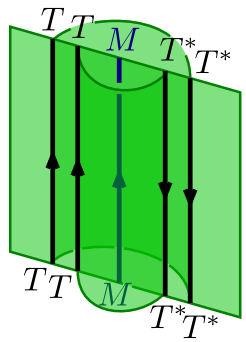
where in step $(*)$ we used that p is a morphism in $\mathcal{C}^1_{\mathbb{A}}$. The argument that e and r are morphisms in $\mathcal{C}^1_{\mathbb{A}}$ is similar. (S, e, r) is therefore a retract in $\mathcal{C}^1_{\mathbb{A}}$. □

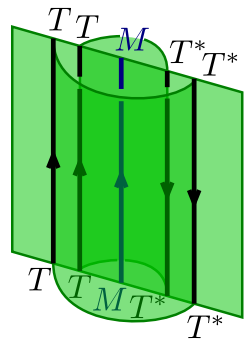
Combining Propositions 5.10 and 5.13 one gets

Corollary 5.14. $\mathcal{C}_{\mathbb{A}}$ is a ribbon multifusion category.

Remark 5.15. Note that the diagram in (5.31) commutes with identity natural isomorphism, $U := U_1 \circ U_{21} = U_2 \circ U_{12} : \mathcal{C}_{\mathbb{A}} \to {}_A\mathcal{C}_A$, as each path sends an object in $\mathcal{C}_{\mathbb{A}}$ to its underlying bimodule. By Proposition 5.12, both $H_{21} \circ H_1$ and $H_{12} \circ H_2$ are biadjoint to U, and hence in particular naturally isomorphic. Thus the diagram in (5.24) commutes as well. Indeed the two functors $P_1, P_2\colon {}_A\mathcal{C}_A \to \mathcal{C}_{\mathbb{A}}$ obtained

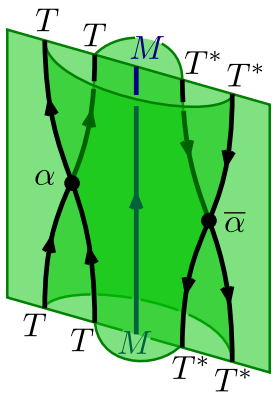
from it are explicitly defined on a bimodule $M \in {}_A\mathcal{C}_A$ by

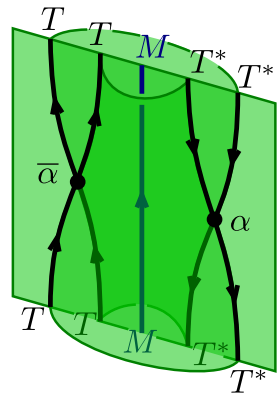
$P_1(M) :=$ 

, $P_2(M) :=$ 

, (5.39)

and the natural isomorphism $\varphi\colon P_1 \Rightarrow P_2$ is defined by

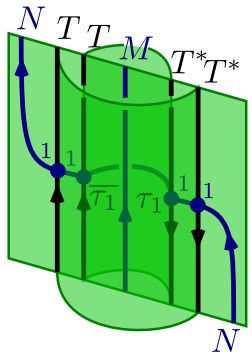
$\varphi_M :=$ 

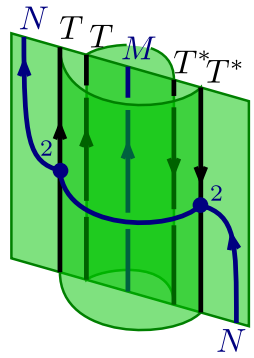
, $\varphi_M^{-1} :=$ 

. (5.40)

Below we will work exclusively with the functor $P := P_2$ and will refer to it as the *pipe functor*.

The braiding morphisms (5.18) involving a pipe object $P(M)$, $M \in {}_A\mathcal{C}_A$ can be simplified, for example for $N \in \mathcal{C}_{\mathbb{A}}$ one has

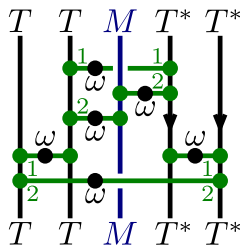
$c_{PM,N} =$ 

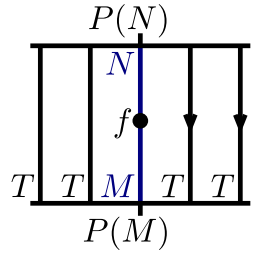
, $c_{N,PM}^{-1} =$ 

. (5.41)

Remark 5.16. For a bimodule $M \in {}_A\mathcal{C}_A$ and a bimodule morphism $[f\colon M \to N] \in {}_A\mathcal{C}_A$, the definitions of the underlying bimodule of $P(M) \in \mathcal{C}_{\mathbb{A}}$, as well as the

morphism $P(f) \in \mathcal{C}_{\mathbb{A}}$ can be stated more algebraically in terms of string diagrams as follows:

$P(M) :=$ 

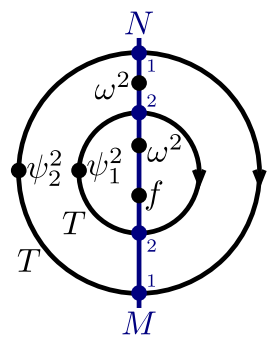
, $P(f) :=$ 

. (5.42)

5.5. Modularity

In this section we will in addition assume that $\mathbb{A}$ is a simple orbifold datum (see Definition 5.5), so that by Corollary 5.14, $\mathcal{C}_{\mathbb{A}}$ is a ribbon fusion category. We will show that $\mathcal{C}_{\mathbb{A}}$ is in fact modular.

Let us start with some tools which will be helpful when performing computations in $\mathcal{C}_{\mathbb{A}}$. For objects $M, N \in \mathcal{C}_{\mathbb{A}}$ and a morphism $[f\colon M \to N] \in \mathcal{C}$ between the underlying objects, define the averaged morphism $\overline{f}\colon M \to N$ to be the A-A-bimodule morphism

$\overline{f} := \phi^4 \cdot$ 

. (5.43)

One can check that $\overline{f}$ is a morphism in $\mathcal{C}_{\mathbb{A}}$. Moreover, if $f \in \mathcal{C}_{\mathbb{A}}$, then $f = \overline{f}$, i.e. averaging is an idempotent on the morphism spaces of $\mathcal{C}$, projecting onto the morphism spaces of $\mathcal{C}_{\mathbb{A}}$ (much like the one in (2.42), projecting onto the morphism spaces of ${}_A\mathcal{C}$).

Remark 5.17. Similar averaging maps can be defined to project on the morphism spaces of categories $\mathcal{C}^1_{\mathbb{A}}, \mathcal{C}^2_{\mathbb{A}}$. We list them, as well as (5.43) in 3-dimensional notation in Figure 5.8.

The next tool allows one to compute traces of endomorphisms and dimensions of objects in $\mathcal{C}_{\mathbb{A}}$ in terms of the underlying MFC $\mathcal{C}$.

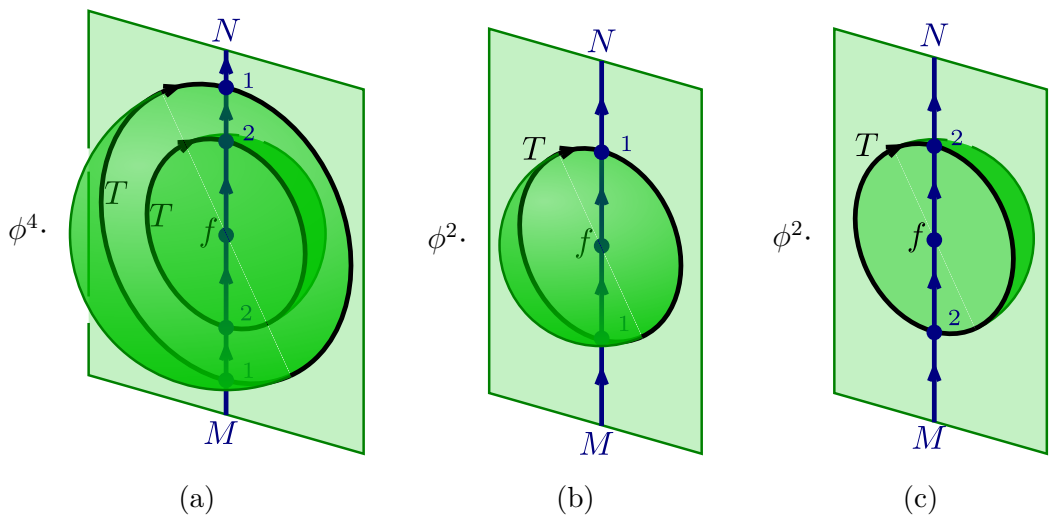


Figure 5.8: Projecting a morphism $[f\colon M \to N] \in \mathcal{C}$ on (a) the space $\mathcal{C}_{\mathbb{A}}(M,N)$ for $M,N \in \mathcal{C}_{\mathbb{A}}$; (b) the space $\mathcal{C}^1_{\mathbb{A}}(M,N)$ for $M,N \in \mathcal{C}^1_{\mathbb{A}}$; (c) the space $\mathcal{C}^2_{\mathbb{A}}(M,N)$ for $M,N \in \mathcal{C}^2_{\mathbb{A}}$.

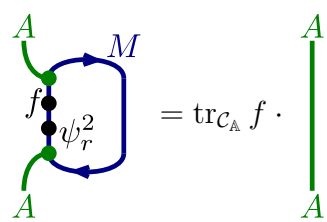
Lemma 5.18. Let $\mathbb{A}$ be a simple orbifold datum in $\mathcal{C}$. For $M \in \mathcal{C}_{\mathbb{A}}$ and $f \in \mathrm{End}_{\mathcal{C}_{\mathbb{A}}}(M)$ one has:

$$\mathrm{tr}_{\mathcal{C}_{\mathbb{A}}} f \cdot \mathrm{tr}_{\mathcal{C}}\, \omega_A^2 = \mathrm{tr}_{\mathcal{C}}(f \circ \omega_M^2)\,, \tag{5.44}$$

where on the right hand side of (5.44), f is treated as a morphism in $\mathrm{End}_{\mathcal{C}}\, M$. In particular, one has

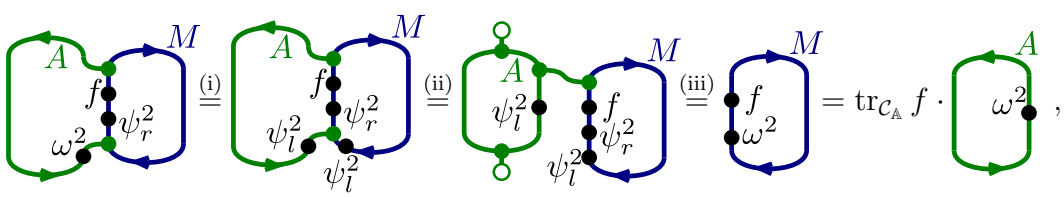
$$\dim_{\mathcal{C}_{\mathbb{A}}} M \cdot \mathrm{tr}_{\mathcal{C}}\, \omega_A^2 = \mathrm{tr}_{\mathcal{C}}\, \omega_M^2\,. \tag{5.45}$$

Proof. Since $\mathbb{1}_{\mathcal{C}_{\mathbb{A}}} := A$ is simple, one has



(5.46)

Precomposing both sides with ω_A^2 and taking the trace in $\mathcal{C}$ one gets



(5.47)

where in step (i) we used the definitions (2.40), in step (ii) the fact that the left action commutes with both f and ψ_r^M, as well as the Frobenius and symmetry properties (2.30) and (2.31) of A, and in step (iii) the separability condition (2.37) of A (as always for $\zeta = \psi^2$, see Convention 2.11). $\square$

To show that $\mathcal{C}_{\mathbb{A}}$ is modular, we will use the condition in Proposition 2.9iv) by relating the Kirby colour $(C_{\mathcal{C}_{\mathbb{A}}}, d_{\mathcal{C}_{\mathbb{A}}})$ in $\mathcal{C}_{\mathbb{A}}$ with the Kirby colour $(C_{\mathcal{C}}, d_{\mathcal{C}})$ in $\mathcal{C}$ and exploiting the modularity of $\mathcal{C}$. This relation is formulated in the following important technical lemma.

Let R be a any morphism in the category $\mathrm{Rib}_{\mathcal{C}_{\mathbb{A}}}$ of $\mathcal{C}_{\mathbb{A}}$-coloured ribbon tangles (for the sake of generality, we do not assume that R is closed, i.e. it can have free incoming/outgoing strands), and let L be an uncoloured oriented framed knot. As in Section 3.2, let $L \sqcup R$ denote a fixed ribbon tangle consisting of possibly entangled, but not intersecting components of L and R. For an object $M \in \mathcal{C}_{\mathbb{A}}$ and a morphism $f \in \mathrm{End}_{\mathcal{C}_{\mathbb{A}}} M$, let $L(M, f)$ denote the colouring of L by the object M with a single f-labelled insertion.

Lemma 5.19. One has the following identity:

$$\begin{aligned} &F_{\mathcal{C}_{\mathbb{A}}}\big(\, L(C_{\mathcal{C}_{\mathbb{A}}}, d_{\mathcal{C}_{\mathbb{A}}}) \sqcup R \,\big) \cdot \mathrm{tr}_{\mathcal{C}}\, \omega_A^2 \\ &= F_{\mathcal{C}_{\mathbb{A}}}\big(\, L\big(\, P(A \otimes C_{\mathcal{C}} \otimes A),\, P((\psi_r^A)^2 \otimes d_{\mathcal{C}} \otimes (\psi_l^A)^2)\,\big) \sqcup R \,\big)\,. \end{aligned} \tag{5.48}$$

Proof. For $M \in \mathcal{C}_{\mathbb{A}}$ and $f \in \mathrm{End}_{\mathcal{C}_{\mathbb{A}}} M$, let us abbreviate

$$Q(M, f) := F_{\mathcal{C}_{\mathbb{A}}}\big(\, L(M, f) \sqcup R \,\big)\,, \quad Q(M) := Q(M, \mathrm{id}_M)\,. \tag{5.49}$$

Note that Q is linear with respect to direct sums on the first argument and arithmetical operations on the second argument. Moreover, since L is a closed loop, it is cyclic with respect to the compositions on the second argument, i.e. for $M, N \in \mathcal{C}_{\mathbb{A}}$, $f \in \mathcal{C}_{\mathbb{A}}(M, N)$, $g \in \mathcal{C}_{\mathbb{A}}(N, M)$ one has

$$Q(M,\, g \circ f) = Q(N,\, f \circ g)\,. \tag{5.50}$$

The biadjunction between the pipe functor $P\colon {}_A\mathcal{C}_A \to \mathcal{C}_{\mathbb{A}}$ and the forgetful functor $U\colon \mathcal{C}_{\mathbb{A}} \to {}_A\mathcal{C}_A$ yield the following decompositions for a pair of simple objects $\Delta \in \mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}}$, $\mu \in \mathrm{Irr}_{{}_A\mathcal{C}_A}$:

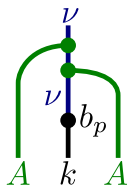
$$\Delta \overset{{}_A\mathcal{C}_A}{\cong} \bigoplus_{\nu \in \mathrm{Irr}_{{}_A\mathcal{C}_A}} \nu \otimes {}_A\mathcal{C}_A(\nu, \Delta) \quad , \qquad P(\mu) \overset{\mathcal{C}_{\mathbb{A}}}{\cong} \bigoplus_{\Lambda \in \mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}}} \Lambda \otimes {}_A\mathcal{C}_A(\mu, \Lambda)\,, \tag{5.51}$$

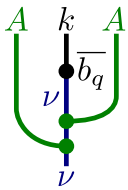
where the label over the isomorphism sign indicates the category in which it holds and tensor products with vector spaces is to be interpreted as multiplicities in the

decomposition. One has:

$$Q(C_{\mathcal{C}_{\mathbb{A}}}, d_{\mathcal{C}_{\mathbb{A}}}) \cdot \mathrm{tr}_{\mathcal{C}}\, \omega_A^2 = \sum_{\Lambda \in \mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}}} \dim_{\mathcal{C}_{\mathbb{A}}} \Lambda \cdot \mathrm{tr}_{\mathcal{C}}\, \omega_A^2 \cdot Q(\Lambda) \overset{(5.45)}{=} \sum_{\Lambda \in \mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}}} \mathrm{tr}_{\mathcal{C}}\, \omega_\Lambda^2 \cdot Q(\Lambda)$$
$$\overset{(5.51)}{=} \sum_{\substack{\Lambda \in \mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}} \\ \nu \in \mathrm{Irr}_{{}_A\mathcal{C}_A}}} \mathrm{tr}_{\mathcal{C}}\, \omega_\nu^2 \cdot \dim {}_A\mathcal{C}_A(\nu, \Lambda) \cdot Q(\Lambda) \overset{(5.51)}{=} \sum_{\nu \in \mathrm{Irr}_{{}_A\mathcal{C}_A}} \mathrm{tr}_{\mathcal{C}}\, \omega_\nu^2 \cdot Q\big(P(\nu)\big)\,. \quad (5.52)$$

To proceed, let us fix for each pair of simple objects $k \in \mathrm{Irr}_{\mathcal{C}}$, $\nu \in \mathrm{Irr}_{{}_A\mathcal{C}_A}$ a basis $\{b_p\}$ of the space $\mathcal{C}(k, \nu)$, along with the dual basis $\{\overline{b_p}\}$ of $\mathcal{C}(\nu, k)$ with respect to the composition pairing, i.e. $\overline{b_q} \circ b_p = \delta_{pq} \cdot \mathrm{id}_k$. Furthermore, define the following morphisms in ${}_A\mathcal{C}_A(A \otimes k \otimes A, \nu)$ and ${}_A\mathcal{C}_A(\nu, A \otimes k \otimes A)$:

$\beta_p :=$ 

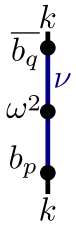
, $\overline{\beta_q} :=$ 

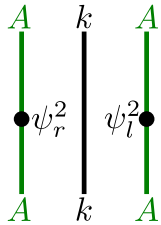
. (5.53)

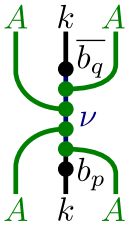
One obtains the relation

$$\beta_p \circ \overline{\beta_q} = \delta_{pq} \cdot \frac{\dim_{\mathcal{C}} k}{\mathrm{tr}_{\mathcal{C}}\, \omega_\nu^2} \cdot \mathrm{id}_\nu\,, \quad (5.54)$$

from a similar computation as in the proof of Proposition 2.19 (by the statement of which one also has $\mathrm{tr}_{\mathcal{C}}\, \omega_\nu^2 \neq 0$ so that the relation above is well defined). By Proposition 2.12, the induction functor $\mathrm{Ind}_{AA} \colon \mathcal{C} \to {}_A\mathcal{C}_A$ and the forgetful functor $U_{AA} \colon {}_A\mathcal{C}_A \to \mathcal{C}$ are biadjoint, so one has the isomorphism ${}_A\mathcal{C}_A(A \otimes k \otimes A, \nu) \cong \mathcal{C}(k, \nu)$. The relation (5.54) therefore shows that $\{\beta_p\}$ and $\{\overline{\beta_p}\}$ constitute bases of the spaces ${}_A\mathcal{C}_A(A \otimes k \otimes A, \nu)$ and ${}_A\mathcal{C}_A(\nu, A \otimes k \otimes A)$ respectively. Using this, let us introduce the scalars $\Omega_{pq}^{k,\nu}, \Psi_{\nu pq}^k \in \Bbbk$ such that



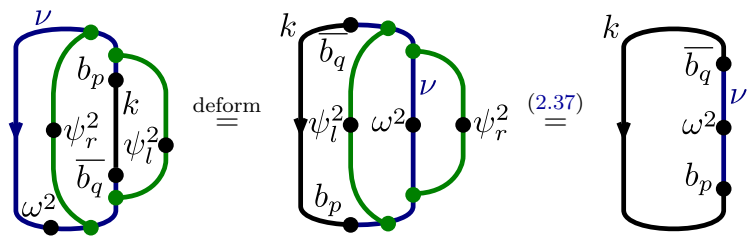
$= \Omega_{pq}^{k,\nu} \cdot \mathrm{id}_k$, 

$= \sum_{\nu \in \mathrm{Irr}_{{}_A\mathcal{C}_A}} \sum_{p,q} \Psi_{\nu pq}^k \cdot$ 

, (5.55)

which is allowed since $(\psi_r^A)^2 \otimes \mathrm{id}_k \otimes (\psi_l^A)^2$ is an A-A-bimodule morphism. A relation between these sets of scalars can be determined as follows: pre- and postcomposing

the left-hand side of the second identity in (5.55) with $\overline{\beta_q} \circ \omega_\nu^2$ and β_p respectively and taking the trace in $\mathcal{C}$ one obtains



$$= \Omega_{pq}^{k,\nu} \cdot \dim_{\mathcal{C}} k\,. \tag{5.56}$$

Doing the same with the right-hand side and using (5.54) yields $\Psi_{\nu pq}^k \cdot \frac{(\dim_{\mathcal{C}} k)^2}{\operatorname{tr}_{\mathcal{C}} \omega_\nu^2}$, so that one has

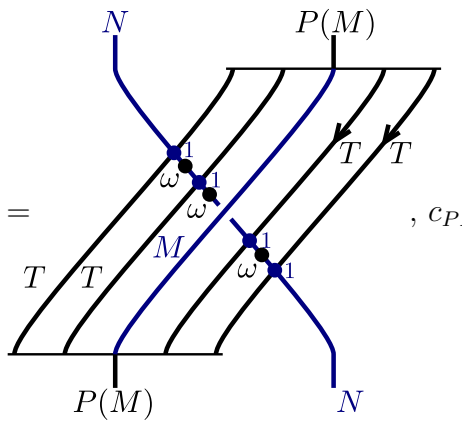
$$\Omega_{pq}^{k,\nu} = \frac{\dim_{\mathcal{C}} k}{\operatorname{tr}_{\mathcal{C}} \omega_\nu^2} \cdot \Psi_{\nu pq}^k\,. \tag{5.57}$$

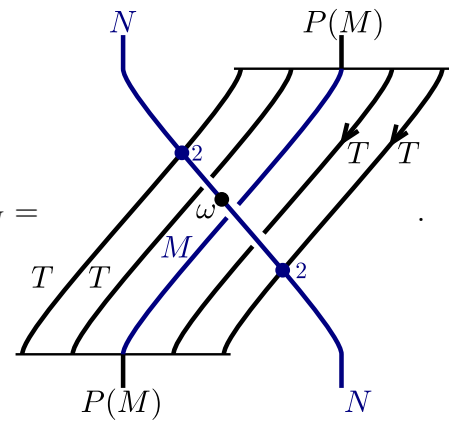
The computation (5.52) can now be continued as follows:

$$\begin{aligned}
&\sum_{\nu \in \operatorname{Irr}_A \mathcal{C}_A} \operatorname{tr}_{\mathcal{C}} \omega_\nu^2 \cdot Q\big(\, P(\nu)\,\big) \overset{(5.55)}{=} \sum_{\substack{\nu \in \operatorname{Irr}_A \mathcal{C}_A \\ k \in \operatorname{Irr}_{\mathcal{C}}}} \sum_p \Omega_{pp}^{k,\nu} \cdot \dim_{\mathcal{C}} k \cdot Q\big(\, P(\nu)\,\big) \\
&\overset{(5.57)}{=} \sum_{\substack{\nu \in \operatorname{Irr}_A \mathcal{C}_A \\ k \in \operatorname{Irr}_{\mathcal{C}}}} \sum_p \frac{(\dim_{\mathcal{C}} k)^2}{\operatorname{tr}_{\mathcal{C}} \omega_\nu^2} \cdot \Psi_{\nu pp}^k \cdot Q\big(\, P(\nu)\,\big) \\
&\overset{(5.54)}{=} \sum_{\substack{\nu \in \operatorname{Irr}_A \mathcal{C}_A \\ k \in \operatorname{Irr}_{\mathcal{C}}}} \sum_{p,q} \dim_{\mathcal{C}} k \cdot \Psi_{\nu pq}^k \cdot Q\big(\, P(\nu)\,, P(\beta_p \circ \overline{\beta_q})\,\big) \\
&\overset{(5.50)}{=} \sum_{\substack{\nu \in \operatorname{Irr}_A \mathcal{C}_A \\ k \in \operatorname{Irr}_{\mathcal{C}}}} \sum_{p,q} \dim_{\mathcal{C}} k \cdot \Psi_{\nu pq}^k \cdot Q\big(\, P(A \otimes \mathrm{id}_k \otimes A)\,, P(\overline{\beta_q}) \circ P(\beta_p)\,\big) \\
&\overset{(5.55)}{=} \sum_{k \in \operatorname{Irr}_{\mathcal{C}}} \dim_{\mathcal{C}} k \cdot Q\big(\; P(A \otimes \mathrm{id}_k \otimes A)\,,\; P((\psi_r^A)^2 \otimes \mathrm{id}_k \otimes (\psi_l^A)^2))\;\big) \\
&= Q\big(\; P(A \otimes \mathrm{id}_k \otimes A)\,,\; P((\psi_r^A)^2 \otimes d_{\mathcal{C}} \otimes (\psi_l^A)^2))\;\big)\,.
\end{aligned} \tag{5.58}$$

□

We are now in a position to prove the most important result of this chapter. To keep a better track of the ψ-insertions, let us write down the braidings (5.41) in terms of string diagrams using Convention 2.16:

$c_{PM,N} =$ 

, $c_{PM,N} =$ 

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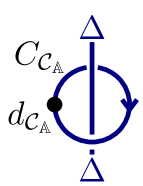
(5.59)

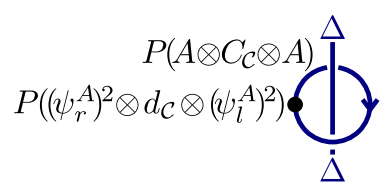
Theorem 5.20. Let $\mathbb{A}$ be a simple orbifold datum in $\mathcal{C}$. Then

i) $\mathcal{C}_{\mathbb{A}}$ is a modular fusion category;

ii) $\operatorname{tr}_{\mathcal{C}} \omega_A^2 \neq 0$ and $\operatorname{Dim} \mathcal{C}_{\mathbb{A}} = \dfrac{\operatorname{Dim} \mathcal{C}}{\phi^8 \cdot (\operatorname{tr}_{\mathcal{C}} \omega_A^2)^2}$.

Proof. Let $\Delta \in \operatorname{Irr}_{\mathcal{C}_{\mathbb{A}}}$ be a simple object of $\mathcal{C}_{\mathbb{A}}$. Following Proposition 2.9iv), let us introduce a morphism $L_\Delta \in \operatorname{End}_{\mathcal{C}_{\mathbb{A}}} \Delta$ defined as follows (the string diagrams are understood to be in $\mathcal{C}_{\mathbb{A}}$):

$L_\Delta :=$ 

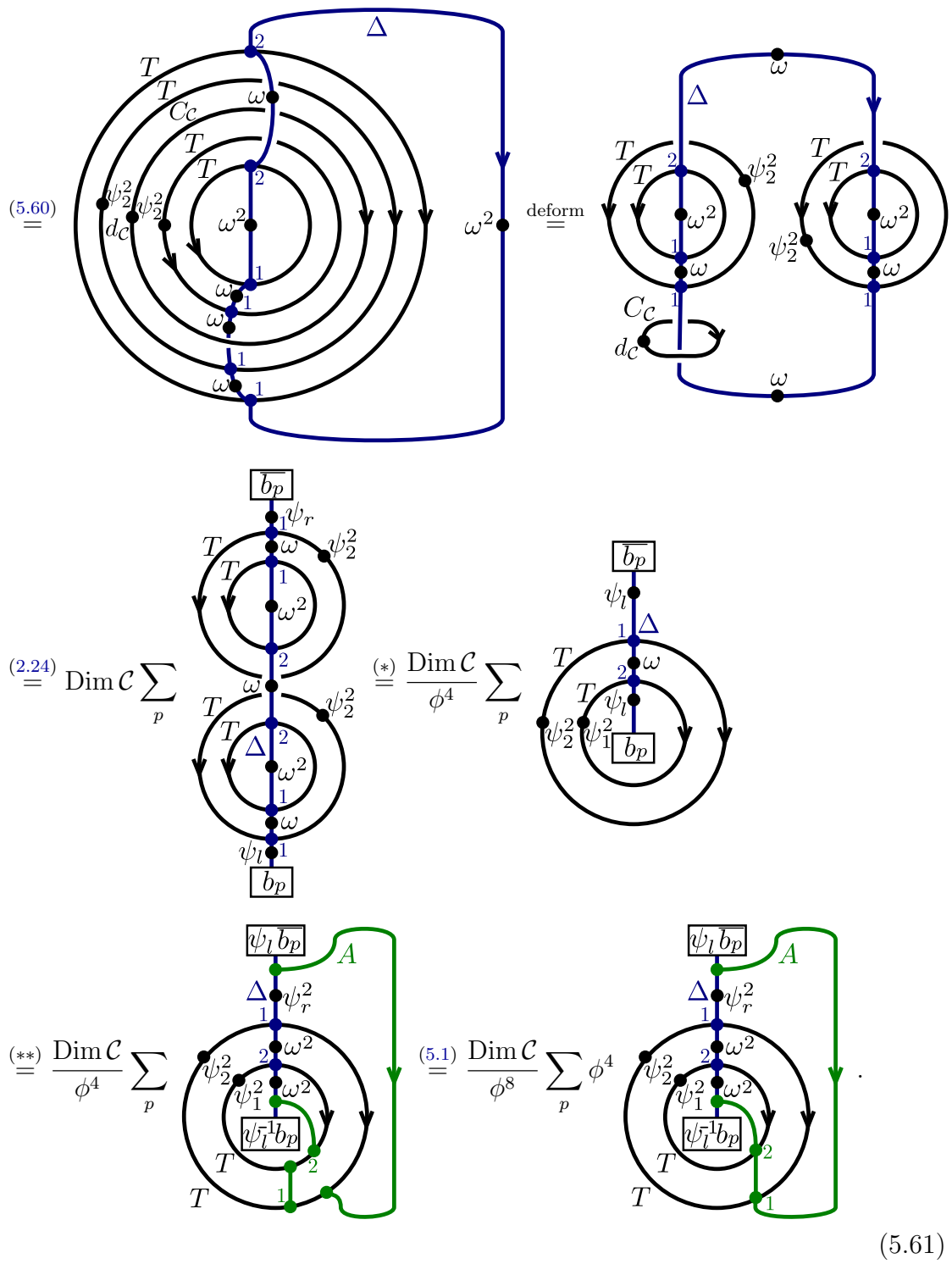
, so that $L_\Delta \cdot \operatorname{tr}_{\mathcal{C}} \omega_A^2 \overset{\text{Lem.5.19}}{=}$ 

.

(5.60)

Let $\{b_p\}$ be a basis of $\mathcal{C}(\mathbb{1}_{\mathcal{C}}, \Delta)$ with the dual basis $\{\overline{b_p}\}$ of $\mathcal{C}(\Delta, \mathbb{1})$ with respect to the composition pairing. Using (5.41) one has:

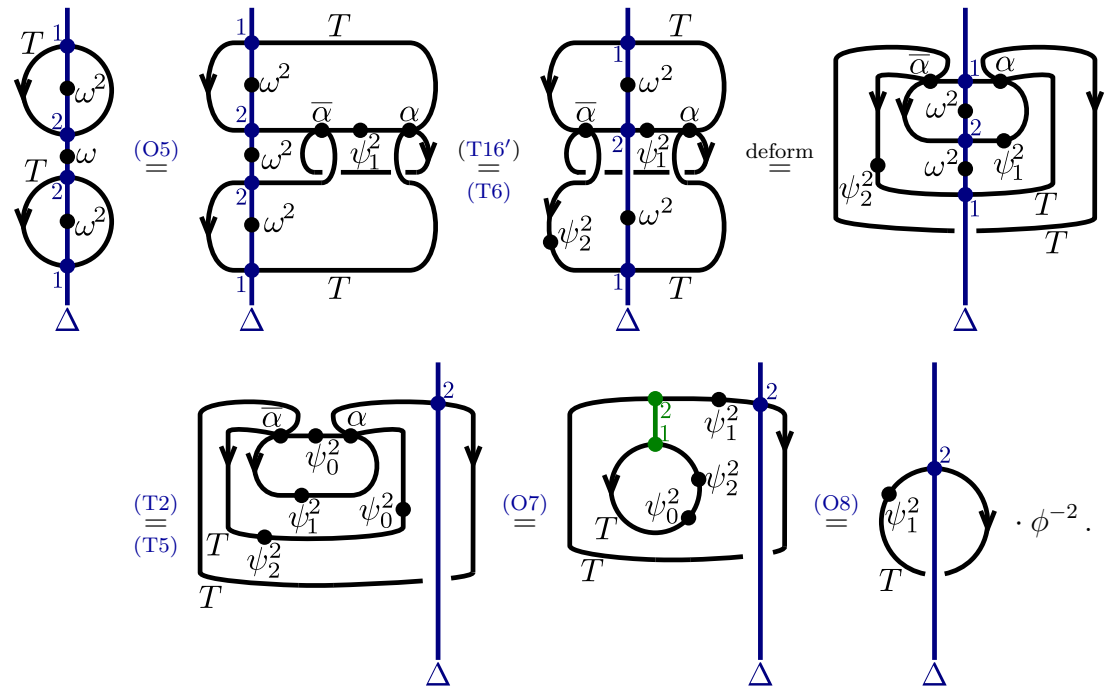
$$\operatorname{tr}_{\mathcal{C}_{\mathbb{A}}} L_\Delta \cdot (\operatorname{tr}_{\mathcal{C}} \omega_A^2)^2 = \operatorname{tr}_{\mathcal{C}}(L_\Delta \circ \omega_\Delta^2 \cdot \operatorname{tr}_{\mathcal{C}} \omega_A^2)$$



(5.61)

Step (∗∗) is best checked in reverse: the A-strings can be removed using the in-

tertwining properties of τ_i and the separability property of A. In step $(*)$ we have changed the ψ_r^Δ insertion into ψ_l^Δ by moving it along the top T-loop and then performed two computations, each of which combines two of the four T-loops into one. We will only show the first:



The last term in (5.61) contains the average of a morphism as defined in (5.43) which projects onto $\mathcal{C}_{\mathbb{A}}(A,\Delta)$. The sum over p therefore computes the trace of this projection and one has

$$\mathrm{tr}_{\mathcal{C}}(L_\Delta \circ \omega_\Delta^2 \cdot \mathrm{tr}_{\mathcal{C}}\,\omega_A^2) = \frac{\mathrm{Dim}\,\mathcal{C}}{\phi^8} \cdot \dim \mathcal{C}_{\mathbb{A}}(A,\Delta) = \frac{\mathrm{Dim}\,\mathcal{C}}{\phi^8} \cdot \delta_{A,\Delta}\,. \tag{5.62}$$

It follows that $\mathrm{tr}_{\mathcal{C}}\,\omega_A^2 \neq 0$, as the right hand side is non-zero for $A = \Delta$. This proves the first claim in part (ii) of the theorem.

Recall from Corollary 2.18 that since $\mathcal{C}_{\mathbb{A}}$ is fusion, $\dim_{\mathcal{C}_{\mathbb{A}}} \Delta \neq 0$ for all simple $\Delta \in \mathcal{C}_{\mathbb{A}}$. Using this, we finally get

$$\begin{aligned} L_\Delta &= \frac{\mathrm{tr}_{\mathcal{C}_{\mathbb{A}}} L_\Delta}{\dim_{\mathcal{C}_{\mathbb{A}}} \Delta} \cdot \mathrm{id}_\Delta = \frac{\mathrm{tr}_{\mathcal{C}_{\mathbb{A}}} L_\Delta \cdot \mathrm{tr}_{\mathcal{C}}\,\omega_A^2}{\dim_{\mathcal{C}_{\mathbb{A}}} \Delta \cdot \mathrm{tr}_{\mathcal{C}}\,\omega_A^2} \cdot \mathrm{id}_\Delta = \frac{\mathrm{tr}_{\mathcal{C}}(L_\Delta \circ \omega_A^2) \cdot \mathrm{tr}_{\mathcal{C}}\,\omega_A^2}{\dim_{\mathcal{C}_{\mathbb{A}}} \Delta \cdot (\mathrm{tr}_{\mathcal{C}}\,\omega_A^2)^2} \cdot \mathrm{id}_\Delta \\ &= \frac{\mathrm{Dim}\,\mathcal{C}}{\dim_{\mathcal{C}_{\mathbb{A}}} \Delta \cdot \phi^8 \cdot (\mathrm{tr}_{\mathcal{C}}\,\omega_A^2)^2} \cdot \delta_{A,\Delta} \cdot \mathrm{id}_\Delta = \frac{\mathrm{Dim}\,\mathcal{C}}{\phi^8 \cdot (\mathrm{tr}_{\mathcal{C}}\,\omega_A^2)^2} \cdot \delta_{A,\Delta} \cdot \mathrm{id}_\Delta\,. \end{aligned} \tag{5.63}$$

Proposition 2.9iv) now implies part (i) and the remaining claim in part (ii). □

Remark 5.21. As we know from Chapter 3, the applications of modular fusion categories to TQFTs require choosing a square root of the global dimension. For $\mathcal{C}$ and $\mathbb{A}$ as in Theorem 5.20, if $\mathcal{C}$ is equipped with a root $\mathcal{D}_{\mathcal{C}} = \sqrt{\operatorname{Dim}\mathcal{C}}$, then there is a natural choice for $\mathcal{C}_{\mathbb{A}}$, namely

$$\mathcal{D}_{\mathcal{C}_{\mathbb{A}}} := \frac{\mathcal{D}_{\mathcal{C}}}{\phi^4 \cdot \operatorname{tr}_{\mathcal{C}} \omega_A^2} \,. \tag{5.64}$$

When necessary, we will always assume $\mathcal{C}_{\mathbb{A}}$ to be equipped with this root.

Our main interest in an orbifold datum $\mathbb{A}$ in a MFC $\mathcal{C}$ is going to be the MFC $\mathcal{C}_{\mathbb{A}}$ it produces. To this end, let us introduce two (rather strong) notions of an isomorphism of orbifold data. Firstly, a *T-compatible isomorphism* ([CRS3, Def. 3.12]) of $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ to $\widetilde{\mathbb{A}} = (A, \widetilde{T}, \widetilde{\alpha}, \widetilde{\overline{\alpha}}, \psi, \phi)$ is given by an isomorphism $\rho\colon T \to \widetilde{T}$ of A-$A \otimes A$-bimodules, such that

$$(\rho \otimes \rho) \circ \alpha = \widetilde{\alpha} \circ (\rho \otimes \rho) \,. \tag{5.65}$$

Secondly, given a scalar $\xi \in \Bbbk^\times$, one can define a new orbifold datum

$$\mathbb{A}_\xi = (A_\xi, T_\xi, \xi\alpha, \xi\overline{\alpha}, \xi^{-1/2}\psi, \xi^{1/2}\phi) \,, \tag{5.66}$$

where the algebra A_ξ has the same underlying object $A \in \mathcal{C}$ and the (co)multimplication and (co)unit morphisms $(\mu, \eta, \Delta, \varepsilon)$ replaced by $(\xi^{1/2}\mu, \xi^{-1/2}\eta, \xi^{1/2}\Delta, \xi^{-1/2}\varepsilon)$ and $T_\xi = T$ with the actions of A_ξ on T_ξ being those of A on T, multiplied by $\xi^{1/2}$. It is easy to check that $\mathbb{A}_\xi$ again satisfies the conditions (O1)-(O8). We will refer to $\mathbb{A}_\xi$ as a *rescaling* of $\mathbb{A}$.

Proposition 5.22. Let $\mathbb{A}$, $\widetilde{\mathbb{A}}$ and $\mathbb{A}_\xi$ be orbifold data in $\mathcal{C}$, such that $\widetilde{\mathbb{A}}$ and $\mathbb{A}$ are related by a T-compatible isomorphism, and such that $\mathbb{A}_\xi$ is a rescaling of $\mathbb{A}$ for some $\xi \in \Bbbk^\times$. Then $\mathcal{C}_{\mathbb{A}}$, $\mathcal{C}_{\widetilde{\mathbb{A}}}$ and $\mathcal{C}_{\mathbb{A}_\xi}$ are equivalent as ribbon fusion categories.

Proof. Given a T-compatible isomorphism $\rho : T \to \widetilde{T}$, define the functor $F : \mathcal{C}_{\mathbb{A}} \to \mathcal{C}_{\widetilde{\mathbb{A}}}$ as $(M, \tau_1^M, \tau_2^M) \mapsto (M, \widetilde{\tau}_1^M, \widetilde{\tau}_2^M)$, where

$$\widetilde{\tau}_i^M := [M \otimes_A \widetilde{T} \xrightarrow{\mathrm{id}_M \otimes \rho^{-1}} M \otimes_A T \xrightarrow{\tau_i} T \otimes_i M \xrightarrow{\rho \otimes \mathrm{id}_M} \widetilde{T} \otimes_i M], \quad i = 1, 2 \,. \tag{5.67}$$

On morphisms, F acts as the identity. One checks that $\widetilde{\tau}_i$, $i = 1, 2$ are indeed $\widetilde{T}$-crossings. For $M, N \in \mathcal{C}_{\mathbb{A}}$ the objects $F(M \otimes N)$ and $F(M) \otimes F(N)$ are equal, giving F a natural monoidal structure, which preserves braidings and twists.

Similarly the functor $\mathcal{C}_{\mathbb{A}} \to \mathcal{C}_{\mathbb{A}_\xi}$, $(M, \tau_1^M, \tau_2^M) \mapsto (M, \xi\tau_1^M, \xi\tau_2^M)$ gives a ribbon equivalence between $\mathcal{C}_{\mathbb{A}}$ and $\mathcal{C}_{\mathbb{A}_\xi}$. $\square$

6. Reshetikhin-Turaev orbifold graph TQFT

Let $\mathcal{C}$ be a modular fusion category and $\mathbb{A}$ an orbifold datum in it (see Chapter 5). In this chapter we define the Reshetikhin-Turaev orbifold graph TQFT

$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}} \colon \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}}) \to \mathrm{Vect}_{\Bbbk}\,, \tag{6.1}$$

obtained by performing an internal 3-dimensional state sum construction in the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ introduced in Chapter 4. The construction is reminiscent to that of 3-dimensional state sum TQFTs due to Turaev-Viro-Barrett-Westburry, which in fact is a special case of (6.1), as was shown in [CRS3, Sec. 4]. We also prove that $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ is isomorphic to the Reshetikhin-Turaev graph TQFT $Z_{\mathcal{C}_{\mathbb{A}}}^{\mathrm{RT}}$ built from the modular fusion category $\mathcal{C}_{\mathbb{A}}$ associated to $\mathbb{A}$.

This chapter is based on [CRS1, CMRSS1], where graph TQFTs were first introduced as an internal state sum construction in an arbitrary 3-dimensional defect TQFT, and on [CRS3, CMRSS2], where the specialisation to the case of the Reshetikhin-Turaev defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ was explored.

6.1. Foamification

As will become apparent below, the definition of the orbifold graph TQFT $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ in terms of the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ is analogous to the definition of $Z_{\mathcal{C}}^{\mathrm{def}}$ in terms of the graph TQFT $Z_{\mathcal{C}}^{\mathrm{RT}}$. We start with the notion of an admissible 2-skeleton (cf. Definition 4.4):

Definition 6.1. An *admissible 2-skeleton* of a bordism $M \in \mathrm{Bord}_3$ is a stratification T of M such that

- each 3-stratum is diffeomorphic to $\mathbb{R}^3$ or $\mathbb{R}^2 \times [0,1)$ (in particular contractible and does not intersect ∂M in more than one connected component);
- each point of $\mathrm{T}^{(2)}$ has a neighbourhood isomorphic (as oriented stratified manifold) to one of the local models in Figure 6.1.

The qualifier "admissible" refers to the restriction on orientations of adjacent strata of T. Note that an admissible 2-skeleton restricts to an admissible 1-skeleton on the boundary ∂M[10].

One can modify a given 2-skeleton as stated in the following (cf. Lemma 4.6):

[10] Except for, as evident from comparing Figures 4.5 and 6.1, the orientations of the 0-strata on ∂M. This is due to notational differences in [CRS1] and [CRS3], we will ignore them in this chapter and address them in Appendix B.2.

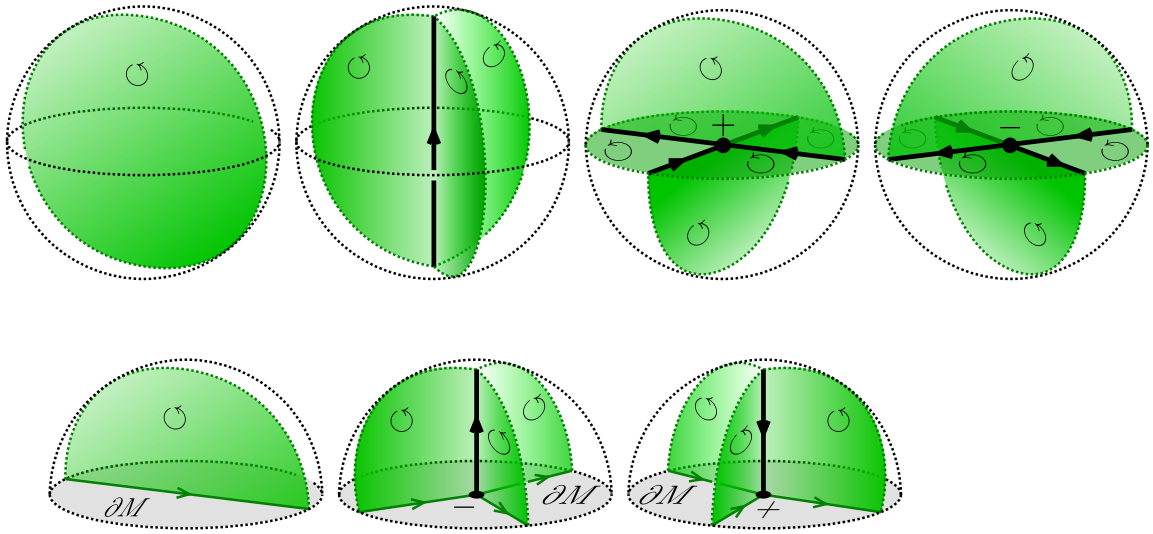


Figure 6.1: Local models of a point on a 0-, 1- or 2-stratum of an admissible 2-skeleton of a bordism $M \in \mathrm{Bord}_3$.

Lemma 6.2. Let $M \in \mathrm{Bord}_3$.

i) Two admissible 2-skeleta T_1, T_2 of M, which coincide on the boundary ∂M, are related by a finite number of the so-called *admissible BLT* moves shown in Figure 6.2.

ii) Any admissible 1-skeleton of ∂M can be extended to an admissible 2-skeleton of M.

As was the case in dimension 2, one can prove Lemma 6.2 by reducing an arbitrary 2-skeleton to a one obtained as the Poincaré dual of a triangulation. The unoriented BLT-moves imply the (dual) Pachner moves, so once again it is enough to consider the admissibility constraints on the orientations. We briefly sketch some details of the proof in the appendix section B.2, mostly by adapting the proof in dimension 2.

Anticipating the definition of the graph TQFT (6.1), let us introduce a generalisation of admissible 2-skeleta, allowing one to present bordisms with embedded ribbon graphs as well. The notions in the definition below are borrowed from [TV, Sec. 14].

Definition 6.3. Let $M = (M, R) \in \mathrm{Bord}_3^{\mathrm{rib}}$. A *(positive, admissible) ribbon diagram* of M is a pair $(\mathrm{T}, \imath)$, where T is an admissible 2-skeleton of M and $\imath\colon R \hookrightarrow \mathrm{T}^{(2)}$ is an embedding such that:

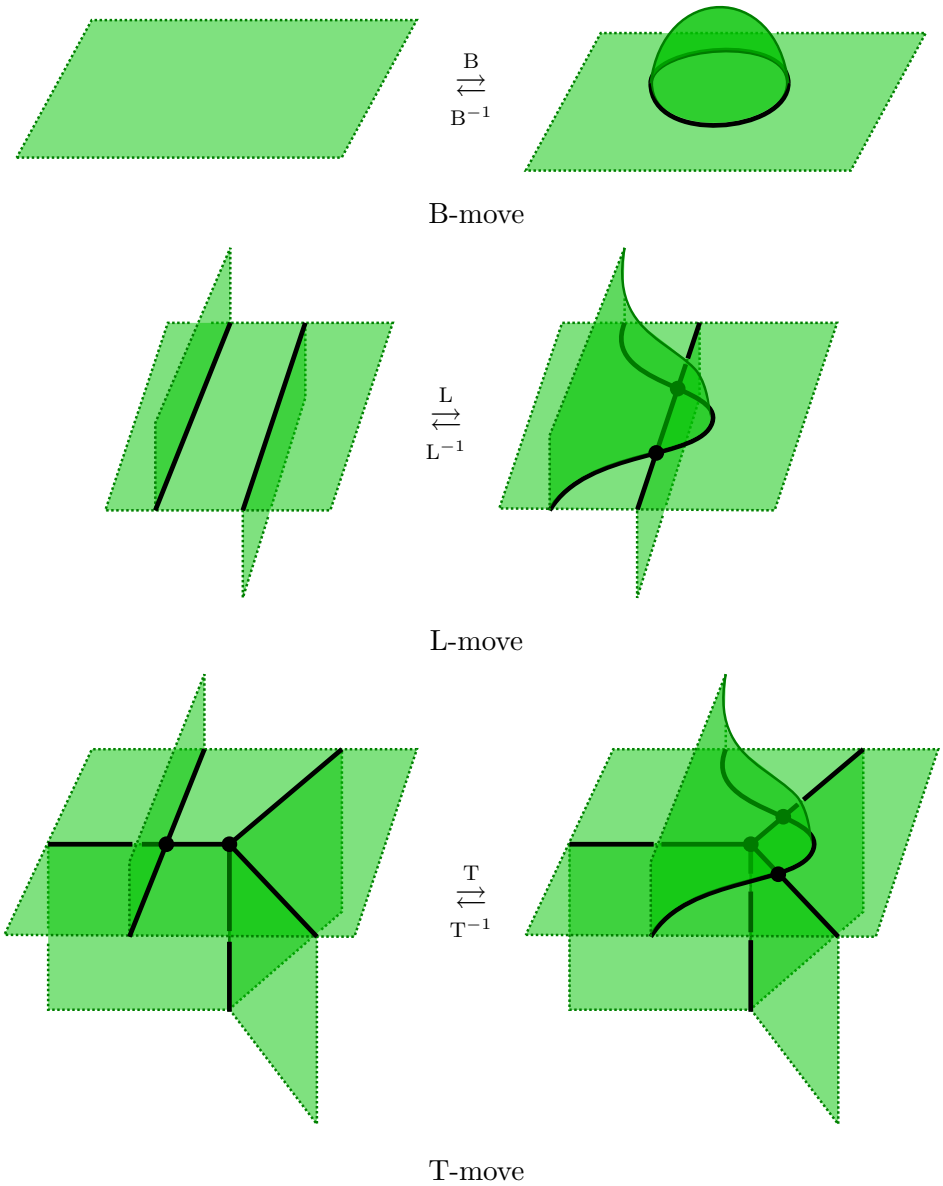
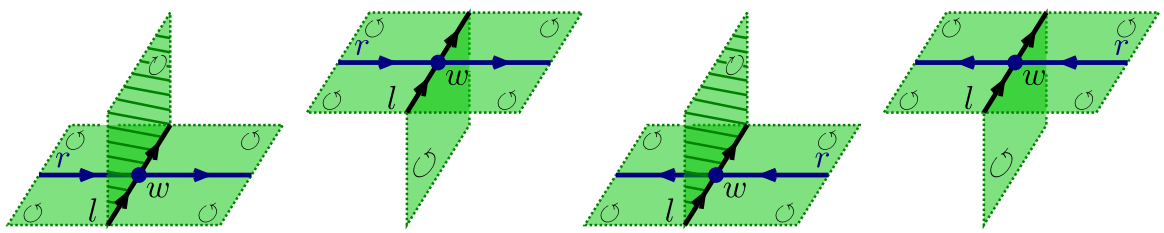


Figure 6.2: BLT-moves on an admissible 2-skeleton M. The orientations are not indicated; each move can be applied if the orientations on both sides are chosen to be admissible (i.e. compatible with the local models in Figure 6.1). This can sometimes prevent an inverse L-move, the rest of the moves are always possible if the orientation of the newly created strata are chosen admissibly.

- $\imath(R) \cap \mathrm{T}^{(0)} = \varnothing$;
- $\imath\big|_{\partial R} = \mathrm{id}_{\partial R}$;
- the coupons of R are embedded into the 2-strata of T; the restriction of $\imath$ on $\mathrm{T}^{(2)} \setminus \mathrm{T}^{(1)}$ is smooth with framings of strands and orientations of coupons in $\imath(R)$ determined by the 2-strata they lie in; the ribbon graph obtained from $\imath(R)$ by smoothing out the intersection points with $\mathrm{T}^{(1)}$ is isotopic to R;
- a point w at which a strand r of $\imath(R)$ intersects a 1-stratum l of T has a neighbourhood isomorphic to one of the following local models:



(6.2)

We call such points w the *(positive) switches* of the ribbon diagram T and denote the set of them by T_{sw}.

