Tropical correspondence for smooth del Pezzo log Calabi-Yau pairs

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To my family.

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Publications

The results of this thesis are contained in the following publication:

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Abstract

Consider a log Calabi-Yau pair (X, D) consisting of a smooth del Pezzo surface X of degree ≥ 3 and a smooth anticanonical divisor D. Let N_β be the logarithmic Gromov-Witten invariant of stable log maps to X of genus 0 and effective curve class β intersecting D in a single (unspecified) point with maximal tangency. The main result of this thesis is a correspondence between the invariants N_β and the consistent wall structure $\mathscr{S}_\infty$ appearing in the dual intersection complex of (X, D) from the Gross-Siebert reconstruction algorithm: The logarithm of the product of functions attached to unbounded walls in $\mathscr{S}_\infty$ yields a generating function for the N_β. In the case of $(\mathbb{P}^2, E)$, with E an elliptic curve, this correspondence respects the group law on E with identity given by a flex point. This yields a refined correspondence for logarithmic Gromov-Witten invariants $N_{d,k}$ of stable log maps to $\mathbb{P}^2$ of degree d meeting E in a single fixed point of order $3k$. In the Gross-Siebert program of mirror symmetry, the wall structure $\mathscr{S}_\infty$ is used to construct the Landau-Ginzburg model mirror dual to (X, D). So the above correspondence can be interpreted as an explicit manifestation of the relation between holomorphic curves and deformations of complex structures predicted by mirror symmetry.

Zusammenfassung

Sei (X, D) ein log Calabi-Yau Paar bestehend aus einer glatten del Pezzo Fläche X vom Grad ≥ 3 und einem glatten antikanonischen Divisor D. Definiere N_β als die logarithmische Gromov-Witten-Invariante der stabilen log Abbildungen nach X vom Geschlecht 0 und effektiver Kurven-Klasse β, die D in exakt einem (unbestimmten) Punkt treffen. Das Hauptresultat dieser Arbeit ist eine Korrespondenz zwischen den Invarianten N_β und der konsistenten Wall-Struktur $\mathscr{S}_\infty$, die auf dem dualen Schnittkomplex von (X, D) durch Anwendung des Gross-Siebert-Rekonstruktions-Algorithmus entsteht: Der Logarithmus des Produktes von Funktionen der unbeschränkten Wälle in $\mathscr{S}_\infty$ ist eine Erzeugendenfunktion für die N_β. Im Fall $(\mathbb{P}^2, E)$, mit einer elliptischen Kurve E, respektiert obige Korrespondenz die Gruppenstruktur auf E mit Wendepunkt als Identität. So erhalten wir eine feinere Korrespondenz für logarithmische Gromov-Witten-Invarianten $N_{d,k}$ von stabilen log Abbildungen nach X vom Grad d, die E in einem festen Punkt der Ordnung $3k$ treffen. Das Gross-Siebert Programm der Spiegelsymmetrie nutzt die Wall-Struktur $\mathscr{S}_\infty$, um das zu (X, D) spiegelsymmetrisch duale Landau-Ginzburg-Modell zu konstruieren. So kann die obige Korrespondenz als eine explizite Manifestation der Verbindung zwischen holomorphen Kurven und Deformationen der komplexen Struktur in der Spiegelsymmetrie aufgefasst werden.

Contents

Introduction

Over the last decades there has been much progress in enumerative algebraic geometry using ideas from string theory (*mirror symmetry*) and combinatorics (*tropical geometry*). The main result of this thesis, the *tropical correspondence theorem for smooth del Pezzo log Calabi-Yau pairs* (Theorem 1) fits into these developments. In this introduction I shortly summarize some ideas of mirror symmetry and tropical geometry with a view towards applications to enumerative questions.

String theory was invented in the 1960s in order to describe the strong interaction in quantum field theory and has evolved into a candidate for a theory unifying general relativity and quantum field theory. The central idea is that the fundamental objects in physics are one-dimensional strings rather than zero-dimensional point particles. As a string moves in spacetime it sweeps out a two-dimensional world sheet rather than a one-dimensional world line. This resolves some of the infinities in the theory, naively following from the simple observation that the world line of a particle splitting into two particles is singular, whereas the world sheet of a string splitting into two strings is a smooth surface. Even more difficulties can be eliminated by assuming supersymmetry – a symmetry exchanging particles describing matter (fermions) and particles describing interactions (bosons). It turns out that a consistent supersymmetric string theory requires ten spacetime dimensions. Four of them are our ordinary Minkowskian spacetime and the other six are assumed to be compactified at the order of the planck length $\sim 10^{-35}m$. In order to solve the hierarchy problem – the unification of the fundamental forces at the Planck scale – the compactified dimensions need to form a complex 3-dimensional manifold X with $SU(3)$-holonomy. Such a manifold is called a *Calabi-Yau manifold*. In terms of algebraic geometry this is equivalent to the condition that X has trivial canonical class K_X. In particular, X has a Ricci-flat metric or, equivalently, vanishing first Chern class $c_1(X) = 0$. Sometimes manifolds satisfying this weaker condition are also called Calabi-Yau manifolds. Since its invention, physicists constructed five different consistent supersymmetric string theories compactified on Calabi-Yau 3-folds.