On the boundary ∂M, a ribbon diagram restricts to what we will call an *(admissible) 1-skeleton of a punctured surface* Σ. It consists of an admissible 1-skeleton t, such that the punctures of Σ lie in the 1-strata of t and the framings of punctures are determined by the orientations of 1-strata. □

In order to relate two ribbon diagrams of a ribbon bordism $(M, R) \in \mathrm{Bord}_3^{\mathrm{rib}}$, the BLT moves need to be supplemented by the *(admissible) ω-moves* depicted in Figure 6.3. They are versions of the analogous moves that appear in [TV, Sec. 14.3] for the case of 2-skeleta with unoriented 0- and 1-strata (where the moves corresponding to ω_9 and ω_{10} are denoted by $\omega_{9,0,1}$ and $\omega_{9,1,0}$ respectively). The setting of [TV] also allowed the embedding $\imath$ of a ribbon diagram $(\mathrm{T}, \imath)$, representing the ribbon graph R in M, to have double points, marked as overcrossings and undercrossings. In our setting this will not be necessary, mostly because we use the stratifications as in (5.18) to define the braiding in the category $\mathcal{C}_{\mathbb{A}}$. This could be utilised to simplify the set of ω-moves, but would also make us digress more from the reference [TV], on which this approach is based. Note that with the braiding (5.18) in mind, the moves ω_1, ω_2, ω_3 can be recognised as the framed Reidemeister moves.

Lemma 6.4. Let $M = (M, R) \in \mathrm{Bord}_3^{\mathrm{rib}}$.

i) Two positive admissible ribbon diagrams T_1, T_2 of M which coincide on the boundary ∂M are related by a finite number of admissible BLT- and ω-moves.

ii) Any admissible 1-skeleton of the punctured surface ∂M can be extended to a positive admissible ribbon diagram of M.

Proof. The BLT moves are the only ones that change a ribbon diagram $(\mathrm{T}, \imath)$ away from the embedding $\imath(R)$. Hence when restricting to ribbon diagrams T_1, T_2, which differ only away from the embeddings $\imath_1(R)$, $\imath_2(R)$, the statement follows from Lemma 6.2.

In [TV], the notion of a 2-skeleton comes with orientations for 2-strata, while 0- and 1-strata do not carry orientations (contrary to our setting). Let us refer to the original variant of ω-moves in [TV, Sec. 14.3.1] as ω^{TV}-moves. Lemma 6.4 then follows if in the proofs of [TV], we can restrict to ω^{TV}-moves which lift to ω-moves. This is indeed the case: whenever a new 2-stratum appears in the construction of [TV, Sec. 14.4-14.7] (i.e. when "attaching a bubble", c.f. Lemma 14.7 and Figure 14.13 of loc. cit.), there is a choice of orientation for this 2-stratum, and upon close inspection we notice that one of these choices is compatible with a (unique) choice of orientations for the new 0- and 1-strata, which makes the entire ribbon diagram admissibly oriented. □

We now introduce a rather predictable labelling convention for admissible ribbon diagrams. Let $\mathcal{C}$ be a MFC and $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ an orbifold datum in $\mathcal{C}$. Recall from Section 5.2 that the components of $\mathbb{A}$ and objects of $\mathcal{C}_{\mathbb{A}}$ have a natural interpretation as defect labels for the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$. The local neighbourhoods of such defects (see Figures 5.3 and 5.5) correspond precisely to the local models of a ribbon diagram (listed in Figure 6.1 and (6.2)).

Definition 6.5. Let $M = (M, R) \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}})$ be a $\mathcal{C}_{\mathbb{A}}$-coloured ribbon bordism. An $\mathbb{A}$-*coloured (admissible positive) ribbon diagram* is a ribbon diagram $\mathrm{T} = (\mathrm{T}, \imath)$ of the underlying unlabelled ribbon bordism (M, R), together with the following labellings and additional points:

- the 2- and 1-strata of the underlying admissible skeleton T are labelled with A and T respectively, the positively (resp. negatively) oriented 0-strata as in Figure 6.1 by α (resp. $\overline{\alpha}$);

- the switch between an $(N, \tau_1, \tau_2) \in \mathcal{C}_{\mathbb{A}}$ labelled strand of $\imath(R)$ and a T-labelled 1-stratum of T as depicted in each case in (6.2) is labelled correspondingly by τ_1, τ_2, $\overline{\tau_1}$ and $\overline{\tau_2}$;

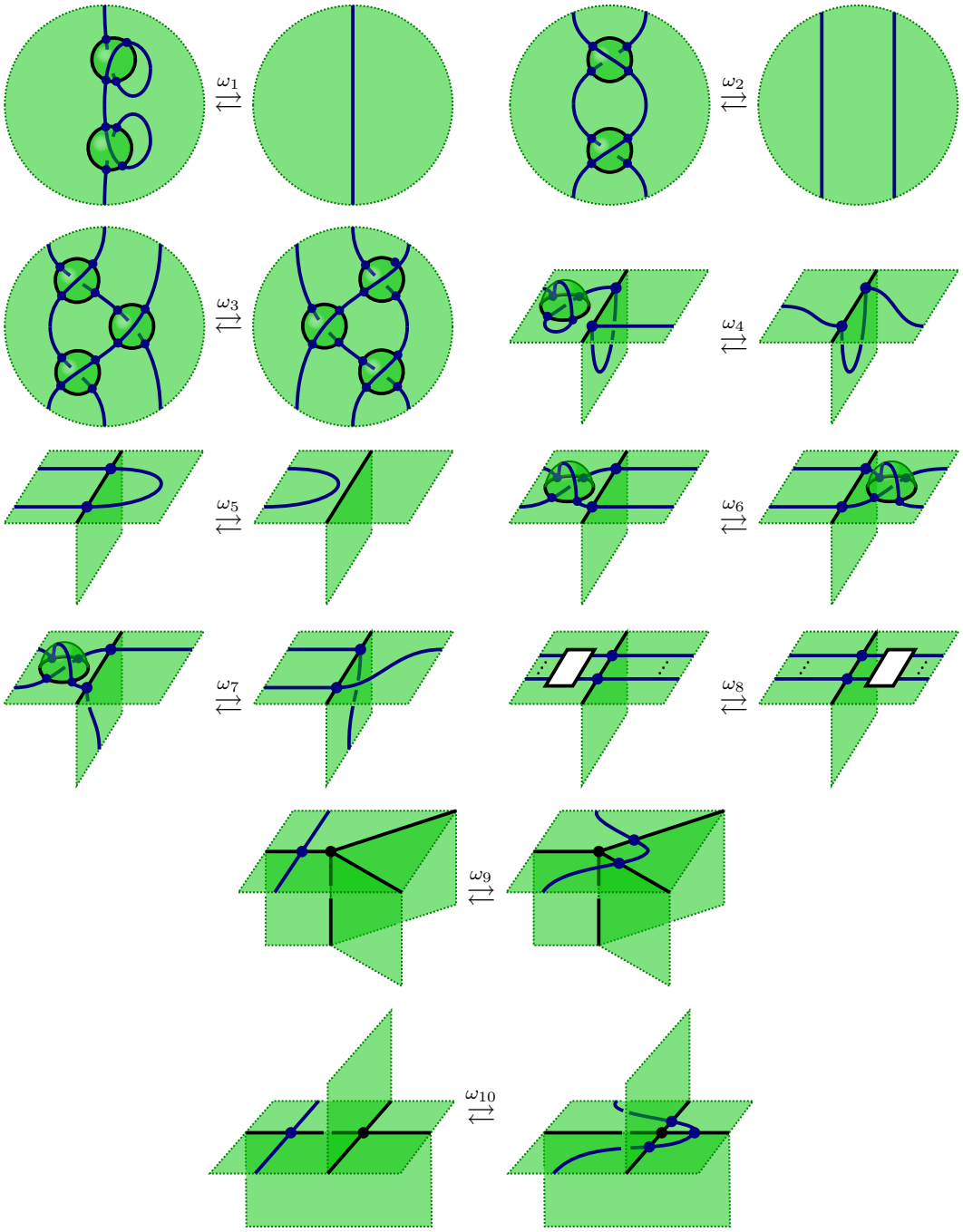


Figure 6.3: ω-moves. The orientations are omitted, but assumed to be admissible (i.e. compatible with the local models in Figure 6.1 and (6.2)) and agree at the boundary on both sides of each move.

- each 3-stratum u of T is assigned a *special point* $p \in u \setminus \partial M$, labelled by ϕ if u intersects the boundary and ϕ^2 otherwise; the set of special points of T will be denoted by T_ϕ.

In case $R = \varnothing$, one also talks about an admissible $\mathbb{A}$-coloured 2-skeleton T of M. On the boundary, an $\mathbb{A}$-coloured ribbon diagram restricts to an $\mathbb{A}$-*coloured 1-skeleton of punctured surfaces.* □

The final ingredient in defining the orbifold graph TQFT (6.1) is the "foamification" map

$$F\colon \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}}) \to \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})\,, \tag{6.3}$$

which is an analogue of the ribbonisation map (4.9) used to define the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ (in particular it is also not a functor).

For an object $\Sigma = (\Sigma, P) \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}})$, where P denotes the set of punctures, and a choice of an $\mathbb{A}$-coloured admissible 1-skeleton t of Σ we define

$$F(\Sigma, \mathrm{t}) \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) \tag{6.4}$$

to be the surface the underlying surface Σ with the stratification:

$$\varnothing = \mathrm{t}^{(-1)} \;\subseteq\; (\mathrm{t}^{(0)} \cup P) \;\subseteq\; \mathrm{t}^{(1)} \;\subseteq\; \mathrm{t}^{(2)} = \Sigma\,, \tag{6.5}$$

i.e. obtained from t by adding the punctures as new 0-strata. Note that the strata of $F(\Sigma, \mathrm{t})$ are labelled according to the defect datum $\mathbb{D}^{\mathcal{C}}$: the 1-strata by the algebra $A \in D_1^{\mathcal{C}}$, the 0-strata by either the A-$A \otimes A$-bimodule $T \in D_1^{\mathcal{C}}$ or by the underlying A-A-bimodules of the objects of $\mathcal{C}_{\mathbb{A}}$.

For a bordism $M = [(M, R)\colon \Sigma \to \Sigma'] \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}})$ and an $\mathbb{A}$-coloured ribbon diagram $\mathrm{T} = (\mathrm{T}, \imath)$ for (M, R), which restricts to $\mathbb{A}$-coloured 1-skeleta t, t$'$ of punctured surfaces Σ, Σ', we define

$$[F(M, \mathrm{T})\colon F(\Sigma, \mathrm{t}) \to F(\Sigma', \mathrm{t}')] \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) \tag{6.6}$$

to be the underlying bordism M with the stratification

$$\varnothing = \mathrm{T}^{(-1)} \;\subseteq\; \big(\mathrm{T}^{(0)} \cup \imath(R^{(0)}) \cup \mathrm{T}_{sw} \cup \mathrm{T}_\phi\big) \;\subseteq\; \big(\mathrm{T}^{(1)} \cup R\big) \;\subseteq\; \mathrm{T}^{(2)} \;\subseteq\; \mathrm{T}^{(3)} = M\,, \tag{6.7}$$

i.e. obtained from T by adding the (segments of) strands of R as new 1-strata and the coupons of R, as well as the switches and the special points of T as new 0-strata. Note once again, that the strata of $F(M, \mathrm{T})$ are correctly labelled according to the defect datum $\mathbb{D}^{\mathcal{C}}$.

6.2. Definition of the orbifold graph TQFT

We repeat the steps in Section 4.4, starting with the independence of $Z_{\mathcal{C}}^{\mathrm{def}}$ on the foamification.

Lemma 6.6. Let $[M\colon \Sigma \to \Sigma'] \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}})$ and let T_1, T_2 be two $\mathbb{A}$-coloured admissible ribbon diagrams of M which restrict to the same $\mathbb{A}$-coloured 1-skeleta of the punctured surfaces Σ, Σ'. Then one has

$$Z_{\mathcal{C}}^{\mathrm{def}}\big(\,F(M,\mathrm{T}_1)\,\big) = Z_{\mathcal{C}}^{\mathrm{def}}\big(\,F(M,\mathrm{T}_2)\,\big)\,. \tag{6.8}$$

Proof. We need to check that upon evaluation with $Z_{\mathcal{C}}^{\mathrm{def}}$ one can perform the BLT- and the ω-moves. All of them are implied by the identities (O1)-(O8) and (T1)-(T7) (or more conveniently, their 3-dimensional form in Figures 5.6 and 5.7). Indeed, the 3 B-moves (when counting the orientations) correspond to (O8), 6 of the 9 lune moves correspond to (O2)-(O7), one of the 19 T-moves corresponds to (O1). Showing that (O1)-(O8) imply the BLT moves with other orientations as well can be checked by hand with the methods remarked in the appendix section B.2 (see also [CRS1, Lem. 3.15, Prop. 3.18], [CMRSS1, Lem. 2.11]).

The ω-moves follow from (T1)-(T7) even more directly: The Reidemeister moves ω_1-ω_3 follow from defining the braiding in $\mathcal{C}_{\mathbb{A}}$ by (5.18) and Proposition 5.9. Showing ω_4 is similar to the proof of Proposition 5.10. ω_5 and ω_6 are implied by the fact that the (co)-evaluation morphisms (5.15), (5.16) and the braiding morphisms (5.18) satisfy (M). ω_7 is implied by Lemma 5.8. ω_8 is a restatement of (M). Finally, the moves ω_9, ω_{10} follow from the identities (T1)-(T3) and their variations. □

Let $\Sigma \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C})$ be a punctured surface and let t, t' be two admissible $\mathbb{A}$-coloured 1-skeleta. Consider the cylinder $C_\Sigma := \Sigma \times [0,1]$ and the linear map

$$\Psi_{\mathrm{t}}^{\mathrm{t}'} := \Big[\,Z_{\mathcal{C}}^{\mathrm{def}}\big(\,F(C_\Sigma,\mathrm{T})\,\big)\colon Z_{\mathcal{C}}^{\mathrm{def}}\big(\,F(\Sigma,\mathrm{t})\,\big) \to Z_{\mathcal{C}}^{\mathrm{def}}\big(\,F(\Sigma,\mathrm{t}')\,\big)\Big]\,, \tag{6.9}$$

where T is an arbitrary $\mathbb{A}$-coloured ribbon diagram of C_Σ, restricting to t, t' on $\Sigma \times \{0\}$, $\Sigma \times \{1\}$ (by Lemma 6.6 the map $\Psi_{\mathrm{t}}^{\mathrm{t}'}$ is independent of the choice of T). For three $\mathbb{A}$-coloured 1-skeleta t, t', t'' of Σ one has

$$\Psi_{\mathrm{t}'}^{\mathrm{t}''} \circ \Psi_{\mathrm{t}}^{\mathrm{t}'} = \Psi_{\mathrm{t}}^{\mathrm{t}''}\,. \tag{6.10}$$

In particular, each map $\Psi_{\mathrm{t}}^{\mathrm{t}}$ is an idempotent.

Construction 6.7. Let $\mathcal{C}$ be a modular fusion category and $\mathbb{A}$ an orbifold datum in it. The Reshetikhin-Turaev orbifold graph TQFT

$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}\colon \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}}) \to \mathrm{Vect}_{\Bbbk} \tag{6.11}$$

is defined as follows:

1. For an object $\Sigma \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}})$, we set
$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}(\Sigma) := \mathrm{colim}\big\{\Psi_{\mathrm{t}}^{\mathrm{t}'}\big\}\,, \tag{6.12}$$
where t, t′ range over all sets of $\mathbb{A}$-coloured admissible 1-skeleta for the punctured surface Σ.

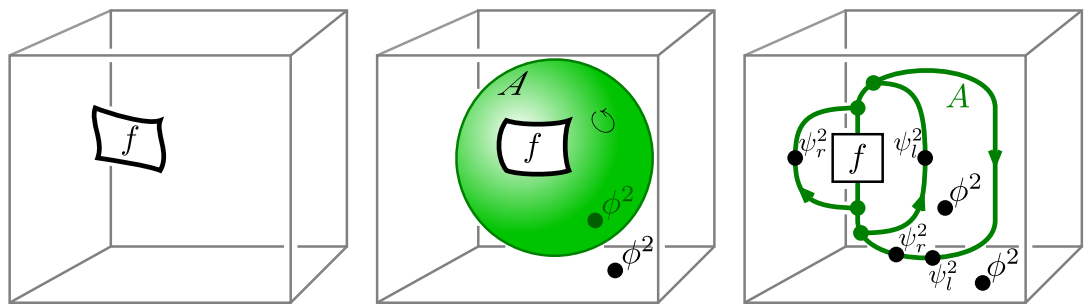
2. For a morphism $[M\colon \Sigma \to \Sigma'] \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}})$, we set $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}(M)$ to be
$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}(\Sigma) \hookrightarrow Z_{\mathcal{C}}^{\mathrm{def}}\big(F(\Sigma,\mathrm{t})\big) \xrightarrow{Z_{\mathcal{C}}^{\mathrm{def}}(F(M,\mathrm{T}))} Z_{\mathcal{C}}^{\mathrm{def}}\big(F(\Sigma',\mathrm{t}')\big) \twoheadrightarrow Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}(\Sigma')\,, \tag{6.13}$$
where T is an arbitrary $\mathbb{A}$-coloured ribbon diagram of M that restricts to $\mathbb{A}$-coloured 1-skeleta t and t′ of the punctured surfaces Σ and Σ' respectively. As in (4.19), the inclusion and the surjection maps are given by the data and the universal property of the colimit.

Remark 6.8. As in (4.20), the state spaces of $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ are isomorphic to the images of the idempotents $\Psi_{\mathrm{t}}^{\mathrm{t}}$,
$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}(\Sigma) \cong \mathrm{im}\,\Psi_{\mathrm{t}}^{\mathrm{t}}\,. \tag{6.14}$$

6.3. Properties of the orbifold graph TQFT

Property 6.9. The orbifold graph TQFT $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ (like the Reshetikhin-Turaev graph TQFT $Z_{\mathcal{C}_{\mathbb{A}}}^{\mathrm{RT}}$, see Property 3.11) is a regular graph TQFT, i.e. upon evaluation one can perform graphical calculus on the embedded ribbon graphs. This follows from Property 4.13 of the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$, stating that it is D_0-complete and the composition of point insertions corresponds to the composition of the multimodule morphisms labelling them. Indeed, upon choosing an admissible $\mathbb{A}$-coloured ribbon diagram and evaluating with $Z_{\mathcal{C}}^{\mathrm{def}}$, without loss of generality one can assume e.g. that the two coupons being composed lie in the same A-labelled 2-stratum and interpret them as 0-strata labelled by the corresponding A-A-bimodule morphisms.

Property 6.10. After applying the foamification procedure, the chosen stratification by an admissible $\mathbb{A}$-coloured ribbon diagram can be further modified upon evaluating the resulting defect bordism with $Z_{\mathcal{C}}^{\mathrm{def}}$. In particular, some $\mathcal{C}_{\mathbb{A}}$-coloured strands/coupons of an embedded ribbon graph can be exchanged for more complicated configurations of defects. For example, a strand labelled by an object $P(M)$, obtained from applying the pipe functor $P\colon {}_A\mathcal{C}_A \to \mathcal{C}_{\mathbb{A}}$ (see Remark 5.15) to a bimodule $M \in {}_A\mathcal{C}_A$, can be exchanged for the "pipe" stratification made of four T-labelled 1-strata with an M-labelled line in the middle as depicted in (5.39) (for $P := P_2$). Similarly, coupons labelled with braiding morphisms of two objects $M, N \in \mathcal{C}_{\mathbb{A}}$ can be exchanged for the stratifications in (5.18) (with the ϕ^2-factor either as a prefactor to the resulting invariant, or made into a point insertion in the interior of the bubble). Braidings with an object $P(M)$ can be instead exchanged for the stratifications in (5.41).



(a) (b) (c)

Figure 6.4: (a) The 3-sphere with an embedded $f \in \mathrm{End}_{\mathcal{C}_{\mathbb{A}}}(A)$ labelled coupon S^3_f, depicted as a cube with boundary representing the point at infinity. (b) The admissible $\mathbb{A}$-coloured ribbon diagram T for S^3_f. (c) A ribbonisation for the defect 3-manifold $F(S^3_f, \mathrm{T})$.

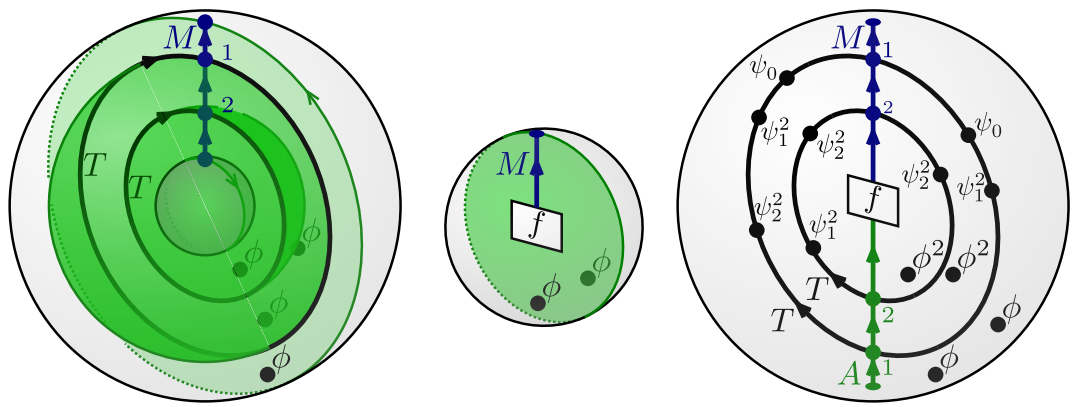
Example 6.11. Let $f \in \mathrm{End}_{\mathcal{C}_{\mathbb{A}}}(A)$ and let $S^3_f = [(S^3_f, 0)\colon \varnothing \to \varnothing]$ be the morphism in $\widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C}_{\mathbb{A}})$ represented by the 3-sphere with a single embedded f-labelled coupon (with the entry $0 \in \mathbb{Z}$ referring to the signature, see Definition 3.4). Since A is the tensor unit in $\mathcal{C}_{\mathbb{A}}$, the coupon does not need to have adjacent strands, see Figure 6.4a. We compute the invariant $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(S^3_f) \in \Bbbk$. Let T be an admissible $\mathbb{A}$-decorated ribbon diagram consisting of an embedded 2-sphere containing the coupon. The resulting foamification $F(S^3_f, \mathrm{T})$ is depicted in Figure 6.4b. Note that T has two 3-strata (the "inside" and the "outside" of the defect sphere), each of which gets a ϕ^2-insertion as prescribed by adding the special points in Definition 6.5.

Next we consider the ribbonisation $R(\,F(S^3_f, \mathrm{T}),\, t\,)$ where t is the 1-skeleton of the defect sphere with an f-insertion as shown in Figure 6.4c. By the Property 3.14, the invariant assigned by $Z^{\mathrm{RT}}_{\mathcal{C}}$ to the 3-sphere S^3 is $\mathcal{D}^{-1}_{\mathcal{C}}$, the inverse of the square root of the global dimension of $\mathcal{C}$, see (2.16), (2.27). One therefore has:

$$\begin{aligned} Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(S^3_f) &= Z^{\mathrm{def}}_{\mathcal{C}}\big(\,F(S^3_f, \mathcal{T})\,\big) = Z^{\mathrm{RT}}_{\mathcal{C}}\big(\,R\big(\,F(S^3_f, \mathcal{T}),\, t\,\big)\,\big) \\ &= \phi^4 \cdot \mathrm{tr}_{\mathcal{C}}(f \circ \omega_A^2) \cdot Z^{\mathrm{RT}}_{\mathcal{C}}(S^3) = \phi^4 \cdot \mathrm{tr}_{\mathcal{C}}(f \circ \omega_A^2) \cdot \mathcal{D}^{-1}_{\mathcal{C}}\,, \end{aligned} \tag{6.15}$$

where in the third equality we collected the ϕ-insertions into a single prefactor and simplified the ribbon graph into an A-labelled loop with an $f \circ \omega_A^2\colon A \to A$ insertion. If $\mathbb{A}$ is in addition a simple orbifold datum (i.e. if $\mathcal{C}_{\mathbb{A}}$ is fusion, see Definition 5.5), one can identify the scalar $f \in \mathrm{End}_{\mathcal{C}_{\mathbb{A}}}(A) \cong \Bbbk$ with the trace $\mathrm{tr}_{\mathcal{C}_{\mathbb{A}}} f$. Applying the expression (5.64) then yields

$$Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(S^3_f) = \phi^4 \cdot \mathrm{tr}_{\mathcal{C}}\, \omega_A^2 \cdot \mathcal{D}^{-1}_{\mathcal{C}} \cdot f = \mathcal{D}^{-1}_{\mathcal{C}_{\mathbb{A}}} \cdot f\,. \tag{6.16}$$



(a) (b) (c)

Figure 6.5: (a) An admissible $\mathbb{A}$-coloured ribbon diagram T for the cylinder $C = S^2_M \times [0,1]$ depicted as a closed ball with an open ball removed. On the boundary T restricts to a 1-skeleton t of the punctured surface S^2_M consisting of the M-labelled puncture with an adjacent A-labelled 1-stratum. In the interior of T, the two T-labelled 1-strata have adjacent A-coloured 2-strata forming "bubbles" between the two boundary components. (b) The bordism $B_f \colon \varnothing \to S^2_{\mathrm{t}}$. (c) Ribbonisation of $C_{\mathrm{T}} \circ B_f \colon \varnothing \to S^2_{\mathrm{t}}$.

The resulting invariant is therefore precisely $Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}(S^3_f)$. In fact, since any $\mathcal{C}_{\mathbb{A}}$-coloured ribbon graph R in S^3 can be simplified into a single coupon, this shows that more generally the equality

$$Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}\big((S^3, R),\, 0\big) = Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}\big((S^3, R),\, 0\big) \tag{6.17}$$

holds. In Section 6.4 this will play a role in showing that the two graph TQFTs are isomorphic. Note that at this point we have not yet shown that the equality (6.17) holds when the signatures are not 0 as we have not shown that the anomalies of $\mathcal{C}$ and $\mathcal{C}_{\mathbb{A}}$ coincide. This is done in Lemma 6.17 below.

Example 6.12. Let $S^2_M \in \widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C}_{\mathbb{A}})$ be the 2-sphere with a single $M \in \mathcal{C}_{\mathbb{A}}$ labelled puncture. We compute the vector space $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(S^2_M)$. Let T be the admissible $\mathbb{A}$-coloured ribbon diagram of the cylinder $C = S^2_M \times [0,1]$ restricting to a 1-skeleton t on the boundary as depicted in Figure 6.5a, and let

$$\big[C_{\mathrm{T}} \colon S^2_{\mathrm{t}} \to S^2_{\mathrm{t}}\big] := F(C, \mathrm{T}) \in \widehat{\mathrm{Bord}}^{\mathrm{def}}_3(\mathbb{D}^{\mathcal{C}}) \tag{6.18}$$

be the corresponding foamification. By definition one has

$$Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(S^2_M) \cong \operatorname{im} \Psi^{\mathrm{t}}_{\mathrm{t}}\,, \quad \text{where } \Psi^{\mathrm{t}}_{\mathrm{t}} = \big[Z^{\mathrm{def}}_{\mathcal{C}}(C_{\mathrm{T}}) \colon Z^{\mathrm{def}}_{\mathcal{C}}(S^2_{\mathrm{t}}) \to Z^{\mathrm{def}}_{\mathcal{C}}(S^2_{\mathrm{t}})\big]\,. \tag{6.19}$$

The vector space $Z_{\mathcal{C}}^{\mathrm{def}}(S^2_{\mathrm{t}})$ can be identified with ${}_A\mathcal{C}_A(A,M)$ by sending an A-A-bimodule morphism $f\colon A \to X$ to $Z_{\mathcal{C}}^{\mathrm{def}}(B_f)$ where $B_f\colon \varnothing \to S^2_{\mathrm{t}}$ is the stratified closed ball bordism as in Figure 6.5b. One has the equality

$$\Psi^{\mathrm{t}}_{\mathrm{t}}(f) = \overline{f}\,, \tag{6.20}$$

where $[\overline{f}\colon A \to M] \in \mathcal{C}_{\mathbb{A}}$ is the averaged morphism introduced in (5.43) (with the 3-dimensional representation in Figure 5.8a). Indeed, $\Psi^{\mathrm{t}}_{\mathrm{t}}(f)$ corresponds to the evaluation $Z_{\mathcal{C}}^{\mathrm{def}}(C_{\mathrm{T}} \circ B_f)$ which can be computed by ribbonising the argument and rearranging the result as in Figure 6.5c. We conclude that

$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}(S^2_M) \cong \mathcal{C}_{\mathbb{A}}(A,M) \cong Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}(S^2_M)\,. \tag{6.21}$$

Note that for a morphism $f \in \mathcal{C}_{\mathbb{A}}(A,M)$, the defect bordism B_f is a choice of foamification of the bordism $B_f\colon \varnothing \to S^2_M$ as introduced in (3.20).

6.4. Isomorphism to graph TQFT of RT type

In this section we prove the TQFTs of Reshetikhin-Turaev type close under orbifolds, or more precisely, that the following holds:

Theorem 6.13. There is an isomorphism between the graph TQFTs

$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}},\ Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}\colon \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}}) \to \mathrm{Vect}_{\Bbbk}\ . \tag{6.22}$$

The proof is similar to that of the isomorphism between the TQFTs of Turaev-Viro type obtained from a spherical fusion category $\mathcal{S}$ with $\dim \mathcal{S} \neq 0$ and the Reshetikhin-Turaev TQFTs obtained from the Drinfeld centre $\mathcal{Z}(\mathcal{S})$, see [TV, Ch. 17]. It relies on the following technical lemma [TV, Lem. 17.2]:

Lemma 6.14. Let $\mathcal{M}$ be a monoidal category and let $F, G\colon \mathcal{M} \to \mathrm{Vect}_{\Bbbk}$ be monoidal functors. Moreover, suppose that

i) F is *non-degenerate*, i. e. for all objects $X \in \mathcal{M}$ one has

$$F(X) \cong \mathrm{span}_{\Bbbk}\, F\big(\ \mathcal{M}(\mathbb{1}, X)\ \big)\,; \tag{6.23}$$

ii) for all $X \in \mathcal{M}$ one has

$$\dim F(X) \geq \dim G(X)\,; \tag{6.24}$$

iii) for all $\varphi \in \mathrm{End}_{\mathcal{M}}(\mathbb{1})$ one has

$$F(\varphi) = G(\varphi) \tag{6.25}$$

(where we omitted the structural isomorphisms $F(\mathbb{1}_{\mathcal{M}}) \cong \Bbbk \cong G(\mathbb{1}_{\mathcal{M}})$).

Then there is a unique monoidal natural isomorphism between F and G.

This lemma will be applied for $\mathcal{M} = \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}})$, $F = Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}$ and $G = Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}$. For this one needs to make sure that the conditions i), ii), iii) hold. Note that i) is already fulfilled by definition of the Reshetikhin-Turaev TQFT (recall that by definition the vector space $Z^{\mathrm{RT}}_{\mathcal{C}}(\Sigma)$ for a punctured surface Σ is defined as a quotient of the linear span of the bordisms of type $[\varnothing \to \Sigma]$, see (3.17)). Condition iii) translates into the statement that the TQFTs $Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}$ and $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}$ yield equal invariants of closed $\mathcal{C}_{\mathbb{A}}$-coloured ribbon 3-manifolds, which is verified below. Finally, we show that the condition ii) holds in Section 6.5. Note that the condition ii) is only relevant for the case $\operatorname{char}\Bbbk \neq 0$, since otherwise, for all $\Sigma \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}})$, the invariants of $\Sigma \times S^1$ already yield the dimensions of the corresponding state space, see (3.27).

For the rest of the section we focus on showing, that for a closed $\mathcal{C}_{\mathbb{A}}$-coloured ribbon manifold $M = [(M, R, n)\colon \varnothing \to \varnothing] \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}})$, with the embedded $\mathcal{C}_{\mathbb{A}}$-coloured ribbon graph R and signature $n \in \mathbb{Z}$, one has

$$Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(M) = Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}(M)\,. \tag{6.26}$$

For this one needs to relate the presentations of M in terms of surgery on S^3 and as an admissible $\mathbb{A}$-coloured ribbon diagram, which is achieved in the following

Construction 6.15. For a choice L of a surgery link for M we construct an admissible $\mathbb{A}$-coloured ribbon diagram T_L for (M, R) as follows:

i) Following Section 3.2, let $L \sqcup R$ be the ribbon graph in $S^3 \simeq \mathbb{R}^3 \cup \{\infty\}$ representing (M, R) i.e. consisting of possibly linked components of L and R such that surgery along L yields (M, R). Project $L \sqcup R$ on the outwards oriented unit 2-sphere so that the coupons do not intersect the strands, the intersections of strands are pairwise distinct and transversal, and the framings of components of $L \sqcup \mathcal{R}$ agree with the orientation of the 2-sphere. The intersection points are marked to distinguish between "overcrossings" and "undercrossings". For simplicity we assume that there are no crossings between the strands of R, which can be achieved by exchanging them with coupons labelled with braiding morphisms. Below we identify L and R with their projection images in S^2.

ii) Surround each strand of L by additional 1- and 2-strata as shown in Figure 6.6a. As we will see later, this is done intentionally to mimic the stratification corresponding to a "pipe" object of $\mathcal{C}_{\mathbb{A}}$ as explained in Remark 5.15. The crossings between the components of L and R are exchanged for the stratifications as in Figures 6.6b and 6.6c, analogous to those corresponding to the braiding morphisms involving the pipe objects. The crossings between the components of L are handled in the same way, which results in a more intricate stratification shown in Figure 6.6d.

iii) Perform surgery along L (see Section 3.2): for each component L_i of L let U_i be its tubular neighbourhood and let $\varphi_i\colon S^1\times S^1 \to -\partial(S^3\setminus U_i)$ be the diffeomorphism along which the gluing with a solid torus $B^2 \times S^1$ is performed, i.e. such that $\varphi_i(S^1 \times \{1\})$ coincides with the framing of L_i. We make this into a gluing of two stratified manifolds as follows: Provide the solid tori with stratifications by two oriented meridian discs

$$d_1 = B^2 \times \{\mathrm{e}^{i\pi/2}\} \quad \text{and} \quad d_2 = (-B^2) \times \{\mathrm{e}^{-i\pi/2}\}\,. \tag{6.27}$$

Since the 1-strata on ∂U_i run parallel to the curve left by the framing of L_i, we can assume that φ_i is an isomorphism of stratified surfaces, i.e. φ_i maps $S^1 \times \{\mathrm{e}^{\pm i\pi/2}\}$ to the 1-strata of ∂U_i, see Figure 6.7a. This yields the manifold M together with a stratification T_L, whose strata (besides those belonging to R) are unlabelled.

iv) To show that T_L is a ribbon diagram for (M, R), one needs to argue that all its 3-strata are contractible. For a component L_i of L, let V_1, V_2 be the 3-strata inside the pipe stratification added in step ii). Before the surgery, V_1 and V_2 are diffeomorphic to open solid tori. After the surgery their topology is changed: one can move the pipe stratification together with V_1, V_2 across the diffeomorphism φ_i into the interior of the solid torus where they look like open solid cylinders and are evidently contractible (see Figures 6.7b and 6.7c).

v) Define the admissible $\mathbb{A}$-coloured ribbon diagram T_L by labelling the 2-, 1- and 0-strata of T_L with A, T and $\alpha/\overline{\alpha}$ and adding the appropriate ϕ-insertions as in Definition 6.5.

The number of 3-strata in T_L is $2 + 2|L|$, where $|L|$ is the number of components of L. Indeed, the 2-sphere in step i) creates two 3-strata and for each component of L the pipe stratification creates two more 3-strata. $\square$

Let us now use the admissible $\mathbb{A}$-coloured ribbon diagram T_L for (M, R) to compute the invariant $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}(M, R, n)$. In parallel we illustrate the computation with the explicit example of the invariant of the ribbon 3-manifold $(S^3, \varnothing, 0)$, for which we use the surgery link W having one component with a single twist, i.e.

$$W = \tag{6.28}$$

This example will also later help us compute the anomaly of the MFC $\mathcal{C}_{\mathbb{A}}$.

Recall the notation, introduced before the statement of Lemma 5.19: for an object $M \in \mathcal{C}_{\mathbb{A}}$ and a morphism $f \in \mathrm{End}_{\mathcal{C}_{\mathbb{A}}}(M)$, $L(M, f) \sqcup R$ will mean the ribbon

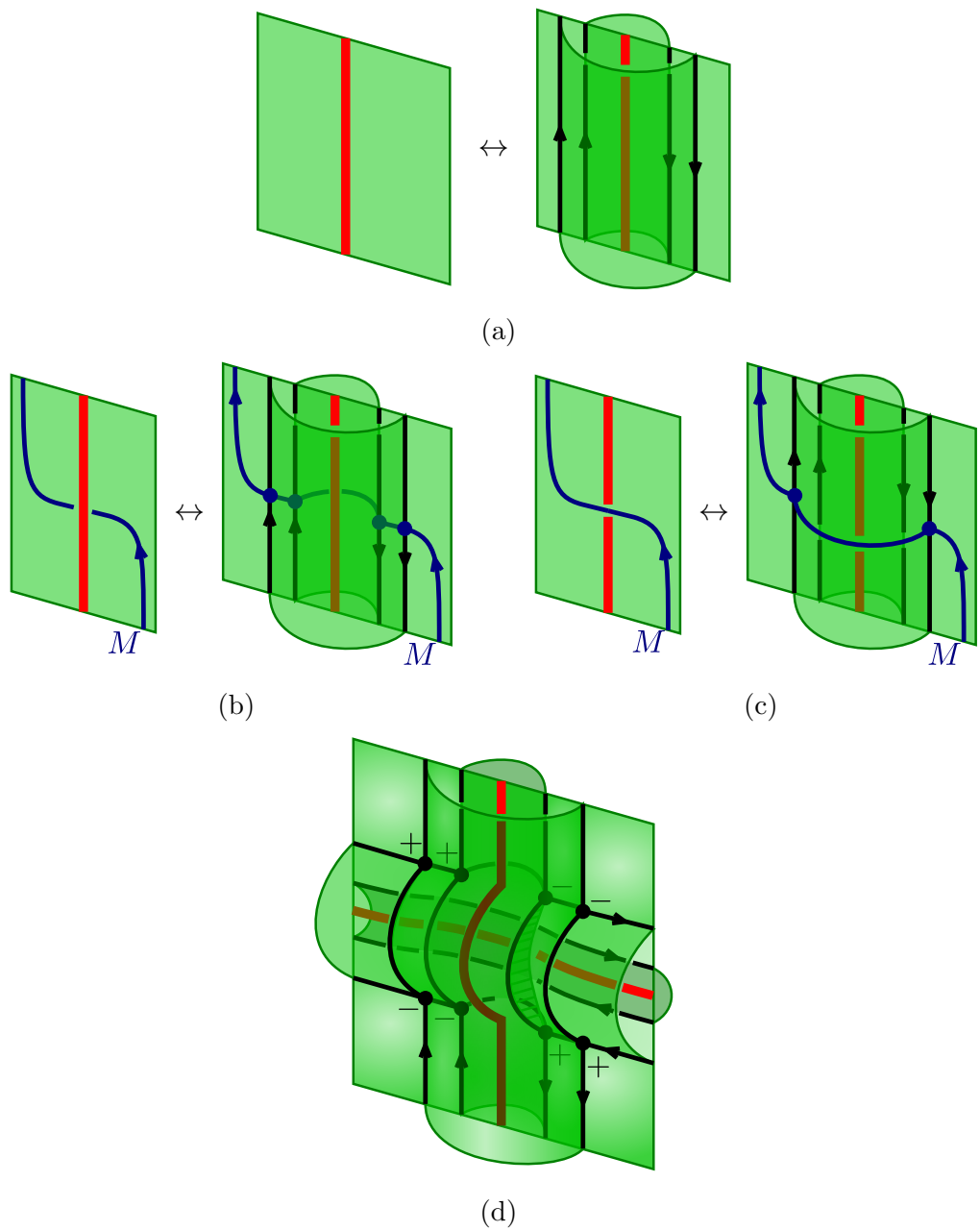


Figure 6.6: Exchanging surgery lines with tubes. Each 2-stratum has the paper-plane orientation, 1-strata have orientations as indicated. Strands that belong to the ribbon graph R carry a label of some object $M = (M, \tau_1, \tau_2) \in \mathcal{C}_{\mathbb{A}}$.

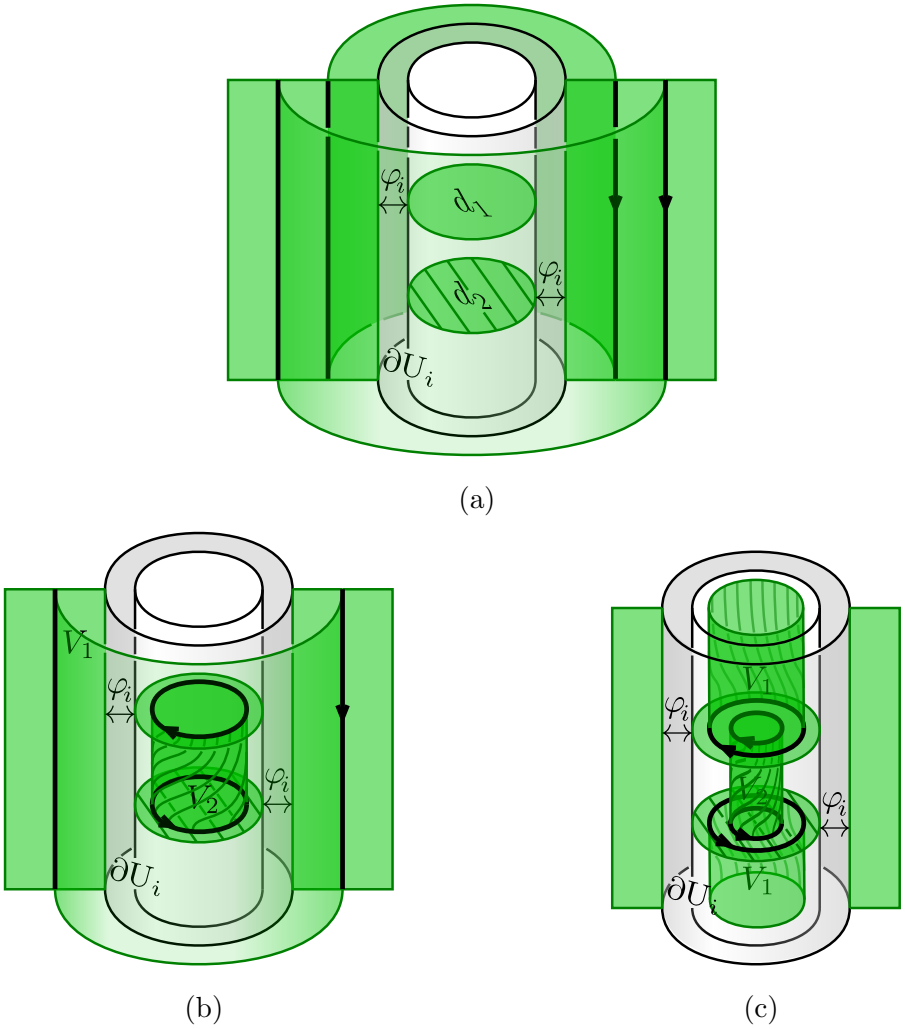


Figure 6.7: For each component L_i, the 3-strata V_1, V_2 in the interior of the surrounding pipe become contractible after surgery. The pictures only show a strip of the pipe: it is actually closed and can be adjacent to other 2-strata or the strands of R (which are ignored in this illustration). The inner white tube represents the solid torus with the two meridian disks d_1, d_2, that is being glued along φ_i during surgery (for this tube the top and the bottom can be thought of as identified). Moreover, φ_i maps the boundaries of the discs d_1, d_2 in ∂U_i to the meridian lines on the boundary of the solid torus. The pictures (a), (b) and (c) all depict the same configuration, in (b) and (c) the pipe stratification is moved inside the solid torus along φ_i.

graph, obtained from $L \sqcup R$ by colouring the components of L with M together with a single f-insertion. For brevity, let us denote

$$P := P(A \otimes C_{\mathcal{C}} \otimes A)\,, \quad d_P := P((\psi_r^A)^2 \otimes d_{\mathcal{C}} \otimes (\psi_l^A)^2)\,\big)\,, \tag{6.29}$$

where the object/morphism pair $(C_{\mathcal{C}}, d_{\mathcal{C}})$ is the Kirby colour for the MFC $\mathcal{C}$, see (2.18).

Lemma 6.16. Let $\mathbb{A}$ be a simple orbifold datum in a MFC $\mathcal{C}$ and $M = (M, R, n)$, L as above. Then the following identity holds:

$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}(M, R, n) = \delta_{\mathcal{C}}^{n-\sigma(L)} \cdot \mathcal{D}_{\mathcal{C}}^{-|L|-1} \cdot \phi^{4|L|} \cdot F_{\mathcal{C}_{\mathbb{A}}}\big(\ L(P, d_P) \sqcup R\ \big)\,, \tag{6.30}$$

where $F_{\mathcal{C}_{\mathbb{A}}}\colon \mathrm{Rib}_{\mathcal{C}_{\mathbb{A}}} \to \mathrm{Vect}_{\Bbbk}$ is the Reshetikhin-Turaev functor, and one uses the isomorphism $\mathrm{End}_{\mathcal{C}_{\mathbb{A}}}(A) \cong \Bbbk$ to read $F_{\mathcal{C}_{\mathbb{A}}}\big(\ L(P, d_P) \sqcup R\ \big)$ as a scalar.

Proof. For the invariants of closed manifolds one has by definition:

$$Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}(M) = Z_{\mathcal{C}}^{\mathrm{def}}\big(\,F(M, \mathrm{T}_L)\,\big)\,, \tag{6.31}$$

where F is the foamification map (6.3). We apply the properties of the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ to modify the stratification T_L in several steps:

i) For each component L_i of L, the meridian discs of the solid tori glued in during surgery (i. e. d_1, d_2 as in (6.27)) extend to contractible 2-strata of $F(M, \mathrm{T}_L)$. By Property 4.16, upon evaluating with $Z_{\mathcal{C}}^{\mathrm{def}}$ they can be removed, leaving ψ^2-insertions on the adjacent T-labelled 1-strata. The rest of T_L does not intersect the link L, i.e. it can be moved away to not intersect the boundaries of the solid tori, along which the surgery is performed.

ii) By definition of $Z_{\mathcal{C}}^{\mathrm{def}}$ via the Reshetikhin-Turaev TQFT $Z_{\mathcal{C}}^{\mathrm{RT}}$ and the formula (3.26) for the invariants of closed manifolds, one can further proceed by replacing the link L with its coloured version $L(C_{\mathcal{C}}, d_{\mathcal{C}})$ and adding the overall factor $\delta_{\mathcal{C}}^{n-\sigma(L)} \cdot \mathcal{D}_{\mathcal{C}}^{-|L|}$. The resulting stratification of S^3 for the example surgery link W in (6.28) is depicted in Figure 6.8b.

iii) By Property 6.10, the components of the labelled surgery link $L(C_{\mathcal{C}}, d_{\mathcal{C}})$, along with the pipe stratifications surrounding them, can be exchanged for the 1-strata labelled by the objects $P \in \mathcal{C}_{\mathbb{A}}$, i.e. by the pipe objects obtained from the induced bimodule $A \otimes C_{\mathcal{C}} \otimes A \in {}_A\mathcal{C}_A$. As an intermediate step in this change one uses the Property 4.15 and the isomorphisms $T \otimes_2 A \cong T$ of A-$A \otimes A$-bimodules in order to apply the definition (5.42) with $M = A \otimes C_{\mathcal{C}} \otimes A$. For each component of $L(C_{\mathcal{C}}, d_{\mathcal{C}})$, the $d_{\mathcal{C}}$-insertion and the two ψ^2-insertions in the surrounding stratification obtained in step i) can

be collected into a single d_P-insertion. The stratifications that replaced the crossings of strands in the construction of T_L (i. e. the ones depicted in Figures 6.6b, 6.6c and 6.6d) can in turn be replaced with coupons labelled by the braiding morphisms for P.

iv) The previous step yields an overall factor $\phi^{4|L|}$ (which previously served as ϕ^2-insertions for the 3-strata inside the pipe stratifications surrounding the components of L) and simplifies the stratification of S^3 to an A-labelled 2-sphere with the graph $L(P, d_P) \sqcup R$ projected on it as well as the two remaining ϕ^2-insertions in its 3-strata (see Figure 6.8c for how it looks for the example surgery link W). The ribbon graph $L(P, d_P) \sqcup R$ can be further exchanged for a single coupon labelled with $f = [F_{\mathcal{C}_\mathbb{A}}(L(P, d_P) \sqcup R) \colon A \to A]$. At this point one recognises the remaining stratification of S^3 as the ribbon diagram used in Example 6.11. Using the computation (6.16) and collecting all prefactors, one obtains the formula (6.30).

□

We now proceed to compare the expression (6.30) with the invariant $Z^{\mathrm{RT}}_{\mathcal{C}_\mathbb{A}}(M, R, n)$, which is obtained by adapting (3.12) for the MFC $\mathcal{C}_\mathbb{A}$. One of the main differences in the two formulas is using the colouring (P, d_P) of the surgery link L for $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}$ and the colouring $(C_{\mathcal{C}_\mathbb{A}}, d_{\mathcal{C}_\mathbb{A}})$ for $Z^{\mathrm{RT}}_{\mathcal{C}_\mathbb{A}}$. The two colourings are however already related in Lemma 5.19. We apply it first to compare the anomalies (see (2.28)):

Lemma 6.17. The modular fusion categories $\mathcal{C}$ and $\mathcal{C}_\mathbb{A}$ have the same anomaly, i.e. $\delta_{\mathcal{C}} = \delta_{\mathcal{C}_\mathbb{A}}$.

Proof. We compare two computations of the invariant $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(S^3, \varnothing, 0)$, one using the empty surgery link as in the Example 6.11 and the other using the surgery link W in (6.28). The former one is obtained from (6.17) and (3.24). For the latter, note that one has $\sigma(W) = 1$ and it follows from (2.19) that $F_{\mathcal{C}_\mathbb{A}}(W(C_{\mathcal{C}_\mathbb{A}}, d_{\mathcal{C}_\mathbb{A}})) = p^+_{\mathcal{C}_\mathbb{A}}$. One uses the formula (6.30) together with Lemma 5.19 to compute:

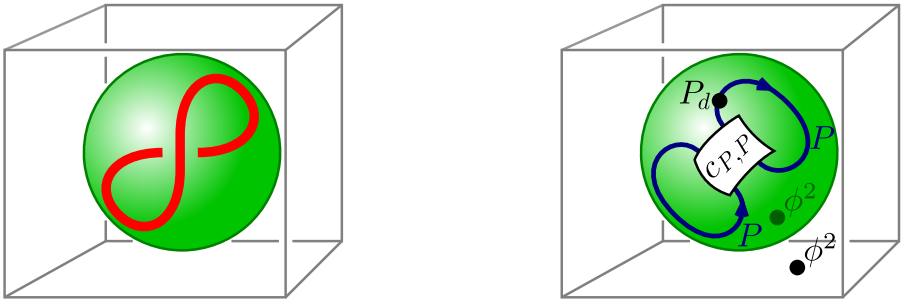
$$\begin{aligned} \mathcal{D}^{-1}_{\mathcal{C}_\mathbb{A}} = Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(S^3, \varnothing, 0) \;&=\; \delta^{-1}_{\mathcal{C}} \cdot \mathcal{D}^{-1}_{\mathcal{C}} \cdot \phi^4 \cdot \mathcal{D}^{-1}_{\mathcal{C}_\mathbb{A}} \cdot F_{\mathcal{C}_\mathbb{A}}\big(\, W(P, d_P) \,\big) \\ &= \delta^{-1}_{\mathcal{C}} \cdot \left(\frac{\mathcal{D}_{\mathcal{C}}}{\phi^4 \cdot \mathrm{tr}_{\mathcal{C}}\, \omega_A^2} \right)^{-1} \cdot \mathcal{D}^{-1}_{\mathcal{C}_\mathbb{A}} \cdot p^+_{\mathcal{C}_\mathbb{A}} \overset{(5.64),(2.28)}{=} \delta^{-1}_{\mathcal{C}} \cdot \mathcal{D}^{-1}_{\mathcal{C}_\mathbb{A}} \cdot \delta_{\mathcal{C}_\mathbb{A}} \,, \end{aligned} \tag{6.32}$$

which yields the result. □

Finally, a similar computation as in the proof of Lemma 6.17 can be performed to prove

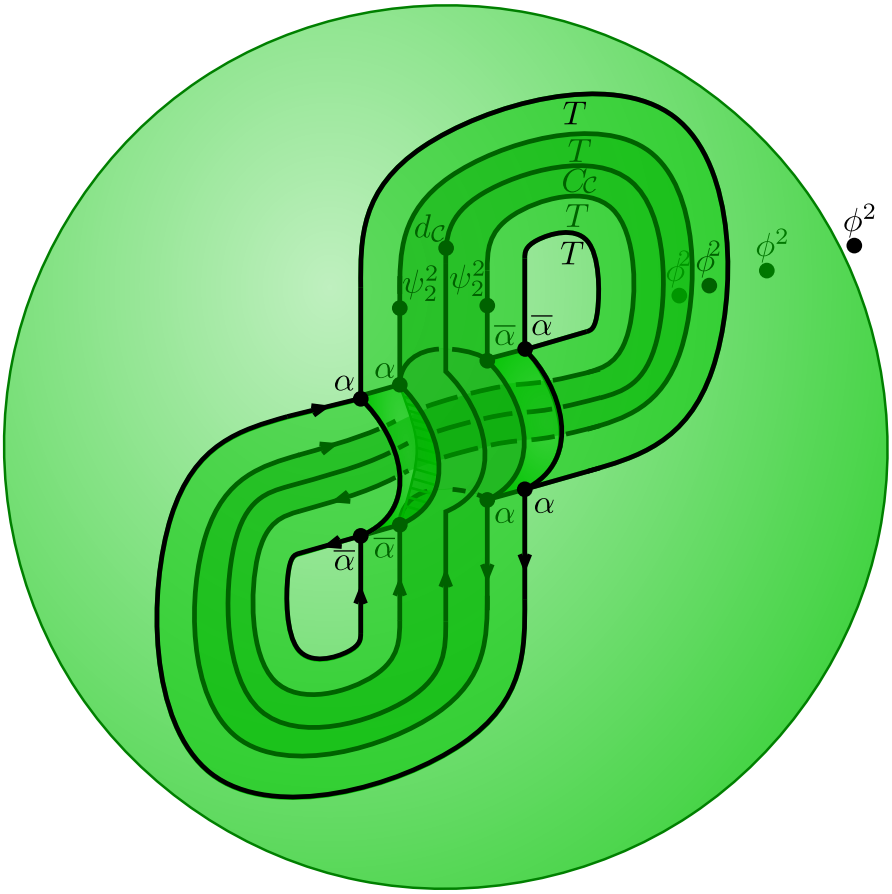
Lemma 6.18. Let $[(M, R, n) \colon \varnothing \to \varnothing] \in \widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C}_\mathbb{A})$ be a closed $\mathcal{C}_\mathbb{A}$-coloured ribbon 3-manifold with the embedded ribbon graph R and signature $n \in \mathbb{Z}$. One has:

$$Z^{\mathrm{RT}}_{\mathcal{C}_\mathbb{A}}(M, R, n) = Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(M, R, n)\,. \tag{6.33}$$



(a)

(c)



(b)

Figure 6.8: Illustration of the proof of Lemma 6.16 for the case of surgery link W in (6.28): (a) Projecting W on a 2-sphere in S^3 as in step i) of Construction 6.15. (b) Surrounding W with pipe stratifications and adding labels. (c) Exchanging the pipes with P-labelled strands with P_d-insertions and replacing crossings with coupons.

Proof. Since both invariants are multiplicative with respect to connected components, it is enough to consider the case when M is connected. One has:

$$\begin{aligned}
Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}(M,R,n) &\overset{(3.12)}{=} \delta^{n-\sigma(L)}_{\mathcal{C}_{\mathbb{A}}} \cdot \mathcal{D}^{-|L|-1}_{\mathcal{C}_{\mathbb{A}}} \cdot F_{\mathcal{C}_{\mathbb{A}}}\big(\, L(C_{\mathcal{C}_{\mathbb{A}}}, d_{\mathcal{C}_{\mathbb{A}}}) \sqcup R \,\big) \\
&\underset{\text{Lem.5.19}}{\overset{\text{Lem.6.17}}{=}} \delta^{n-\sigma(L)}_{\mathcal{C}} \cdot \mathcal{D}^{-|L|-1}_{\mathcal{C}_{\mathbb{A}}} \cdot \frac{1}{(\mathrm{tr}_{\mathcal{C}}\,\omega_A^2)^{|L|}} \cdot F_{\mathcal{C}_{\mathbb{A}}}\big(\, L(P, d_P) \sqcup R \,\big) \\
&\overset{(6.30)}{=} \mathcal{D}^{-|L|}_{\mathcal{C}_{\mathbb{A}}} \cdot \frac{1}{(\mathrm{tr}_{\mathcal{C}}\,\omega_A^2)^{|L|}} \cdot \mathcal{D}^{|L|}_{\mathcal{C}} \cdot \phi^{-4|L|} \cdot Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(M,R,n) \\
&= \mathcal{D}^{-|L|}_{\mathcal{C}_{\mathbb{A}}} \cdot \left(\frac{\mathcal{D}_{\mathcal{C}}}{\phi^4 \cdot \mathrm{tr}_{\mathcal{C}}\,\omega_A^2}\right)^{|L|} \cdot Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(M,R,n) \\
&\overset{(5.64)}{=} Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(M,R,n)\,. \qquad (6.34)
\end{aligned}$$

□

Having proved this, we have shown that the condition iii) of Lemma 6.14 for $F = Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}$ and $G = Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}$ is satisfied. The remaining condition ii) is addressed in the following section.

6.5. State spaces

We now compare the vector spaces assigned to a punctured surface $\Sigma \in \widehat{\mathrm{Bord}}^{\mathrm{rib}}_3(\mathcal{C}_{\mathbb{A}})$ by the graph TQFTs $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}$ and $Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}$. It is enough to consider the case of Σ having a single $M = (M, \tau_1, \tau_2) \in \mathcal{C}_{\mathbb{A}}$ labelled puncture (this follows from [TV, Lem. 15.1] and the fact that upon evaluation both TQFTs allow to compose the coupons of embedded ribbon graphs). In Example 6.11 we already looked at the case when Σ has genus 0 so here we will assume Σ to have genus $g > 0$.

Recall from Property 3.13 that the vector space $Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}(\Sigma)$ is spanned by the (classes of) $\mathcal{C}_{\mathbb{A}}$-coloured ribbon bordisms $H_g = [(H_g, R)\colon \varnothing \to \Sigma]$, where H_g is a solid handlebody of genus g and R is a ribbon graph, positioned at the core of H_g with strands labelled by the simple objects of $\mathcal{C}_{\mathbb{A}}$ (except for the one labelled with $M \in \mathcal{C}_{\mathbb{A}}$, which may or may not be simple, see (3.22)).

On the other hand, to compute the vector space $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(\Sigma)$ we will use the formula (6.14). The admissible $\mathbb{A}$-coloured ribbon diagram T for the cylinder $C_\Sigma = \Sigma \times [0,1]$ that we utilise is depicted in Figure 6.9, where we use a similar presentation as in (3.22): each of the g handles is depicted as a vertical cylinder with the ends identified. To arrive at the ribbon diagram T, we proceed in three steps illustrated in Figure 6.10, where we show only the part involving the leftmost handle:

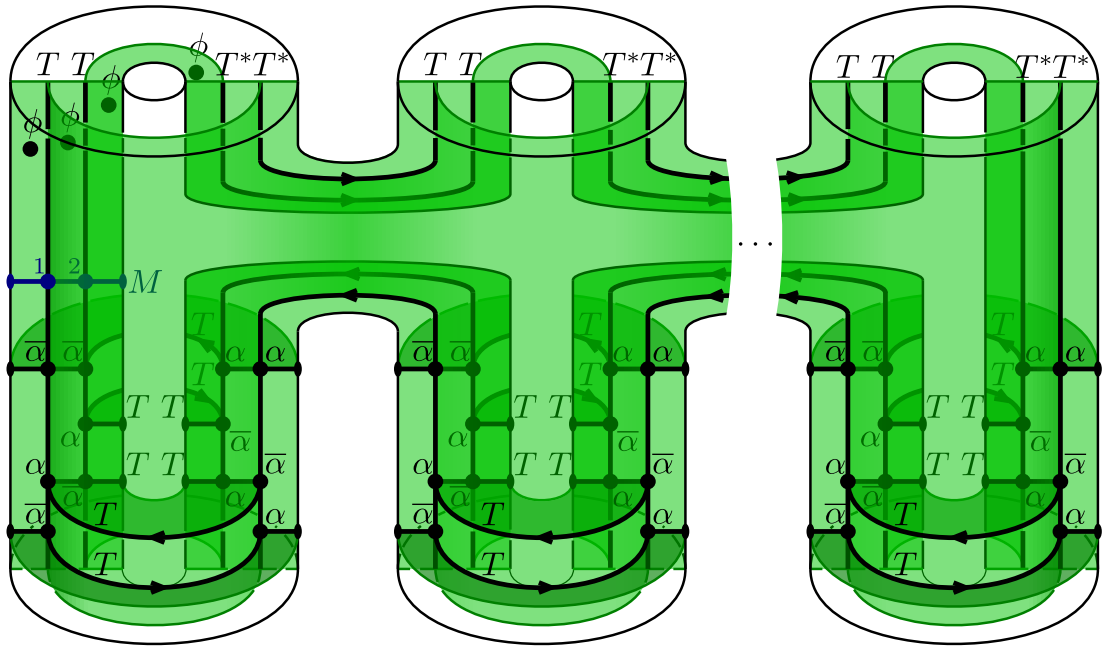


Figure 6.9: An admissible $\mathbb{A}$-coloured ribbon diagram T for the cylinder $C_\Sigma = \Sigma \times [0,1]$ over an object $\Sigma \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_\mathbb{A})$ with a single puncture labelled by $M \in \mathcal{C}_\mathbb{A}$.

i) one starts by adding to C_Σ vertical A-labelled 2-strata connecting the boundary components as shown in Figure 6.10a;

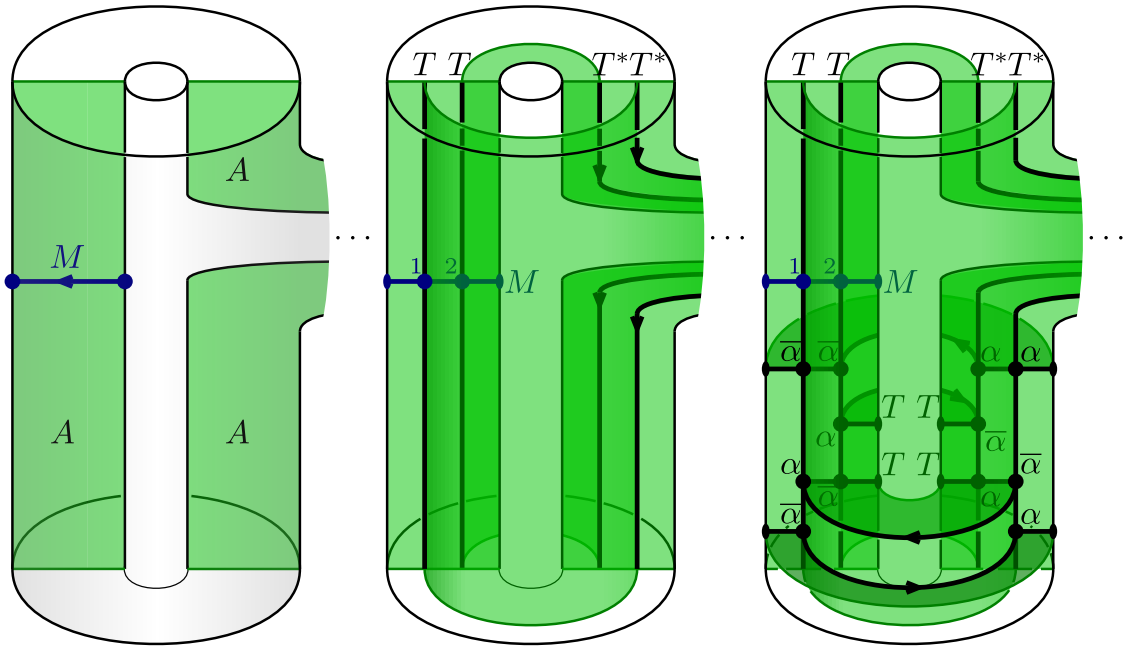
ii) the two boundary components are then separated by further stratification with A-labelled 2-strata, T-labelled 1-strata, and 0-strata labelled with the appropriate T-crossings of M, resembling the one in the definition of the pipe functor as shown in Figure 6.10b;

iii) finally one adds the horizontal A-labelled strata as in Figure 6.10c, to make the 3-strata of C_Σ contractible.

T has four 3-strata, each of which touches a boundary component of C_Σ and therefore receives a ϕ-insertion.

The steps needed to compute the spanning set of the space $Z_\mathcal{C}^{\mathrm{orb}\,\mathbb{A}}(\Sigma)$ are schematically shown in Figure 6.11, and we now proceed to explain them. Let t be the restriction of the ribbon diagram T to Σ, and denote the foamifications of Σ and C_Σ by $\Sigma^{\mathrm{t}} := F(\Sigma, \mathrm{t})$ and $C_\Sigma^{\mathrm{T}} := F(C_\Sigma, \mathrm{T})$, respectively. We need to compute the image of the idempotent

$$\Psi_{\mathrm{t}}^{\mathrm{t}} = \left[Z_\mathcal{C}^{\mathrm{def}}(C_\Sigma^{\mathrm{T}}) \colon Z_\mathcal{C}^{\mathrm{def}}(\Sigma^{\mathrm{t}}) \to Z_\mathcal{C}^{\mathrm{def}}(\Sigma^{\mathrm{t}}) \right]. \tag{6.35}$$



(a) (b) (c)

Figure 6.10: Constructing the $\mathbb{A}$-coloured ribbon diagram of C_Σ in Figure 6.9; only the leftmost handle is shown.

From Property 4.17 of the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$ we know that the space $Z_{\mathcal{C}}^{\mathrm{def}}(\Sigma^{\mathrm{t}})$ is spanned by the (classes of) genus g defect handlebodies $[H_g^{\mathrm{t}}\colon \varnothing \to \Sigma^{\mathrm{t}}]$ in $\widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$, where the stratification of H_g^{t} is obtained by retracting the stratification t on Σ to the core of the handlebody. One has:

$$\Psi_{\mathrm{t}}^{\mathrm{t}}(H_g^{\mathrm{t}}) = C_\Sigma^{\mathrm{T}} \circ H_g^{\mathrm{t}} \,. \tag{6.36}$$

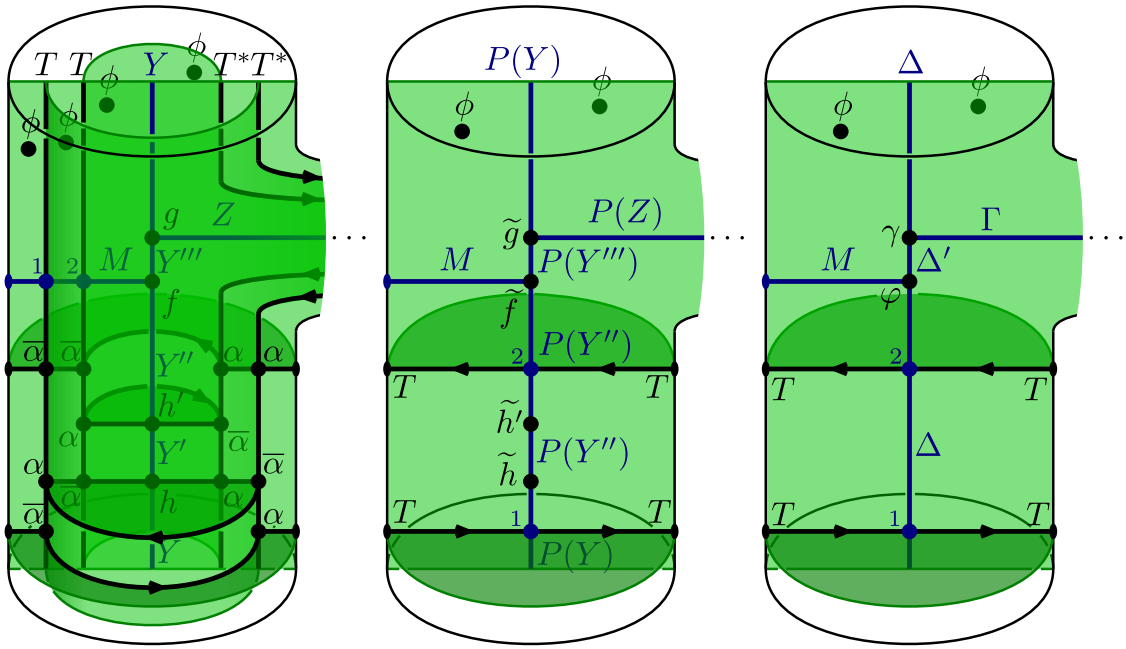
The defect handlebody on the right-hand side of (6.36) is shown in Figure 6.11a, where, as in Figure 6.10, only the first handle is depicted. The labels consist of arbitrary A-A-bimodules Y, Y', Y'', Y''', Z, as well as A-A-bimodule morphisms

$$f\colon Y \to X \otimes_A Y''' \,, \qquad g\colon Y''' \to Y \otimes_A Z \tag{6.37}$$

and A-$A \otimes A$-bimodule morphisms

$$h\colon T \otimes_1 Y \to Y' \otimes_0 T \,, \qquad h'\colon Y' \otimes_0 T \to T \otimes_2 Y'' \tag{6.38}$$

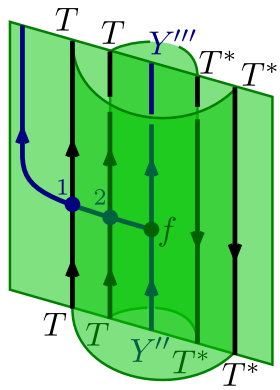
(similarly in the handles not depicted in the figure). Using the pipe functor $P\colon {}_A\mathcal{C}_A \to \mathcal{C}_{\mathbb{A}}$, the stratification inside the handlebody can be exchanged for the

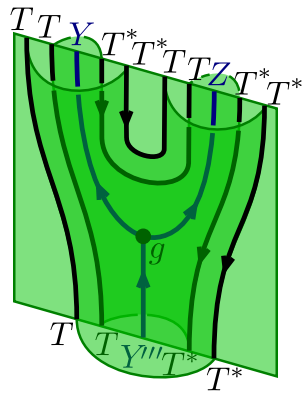


(a) (b) (c)

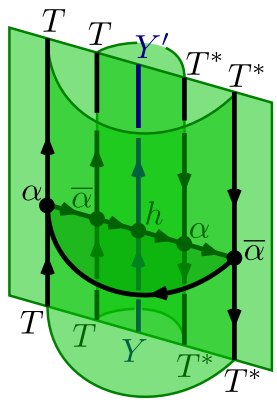
Figure 6.11: Sets of handlebodies, spanning the vector space $\operatorname{im}\Psi_{\mathrm{t}}^{\mathrm{t}} \cong Z_{\mathcal{C}}^{\operatorname{orb}\mathbb{A}}(\Sigma)$.

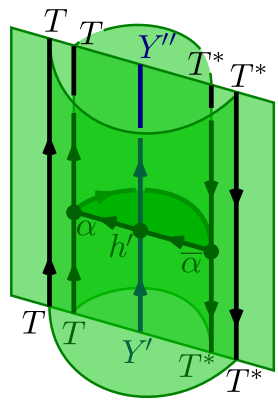
one depicted in Figure 6.11b, where the points are labelled by morphisms

$\widetilde{f} =$ 

, $\widetilde{g} =$ 

, (6.39)

$\widetilde{h} = \phi \cdot$ 

, $\widetilde{h'} = \phi \cdot$ 

, (6.40)

all of which satisfy (M) and are therefore morphisms in $\mathcal{C}_{\mathbb{A}}$. By the semisimplicity of $\mathcal{C}_{\mathbb{A}}$, one can decompose $\Psi_{\mathrm{t}}^{\mathrm{t}}(H_g^{\mathrm{t}})$ into a linear combination of handlebodies with stratification as in Figure 6.11c, where Δ, Δ', Γ are simple objects of $\mathcal{C}_{\mathbb{A}}$, and $\varphi\colon \Delta \to M \otimes_A \Delta'$ and $\gamma\colon \Delta' \to \Delta \otimes_A \Gamma$ are morphisms in $\mathcal{C}_{\mathbb{A}}$. Note that such handlebodies are exactly admissible ribbon diagrams of the ones spanning the space $Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}(\Sigma)$. One therefore concludes that

$$\dim Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}(\Sigma) \geq \dim Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(\Sigma) \tag{6.41}$$

and so the condition ii) of Lemma 6.14 for $F = Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}$ and $G = Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}$ is satisfied. Together with Lemma 6.18, this proves Theorem 6.13. The isomorphism of the two TQFTs of course implies that the two state spaces are actually isomorphic:

$$Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}(\Sigma) \cong Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(\Sigma)\,. \tag{6.42}$$

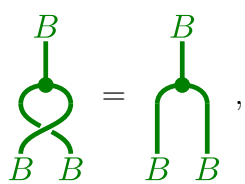
7. Examples of orbifold data

In this chapter we analyse two examples of modular fusion categories, associated to orbifold data. The first example, laid out in Section 7.1, explores the orbifold datum $\mathbb{B}$, obtained from a condensable algebra B in a modular fusion category $\mathcal{C}$ (see Definition 7.1 below). In this case $\mathcal{C}_{\mathbb{B}}$ is equivalent to the category of local modules of B. The second example, presented in Section 7.2, deals with the orbifold datum $\mathbb{A}^{\mathcal{S}}$ in $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$, built from a spherical fusion category $\mathcal{S}$ with non-vanishing global dimension $\mathrm{Dim}\,\mathcal{S}$. The associated modular fusion category $\mathcal{C}_{\mathbb{A}^{\mathcal{S}}}$ then turns out to be equivalent to the Drinfeld centre $\mathcal{Z}(\mathcal{S})$.

The orbifold data in both of the examples were introduced in [CRS3] and their associated modular fusion categories were studied in [MR1].

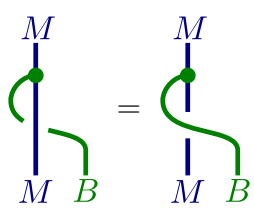
7.1. Condensations

Recall that an algebra B in a braided category $\mathcal{C}$ is called *commutative* if one has



(7.1)

or alternatively, if $B = B^{\mathrm{op}}$. We call a (left) B-module M *local* (or *dyslectic*) if



(7.2)

Here we will focus on commutative algebras in a MFC $\mathcal{C}$.

Definition 7.1. A *condensable algebra* in a MFC $\mathcal{C}$ is a symmetric Δ-separable haploid Frobenius algebra $B \in \mathcal{C}$. The full subcategory of ${}_B\mathcal{C}$ of local B-modules will be denoted by $\mathcal{C}_B^{\mathrm{loc}}$ and referred to as the *B-condensation* of $\mathcal{C}$.

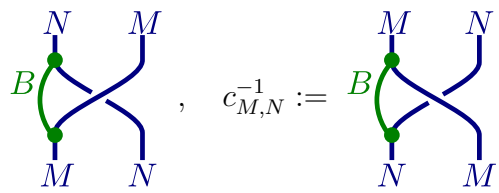
Remark 7.2. The term "condensable algebra" is borrowed from applications of fusion categories to condensed matter physics, see e.g. [Ko, Def. 2.6], [CZW, Ex. 3.2.4]. Sometimes it is also used to refer to an *étale algebra* in a non-degenerate braided fusion category, which is a haploid (or connected) commutative separable algebra, i.e. without an assigned Frobenius structure (see e.g. [DMNO, Def. 3.1 & Ex. 3.3ii)]). An étale algebra A is automatically Frobenius and symmetric if $\dim A \neq 0$, see [FRS1, Lem. 3.7, Cor. 3.10, Lem. 3.11].

Let us review some of the basic properties of the B-condensation of $\mathcal{C}$. We refer to [FFRS] for more details.

- Since B is commutative, for any B-module M the morphisms on both sides of (7.2) define right B-actions on M, which yields two bimodules M_+ and M_-. Local modules are precisely those for which one has $M_+ = M_-$; when talking about the right action on $M \in \mathcal{C}_B^{\mathrm{loc}}$ we will always mean the one given by (7.2).

- One uses the tensor product of ${}_B\mathcal{C}_B$ to equip $\mathcal{C}_B^{\mathrm{loc}}$ with tensor product and duals. Furthermore, the braiding on $\mathcal{C}$ induces a braiding on $\mathcal{C}_B^{\mathrm{loc}}$, for all $M, N \in \mathcal{C}_B^{\mathrm{loc}}$ defined by (with omitted inclusions/projections onto the relative tensor products according to Convention 2.16)

$$c_{M,N} := \quad , \quad c_{M,N}^{-1} := \quad . \tag{7.3}$$



  Because of the identity (7.2), the twist of an arbitrary object $M \in \mathcal{C}_B^{\mathrm{loc}}$ coincides with the twist of the underlying object in $\mathcal{C}$, which makes $\mathcal{C}_B^{\mathrm{loc}}$ into a ribbon category. Note in particular, that the twist of the underlying object of the condensable algebra $B \in \mathcal{C}$ is trivial, i.e. $\theta_B = \mathrm{id}_B$.

- Since B is Δ-separable, the condensation $\mathcal{C}_B^{\mathrm{loc}}$ is finitely semisimple. Moreover, since B is haploid, the tensor unit $\mathbb{1}_{\mathcal{C}_B^{\mathrm{loc}}} := B$ is a simple object, so $\mathcal{C}_B^{\mathrm{loc}}$ is fusion. It was proven in [KO] and [DMNO, Cor. 3.30] that the B-condensation $\mathcal{C}_B^{\mathrm{loc}}$ is in fact a MFC.

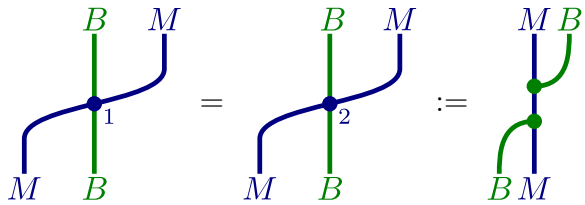
For the remainder of the section we show that the B-condensation can be obtained as a MFC associated to an orbifold datum $\mathbb{B}$ in $\mathcal{C}$ with

$$\mathbb{B} = \big(B,\, {}_BB_{BB},\, \alpha = \overline{\alpha} = \Delta \circ \mu,\, \psi = \eta,\, \phi = 1 \big)\,, \tag{7.4}$$

where the commutativity of the algebra B allows one to treat B as a B-$B \otimes B$-bimodule and the composition $[\Delta \circ \mu \colon AA \to AA]$ of the product and the coproduct of B as a B-$B \otimes B \otimes B$-bimodule morphism. It was shown in [CRS3] that $\mathbb{B}$ is indeed an orbifold datum, i.e. satisfies the identities (O1)-(O8).

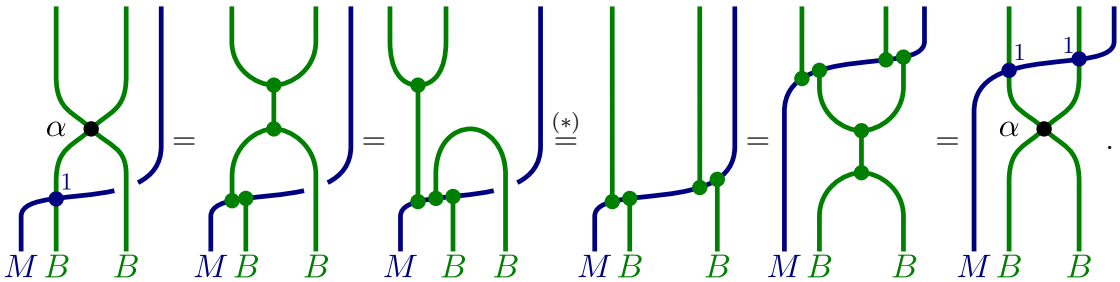
Theorem 7.3. Let B be a condensable algebra in a MFC $\mathcal{C}$ and $\mathbb{B}$ the orbifold datum as in (7.4). Then $\mathbb{B}$ is simple and one has $\mathcal{C}_{\mathbb{B}} \simeq \mathcal{C}_B^{\mathrm{loc}}$ as $\Bbbk$-linear ribbon categories.

Proof. Define a functor $F\colon \mathcal{C}_B^{\mathrm{loc}} \to \mathcal{C}_{\mathbb{B}}$ as follows: Given a local module M, equip it with the canonical bimodule structure and define the T-crossings (for $T = B$ as in (7.4)) to be



(7.5)

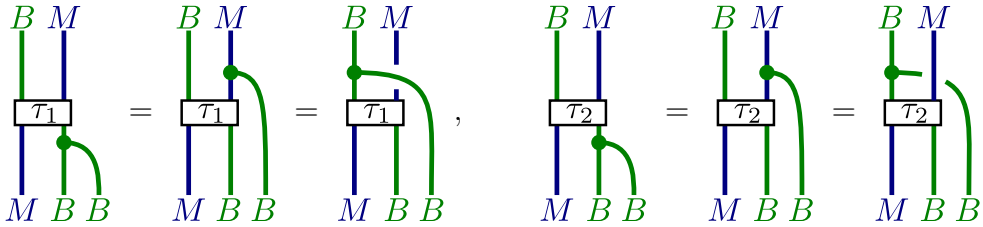
All of the identities (T1)-(T7) then hold and are easy to check, e.g. for (T1) one has:



In $(*)$ one uses the fact that the right action of M comes from (7.2). A morphism in $\mathcal{C}_{\mathbb{B}}$ is precisely a B-module morphism, i.e. F is fully faithful. Since B is simple as a left module over itself (because B is haploid), this shows that the orbifold datum $\mathbb{B}$ is simple.

It is easy to check that F preserves tensor products, braidings and twists, hence it only remains to check that it is an equivalence. We show that F is essentially surjective.

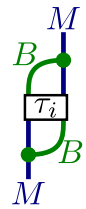
Let $(M, \tau_1, \tau_2) \in \mathcal{C}_{\mathbb{B}}$. Since τ_1, τ_2 are B-$B \otimes B$-bimodule morphisms, one has:



(7.6)

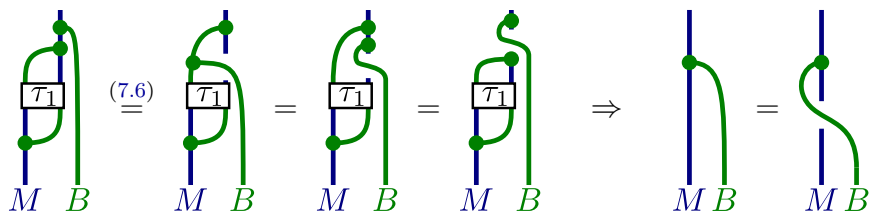
For example in the first equality for τ_1 we think of the right B-action as the action of the first tensor factor of $B \otimes B$ and in the second equality as the action of the second tensor factor. Since $M \cong B \otimes_B M \cong M \otimes_B B$, the T-crossings τ_1, τ_2 can

be recovered from the following invertible B-module morphisms $\widehat{\tau}_1, \widehat{\tau}_2 : M \to M$:

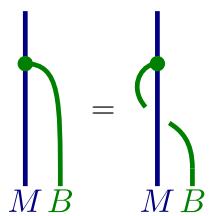
$\widehat{\tau}_i :=$ 

, $\quad i = 1, 2.$ (7.7)

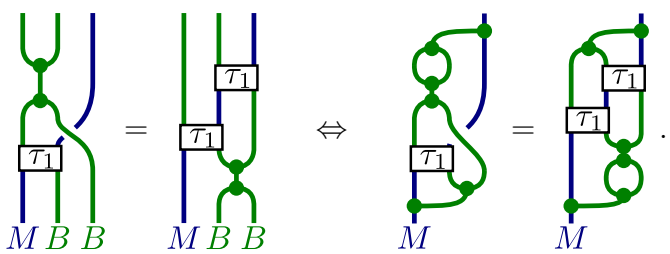
We can then relate the left and right action on M as follows:



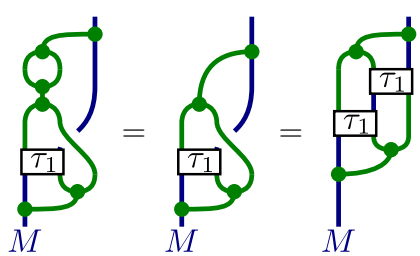
Similarly, the identities for τ_2 in (7.6) imply



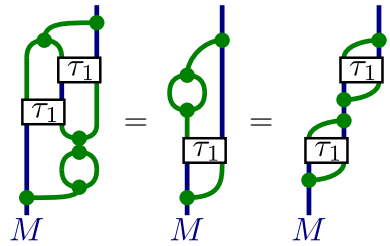
Hence M is a local module with the canonical bimodule structure. It remains to show that the T-crossings are as in (7.5). Using the identity (T1) one has



Examining both sides of the last equality gives:

left-hand side: 

$= \widehat{\tau}_1,$

right-hand side: 

$= \widehat{\tau}_1 \circ \widehat{\tau}_1.$

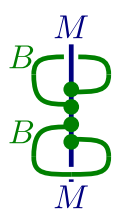
Hence one has $\widehat{\tau}_1 = \widehat{\tau}_1 \circ \widehat{\tau}_1$ and since it is invertible, $\widehat{\tau}_1 = \mathrm{id}_M$, which in turn implies that τ_1 is precisely as in (7.5). The identity (T3) implies the same for τ_2. □

Combining the above result with Theorem 5.20 gives an independent proof that $\mathcal{C}_B^{\mathrm{loc}}$ is modular. For the orbifold datum (7.4) one has $\mathrm{tr}_{\mathcal{C}}\, \omega_A^2 = \dim_{\mathcal{C}} B$ and $\phi = 1$, so the second part of Theorem 5.20 yields

$$\mathrm{Dim}\, \mathcal{C}_B^{\mathrm{loc}} = \frac{\mathrm{Dim}\, \mathcal{C}}{(\dim_{\mathcal{C}} B)^2}\,. \tag{7.8}$$

Both modularity and the above dimension formula are already known from [KO].

Remark 7.4. For a B-B-bimodule $M \in {}_B\mathcal{C}_B$ and an object $X \in \mathcal{C}$, applying the pipe functor $P\colon {}_B\mathcal{C}_B \to \mathcal{C}_{\mathbb{B}}$ on M and on the induced bimodule $B \otimes X \otimes B$ yields the objects

$P(M) \cong \mathrm{im}$ 

, $P(B \otimes X \otimes B) \cong \mathrm{im}$

, (7.9)

where the horizontal B-lines play the role of the idempotent (2.50), which commute with the actions of B because of commutativity and enforce the relation (7.1) on the images. The isomorphisms (7.9) follow from the definition (5.42) and using the obvious isomorphisms $B \otimes_B M \cong M$, etc.

Remark 7.5. A condensable algebra $B \in \mathcal{C}$ can be used to label 2-strata in the defect TQFT $Z_{\mathcal{C}}^{\mathrm{def}}$. Because of the commutativity condition (7.1), such surface defects can be thought of as having no orientation assigned, see Property 4.12. Moreover, B can be considered as a multimodule over any number of copies of B, and therefore also used to label 1-strata having any number of B-labelled 2-strata adjacent to it. Since the twist of the object $B \in \mathcal{C}$ is trivial, B-labelled 1-strata can be interpreted as having no framing assigned. This makes the resulting orbifold graph TQFT $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{B}}$ particularly easy to handle, which will be exploited in Chapter 9 to introduce domain walls between different orbifold theories of type (7.4).

7.2. Drinfeld centres

In this section, let $\mathcal{S}$ be a spherical fusion category such that $\operatorname{Dim}\mathcal{S} \neq 0$ (this condition is only relevant if $\operatorname{char}\Bbbk \neq 0$). Recall from Proposition 2.7, that in this case the Drinfeld centre $\mathcal{Z}(\mathcal{S})$ is a MFC. We show that it can be obtained as a MFC associated to a certain orbifold datum $\mathbb{A}^{\mathcal{S}}$ in the trivial modular fusion category $\mathrm{Vect}_{\Bbbk}$ of finite dimensional $\Bbbk$-vector spaces. This orbifold datum appeared in [CRS3, Sec. 4.2] and will be presented in the setting of Δ-separable Frobenius algebras and Euler completion summarised in Appendix C.

Let us define the natural transformation $\psi\colon \mathrm{Id}_{\mathcal{S}} \Rightarrow \mathrm{Id}_{\mathcal{S}}$ by taking for each $X \in \mathcal{S}$

$$\psi_X := \sum_{i,\,\pi} \dim_{\mathcal{S}}^{1/2}(i) \cdot [X \xrightarrow{\pi} i \xrightarrow{\bar{\pi}} X]\,. \tag{7.10}$$

Here, i in the sum ranges over $\mathrm{Irr}_{\mathcal{S}}$, π over a basis of $\mathcal{S}(X,i)$, and $\bar{\pi}$ is the corresponding element of the dual basis of $\mathcal{S}(i,X)$ with respect to the composition pairing $\mathcal{S}(i,X) \otimes_{\Bbbk} \mathcal{S}(X,i) \to \mathcal{S}(i,i) \cong \Bbbk$. We now define $\mathbb{A}^{\mathcal{S}} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ with:

$$\begin{aligned}
A &= \bigoplus_{i\in\mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(i,i) \cong \bigoplus_{i\in\mathrm{Irr}_{\mathcal{S}}} \Bbbk, \qquad T = \bigoplus_{l,i,j\in\mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l,ij),\\
\alpha :\ & \underbrace{\bigoplus_{l,a,i,j,k} \mathcal{S}(l,ia)\otimes_{\Bbbk}\mathcal{S}(a,jk)}_{\cong\bigoplus_{l,i,j,k}\mathcal{S}(l,\ i(jk))} \longrightarrow \underbrace{\bigoplus_{l,b,i,j,k} \mathcal{S}(l,bk)\otimes_{\Bbbk}\mathcal{S}(b,ij)}_{\cong\bigoplus_{l,i,j,k}\mathcal{S}(l,\ (ij)k)},\\
&\qquad \Big[l \xrightarrow{f} i(jk)\Big] \longmapsto \Big[l \xrightarrow{f} i(jk) \xrightarrow{a^{-1}_{i,j,k}} (ij)k \xrightarrow{\psi^{-2}_{ij}\otimes\mathrm{id}_k} (ij)k\Big],\\
\overline{\alpha} :\ & \underbrace{\bigoplus_{l,b,i,j,k} \mathcal{S}(l,bk)\otimes_{\Bbbk}\mathcal{S}(b,ij)}_{\cong\bigoplus_{l,i,j,k}\mathcal{S}(l,\ (ij)k)} \longrightarrow \underbrace{\bigoplus_{l,a,i,j,k} \mathcal{S}(l,ia)\otimes_{\Bbbk}\mathcal{S}(a,jk)}_{\cong\bigoplus_{l,i,j,k}\mathcal{S}(l,\ i(jk))},\\
&\qquad \Big[l \xrightarrow{g} (ij)k\Big] \longmapsto \Big[l \xrightarrow{g} (ij)k \xrightarrow{a_{i,j,k}} i(jk) \xrightarrow{\mathrm{id}_i\otimes\psi^{-2}_{jk}} i(jk)\Big],\\
\psi :\ & \Big[i \xrightarrow{f} i\Big] \longmapsto \Big[i \xrightarrow{f} i \xrightarrow{\psi_i} i\Big] \quad , \quad \phi^2 = \frac{1}{\operatorname{Dim}\mathcal{S}} = \left(\sum_{i\in\mathrm{Irr}_{\mathcal{S}}} \dim_{\mathcal{S}}^2(i)\right)^{-1} .
\end{aligned} \tag{7.11}$$

Here, we abuse notation by denoting the morphism $\psi\colon A \to A$ (which labels the point insertions in the setting of Appendix C) in the orbifold datum and the natural transformation $\psi\colon \mathrm{Id}_{\mathcal{S}} \Rightarrow \mathrm{Id}_{\mathcal{S}}$ from (7.10) with the same symbol. The left action of $[f : k \to k] \in A$ on $[m : l \to ij] \in T$, $i,j,k,l \in \mathrm{Irr}_{\mathcal{S}}$ is precomposition and the first (resp. second) right action is postcomposition with $(f \otimes \mathrm{id}_j)$ (resp. $(\mathrm{id}_i \otimes f)$) (if the

composition is undefined, the corresponding action is by 0). The isomorphisms in the definitions of α, $\overline{\alpha}$ given by composition. For example, in the source object of α, the explicit form of the isomorphism is $f \otimes_{\Bbbk} g \mapsto (\mathrm{id}_i \otimes g) \circ f$.

Theorem 7.6. Let $\mathcal{S}$ be a spherical fusion category with $\mathrm{Dim}\,\mathcal{S} \neq 0$, $\mathbb{A}^{\mathcal{S}}$ the orbifold datum as in (7.11) and $\mathcal{C} = \mathrm{Vect}_{\Bbbk}$. Then $\mathbb{A}^{\mathcal{S}}$ is simple and $\mathcal{C}_{\mathbb{A}^{\mathcal{S}}} \cong \mathcal{Z}(\mathcal{S})$ as $\Bbbk$-linear ribbon categories.

Together with Theorem 5.20 this gives an alternative proof that $\mathcal{Z}(\mathcal{S})$ is modular. Furthermore, for the orbifold datum (7.11) one has $\mathrm{tr}_{\mathcal{C}}\, \omega_A^2 = \mathrm{Dim}\,\mathcal{S}$, so the second part of Theorem 5.20 yields

$$\mathrm{Dim}\,\mathcal{Z}(\mathcal{S}) = (\mathrm{Dim}\,\mathcal{S})^2 \ . \tag{7.12}$$

Modularity and the dimension of $\mathcal{Z}(\mathcal{S})$ are of course already known from [Mü2].

Remark 7.7. It was shown in [CRS3, Sec. 4] that the TQFT, obtained as the generalised orbifold of the trivial TQFT (i.e. obtained from the trivial MFC $\mathrm{Vect}_{\Bbbk}$) using the orbifold datum $\mathbb{A}^{\mathcal{S}}$ (without embedded ribbon graphs) coincides with the TQFT of Turaev-Viro-Barrett-Westbury (TVBW) type, obtained from the spherical fusion category $\mathcal{S}$ [TViro, BW2, TVire, TV]. The orbifold graph TQFT from Chapter 6 in principle allows one to generalise this result to bordisms with embedded ribbon graphs: By Theorem 6.13, both graph TQFTs are isomorphic to the Reshetikhin-Turaev (RT) TQFT, obtained from the Drinfeld centre $\mathcal{Z}(\mathcal{S})$. In this sense, Theorem 6.13 generalises the known equivalence between the TQFTs of TVBW and RT types for Drinfeld centres, proved by Turaev-Viralizier [TVire, TV] and Balsam-Kirillov [BalK1, BalK2, BalK3] (the latter proof also applies to *extended* TQFTs, which we do not discuss in this work). Namely, Theorem 6.13 proves a similar result for an orbifold datum $\mathbb{A}$ in a MFC $\mathcal{C}$ also when $\mathcal{C}_{\mathbb{A}}$ is not necessarily a Drinfeld centre.

The proof of Theorem 7.6 is somewhat lengthy and technical. It is presented in the rest of the section and organised as follows: In Section 7.2.1 we define an auxiliary category $\mathcal{A}(\mathcal{S})$ which is proved to be equivalent to the centre $\mathcal{Z}(\mathcal{S})$ as a linear category. Then in Sections 7.2.2 and 7.2.3 we show that $\mathcal{C}_{\mathbb{A}^{\mathcal{S}}} \cong \mathcal{A}(\mathcal{S})$ as linear categories, and that the orbifold datum $\mathbb{A}^{\mathcal{S}}$ is simple. Composing the two equivalences gives a linear equivalence $F\colon \mathcal{Z}(\mathcal{S}) \to \mathcal{C}_{\mathbb{A}^{\mathcal{S}}}$. In Section 7.2.4 we equip F with a monoidal structure and show that it preserves braidings and twists.

We emphasise again, that the orbifold datum $\mathbb{A}^{\mathcal{S}}$ is presented in the setting of Δ-separable algebras and Euler completion. This means that the identities (O1)-(O8) for $\mathbb{A}^{\mathcal{S}}$ and the identities (T1)-(T7) will be exchanged for the identities ($\mathrm{O1}_\Delta$)-($\mathrm{O8}_\Delta$) and ($\mathrm{T1}_\Delta$)-($\mathrm{T7}_\Delta$) listed in Appendix C. If necessary, their 3-dimensional presentations can be ribbonised as explained in Section 5.2 and illustrated in Figure 5.4.

7.2.1. Auxiliary category $\mathcal{A}(\mathcal{S})$ and equivalence to $\mathcal{Z}(\mathcal{S})$

Definition 7.8. Define the category $\mathcal{A}(\mathcal{S})$ to have

• *objects*: triples (X, t^X, b^X), where $X \in \mathcal{S}$ and $t^X : X \otimes (- \otimes -) \Rightarrow (X \otimes -) \otimes -$, $b^X : X \otimes (- \otimes -) \Rightarrow - \otimes (X \otimes -)$ are natural transformations between endofunctors of $\mathcal{S} \times \mathcal{S}$, such that the following diagrams commute for all $U, V, W \in \mathcal{S}$:

$$\begin{tikzcd}
 & (XU)(VW) \arrow[dr, "a^{-1}_{XU,V,W}"] & \\
X(U(VW)) \arrow[ur, "t^X_{U,VW}"] \arrow[d, "\mathrm{id}_X \otimes a^{-1}_{U,V,W}"'] & & ((XU)V)W \\
X((UV)W) \arrow[rr, "t^X_{UV,W}"'] & & (X(UV))W \arrow[u, "t^X_{U,V} \otimes \mathrm{id}_W"']
\end{tikzcd} , \tag{7.13}$$

$$\begin{tikzcd}
U(X(VW)) \arrow[r, "\mathrm{id}_U \otimes t^X_{V,W}"] & U((XV)W) \arrow[d, "a^{-1}_{U,XV,W}"] \\
X(U(VW)) \arrow[u, "b^X_{U,VW}"] \arrow[d, "\mathrm{id}_X \otimes a^{-1}_{U,V,W}"'] & (U(XV))W \\
X((UV)W) \arrow[r, "t^X_{UV,W}"'] & (X(UV))W \arrow[u, "b^X_{U,V} \otimes \mathrm{id}_W"']
\end{tikzcd} ,
\qquad
\begin{tikzcd}
U(X(VW)) \arrow[r, "\mathrm{id}_U \otimes b^X_{V,W}"] & U(V(XW)) \arrow[d, "a^{-1}_{U,V,XW}"] \\
X(U(VW)) \arrow[u, "b^X_{U,VW}"] \arrow[dr, "\mathrm{id}_X \otimes a^{-1}_{U,V,W}"'] & (UV)(XW) \\
 & X((UV)W) \arrow[u, "b^X_{UV,W}"']
\end{tikzcd} ; \tag{7.14}$$

• *morphisms:* $\varphi : (X, t^X, b^X) \to (Y, t^Y, b^Y)$ is a natural transformation $\varphi : X \otimes - \Rightarrow Y \otimes -$, such that the following diagrams commute for all $U, V \in \mathcal{S}$

$$\begin{tikzcd}
 & Y(UV) \arrow[dr, "t^Y_{U,V}"] & \\
X(UV) \arrow[ur, "\varphi_{UV}"] \arrow[dr, "t^X_{U,V}"'] & & (YU)V \\
 & (XU)V \arrow[ur, "\varphi_U \otimes \mathrm{id}_V"'] &
\end{tikzcd} ,
\qquad
\begin{tikzcd}
 & Y(UV) \arrow[dr, "b^Y_{U,V}"] & \\
X(UV) \arrow[ur, "\varphi_{UV}"] \arrow[dr, "b^X_{U,V}"'] & & U(YV) \\
 & U(XV) \arrow[ur, "\mathrm{id}_U \otimes \varphi_V"'] &
\end{tikzcd} . \tag{7.15}$$

□

Proposition 7.9. The functor $E : \mathcal{Z}(\mathcal{S}) \to \mathcal{A}(\mathcal{S})$, acting

- *on objects:* $E(X,\gamma) := (X, t^X, b^X)$, where for all $U, V \in \mathcal{S}$

$$t^X_{U,V} := \left[X(UV) \xrightarrow{a^{-1}_{X,U,V}} (XU)V \right] ,$$
$$b^X_{U,V} := \left[X(UV) \xrightarrow{a^{-1}_{X,U,V}} (XU)V \xrightarrow{\gamma_U \otimes \mathrm{id}_V} (UX)V \xrightarrow{a_{U,X,V}} U(XV) \right] ; \quad (7.16)$$

- *on morphisms:* $E\big([(X,\gamma) \xrightarrow{f} (Y,\delta)] \big) := \{ X \otimes U \xrightarrow{f \otimes \mathrm{id}_U} Y \otimes U \}_{U \in \mathcal{S}}$.

is a linear equivalence.

Proof. It is easy to see that $E(X,\gamma)$ is indeed an object in $\mathcal{A}(\mathcal{S})$ and that $E(f)$ is a morphism in $\mathcal{A}(\mathcal{S})$. In the remainder of the proof we show that E is essentially surjective and fully faithful.

As a preparation, given an object $(X,t,b) \in \mathcal{A}(\mathcal{S}) \in \mathcal{A}(\mathcal{S})$ we derive some properties of t and b. For $V, W \in \mathcal{S}$, consider the following diagram, whose ingredients we proceed to explain:

$$\begin{tikzcd}
 & & X(VW) \arrow[d, "\wr"] \arrow[drr, "a^{-1}_{X,V,W}"] & & \\
 & & (X\mathbb{1})(VW) \arrow[dr, "a^{-1}_{X\mathbb{1},V,W}"] & & \\
X(VW) \arrow[r, "\sim"] \arrow[uurr, "\widehat{t_{\mathbb{1},VW}}"] \arrow[ddr, "\mathrm{id}"'] & X(\mathbb{1}(VW)) \arrow[ur, "t_{\mathbb{1},VW}"] \arrow[dr, "\mathrm{id}_X \otimes a^{-1}_{\mathbb{1},V,W}"] & & ((X\mathbb{1})V)W \arrow[r, "\sim"] & (XV)W \\
 & & X((\mathbb{1}V)W) \arrow[r, "t_{\mathbb{1}V,W}"'] \arrow[d, "\wr"] & (X(\mathbb{1}V))W \arrow[ur, "t_{\mathbb{1},V} \otimes \mathrm{id}_W"'] \arrow[d, "\wr"] & \\
 & & X(VW) \arrow[r, "t_{V,W}"'] & (XV)W \arrow[uur, "\widehat{t_{\mathbb{1},V}} \otimes \mathrm{id}_W"'] &
\end{tikzcd} . \quad (7.17)$$

We abbreviate t^X by t, and we use the following notation for all $U \in \mathcal{S}$:

$$\begin{aligned}
\widehat{t_{\mathbb{1},U}} &:= \left[XU \xrightarrow{\mathrm{id}_X \otimes l_U^{-1}} X(\mathbb{1}U) \xrightarrow{t_{\mathbb{1},U}} (X\mathbb{1})U \xrightarrow{r_X \otimes \mathrm{id}_U} XU \right] , \\
\widehat{t_{U,\mathbb{1}}} &:= \left[XU \xrightarrow{\mathrm{id}_X \otimes r_U^{-1}} X(U\mathbb{1}) \xrightarrow{t_{U,\mathbb{1}}} (XU)\mathbb{1} \xrightarrow{r_{XU}} XU \right] , \qquad (7.18) \\
\widehat{b_{\mathbb{1},U}} &:= \left[XU \xrightarrow{\mathrm{id}_X \otimes l_U^{-1}} X(\mathbb{1}U) \xrightarrow{b_{\mathbb{1},U}} \mathbb{1}(XU) \xrightarrow{l_{XU}} XU \right] , \\
\widehat{b_{U,\mathbb{1}}} &:= \left[XU \xrightarrow{\mathrm{id}_X \otimes r_U^{-1}} X(U\mathbb{1}) \xrightarrow{b_{U,\mathbb{1}}} U(X\mathbb{1}) \xrightarrow{\mathrm{id}_U \otimes r_X} UX \right] .
\end{aligned}$$

This notation will be used in the remainder of this section, too.

By taking $U = \mathbb{1}$ in (7.13), the inner pentagon in (7.17) commutes and all squares commute by definition of $\widehat{t_{\mathbb{1},U}}$, by naturality or by monoidal coherence, and hence the outer pentagon commutes as well. Leaving out the identity edge, we get the following commutative diagram for all $V, W \in \mathcal{S}$:

$$\begin{tikzcd}
 & X(VW) \arrow[dr, "a^{-1}_{X,V,W}"] & \\
X(VW) \arrow[ur, "\widehat{t_{\mathbb{1},VW}}"] \arrow[dr, "t_{V,W}"'] & & (XV)W \\
 & (XV)W \arrow[ur, "\widehat{t_{\mathbb{1},V}} \otimes \mathrm{id}_W"'] &
\end{tikzcd} . \tag{7.19}$$

Similarly, by taking $V = \mathbb{1}$ and $W = \mathbb{1}$ in (7.13) one in the end gets the following two commuting diagrams:

$$\forall U, W \in \mathcal{S} : \begin{tikzcd}
X(UW) \arrow[r, "t_{U,W}"] \arrow[d, "t_{U,W}"'] & (XU)W \arrow[dl, "\widehat{t_{U,\mathbb{1}}} \otimes \mathrm{id}_W"] \\
(XU)W &
\end{tikzcd} , \forall U, V \in \mathcal{S} : \begin{tikzcd}
X(UV) \arrow[r, "t_{U,V}"] \arrow[dr, "\widehat{t_{UV,\mathbb{1}}}"'] & (XU)V \\
 & X(UV) \arrow[u, "t_{U,V}"']
\end{tikzcd} . \tag{7.20}$$

These diagrams imply that for all $U \in \mathcal{S}$ one has $\widehat{t_{U,1}} = \mathrm{id}_U$.