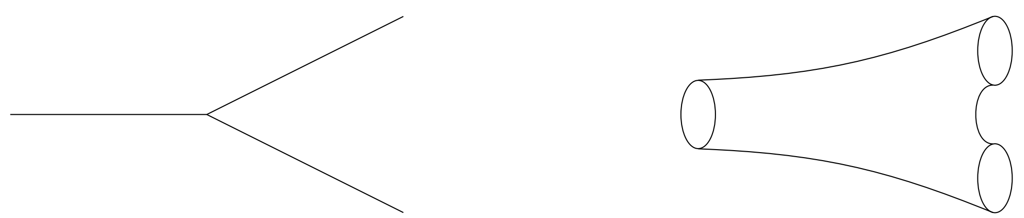


Figure 0.1: The world line (left) and world sheet (right) of a particle splitting into two particles. The horizontal dimension is time.

In the 1990s physicists noticed that some Calabi-Yau 3-folds come in pairs $(X, \check{X})$ such that one string theory compactified on X is equivalent to another one compactified on $\check{X}$. This duality of Calabi-Yau 3-folds was called *mirror symmetry* as it implies a symmetry in Hodge numbers $h^{p,q}(X) = h^{p,3-q}(\check{X})$, where $h^{p,q}(X) := \dim H^q(X, \Omega^p_X)$. While the above formulation of mirror symmetry is in terms of physics, there are several mathematically rigorous definitions. Kontsevich's *homological mirror symmetry (HMS) conjecture* [Kon] states mirror symmetry as an equivalence between the bounded derived category of coherent sheaves on X and the Fukaya category of special Lagrangian submanifolds of $\check{X}$. The *Strominger-Yau-Zaslow (SYZ) conjecture* [SYZ] seeks for the geometric reason behind mirror symmetry. It states that mirror dual Calabi-Yau 3-folds admit special Lagrangian fibrations $X \to B$ and $\check{X} \to B$ over the same base B such that if a fiber X_b over some $b \in B$ is regular, then X_b is an algebraic torus and $\check{X}_b$ is the dual torus, meaning $X_b = H^1(\check{X}_b, \mathbb{R}/\mathbb{Z})$ and $\check{X}_b = H^1(X_b, \mathbb{R}/\mathbb{Z})$. Such a fibration $X \to B$ yields two different natural affine structures on the subset B_0 of B with regular fibers, and these affine structures are exchanged by replacing X with $\check{X}$.

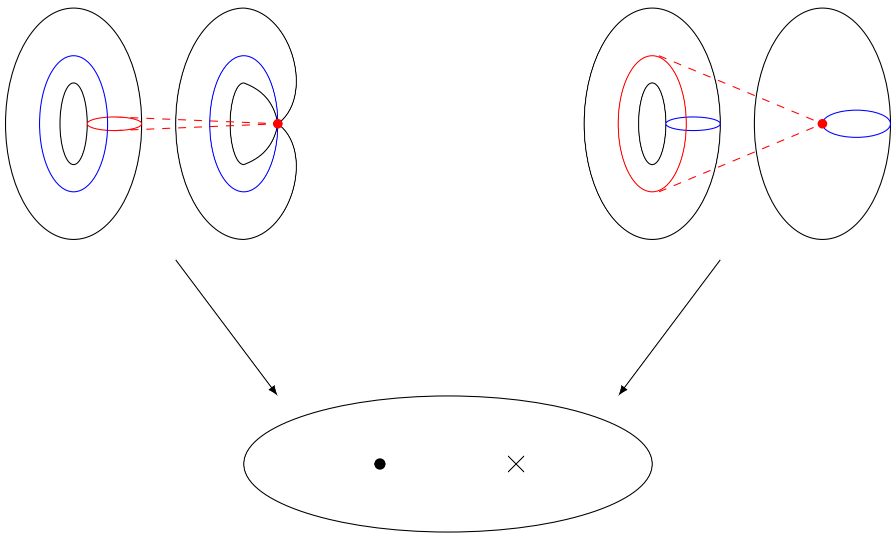


Figure 0.2: Mirror dual special Lagrangian torus fibrations. For each fibration a regular fiber and a singular fiber are shown. Two different tori and their limits are shown in red and blue, respectively. These are exchanged by mirror symmetry.