Repeating the above procedure of setting individual objects to $\mathbb{1}$ also for two diagrams in (7.14) yields three more conditions. Namely, for all $U, V, W \in \mathcal{S}$ one has $\widehat{b_{\mathbb{1},U}} = \mathrm{id}_U$ and the following diagrams commute:

$$\begin{tikzcd}
 & U(XW) \arrow[r, "\mathrm{id}_U \otimes \widehat{t_{\mathbb{1},W}}"] & U(XW) \arrow[dr, "a^{-1}_{U,X,W}"] & \\
X(UW) \arrow[ur, "b_{U,W}"] \arrow[dr, "t_{U,W}"'] & & & (UX)W \\
 & (XU)W \arrow[urr, "\widehat{b_{U,\mathbb{1}}} \otimes \mathrm{id}_W"'] & &
\end{tikzcd} , \begin{tikzcd}
X(UV) \arrow[r, "b_{U,V}"] \arrow[d, "\widehat{b_{UV,\mathbb{1}}}"] & U(XV) \arrow[d, "\mathrm{id}_U \otimes \widehat{b_{V,\mathbb{1}}}"] \\
(UV)X & U(VX) \arrow[l, "a^{-1}_{U,V,X}"]
\end{tikzcd} . \tag{7.21}$$

Let $\eta_U := [XU \xrightarrow{\widehat{t_{\mathbb{1},U}}} XU]$ and $\gamma_U := [XU \xrightarrow{\eta_U^{-1}} XU \xrightarrow{\widehat{b_{U,\mathbb{1}}}} UX]$ for all $U \in \mathcal{S}$. We claim that γ is a half-braiding for X and that η is an isomorphism $(X, t, b) \to E(X, \gamma)$ in $A(\mathcal{S})$.

We start by showing that η is indeed a morphism in $A(\mathcal{S})$. First note that (7.19) now reads

$$t_{U,V} = \left[X(UV) \xrightarrow{\eta_{UV}} X(UV) \xrightarrow{a^{-1}_{X,U,V}} (XU)V \xrightarrow{\eta_U^{-1} \otimes \mathrm{id}_V} (XU)V \right] . \tag{7.22}$$

Together with the definition of $E(X,\gamma)$ in (7.16) we see that this is precisely the first condition in (7.15). Plugging (7.22) into the first diagram in (7.21) results in

$$b_{U,V} = \left[\begin{array}{l} X(UV) \xrightarrow{\eta_{UV}} X(UV) \xrightarrow{a^{-1}_{X,U,V}} (XU)V \xrightarrow{\gamma_U \otimes \mathrm{id}_V} (UX)V \\ \qquad\qquad \xrightarrow{a_{U,X,V}} U(XV) \xrightarrow{\mathrm{id}_U \otimes \eta_V^{-1}} U(XV) \end{array} \right] . \tag{7.23}$$

This is precisely the second condition in (7.15).

Checking that γ satisfies the hexagon condition is now a direct consequence of plugging (7.23) into the second diagram in (7.21).

Altogether, this shows that E is essentially surjective.

To get that E is fully faithful, let $\varphi : E(X,\gamma) \to E(Y,\delta)$ be a morphism in $A(\mathcal{S})$. Setting $U = \mathbb{1}$ in the first condition in (7.15) yields that for all $V \in \mathcal{S}$ one has $\varphi_V = \widehat{\varphi_{\mathbb{1}}} \otimes \mathrm{id}_V$, where

$$\widehat{\varphi_{\mathbb{1}}} := \left[X \xrightarrow{r_X^{-1}} X\mathbb{1} \xrightarrow{\varphi_{\mathbb{1}}} Y\mathbb{1} \xrightarrow{r_Y} Y \right] . \tag{7.24}$$

Setting $V = \mathbb{1}$ in the second condition in (7.15) shows that $\widehat{\varphi_{\mathbb{1}}}$ commutes with the half-braidings γ and δ. Altogether, φ is in the image of E. □

7.2.2. The functor D from $\mathcal{A}(\mathcal{S})$ to $\mathcal{C}_{\mathbb{A}^{\mathcal{S}}}$

In this subsection we define a functor $D : \mathcal{A}(\mathcal{S}) \to \mathcal{C}_{\mathbb{A}^{\mathcal{S}}}$. We start by defining D on objects. Let $(X,t,b) \in \mathcal{A}(\mathcal{S})$ and denote the components of $D(X,t,b)$ by

$$D(X,t,b) =: (M, \tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2}) . \tag{7.25}$$

We will go through the definition of the constituents step by step, starting with the A-A-bimodule M.

For $n \geq 1$, an $A^{\otimes n}$-module is an $\mathrm{Irr}_{\mathcal{S}}^{\times n}$-graded vector space and a morphism between modules is a grade-preserving linear map. In particular, an A-A-bimodule M is a vector space with a decomposition $M = \bigoplus_{i,j \in \mathrm{Irr}_{\mathcal{S}}} M_{ij}$, where for M_{ij} only the $\mathcal{S}(i,i)$-$\mathcal{S}(j,j)$ action is non-trivial. For the bimodule M in (7.25) we set

$$M = \bigoplus_{i,j \in \mathrm{Irr}_{\mathcal{S}}} M_{ij} \quad \text{with} \quad M_{ij} = \mathcal{S}(i, Xj) , \tag{7.26}$$

with action of $\mathcal{S}(i,i)$ (from the left) and $\mathcal{S}(j,j)$ (from the right) given by pre- and postcomposition, respectively.

Next we turn to defining τ_i and $\overline{\tau_i}$. We will need two ingredients. The first are certain A-AA-bimodule isomorphisms σ_x, $x = 0,1,2$, which are defined as

$$M \otimes_0 T = \bigoplus_{l,i,j,a \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, Xa) \otimes_{\Bbbk} \mathcal{S}(a, ij) \xrightarrow{\sigma_0} \bigoplus_{l,i,j \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, X(ij))$$

$$[l \xrightarrow{f} Xa] \otimes_{\Bbbk} [a \xrightarrow{g} ij] \;\longmapsto\; [l \xrightarrow{(\mathrm{id}_X \otimes g)\circ f} X(ij)] \;,$$

$$T \otimes_1 M = \bigoplus_{l,i,j,a \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, aj) \otimes_{\Bbbk} \mathcal{S}(a, Xi) \;\xrightarrow{\sigma_1}\; \bigoplus_{l,i,j \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, (Xi)j)$$

$$[l \xrightarrow{f} aj] \otimes_{\Bbbk} [a \xrightarrow{g} Xi] \;\longmapsto\; [l \xrightarrow{(g \otimes \mathrm{id}_j)\circ f} (Xi)j)] \;,$$

$$T \otimes_2 M = \bigoplus_{l,i,j,a \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, ia) \otimes_{\Bbbk} \mathcal{S}(a, Xj) \;\xrightarrow{\sigma_2}\; \bigoplus_{l,i,j \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, i(Xj))$$

$$[l \xrightarrow{f} ia] \otimes_{\Bbbk} [a \xrightarrow{g} Xj] \;\longmapsto\; [l \xrightarrow{(\mathrm{id}_i \otimes g)\circ f} i(Xj))] \;. \tag{7.27}$$

To describe the second ingredient, it will be useful to relate linear maps between morphism spaces in $\mathcal{S}$ to actual morphisms in $\mathcal{S}$ as described in the following remark.

Remark 7.10. Let $\mathcal{T}^n_B : \mathcal{S} \times \cdots \times \mathcal{S} \to \mathcal{S}$ be the functor which takes the n-fold tensor product with a given bracketing B. Consider the $\mathrm{Irr}_{\mathcal{S}}^{\times(n+1)}$-graded vector space $V_B := \bigoplus_{l,i_1,\ldots,i_n} \mathcal{S}(l, \mathcal{T}^n_B(i_1,\ldots,i_n))$. For two bracketings B, B', one has a linear isomorphism

$$\big\{\text{natural transformations } \mathcal{T}^n_B \Rightarrow \mathcal{T}^n_{B'}\big\} \;\xrightarrow{\sim}\; \big\{\text{graded linear maps } V_B \to V_{B'}\big\} \;,$$

given by postcomposition. That is, it takes a natural transformation φ to the graded linear map

$$\Big[l \xrightarrow{f} \mathcal{T}^n_B(i_1,\ldots,i_n)\Big] \;\longmapsto\; \Big[l \xrightarrow{f} \mathcal{T}^n_B(i_1,\ldots,i_n) \xrightarrow{\varphi_{i_1,\ldots,i_n}} \mathcal{T}^n_{B'}(i_1,\ldots,i_n)\Big] \;. \tag{7.28}$$

This is easily generalised for functors obtained from $\mathcal{T}^n_B$ by fixing some of the arguments.

Recall the natural transformations t, b that form part of the object (X, t, b) on which we are defining the functor D. The second ingredient needed to define τ_i, $\overline{\tau_i}$ are four families of morphisms $(\tau_i')_{UV}$, $(\overline{\tau_i}')_{UV}$ in $\mathcal{S}$ which are natural in $U, V \in \mathcal{S}$:

$$\begin{aligned}
\big[X(UV) \xrightarrow{(\tau_1')_{UV}} (XU)V\big] \;&:=\; \big[X(UV) \xrightarrow{t_{UV}} (XU)V \xrightarrow{\psi_{XU}^{-2} \otimes \mathrm{id}_V} (XU)V\big] \;,\\
\big[X(UV) \xrightarrow{(\tau_2')_{UV}} U(XV)\big] \;&:=\; \big[X(UV) \xrightarrow{b_{UV}} U(XV) \xrightarrow{\mathrm{id}_U \otimes \psi_{XV}^{-2}} U(XV)\big] \;,\\
\big[(XU)V \xrightarrow{(\overline{\tau_1}')_{UV}} X(UV)\big] \;&:=\; \big[(XU)V \xrightarrow{t_{UV}^{-1}} X(UV) \xrightarrow{\mathrm{id}_X \otimes \psi_{UV}^{-2}} X(UV)\big] \;,\\
\big[U(XV) \xrightarrow{(\overline{\tau_2}')_{UV}} X(UV)\big] \;&:=\; \big[U(XV) \xrightarrow{b_{UV}^{-1}} X(UV) \xrightarrow{\mathrm{id}_X \otimes \psi_{UV}^{-2}} X(UV)\big] \;.
\end{aligned} \tag{7.29}$$

Combining these two ingredients, we define τ_i, $\overline{\tau_i}$ in (7.25) to be:

$$\begin{aligned}
\tau_1 &:= \left[M \otimes_0 T \xrightarrow{\sigma_0} \mathcal{S}(l, X(ij)) \xrightarrow{(\tau_1')_{ij} \circ (-)} \mathcal{S}(l, (Xi)j) \xrightarrow{\sigma_1^{-1}} T \otimes_1 M \right] , \\
\tau_2 &:= \left[M \otimes_0 T \xrightarrow{\sigma_0} \mathcal{S}(l, X(ij)) \xrightarrow{(\tau_2')_{ij} \circ (-)} \mathcal{S}(l, i(Xj)) \xrightarrow{\sigma_2^{-1}} T \otimes_2 M \right] , \\
\overline{\tau_1} &:= \left[T \otimes_1 M \xrightarrow{\sigma_1} \mathcal{S}(l, (Xi)j) \xrightarrow{(\overline{\tau_1}')_{ij} \circ (-)} \mathcal{S}(l, X(ij)) \xrightarrow{\sigma_0^{-1}} M \otimes_0 T \right] , \\
\overline{\tau_2} &:= \left[T \otimes_2 M \xrightarrow{\sigma_2} \mathcal{S}(l, i(Xj)) \xrightarrow{(\overline{\tau_2}')_{ij} \circ (-)} \mathcal{S}(l, X(ij)) \xrightarrow{\sigma_0^{-1}} M \otimes_0 T \right] . \qquad (7.30)
\end{aligned}$$

The verification that these morphisms satisfy (T1$_\Delta$)-(T7$_\Delta$) will be part of the proof of Proposition 7.11 below.

The action of D on a morphism $\varphi : (X, t^X, b^X) \to (Y, t^Y, b^Y)$ in $\mathcal{A}(\mathcal{S})$ is

$$D(\varphi) := \left[D(X, t^X, b^X) = \bigoplus_{i,j \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(i, Xj) \xrightarrow{\varphi_j \circ (-)} \bigoplus_{i,j \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(i, Yj) = D(Y, t^Y, b^Y) \right] . \qquad (7.31)$$

Proposition 7.11. The functor $D : \mathcal{A}(\mathcal{S}) \to \mathcal{C}_{\mathbb{A}^S}$ is well-defined and a linear equivalence.

Proof. The proof that $D(X, t, b)$ is indeed an object in $\mathcal{C}_{\mathbb{A}^S}$ is a little tedious and will be given in Subsection 7.2.3 below. For now we assume that this has been done and continue with the remaining points.

To see that $D(\varphi) : D(X, t^X, b^X) \to D(Y, t^Y, b^Y)$ is a morphism in $\mathcal{C}_{\mathbb{A}^S}$ we have to verify the identities in (M). We will demonstrate this for τ_1 as an example. Denote the underlying A-A-bimodules of $D(X, t^X, b^X)$ and $D(Y, t^Y, b^Y)$ as M and N, respectively, and consider the following diagram:

$$\begin{tikzcd}
M \otimes_0 T \arrow[rrr, "\tau_1^M"'] \arrow[dr, "\sigma_0"] \arrow[dd, "D(\varphi)\otimes_0 \mathrm{id}"'] & & & T \otimes_1 M \arrow[dl, "\sigma_1"'] \arrow[dd, "\mathrm{id} \otimes_1 D(\varphi)"] \\
& \oplus\mathcal{S}(l, X(ij)) \arrow[r, "\oplus(t^X_{ij})_*"] \arrow[d, "\oplus(\varphi_{ij})_*"'] & \oplus\mathcal{S}(l, (Xi)j) \arrow[r, "\oplus(\psi^{-2}_{Xi}\otimes\mathrm{id})_*"] \arrow[d, "\oplus(\varphi_i\otimes\mathrm{id})_*"'] & \oplus\mathcal{S}(l, (Xi)j) \arrow[d, "\oplus(\varphi_i\otimes\mathrm{id})_*"'] \\
& \oplus\mathcal{S}(l, Y(ij)) \arrow[r, "\oplus(t^Y_{ij})_*"'] & \oplus\mathcal{S}(l, (Yi)j) \arrow[r, "\oplus(\psi^{-2}_{Yi}\otimes\mathrm{id})_*"'] & \oplus\mathcal{S}(l, (Yi)j) \\
N \otimes_0 T \arrow[rrr, "\tau_1^N"] \arrow[ur, "\sigma_0"] & & & T \otimes_1 N \arrow[ul, "\sigma_1"']
\end{tikzcd} \qquad (7.32)$$

Here, all direct sums run over $i, j, l \in \mathrm{Irr}_{\mathcal{S}}$. The notation $(-)_*$ stands for postcomposition with the corresponding morphism. The left innermost square commutes by (7.15), and the right innermost square commutes by naturality of ψ. The top

and bottom squares are just the definition of τ_1 in (7.29) and (7.30). That the rightmost square commutes is immediate from the definition of σ_1 in (7.27), while for the leftmost square one needs to invoke in addition the naturality of φ.

So far we have shown that the functor D is well-defined. We now check that it is essentially surjective and fully faithful.

Let $(M, \tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2})$ be an arbitrary object in $\mathcal{C}_{\mathbb{A}^{\mathcal{S}}}$. As above, we decompose $M = \bigoplus_{i,j\in \mathrm{Irr}_{\mathcal{S}}} M_{ij}$, where for M_{ij} only the $\mathcal{S}(i,i)$-$\mathcal{S}(j,j)$ action is non-trivial. Since τ_1 is an A-AA-bimodule isomorphism $M \otimes_0 T \xrightarrow{\sim} T \otimes_1 M$, we have a graded linear isomorphism

$$\tau_1 : \bigoplus_{i,j,l,a\in \mathrm{Irr}_{\mathcal{S}}} M_{la} \otimes_{\Bbbk} \mathcal{S}(a, ij) \xrightarrow{\sim} \bigoplus_{i,j,l,b\in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, bj) \otimes_{\Bbbk} M_{bi} \ , \tag{7.33}$$

Specialising to $i = \mathbb{1}$ gives linear isomorphisms, for all $l, j \in \mathrm{Irr}_{\mathcal{S}}$,

$$\underbrace{\bigoplus_{a\in \mathrm{Irr}_{\mathcal{S}}} M_{la} \otimes_{\Bbbk} \mathcal{S}(a, \mathbb{1}j)}_{\cong M_{lj}} \xrightarrow{\sim} \bigoplus_{b\in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, bj) \otimes_{\Bbbk} M_{b\mathbb{1}} \ , \tag{7.34}$$

Setting $X = \bigoplus_{b\in \mathrm{Irr}_{\mathcal{S}}} b \otimes M_{b\mathbb{1}} \in \mathcal{S}$, we see that this implies $M \cong \bigoplus_{l,j} \mathcal{S}(l, Xj)$ as A-A-bimodules. We may thus assume without loss of generality that in fact $M = \bigoplus_{l,j} \mathcal{S}(l, Xj)$ for some $X \in \mathcal{S}$.

Define t, b by inverting the first two defining relations in each of (7.29) and (7.30) (this is possible by Remark 7.10). We need to verify that t, b satisfy the conditions in (7.13) and (7.14).

Consider condition ($\mathrm{T1}_\Delta$) satisfied by τ_1. Along the same lines as was done in (7.32), one can translate ($\mathrm{T1}_\Delta$) into an equality of two graded linear maps $\bigoplus \mathcal{S}(l, X(i(jk))) \to \bigoplus \mathcal{S}(l, ((Xi)j)k)$. Both of these maps are given by postcomposition, resulting in a commuting diagram of morphisms in $\mathcal{S}$, for all i, j, k shown in Figure 7.1. Since t, a and ψ are natural transformations, one can cancel all arrows with ψ, which then yields precisely the diagram (7.13). Similarly, ($\mathrm{T2}_\Delta$), ($\mathrm{T3}_\Delta$) give the two diagrams in (7.14).

It remains to show that D is fully faithful. Faithfulness is clear from (7.31). For fullness, let $f : D(X, t^X, b^X) \to D(Y, t^Y, b^Y)$ be a morphism in $\mathcal{C}_{\mathbb{A}^{\mathcal{S}}}$. By Remark 7.10, f is given by postcomposition with a natural transformation $\varphi : X \otimes - \Rightarrow Y \otimes -$. The identities (M) impose that the two diagrams in (7.15) commute. Thus φ is a morphism in $\mathcal{A}(\mathcal{S})$ and $f = D(\varphi)$. □

Corollary 7.12. The orbifold datum $\mathbb{A}^{\mathcal{S}}$ is simple.

Proof. By Proposition 7.11, the functor $D : \mathcal{A}(\mathcal{S}) \to \mathcal{C}_{\mathbb{A}^{\mathcal{S}}}$ is a linear equivalence. Since $\mathcal{C}_{\mathbb{A}^{\mathcal{S}}}$ is semisimple (Proposition 5.13), so is $\mathcal{A}(\mathcal{S})$. Any object of the form

$$
\begin{tikzcd}
 & & (Xi)(jk) \arrow[dr, "a^{-1}_{Xi,j,k}"] & & \\
 & (Xi)(jk) \arrow[ur, "\psi^{-2}_{Xi}\otimes \mathrm{id}_{jk}"] & & ((Xi)j)k \arrow[dr, "\psi^{-2}_{(Xi)j}\otimes \mathrm{id}_k"] & \\
X(i(jk)) \arrow[ur, "t_{i,jk}"] \arrow[d, "\mathrm{id}_X\otimes a^{-1}_{ijk}"'] & & & & ((Xi)j)k \\
X((ij)k) \arrow[d, "\mathrm{id}_X\otimes(\psi^{-2}_{ij}\otimes \mathrm{id}_k)"'] & & & & ((Xi)j)k \arrow[u, "(\psi^{-2}_{Xi}\otimes \mathrm{id}_j)\otimes \mathrm{id}_k"'] \\
X((ij)k) \arrow[r, "\mathrm{id}_X\otimes(\psi^{2}_{ij}\otimes \mathrm{id}_k)"] & X((ij)k) \arrow[r, "t_{(ij),k}"] & (X(ij))k \arrow[r, "\psi^{-2}_{X(ij)}\otimes \mathrm{id}_k"] & (X(ij))k \arrow[u, "t_{ij}\otimes \mathrm{id}_k"'] &
\end{tikzcd}
$$

Figure 7.1: Condition (T1$_\Delta$) on an A-A bimodule $M = \bigoplus_{l,j\in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, Xj)$, $X \in \mathcal{S}$.

$(\mathbb{1}_{\mathcal{S}}, t, b)$ is simple in $\mathcal{A}(\mathcal{S})$, as $\mathbb{1}_{\mathcal{S}}$ is simple in $\mathcal{S}$. For an appropriate choice of t, b we have $D(\mathbb{1}_{\mathcal{S}}, t, b) \cong \mathbb{1}_{\mathcal{C}_{\mathbb{A}\mathcal{S}}}$, the tensor unit of $\mathcal{C}_{\mathbb{A}\mathcal{S}}$. Using once more that D is an equivalence, we conclude that $\mathbb{1}_{\mathcal{C}_{\mathbb{A}\mathcal{S}}}$ is simple in $\mathcal{C}_{\mathbb{A}\mathcal{S}}$. □

7.2.3. Conditions on T-crossings

Here we complete the proof of Proposition 7.11 by showing that $D(X, t, b)$ from (7.25) satisfies conditions (T1$_\Delta$)–(T7$_\Delta$).

For condition (T1$_\Delta$), the computation is the same as in the proof of essential surjectivity of D, just in the opposite direction, i.e. one starts by writing (7.13) as the diagram in Figure 7.1. Analogously, (7.14) produces (T2$_\Delta$), (T3$_\Delta$).

Conditions (T4$_\Delta$) and (T5$_\Delta$) are straightforward to check from the definitions (7.29) and (7.30).

Since (T6$_\Delta$) and (T7$_\Delta$) involve duals, it is helpful to express the (vector space) dual bimodule M^* of $M = \bigoplus_{l,a} \mathcal{S}(l, Xa)$ in terms of the bimodule $M^\vee := \bigoplus_{l,a} \mathcal{S}(l, X^*a)$. Given a basis $\{\mu\}$ of $\mathcal{S}(l, Xa)$, we get the basis $\{\mu^*\}$ of the dual vector space $\mathcal{S}(l, Xa)^*$ and the basis $\{\bar{\mu}\}$ of $\mathcal{S}(Xa, l)$, which is dual to $\{\mu\}$ with respect to the composition pairing. Let us fix an isomorphism $M^* \to M^\vee$ as follows:

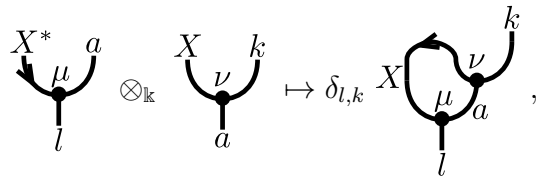
$$
\zeta : M^* \longrightarrow M^\vee \ , \quad \mu^* \longmapsto \frac{\dim_{\mathcal{S}} a}{\dim_{\mathcal{S}} l}\big[a \xrightarrow{\sim} \mathbb{1}a \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}} (XX^*)a \xrightarrow{\sim} X(X^*a) \xrightarrow{\bar{\mu}} Xl\big] \tag{7.35}
$$

Using ζ, one can translate the evaluation and coevaluation maps from vector space duals to the new duals $M^\vee$. For example,

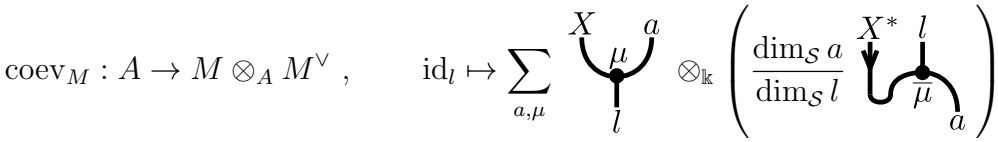
$$\begin{aligned} \mathrm{coev}_M &:= \left[A \longrightarrow M \otimes_A M^* \xrightarrow{\mathrm{id}\otimes\zeta} M \otimes_A M^\vee\right] \ , \\ \widetilde{\mathrm{coev}}_M &:= \left[A \longrightarrow M^* \otimes_A M \xrightarrow{\zeta\otimes\mathrm{id}} M^\vee \otimes_A M\right] \ , \end{aligned} \tag{7.36}$$

where the unlabelled arrow is the canonical coevaluation in vector spaces. Explicitly, this gives the A-A-bimodule maps

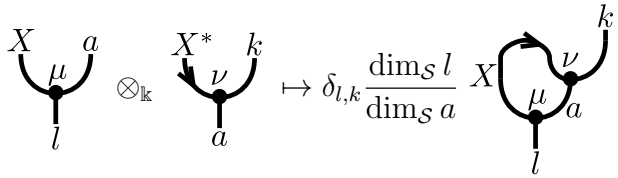
$\mathrm{ev}_M : M^\vee \otimes_A M \to A$,



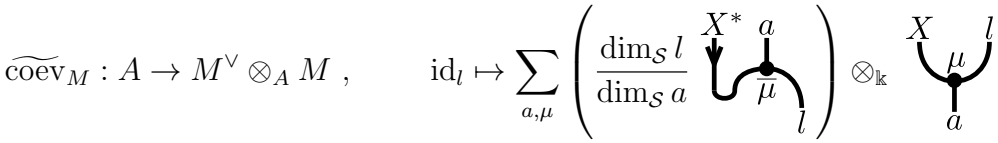
$\mathrm{coev}_M : A \to M \otimes_A M^\vee$,



$\widetilde{\mathrm{ev}}_M : M \otimes_A M^\vee \to A$,



$\widetilde{\mathrm{coev}}_M : A \to M^\vee \otimes_A M$,



(7.37)

The choice (7.35) makes the expression for ev_M simpler but the other three duality maps still contain the dimension factors. Using isomorphisms given by composition similar to those in (7.27), one can also write these maps as (by abuse of notation we keep the same names for the maps)

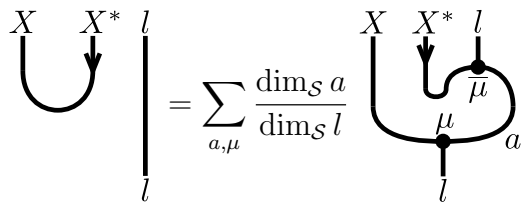
$$\mathrm{ev}_M : \mathcal{S}(l, X^*(Xl)) \to \mathcal{S}(l,l), \qquad \lambda \mapsto \left[\begin{array}{l} l \xrightarrow{\lambda} X^*(Xl) \xrightarrow{a^{-1}_{X^*,X,l}} (X^*X)l \\ \qquad\qquad\qquad\qquad \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_l} \mathbb{1}l \xrightarrow{l_l} l \end{array}\right],$$

$$\mathrm{coev}_M : \mathcal{S}(l,l) \to \mathcal{S}(l, X(X^*l)), \quad \mu \mapsto \left[\begin{array}{l} l \xrightarrow{\mu} l \xrightarrow{l_l^{-1}} \mathbb{1}l \xrightarrow{\mathrm{coev}_X} (XX^*)l \\ \qquad\qquad\qquad\qquad \xrightarrow{a_{X,X^*,l}} X(X^*l) \end{array}\right],$$

$$\widetilde{\mathrm{ev}}_M : \mathcal{S}(l, X(X^*l)) \to \mathcal{S}(l,l), \qquad \nu \mapsto \left[\begin{array}{l} l \xrightarrow{\nu} X(X^*l) \xrightarrow{\mathrm{id}_X \otimes \psi^{-2}_{X^*l}} X(X^*l) \\ \qquad \xrightarrow{a^{-1}_{X,X^*,l}} (XX^*)l \xrightarrow{\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_l} \mathbb{1}l \\ \qquad\qquad \xrightarrow{l_l} l \xrightarrow{\psi^2_l} l \end{array}\right],$$

$$\widetilde{\mathrm{coev}}_M : \mathcal{S}(l,l) \to \mathcal{S}(l, X^*(Xl)), \quad \xi \mapsto \left[\begin{array}{l} l \xrightarrow{\xi} l \xrightarrow{l_l^{-1}} \mathbb{1}l \xrightarrow{\widetilde{\mathrm{coev}}_X} (X^*X)l \\ \xrightarrow{a_{X^*,X,l}} X^*(Xl) \xrightarrow{\mathrm{id}_{X^*}\, \psi^{-2}_{Xl}} X^*(Xl) \\ \qquad\qquad \xrightarrow{\mathrm{id}_{X^*} \otimes (\mathrm{id}_X \otimes \psi^2_l)} X^*(Xl) \end{array}\right]. \tag{7.38}$$

For example to get the expression for coev_M one uses the identity:



(7.39)

Note that these are dualities in ${}_A\mathcal{C}_A$. To obtain the dualities in $\mathcal{C}_{\mathbb{A}^{\mathcal{S}}}$ some extra ψ-insertions are needed, see (C.2), (C.3).

Given this reformulation of the duality morphisms, the verification of ($\mathrm{T6}_\Delta$), ($\mathrm{T7}_\Delta$) works along the same lines as ($\mathrm{T1}_\Delta$)-($\mathrm{T3}_\Delta$).

7.2.4. Ribbon structure of the composed functor

Denoting the composed functor by $F := D \circ E$, we obtain the following corollary to Propositions 7.9 and 7.11.

Corollary 7.13. The functor $F : \mathcal{Z}(\mathcal{S}) \to \mathcal{C}_{\mathbb{A}^{\mathcal{S}}}$, acting

- *on objects:* $F(X,\gamma) := (\bigoplus_{k,l \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, Xk), \tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2})$, where for all $i,j,l \in \mathrm{Irr}_{\mathcal{S}}$ the T-crossings and their pseudo-inverses are (we omit writing out the isomorphisms σ_i from (7.27) explicitly)

$$\tau_1 : \mathcal{S}(l, X(ij)) \to \mathcal{S}(l, (Xi)j), \quad \lambda \mapsto \left[l \xrightarrow{\lambda} X(ij) \xrightarrow{a^{-1}_{X,i,j}} (Xi)j \xrightarrow{\psi^{-2}_{Xi} \otimes \mathrm{id}_j} (Xi)j \right]$$

$$\tau_2 : \mathcal{S}(l, X(ij)) \to \mathcal{S}(l, i(Xj)), \quad \mu \mapsto \left[\begin{array}{l} l \xrightarrow{\mu} X(ij) \xrightarrow{a^{-1}_{X,i,j}} (Xi)j \xrightarrow{\gamma_i \otimes \mathrm{id}_j} (iX)j \\ \qquad\qquad \xrightarrow{a_{i,X,j}} i(Xj) \xrightarrow{\mathrm{id}_i \otimes \psi^{-2}_{Xj}} i(Xj) \end{array}\right]$$

$$\overline{\tau_1} : \mathcal{S}(l, (Xi)j) \to \mathcal{S}(l, X(ij)), \quad \nu \mapsto \left[l \xrightarrow{\nu} (Xi)j \xrightarrow{a_{X,i,j}} X(ij) \xrightarrow{\mathrm{id}_X \otimes \psi^{-2}_{ij}} X(ij) \right]$$

$$\overline{\tau_2} : \mathcal{S}(l, X(ij)) \to \mathcal{S}(l, i(Xj)), \quad \xi \mapsto \left[\begin{array}{l} l \xrightarrow{\xi} i(Xj) \xrightarrow{a^{-1}_{i,X,j}} (iX)j \xrightarrow{\gamma_i^{-1} \otimes \mathrm{id}_j} (Xi)j \\ \qquad\qquad \xrightarrow{a_{X,i,j}} X(ij) \xrightarrow{\mathrm{id}_X \otimes \psi_{ij}^{-2}} X(ij) \end{array} \right];$$

- *on morphisms:* $F([(X,\gamma) \xrightarrow{f} (Y,\delta)]) := \left[\begin{array}{lcl} \mathcal{S}(l, Xk) & \to & \mathcal{S}(l, Yk) \quad \forall k, l \in \mathrm{Irr}_{\mathcal{S}} \\ [l \xrightarrow{g} Xk] & \mapsto & [l \xrightarrow{g} Xk \xrightarrow{f \otimes \mathrm{id}_k} Yk] \end{array} \right]$

is a linear equivalence.

Recall from Section 2.1 that a monoidal structure on the functor F consists of an isomorphism

$$F_0 : \mathbb{1}_{\mathcal{C}_{\mathbb{A}\mathcal{S}}} \xrightarrow{\sim} F(\mathbb{1}_{\mathcal{Z}(\mathcal{S})}), \tag{7.40}$$

in $\mathcal{C}_{\mathbb{A}\mathcal{S}}$ as well as a collection of isomorphisms

$$F_2((X,\gamma),(Y,\delta)) : F(X,\gamma) \otimes_{\mathcal{C}_{\mathbb{A}\mathcal{S}}} F(Y,\delta) \to F(X \otimes Y, \Gamma), \tag{7.41}$$

in $\mathcal{C}_{\mathbb{A}\mathcal{S}}$ (where the halfbraiding Γ of $X \otimes Y$ is as in (2.6)), natural in $(X,\gamma),(Y,\delta) \in \mathcal{Z}(\mathcal{S})$, satisfying the usual compatibility conditions (see e.g. [TV, Sec. 1.4]). We set:

$$F_0 : \bigoplus_{i \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(i,i) \to \bigoplus_{i \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(i, \mathbb{1}i), \quad \left[i \xrightarrow{f} i \right] \mapsto \left[i \xrightarrow{f} i \xrightarrow{\psi_i^{-1}} i \xrightarrow{l_i^{-1}} \mathbb{1}i \right]. \tag{7.42}$$

As in Section 7.2.2 we get the isomorphisms

$$F(X,\gamma) \otimes_{\mathcal{C}_{\mathbb{A}\mathcal{S}}} F(Y,\delta) \cong \bigoplus_{l,r \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, X(Yr)), \quad F(X \otimes Y, \Gamma) \cong \bigoplus_{l,r \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, (XY)r). \tag{7.43}$$

For all $l, r \in \mathrm{Irr}_{\mathcal{S}}$, set

$$F_2((X,\gamma),(Y,\delta)) : \left[l \xrightarrow{f} X(Yr) \right] \mapsto \left[l \xrightarrow{f} X(Yr) \xrightarrow{\mathrm{id}_X \otimes \psi_{Yr}} X(Yr) \xrightarrow{a^{-1}_{X,Y,r}} (XY)r \right]. \tag{7.44}$$

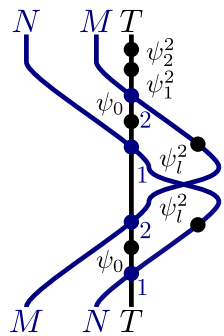
One can check that they are indeed morphisms in $\mathcal{C}_{\mathbb{A}\mathcal{S}}$ and satisfy the compatibilities. $F = (F, F_0, F_2)$ is therefore a monoidal equivalence.

Recall, that F is a braided functor if it preserves the braiding morphisms, i.e.

$$F_2((Y,\delta),(X,\gamma)) \circ c_{F(X,\gamma),F(Y,\delta)} = F(c_{(X,\gamma),(Y,\delta)}) \circ F_2((X,\gamma),(Y,\delta)). \tag{7.45}$$

For $M = \bigoplus_{l,r \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, Xr)$, $N = \bigoplus_{l,r \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(l, Yr)$ with T-crossings τ_i^M, τ_i^N, $i = 1, 2$, let us calculate the braiding morphism $c_{M,N} \in \mathcal{C}_{\mathbb{A}\mathcal{S}}$ explicitly.

Let us consider the morphism

$\Omega := \phi^2 \cdot$ 

, (7.46)

where the string diagram is to be read in $\mathrm{Vect}_{\Bbbk}$, with M, N, T, ψ, ϕ as above. The braiding in $\mathcal{C}_{\mathbb{A}\mathcal{S}}$ is obtained by taking the partial trace of the morphism Ω with respect to T (so that (7.46) yields the expression for $c_{M,N}$ in (C.4)). This amounts to a family of linear maps $\mathcal{S}(l, X(Y(ij))) \to \mathcal{S}(l, Y(X(ij)))$, $i, j, l \in \mathrm{Irr}_{\mathcal{S}}$, which are postcompositions with

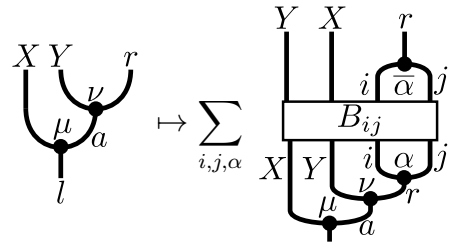
$$
\begin{aligned}
B_{ij} \;:=\; & \frac{1}{\mathrm{Dim}\,\mathcal{S}} \cdot \\
& \Big[X(Y(ij)) \xrightarrow{\mathrm{id}_X \otimes \psi_{Y(ij)}} X(Y(ij)) \xrightarrow{\mathrm{id}_X \otimes ({\tau_1'}^{Y})_{ij}} X((Yi)j) \xrightarrow{({\tau_2'}^{X})_{Yi,j}} (Yi)(Xj) \\
& \xrightarrow{\psi^2_{Yi} \otimes \psi^2_{Xj}} (Yi)(Xj) \xrightarrow{(\mathrm{id}_Y \otimes \psi_i^2) \otimes (\mathrm{id}_X \otimes \psi_j^2)} (Yi)(Xj) \xrightarrow{({\overline{\tau}_1'}^{Y})_{i,Xj}} Y(i(Xj)) \\
& \xrightarrow{\mathrm{id}_Y \otimes ({\overline{\tau}_2'}^{X})_{ij}} Y(X(ij)) \xrightarrow{\mathrm{id}_Y \otimes \psi_{X(ij)}} Y(X(ij)) \Big] \,.
\end{aligned}
$$

We now need to trace the above morphism over T, for which we need the dual T^*. Similar to Section 7.2.3 it is useful to work with $T^\vee := \bigoplus_{i,j,r \in \mathrm{Irr}_{\mathcal{S}}} \mathcal{S}(ij, r)$ instead. Given a basis $\{\alpha\}$ of $\mathcal{S}(r, ij)$, the basis $\{\alpha^*\}$ of the dual vector space $\mathcal{S}(r, ij)^*$ and the composition-dual basis $\{\bar{\alpha}\}$ of $\mathcal{S}(ij, r)$, we fix the isomorphism $T^* \to T^\vee$, $\alpha^* \mapsto \bar{\alpha}$. Using this isomorphism, the relevant evaluation and coevaluation maps are

$$
\begin{aligned}
[\widetilde{\mathrm{ev}}_T : T \otimes_{1,2} T^\vee \to A] \quad &= \left[\begin{array}{ccccccc} \bigoplus_{i,j \in \mathrm{Irr}_{\mathcal{S}}} & \mathcal{S}(l, ij) & \otimes_{\Bbbk} & \mathcal{S}(ij, r) & \to & \mathcal{S}(l, l) \\ & f & \otimes_{\Bbbk} & g & \mapsto & \delta_{k,r}\, g \circ f \end{array} \right], \\
[\mathrm{coev}_T : A \to T \otimes_{1,2} T^\vee] \quad &= \left[\begin{array}{ccccccc} \mathcal{S}(l, l) & \to & \bigoplus_{i,j \in \mathrm{Irr}_{\mathcal{S}}} & \mathcal{S}(l, ij) & \otimes_{\Bbbk} & \mathcal{S}(ij, l) \\ \mathrm{id}_l & \mapsto & \sum_{i,j,\alpha} & \alpha & \otimes_{\Bbbk} & \bar{\alpha} \end{array} \right].
\end{aligned}
$$

All in all, we get the braiding to be the map

$$\mathcal{S}(l, X(Yr)) \to \mathcal{S}(l, Y(Xr)),$$



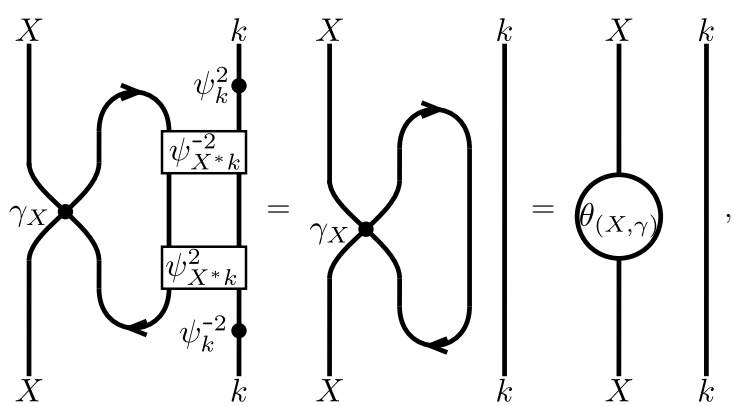
$$\mapsto \sum_{i,j,\alpha} \quad . \tag{7.47}$$

For $M = F(X, \gamma)$, $N = F(Y, \delta)$, the T-crossings are as given in Corollary 7.13. Using these expressions, the braiding (7.47) and the monoidal structure (7.44), one concludes that the left hand side of (7.45) is a family of linear maps $\mathcal{S}(l, X(Yr)) \to \mathcal{S}(l, (YX)r)$, $i, j, l \in \mathrm{Irr}_{\mathcal{S}}$, obtained from postcomposition with morphisms $X(Yr) \to (YX)r$, which in graphical calculus are as shown in Figure 7.2. In the last equality there we used

$$\sum_{i,j} \dim_{\mathcal{S}} i \cdot \dim_{\mathcal{S}} j \cdot N_{ij}^{r} = \sum_{i,j} \dim_{\mathcal{S}} i \cdot \dim_{\mathcal{S}} j^* \cdot N_{ir^*}^{j^*} = \sum_{i} \dim_{\mathcal{S}} i \cdot \dim_{\mathcal{S}} i \cdot \dim_{\mathcal{S}} r$$
$$= \mathrm{Dim}\, \mathcal{S} \cdot \dim_{\mathcal{S}} r\,. \tag{7.48}$$

Substituting the braiding $c_{(X,\gamma),(Y,\delta)} := \gamma_Y$ of two objects $(X, \gamma), (Y, \delta)\mathcal{Z}(\mathcal{S})$, one immediately finds the right hand side of (7.45) to be given by postcomposition with the morphism in the last diagram of Figure 7.2. The condition (7.45) then holds and hence F is a braided equivalence.

Finally, recall that the twist of $M \in \mathcal{C}_{\mathbb{A}\mathcal{S}}$ is given by adapting the morphism θ_M in (C.4). Using the calculation in Figure 7.2 and the expressions (7.38) for (co-)evaluation maps one computes that the twist $\theta_{F(X,\gamma)}$ is a family of maps $\mathcal{S}(l, Xk) \to \mathcal{S}(l, Xk)$, obtained from postcomposition with



$$, \tag{7.49}$$

where in the last equality one adapts (2.12) to the category $\mathcal{Z}(\mathcal{S})$. This is the same morphism as $F(\theta_{(X,g)})$ and therefore F is a ribbon equivalence and with that the proof of Theorem 7.6 is complete.

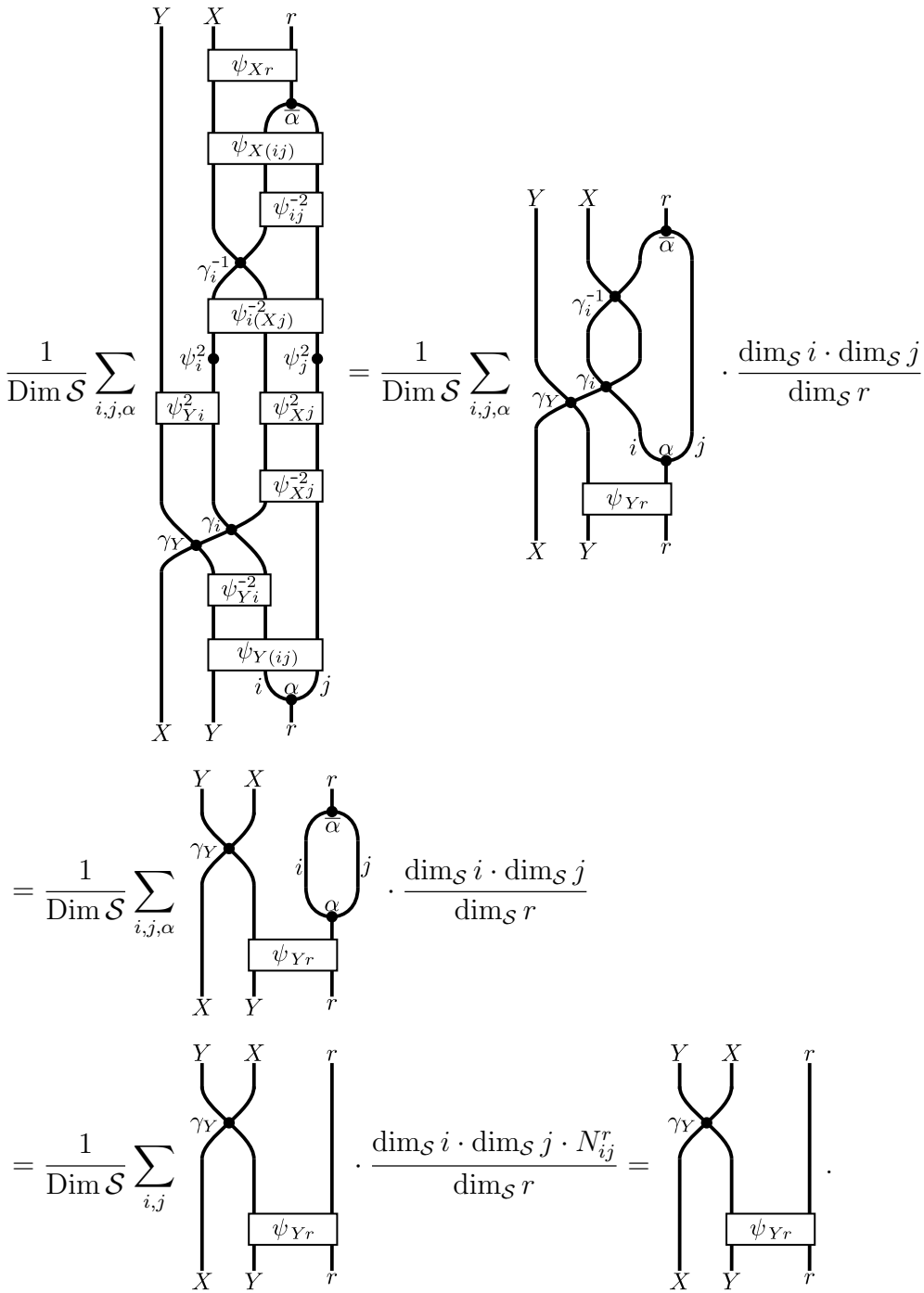


Figure 7.2: Left hand side of (7.45). Here, in the first equality one uses the natural transformation property of ψ and the half-braidings, in the third we abbreviate $N_{ij}^r = \dim \mathcal{S}(r, ij)$.

.

8. Examples in categories of Ising type

As was shown in Chapter 5, orbifold data can be used to construct new modular fusion categories out of a given one. In this chapter, we look at orbifold data $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ in a modular fusion category $\mathcal{C}$ such that several simplifying assumptions are fulfilled, namely:

(A1) The category $\mathcal{C}$ is multiplicity-less, i.e. for $i, j, k \in \mathrm{Irr}_{\mathcal{C}}$ one has

$$N_{ij}^k := \dim \mathcal{C}(i \otimes j, k) \in \{0, 1\} \,. \tag{8.1}$$

(A2) $\mathbb{A}$ is given in the setting of Δ-separable Frobenius algebras and Euler completion (see Appendix C) with the algebra A and the bimodule morphism $\psi\colon A \to A$ having the form

$$A = \bigoplus_{a \in B} \mathbb{1}_a \,, \qquad \psi = \bigoplus_{a \in B} \psi_a \cdot \mathrm{id}_{\mathbb{1}_a} \,, \quad \psi_a \in \Bbbk^\times \tag{8.2}$$

where B is a finite set and an index $a \in B$ is used to distinguish different copies of $\mathbb{1}$, i.e. $\mathbb{1}_a = \mathbb{1}$.

(A3) The A-$A \otimes A$-bimodule T decomposes as

$$T = \bigoplus_{a,b,c \in B} {}_a t_{bc} \,, \quad \text{each } {}_a t_{bc} \text{ is either simple or } 0. \tag{8.3}$$

The bimodule structure of T is such that ${}_a t_{bc}$ is an $\mathbb{1}_a$-$(\mathbb{1}_b \otimes \mathbb{1}_c)$-bimodule, where the action of the corresponding summand $\mathbb{1}_d$, $d \in B$ is via the unitor morphisms in $\mathcal{C}$ and the other summands act by zero.

(A4) There is a distinguished element $\iota \in B$, such that

$${}_a t_{\iota b} = {}_a t_{b\iota} = \begin{cases} \mathbb{1} \ , \text{ if } a = b \\ 0 \ , \text{ if } a \neq b \end{cases} \tag{8.4}$$

The orbifold datum $\mathbb{A}$ can then be described by a set of polynomial equations, much like the pentagon equation for the associator of a fusion category. We list them in Section 8.1. Then in Section 8.3 we look at the concrete example when $\mathcal{C}$ is a modular category of Ising type over $\Bbbk = \mathbb{C}$, i.e. has three simples $\mathbb{1}$, σ, ε with fusion rules

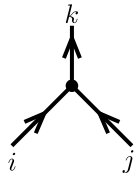
$$\varepsilon \otimes \varepsilon \cong \mathbb{1} \quad , \quad \varepsilon \otimes \sigma \cong \sigma \quad , \quad \sigma \otimes \sigma \cong \mathbb{1} \oplus \varepsilon \,. \tag{8.5}$$

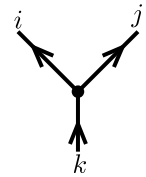
In this case the polynomial equations describing an orbifold datum can be solved with the help of computer algebra. We also demonstrate how the knowledge on the construction of the category $\mathcal{C}_{\mathbb{A}}$ and its relations to the Reshetikhin-Turaev defect TQFT allow one to indirectly compute some of its properties.

This chapter is based on the results in [MR2].

8.1. Orbifold data via polynomial equations

Let $\mathcal{C}$ be a MFC satisfying the assumption (A1). It is going to be useful to fix a non-zero element (i.e. a basis) of those spaces $\mathcal{C}(i \otimes j, k)$, $i, j, k \in \mathrm{Irr}_{\mathcal{C}}$ which are 1-dimensional (i.e. non-zero). We will denote this element, as well as its dual in $\mathcal{C}(k, i \otimes j)$ with respect to the composition pairing by

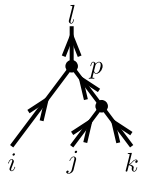
$\lambda_{(ij)k} =$ 

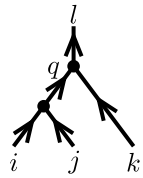
, $\lambda^{(ij)k} =$ 

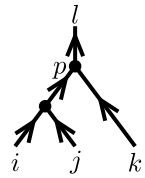
, (8.6)

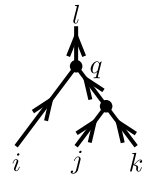
If $i = \mathbb{1}$ and $k = j$ we choose $\lambda_{(\mathbb{1}j)j}$ to be the left unitor $[l_j \colon \mathbb{1} \otimes j \to j]$ of $\mathcal{C}$. In the same way, we choose $\lambda_{(i\mathbb{1})i}$ to be the right unitor $[r_i \colon i \otimes \mathbb{1} \to i]$.

The associator morphisms are encoded in the F-matrix and its inverse G, whose elements are indexed by $i, j, k, l, p, q \in \mathrm{Irr}_{\mathcal{C}}$. They are defined by the following relations:



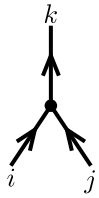
$= \sum_{q \in \mathrm{Irr}_{\mathcal{C}}} F_{pq}^{(ijk)l}$ 

, 

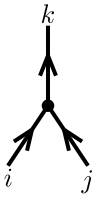
$= \sum_{q \in \mathrm{Irr}_{\mathcal{C}}} G_{pq}^{(ijk)l}$ 

. (8.7)

Since $\lambda_{(\mathbb{1}x)x}$ and $\lambda_{(x\mathbb{1})x}$ are unitors, if at least one of i, j, k is $\mathbb{1}$ and the corresponding F-matrix element is not automatically zero by the fusion rules, we have $F_{pq}^{(ijk)l} = 1$. Similarly, the braiding morphisms are given by the R-matrix and its inverse, whose elements are, for $i, j, k \in \mathrm{Irr}_{\mathcal{C}}$:

$= R^{(ij)k}$ 

,

$= R^{-(ij)k}$ 

. (8.8)

For an orbifold datum $\mathbb{A}$ in $\mathcal{C}$ such that the assumptions (A1)-(A3) are satisfied one gets decompositions $\alpha = \bigoplus_{a,b,c,d \in B} \alpha_{bcd}^a$, $\overline{\alpha} = \bigoplus_{a,b,c,d \in B} \overline{\alpha}_{bcd}^a$, where

$$\alpha_{bcd}^a : \bigoplus_{p \in B} {}_a t_{bp} \otimes {}_p t_{cd} \longrightarrow \bigoplus_{q \in B} {}_a t_{qd} \otimes {}_q t_{bc} \ , \tag{8.9}$$

$$\overline{\alpha}^a_{bcd} : \bigoplus_{p\in B} {}_a t_{pd} \otimes {}_p t_{bc} \longrightarrow \bigoplus_{q\in B} {}_a t_{bq} \otimes {}_q t_{cd} \ . \tag{8.10}$$

The conditions these data must satisfy are somewhat simpler if one uses the following rescaled version of morphisms α^a_{bcd} instead:

$$\alpha^a_{bcd} = \frac{1}{\psi_c^2}\, f^a_{bcd} \ , \quad a,b,c,d \in B \ , \tag{8.11}$$

where f^a_{bcd} are morphisms having the same source and target as α^a_{bcd} in (8.10). Similarly, we denote by g^a_{bcd} a rescaling of $\overline{\alpha}^a_{bcd}$ (we will see later that this is the inverse of f^a_{bcd}):

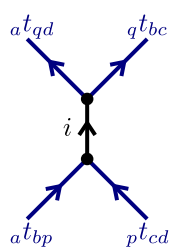
$$\overline{\alpha}^a_{bcd}\big|_{pq} = \frac{\psi_c^2}{\psi_p^2\psi_q^2}\, g^a_{bcd}\big|_{pq} \ , \tag{8.12}$$

where we use the notation

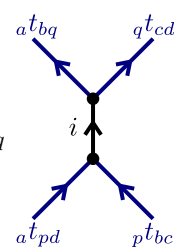
$$\begin{aligned} \alpha^a_{bcd}\big|_{pq},\ f^a_{bcd}\big|_{pq} &: {}_a t_{bp} \otimes {}_p t_{cd} \longrightarrow {}_a t_{qd} \otimes {}_q t_{bc} \ , \\ \overline{\alpha}^a_{bcd}\big|_{pq},\ g^a_{bcd}\big|_{pq} &: {}_a t_{pd} \otimes {}_p t_{bc} \longrightarrow {}_a t_{bq} \otimes {}_q t_{cd} \ , \end{aligned} \tag{8.13}$$

for the restrictions of α^a_{bcd}, f^a_{bcd}, $\overline{\alpha}^a_{bcd}$ and g^a_{bcd} to the corresponding direct summands. For $i \in \mathrm{Irr}_{\mathcal{C}}$, we introduce scalars $f^{a,\ i}_{bcd,\ pq}$ and $g^{a,\ i}_{bcd,\ pq}$ such that

$$f^a_{bcd}\big|_{pq} = \sum_{i\in\mathrm{Irr}_{\mathcal{C}}} f^{a,\ i}_{bcd,\ pq}$$



$$, \quad g^a_{bcd}\big|_{pq} = \sum_{i\in\mathrm{Irr}_{\mathcal{C}}} g^{a,\ i}_{bcd,\ pq}$$



$$. \tag{8.14}$$

One can now translate the conditions (O1$_\Delta$)-(O8$_\Delta$) into equations for these scalars. The result is:

Proposition 8.1. Under the assumptions (A1), (A2), (A3), giving an orbifold datum in $\mathcal{C}$ is equivalent to giving a set of scalars

$$f^{a,\ i}_{bcd,\ pq}, \quad g^{a,\ i}_{bcd,\ pq}, \quad \psi_a, \quad \phi, \quad a,b,c,d,p,q \in B, \quad i \in \mathrm{Irr}_{\mathcal{C}}, \tag{8.15}$$

which satisfy the equations (P1)-(P8) in Table 8.1.