The *Gross-Siebert program* is an algebro-geometric version of the SYZ conjecture. It is stated as a duality of *toric degenerations* of (log) Calabi-Yau varieties – degenerations $\mathfrak{X} \to T$ with central fiber X_0 a union of toric varieties glued along toric divisors and such that the family is strictly semistable away from a codimension 2 subset on X_0. From such a degeneration one can construct two natural affine manifolds with singularities B resp. $\check{B}$, together with polyhedral decompositions $\mathscr{P}$ resp. $\check{\mathscr{P}}$ and multi-valued piecewise affine functions φ resp $\check{\varphi}$. The *dual intersection complex* B is obtained by gluing the fans corresponding to the toric irreducible components of X_0. The *intersection complex* $\check{B}$ arises by gluing the momentum polytopes of the components of X_0. There is an involution mapping the intersection complex of a toric degeneration to its dual intersection complex and vice versa, called *discrete Legendre transform.* The idea of the Gross-Siebert program is that the intersection complex and its dual are exchanged by replacing a toric degeneration with its mirror dual toric degeneration. Now to construct the mirror to a given toric degeneration $\mathfrak{X}$ one could pass to its dual intersection complex $(B, \mathscr{P}, \varphi)$ and try to construct another (formal) toric degeneration with intersection complex $(B, \mathscr{P}, \varphi)$. Indeed, this reconstruction problem was solved in [GS3] using *scattering diagrams* and *wall structures* – objects carrying combinatorial and algebraic data about local models of the toric degeneration and their gluings.

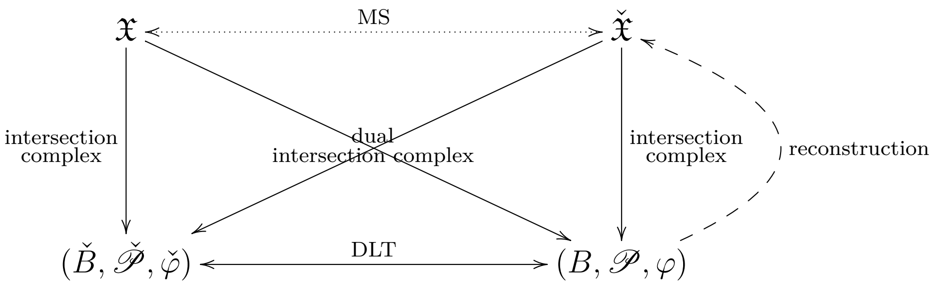


Figure 0.3: Schematic picture of the Gross-Siebert program.

Mirror symmetry attracted the attention of enumerative algebraic geometers with the work of Candelas, de la Ossa, Green and Parkes [COGP]. Using period calculations on the mirror, they predicted the number of rational degree d curves on a general quintic threefold for large degrees. Before, this number had only been known up to degree 3 due to the work of Ellingsrud and Strømme [ES].

In a different direction, Kontsevich and Manin [KMa] established a recursion relation for the Gromov-Witten invariant $N_{0,3d-1}(d)$ of rational degree d curves through $3d-1$ general curves in $\mathbb{P}^2$, and similarly for other del Pezzo surfaces and projective spaces. This formula can be obtained from the associativity of quantum cohomology.

In [Mik] Mikhalkin showed that the number $N_{0,3d-1}(d)$ and its counterpart for other toric surfaces can be computed via combinatorial methods, using piecewise linear objects (tropical curves). The recursion relation was reproven via tropical methods by Gathmann and Markwig [GM]. Nishinou and Siebert generalized Mikhalkin's tropical correspondence theorem to toric varieties of any dimension. The idea of the proof is to construct a toric degeneration such that curves (stable log maps) on the central fiber split into a chain of lines that are sufficiently general (torically transverse). The combinatorics of such chains of lines can be described by tropical curves. Mandel and Ruddat [MR] generalized the tropical correspondence to more general tangency conditions (ψ-classes) and Bousseau [Bou1] generalized it to higher genus.

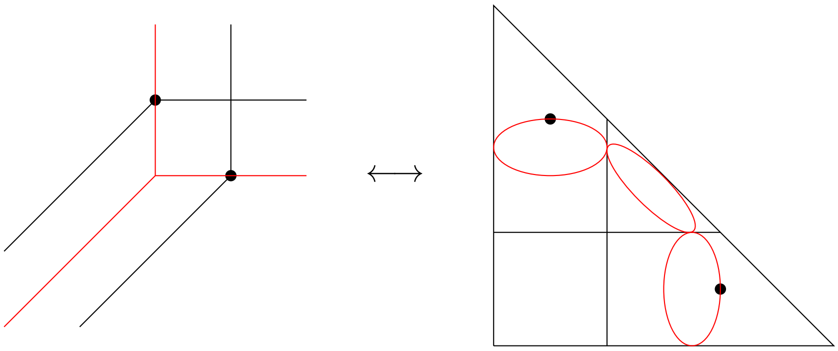


Figure 0.4: The intersection complex (right) and its dual (left) of a toric degeneration of $\mathbb{P}^2$ such that the line through two points is torically transverse on the central fiber.

An enumerative interpretation of scattering diagrams was established by Gross, Pandharipande and Siebert in [GPS]. They showed that the logarithm of the functions attached to rays in a scattering diagram give generating functions for certain log Gromov-Witten invariants with a maximal tangency condition, meaning that curves are required to meet a given divisor in exactly one unspecified point with maximal tangency. Again, this was generalized by Bousseau to higher genus [Bou2].

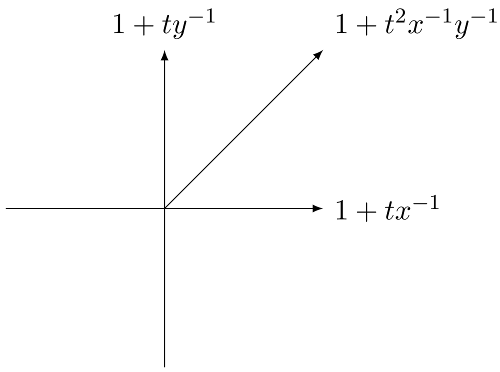


Figure 0.5: A scattering diagram is a set of rays with attached functions.

In [GHK] Gross, Hacking and Keel construct the mirror to a log Calabi-Yau surface (X,D) with maximal boundary, i.e., D is a cycle of rational curves, as a formal smoothing of the n-vertex $\mathbb{V}_n = \mathbb{A}^2_{x_1x_2} \cup \mathbb{A}^2_{x_2x_3} \cup \ldots \cup \mathbb{A}^2_{x_nx_1}$. They use the correspondence of [GPS] to define a canonical consistent scattering diagram from the enumerative geometry of (X,D). There is an affine singularity at the vertex of this scattering diagram. Hence, the scattering diagram only gives an open subscheme of the mirror as a flat deformation of $\mathbb{V}_n^\circ = \mathbb{V}_n \setminus \{0\}$. To obtain the whole mirror they use broken lines to construct theta functions – certain canonical global sections of line bundles on $\check{X}^\circ$. This gives enough functions to define an embedding of $\check{X}^\circ$ into an affine space. Taking the closure gives the mirror to (X,D). It can be defined explicitly as the spectrum of an explicit algebra generated by theta functions and with multiplication rule defined by the enumerative geometry of (X,D).

This has led to the modern viewpoint of *intrinsic mirror symmetry* [GS6][GS7]. It circumvents the constructions of scattering diagrams and broken lines and directly defines the mirror to (X, D) as the spectrum of an algebra with multiplication rule defined by certain *punctured Gromov-Witten invariants* of (X, D) [GS6][ACGS2].

Outline

A smooth projective surface X over the complex numbers $\mathbb{C}$ together with a reduced effective anticanonical divisor D forms a *log Calabi-Yau pair* (X, D), meaning that $K_X + D$ is numerically trivial. The case where $D = D_1 + \ldots + D_m$ is a cycle of smooth rational curves (*maximal boundary*) has been studied in [GHK]. In [GPS] it was shown that generating functions of logarithmic Gromov-Witten invariants of X with maximal tangency at a single point on D in this case can be read off from a certain *scattering diagram*. The statement of [GPS] was generalized in [Bou2] to q-refined scattering diagrams and higher genus Gromov-Witten invariants.