The reformulation of conditions (O1$_\Delta$)-(O8$_\Delta$) in terms of the scalars defining the orbifold datum under assumptions (A1)–(A3) is tedious but straightforward. As an example, the computation for the identity (O1$_\Delta$) is given in Appendix D.1.

$$\sum_{i,j\in\mathrm{Irr}_{\mathcal{C}}} F_{k\,i}^{({}_a t_{re}\ {}_r t_{sd}\ {}_s t_{bc})g}\ G_{j\,m}^{\left({}_a t_{bp}\ {}_p t_{cq}\ {}_q t_{de}\right)g}\ F_{i\,j}^{({}_s t_{bc}\ {}_a t_{sq}\ {}_q t_{de})g}\ R^{({}_a t_{sq}\ {}_s t_{bc})j}\ R^{-({}_s t_{bc}\ i)g} \cdot f_{sde,\ qr}^{a,\ i}\ f_{bcq,\ ps}^{a,\ j} = \sum_{x\in B, l\in\mathrm{Irr}_{\mathcal{C}}} F_{k\,l}^{({}_a t_{re}\ {}_r t_{bx}\ {}_x t_{cd})g}\ G_{l\,m}^{\left({}_a t_{bp}\ {}_p t_{xe}\ {}_x t_{cd}\right)g}\ f_{cde,\ qx}^{p,\ m}\ f_{bxe,\ pr}^{a,\ l}\ f_{bcd,\ xs}^{r,\ k} \tag{P1}$$

$$\sum_{q\in B} f_{bcd,\ qr}^{a,\ i} \cdot g_{bcd,\ pq}^{a,\ i} = \delta_{pr}\ N_{{}_a t_{pd}\ {}_p t_{bc}}^{i} \tag{P2}$$

$$\sum_{q\in B} g_{bcd,\ qr}^{a,\ i} \cdot f_{bcd,\ pq}^{a,\ i} = \delta_{pr}\ N_{{}_a t_{bp}\ {}_p t_{cd}}^{i} \tag{P3}$$

$$\sum_{d\in B, i,j\in\mathrm{Irr}_{\mathcal{C}}} \frac{\psi_b^2\psi_d^2}{\psi_q^2\psi_p^2}\ f_{bcd,\ pq}^{a,\ j}\ g_{b'cd,\ qp}^{a,\ i}\ F_{{}_a t_{bp}\ {}_a t_{qd}}^{({}_p t_{cd}\ m\ {}_q t_{bc})j}\ G_{{}_a t_{qd}\ {}_a t_{b'p}}^{({}_p t_{cd}\ m\ {}_q t_{b'c})i} \cdot R^{\left({}_p t_{cd}\ {}_a t_{bp}\right)j}\ R^{-\left({}_a t_{b'p}\ {}_p t_{cd}\right)i}\ \frac{\dim j}{\dim {}_a t_{qd}} \frac{\dim i}{\dim {}_a t_{bp}} = \delta_{bb'}\ N_{m\ {}_q t_{bc}}^{{}_a t_{bp}} \tag{P4}$$

$$\sum_{b\in B, i,j\in\mathrm{Irr}_{\mathcal{C}}} \frac{\psi_b^2\psi_d^2}{\psi_p^2\psi_q^2}\ g_{bcd,\ pq}^{a,\ j}\ f_{bcd',\ qp}^{a,\ i}\ F_{{}_a t_{pd}\ {}_a t_{bq}}^{({}_p t_{bc}\ m\ {}_q t_{cd})j}\ G_{{}_a t_{bq}\ {}_a t_{pd'}}^{({}_p t_{bc}\ m\ {}_q t_{cd'})i} \cdot R^{-\left({}_p t_{bc}\ {}_a t_{pd}\right)j}\ R^{\left({}_a t_{pd'}\ {}_p t_{bc}\right)i}\ \frac{\dim j}{\dim {}_a t_{bq}} \frac{\dim i}{\dim {}_a t_{pd}} = \delta_{dd'}\ N_{m\ {}_q t_{cd}}^{{}_a t_{pd}} \tag{P5}$$

$$\sum_{c\in B, i,j\in\mathrm{Irr}_{\mathcal{C}}} \frac{\psi_a^2\psi_c^2}{\psi_q^2\psi_p^2}\ f_{bcd,\ pq}^{a,\ j}\ g_{bcd,\ qp}^{a',\ i}\ G_{{}_a t_{bp}\ {}_q t_{bc}}^{\left({}_a t_{qd}\ m\ {}_p t_{cd}\right)j}\ F_{{}_q t_{bc}\ {}_{a'} t_{bp}}^{\left({}_{a'} t_{qd}\ m\ {}_p t_{cd}\right)i} \cdot \frac{\dim j}{\dim {}_q t_{bc}} \frac{\dim i}{\dim {}_a t_{bp}} = \delta_{aa'}\ N_{{}_a t_{qd}\ m}^{{}_a t_{bp}} \tag{P6}$$

$$\sum_{a\in B, i,j\in\mathrm{Irr}_{\mathcal{C}}} \frac{\psi_a^2\psi_c^2}{\psi_p^2\psi_q^2}\ g_{bcd,\ pq}^{a,\ j}\ f_{bc'd,\ qp}^{a,\ i}\ F_{{}_p t_{bc}\ {}_a t_{bq}}^{\left({}_a t_{pd}\ m\ {}_q t_{cd}\right)j}\ G_{{}_a t_{bq}\ {}_p t_{bc'}}^{\left({}_a t_{pd}\ m\ {}_q t_{c'd}\right)i} \cdot \frac{\dim j}{\dim {}_a t_{bq}} \frac{\dim i}{\dim {}_p t_{bc}} = \delta_{cc'}\ N_{m\ {}_q t_{cd}}^{{}_p t_{bc}} \tag{P7}$$

$$\sum_{b,c\in B} \psi_b^2\psi_c^2 \dim {}_a t_{bc} = \sum_{b,c\in B} \psi_b^2\psi_c^2 \dim {}_b t_{ca} = \sum_{b,c\in B} \psi_b^2\psi_c^2 \dim {}_c t_{ab} = \psi_a^2\phi^{-2} \tag{P8}$$

Table 8.1: Polynomial equations defining an orbifold datum $\mathbb{A}$ in a modular fusion category $\mathcal{C}$ under assumptions (A1)–(A3). The sum over $d \in B$ in (P4) is restricted to those d for which ${}_a t_{qd} \neq 0$. Analogous restrictions apply to the sums over B in (P5)–(P7).

Remark 8.2. The identities (P2) and (P3) show that

$$f^a_{bcd} \;:\; \bigoplus_{p\in B} {}_a t_{bp} \otimes {}_p t_{cd} \;\longleftrightarrow\; \bigoplus_{q\in B} {}_a t_{qd} \otimes {}_q t_{bc} \;:\; g^a_{bcd} \tag{8.16}$$

are indeed inverse to each other. In particular, the scalars $g^{a,\ i}_{bcd,\ pq}$ are uniquely determined by the scalars $f^{a,\ i}_{bcd,\ pq}$.

Our interest in orbifold data $\mathbb{A}$ in a MFC $\mathcal{C}$ is only up to a ribbon equivalence of the resulting associated MFC $\mathcal{C}_{\mathbb{A}}$. In Section 5.5 we showed two ways to obtain from $\mathbb{A}$ an equivalent orbifold datum: a rescaling $\mathbb{A}_\xi$ for $\xi \in \Bbbk^\times$ and a T-compatible isomorphism $\rho\colon T \to \widetilde{T}$. One can use them to eliminate some variables in the equations (P1)–(P8).

Consider a T-compatible isomorphism $\rho\colon T \to T$ of orbifold data satisfying the assumptions (A1)–(A3). In this case one has:

$$\rho = \bigoplus_{a,b,c\in B} {}_a\lambda_{bc} \cdot \mathrm{id}_{{}_a t_{bc}} \;, \tag{8.17}$$

where there is one scalar ${}_a\lambda_{bc} \in \Bbbk^\times$ for each non-zero ${}_a t_{bc}$. It follows from (5.65) that α and $\widetilde{\alpha}$ are related by

$$\widetilde{\alpha}^a_{bcd}\big|_{pq} = \frac{{}_a\lambda_{qd}\ {}_q\lambda_{bc}}{{}_a\lambda_{bp}\ {}_p\lambda_{cd}} \cdot \alpha^a_{bcd}\big|_{pq}, \quad a,b,c,d,p,q \in B \;, \tag{8.18}$$

or equivalently, the sets of scalars $f^{a,\ i}_{bcd,\ pq}$, $g^{a,\ i}_{bcd,\ pq}$, $\widetilde{f}^{a,\ i}_{bcd,\ pq}$, $\widetilde{g}^{a,\ i}_{bcd,\ pq}$ are related by

$$\widetilde{f}^{a,\ i}_{bcd,\ pq} = \frac{{}_a\lambda_{qd}\ {}_q\lambda_{bc}}{{}_a\lambda_{bp}\ {}_p\lambda_{cd}} \cdot f^{a,\ i}_{bcd,\ pq} \;, \quad \widetilde{g}^{a,\ i}_{bcd,\ pq} = \frac{{}_a\lambda_{bq}\ {}_q\lambda_{cd}}{{}_a\lambda_{pd}\ {}_p\lambda_{bc}} \cdot g^{a,\ i}_{bcd,\ pq} \;. \tag{8.19}$$

Evidently, if $f^{a,\ i}_{bcd,\ pq}$, $g^{a,\ i}_{bcd,\ pq}$, ψ_a, ϕ solve the equations in Table 8.1, then so do $\widetilde{f}^{a,\ i}_{bcd,\ pq}$, $\widetilde{g}^{a,\ i}_{bcd,\ pq}$, ψ_a, ϕ. We exploit this invariance to simplify the search for solutions by imposing the additional unitality assumption (A4), so that one has the following

Lemma 8.3. Suppose assumptions (A1)–(A4) hold. Let $a,b,c \in B$ be such that ${}_a t_{bc} \neq 0$. Then the morphisms $f^a_{\iota bc}$, $f^a_{b\iota c}$, $f^a_{bc\iota}$ are determined by the scalars $f^{a,\ {}_a t_{bc}}_{\iota bc,\ ab}$, $f^{a,\ {}_a t_{bc}}_{b\iota c,\ cb}$, $f^{a,\ {}_a t_{bc}}_{bc\iota,\ ca}$, all of which are non-zero. Via a suitable transformation of the form (8.19), we can achieve that

$$f^{a,\ {}_a t_{bc}}_{\iota\, b\, c,\ ab} \;=\; f^{a,\ {}_a t_{bc}}_{b\,\iota\, c,\ cb} \;=\; f^{a,\ {}_a t_{bc}}_{b\, c\, \iota,\ ca} \;=\; 1 \;. \tag{8.20}$$

Proof. One quickly determines that the expression in (8.14) e.g. for $f^{a}_{bc\iota}$ has a single term when $p = c$, $q = a$ and vanishes otherwise. That the corresponding scalars are non-zero is implied by the invertibility conditions (P2) and (P3).

The normalisation can be shown as follows: Take $c = d = \iota$, $p = q = e$, $r = s = b$, $m = k = \mathbb{1}$, $g = {}_a t_{be}$ in the condition (P1). Then assuming ${}_a t_{be} \neq 0$ one can simplify the resulting equation to

$$f^{a,\ {}_a t_{be}}_{b\iota e,\ eb} = f^{e,\ \mathbb{1}}_{\iota\iota e,\ e\iota} \cdot f^{b,\ \mathbb{1}}_{b\iota\iota,\ \iota b}. \tag{8.21}$$

Any transformation (8.19) such that for all $b, e \in B$ one has

$${}_b\lambda_{b\iota} = \left(f^{b,\ \mathbb{1}}_{b\iota\iota,\ \iota b}\right)^{-1}, \quad {}_e\lambda_{\iota e} = f^{e,\ \mathbb{1}}_{\iota\iota e,\ e\iota} \tag{8.22}$$

then results in $\widetilde{f}^{a,\ {}_a t_{be}}_{b\iota e,\ eb} = 1$. After this transformation, setting $b = c = s = \iota$, $p = q = a$, $r = d$, $k = \mathbb{1}$, $m = g = {}_a t_{de}$, in (P1) and assuming ${}_a t_{de} \neq 0$ one finds that $\widetilde{f}^{a,\ {}_a t_{de}}_{\iota de,\ ad} = 1$ already holds. Similarly, by taking $d = e = q = \iota$, $r = s = a$, $p = c$, $m = \mathbb{1}$, $k = g = {}_a t_{bc}$ and assuming ${}_a t_{bc} \neq 0$ one finds that $\widetilde{f}^{a,\ {}_a t_{bc}}_{bc\iota,\ ca} = 1$ holds too. □

Let us look at some simple properties of the category $\mathcal{C}_{\mathbb{A}}$ which can be determined from having an orbifold datum $\mathbb{A}$ presented as a solution to the equations (P1)–(P8). We start with a criterion for the simplicity of $\mathbb{A}$, which is obtained by applying the projector (5.43) on A-A-bimodule endomorphisms of A:

Proposition 8.4. Under the assumptions (A1)–(A3) we have

$$\phi^4 \sum_{a,b,d,p\in B} \psi_b^2\, \psi_d^2 \,\dim {}_a t_{pd} \,\dim {}_p t_{ba} \;=\; \dim \mathcal{C}_{\mathbb{A}}(A,A) \mod \operatorname{char} \Bbbk. \tag{8.23}$$

In particular, if $\operatorname{char}(\Bbbk) = 0$, $\mathbb{A}$ is simple if and only if the left-hand side yields 1.

Next we turn to the global dimension of $\mathcal{C}_{\mathbb{A}}$: Using the assumptions (A1)-(A3) and substituting the scalars in Proposition 8.1, by Theorem 5.20 one gets

$$\operatorname{Dim} \mathcal{C}_{\mathbb{A}} \;=\; \frac{\sum_{i\in \mathrm{Irr}_{\mathcal{C}}} (\dim i)^2}{\phi^8 \cdot \left(\sum_{b\in B} \psi_b^4\right)^2}\,. \tag{8.24}$$

Finally, we turn to computing the number $|\mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}}|$ of (isomorphism classes of) simple objects in $\mathcal{C}_{\mathbb{A}}$ in case $\mathbb{A}$ is simple. We will do this with the help of the equivalence of TQFTs $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}} \cong Z^{\mathrm{RT}}_{\mathcal{C}_{\mathbb{A}}}$ (see Theorem 6.13). Recall from Property 3.14 that if $\operatorname{char} \Bbbk = 0$, the Reshetikhin-Turaev invariant $Z^{\mathrm{RT}}_{\mathcal{C}}(T^3)$ of the 3-torus $T^3 = S^1 \times S^1 \times S^1$ is precisely $|\mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}}|$. One has:

Lemma 8.5. For a modular fusion category $\mathcal{C}$ and a simple orbifold datum $\mathbb{A}$ satisfying assumptions (A1)–(A3), we have

$$
\begin{aligned}
&Z_{\mathcal{C}}^{\mathrm{orb},\mathbb{A}}(T^3) = \\
&\quad \phi^2 \cdot \sum_{\substack{a,b,c,d,e,f,g\in B\\ r,s,t,u,v,w\in \mathrm{Irr}_{\mathcal{C}}}} \frac{\psi_a^2\psi_c^2\psi_e^2\psi_f^2}{\psi_b^2\psi_d^2\psi_g^2}\, f_{afc,\ bg}^{e,\ r}\, f_{caf,\ gd}^{e,\ s}\, f_{fca,\ db}^{e,\ u}\, g_{acf,\ db}^{e,\ t}\, g_{fac,\ gd}^{e,\ v}\, g_{cfa,\ bg}^{e,\ w} \\
&\qquad \cdot \sum_{x,y,z,k,l,m\in \mathrm{Irr}_{\mathcal{C}}} L_{eba,\ ur}^{bcf,\ wt}\Big|_x\, L_{efd,\ vt}^{dca,\ us}\Big|_y\, L_{ecg,\ wv}^{gaf,\ sr}\Big|_k \\
&\qquad\qquad \cdot \frac{\dim z}{\dim l}\,\frac{\dim v}{\dim k}\, F_{k\,u}^{(r\,z\,y)v}\, F_{l\,w}^{(s\,k\,x)u}\, G_{r\,m}^{(t\,x\,z)u}\, G_{t\,l}^{(s\,y\,m)u} \\
&\qquad\qquad \cdot T_{xyz,\ klm}\ ,
\end{aligned}
\tag{8.25}
$$

where

$$
\begin{aligned}
L_{eda,\ ur}^{bcf,\ wt}\Big|_x &= N_{e\,t_{da}\ \ b\,t_{cf}}^{w}\, N_{e\,t_{ad}\ \ b\,t_{cf}}^{t}\, \frac{\dim x}{\dim {}_b t_{fc}}\, F_{b\,t_{fc}\ \ t}^{(e\,t_{ad}\ \ b\,t_{cf}\ \ x)r}\, G_{w\ \ b\,t_{fc}}^{(e\,t_{da}\ \ b\,t_{cf}\ \ x)u}\ , \\
T_{xyz,\ klm} &= \sum_{p,j\in \mathrm{Irr}_{\mathcal{C}}} \frac{\dim j}{\dim p}\,\frac{\theta_j}{\theta_p\theta_x}\, R^{-(z\,x)m}\, G_{p\,k}^{(p\,z\,y)p}\, G_{p\,l}^{(p\,k\,x)j}\, F_{l\,p}^{(p\,y\,m)j}\, F_{m\,p}^{(p\,z\,x)j},
\end{aligned}
$$

and the expression for the L-symbol holds when ${}_b t_{fc} \neq 0$ and is set to zero otherwise.

The proof is a rather technical computation which we present in Appendix D.2. In light of the previous remark, one gets

Proposition 8.6. Suppose that $\operatorname{char} \Bbbk = 0$. For a modular fusion category $\mathcal{C}$ and a simple orbifold datum $\mathbb{A}$ satisfying assumptions (A1)–(A3), the number of isomorphism classes of simple objects in $\mathcal{C}_{\mathbb{A}}$ is given by (8.25).

8.2. Ising-type modular categories

For the rest of the chapter, we work over the field of complex numbers, $\Bbbk = \mathbb{C}$.

A braided fusion category of Ising-type is a special case of a Tambara-Yamagami category, where the underlying abelian group is $\mathbb{Z}_2$ [TY, Si]. Including also the ribbon twist, there are 16 different modular fusion categories of Ising-type [DGNO, App. B],

$$
\mathcal{I}_{\zeta,\epsilon} \quad , \qquad \text{where} \quad \zeta^8 = -1\ , \quad \epsilon \in \{\pm 1\}\ . \tag{8.26}
$$

We will use the abbreviation

$$
\lambda = \zeta^2 + \zeta^{-2}\ , \tag{8.27}
$$

which implies that $\lambda^2 = 2$. The category $\mathcal{I}_{\zeta,\epsilon}$ has three simple objects, $I = \{\mathbb{1}, \varepsilon, \sigma\}$. Here, $\mathbb{1}, \varepsilon$ form the $\mathbb{Z}_2$-subgroup of the fusion ring, and $\sigma \otimes \sigma \cong \mathbb{1} \oplus \varepsilon$. The quantum dimensions and twist eigenvalues of the simple objects are

$$\begin{array}{rclrclrcl} \dim(\mathbb{1}) &=& 1\ , & \dim(\varepsilon) &=& 1\ , & \dim(\sigma) &=& \epsilon\lambda\ , \\ \theta_{\mathbb{1}} &=& 1\ , & \theta_\varepsilon &=& -1\ , & \theta_\sigma &=& \epsilon\zeta^{-1}\ . \end{array} \tag{8.28}$$

The R-matrices of $\mathcal{I}_{\zeta,\epsilon}$ are given by

$$R^{(\varepsilon\varepsilon)\mathbb{1}} = -1\ ,\quad R^{(\sigma\sigma)\mathbb{1}} = \zeta\ ,\quad R^{(\sigma\sigma)\varepsilon} = \zeta^{-3}\ ,\quad R^{(\sigma\varepsilon)\sigma} = R^{(\varepsilon\sigma)\sigma} = \zeta^4\ . \tag{8.29}$$

The F-matrices with one internal channel are

$$\begin{array}{lll} F_{\mathbb{1}\mathbb{1}}^{(\varepsilon\varepsilon\varepsilon)\varepsilon} = 1\ , & & \\ F_{\sigma\mathbb{1}}^{(\varepsilon\varepsilon\sigma)\sigma} = 1\ , & F_{\sigma\sigma}^{(\varepsilon\sigma\varepsilon)\sigma} = -1\ , & F_{\mathbb{1}\sigma}^{(\sigma\varepsilon\varepsilon)\sigma} = 1\ , \\ F_{\sigma\mathbb{1}}^{(\sigma\sigma\varepsilon)\varepsilon} = 1\ , & F_{\sigma\sigma}^{(\sigma\varepsilon\sigma)\varepsilon} = -1\ , & F_{\mathbb{1}\sigma}^{(\varepsilon\sigma\sigma)\varepsilon} = 1\ , \end{array} \tag{8.30}$$

as well as $F_{ik}^{(ijk)\mathbb{1}} = 1$ for $i, j, k \in I$ whenever the F-matrix is allowed by fusion, i.e. when $\mathbb{1}$ is a summand of $i \otimes j \otimes k$. Finally, the only F-matrix with two internal channels is

$$F_{\mathbb{1}\mathbb{1}}^{(\sigma\sigma\sigma)\sigma} = F_{\mathbb{1}\varepsilon}^{(\sigma\sigma\sigma)\sigma} = F_{\varepsilon\mathbb{1}}^{(\sigma\sigma\sigma)\sigma} = \frac{1}{\lambda}\ ,\quad F_{\varepsilon\varepsilon}^{(\sigma\sigma\sigma)\sigma} = -\frac{1}{\lambda}\ . \tag{8.31}$$

The G-matrix in this case is obtained from the relation $G_{pq}^{(ijk)l} = F_{pq}^{(kji)l}$ (see e.g. [FRS1, Eqn. (2.61)]).

The global dimension of $\mathcal{I}_{\zeta,\epsilon}$ and its anomaly are given by

$$\begin{aligned} \mathrm{Dim}(\mathcal{I}_{\zeta,\epsilon}) &= \sum_{i\in I} \dim(i)^2 = 4\ , \\ \delta_{\mathcal{I}_{\zeta,\epsilon}} &= \frac{1}{\sqrt{\mathrm{Dim}(\mathcal{I}_{\zeta,\epsilon})}} \sum_{i\in I} \dim(i)^2\, \theta_i = \epsilon\zeta^{-1}\ . \end{aligned} \tag{8.32}$$

8.3. Fibonacci-type solutions inside Ising categories

Here we find all solutions for orbifold data in Ising-type categories for a particular ansatz for A and T.

Fix ζ and ϵ as in Section 8.2. We will work in the modular fusion category $\mathcal{I}_{\zeta,\epsilon}$. We make the ansatz $B = \{\iota, \varphi\}$ and

$$A = \mathbb{1}_\iota \oplus \mathbb{1}_\varphi\ ,\quad {}_a t_{bc} = \begin{cases} \mathbb{1} & ;\ \text{either 0 or 2 of } a,b,c \text{ are } \varphi \\ \sigma & ;\ \text{all of } a,b,c \text{ are } \varphi \\ 0 & ;\ \text{else} \end{cases} \tag{8.33}$$

This mimics the fusion rules of a Fibonacci category in that both ${}_\iota t_{\varphi\varphi}$ and ${}_\varphi t_{\varphi\varphi}$ are non-zero (i.e. reminiscent of "$\varphi \otimes \varphi \cong \iota \oplus \varphi$"). We therefore call the solutions below of Fibonacci type.

Let $h \in \mathbb{C}$ satisfy

$$h^3 = \zeta \qquad \text{and} \qquad h \text{ is a primitive } 48^{\text{th}} \text{ root of unity .} \tag{8.34}$$

We fix the following values for f, ψ, ϕ^2:

$$\begin{aligned}
&\psi_\iota^2 \phi^2 = \frac{1}{3 - h^4 - h^{-4}} \quad , \quad \psi_\varphi^2 \phi^2 = -\frac{h^{10} + h^{-10}}{3 - h^4 - h^{-4}} \cdot \epsilon \ , \\
&f^{\iota,\ \sigma}_{\varphi\varphi\varphi,\ \varphi\varphi} = h \quad , \quad f^{\varphi,\ \varepsilon}_{\varphi\varphi\varphi,\ \varphi\varphi} = h^5 \quad , \quad f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\iota} = \frac{1}{h^{12}(h^2 - h^{-2})} \quad , \\
&f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \varphi\varphi} = -h^{-1} f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\iota} \quad , \quad f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\varphi} f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \varphi\iota} = \frac{\lambda}{h} f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\iota} \ .
\end{aligned} \tag{8.35}$$

The value of $\phi^2 \in \mathbb{C}^\times$ can be chosen arbitrarily and then fixes those of ψ_ι^2, ψ_φ^2. Similarly, the value of, say, $f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\varphi} \in \mathbb{C}^\times$ is arbitrary, fixing that of $f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \varphi\iota}$.

Theorem 8.7. Every orbifold datum in $\mathcal{I}_{\zeta,\epsilon}$ which has A, T as specified in (8.33) and satisfies the normalisation condition (8.20) is given by (8.35) with h subject to (8.34). Different choices for $\phi^2, f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\varphi} \in \mathbb{C}^\times$ are related by rescalings and by T-compatible isomorphisms.

Proof. Let us first note that by Lemma 8.3 one can set

$$f^{a,\ i}_{bcd,\ pq} = N^i_{a t_{bp}\ p t_{cd}}\, N^i_{a t_{qd}\ q t_{bc}} \ , \quad a, b, c, d, p, q \in B \ , \quad i \in I \tag{8.36}$$

whenever at least one of b, c, d is $\iota \in B$. For the rest of the f-coefficients which are not automatically zero by the fusion rules we use the abbreviations

$$\begin{gathered}
f^{\iota,\ \sigma}_{\varphi\varphi\varphi,\ \varphi\varphi} = h \ , \quad f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\iota} = f_{\iota\iota} \ , \quad f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\varphi} = f_{\iota\varphi} \ , \quad f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \varphi\iota} = f_{\varphi\iota} \ , \\
f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \varphi\varphi} = f^{\mathbb{1}}_{\varphi\varphi} \ , \quad f^{\varphi,\ \varepsilon}_{\varphi\varphi\varphi,\ \varphi\varphi} = f^{\varepsilon}_{\varphi\varphi} \ .
\end{gathered} \tag{8.37}$$

Next we consider some of the equations implied by (P1):

a	b	c	d	e	p	q	r	s	g	m	k	equation	
ι	φ	φ	φ	φ	φ	ι	φ	ι	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$f_{\iota\iota}^2 + h f_{\iota\varphi} f_{\varphi\iota} = 1$	(a)
ι	φ	φ	φ	φ	φ	ι	φ	φ	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$f_{\iota\iota} f_{\iota\varphi} = -h f_{\iota\varphi} f^{\mathbb{1}}_{\varphi\varphi}$	(b)
ι	φ	φ	φ	φ	φ	φ	φ	φ	ε	ε	ε	$h(f^{\varepsilon}_{\varphi\varphi})^2 = h^2 \zeta^3$	(c)
φ	φ	φ	φ	φ	ι	φ	ι	φ	σ	σ	σ	$h^2 f_{\iota\iota} = \frac{\zeta f_{\iota\varphi} f_{\varphi\iota}}{\lambda}$	(d)
φ	φ	φ	φ	φ	ι	φ	φ	ι	σ	σ	$\mathbb{1}$	$f_{\iota\iota} = \frac{h f_{\iota\varphi} f_{\varphi\iota}}{\lambda}$	(e)
φ	φ	φ	φ	φ	ι	φ	φ	φ	σ	σ	$\mathbb{1}$	$\frac{f_{\iota\varphi} f^{\varepsilon}_{\varphi\varphi}}{\zeta^3 \lambda^2} = \frac{f_{\iota\varphi} f^{\mathbb{1}}_{\varphi\varphi}}{\lambda} \left(h - \frac{\zeta}{\lambda}\right)$	(f)

(8.38)

Let us denote $z = hf_{\iota\varphi}f_{\varphi\iota}$. Substituting (8.38 e) into (8.38 a) yields $z^2 + \lambda^2 z = \lambda^2$. Substituting (8.27) gives $z^2 + 2z = 2$, which implies

$$z = -1 + \delta\sqrt{3}\ , \quad \delta = \pm 1\ . \tag{8.39}$$

In particular, h, $f_{\iota\varphi}$, $f_{\varphi\iota}$ are all non-zero. Other constants can then also be expressed in terms of z as follows:

$$f_{\iota\iota} \overset{(8.38\,\mathrm{e})}{=} \frac{z}{\lambda}\ , \quad f^{\mathbb{1}}_{\varphi\varphi} \overset{(8.38\,\mathrm{b})}{=} -\frac{z}{\lambda h}\ , \quad f^{\varepsilon}_{\varphi\varphi} \overset{(8.38\,\mathrm{f})}{=} \frac{\zeta^3}{h}\left(\frac{\zeta}{\lambda} - h\right)\cdot z\ . \tag{8.40}$$

In particular, $f_{\iota\iota}$ and $f^{\mathbb{1}}_{\varphi\varphi}$ are non-zero. Substituting (8.40) into (8.38 d) yields the relation $\zeta = h^3$. Using this, one can conclude that also $f^{\varepsilon}_{\varphi\varphi} \neq 0$.

Conditions (P2) and (P3) are equivalent to expressing the constants $g^{a,\ i}_{bcd,\ pq}$, $a, b, c, d, p, q \in B$, $i \in I$ in terms of the ones in (8.36) and (8.37):

$$\begin{gathered} g^{a,\ i}_{bcd,\ pq} = N^i_{a t_{pd}\ p t_{bc}} N^i_{a t_{bq}\ q t_{cd}}, \quad \text{if } \iota \in \{b, c, d\}\ , \\ g^{\iota,\ \sigma}_{\varphi\varphi\varphi,\ \varphi\varphi} = \frac{1}{h}\ , \quad g^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\iota} = -hf^{\mathbb{1}}_{\varphi\varphi}\ , \quad g^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\varphi} = hf_{\iota\varphi}, \quad g^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \varphi\iota} = hf_{\varphi\iota}\ , \\ g^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \varphi\varphi} = -hf_{\iota\iota}\ , \quad g^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \varphi\varphi} = \frac{1}{f^{\varepsilon}_{\varphi\varphi}}\ , \end{gathered} \tag{8.41}$$

the rest of them being automatically zero.

Among the equations given by the condition (P8) only two are distinct, namely

$$\psi_\iota^4 + \psi_\varphi^4 = \frac{\psi_\iota^2}{\phi^2}\ , \quad 2\psi_\iota^2\psi_\varphi^2 + \epsilon\lambda\psi_\varphi^2 = \frac{\psi_\varphi^2}{\phi^2}\ . \tag{8.42}$$

The solutions to (P8) therefore are

$$\psi_\iota^2 = \frac{1}{2\phi^2}\left(1 - \frac{\nu\epsilon\lambda}{\sqrt{6}}\right), \quad \psi_\varphi^2 = \frac{\nu}{\phi^2\sqrt{6}}, \qquad \nu = \pm 1\ . \tag{8.43}$$

At this point the solutions are collected in (8.40), (8.39) and (8.43) and parametrised by (h, δ, ν), where $h^{24} = -1$ (implied by (8.26)) and $\delta, \nu \in \{1, -1\}$. Plugging them into (P4)-(P7) and into the remaining equations implied by (P1), one finds[11] that h must be a primitive 48th root of unity and that δ, ν are determined by h and ϵ as follows, writing $h = \exp\left(\frac{\pi n i}{24}\right)$,

n	1	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47
δ	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$
$\epsilon\nu$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$

[11] For most of these computations we used the computer algebra system Mathematica.

The signs can be expressed explicitly in terms of h and ϵ as

$$\delta = \frac{2h^4 - h^{12}}{\sqrt{3}} \ , \quad \nu = \frac{h^6 + h^{-6}}{\sqrt{2}} \delta\epsilon \ . \tag{8.44}$$

With the expressions derived above one can now verify[11] that (8.35) indeed gives all solutions to (P1)–(P8).

Different choices for ϕ^2 are trivially related by a rescaling as in (5.66) (if $\mathbb{A}$ is defined via the Δ-separable setting and Euler completion, the rescaling is $\mathbb{A}_\xi = (A, T, \xi\alpha, \xi\overline{\alpha}, \xi^{-1/2}\psi, \xi^{1/2}\phi)$, i.e. in contrast with (5.66), the algebra A and the module T are left unmodified).

It remains to show that different choices of $f_{\iota\varphi}$, $f_{\varphi\iota}$ such that $hf_{\iota\varphi}f_{\varphi\iota} = \lambda f_{\iota\iota}$ give orbifold data that are related by T-compatible isomorphisms. To see this, take all constants ${}_a\lambda_{bc}$ in (8.17) (which correspond to a non-zero ${}_at_{bc}$) to be equal to one, except for ${}_\varphi\lambda_{\varphi\varphi}$. Using (8.19) to compute the effect on the f-coefficients shows that all but $f_{\iota\varphi}$ and $f_{\varphi\iota}$ are invariant, and that the latter two get multiplied by $({}_\varphi\lambda_{\varphi\varphi})^2$ and $({}_\varphi\lambda_{\varphi\varphi})^{-2}$, respectively. □

Let us denote the orbifold datum obtained in Theorem 8.7 by $\mathbb{A}_{h,\epsilon}$. By Proposition 5.22 (and Theorem 8.7), different choices for $\phi^2, f^{\varphi,\ \mathbb{1}}_{\varphi\varphi\varphi,\ \iota\varphi} \in \mathbb{C}^\times$ lead to equivalent ribbon fusion categories $(\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}$. Hence it makes sense not to include these choices into the notation for the orbifold datum $\mathbb{A}_{h,\epsilon}$.

Starting from the data in (8.33)–(8.35), by direct computation[11] from (8.23) and Proposition 8.6, and by Theorem 5.20 one obtains:

Proposition 8.8. The orbifold datum $\mathbb{A}_{h,\epsilon}$ is simple. The category $(\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}$ is a modular fusion category with 11 simple objects and has global dimension

$$\mathrm{Dim}\big((\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}\big) \ = \ 24\left(h^2 + h^{-2}\right)^{-2} \ . \tag{8.45}$$

Remark 8.9. The ansatz (8.33) was obtained by trying the to invert a condensation of a MFC (see Section 7.1). In particular, one takes the MFC $\mathcal{C}(sl(2), 10)$ of integrable highest weight representations of the affine Lie algebra $\widehat{sl}(2)_{10}$. It has 11 simple objects, denoted by the Dynkin labels $\underline{0}, \underline{1}, \ldots, \underline{10}$ and has a condensable algebra $A = \underline{0} \oplus \underline{6}$, the so-called E_6 algebra. The resulting condensation is known to be $\mathcal{C}(sp(4), 1)$ (see e.g. [DMS, Ch. 17.5] and [Os]), which is a category of Ising type (see [DMNO, Sec. 6.4]). The resulting ansatz (8.33) is inspired by some observations, suggesting that the Morita class of the algebra $A \otimes A$ is related to inverting condensations and the fact that for the E_6 algebra A, the algebras $A \otimes A$ and $A \oplus A$ are Morita equivalent, the latter being two copies of the tensor unit in the condensation $\mathcal{C}_A^{\mathrm{loc}}$. In the present example we "pretend" not to know this information, illustrating what one can feasibly achieve by exploring the category $\mathcal{C}_{\mathbb{A}}$ knowing only the orbifold datum $\mathbb{A}$.

8.4. Analysing $\mathcal{C}_{\mathbb{A}}$ via pipe functors

Let $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ be an orbifold datum in a modular fusion category $\mathcal{C}$. Finding the simple objects and tensor products of $\mathcal{C}_{\mathbb{A}}$ requires analysing the possible T-crossings, which can be complicated. In this section we will illustrate that one can obtain some of that information already from looking only at the underlying bimodules. In particular, we exploit the properties of the pipe functor $P\colon {}_A\mathcal{C}_A \to \mathcal{C}_{\mathbb{A}}$ defined in Section 5.4. Recall that it is biadjoint to the forgetful functor $U\colon {}_A\mathcal{C}_A \to \mathcal{C}_{\mathbb{A}}$.

For simple bimodules $\mu, \nu \in \mathrm{Irr}_{{}_A\mathcal{C}_A}$ we define the constants

$$X_{\mu\nu} = \dim \mathcal{C}_{\mathbb{A}}(P(\mu), P(\nu)) = \sum_{\Lambda \in \mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}}} \dim \mathcal{C}_{\mathbb{A}}(P(\mu), \Lambda) \dim \mathcal{C}_{\mathbb{A}}(\Lambda, P(\nu)) \ . \tag{8.46}$$

Applying the adjunction, this can be rewritten as (we do not write out the forgetful functor)

$$X_{\mu\nu} = \dim {}_A\mathcal{C}_A(P(\mu), \nu) = \sum_{\Lambda \in \mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}}} \dim {}_A\mathcal{C}_A(\mu, \Lambda) \dim {}_A\mathcal{C}_A(\Lambda, \nu) \ . \tag{8.47}$$

In particular, the first equality shows that $X_{\mu\nu}$ describes how the underlying bimodule of $P(\mu)$ decomposes into simple bimodules,

$$P(\mu) \overset{{}_A\mathcal{C}_A}{\cong} \bigoplus_{\nu \in \mathrm{Irr}_{{}_A\mathcal{C}_A}} X_{\mu\nu}\ \nu \ . \tag{8.48}$$

Every simple object of $\mathcal{C}_{\mathbb{A}}$ appears as direct summand of some $P(\mu)$ (use the unit or counit of the adjunction to see this). Thus one can attempt to find the simple objects of $\mathcal{C}_{\mathbb{A}}$ by using the above information about Hom spaces to decompose all the $P(\mu)$.

Under assumptions (A1)–(A3), the simple objects of ${}_A\mathcal{C}_A$ are of the form ${}_a x_b$, where $x \in I$ and $a, b \in B$, and the left and right actions of A are by the unitors of $\mathbb{1}_a$ and $\mathbb{1}_b$ respectively. By specialising (5.42) one gets (if $\mathbb{A}$ is defined using the Δ-separable setting and Euler completion, there are no ψ- (hence also ω-) insertions in (5.42)):

$$P({}_a x_b) = \bigoplus_{p,q,r,s,v \in B} {}_p t_{rv} \otimes {}_r t_{ua} \otimes x \otimes {}_s t^*_{ub} \otimes {}_q t^*_{sv} \ . \tag{8.49}$$

Let us now specialise these general considerations to the concrete example of the orbifold datum $\mathbb{A} = \mathbb{A}_{h,\epsilon}$ in $\mathcal{C} = \mathcal{I}_{h^3,\epsilon}$ as given in Section 8.3. Note that the precise values of h and ϵ are immaterial as we only need the fusion rules of $\mathcal{I}_{\zeta,\epsilon}$ and the expressions for A and T in (8.33). Applying the fusion rules of $\mathcal{I}_{\zeta,\epsilon}$ gives

$$
\begin{aligned}
X_{{}_a x_b,\ {}_p y_q} &= \dim {}_A\mathcal{C}_A\,(\ P({}_a x_b),\ {}_p y_q\ ) \\
&= \sum_{i,j,k\in I}\ \sum_{r,s,u,v\in B} N^i_{{}_p t_{rv},\ {}_r t_{ua}}\ N^j_{i,x}\ N^k_{j,\ {}_s t^*_{ub}}\ N^y_{k,\ {}_q t^*_{sv}}\ . \qquad (8.50)
\end{aligned}
$$

The simple A-A-bimodules are

$$
\underbrace{{}_\iota\mathbb{1}_\iota\ ,\ {}_\varphi\mathbb{1}_\varphi\ ,\ {}_\iota\varepsilon_\iota\ ,\ {}_\varphi\varepsilon_\varphi\ ,\ {}_\iota\sigma_\varphi\ ,\ {}_\varphi\sigma_\iota}_{\text{grade 0}}\quad ,\quad \underbrace{{}_\iota\mathbb{1}_\varphi\ ,\ {}_\varphi\mathbb{1}_\iota\ ,\ {}_\iota\varepsilon_\varphi\ ,\ {}_\varphi\varepsilon_\iota\ ,\ {}_\iota\sigma_\iota\ ,\ {}_\varphi\sigma_\varphi}_{\text{grade 1}}\ . \qquad (8.51)
$$

The grading indicated above is respected by the tensor product $\otimes_A$, and it turns out that the matrix X is block-diagonal with respect to this grading.[12] Indeed, a straightforward computation shows that, in the above ordering of the simple bimodules,

$$
\big(X_{{}_a x_b,\ {}_p y_q}\big) = \begin{pmatrix} 2 & 3 & 0 & 1 & 1 & 1 \\ 3 & 8 & 1 & 6 & 5 & 5 \\ 0 & 1 & 2 & 3 & 1 & 1 \\ 1 & 6 & 3 & 8 & 5 & 5 \\ 1 & 5 & 1 & 5 & 4 & 4 \\ 1 & 5 & 1 & 5 & 4 & 4 \end{pmatrix} \oplus \begin{pmatrix} 3 & 3 & 1 & 1 & 1 & 5 \\ 3 & 3 & 1 & 1 & 1 & 5 \\ 1 & 1 & 3 & 3 & 1 & 5 \\ 1 & 1 & 3 & 3 & 1 & 5 \\ 1 & 1 & 1 & 1 & 2 & 4 \\ 5 & 5 & 5 & 5 & 4 & 14 \end{pmatrix}\ . \qquad (8.52)
$$

Let us first focus on the grade-0 block of X. We already know a simple object of $\mathcal{C}_{\mathbb{A}}$, namely the tensor unit $A = {}_\iota\mathbb{1}_\iota \oplus {}_\varphi\mathbb{1}_\varphi$. Hence, we may as well decompose each $P(\mu)$ as

$$
P(\mu) \cong \underbrace{\mathcal{C}_{\mathbb{A}}(P(\mu), A)}_{={}_A\mathcal{C}_A(\mu,A)}\, A \oplus P^{(1)}(\mu)\ , \qquad (8.53)
$$

where now $P^{(1)}(\mu)$ does no longer contain A as a direct summand. Equivalently,

$$
P^{(1)}(\mu) \cong \bigoplus_{\Lambda\in\mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}},\Lambda\neq A} {}_A\mathcal{C}_A(\mu,\Lambda)\,\Lambda\ . \qquad (8.54)
$$

In terms of the $P^{(1)}(\mu)$ we can define a new matrix $X^{(1)}_{\mu\nu} = \dim \mathcal{C}_{\mathbb{A}}(P^{(1)}(\mu), P^{(1)}(\nu))$, which can be written as

$$
\begin{aligned}
X^{(1)}_{\mu\nu} &= \sum_{\Lambda\in\mathrm{Irr}_{\mathcal{C}_{\mathbb{A}}},\Lambda\neq A} \dim {}_A\mathcal{C}_A(\mu,\Lambda)\dim {}_A\mathcal{C}_A(\Lambda,\nu) \\
&= X_{\mu\nu} - \dim {}_A\mathcal{C}_A(\mu,A)\dim {}_A\mathcal{C}_A(A,\nu)\ . \qquad (8.55)
\end{aligned}
$$

[12] Actually, ${}_A\mathcal{C}_A$ is graded by $\mathbb{Z}_2\times\mathbb{Z}_2$, but only the indicated $\mathbb{Z}_2$ is respected by X.

Explicitly, the grade-0 summand $X^{(1)}\big|_{\text{grade }0}$ is given by

$$\begin{pmatrix} 2 & 3 & 0 & 1 & 1 & 1 \\ 3 & 8 & 1 & 6 & 5 & 5 \\ 0 & 1 & 2 & 3 & 1 & 1 \\ 1 & 6 & 3 & 8 & 5 & 5 \\ 1 & 5 & 1 & 5 & 4 & 4 \\ 1 & 5 & 1 & 5 & 4 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 1 \\ 2 & 7 & 1 & 6 & 5 & 5 \\ 0 & 1 & 2 & 3 & 1 & 1 \\ 1 & 6 & 3 & 8 & 5 & 5 \\ 1 & 5 & 1 & 5 & 4 & 4 \\ 1 & 5 & 1 & 5 & 4 & 4 \end{pmatrix} . \tag{8.56}$$

The first diagonal entry reads $\dim \mathcal{C}_{\mathbb{A}}(P^{(1)}({}_\iota\mathbb{1}_\iota), P^{(1)}({}_\iota\mathbb{1}_\iota)) = 1$, which means that $P^{(1)}({}_\iota\mathbb{1}_\iota) =: \Delta$ is itself a simple object of $\mathcal{C}_{\mathbb{A}}$.

Note that we can write $X^{(1)}_{\mu\nu} = \dim \mathcal{C}_{\mathbb{A}}(P^{(1)}(\mu), P(\nu)) = \dim {}_A\mathcal{C}_A(P^{(1)}(\mu), \nu)$, and so can still read off the decomposition into simple bimodules form the first row of the above matrix:

$$\Delta = {}_\iota\mathbb{1}_\iota \;\oplus\; {}_\varphi\mathbb{1}_\varphi^{\oplus 2} \;\oplus\; {}_\varphi\varepsilon_\varphi \;\oplus\; {}_\iota\sigma_\varphi \;\oplus\; {}_\varphi\sigma_\iota \;. \tag{8.57}$$

procedure, we now define $P^{(2)}$ and $X^{(2)}$ by excluding the simple objects A and Δ from the sum. One finds that $X^{(2)}\big|_{\text{grade }0}$ is given by

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 1 \\ 2 & 7 & 1 & 6 & 5 & 5 \\ 0 & 1 & 2 & 3 & 1 & 1 \\ 1 & 6 & 3 & 8 & 5 & 5 \\ 1 & 5 & 1 & 5 & 4 & 4 \\ 1 & 5 & 1 & 5 & 4 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 1 \\ 2 & 4 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 4 & 3 & 3 \\ 0 & 1 & 2 & 3 & 1 & 1 \\ 0 & 4 & 3 & 7 & 4 & 4 \\ 0 & 3 & 1 & 4 & 3 & 3 \\ 0 & 3 & 1 & 4 & 3 & 3 \end{pmatrix} . \tag{8.58}$$

The third diagonal entry shows that $P^{(2)}({}_\iota\varepsilon_\iota)$ is a direct sum of two non-isomorphic simple objects in $\mathcal{C}_{\mathbb{A}}$, which we denote by E_1 and E_2. The corresponding row again gives the decomposition into bimodules as

$$E_1 \oplus E_2 = {}_\varphi\mathbb{1}_\varphi \;\oplus\; {}_\iota\varepsilon_\iota^{\oplus 2} \;\oplus\; {}_\varphi\varepsilon_\varphi^{\oplus 3} \;\oplus\; {}_\iota\sigma_\varphi \;\oplus\; {}_\varphi\sigma_\iota \;. \tag{8.59}$$

From (8.47) – applied to $P^{(2)}$ – one obtains a constraint on how to distribute the bimodules between E_1 and E_2. Namely, for each μ we have

$$X^{(2)}_{\mu\mu} \;\geq\; \big(\dim {}_A\mathcal{C}_A(\mu, E_1)\big)^2 + \big(\dim {}_A\mathcal{C}_A(\mu, E_2)\big)^2 \;. \tag{8.60}$$

It follows that each E_i must contribute one copy of ${}_\iota\varepsilon_\iota$, and that no one of the E_i can contain all three copies of ${}_\varphi\varepsilon_\varphi$. If we denote the summand that contains ${}_\varphi\mathbb{1}_\varphi$ by E_1, the remaining possibilities are

$$\begin{aligned} E_1 &= {}_\varphi\mathbb{1}_\varphi \;\oplus\; {}_\iota\varepsilon_\iota \;\oplus\; {}_\varphi\varepsilon_\varphi^{\oplus(1+u)} \;\oplus\; {}_\iota\sigma_\varphi^{\oplus x} \;\oplus\; {}_\varphi\sigma_\iota^{\oplus y} \\ E_2 &= {}_\iota\varepsilon_\iota \;\oplus\; {}_\varphi\varepsilon_\varphi^{\oplus(2-u)} \;\oplus\; {}_\iota\sigma_\varphi^{\oplus(1-x)} \;\oplus\; {}_\varphi\sigma_\iota^{\oplus(1-y)} \end{aligned} \quad , \quad u, x, y = 0, 1 \;. \tag{8.61}$$

If we denote by $X^{(3)}$ the matrix obtained by the sum in (8.47) with simple objects A, Δ, E_1, E_2 omitted, we find

$$X^{(3)}\big|_{\text{grade } 0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3-u & 3-x & 3-y \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3-u & 0 & 2 & 1+\delta_{ux} & 1+\delta_{uy} \\ 0 & 3-x & 0 & 1+\delta_{ux} & 2 & 1+\delta_{xy} \\ 0 & 3-y & 0 & 1+\delta_{uy} & 1+\delta_{xy} & 2 \end{pmatrix} . \tag{8.62}$$

From the second row we deduce that there are two further non-isomorphic simple objects Φ_1, Φ_2, such that

$$\Phi_1 \oplus \Phi_2 = {}_\varphi\mathbb{1}_\varphi^{\oplus 2} \ \oplus \ {}_\varphi\varepsilon_\varphi^{\oplus(3-u)} \ \oplus \ {}_\iota\sigma_\varphi^{\oplus(3-x)} \ \oplus \ {}_\varphi\sigma_\iota^{\oplus(3-y)} . \tag{8.63}$$

The only way to satisfy the bound given by the diagonal entries is to have $u = x = y = 1$ and to distribute the bimodule direct summands equally between Φ_1 and Φ_2. Thus $\Phi_1 = \Phi_2$ as bimodules (but not as objects in $\mathcal{C}_\mathbb{A}$).

At this point we have found the bimodule part of all simple objects of grade 0. One can repeat the procedure to obtain those of grade 1 as well. In this case two solutions are possible: one yielding 6 simple objects, and the other 5. From Proposition 8.8 we already know that there are 11 simple objects in total, which eliminates the first solution.

In order to study the tensor products of these 11 simple objects of $\mathcal{C}_\mathbb{A}$ via their underlying bimodules, it is helpful to use a matrix notation for the direct sum decomposition:

$$M = {}_\iota W_\iota \oplus {}_\iota X_\varphi \oplus {}_\varphi Y_\iota \oplus {}_\varphi Z_\varphi \quad \rightsquigarrow \quad M = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} . \tag{8.64}$$

The tensor product $\otimes_A$ is then given by matrix multiplication. In this notation, the decomposition of the simple objects in $\mathcal{C}_\mathbb{A}$ into simple bimodules is[13]

$$\begin{array}{cc} \text{grade } 0 & \text{grade } 1 \\ A = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} & S_1 = \begin{pmatrix} \sigma & \mathbb{1} \\ \mathbb{1} & 2\sigma \end{pmatrix} \\ \Delta = \begin{pmatrix} \mathbb{1} & \sigma \\ \sigma & 2\mathbb{1}+\varepsilon \end{pmatrix} & S_2 = \begin{pmatrix} \sigma & \varepsilon \\ \varepsilon & 2\sigma \end{pmatrix} \\ E_1 = \begin{pmatrix} \varepsilon & \sigma \\ \sigma & \mathbb{1}+2\varepsilon \end{pmatrix} & \Psi_1 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & \sigma \end{pmatrix} \end{array} \tag{8.65}$$

[13]Here we write multiplicities as multiplying by integers and "+" instead of "$\oplus$" for better readability, that is, we work with entries in the Grothendieck ring of $\mathcal{C}$.

dim	1	1.93..	2.73..	3.34..	3.73..	3.86..
$\mathcal{C}(sl(2),10)$	$\underline{0}$, $\underline{10}$	$\underline{1}$, $\underline{9}$	$\underline{2}$, $\underline{8}$	$\underline{3}$, $\underline{7}$	$\underline{4}$, $\underline{6}$	$\underline{5}$
$(\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}$	A, E_2	Ψ_1, L	Φ_1, Φ_2	S_1, S_2	Δ, E_1	Ψ_2

Table 8.2: Simple objects of $\mathcal{C}(sl(2),10)$ and of $(\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}$ for $h=\exp(\pi i\frac{19}{24})$, sorted by quantum dimension. For $\mathcal{C}(sl(2),10)$ the Dynkin label $\underline{0},\underline{1},\dots,\underline{10}$ is used to denote the simple objects, and for $(\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}$ the notation in (8.65) is used.

$$E_2=\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix} \qquad \Psi_2=\begin{pmatrix}0 & \mathbb{1}+\varepsilon\\ \mathbb{1}+\varepsilon & 2\sigma\end{pmatrix}$$
$$\Phi_1=\Phi_2=\begin{pmatrix}0 & \sigma\\ \sigma & \mathbb{1}+\varepsilon\end{pmatrix} \qquad L=\begin{pmatrix}0 & \varepsilon\\ \varepsilon & \sigma\end{pmatrix}$$

Thus, in this example all simple objects of $\mathcal{C}_{\mathbb{A}}$ except for Φ_1, Φ_2 are uniquely characterised by their underlying bimodule. Since taking duals is compatible with the underlying bimodule, in particular all simple objects except for possibly $\Phi_{1/2}$ are self-dual.

The underlying bimodules determine the quantum dimensions of objects in $\mathcal{C}_{\mathbb{A}}$ (in this case using (C.2),(C.3), since the orbifold datum $\mathbb{A}=\mathbb{A}_{h,\epsilon}$ is given in terms of Δ-separable algebras and Euler completion). Namely, if $M\in\mathcal{C}_{\mathbb{A}}$ and $M=\sum_{a,b\in B}{}_aM_b$ as an A-A-bimodule with ${}_aM_b\in\mathcal{C}$, then, for $a\in B$,

$$\dim_{\mathcal{C}_{\mathbb{A}}}(M)=\sum_{b\in B}\frac{\psi_b^2}{\psi_a^2}\dim_{\mathcal{C}}({}_aM_b)\ . \tag{8.66}$$

In particular, it follows that the above expression is independent of the choice of $a\in B$ (for $\mathbb{A}$ simple), which in itself is a non-trivial condition if one tries to understand which bimodules can appear in objects of $\mathcal{C}_{\mathbb{A}}$.

We can therefore use the expressions in (8.65) to compute the quantum dimensions of the 11 simple objects. We have done that for $h=\exp(\pi i\frac{19}{24})$ and recovered the quantum dimensions of $\mathcal{C}(sl(2),10)$, see Table 8.2 and Remark 8.9.

Direct sum decompositions in $\mathcal{C}_{\mathbb{A}}$ cannot be uniquely identified by the underlying bimodules, not even up to the ambiguity of Φ_1 vs. Φ_2. For example,

$$\Psi_1\otimes_A\Psi_1\cong\begin{pmatrix}\mathbb{1} & \sigma\\ \sigma & 2\mathbb{1}+\varepsilon\end{pmatrix}\ . \tag{8.67}$$

In this case, the right hand side could be the underlying bimodule of Δ or of $A\oplus\Phi_{1/2}$. (However, since Ψ_1 is self-dual, its tensor square in $\mathcal{C}_{\mathbb{A}}$ has to contain the

tensor unit A of $\mathcal{C}_{\mathbb{A}}$ as a direct summand, and so the second decomposition is the correct one in $\mathcal{C}_{\mathbb{A}}$.) We verified that – taking Ψ_1 has the generator – the iterated tensor products of Ψ_1 are compatible with those of $\mathcal{C}(sl(2),10)$.

The quantum dimensions of simple objects are also useful in showing the following result.

Proposition 8.10. The 32 modular fusion categories $(\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}$ for the 16 possible values of h and $\epsilon \in \{\pm 1\}$ are pairwise non-equivalent as $\mathbb{C}$-linear ribbon categories.

Proof. A ribbon equivalence preserves the global dimension and the anomaly. It is shown in Lemma 6.17 that the MFCs $\mathcal{C}$ and $\mathcal{C}_{\mathbb{A}}$ have equal anomalies. Hence the anomaly δ of $(\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}$ is equal to that of $\mathcal{I}_{h^3,\epsilon}$ as stated in (8.32).

A ribbon equivalence also preserves the quantum dimension of simple objects, and from this one can verify that any equivalence must map Ψ_1 for one choice of (h,ϵ) to either Ψ_1 or L for any other choice (h',ϵ').

Abbreviating $\mathcal{D} = (\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}$, altogether we see that the triple of numbers

$$\big(\, \mathrm{Dim}(\mathcal{D})\,,\, \delta_{\mathcal{D}}\,,\, \dim_{\mathcal{D}}(\Psi_1)\,\big) \tag{8.68}$$

is a ribbon invariant. Note that $\dim_{\mathcal{D}}(\Psi_1) = \dim_{\mathcal{D}}(L)$, so it does not matter whether we use Ψ_1 or L. From (8.45), (8.32) and (8.35) we read off the explicit values to be

$$\big(\, 24\left(h^2+h^{-2}\right)^{-2}\,,\, \epsilon h^{-3}\,,\, -\epsilon(h^{10}+h^{-10})\,\big)\ . \tag{8.69}$$

It is straightforward to check that this distinguishes all 32 possibilities. □

Remark 8.11. The above computation does not prove that one of the MFCs $(\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}$ is equivalent to $\mathcal{C}(sl(2),10)$, it merely serves to illustrate how orbifold data can be used to construct new MFCs and to indirectly get some of their properties. As mentioned in Remark 8.9, the ansatz (8.33) was obtained from an heuristic reasoning why it should invert the E_6-algebra condensation in $\mathcal{C}(sl(2),10)$. Together with the outcome of the above computations for the number of simple objects and their quantum dimensions, one can conjecture that for different values of h, the MFCs $(\mathcal{I}_{h^3,\epsilon})_{\mathbb{A}_{h,\epsilon}}$ constitute $\mathcal{C}(sl(2),10)$ and its Galois conjugates.

9. Domain walls

In Chapter 4 we reviewed the construction of the Reshetikhin-Turaev defect TQFT $Z^{\mathrm{def}}_{\mathcal{C}}$, obtained from a modular fusion category $\mathcal{C}$. By definition, the label set $D^{\mathcal{C}}_3$ for bulk phases consisted only of a single element, namely that describing the Reshetikhin-Turaev TQFT obtained from $\mathcal{C}$. In this chapter we provide a generalisation of $Z^{\mathrm{def}}_{\mathcal{C}}$, which includes different labels for bulk phases, in particular they are labelled by the condensable algebras in $\mathcal{C}$. Equivalently the bulk phases can be thought of as labelled by the instances of orbifold data discussed in Section 7.1, or by the Reshetikhin-Turaev theories, obtained from condensations $\mathcal{C}^{\mathrm{loc}}_A$ for condensable algebras $A \in \mathcal{C}$. In addition, we investigate the bicategory of domain walls between two bulk phases and compare it to the one obtained from the model-independent analysis in [FSV], which we review in Section 9.2. We find that the two bicategories agree.

The material in this chapter is based on [KMRS].

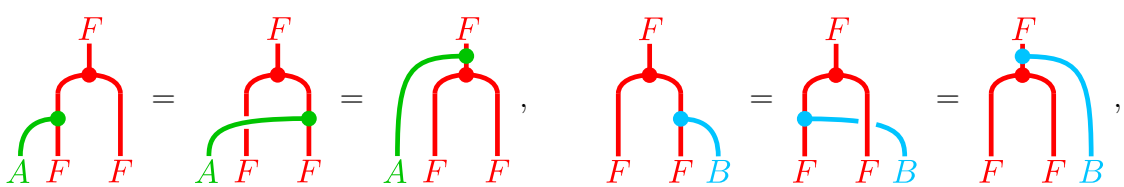
9.1. Bulk phases from condensable algebras

Throughout this section, let $\mathcal{C}$ be a MFC. Below we describe a generalisation[14]

$$Z^{\mathrm{def}}_{\mathcal{C},w} \colon \widehat{\mathrm{Bord}}^{\mathrm{def}}_3(\mathbb{D}^{\mathcal{C},w}) \to \mathrm{Vect}_{\Bbbk} \tag{9.1}$$

of the defect TQFT $Z^{\mathrm{def}}_{\mathcal{C}}$, which extends the label sets of the defect datum introduced in Section 4.2, i.e. one has $D^{\mathcal{C}}_i \subseteq D^{\mathcal{C},w}_i$, for $i = 0, 1, 2, 3$. We start with some definitions needed to define the elements of the sets of $\mathbb{D}^{\mathcal{C},w}$. Recall Definition 7.1 of a condensable algebra in $\mathcal{C}$.

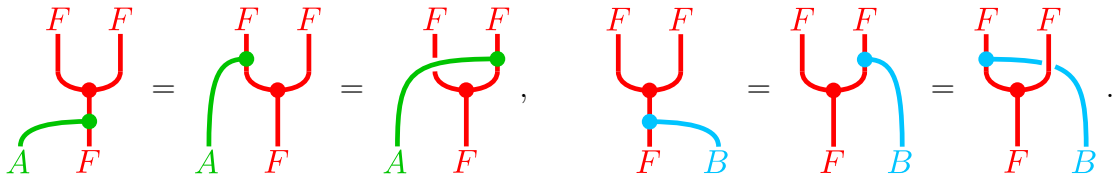
Definition 9.1. Let $\mathcal{C}$ be a MFC and $A, B \in \mathcal{C}$ two condensable algebras. A (*symmetric, separable*) *Frobenius algebra over* (A, B) is a (symmetric, separable) Frobenius algebra $F \in \mathcal{C}$, which is simultaneously an A-B-bimodule, such that



Note that $A \otimes B$ is an example of such an algebra.

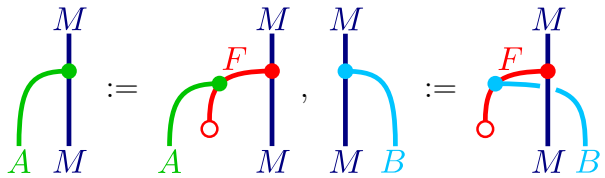
[14] The symbol w in the notations $Z^{\mathrm{def}}_{\mathcal{C},w}$ and $\mathbb{D}^{\mathcal{C},w}$ is there to emphasise that the defect TQFT admits domain ***w***alls between different bulk theories.

Remark 9.2. i) Because of the Frobenius property (2.30), the compatibility conditions of A- and B-actions with the product of F in the above definition imply similar identities with the coproduct, i.e. one has

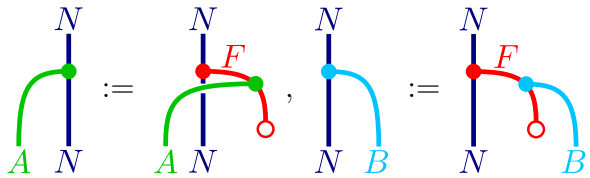


ii) In what follows, symmetric separable Frobenius algebras $F \in \mathcal{C}$ over (A, B) will be used to label domain walls separating two bulk theories labelled by A and B. The crossings are chosen in such a way that the A-phase is "above" the defect, while the B-phase is "below", assuming that the defect itself is oriented towards the A-phase.

Let A, B, F be as in the Definition 9.1 above. A left F-module $M \in \mathcal{C}$ and a right F-module $N \in \mathcal{C}$ are automatically A-B-bimodules with actions

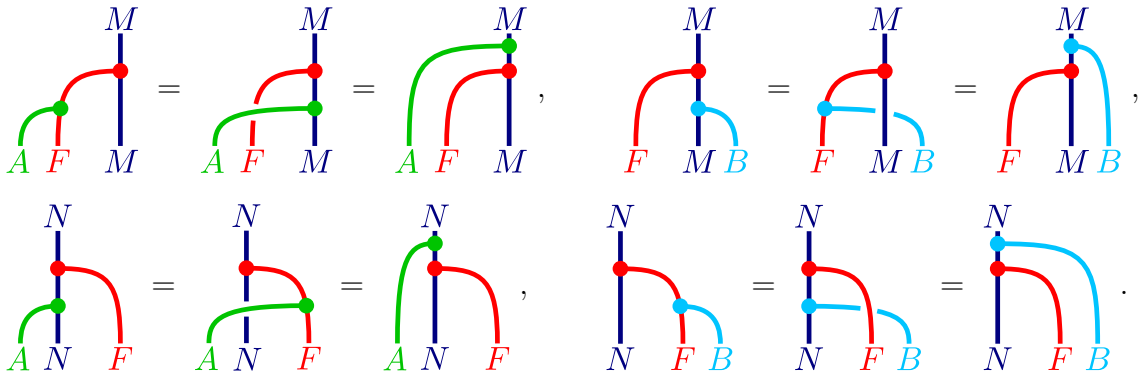


(9.2)



(9.3)

It is easy to check that these are the unique A- and B- actions on M and N such that the following identities hold:



(9.4)

A morphism of (left, right) modules M, N is a morphism $f\colon M \to N$, commuting with the F- (and therefore also A-, B-) actions. Because of the above argument, there is no need to introduce "left or right modules over (A, B)". However, the situation is different for bimodules, where one obtains two a priori different A-B-bimodule structures. Since the application to line defects requires all of the identities in (9.4) to hold simultaneously for a bimodule $M = N$, we define:

Definition 9.3. Let $F_1, F_2 \in \mathcal{C}$ be algebras over (A, B). An F_1-F_2-*bimodule over* (A, B) is an F_1-F_2-bimodule M such that the left A-actions and right B-actions induced via (9.2) and (9.3) by F_1 and F_2 respectively, are equal.

Like in Section 4.2, we introduce the notion of a multimodule of the algebras as in Definition 9.1, which will be used to label line defects. To this end, let $A_0, A_1, \ldots, A_n$ and $B_0, B_1, \ldots, B_m$ be a condensable algebras in $\mathcal{C}$ with $A_0 = B_0$, $A_n = B_m$ and let F_i, $i = 1, \ldots, n$ and G_k, $k = 1, \ldots, m$ be symmetric separable Frobenius algebras over (A_i, A_{i-1}) and (B_k, B_{k-1}) respectively. By an $F_1 \cdots F_n$-$G_1 \cdots G_m$-*multimodule* we will mean a multimodule M of the underlying Frobenius algebras (meaning the actions of F_i and G_k commute as in (4.8)) such that

i) for $i = 1, \ldots, n-1$, (resp. $k = 1, \ldots, m-1$) the two A_i-actions obtained from F_i and F_{i+1} (resp. B_k-actions obtained from G_k and G_{k+1}) by analogy to (9.2) and (9.3) are equal;

ii) for $i = k = 0$ (resp. $i = n$, $k = m$), the two $A_i = B_k$ actions obtained from F_0 and G_0 (resp. from F_n and G_m) are equal.

One can also talk about relative tensor products of such multimodules

The defect datum $\mathbb{D}^{\mathcal{C},w}$ can now be defined as follows (cf. Section 4.2):

- $D_3^{\mathcal{C},w}$: 3-strata are labelled by condensable algebras in $\mathcal{C}$.
- $D_2^{\mathcal{C},w}$: 2-strata separating two 3-strata labelled by $A, B \in D_3^{\mathcal{C},w}$ and oriented towards the one labelled by A are labelled by symmetric separable Frobenius algebras over (A, B).
- $D_1^{\mathcal{C},w}$: a framed 1-stratum that has no adjacent 2-strata is labelled by a local module $M \in C_A^{\mathrm{loc}}$ (see (7.2)), where $A \in D_3^{\mathcal{C},w}$ labels the adjacent 3-stratum.

  Suppose a 1-stratum l has $n + m > 0$ adjacent 2-strata. We require l to have a neighbourhood isomorphic to the one shown in Figure 9.1 with $F_1, \ldots F_n, G_1, \ldots, G_m \in D_2^{\mathcal{C},w}$ labelling the adjacent 2-strata and $A_0, \ldots A_n$, $B_0, \ldots, B_m \in D_3^{\mathcal{C},w}$, $A_0 = B_0$, $A_n = B_m$ labelling the adjacent 3-strata. We then label l with an $F_1 \cdots F_n$-$G_1 \cdots G_m$-multimodule $M \in \mathcal{C}$.

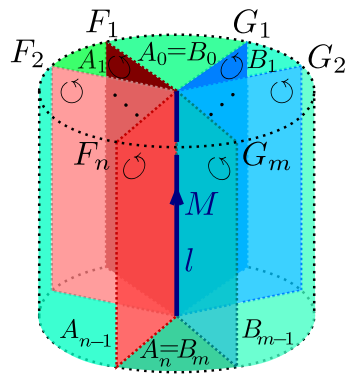


Figure 9.1: Neighbourhood of a line defect l with several adjacent 2-strata.

- $D_0^{\mathcal{C},w}$: point insertions on an $F \in D_2^{\mathcal{C},w}$ labelled 2-stratum are labelled by F-F-bimodule morphisms, while 0-strata that have adjacent 1-strata are labelled by the morphisms of (relative tensor products of) multimodules.

We do not state the explicit definition of the defect TQFT $Z^{\mathrm{def}}_{\mathcal{C},w}$, but rather remark that it is defined in terms of $Z^{\mathrm{def}}_{\mathcal{C}}$ in a very similar way the orbifold graph TQFT $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}$ was defined in terms of $Z^{\mathrm{def}}_{\mathcal{C}}$ in Chapter 6. For example, let $M \in \widehat{\mathrm{Bord}}^{\mathrm{def}}_3(\mathbb{D}^{\mathcal{C},w})$ be a closed $\mathbb{D}^{\mathcal{C},w}$-coloured defect 3-manifold, which for simplicity is assumed to not have 0- or 1-strata. The steps to compute the invariant $Z^{\mathrm{def}}_{\mathcal{C},w}(M)$ are then the following:

- Let s be a 2-stratum of M, labelled by a symmetric separable Frobenius algebra $F \in D_2^{\mathcal{C},w}$ over (A, B), where $A, B \in D_3^{\mathcal{C},w}$. Pick an F-coloured admissible 1-skeleton t for s. The lines and points of t can then be thought of as defects in $Z^{\mathrm{def}}_{\mathcal{C}}$, having adjacent A and B labelled lines and surfaces (see Figure 9.2a and Remark 7.5). Do this for all 2-strata of M.

- Let u and v be the A- and B-labelled 3-strata adjacent to s. Since s can be thought of as a "boundary component" of both u and v, one can extend t (as well the 1-skeleta of the other 2-strata adjacent to u and v) to admissible 2-skeleta T_u, T_v of u, v (see Lemma 6.2), which can in turn be coloured by the orbifold data corresponding to A and B discussed in Section 7.1. Doing this for all 3-strata results in the analogue of the foamification procedure of M, producing a closed $\mathbb{D}^{\mathcal{C}}$-coloured defect 3-manifold M_{f}.

- One defines:
$$Z^{\mathrm{def}}_{\mathcal{C},w}(M) := Z^{\mathrm{def}}_{\mathcal{C}}(M_{\mathrm{f}})\,. \tag{9.5}$$
The independence on the choices of the 1- and 2-skeleta for the 2- and 3-strata of M can be shown as follows: In the interior of the 3-strata one can apply the

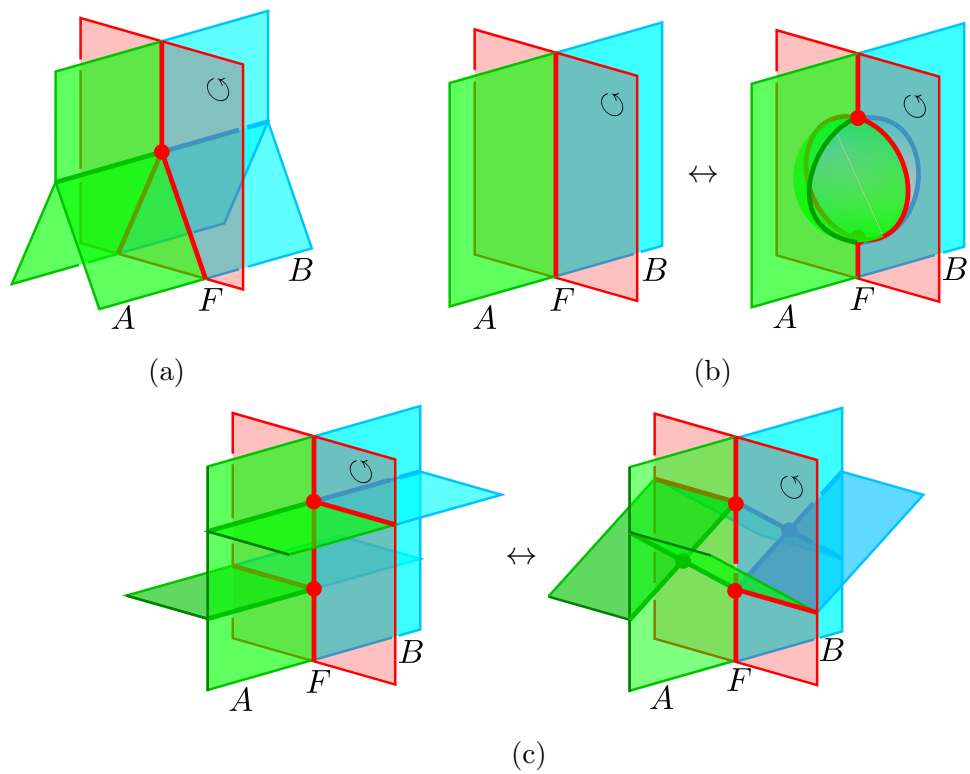


Figure 9.2: (a) The multiplication morphism of an algebra F over (A, B) as point defect in $Z_{\mathcal{C}}^{\text{def}}$. F can be used to label both a 2-stratum and 1-strata with four adjacent 2-strata, labelled by A, F, B, F. (b) The separability property of F implements the b-move at the boundary of A- and B-labelled 2-skeleta. (c) The Frobenius property of F implements the l-move.

BLT moves (see Figure 6.2, Lemma 6.2) upon evaluating with $Z_{\mathcal{C}}^{\text{def}}$. On the 2-strata, for example s, the properties of the symmetric separable Frobenius algebra F allow one to change the 1-skeleton into any other 1-skeleton using the bl moves, which, because of the A and B actions on F, also change the 2-skeleta of u and v (see Figures 9.2b and 9.2c).

With some effort the evaluation procedure above can be generalised to arbitrary stratifications of M. In particular, the use of local modules in $D_1^{\mathcal{C},w}$ to label the 1-strata in the bulk theories allows one to include such lines in the construction by using the admissible ribbon diagrams instead of 2-skeleta. The bordisms with non-empty boundary can be handled by taking the limit over 1-skeleta on the boundary as it was done in Construction 6.7.

Remark 9.4. i) The defect TQFT $Z_{\mathcal{C},w}^{\text{def}}$ has many of the properties of $Z_{\mathcal{C}}^{\text{def}}$. For example it is also D_0-complete and Euler complete with respect to surfaces (see

Properties 4.13, 4.14). If $F \in D_2^{\mathcal{C},w}$ is a Frobenius algebra over (A, B), its opposite F^{op} is a Frobenius algebra over (B, A) and can be used to reverse the orientation of an F-labelled 2-stratum.

ii) $Z^{\mathrm{def}}_{\mathcal{C},w}$ is indeed a generalisation of $Z^{\mathrm{def}}_{\mathcal{C}}$ as restricting $\mathbb{D}_3^{\mathcal{C},w}$ to the trivial condensable algebra $\mathbb{1}_{\mathcal{C}}$ yields $Z^{\mathrm{def}}_{\mathcal{C}}$ by definition. The bulk theory for a general condensable algebra $A \in D_3^{\mathcal{C},2}$ is isomorphic to $Z^{\mathrm{RT}}_{\mathcal{C}^{\mathrm{loc}}_A}$, the Reshetikhin-Turaev theory obtained from the condensation $\mathcal{C}^{\mathrm{loc}}_A$. Indeed, without having 2-strata, the definition of $Z^{\mathrm{def}}_{\mathcal{C},w}$ simplifies to that of the orbifold graph TQFT $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}$, with the orbifold datum $\mathbb{A}$ obtained from the condensable algebra A as in Section 7.1. The claim then follows from Theorems 7.3 and 6.13. The defect TQFT $Z^{\mathrm{def}}_{\mathcal{C},w}$ is therefore able to handle domain walls, between the Reshetikhin-Turaev theories labelled by MFCs $\mathcal{C}^{\mathrm{loc}}_A$ and $\mathcal{C}^{\mathrm{loc}}_B$ for two condensable algebras $A, B \in \mathcal{C}$.

Let us apply the construction of the bicategory of surface defects to the defect TQFT $Z^{\mathrm{def}}_{\mathcal{C},w}$ mentioned in Section 4.1. Assuming that the adjacent bulk theories are labelled with algebras $A, B \in D_3^{\mathcal{C},w}$, it readily yields the bicategory in the following

Definition 9.5. Let $A, B \in \mathcal{C}$ be condensable algebras. We denote by $\mathrm{FrobAlg}^{\mathrm{ssep}}_{\mathcal{C},A,B}$ the bicategory

- having symmetric separable Frobenius algebras over (A, B) in $\mathcal{C}$ as objects,
- $\mathrm{FrobAlg}^{\mathrm{ssep}}_{\mathcal{C},A,B}(F_1, F_2)$ being the category of F_1-F_2-bimodules over (A, B) and their morphisms,
- the composition of two bimodules ${}_{F_1}M_{F_2}$ and ${}_{F_2}M_{F_3}$ being the tensor product $M \otimes_{F_2} N$ over the respective algebra,
- for each object F, the unit being F seen as a bimodule over itself.

9.2. Domain walls between Reshetikhin-Turaev theories

In the remainder of this chapter we prefer to sometimes use the term *Wilson line* to refer to a (framed) 1-stratum in a bulk and *defect Wilson line* to refer to a 1-stratum within a surface defect. This terminology is used in [FSV], which we summarise in this section and rely upon in later sections. For the rest of the chapter we will assume that the characteristic of the ambient field $\Bbbk$ is 0[15].

[15] This is mostly to use the results of [DMNO] on Witt equivalence of MFCs, and those of [Schm] on module traces. Both references assume $\mathrm{char}\,\Bbbk = 0$, and even though the generalisations to arbitrary fields seem possible, one might expect some nuances to arise.

Recall from Section 4.1 that surface defects between two bulk theories in a 3-dimensional defect TQFT can be collected into a bicategory which has such surface defects as objects, Wilson lines separating two surface defects as 1-morphisms and point insertions as 2-morphisms. The bicategory of surface defects between two bulk theories of Reshetikhin-Turaev type was studied from this point of view in [FSV]. The surface defects in question were only considered locally, i.e. not as part of a stratification of a compact manifold and without providing a construction of a defect TQFT. Indeed, one can define a natural bicategory of surface defects separating two bulk theories labelled by MFCs $\mathcal{C}$ and $\mathcal{D}$, even before an exact construction of a complete defect TQFT is known. Concretely, the algebraic description given in [FSV] is: Let $\mathcal{W}$ be a pivotal fusion category such that there is a braided equivalence

$$F\colon \mathcal{C} \boxtimes \widetilde{\mathcal{D}} \xrightarrow{\sim} \mathcal{Z}(\mathcal{W})\,. \tag{9.6}$$

Then the bicategory of surface defects is $\mathcal{W}$-Mod, the bicategory of $\mathcal{W}$-module categories, module functors and natural transformations, see Section 2.7. A defect between the two bulk theories can therefore only exist if the modular categories $\mathcal{C}$ and $\mathcal{D}$ describing them are Witt equivalent (see Definition 9.7 below).

Let us review the argument of [FSV]. Having a surface defect s, the labels for defect Wilson lines within it form a category $\mathcal{W}$. The topological nature of Wilson lines implies that $\mathcal{W}$ is monoidal and pivotal; we also assume it to be fusion. Having an $X \in \mathcal{C}$ labelled Wilson line in the bulk, one can bring it next to s so that it becomes a defect Wilson line $F_{\rightarrow}(X) \in \mathcal{W}$. Since it is merely "hovering" next to s, it can cross to the other side of any defect Wilson line $W \in \mathcal{W}$, i.e. one has a family of morphisms $F_{\rightarrow}(X) \otimes W \to W \otimes F_{\rightarrow}(X)$, which assemble into a half-braiding for $F_{\rightarrow}(X)$. This implies the existence of a functor of braided categories $F_{\rightarrow}\colon \mathcal{C} \to \mathcal{Z}(\mathcal{W})$. An analogous argument then gives a functor $F_{\leftarrow}\colon \widetilde{\mathcal{D}} \to \mathcal{Z}(\mathcal{W})$. For $X \in \mathcal{C}$, $Y \in \mathcal{D}$ we then define the functor in (9.6) by $F(X \otimes Y) := F_{\rightarrow}(X) \otimes F_{\leftarrow}(Y)$ and assume it to be an equivalence.

One can then consider two parallel defect Wilson lines. One of them has defect condition s to both sides. According to the preceding discussion, it is labelled by an object $W \in \mathcal{W}$. The other defect Wilson line separates defect conditions s and s' and is labelled by an object W' of a category $\mathcal{W}_{s'}$. Fusing the two Wilson lines yields a new Wilson line which must be labelled by an object $W \triangleright W' \in \mathcal{W}_{s'}$. Since this should also be compatible with point insertions on the Wilson lines, we get an action $\mathcal{W} \times \mathcal{W}_{s'} \to \mathcal{W}_{s'}$, so that $\mathcal{W}_{s'}$ gets the structure of a $\mathcal{W}$-module category. Thus, the bicategory of defect conditions is equivalent to the bicategory of $\mathcal{W}$-module categories.

Let s and s' be two surface defects as in the setting above with the corresponding categories $\mathcal{W}$ and $\mathcal{V}$ of surface Wilson lines. Note that by the argument above

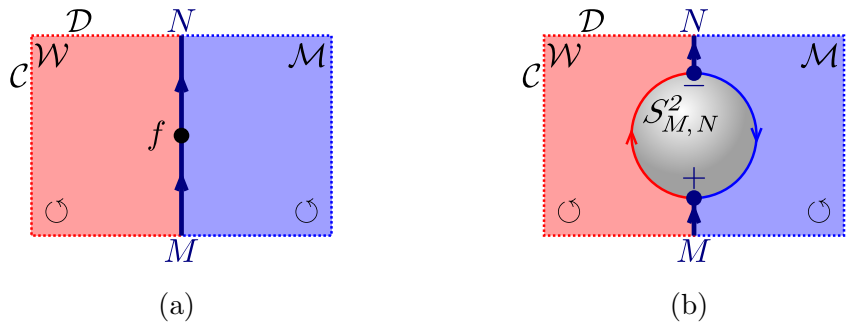


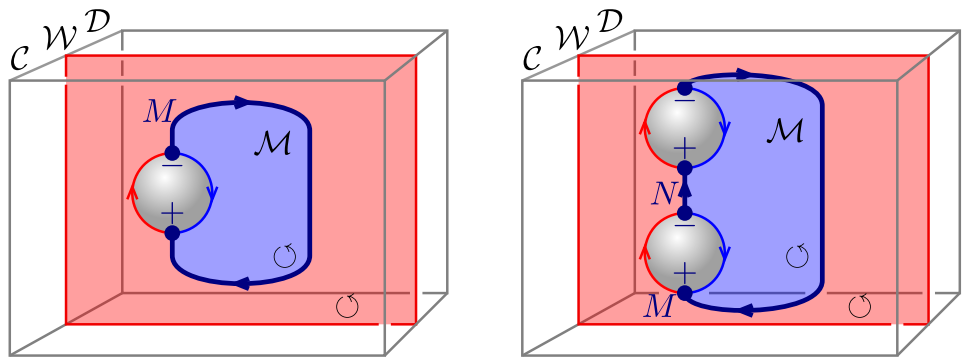
Figure 9.3: (a) Point defect labelled by a morphism $f \in \mathcal{M}(M, N)$. (b) Removing a small neighbourhood of the defect point produces a boundary component $S^2_{M,N}$.

one must have a braided equivalence $\mathcal{Z}(\mathcal{W}) \simeq \mathcal{Z}(\mathcal{V})$. By Proposition 2.22 this implies that the bicategories $\mathcal{W}$-Mod, $\mathcal{V}$-Mod of module categories of $\mathcal{W}$ and $\mathcal{V}$ are equivalent. The choice of $\mathcal{W}$ and the equivalence as in (9.6) therefore serves as a choice of "coordinates" which help describing the abstract bicategory of surface defects in a more concrete way.

9.3. Module traces from sphere defects

Let us now extend the treatment in [FSV] by considering a hypothetical defect TQFT Z, whose bulk theories are of Reshetikhin-Turaev type, whose defects are described as in Section 9.2. As we have seen, surface defects separating theories labelled by $\mathcal{C}$ and $\mathcal{D}$ are described by module categories of a fusion category $\mathcal{W}$, for which there is an equivalence $\mathcal{C} \boxtimes \widetilde{\mathcal{D}} \simeq \mathcal{Z}(\mathcal{W})$. We show that under a reasonable assumption on Z, the module categories in question are of a particular type, namely they have a module trace (see Definition 2.23).

Before proceeding, let us focus for a moment on possible labels for point insertions on a surface defect. In particular, consider a point insertion separating two lines between two surface defects, one labelled with $\mathcal{W}$ (as a module category over itself), and the other by a $\mathcal{W}$-module category $\mathcal{M}$. By the previous section, the lines are labelled by module functors $\mathcal{W} \to \mathcal{M}$ (equivalently, objects $M, N \in \mathcal{M}$ which correspond to module functors $- \triangleright M, - \triangleright N \colon \mathcal{W} \to \mathcal{M}$, see Proposition 2.21). The point insertion is labelled by a natural transformation between the module functors (equivalently, a morphism in $f \colon M \to N$ which corresponds to the natural transformation $\{\mathrm{id}_W \triangleright f\}_{W \in \mathcal{W}}$), see Figure 9.3a. Another way of labelling such point insertions is the D_0-completion mentioned in Section 4.1: Remove a small open ball surrounding the point in question. It leaves a boundary component which is a stratified 2-sphere $S^2_{M,N}$, to which the defect TQFT Z assigns a vector space $Z(S^2_{M,N})$ (see Figure 9.3b). The point insertions can then be labelled by vectors

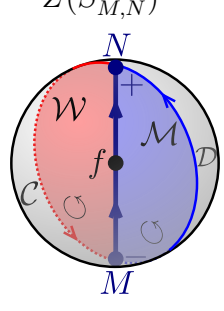


(a) (b)

Figure 9.4: (a) The manifold B^3_M: The boundary of the cube represents a single point in $S^3 \simeq \mathbb{R}^3 \cup \{\infty\}$, the rest of which corresponds to the interior of the cube. The sphere in the middle represents the incoming boundary component $S^2_{M,M}$. (b) The manifold $P_{M,N}$: Similar to (a), but has two boundary components $S^2_{M,N}$ and $S^2_{N,M}$. Note that as a stratified 3-manifold it is isomorphic to the cylinder $S^2_{M,N} \times [0,1]$.

in this vector space. The assumption we make on Z is that it is D_0-complete, i.e. the map

$$\begin{array}{ccc} \mathcal{M}(M,N) & \to & Z(S^2_{M,N}) \\ f & \mapsto & Z\Bigg(\end{array}$$



$$\Bigg), \tag{9.7}$$

is an isomorphism of vector spaces. Here the image of $f \in \mathcal{M}(M,N)$ is to be understood as follows: The picture in the argument of Z represents a stratified solid ball, seen as a bordism $\varnothing \to S^2_{M,N}$. Consequently, evaluation gives a linear map $\Bbbk \to Z(S^2_{M,N})$ whose image of $1 \in \Bbbk$ produces a vector in $Z(S^2_{M,N})$. We remark that in the case $\mathcal{C} = \mathcal{D} = \mathcal{M} = \mathcal{W}$, i.e. when there is no surface defect, the map is indeed an isomorphism and one recovers the state space that the RT TQFT assigns to a 2-sphere with two punctures (see Property 3.12).

We now define a module trace Θ on a $\mathcal{W}$-module category $\mathcal{M}$ describing a surface defect as follows: For $M \in \mathcal{M}$, let B^3_M be the stratified sphere S^3 with a removed open ball as in Figure 9.4a. It has a single boundary component $S^2_{M,M}$, which we

assume to be an incoming boundary. Evaluating with the TQFT we obtain the linear map

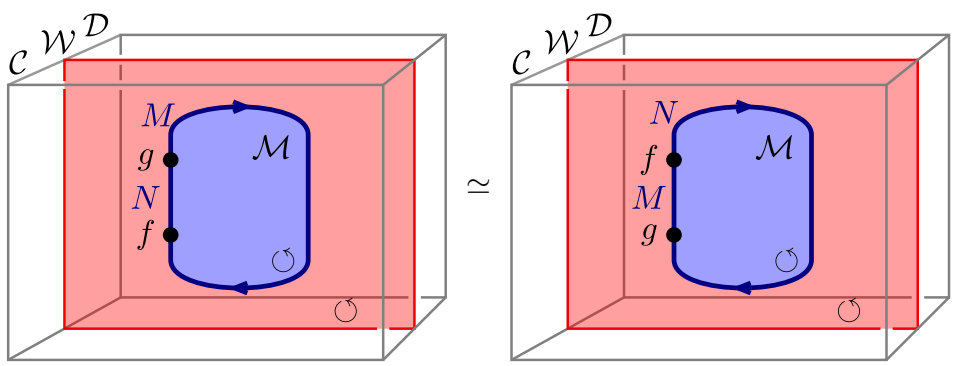
$$Z(B^3_M)\colon Z(S^2_{M,M}) \to Z(\varnothing)\ , \tag{9.8}$$

which by precomposing with (9.7) can be seen as a map $\mathrm{End}_{\mathcal{M}} M \to \Bbbk$.

Proposition 9.6. The collection of maps $\Theta_M := Z(B^3_M)$, $M \in \mathcal{M}$ is a module trace on $\mathcal{M}$.

Proof. The properties of a module trace can be shown by using the fact that upon evaluation with Z only the topological configuration of defects is important:

i) A simple deformation yields an isomorphism of stratified manifolds



. (9.9)

By using (9.7) on morphisms $g \circ f$ and $f \circ g$ one gets $\Theta_M(g \circ f) = \Theta_N(f \circ g)$.

ii) To show that the pairing $\omega_{M,N}$ as in (2.66) is non-degenerate, it is enough to provide a copairing

$$\Omega_{M,N}\colon \Bbbk \to \mathcal{M}(N,M) \otimes_{\Bbbk} \mathcal{M}(M,N), \tag{9.10}$$

such that

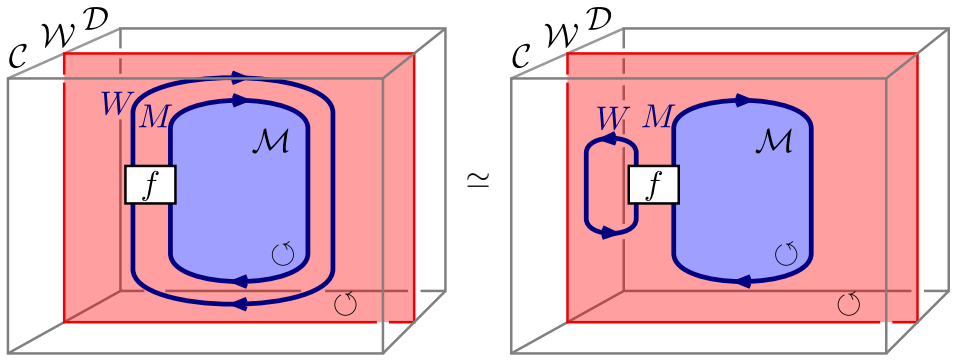
$$\begin{aligned} (\omega_{M,N} \otimes_{\Bbbk} \mathrm{id}_{M,N}) \circ (\mathrm{id}_{M,N} \otimes_{\Bbbk} \Omega_{M,N}) &= \mathrm{id}_{M,N}\ , \\ (\mathrm{id}_{N,M} \otimes_{\Bbbk} \omega_{M,N}) \circ (\Omega_{M,N} \otimes_{\Bbbk} \mathrm{id}_{N,M}) &= \mathrm{id}_{N,M}\ . \end{aligned} \tag{9.11}$$

Let $P_{M,N}$ be the stratified manifold as in Figure 9.4b. Interpreting both its boundary components as incoming, one gets a bordism $P_\omega\colon S^2_{M,N} \sqcup S^2_{N,M} \to \varnothing$. Together with the identification in (9.7), the evaluation with the TQFT $Z(P_\omega)$ gives a pairing $\mathcal{M}(N,M) \otimes_{\mathbb{C}} \mathcal{M}(M,N) \to \Bbbk$ which by (2.66) and the definition of Θ is equal to $\omega_{M,N}$. Similarly, interpreting the boundary components of $P_{M,N}$ as outgoing one gets a bordism $P_\Omega\colon \varnothing \to S^2_{N,M} \sqcup S^2_{M,N}$. We define $\Omega_{M,N} := Z(P_\Omega)$. It remains to show that the identities (9.11) hold. They follow from the functoriality of the TQFT Z and isomorphisms

$$P_\omega \cup_{S^2_{M,N}} P_\Omega \simeq S^2_{N,M} \times [0,1]\,, \quad P_\omega \cup_{S^2_{N,M}} P_\Omega \simeq S^2_{M,N} \times [0,1]\,, \tag{9.12}$$

i.e. gluing two copies of $P_{M,N}$ across a suitable boundary component gives a cylinder.

iii) The partial trace condition (2.67) follows from the isomorphism of stratified manifolds obtained by moving the W-line over the point at infinity in S^3:



. (9.13)

□

A noteworthy consequence of this result is that the fusion category $\mathcal{W}$ is spherical. Indeed, $\mathcal{W}$ as an indecomposable module category over itself has a module trace given by the left categorical trace, which is then proportional to the one in Proposition 9.6 due to Proposition 2.24. The isomorphism (9.13) for $\mathcal{M} = \mathcal{W}$ and $M = \mathbb{1}_{\mathcal{W}}$ yields

$$\Theta_W f = \Theta_{W^*} f^* \ , \quad W \in \mathcal{W} \ , \ f \in \mathrm{End}_{\mathcal{W}} W \ . \tag{9.14}$$

9.4. Witt equivalence of modular categories

Definition 9.7. Two MFCs $\mathcal{C}$ and $\mathcal{D}$ are *Witt equivalent* if there exists a spherical fusion category $\mathcal{S}$ and a ribbon equivalence $\mathcal{C} \boxtimes \widetilde{\mathcal{D}} \simeq \mathcal{Z}(\mathcal{S})$, which we call a *Witt trivialisation*.

Remark 9.8. i) The notion of Witt equivalence was introduced in [DMNO] for *non-degenerate braided* fusion categories, i.e. without an assigned ribbon structure. There are hence two Witt groups: that of modular fusion categories and that of non-degenerate braided fusion categories. For the application in this work, the version with ribbon structure is the relevant one.

ii) Witt equivalence is indeed an equivalence relation on MFCs. The set of equivalence classes forms the so-called *Witt group* whose multiplication is induced by the Deligne product, the unit is given by the class consisting of Drinfeld centres and the inverses are given by braiding reversal due to existence of the equivalence (2.20), see [DMNO].

As was already mentioned in Section 9.2, the notion of Witt equivalence turns out to be of central importance in the analysis of surface defects in 3-dimensional TQFTs. The following characterisation of Witt equivalence is formulated in [DMNO, Prop. 5.15] for non-degenerate braided fusion categories; we recall the proof to show that the argument applies in the ribbon case:

Proposition 9.9. Two MFCs $\mathcal{D}$, $\mathcal{E}$ are Witt equivalent iff there exists a modular fusion category $\mathcal{C}$ and two condensable algebras $A, B \in \mathcal{C}$ such that $\mathcal{D} \simeq \mathcal{C}_A^{\mathrm{loc}}$ and $\mathcal{E} \simeq \mathcal{C}_B^{\mathrm{loc}}$ as ribbon fusion categories.

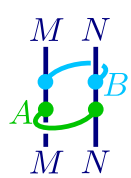
Proof. Having a Witt trivialisation $\mathcal{D} \boxtimes \widetilde{\mathcal{E}} \xrightarrow{\sim} \mathcal{Z}(\mathcal{S})$ for some spherical fusion category $\mathcal{S}$, one can take the Deligne product of both sides with $\mathcal{E}$ and use the equivalence (2.20) to get a ribbon equivalence $F\colon \mathcal{D} \boxtimes \mathcal{Z}(\mathcal{E}) \to \mathcal{E} \boxtimes \mathcal{Z}(\mathcal{S})$ and then set $\mathcal{C} := \mathcal{E} \boxtimes \mathcal{Z}(\mathcal{S})$. As a Drinfeld centre, $\mathcal{Z}(\mathcal{S})$ possesses a *Lagrangian algebra*, i.e. a commutative separable haploid algebra B', such that $\mathcal{Z}(\mathcal{S})_{B'}^{\mathrm{loc}} \simeq \mathrm{Vect}_{\Bbbk}$ as braided fusion categories. By [BalK1, Thm. 2.3], [DMNO, Lem. 3.5], the algebra B' has the underlying object $B' \cong \bigoplus_{i \in \mathrm{Irr}_{\mathcal{S}}} i \otimes i^*$ in $\mathcal{S}$ and therefore

$$\dim_{\mathcal{Z}(\mathcal{S})} B' = \dim_{\mathcal{S}} B' = \mathrm{Dim}\, \mathcal{S} \neq 0\,. \tag{9.15}$$

By [FRS1, Cor. 3.10], B' is Frobenius. One sets $B = \mathbb{1}_{\mathcal{E}} \boxtimes B'$. Similarly, one picks a Lagrangian algebra A' in $\mathcal{Z}(\mathcal{E})$ and sets $A = F(\mathbb{1}_{\mathcal{D}} \boxtimes A')$. □

For the rest of the section, let $\mathcal{C}$, A, B be as in the proposition above. We will look for an explicit Witt trivialisation of $\mathcal{C}_A^{\mathrm{loc}} \boxtimes \widetilde{\mathcal{C}_B^{\mathrm{loc}}}$. Let us consider the semisimple category ${}_A\mathcal{C}_B$ with A-B-bimodules in $\mathcal{C}$ as objects and bimodule maps as morphisms. We equip it with the following monoidal product: for each $M, N \in {}_A\mathcal{C}_B$ we set

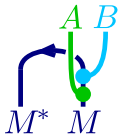
$$M\, {}_A\!\otimes_B N := \mathrm{im}$$



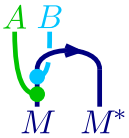
. (9.16)

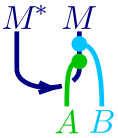
Note that the monoidal unit $\mathbb{1}_{{}_A\mathcal{C}_B} := A \otimes B$ is in general not a simple object and ${}_A\mathcal{C}_B$ is therefore a multifusion category.

The category ${}_A\mathcal{C}_B$ has a natural pivotal structure with the (co)evaluation morphisms for each $M \in {}_A\mathcal{C}_B$ being

$\mathrm{ev}_M =$ 

, $\mathrm{coev}_M =$

, $\widetilde{\mathrm{ev}}_M =$ 

, $\widetilde{\mathrm{coev}}_M =$ 

. (9.17)

Remark 9.10. i) In the case $A = B$, the category ${}_A\mathcal{C}_A$ has A-A-bimodules as objects but its tensor product is *not* the tensor produce over A. Since the latter is the more usual tensor product on ${}_A\mathcal{C}_A$, we stress this point in order to avoid confusion.

ii) ${}_A\mathcal{C}_B$ is equivalent to the category $\mathcal{C}_{A\otimes B}$ of right modules of $A \otimes B$ in $\mathcal{C}$. In general, to define a tensor product of right modules of a separable Frobenius algebra it needs a half-braiding with respect to which it is commutative. In the case of the algebra $A \otimes B \in \mathcal{C}$ it is given by the "dolphin" half-braiding (2.21). Note that $A \otimes B$ is in general not commutative with respect to the braiding of $\mathcal{C}$.

The multifusion category ${}_A\mathcal{C}_B$ is defined in such a way that algebras and their bimodules in it correspond to algebras and their bimodules in $\mathcal{C}$ over (A, B) in the sense of Definitions 9.1 and 9.3. In particular we have (see also Definitions 9.5 and 2.26):

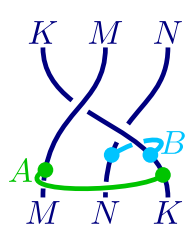
Lemma 9.11. There is an equivalence of bicategories $\mathrm{FrobAlg}^{\mathrm{ssep}}_{\mathcal{C},A,B} \simeq \mathrm{FrobAlg}^{\mathrm{ssep}}_{{}_A\mathcal{C}_B}$.

Proof. Let $F \in {}_A\mathcal{C}_B$ be an algebra. Then F is also an object in $\mathcal{C}$ and it is equipped with the multiplication given by $[F \otimes F \to F_A\otimes_B F \to F]$, where the first morphism is the projection to the tensor product in ${}_A\mathcal{C}_B$ and the second morphism is the multiplication of F in ${}_A\mathcal{C}_B$. This multiplication in $\mathcal{C}$ satisfies the relations in Definition 9.1 because of how the tensor product in ${}_A\mathcal{C}_B$ is defined (see (9.16)). One then takes the morphism $[\mathbb{1}_{\mathcal{C}} \xrightarrow{\eta_A\otimes\eta_B} A \otimes B \xrightarrow{\eta_F} F]$ as the unit, where η_A, η_B are the units of A and B and η_F is the unit of F in ${}_A\mathcal{C}_B$. To compare the bimodules it is enough to notice that the tensor product in ${}_A\mathcal{C}_B$ ensures that the two A- and the two B-actions as in (9.2), (9.3) coincide. A similar argument applies to (symmetric, separable) Frobenius algebras as well. □

Proposition 9.12. The functor

$$\mathcal{C}^{\mathrm{loc}}_A \boxtimes \widetilde{\mathcal{C}^{\mathrm{loc}}_B} \to \mathcal{Z}({}_A\mathcal{C}_B), \qquad M \boxtimes N \mapsto (M \otimes N,\ \gamma^{\mathrm{dol}}_{M,N}), \tag{9.18}$$

is a ribbon equivalence. Here $\gamma^{\mathrm{dol}}_{M,N}$ is an analogue of the "dolphin" half-braiding, defined for all $K \in {}_A\mathcal{C}_B$ by

$\gamma^{\mathrm{dol}}_{M,N,K} :=$ 

. (9.19)

Proof. First we show that $\mathcal{C}_A^{\mathrm{loc}} \boxtimes \widetilde{\mathcal{C}_B^{\mathrm{loc}}} \simeq \mathcal{Z}({}_A\mathcal{C}_B)$ as braided multifusion categories. Indeed, one has obvious equivalences of braided categories

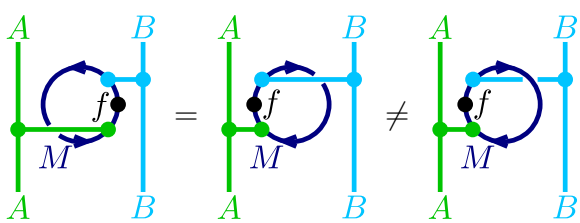
$$\mathcal{C}_A^{\mathrm{loc}} \boxtimes \widetilde{\mathcal{C}_B^{\mathrm{loc}}} \simeq (\mathcal{C} \boxtimes \widetilde{\mathcal{C}})^{\mathrm{loc}}_{A\boxtimes B} \simeq \mathcal{Z}(\mathcal{C})^{\mathrm{loc}}_{A\otimes B} \,. \tag{9.20}$$

It is known (see [Schb, Cor. 4.5]) that for any monoidal category $\mathcal{M}$ and a commutative algebra $(C,\gamma) \in \mathcal{Z}(\mathcal{M})$, one has a braided equivalence $\mathcal{Z}(\mathcal{M})^{\mathrm{loc}}_{(C,\gamma)} \simeq \mathcal{Z}(\mathcal{M}_C)$, where one uses the half-braiding γ of C to define the monoidal product in $\mathcal{M}_C$. Applying this to the right hand side of (9.20) one obtains the result.

Next we show that the explicit functor (9.18) is a braided equivalence. First, using that M and N are local modules one can check that $\gamma^{\mathrm{dol}}_{M,N,K}$ does indeed satisfy the hexagon identity. Next, by the argument above, $\mathcal{Z}({}_A\mathcal{C}_B)$ is fusion and has the same Frobenius-Perron dimension as $\mathcal{C}_A^{\mathrm{loc}} \boxtimes \widetilde{\mathcal{C}_B^{\mathrm{loc}}}$. The functor (9.18) is then a braided functor between non-degenerate braided fusion categories and therefore fully faithful by Proposition 2.8. Hence by Proposition 2.3 it is an equivalence.