In this work we consider the somewhat complementary case with D a smooth irreducible divisor. We restrict to the case where X has very ample anticanonical class $-K_X$, i.e., is a smooth del Pezzo surface of degree ≥ 3. Let $Q \subset \mathbb{R}^2$ be a Fano polytope, that is, a convex lattice polytope containing the origin and with all vertices being primitive integral vectors. From this one can construct a family $(\mathfrak{X}_Q \to \mathbb{A}^2, \mathfrak{D}_Q)$ (see §2) such that fixing one parameter $s \neq 0$ one obtains a toric degeneration of a pair (X, D) as above and fixing $s = 0$ gives a toric degeneration of (X^0, D^0), where X^0 is a smooth nef toric surface admitting a $\mathbb{Q}$-Gorenstein deformation to X and $D^0 = \partial X^0$ is the toric boundary.

Let $(B, \mathscr{P}, \varphi)$ be the *dual intersection complex* of the toric degeneration of (X, D). The affine manifold with singularities B is non-compact without boundary. In [CPS] it is described how to construct a tropical superpotential from such a triple[1], leading to a Landau-Ginzburg model with intersection complex $(B, \mathscr{P}, \varphi)$. This perfectly fits into the picture, since the idea of the *Gross-Siebert program* is that the intersection complex and its dual are exchanged by mirror symmetry, and in fact the mirror of a Fano variety together with a choice of anticanonical divisor is believed to be a Landau-Ginzburg model. The construction involves the scattering calculations described in [GS3], leading to a *consistent wall structure* $\mathscr{S}_\infty$ on $(B, \mathscr{P}, \varphi)$. This is a collection of codimension 1 polyhedral subsets of B (*slabs* and *walls*) with attached functions describing the gluing of canonical thickenings of affine pieces to obtain a toric degeneration with intersection complex $(B, \mathscr{P}, \varphi)$.

[1] In fact, there is additional information captured in so called *gluing data* ([GS1], Definition 2.25). In this thesis we always make the trivial choice, setting $s_e = 1$ for any inclusion $e : \omega \to \tau$ of cells $\omega, \tau \in \mathscr{P}$, and will not mention gluing data.

The statement

Definition. For an effective curve class[2] $\underline{\beta} \in H_2^+(X, \mathbb{Z})$ let β be the class of 1-marked stable log maps to X of genus 0 and class $\underline{\beta}$ meeting D in a single unspecified point with maximal tangency. Let $\mathscr{M}(X, \beta)$ be the moduli space of basic stable log maps of class β. By the results of [GS5] this is a proper Deligne-Mumford stack and admits a virtual fundamental class $[\![\mathscr{M}(X, \beta)]\!]$. Since $\mathscr{M}(X, \beta)$ has virtual dimension zero, $[\![\mathscr{M}(X, \beta)]\!]$ is in the degree zero part of the rational Chow ring of $\mathscr{M}(X, \beta)$. This gives a logarithmic Gromov-Witten invariant by integration, i.e., proper pushforward to a point:

$$N_\beta = \int_{[\![\mathscr{M}(X,\beta)]\!]} 1.$$

Let $\mathscr{S}_\infty$ be the consistent wall structure defined by the dual intersection complex $(B, \mathscr{P}, \varphi)$ of (X, D) via the Gross-Siebert algorithm [GS3] (see §1.6).

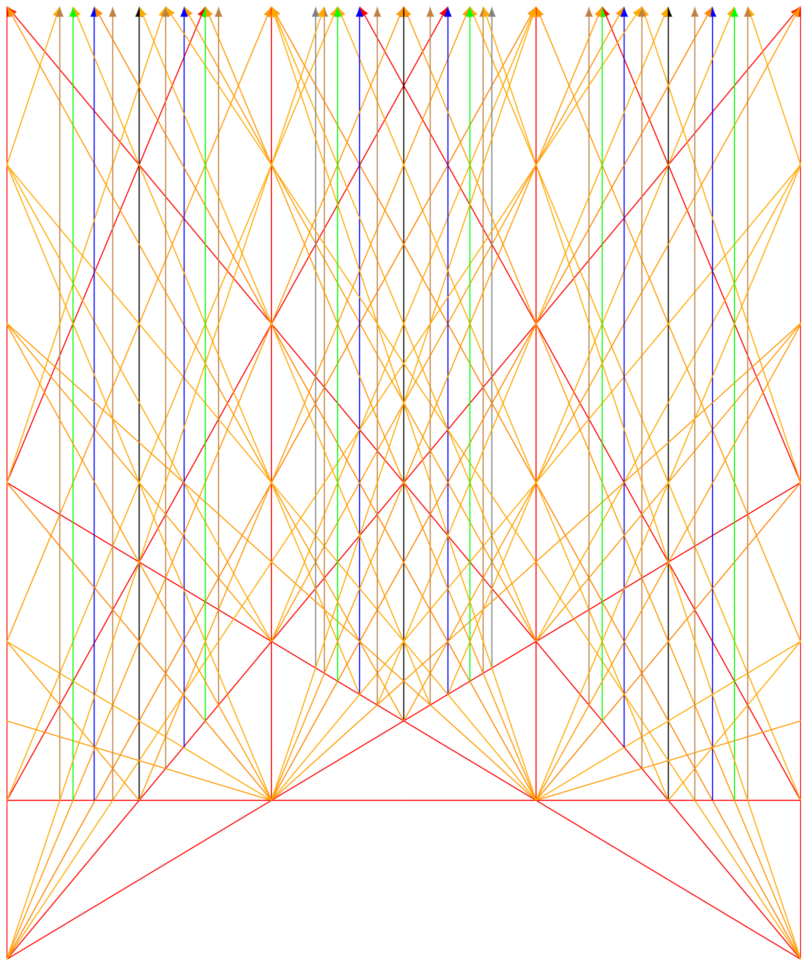


Figure 0.6: The wall structure of $(\mathbb{P}^2, E)$ consistent to order 6. The colors correspond to different orders. For the functions attached to unbounded walls see §8.1.

[2]For del Pezzo surfaces the group of curve classes (1-dimensional Chow group) is isomorphic to the singular homology group $H_2(X, \mathbb{Z})$ (by Kodaira vanishing, Serre and Poicaré duality). We write $H_2^+(X, \mathbb{Z})$ for the submonoid of effective curve classes.

Figure 0.6 shows $\mathscr{S}_\infty$ for $(\mathbb{P}^2, E)$ up to order 6. The unbounded walls in $\mathscr{S}_\infty$ are all parallel in direction $m_{\text{out}} \in \Lambda_B$. Here Λ_B is the sheaf of integral tangent vectors on B and m_{out} is the primitive vector in the unique unbounded direction of B (the upward direction in Figure 0.6). Let f_{out} be the product of all functions attached to unbounded walls in $\mathscr{S}_\infty$, regarded as elements of $\mathbb{C}[\![x]\!]$ for $x := z^{(-m_{\text{out}},0)} \in \mathbb{C}[\Lambda_B \oplus \mathbb{Z}]$. Then the main theorem is the following. It can be interpreted as a *tropical correspondence theorem*, since the wall structure $\mathscr{S}_\infty$ is combinatorial in nature and supported on the dual intersection complex $(B, \mathscr{P}, \varphi)$ of (X, D). Notably, it is a tropical correspondence theorem in a *non-toric* setting, as the smooth divisor D has genus 1 and thus is non-toric. So far, most such theorems have been obtained only in toric cases, a remarkable exception being [Arg].

Theorem 1.

$$\log f_{\text{out}} = \sum_{\underline{\beta} \in H_2^+(X,\mathbb{Z})} (D \cdot \underline{\beta}) \cdot N_\beta \cdot x^{D \cdot \underline{\beta}}.$$

For $(\mathbb{P}^2, E)$, this correspondence respects the torsion points on E: Consider the group law on E with identity a flex point of E. The $3d$-torsion points form a subgroup T_d of $S^1 \times S^1$ isomorphic to $\mathbb{Z}_{3d} \times \mathbb{Z}_{3d}$. The stable log maps contributing to N_d meet E in such a $3d$-torsion point (Lemma 7.1). For $P \in \cup_{d \geq 1} T_d$, let $k(P)$ be the smallest integer such that $P \in T_{k(P)}$. Let $N_{d,k}$ be the logarithmic Gromov-Witten invariant of stable log maps contributing to N_d and intersecting E in a given point P with $k(P) = k$. In §7 we will show that this is well-defined.

Let $s_{k,l}$ be the number of points in $T_d \simeq \mathbb{Z}_{3d} \times \mathbb{Z}_{3d}$ with $k(P) = k$ that are fixed by $M_l = \left(\begin{smallmatrix} 1 & 3l \\ 0 & 1 \end{smallmatrix}\right)$, but not fixed by $M_{l'}$ for any $l' < l$. Let r_l be the number of points on S^1 of order $3l$, defined recursively in Lemma 7.7. For an unbounded wall $\mathfrak{p} \in \mathscr{S}_\infty$ let $l(\mathfrak{p})$ be the smallest integer such that $\log f_{\mathfrak{p}}$ has non-trivial $x^{3l(\mathfrak{p})}$-coefficient. The number of walls in $\mathscr{S}_\infty$ with $l(\mathfrak{p}) = l$ is r_l.