Finally, the explicit equivalence (9.18) implies in particular that $\mathcal{Z}({}_A\mathcal{C}_B)$ is spherical: it is enough to check that the left and the right dimensions of simple objects coincide (see Proposition 2.4). We know that all simple objects are of the form $\mu \otimes \nu$ for $\mu \in \mathrm{Irr}_{\mathcal{C}_A^{\mathrm{loc}}}$, $\nu \in \mathrm{Irr}_{\mathcal{C}_B^{\mathrm{loc}}}$ for which the left/right dimensions in $\mathcal{Z}({}_A\mathcal{C}_B)$ are the product of those of μ and ν and are hence equal. The category $\mathcal{Z}({}_A\mathcal{C}_B)$ is therefore also ribbon (see [TV, Lem. 4.5]). The equivalence (9.18) can also be checked to preserve braidings and twists and is therefore a ribbon equivalence. □

Remark 9.13. The category ${}_A\mathcal{C}_B$ need not be spherical, even though its Drinfeld centre $\mathcal{Z}({}_A\mathcal{C}_B)$ is spherical, as shown in the proof above. Indeed, the left and the right traces of $f \in \mathrm{End}_{{}_A\mathcal{C}_B}(M)$ in general need not be equal:

$\mathrm{tr}_l\, f =$ 

$= \mathrm{tr}_r\, f \,.$ (9.21)

Proposition 9.12 in particular implies that $\mathcal{Z}({}_A\mathcal{C}_B)$ is fusion and hence that ${}_A\mathcal{C}_B$ is indecomposable. Proposition 2.5 then in turn implies the existence of a braided equivalence $\mathcal{Z}({}_A\mathcal{C}_B) \simeq \mathcal{Z}(\mathcal{F})$ where

$$\mathcal{F} := ({}_A\mathcal{C}_B)_{ii} \tag{9.22}$$

is any component category of ${}_A\mathcal{C}_B$, and which is then automatically fusion. An explicit equivalence is given by

$$\mathcal{Z}({}_A\mathcal{C}_B) \to \mathcal{Z}(\mathcal{F}), \quad M \mapsto \mathbb{1}_{\mathcal{F}}\, {}_A\!\otimes_B M \,. \tag{9.23}$$

Indeed, we show in the Appendix E that (9.23) is in fact a ribbon functor. Then, since $\mathcal{Z}({}_A\mathcal{C}_B)$ and $\mathcal{Z}(\mathcal{F})$ are non-degenerate and have equal Frobenius-Perron dimensions, by Propositions 2.3 and 2.8 it is an equivalence.

The composition of the functors (9.18) and (9.23) is then a Witt trivialisation for $\mathcal{C}_A^{\mathrm{loc}} \boxtimes \widetilde{\mathcal{C}_B^{\mathrm{loc}}}$.

9.5. Equivalence of the two descriptions of domain walls

Let $\mathcal{C}$ be a MFC and $A, B \in \mathcal{C}$ condensable algebras. We have looked at two ways to describe domain walls separating two theories of Reshetikhin-Turaev type labelled by MFCs $\mathcal{C}_A^{\mathrm{loc}}$ and $\mathcal{C}_B^{\mathrm{loc}}$. The first one comes from the construction of the defect TQFT $Z^{\mathrm{def}}_{\mathcal{C},w}$ defined in Section 9.1, while the second one is a physics inspired argument from [FSV], which was summarised in Section 9.2 and expanded upon in Section 9.3. Below we argue that both descriptions are essentially the same in the sense that the resulting bicategories of domain walls are equivalent.

We know from Section 9.1 and Lemma 9.11 that the bicategory of domain walls obtained from the defect TQFT $Z^{\mathrm{def}}_{\mathcal{C},w}$ is equivalent to $\mathrm{FrobAlg}^{\mathrm{ssep}}_{{}_A\mathcal{C}_B}$. Following Section 9.4, let $\mathcal{F}$ be a component category of the multifusion category ${}_A\mathcal{C}_B$. The existence of the Witt trivialisation $\mathcal{C}_A^{\mathrm{loc}} \boxtimes \widetilde{\mathcal{C}_B^{\mathrm{loc}}} \xrightarrow{\sim} \mathcal{Z}(\mathcal{F})$ implies that the bicategory of domain walls according to [FSV] is $\mathcal{F}$-$\mathrm{Mod}^{\mathrm{tr}}$ (see Definition 2.26). We have:

Theorem 9.14. One has the following commutative diagram of equivalences, inclusion and forgetful functors between bicategories:

$$\begin{tikzcd}
\mathcal{F}\text{-}\mathrm{Mod}^{\mathrm{tr}} \arrow[d, hook, "\text{include}"] & \mathrm{FrobAlg}^{\mathrm{ssep}}_{\mathcal{F}} \arrow[l, "\sim", "\text{iii)}"'] \arrow[r, "\sim", "\text{iv)}"'] \arrow[d, "\text{forget}"] & \mathrm{FrobAlg}^{\mathrm{ssep}}_{{}_A\mathcal{C}_B} \arrow[d, "\text{forget}"] \\
\mathcal{F}\text{-}\mathrm{Mod} & \mathrm{Alg}_{\mathcal{F}} \arrow[l, "\sim", "\text{i)}"'] \arrow[r, "\sim", "\text{ii)}"'] & \mathrm{Alg}_{{}_A\mathcal{C}_B}
\end{tikzcd} . \tag{9.24}$$

Proof. We exploit various relations between bicategories introduced in Sections 2.7 and 9.4.

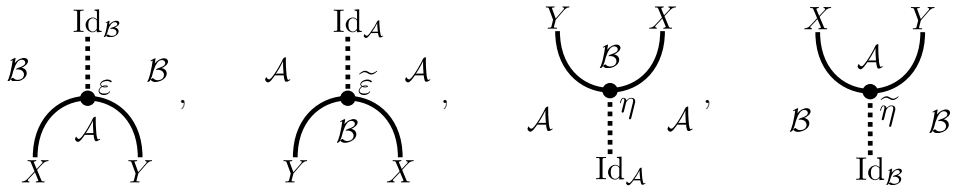
Equivalence i) in (9.24) is given by Proposition 2.21, and equivalence iii) follows from Proposition 2.27. It is clear that the left square commutes (on the nose).

Equivalence ii) is given by inclusion. Indeed, for any indecomposable multifusion category $\mathcal{A}$ and its component category $\mathcal{A}_{ii}$, an algebra $A \in \mathcal{A}_{ii}$ is automatically an algebra in $\mathcal{A}$ by providing it with the unit $[\mathbb{1} \twoheadrightarrow \mathbb{1}_i \xrightarrow{\eta} A]$ where $\mathbb{1}_i \in \mathcal{A}_{ii} \subset \mathcal{A}$ is the restriction of the tensor unit. The inclusion is fully faithful, meaning that upon inclusion of two algebras $A, B \in \mathcal{A}_{ii}$, the corresponding bimodule categories ${}_A(\mathcal{A}_{ii})_B$ and ${}_A\mathcal{A}_B$ are equivalent. Moreover, any simple algebra in $\mathcal{A}$ is Morita equivalent to one in $\mathcal{A}_{ii}$ (see e.g. [KZ2, Rem. 3.9]), which implies the essential

surjectivity of the inclusion. It is easy to check that in case $\mathcal{A}$ is pivotal, the inclusion preserves the structure of a symmetric separable Frobenius algebra, which then gives equivalence iv), as well as commutativity of the right square (again on the nose). $\square$

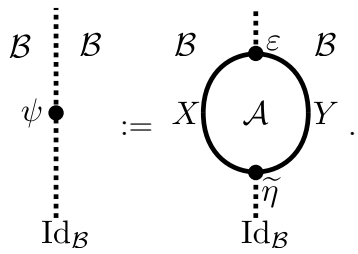
A. Monadicity for separable biadjunctions

Let $\mathcal{A}$, $\mathcal{B}$ be categories, and let $X\colon \mathcal{A} \to \mathcal{B}$, $Y\colon \mathcal{B} \to \mathcal{A}$ be biadjoint functors with units and counits denoted by



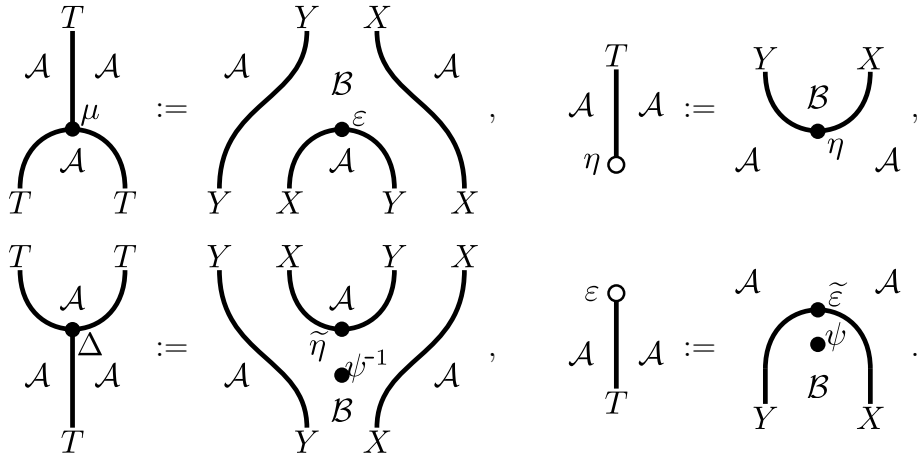
(A.1)

Furthermore we will assume that this biadjunction is separable, i.e. the natural transformation



(A.2)

is invertible. The endofunctor $T := [YX : \mathcal{A} \to \mathcal{A}]$ becomes a Δ-separable Frobenius algebra in the strict monoidal category $\operatorname{End}\mathcal{A}$ via the structure morphisms



(A.3)

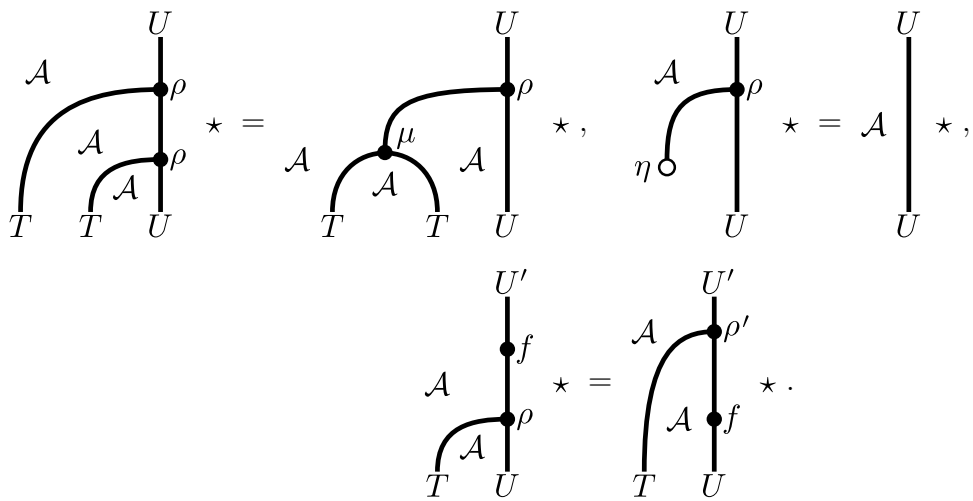
Let $\mathcal{A}^T$ be the category of T-*modules in* $\mathcal{A}$. Its objects are pairs

$$\big(U \in \mathcal{A}\,,\ [\rho : T(U) \to U]\big)$$

and a morphism $(U,\rho) \to (U',\rho')$ is a morphism $[f : U \to U'] \in \mathcal{A}$, such that the following diagrams commute:

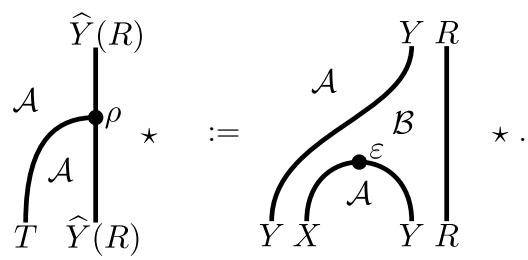
$$\begin{tikzcd} TT(U) \arrow[r, "\mu_U"] \arrow[d, "T(\rho)"] & T(U) \arrow[d, "\rho"] \\ T(U) \arrow[r, "\rho"] & T \end{tikzcd}, \qquad \begin{tikzcd} U \arrow[r, "\eta_U"] \arrow[dr, "\mathrm{id}"'] & T(U) \arrow[d, "\rho"] \\ & U \end{tikzcd}, \qquad \begin{tikzcd} T(U) \arrow[r, "T(f)"] \arrow[d, "\rho"] & T(U') \arrow[d, "\rho'"] \\ U \arrow[r, "f"] & U' \end{tikzcd}. \tag{A.4}$$

Let $\star$ be the category with only one object and only the identity morphism. In what follows, it is going to be useful to identify any category $\mathcal{A}$ with the category of functors $\star \to \mathcal{A}$ and natural transformations in the obvious way. The conditions (A.4) can then be written graphically as



(A.5)

Define the functor $\widehat{Y} : \mathcal{B} \to \mathcal{A}^T$ to be the same as Y, except that the image is equipped with the following T-action:



(A.6)

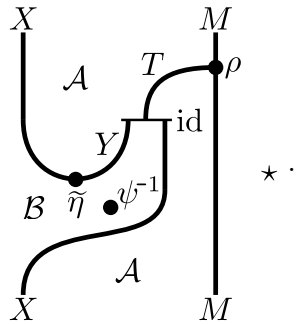
Definition A.1. Let $\mathcal{A}$ be a category.

- An idempotent $[p : U \to U] \in \mathcal{A}$ is called split, if it has a retract, i.e. a triplet (S, e, r) where $S \in \mathcal{A}$, $e : S \to U$, $r : U \to S$, such that e is mono, $r \circ e = \mathrm{id}_S$, $e \circ r = p$.

- $\mathcal{A}$ is called idempotent complete if every idempotent is split.

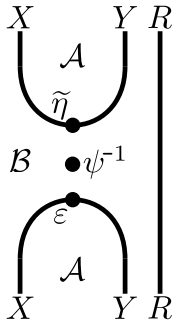
Proposition A.2. If $\mathcal{B}$ is idempotent complete, then $\widehat{Y}$ is an equivalence.

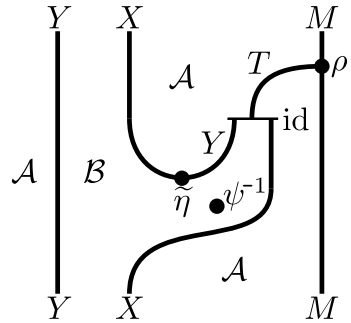
Proof. We will give an inverse $\widehat{X} : \mathcal{A}^T \to \mathcal{B}$. Let $M \in \mathcal{A}^T$. Define the following morphism $[p_M : X(M) \to X(M)] \in \mathcal{B}$:

$p_M :=$ 

$\star$. (A.7)

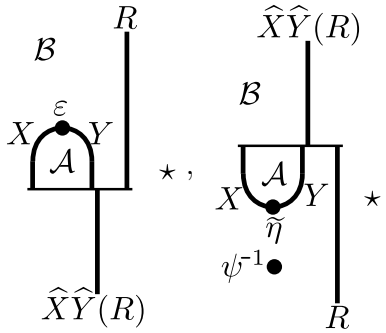
One quickly checks that it is an idempotent. Set $\widehat{X}(M) = \operatorname{im} p$. To prove that it is indeed an inverse, one computes

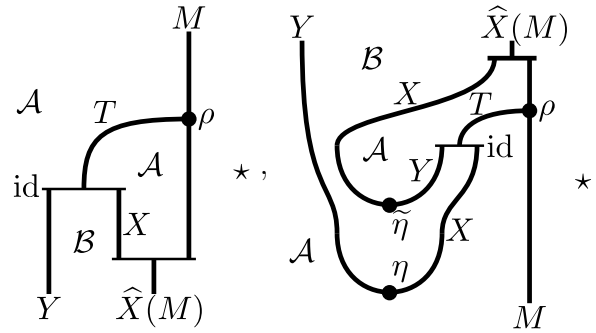
$\widehat{X}\widehat{Y}(R) = \operatorname{im}$ 

$\star$, $\widehat{Y}\widehat{X}(M) = \operatorname{im}$ 

$\star$. (A.8)

The morphisms in pairs



$\star$, $\star$ and 

$\star$, $\star$

are then inverses of each other. □

Now let $\mathcal{A}$, X, Y (and hence also $\widehat{X}$, $\widehat{Y}$) in addition be $\Bbbk$-linear additive categories and functors.

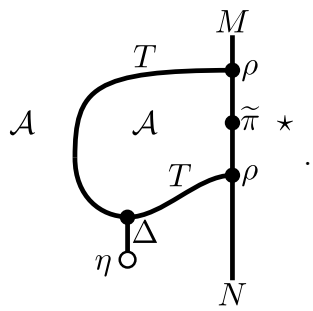
Proposition A.3. Suppose $\mathcal{A}$ is idempotent complete and finitely semisimple. Then so is $\mathcal{A}^T$.

Proof. We first show idempotent completeness of $\mathcal{A}^T$. Given an idempotent $p : M \to M$ in $\mathcal{A}^T$ and a retract $e : S \to M$, $r : M \to S$ in $\mathcal{A}$ with $p = e \circ r$, one can equip S with a T-action as follows,

$$\rho^S = \left[T(S) \xrightarrow{T(e)} T(M) \xrightarrow{\rho^M} M \xrightarrow{r} S\right] . \tag{A.9}$$

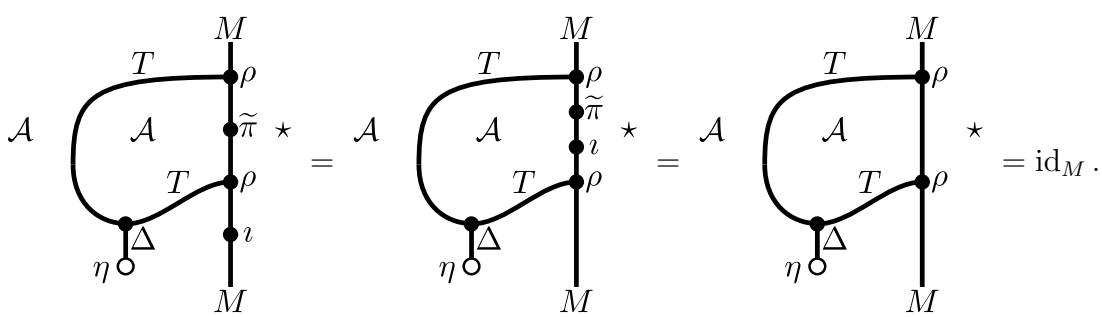
With respect to this action, e and r are morphisms in $\mathcal{A}^T$, so that (S, e, r) becomes a retract in $\mathcal{A}^T$.

Next we show semisimplicity of $\mathcal{A}^T$. Let $M, N \in \mathcal{A}^T$ and let $\imath : M \to N$ be mono in $\mathcal{A}^T$. Since $\mathcal{A}$ is semisimple, there is $\widetilde{\pi} : N \to M$ in $\mathcal{A}$, such that $\widetilde{\pi} \circ \imath = \mathrm{id}_M$. Define

$\pi :=$  . (A.10)

One checks that $\pi : N \to M$ is a morphism in $\mathcal{A}^T$ and

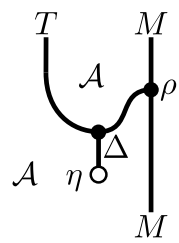
$\pi \circ \imath =$

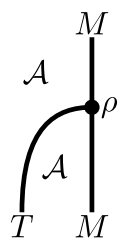


It follows that $N \simeq M \oplus X$ for $X = \ker(\imath \circ \pi)$, and so all subobjects are direct summands. The kernel exists as it is the image of the idempotent $\mathrm{id}_N - \imath \circ \pi$.

For finiteness we show that every T-module $M \in \mathcal{A}^T$ is a submodule of an *induced* T-module, i.e. a one of the form $\mathrm{Ind}(U) := [\star \xrightarrow{U} \mathcal{A} \xrightarrow{T} \mathcal{A}]$ for some object

$U \in \mathcal{A}$. Indeed, pick $U = M$ and the following morphisms: $\Upsilon : M \to \mathrm{Ind}(M)$ and $\Pi : \mathrm{Ind}(M) \to M$

$\Upsilon :=$ 

$\star$, $\Pi :=$ 

$\star$ (A.11)

One can check that Υ and Π are module morphisms and that $\Pi \circ \Upsilon = \mathrm{id}_M$, hence M is indeed a submodule of $\mathrm{Ind}(M)$. Every simple T-module is then a submodule of $\mathrm{Ind}(V)$ where $V \in \mathcal{A}$ is simple and since there are finitely many of those, $\mathcal{A}^T$ must have finitely many simple objects. $\square$

B. Equivalent skeleta for surfaces and 3-manifolds

In this appendix we sketch the proofs of Lemmas 4.6 and 6.2, stating that 1- and 2-skeleta for 2- and 3-manifolds respectively are related by finite sequences of certain moves on them. We note that both of them are just versions of well-known results, the purpose of this exposition is to argue that they also hold for the admissible of 1- and 2-skeleta, which for our purposes are more convenient to use. In doing so we merely reduce this case to the ones that can be found in the literature.

We will use the oriented Pachner moves on the triangulations of manifolds as an intermediate tool. Recall that:

- An *abstract simplicial complex* is a pair (X, Σ) consisting of a set X (the set of *vertices*) and a subset of the power set $\Sigma \subseteq \mathcal{P}X$ (the set of *faces*) such that $\{x\} \in \Sigma$ for all $x \in X$ and if $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ then $\tau \in \Sigma$.

- If X is finite, one can define the *geometric realisation* to be the topological space $|(X, \Sigma)|$ obtained as a union of convex hulls of each face $\sigma \in \Sigma$ in $\operatorname{span}_{\mathbb{R}} X$ with the standard topology. When referring to a *face* σ *of* $|(X, \Sigma)|$ we will mean the convex hull of a face $\sigma \in \Sigma$. If the vertices of a face carry an order, e.g. $\sigma = \{x_1, \ldots, x_{k+1}\}$, we equip it with the orientation given by the vectors $(x_2 - x_1, \ldots, x_{k+1} - x_1)$.

- A *triangulation* of an n-dimensional manifold M is a pair $X = (X, f)$, where $X = (X, \Sigma)$ is a finite abstract simplicial complex such that if $\sigma, \sigma' \in \Sigma$ then $\sigma \cap \sigma'$ is either empty or a face of X, and $f\colon |(X, \Sigma)| \to M$ is a homeomorphism. Note that a triangulation on M induces a triangulation on the boundary ∂M.

- The Poincaré dual of an n-dimensional manifold M with a triangulation (X, Σ) is the stratification of $M \simeq |(X, \Sigma)|$ obtained by taking for each face σ of $|(X, \Sigma)|$ the convex hull of the barycentres of all faces containing σ. This means that each k-face σ gets replaced by an "orthogonal" $(n-k)$-stratum s; If the triangulation is *oriented*, i.e. if the faces of $|(X, \Sigma)|$ have an orientation assigned, we orient the strata of the Poincaré dual by the convention that the ordered pair (s, σ) gives the orientation of M. By the same convention, orienting the strata of the Poincaré dual fixes the orientations of the faces.

- The Poincaré duals of (unoriented) triangulations of 1-, 2- and 3-dimensional manifolds are by definition examples of (unoriented) skeleta. We will call them ∇*-skeleta* to emphasise that not all skeleta are obtained this way. An *admissible triangulation* of a 1-, 2- or 3-dimensional manifold M is an oriented

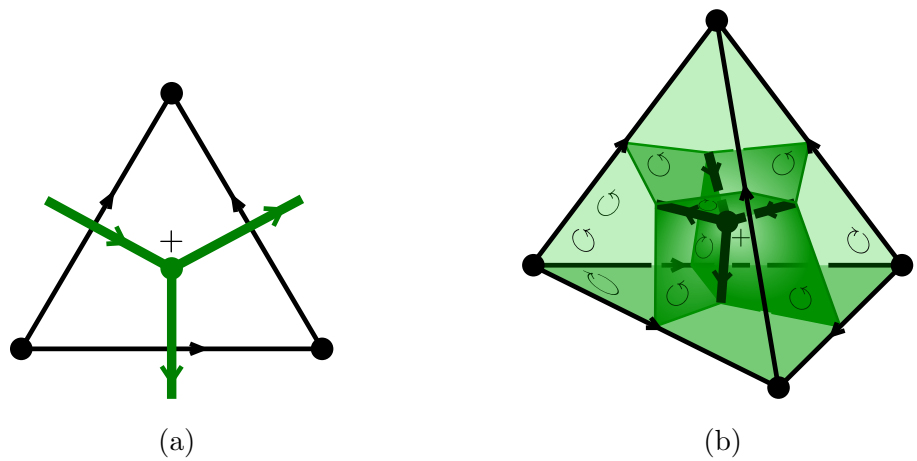


Figure B.1: Examples of orientation conventions for a single globally ordered n-face and its Poincaré dual in cases (a) $n = 2$ and (b) $n = 3$. Recall that the orientations are picked so that the pair (s, σ), consisting of a stratum and a face that it is dual to, yields the orientation of the ambient plane/volume (here in both cases assumed to be right-handed). In dimension 2 the globally ordered ∇-skeleta are admissible, while in dimension 3 there is a purely coincidental discrepancy between the orientations of lines which we address in Section B.2.

triangulation whose Poincaré dual is an admissible ∇-skeleton. A *globally ordered triangulation* of M is a triangulation X with an assigned total order on the set of vertices of X (and is therefore oriented). A globally ordered ∇-skeleton is the Poincaré dual of a globally ordered triangulation. In Figure B.1 we illustrate how the orientations of the faces of a globally ordered triangulation and those of the strata if its Poincaré dual are related.

One has (see [CRS1, Prop. 3.3]):

Proposition B.1. Any two globally oriented triangulations which agree on the boundary are related by a finite sequence of oriented Pachner moves shown in Figure B.2 for dimensions 2 and 3.

It was shown in [CRS1, Prop. 3.18] that the admissible moves bl and BLT on admissible ∇-skeleta imply the oriented Pachner moves on the corresponding triangulations, so Proposition B.1 already implies Lemmas 4.6 and 6.2 for globally oriented ∇-skeleta. Furthermore one has:

Lemma B.2. Let M be an n-dimensional manifold, $n = 2, 3$. An (unoriented) $(n-1)$-skeleton T of M is a ∇-skeleton if and only if

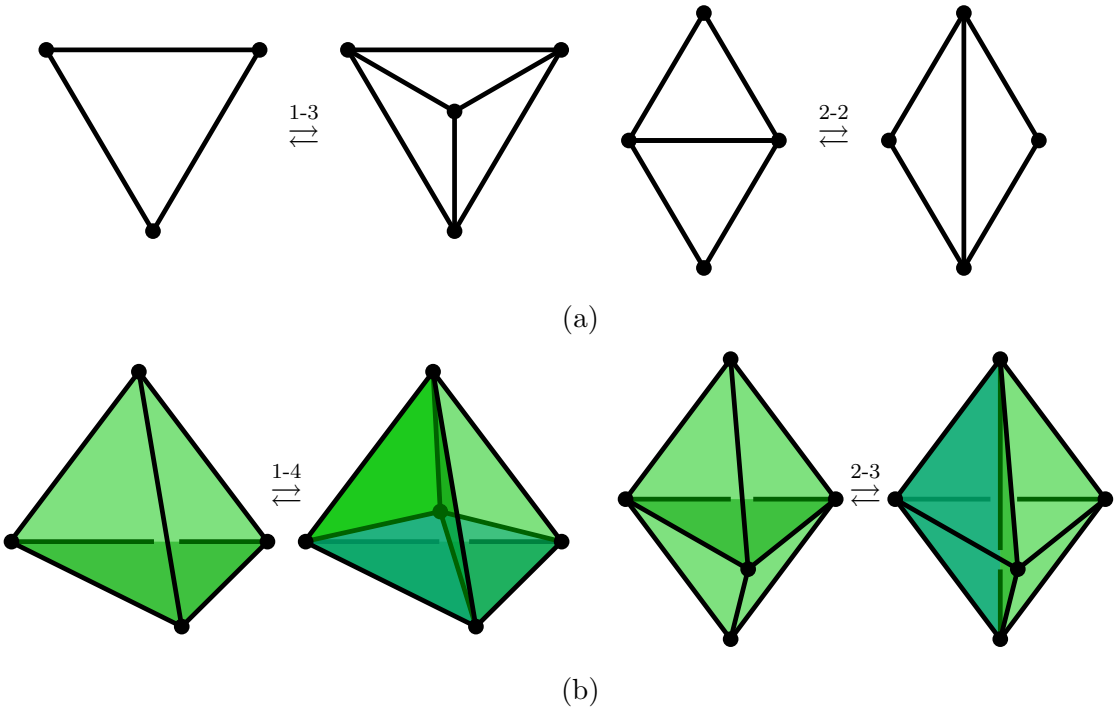


Figure B.2: The Pachner moves for triangulations of n-dimensional manifolds. (a) The moves 1-3 and 2-2 for $n = 2$, (b) the moves 1-4 and 2-3 for $n = 3$. Similarly one defines the oriented Pachner moves, in which case the involved faces have orientations induced by an order on the vertices. The newly created vertex in the moves 1-3 and 1-4 can be assigned an arbitrary new value extending this order, which also fixes the orientations of the newly created faces.

i) every stratum of T is contractible;

ii) for each stratum s of T, each two germs of n-strata adjacent to s strata belong to distinct n-strata;

iii) if two strata s, s' of T are adjacent to the same set of n-strata, then $s = s'$.

Proof. The conditions (i)-(iii) are satisfied for any ∇-skeleton.

Conversely, from a skeleton T that satisfies these conditions we obtain an abstract simplicial complex by taking the set X of vertices to be the set of n-strata of M and with the set of faces Σ defined as follows: From Figures 4.5 (for $n = 2$) and 6.1 (for $n = 3$) we see that each i-stratum is adjacent to exactly $n+1-i$ germs of n-strata. Therefore by condition ii) each i-stratum s of T yields an $(n+1-i)$-element subset of X, which by condition iii) are all distinct. We set Σ to be the collection of these subsets. By construction, (X, Σ) satisfies the conditions of a

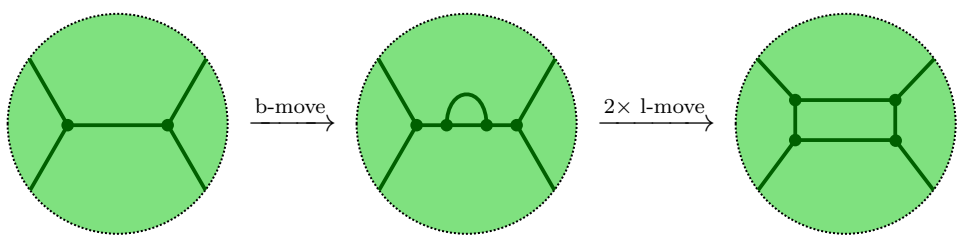
triangulation. The strata of the Poincaré dual of $|(X, \Sigma)|$ are then in canonical bijection with the strata of M. Using the condition i) one can construct a homeomorphism $|(X, \Sigma)| \to M$ preserving the stratifications by using the fact that a self-homeomorphism of the unit sphere can be extended to a self-homeomorphism of the unit ball. $\square$

B.1. 1-skeleta for surfaces

Let Σ be an oriented 2-manifold with boundary. For an admissible 1-skeleton t of Σ we denote by $\underline{t}$ the underlying unoriented 1-skeleton.

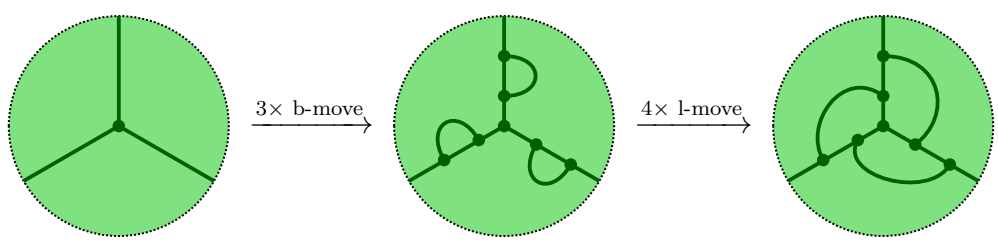
Lemma B.3. Let t be an admissible 1-skeleton for Σ, such that ∂t is an admissible ∇-skeleton. Then there is an admissible ∇-skeleton t' for Σ and a sequence of admissible bl-moves from t to t'.

Proof. We show that one can bring t to a skeleton t' which satisfies the conditions of Lemma B.2 without using the inverse b-move, which ensures that the orientations of the newly created 1-strata can be chosen in such a way that the moves are admissible. Condition i) is achieved by performing the b-move on all non-contractible 1-strata. Condition ii) is achieved by making copies of the 1-strata which are adjacent to the same 2-stratum twice as follows:



, (B.1)

and by surrounding a 0-stratum with new 1-strata:



, (B.2)

Note that the condition ii) cannot be achieved this way for the 1-strata intersecting $\partial\Sigma$, but there is no need, since ∂t is already a ∇-skeleton. The condition iii) can then be achieved similarly. $\square$

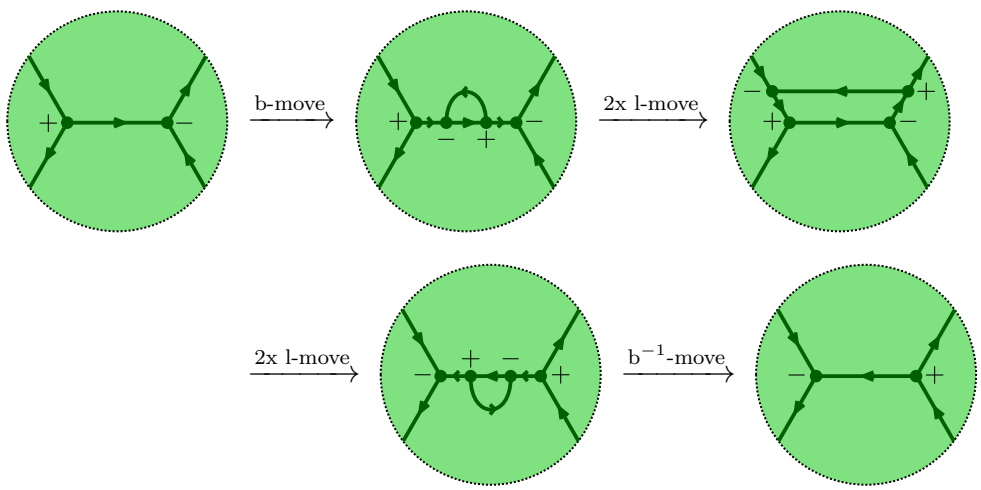
To utilise Proposition B.1 one needs a stronger condition on orientations, namely that the orientations of the ∇-skeleta are given by a global order.

Lemma B.4. Let t be an admissible 1-skeleton of Σ and l a contractible 1-stratum with two distinct adjacent 0-strata p_1 and p_2, such that

1. $l \cap \partial\Sigma = \varnothing$,
2. the other two germs of 1-strata, adjacent to p_i, are not both oriented towards or away from p_i.

Then the orientation of l, p_1 and p_2 can be at once changed by a finite sequence of admissible bl-moves.

Proof. One has for example:



. (B.3)

□

Lemma B.5. Let t, t' be two admissible 1-skeleta for Σ, such that

i) $\underline{t} = \underline{t'}$ is a ∇-skeleton;

ii) $\partial t = \partial t'$.

Then there is a sequence of admissible bl-moves from t to t'.

Proof. Let X and X' be the corresponding admissible triangulations (see Figure B.3a for an illustration). Apply to each 2-face of X a 1-3 Pachner move, orienting each new edge towards the newly created vertex (Figure B.3b), which yields an oriented triangulation $\overline{X}$. Each Pachner move is oriented, so the modification corresponds to a sequence of admissible bl moves on t. The edges that

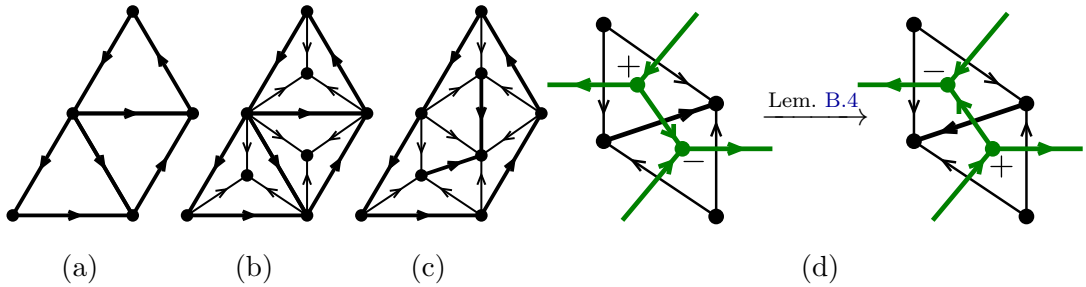


Figure B.3: Fixing orientations with the help of Pachner moves and Lemma B.4.

are both in X and $\overline{X}$ and do not lie in the boundary have exactly two adjacent 2-faces, on which we apply an oriented 2-2 Pachner move, setting the orientation of the new edge arbitrarily (Figure B.3c). This yields an oriented triangulation $\widetilde{X}$, which again is obtained from $\overline{X}$ by a sequence of admissible bl-moves on their Poincaré duals. Applying the same procedure to X' one similarly obtains oriented triangulations $\overline{X'}$ and $\widetilde{X'}$. Note that $\widetilde{X'}$ differs from $\widetilde{X}$ only by the orientations of edges created in the second step. The Poincaré duals of these edges satisfy the conditions of Lemma B.4 and so their orientations can be flipped by a sequence of admissible bl-moves (Figure B.3d), bringing $\widetilde{X}$ to $\widetilde{X'}$. In the end one obtains a sequence of invertible bl-moves $X \leftrightarrow \overline{X} \leftrightarrow \widetilde{X} \leftrightarrow \widetilde{X'} \leftrightarrow \overline{X'} \leftrightarrow X'$. □

Corollary B.6. Let t, t' be two admissible 1-skeleta for Σ such that $\partial t = \partial t'$ is a globally ordered ∇-skeleton. Then there is a finite sequence of admissible bl moves from t to t'.

Proof. One makes t into an admissible ∇-skeleton by the help of Lemma B.3 and then flips the orientations of 1-strata as necessary to make it into a globally ordered ∇-skeleton t_∇. Similarly, t' is made into a globally ordered ∇-skeleton t'_∇. By Proposition B.1, t_∇ can be made into t'_∇. □

At this point all that remains in order to prove Lemma 4.6 is to show that the condition on the boundary of a 1-skeleton in Corollary B.6 can be lifted. To this end, let $M = (M, \mathrm{T}) \in \mathrm{Bord}_3^{\mathrm{def}}$ be a defect bordism and let $s \subseteq M$ be a 2-stratum. The external closure $\overline{\overline{s}}$ (see (4.10)) is then an oriented 2-manifold with boundary. The boundary $\partial\overline{\overline{s}}$ is itself stratified by lines, which are projected on $\mathrm{T}^{(1)}$ and lines that lie in ∂M. Let us call a line of the first kind an *inner line* and of the second kind a *boundary line*. The boundary of a 1-skeleton t of $\overline{\overline{s}}$ must be held unchanged at the boundary lines, while they can be changed at the inner lines by the moves $\partial\,\mathrm{b}$ and $\partial\,\mathrm{l}$. The moves p and l2 allow one to ignore the 0-strata of M and, the choices of 1-skeleta of the other 2-strata of M. To fulfil the condition of Corollary B.6 we

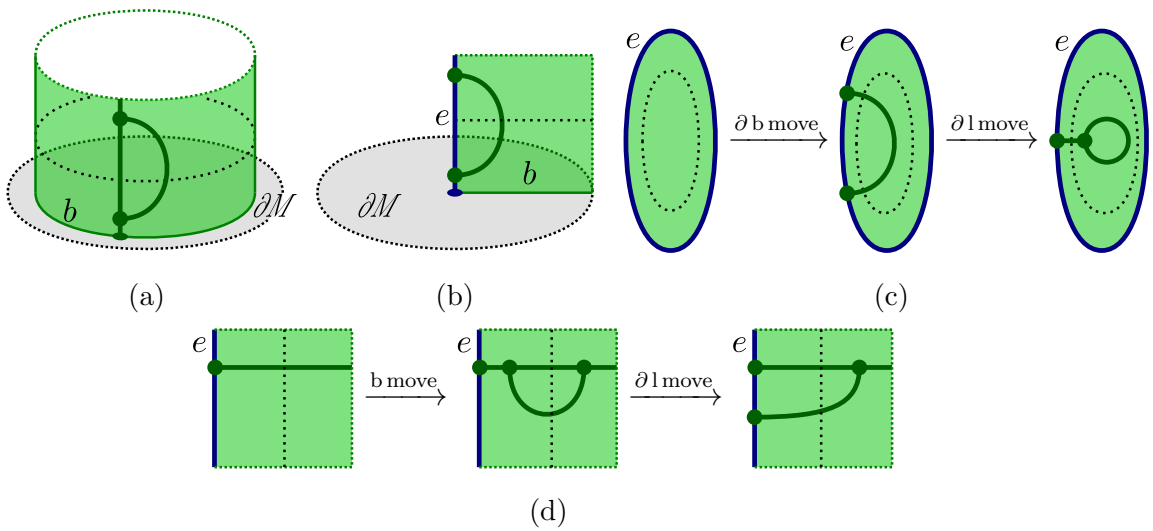


Figure B.4: Modifying a 1-skeleton of a 2-stratum of $M \in \mathrm{Bord}_3^{\mathrm{def}}$ near the boundary. The dotted line on the shaded surface represents the imaginary boundary line, at which the 0-skeleton can be freely modified.

show that, regardless of the stratification of $\partial\overline{\overline{s}}$, t can be changed inside a germ of the boundary $\partial\overline{\overline{s}} \times [0,1) \hookrightarrow \overline{\overline{s}}$ so that t restricts to a globally ordered ∇-skeleton on $\partial\overline{\overline{s}} \times \{1/2\}$, which we will call the *imaginary boundary*. There are several cases to consider

1. Let b be a non-contractible boundary line. Then t has at least one line that ends on b. Apply a b-move on this line near the boundary so that the interior of the new 2-stratum intersects $b \times \{1/2\}$, see Figure B.4a. On the imaginary boundary there are now at least two points, the orientation of one of which can be chosen arbitrarily. One can now repeat the process.

2. Let b be a contractible boundary line. Then at the side of it there must be an adjacent inner line e. Apply a ∂ b-move on e and proceed similarly as in case 1, see Figure B.4b.

3. Let e be an inner line. If $t = \varnothing$, which is possible if $\overline{\overline{s}}$ is contractible, one uses a combination of ∂ b- and ∂ l-moves to add a single line intersecting e as shown in Figure B.4c. Once e has at least one adjacent e line, one can use combinations of b- and ∂ l-moves to add an arbitrary number of lines with arbitrary orientations, see Figure B.4d.

This finally proves Lemma 4.6.

B.2. 2-skeleta for 3-manifolds

In this appendix section we sketch the proof of Lemma 6.2. This can be done by merely repeating the steps in Section B.1 for the admissible 2-skeleta of 3-manifolds and the BLT moves relating them. The detailed proof along these lines is available in [CMRSS1, App. A].

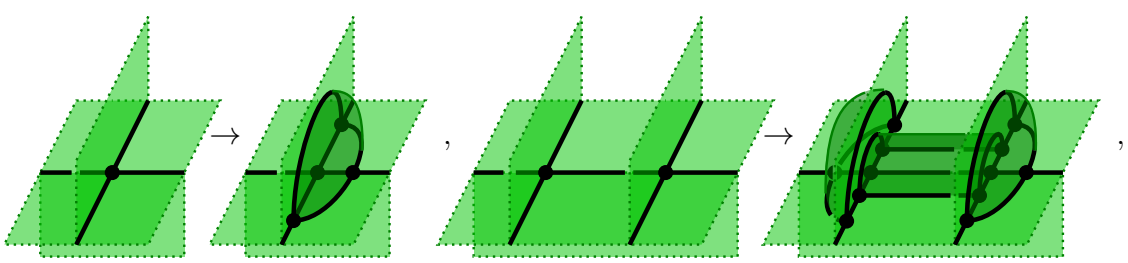
Let us mentioned a small conventional caveat regarding the admissibility of orientations of 2-skeleta. Recall that in Section 6.1, an admissible 2-skeleton of a bordism $M \in \mathrm{Bord}_3$ was defined so that each point has a neighbourhood, corresponding to one of the local models in Figure 6.1. The globally ordered ∇-skeleta would then in principle not be admissible, as their 1-strata are oriented the other way (compare the local models to Figure B.1b). This discrepancy arises due to the difference in conventions used in [CRS1] and [CRS3]: in the main text we follow those of [CRS3], as we rely greatly on the algebraic results presented there. Here we will make the following

Convention B.7. In this section only, the 1-strata of admissible 2-skeleta of 3-manifolds are assumed to have orientations, opposite to the ones listed in the local models in Figure 6.1. This does not change the validity of the proof below.

With this convention, globally ordered ∇-skeleta of M are indeed admissible. Moreover, admissible 2-skeleta induce admissible 1-skeleta on the boundary ∂M, i.e. with the orientations exactly as in Figure 4.5.

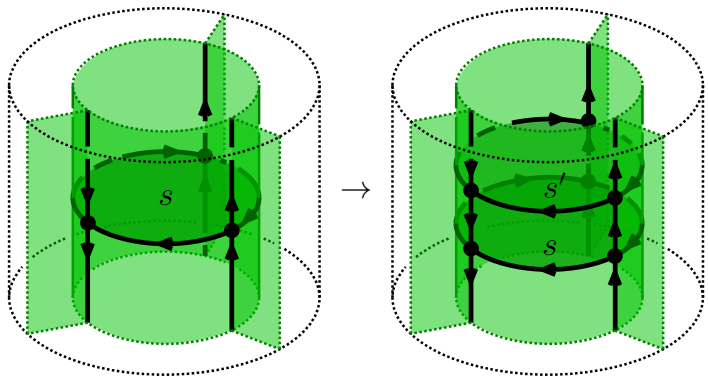
We now turn to proving Lemma 6.2:

- If T is an admissible 2-skeleton for M, such that $\partial\,\mathrm{T}$ is an admissible ∇-skeleton, then there is an admissible ∇-skeleton T' for M and a sequence of admissible BLT moves from T to T' (cf. Lemma B.3). This is because one can use the B- and L-moves to make the strata contractible and use the BLT moves to surround the 1- and 0-strata of T with newly created strata as follows:



  which is then used to fulfil the conditions of Lemma B.2. All this can be done without using the inverse L-move, so the BLT moves can be assumed to be admissible.

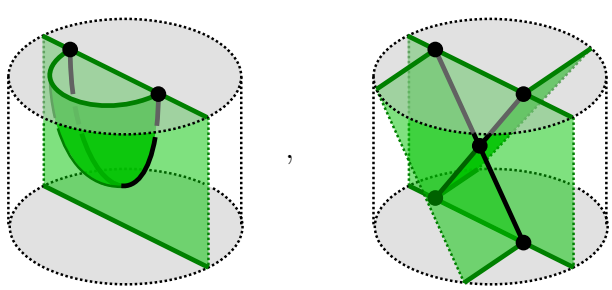
- Let s be a contractible 2-stratum, which does not intersect ∂M and such that for each 1-stratum l, adjacent to s, the other two 2-strata adjacent to l do not both induce the same orientation on l. Then the orientations of s and those of the adjacent to s 0- and 1-strata can be changed by a sequence of admissible BLT moves (cf. Lemma B.4). This is because one make a copy s' of s having the opposite orientation to that of s:



and then apply similar moves in reverse to remove s.

- If T and T' are admissible ∇-skeleta of M, such that $\partial\,\mathrm{T} = \partial\,\mathrm{T}'$ and which otherwise differ only by the orientations (i.e. $\underline{\mathrm{T}} = \underline{\mathrm{T}'}$), then there is a sequence of admissible BLT moves from T to T' (cf. Lemma B.5). To show this one adapts the proof of Lemma B.5: subdivide each 3-face using the Pachner 1-4 move orienting towards the new vertex; apply the Pachner 2-3 move at each inner 2-face; the orientations of the newly created 2-faces can be flipped like in the previous step.

- At this point Lemma B.2 is shown when $\partial\,\mathrm{T} = \partial\,\mathrm{T}'$ is a globally ordered ∇-skeleton of ∂M (cf. Corollary B.6). One can then pre- and postcompose the stratified bordisms (M, T), (M, T') with stratified cylinders $\partial M \times [0, 1]$, which at a patch of ∂M look like



They then implement the admissible bl-moves on the boundary and can therefore be used to bring any admissible 2-skeleton of ∂M to a globally ordered ∇-skeleton.

C. Orbifold data for the non-Euler complete theory

In this appendix section we review the definitions of orbifold data and the associated categories for the version of the defect TQFT $Z^{\mathrm{def}}_{\mathcal{C}}$, which labels surface defects with symmetric Δ-separable Frobenius algebras only, and is therefore not Euler complete with respect to surfaces (see Remark 5.7). This setting is used in some of our main references, for example [CRS2, CRS3, CMRSS1, MR1, MR2]; it has a disadvantage of making the 3-dimensional graphical calculus as introduced in Section 5.2 cluttered with point insertions, and an advantage of making explicit calculations easier in some of the examples. The setting in the main text is in principle more general, as a symmetric Δ-separable Frobenius algebra with an invertible point insertion can be exchanged for a single symmetric separable Frobenius algebra as explained in the Property 4.14.

Let $\mathcal{C}$ be a MFC. In the setting of Δ-separable algebras one defines an orbifold datum in $\mathcal{C}$ to be a tuple $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$, where the entries are as follows:

- A is a symmetric Δ-separable Frobenius algebra (a label for 2-strata);
- T is an A-$A \otimes A$ bimodule (a label for lines with three adjacent 2-strata as in Figure 5.3a);
- $\alpha\colon T \otimes_2 T \leftrightarrow T \otimes_1 T : \overline{\alpha}$ are A-$A \otimes A \otimes A$-bimodule morphisms (labels for points as in Figures 5.3c, 5.3d);
- $\psi\colon A \to A$ is an invertible A-A-bimodule morphism (label for a point insertion on an A-labelled 2-stratum);
- $\phi \in \Bbbk^\times$ (label for a point on a 3-stratum),

and are required to satisfy the conditions in Figure C.1.

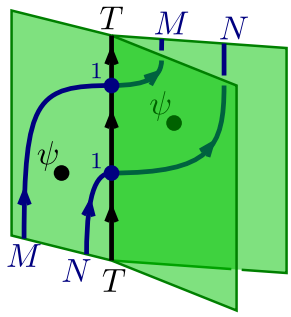
Similarly one defines the category $\mathcal{C}_{\mathbb{A}}$ to have objects $(M, \tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2})$, where

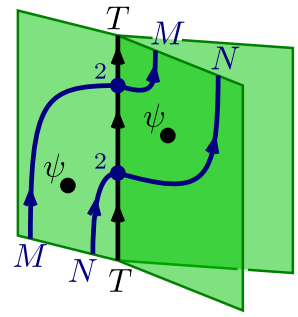
- M is an A-A-bimodule (label for lines on A-labelled 2-strata);
- $\tau_1\colon M \otimes_0 T \leftrightarrow T \otimes_1 M : \overline{\tau_1}$ and $\tau_2\colon M \otimes_0 T \leftrightarrow T \otimes_2 M : \overline{\tau_2}$ are A-$A \otimes A$-bimodule morphisms (labels for points as in Figure 5.5),

which satisfy the conditions in Figure C.2. As in the main text, the morphisms of $\mathcal{C}_{\mathbb{A}}$ are taken to be A-A-bimodule morphisms, such that the identity (M) holds.

The ψ-insertions on 2-strata also appear in defining some structural morphisms in the associated categories $\mathcal{C}_{\mathbb{A}}$, $\mathcal{C}^1_{\mathbb{A}}$, $\mathcal{C}^2_{\mathbb{A}}$, for example:

- The crossing morphisms of the tensor product of two objects $M, N \in \mathcal{C}_{\mathbb{A}}$ are

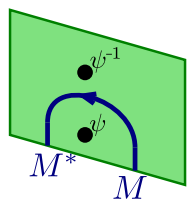
$\tau_1^{M,N} =$ 

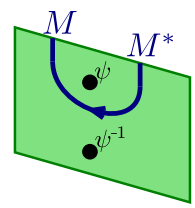
, $\tau_2^{M,N} =$ 

, (C.1)

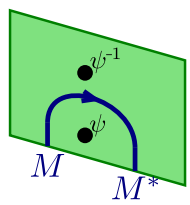
(similarly for $\overline{\tau_1^{M,N}}$, $\overline{\tau_2^{M,N}}$).