Theorem 2. *Let $\mathfrak{p}$ be an unbounded wall in $\mathscr{S}_\infty$ with $l(\mathfrak{p}) = l$. Then*

$$\log f_{\mathfrak{p}} = \sum_{d=1}^{\infty} 3d \left(\sum_{k : l | k | d} \frac{s_{k,l}}{r_l} N_{d,k} \right) x^{3d}.$$

Subtracting multiple cover contributions of curves of smaller degree, one obtains log BPS numbers n_d and $n_{d,k}$ (see §3.4). Some of the $n_{d,k}$ have been calculated in [Tak1]. The logarithmic Gromov-Witten invariants N_d and $N_{d,k}$ and the log BPS numbers n_d and $n_{d,k}$ are calculated, among other invariants, for $d \leq 6$, in §8. Some of these numbers are new:

$n_{4,1} = 14$	$n_{4,2} = 14$	$n_{4,4} = 16$	$n_{6,1} = 927$	$n_{6,2} = 938$	$n_{6,3} = 936$

Remark. The logarithmic Gromov-Witten invariants N_d can be obtained from local Gromov-Witten invariants which in turn can be calculated using local mirror symmetry [CKYZ].

Generalizations and relation to other works

We expect that there is a generalization of Theorem 1 to q-refined wall structures and higher genus logarithmic Gromov-Witten invariants, similar to the maximally degenerated case [Bou2]. The main argument to obtain higher genus statements in [Bou1] and [Bou2], the gluing and vanishing properties of λ-classes, are purely local. So one should be able to use the correspondence of [Bou2] at each joint of the wall structure $\mathscr{S}_\infty$ and obtain a higher genus version of Theorem 1. However, the introduction of q-refinements and λ-classes as well as the higher genus calculations would go beyond the scope of this thesis.

In recent work [BFGW] on the holomorphic anomaly equation for $(\mathbb{P}^2, E)$, Bousseau, Fan, Guo and Wu studied these higher genus invariants. In [Bou3] Pierrick Bousseau establishes a relation between scattering diagrams and wall-crossing phenomena of counts of coherent sheaves on $\mathbb{P}^2$. In [Bou4] he uses this relation and Theorem 2 above to give a proof of [Tak2], Conjecture 1.6.

Very recently, Yu-Shen Lin [Lin] worked out a symplectic analogue of the correspondence described in this thesis.

Motivation

The reason for an enumerative meaning of wall structures is the following. By the Strominger-Yau-Zaslow conjecture [SYZ], mirror dual Calabi-Yau varieties admit mirror dual Lagrangian torus fibrations. To construct the mirror to a given Calabi-Yau, one first constructs the *semi-flat* mirror by dualizing the non-singular torus fibers. Then one corrects the complex structure of the semi-flat mirror such that it extends across the locus of singular fibers. It is expected that these corrections are determined by counts of holomorphic discs in the original variety with boundary on torus fibers [SYZ][Fuk].

Kontsevich and Soibelman [KS] showed that in dimension two and with at most nodal singular fibers in the torus fibration, corrections of the complex structure are determined by algebraic self-consistency constraints which can be encoded by trees of gradient flow lines in the fan picture (dual intersection complex) of the degeneration, with certain automorphisms attached to the edges of the trees. From this they constructed a rigid analytic space from B, in dimension two.

Under the discrete Legendre transform ([GS1], §1.4) the gradient flow lines in the fan picture become straight lines in the cone picture (intersection complex).

This was used by Gross and Siebert to construct a toric degeneration from the cone picture in any dimension. In the cone picture the self-consistency calculations are described by scattering diagrams (locally) and wall structures (globally). The fact that wall structures are used to construct a complex manifold in the cone picture and at the same time give generating functions for curve counts of the fan picture can be seen as an explicit explanation for the connection between deformations and holomorphic curves in mirror symmetry.

Gross, Hacking and Keel [GHK] used this correspondence to construct the mirror to a log Calabi-Yau pair with maximal boundary from its enumerative geometry. This has led to the program of intrinsic mirror symmetry [GS6][GS7].