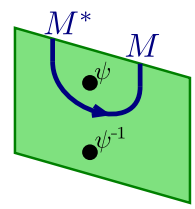
- The (co)evaluation morphisms for $M \in \mathcal{C}_{\mathbb{A}}$ are redefined as

$\mathrm{ev}_M =$ 

, $\mathrm{coev}_M =$ 

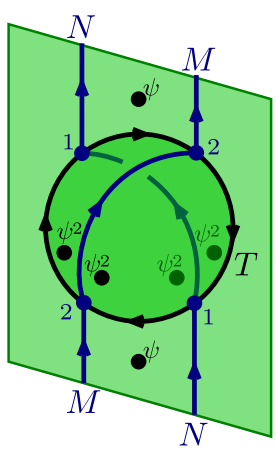
, (C.2)

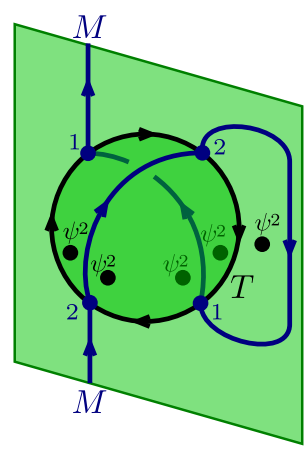
$\widetilde{\mathrm{ev}}_M =$ 

, $\widetilde{\mathrm{coev}}_M =$ 

. (C.3)

- The braiding morphisms for $M, N \in \mathcal{C}_{\mathbb{A}}$, as well as the twist of M become

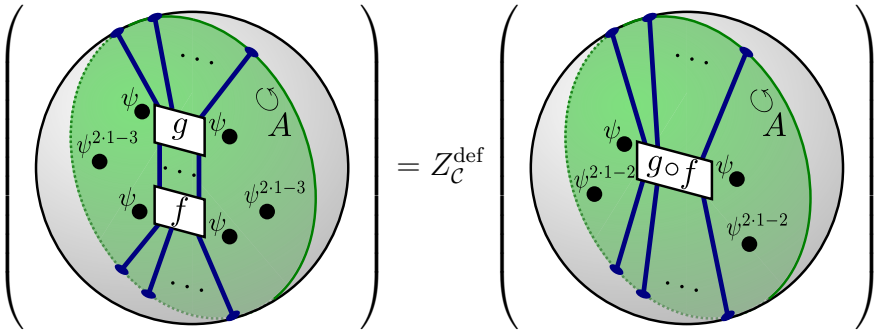
$c_{M,N} = \phi^2 \cdot$ 

, $\theta_M = \phi^2 \cdot$ 

, (C.4)

(similarly for $c_{M,N}^{-1}$, θ_M^{-1}).

The definition of the orbifold graph TQFT $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ remains the same with one exception: In the definition of an admissible $\mathbb{A}$-coloured ribbon diagram of a bordism $M = (M, R) \in \widehat{\mathrm{Bord}}_3^{\mathrm{rib}}(\mathcal{C}_{\mathbb{A}})$, each 2-stratum s of M acquires an additional $\psi^{\chi_{\mathrm{sym}}(s)}$-insertion (where χ_{sym} is the symmetric Euler characteristic, see (4.7)), and for each coupon c of R, the leftmost and the rightmost 2-strata adjacent to c acquire one additional ψ-insertion each; when computing $\chi_{\mathrm{sym}}(s)$ the boundary segments of coupons in R are treated like boundary segments of ∂M. The latter modification is introduced to make sure that, upon evaluating with $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$, the coupons of R can be composed in the same way as A-A-bimodule morphisms. Indeed, by the definition of $Z_{\mathcal{C}}^{\mathrm{orb}\,\mathbb{A}}$ in terms of $Z_{\mathcal{C}}^{\mathrm{def}}$, for two such morphisms f and g, which can be composed into $g \circ f$, then evaluates to the same vector independent on whether their coupons are composed or not:

$Z_{\mathcal{C}}^{\mathrm{def}}$ 

$= Z_{\mathcal{C}}^{\mathrm{def}}$. (C.5)

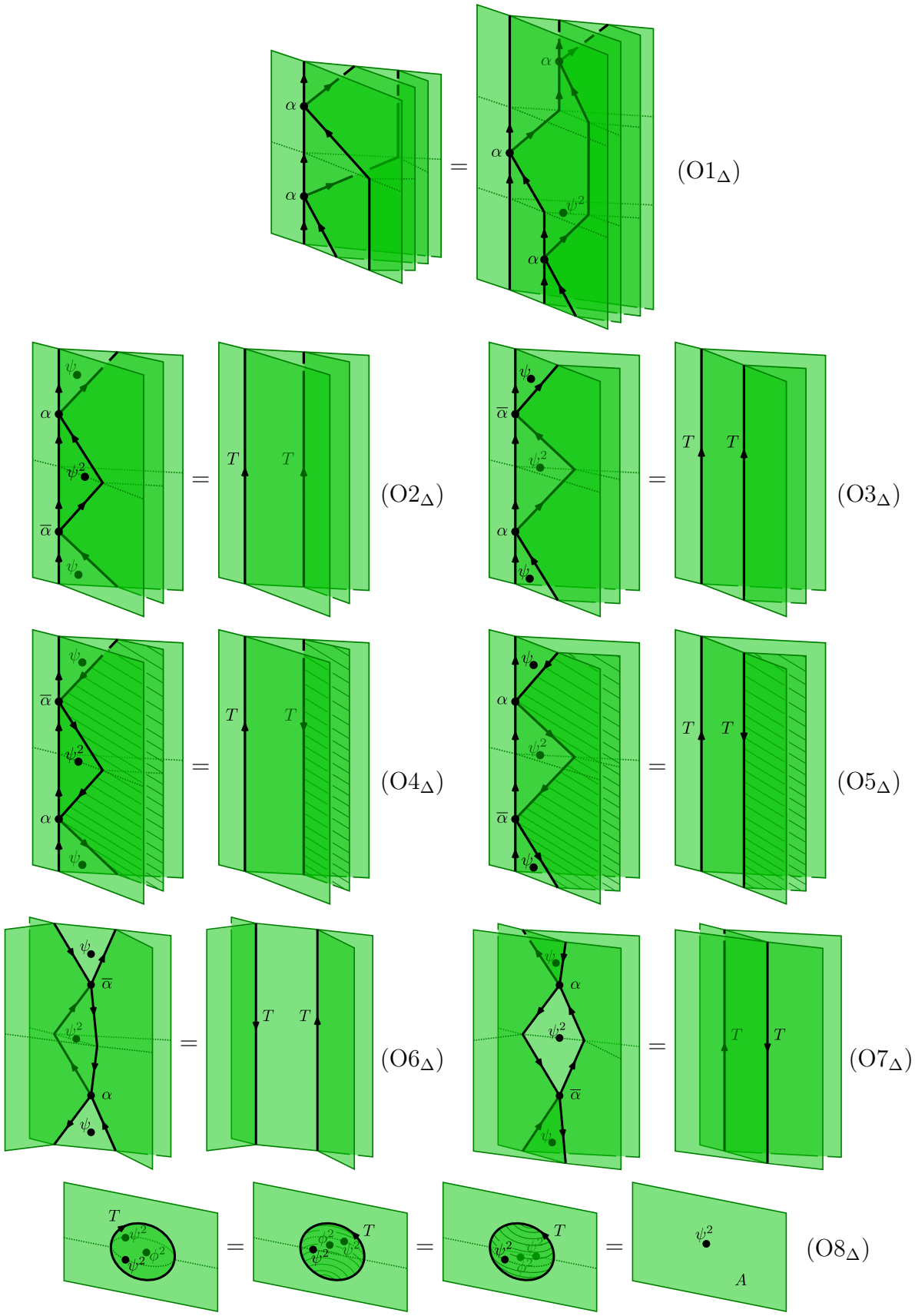


Figure C.1: Conditions on $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ with point insertions

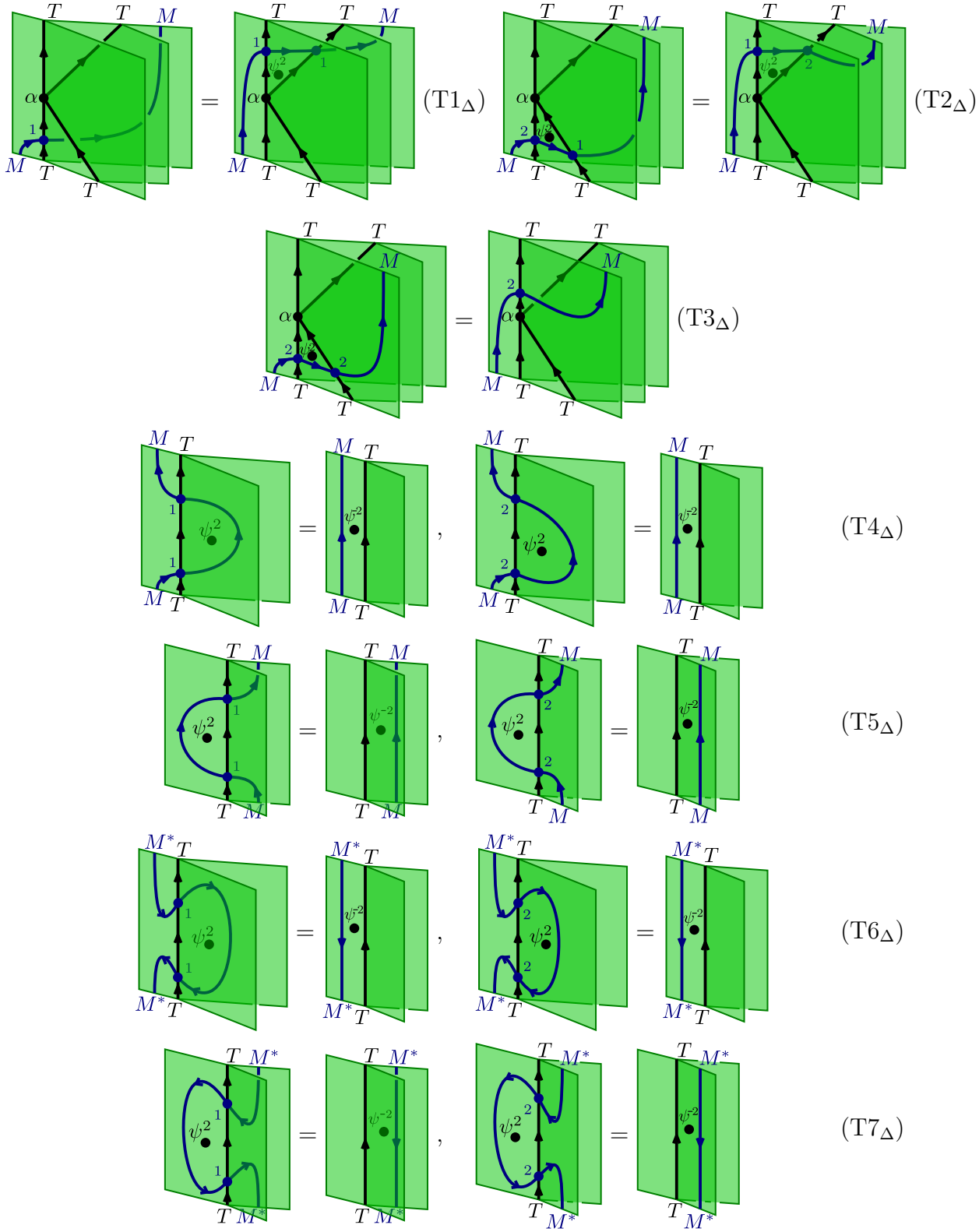


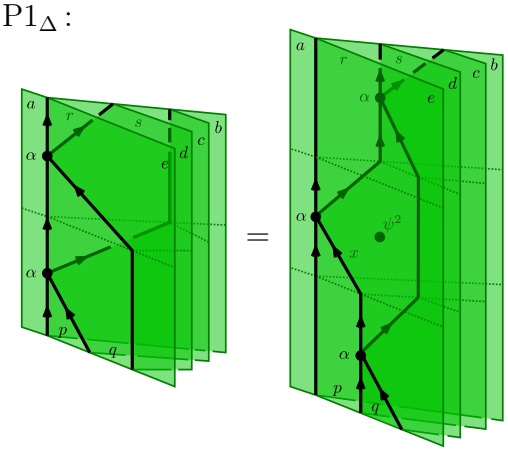
Figure C.2: Conditions on $(M, \tau_1, \tau_2) \in \mathcal{C}_{\mathbb{A}}$ with point insertions

D. Computations for Chapter 8

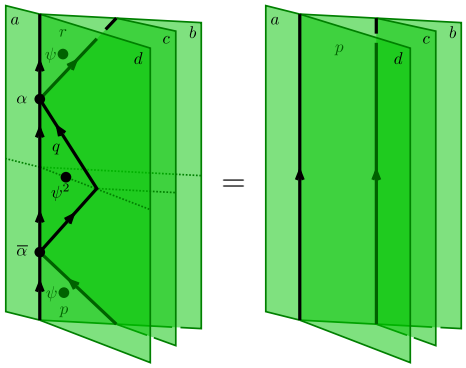
D.1. Deriving the polynomial equations (P1)-(P8)

The polynomial equations (P1)-(P8) are derived from the conditions ($\mathrm{O1}_\Delta$)-($\mathrm{O8}_\Delta$) on an orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ in a MFC $\mathcal{C}$, presented in the setting of a symmetric Δ-separable Frobenius algebra A with an invertible point insertion $\psi\colon A \to A$ (see Appendix C) and satisfying the assumptions (A1)–(A3). By assumption (A2), the algebra A is a direct sum of copies of the tensor unit, indexed by the elements of a set B. In the pictures below, we indicate for each surface the index (which runs through the elements of B) that was used to represent the corresponding A-action when converting the conditions ($\mathrm{O1}_\Delta$)-($\mathrm{O8}_\Delta$) into the polynomial equations (P1)-(P8) in Table 8.1.

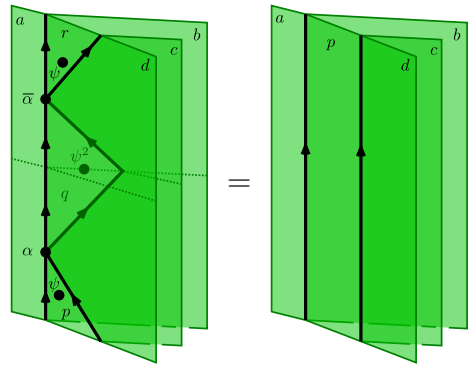
$\mathrm{P1}_\Delta$:



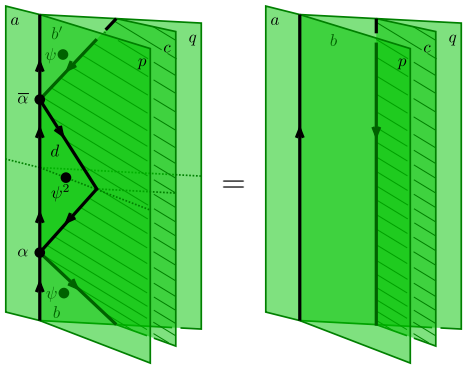
$\mathrm{P2}_\Delta$:



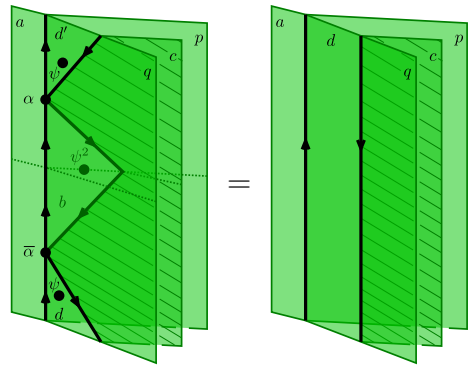
$\mathrm{P3}_\Delta$:



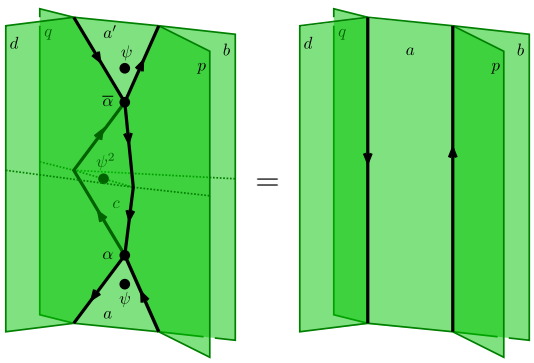
P4$_\Delta$:



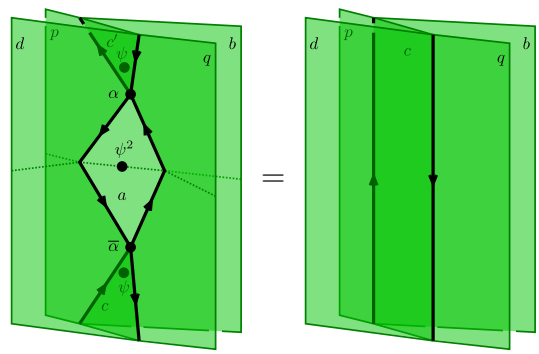
P5$_\Delta$:



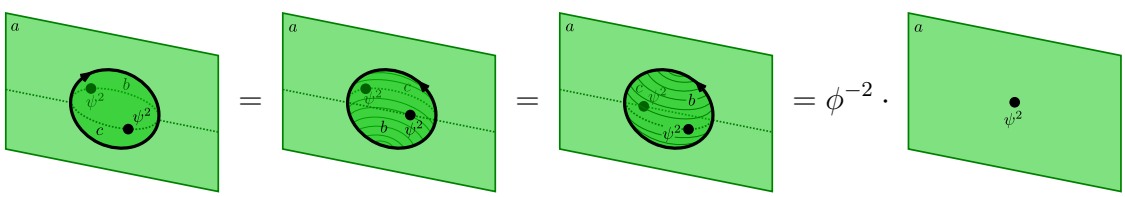
P6$_\Delta$:



P7$_\Delta$:

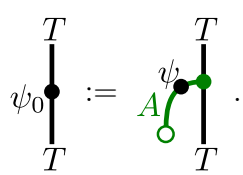


P8$_\Delta$:



Let us now provide the explicit derivation of the equation (P1). The other equations can then be handled similarly.

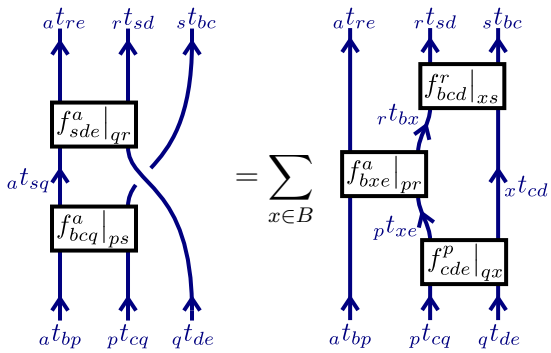
In terms of string diagrams, the equation (O1$_\Delta$) has the same form as the equation (O1) in Figure 5.1, where instead of (5.2) one uses the definition



(D.1)

for the ψ_0-insertion. Under assumptions (A1)-(A3) it is equivalent to the collection

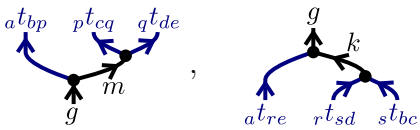
of identities



$$a, b, c, d, e, p, q, r, s \in B \ , \qquad \text{(D.2)}$$

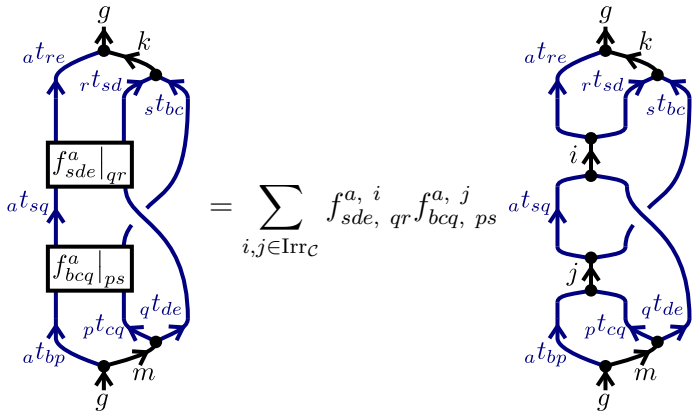
where we used (8.11) and cancelled the appearances of ψ on both sides. Notice that the indexing matches the 3-dimensional presentation (P1$_\Delta$) above. For example, on the left-hand side of (D.2), the object $_at_{bp}$ at the bottom left corresponds in (P1$_\Delta$) to the bottom left line, which is connected to the surface labelled a on the left, and to two surfaces labelled b and p on the right.

Let us decompose the source and target of the morphism in (D.2) into simple objects by composing both sides with



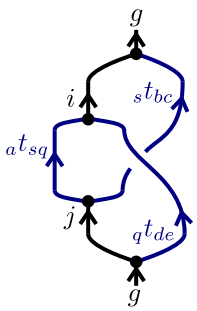
$$g, k, m \in \mathrm{Irr}_{\mathcal{C}} \ . \qquad \text{(D.3)}$$

On the left-hand side one gets:



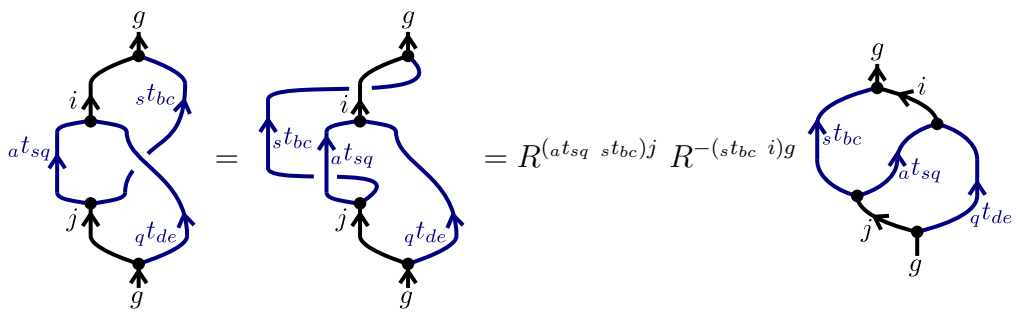
(D.4)

$$= \sum_{i,j\in \mathrm{Irr}_{\mathcal{C}}} f^{a,\ i}_{sde,\ qr}\ f^{a,\ j}_{bcq,\ ps}\ F^{({}_a t_{re}\ \ {}_r t_{sd}\ \ {}_s t_{bc})g}_{k\,i}\ G^{({}_a t_{bp}\ \ {}_p t_{cq}\ \ {}_q t_{de})g}_{j\,m}$$



, (D.5)

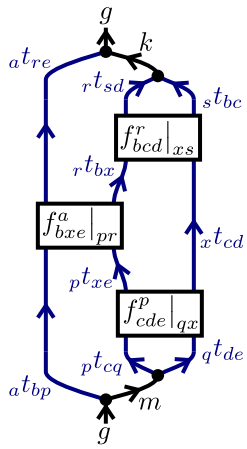
where in the second equality one inserts a projection/inclusion into g from the products $j \otimes {}_q t_{de}$ and $i \otimes {}_s t_{bc}$ and uses (8.7). The remaining diagram can be quickly evaluated as follows:



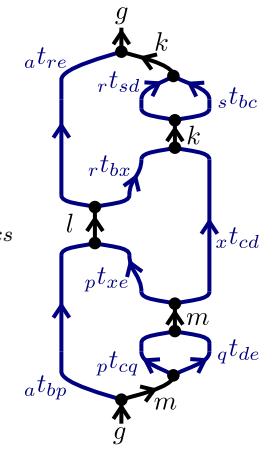
$$= R^{({}_a t_{sq}\ \ {}_s t_{bc})j}\ R^{-({}_s t_{bc}\ \ i)g}\ F^{({}_s t_{bc}\ \ {}_a t_{sq}\ \ {}_q t_{de})g}_{i\,j} \cdot \mathrm{id}_g\ . \tag{D.6}$$

Similarly we compute the right hand side of (D.2) upon composing with the morphisms in (D.3):

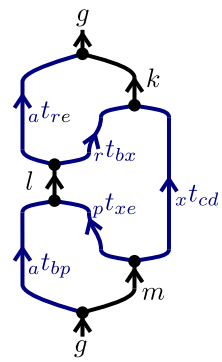
$$\sum_{x\in B}$$



$$= \sum_{x\in B,\ l\in \mathrm{Irr}_{\mathcal{C}}} f^{p,\ m}_{cde,\ qx}\ f^{a,\ l}_{bxe,\ pr}\ f^{r,\ k}_{bcd,\ xs}$$



$$= \sum_{x\in B,\ l\in \mathrm{Irr}_{\mathcal{C}}} f^{p,\ m}_{cde,\ qx}\ f^{a,\ l}_{bxe,\ pr}\ f^{r,\ k}_{bcd,\ xs}\ N^{m}_{{}_p t_{cq}\ {}_q t_{de}}\ N^{k}_{{}_r t_{sd}\ {}_s t_{bc}}$$

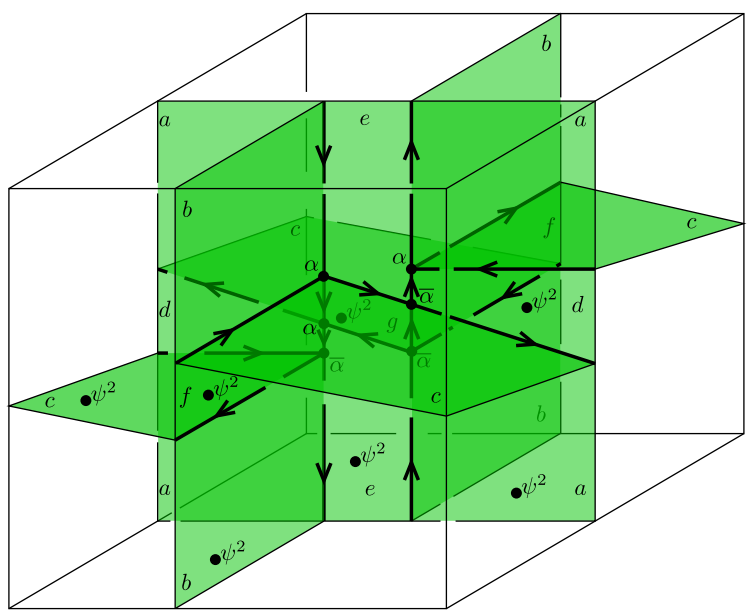


$$= \sum_{x\in B,\ l\in \mathrm{Irr}_{\mathcal{C}}} f^{p,\ m}_{cde,\ qx}\ f^{a,\ l}_{bxe,\ pr}\ f^{r,\ k}_{bcd,\ xs}\ F^{({}_a t_{re}\ {}_r t_{bx}\ {}_x t_{cd})g}_{k\,l}\ G^{\left({}_a t_{bp}\ {}_p t_{xe}\ {}_x t_{cd}\right)g}_{l\,m} \cdot \mathrm{id}_g\ . \tag{D.7}$$

Comparing coefficients in (D.5) and (D.7) gives condition (P1) in Table 8.1.

D.2. Evaluating the T^3-invariant

In this section we compute the invariant $Z^{\mathrm{orb}\,\mathbb{A}}_{\mathcal{C}}(T^3)$ of the 3-torus $T^3 = S^1 \times S^1 \times S^1$ where the orbifold datum $\mathbb{A} = (A, T, \alpha, \overline{\alpha}, \psi, \phi)$ in a MFC $\mathcal{C}$ satisfies the assumptions (A1)-(A3). We use the following admissible $\mathbb{A}$-coloured 2-skeleton for T^3:



. (D.8)

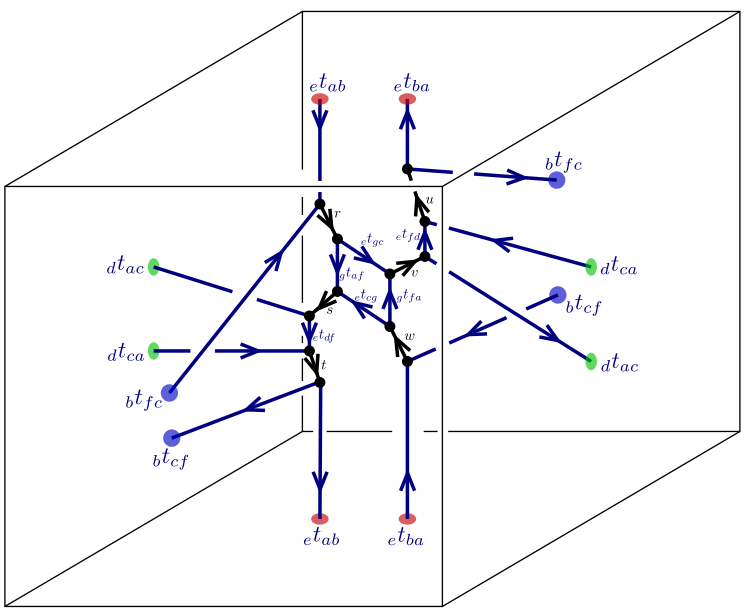
Here T^3 is depicted as a cube with the opposite sides identified, all of the 2-strata have the paper plane orientation and a ψ^2-insertion due to A being a symmetric Δ-separable Frobenius algebra (see Appendix C). The labels $a, b, \ldots, g$ on each

2-stratum match the index of the summand $\mathbb{1}$ in the direct sum decomposition of the corresponding copy of A as used below.

Next one ribbonises the above 1-skeleton and evaluates it with the Reshetikhin-Turaev TQFT $Z_{\mathcal{C}}^{\mathrm{RT}}$ as described in Section 4.4. Using the expressions (8.10), (8.11), (8.12), (8.14) for α and $\overline{\alpha}$ one gets:

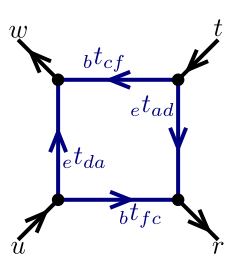
$$Z_{\mathcal{C}}^{\mathrm{orb},\mathbb{A}}(T^3) = \phi^2 \cdot \sum_{\substack{a,b,c,d,e,f,g\in B\\ r,s,t,u,v,w\in \mathrm{Irr}_{\mathcal{C}}}} \frac{\psi_a^2\psi_c^2\psi_e^2\psi_f^2}{\psi_b^2\psi_d^2\psi_g^2}\, f_{afc,\ bg}^{e,\ r}\, f_{caf,\ gd}^{e,\ s}\, f_{fca,\ db}^{e,\ u}\, g_{acf,\ db}^{e,\ t}\, g_{fac,\ gd}^{e,\ v}\, g_{cfa,\ bg}^{e,\ w} \cdot Z_{\mathcal{C}}^{\mathrm{RT}}(T^3_{\mathrm{rib}}) \ . \tag{D.9}$$

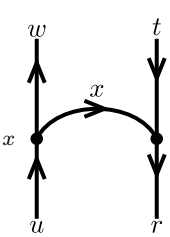
where

$T^3_{\mathrm{rib}} :=$ 

. (D.10)

To evaluate the invariant of the torus (D.10), let us introduce the scalars $L_{eda,\ ur}^{bcf,\ wt}\big|_x$, where $a,b,c,d,e \in B$ and $x,u,r,w,t \in \mathrm{Irr}_{\mathcal{C}}$, such that



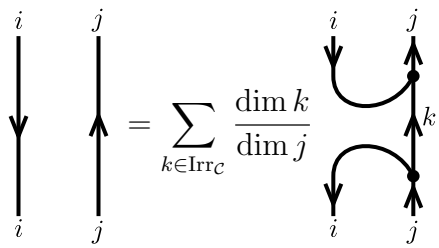
$= \sum_{x\in \mathrm{Irr}_{\mathcal{C}}} L_{eda,\ ur}^{bcf,\ wt}\big|_x$ 

. (D.11)

We claim that the following equality holds:

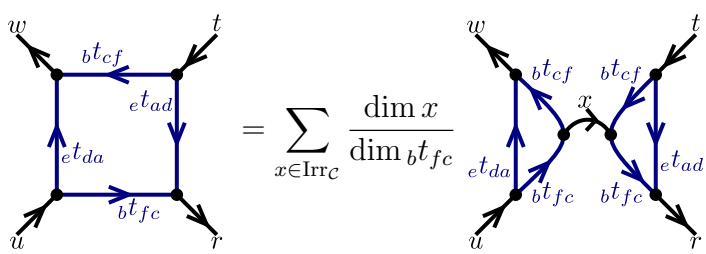
$$L^{bcf,\ wt}_{eda,\ ur}\Big|_x = N^{w}_{{}^{e}t_{da}\ \ {}^{b}t_{cf}}\ N^{t}_{{}^{e}t_{ad}\ \ {}^{b}t_{cf}}\ \frac{\dim x}{\dim {}^{b}t_{fc}}\ F^{({}^{e}t_{ad}\ \ {}^{b}t_{cf}\ \ x)r}_{{}^{b}t_{fc}\ \ t}\ G^{({}^{e}t_{da}\ \ {}^{b}t_{cf}\ \ x)u}_{w\ \ {}^{b}t_{fc}}\ . \tag{D.12}$$

Indeed, use the identity

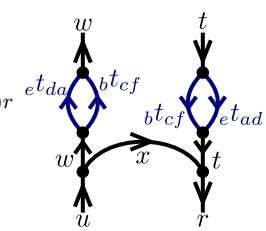


(D.13)

to rewrite the left-hand side of (D.11) as



$$= \sum_{x\in\mathrm{Irr}_{\mathcal{C}}} \frac{\dim x}{\dim {}^{b}t_{fc}} G^{({}^{e}t_{da}\ \ {}^{b}t_{fc}\ \ x)u}_{w\ \ {}^{b}t_{fc}}\ F^{({}^{e}t_{ad}\ \ {}^{b}t_{cf}\ \ x)r}_{{}^{b}t_{fc}\ \ t}$$



. (D.14)

We now focus on evaluating the invariant of the torus T^3_{rib} as in (D.10). Let us denote by $T^3_{(\mathrm{a})}$, $T^3_{(\mathrm{b})}$, …, $T^3_{(\mathrm{f})}$ the 3-tori with embedded ribbon graphs as depicted in Figure D.1. One has:

$$Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{\mathrm{rib}}) = \sum_{x,y,k\in\mathrm{Irr}_{\mathcal{C}}} L^{bcf,\ wt}_{eba,\ ur}\Big|_x\ L^{dca,\ us}_{efd,\ vt}\Big|_y\ L^{gaf,\ sr}_{ecg,\ wv}\Big|_k \cdot Z_{\mathcal{C}}(T^3_{(\mathrm{a})})\ , \tag{D.15}$$

where

$$Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{(\mathrm{a})}) = \sum_{z\in\mathrm{Irr}_{\mathcal{C}}} \frac{\dim z}{\dim u} \cdot Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{(\mathrm{b})})\ ,\quad Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{(\mathrm{b})}) = F^{(r\,z\,y)\,v}_{k\,u} \cdot Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{(\mathrm{c})})\ ,$$

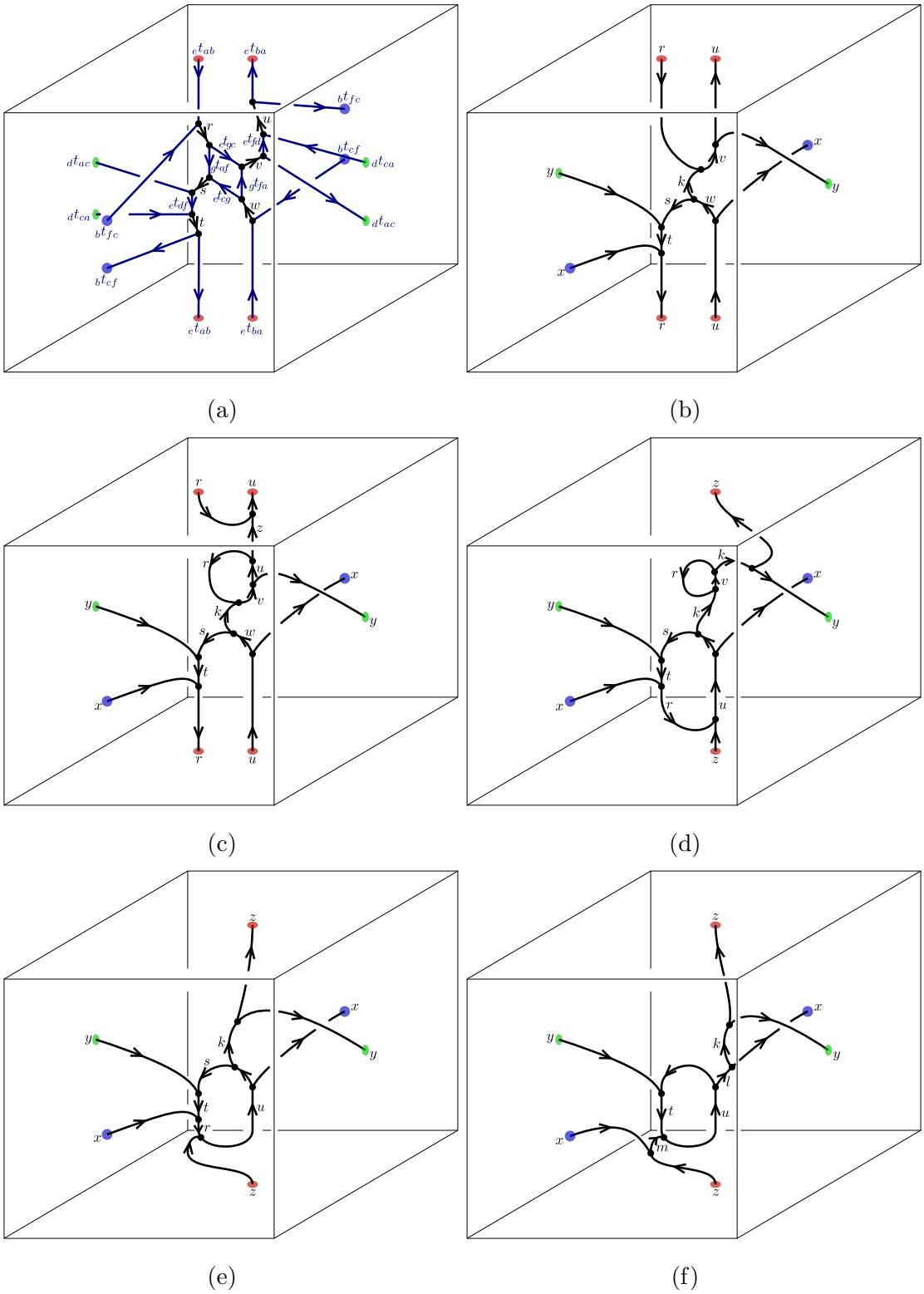


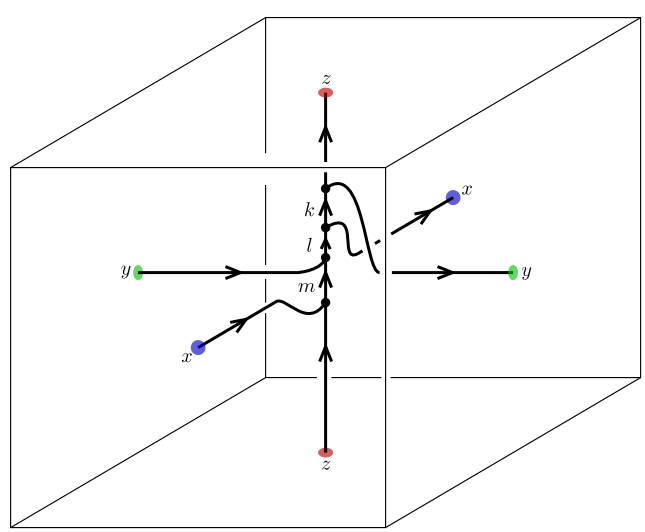
Figure D.1: Three-torus with a series of embedded ribbon graphs as used in the calculation in (D.15).

$$Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{(\mathrm{c})}) = \frac{\dim v}{\dim k} \cdot Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{(\mathrm{d})}) \ , \qquad Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{(\mathrm{d})}) = \sum_{l,m \in \mathrm{Irr}_{\mathcal{C}}} F^{(s\,k\,x)u}_{l\,w}\, G^{(t\,x\,z)u}_{r\,m} \cdot Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{(\mathrm{e})}) \ ,$$

$$Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{(\mathrm{e})}) = G^{(s\,y\,m)u}_{t\,l} \cdot Z_{\mathcal{C}}(T^3_{(\mathrm{f})}) \ , \qquad Z^{\mathrm{RT}}_{\mathcal{C}}(T^3_{(\mathrm{f})}) = \frac{\dim u}{\dim l} \cdot T_{xyz,\ klm} \ ,$$

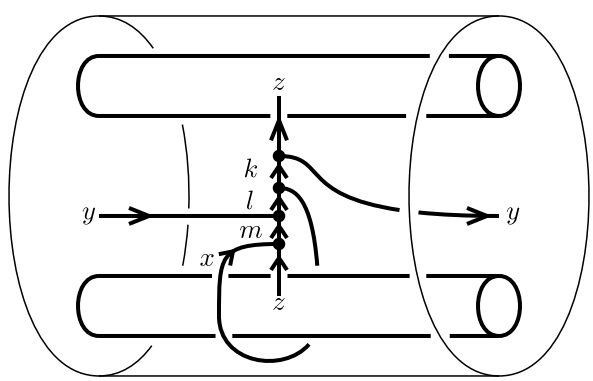
where in the last equation we use the notation

$$T_{xyz,\ klm} := Z^{\mathrm{RT}}_{\mathcal{C}} \Bigg(\quad \Bigg) \ . \tag{D.16}$$



Combining all of the equations in (D.15) one already obtains (8.25). It remains to derive the expression for $T_{xyz,\ klm}$.

By Property 3.14, the invariant of $T^3 = S^1 \times S^1 \times S^1$ can also be computed as the trace of the operator invariant assigned to the cylinder $C = S^1 \times S^1 \times [0,1]$. The same is true if T^3 has an embedded ribbon graph, in which case C can have punctures on its boundary. For the cylinder corresponding to the 3-torus in the argument of $Z^{\mathrm{RT}}_{\mathcal{C}}$ in (D.16) we will use the graphical representation

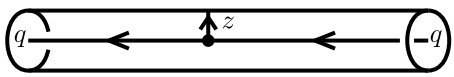


(D.17)

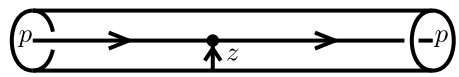
where the outer tube represents the manifold $S^2 \times S^1$ (the ends of the tube on the left and on the right are assumed to be identified and the boundary circle of a

vertical slice corresponds to the point at infinity of S^2). The two small inner tubes represent the boundary components (seen here as obtained by removing two solid tori from $S^2 \times S^1$). The tube at the bottom corresponds to the incoming boundary component, while the one at the top to the outgoing one.

By Property 3.13, the vector space assigned to a 2-torus with a single $z \in \mathrm{Irr}_{\mathcal{C}}$ labelled puncture is isomorphic to $\bigoplus_{q \in \mathrm{Irr}_{\mathcal{C}}} \mathcal{C}(q, q \otimes z)$ and its dual to $\bigoplus_{p \in \mathrm{Irr}_{\mathcal{C}}} \mathcal{C}(p \otimes z, p)$. The image of a basis element $\lambda^{(qz)q}$ and evaluation with the dual basis element $\lambda_{(pz)p}$ are obtained by gluing the solid tori

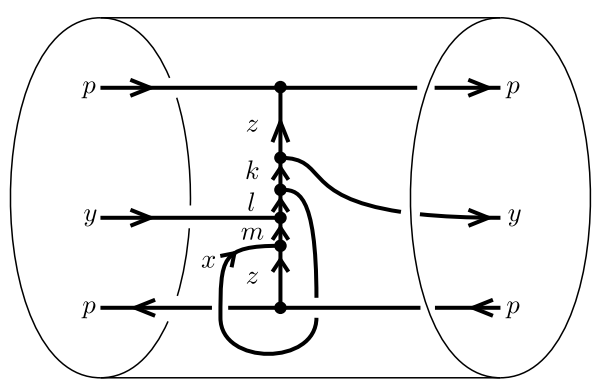


and

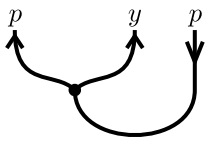


to the incoming and outgoing boundary respectively. One then has:

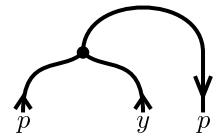
$$T_{xyz,\ klm} = \sum_{p \in \mathrm{Irr}_{\mathcal{C}}} Z_{\mathcal{C}}^{\mathrm{RT}} \Bigg(\quad \Bigg) \quad . \tag{D.18}$$



Now, the invariant of $S^2 \times S^1$ is equal to the trace of the operator invariant of the cylinder $S^2 \times [0,1]$. The vector space assigned to a 2-sphere with three punctures labelled by p, y, p^* as in (D.18) is $\mathcal{C}(\mathbb{1}, p \otimes y \otimes p^*)$ with the dual $\mathcal{C}(p \otimes y \otimes p^*, \mathbb{1})$. Using the basis



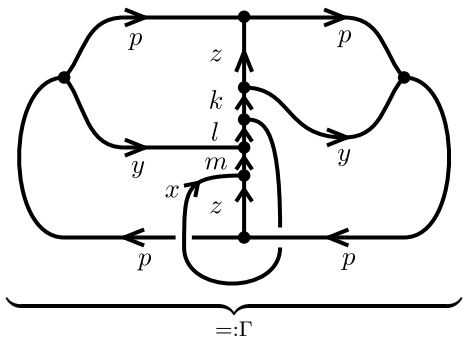
with dual $\dfrac{1}{\dim p}$



(D.19)

one has:

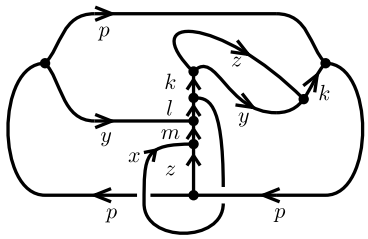
$$T_{xyz,\ klm} = \sum_{p\in\mathrm{Irr}_{\mathcal{C}}} \frac{N_{py}^{p}}{\dim p}$$



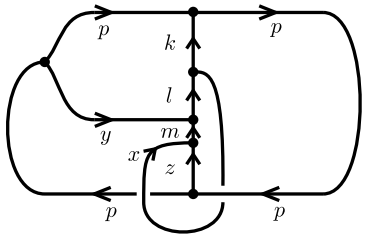
. (D.20)

The scalar represented by the string diagram Γ can be expressed as follows:

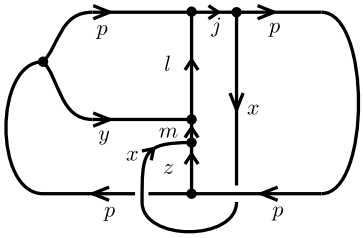
$$\Gamma = G_{p\,k}^{(p\,z\,y)p} \cdot$$



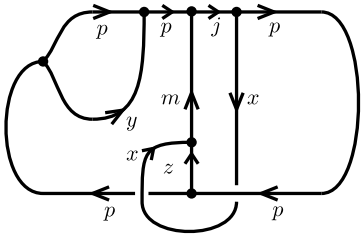
$$= G_{p\,k}^{(p\,z\,y)p} N_{zy}^{k} \cdot$$



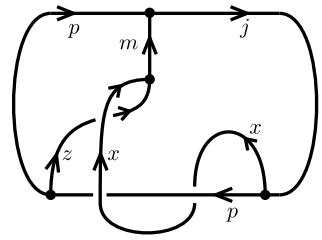
$$\overset{(*)}{=} G_{p\,k}^{(p\,z\,y)p} \sum_{j\in\mathrm{Irr}_{\mathcal{C}}} G_{p\,l}^{(p\,k\,x)j} \cdot$$



$$= G_{p\,k}^{(p\,z\,y)p} \sum_{j\in\mathrm{Irr}_{\mathcal{C}}} G_{p\,l}^{(p\,k\,x)j} F_{l\,p}^{(p\,y\,m)j} \cdot$$

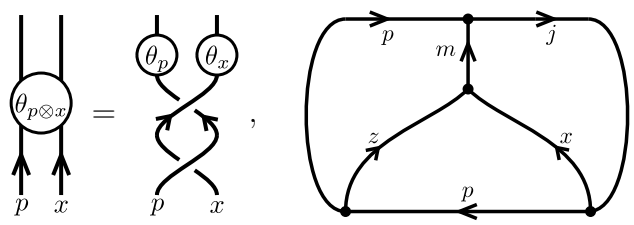


$$= G_{p\,k}^{(p\,z\,y)p} \sum_{j\in \mathrm{Irr}_{\mathcal{C}}} G_{p\,l}^{(p\,k\,x)j} F_{l\,p}^{(p\,y\,m)j} N_{py}^{p} \cdot$$



$$\overset{(**)}{=} G_{p\,k}^{(p\,z\,y)p} \sum_{j\in \mathrm{Irr}_{\mathcal{C}}} G_{p\,l}^{(p\,k\,x)j} F_{l\,p}^{(p\,y\,m)j} R^{-(z\,x)m} \frac{\theta_j}{\theta_x \theta_p} F_{m\,p}^{(p\,z\,x)j} \cdot \dim j \ .$$

In step $(*)$ we omitted N_{zy}^{k} as it is implicit in $G_{pk}^{(pzy)p}$. In step $(**)$ we omitted N_{py}^{p} for the same reason, and we used the identities



$$= F_{m\,p}^{(p\,z\,x)j} \cdot \dim j \ . \qquad \text{(D.21)}$$

Substituting the above expression for Γ into (D.20) yields exactly the expression for $T_{xyz,\ klm}$ in Lemma 8.5.

E. Computations for Chapter 9

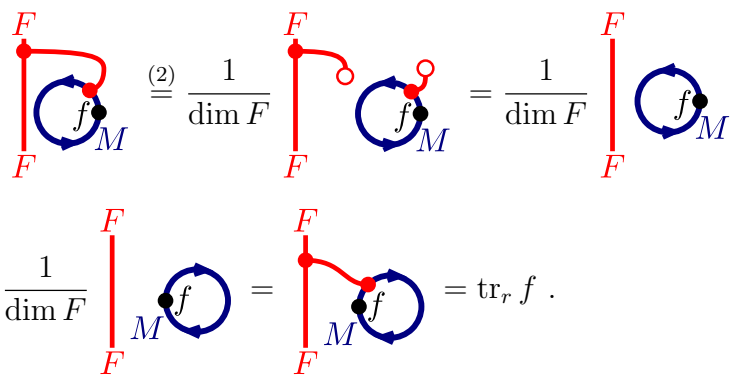
Let $\mathcal{C}$ be a modular fusion category and $A, B \in \mathcal{C}$ condensable algebras. We show here that the component category $\mathcal{F} := ({}_A\mathcal{C}_B)_{ii}$ is spherical and that the functor (9.23) is ribbon.

From Proposition 2.2 we know that one has a decomposition $A \otimes B \cong \bigoplus_i F_i$ where $F_i \in {}_A\mathcal{C}_B$ are simple and mutually non-isomorphic. Each F_i is canonically a symmetric Δ-separable Frobenius algebra in ${}_A\mathcal{C}_B$ and the decomposition of $A \otimes B$ is an isomorphism of algebras in ${}_A\mathcal{C}_B$. Objects of $\mathcal{F}$ can therefore be seen as objects of ${}_A\mathcal{C}_B$ such that only the action of $F := F_i$ is non-trivial. $\mathcal{F}$, as a pivotal category, is hence equivalent to F-F-bimodules in ${}_A\mathcal{C}_B$. By Lemma 9.11, F is automatically a symmetric Δ-separable Frobenius algebra over (A, B) in $\mathcal{C}$ and $\mathcal{F}$ is in turn equivalent to F-F-bimodules over (A, B) in $\mathcal{C}$.

The key observation now is that F is haploid in $\mathcal{C}$, i.e. $\dim \mathcal{C}(\mathbb{1}, F) = 1$. Indeed, according to Proposition 2.2 one has $\dim {}_A\mathcal{C}_B(A \otimes B, F) = 1$ and one can check that the induction/forgetful functors

$$\begin{array}{rcl} \mathrm{Ind}\colon \mathcal{C} & \to & {}_A\mathcal{C}_B \\ X & \mapsto & A \otimes X \otimes B \end{array} \quad , \quad \begin{array}{rcl} U\colon {}_A\mathcal{C}_B & \to & \mathcal{C} \\ M & \mapsto & M \end{array} \tag{E.1}$$

are biadjoint to each other (cf. Proposition 2.12). For a morphism $[f\colon M \to M] \in \mathcal{F}$ we now have:

$\mathrm{tr}_l\, f \overset{(1)}{=}$ 

$\overset{(2)}{=} \frac{1}{\dim F}$
$= \frac{1}{\dim F}$
$\overset{(3)}{=} \frac{1}{\dim F}$
$=$
$= \mathrm{tr}_r\, f$. (E.2)

Here, in step (1) one applies the definition of a left trace (2.9) (as an endomorphism of the tensor unit, i.e. of F) to the category of F-F-bimodules over (A, B). For step (2) one notes that η, $(\dim F)^{-1}\varepsilon$ can be taken for the inclusion/projection morphisms $\mathbb{1} \to F$ and $F \to \mathbb{1}$ (this follows from Proposition 2.19 as F is haploid and therefore simple, as well as Δ-separable, and the relation $\varepsilon \circ \eta = \dim F$, which applies to symmetric Δ-separable Frobenius algebras, see [FRS1, Eq.(3.49)]). In (3) we use that $\mathcal{C}$ is ribbon, hence spherical. $\mathcal{F}$ is therefore a spherical fusion category.

It remains to show that the functor (9.23) is ribbon. Having objects $(M, \gamma) \in \mathcal{Z}({}_A\mathcal{C}_B)$ and $N \in {}_A\mathcal{C}_B$ we note that because of the decomposition $A \otimes B \cong \bigoplus_i F_i$ one can write the half-braiding γ_N as a sum

$$[\gamma_N \colon M {}_A\!\otimes_B N \to N {}_A\!\otimes_B M] = \sum_{i,j} \left[\gamma_N^{ij} \colon M \otimes_{F_i} N \to N \otimes_{F_j} M\right] \ . \tag{E.3}$$

In particular, if N is in $\mathcal{F}$, the sum has a single term γ_N^{ii}, i.e. the half-braiding restricts to the component in $\mathcal{F}$. Since γ consists of isomorphisms, we see that M has only diagonal components in the decomposition ${}_A\mathcal{C}_B \simeq \bigoplus_{ij} F_i \, {}_A\!\otimes_B \, {}_A\mathcal{C}_B \, {}_A\!\otimes_B F_j$. Therefore, braidings and twists in $\mathcal{Z}(\mathcal{F})$ are simply projections of braidings and twists in $\mathcal{Z}({}_A\mathcal{C}_B)$ onto the component $\mathcal{F}$ and the functor (9.23) is precisely this projection.

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