Structure of the thesis

In §1 we give the necessary background enabling the reader to understand the forthcoming material. In §2 we describe how smoothing the boundary of a Fano polytope leads to a family $(\mathfrak{X}_Q \to \mathbb{A}^2, \mathfrak{D}_Q)$ as above. Fixing one parameter $s \neq 0$ gives a toric degeneration $(\mathfrak{X} \to \mathbb{A}^1, \mathfrak{D})$ of (X, D). It contains logarithmic singularities lying on the central fiber, corresponding to affine singularities in the dual intersection complex $(B, \mathscr{P}, \varphi)$. In §3 we describe a small log resolution of these singularities, leading to a log smooth degeneration $(\tilde{\mathfrak{X}} \to \mathbb{A}^1, \tilde{\mathfrak{D}})$ of (X, D). In §4 we describe tropicalizations of stable log maps to the central fiber of $(\tilde{\mathfrak{X}} \to \mathbb{A}^1, \tilde{\mathfrak{D}})$ and show that for each degree there is a finite number of them. The tropicalizations induce a refinement of $\mathscr{P}$ and hence a logarithmic modification of $(\tilde{\mathfrak{X}} \to \mathbb{A}^1, \tilde{\mathfrak{D}})$. This is a degeneration $(\tilde{\mathfrak{X}}_d \to \mathbb{A}^1, \tilde{\mathfrak{D}}_d)$ of (X, D) such that stable log maps to the central fiber are torically transverse. Then we apply the degeneration formula of logarithmic Gromov-Witten theory in §5. It gives a description of N_β in terms of invariants N_V labeled by vertices of the tropical curves found in §4. In §6 we show that the scattering calculations of [GS3] give a similar formula for the logarithm of functions attached to unbounded walls in the consistent wall structure $\mathscr{S}_\infty$. This ultimately leads to a proof of Theorem 1. In §7 we explain that this correspondence respects the torsion points on E, leading to Theorem 2. In §8 we explicitly calculate some invariants for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the cubic surface.

1 Preliminaries

1.1 Affine geometry

In this section we give some basic definitions and constructions from affine geometry needed later. We show how to construct an affine manifold with singularities B from a collection $\mathscr{P}$ of integral convex polyhedra. The pair $(B, \mathscr{P})$ is called a *polyhedral affine manifold*. Moreover, we define a *polarization* of $(B, \mathscr{P})$ to be a multi-valued strictly convex piecewise affine function $\varphi : B \to \mathbb{R}$. For more details see [GS1].

Let $M \simeq \mathbb{Z}^n$ be a lattice, let $N = \mathrm{Hom}(M, \mathbb{Z}) \simeq \mathbb{Z}^n$ be the dual lattice, and let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ be the corresponding vector spaces.

Definition 1.1. A (convex) *polyhedron* $Q \subset M_{\mathbb{R}}$ is the intersection of finitely many closed half-spaces in $M_{\mathbb{R}}$. The *dimension* of Q is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing Q. Its *interior* $\mathrm{Int}(Q)$ is the interior of Q inside this affine space and its *boundary* ∂Q is the complement $Q \setminus \mathrm{Int}(Q)$. If Q is of dimension k, its boundary ∂Q is a union of convex polyhedra of dimensions at most $k-1$, called *faces*. These are given by intersection of Q with affine hyperplanes disjoint from $\mathrm{Int}\, Q$. Faces of dimension $k-1$ are called *facets* and 0-dimensional faces are called *vertices* of Q. A polyhedron Q is called *rational* if the affine half-spaces defining it are given by affine functions with rational coefficients. It is *integral* if it is rational with vertices in M. A bounded polyhedron is called a *polytope*.

Definition 1.2. An *affine manifold* B_0 is a differentiable manifold with an equivalence class of charts with transition functions in $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(\mathbb{R}^2)$. It is *integral* if the transition functions lie in $\mathrm{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \mathrm{GL}(\mathbb{Z}^n)$. An integral affine manifold comes with a sheaf of integral tangent vectors Λ_{B_0}. This is a locally constant sheaf with stalks isomorphic to $\mathbb{Z}^n$. A map between (integral) affine manifolds preserving this structure is called an *(integral) affine map*.

Example 1.3. A polyhedron $Q \subset M_{\mathbb{R}}$ has a natural structure of an affine manifold via its embedding into $M_{\mathbb{R}} \simeq \mathbb{R}^n$.

Definition 1.4. An *(integral) affine manifold with singularities* is a topological manifold B along with a closed set $\Delta \subseteq B$, the *discriminant locus*, which is locally a finite union of locally closed submanifolds of codimension ≥ 2 such that $B_0 = B \setminus \Delta$ is an (integral) affine manifold.

In the following we restrict to polyhedra and affine manifolds that are integral, since these are the affine manifolds arising in toric geometry. Moreover, we require polyhedra to have at least one vertex.

Definition 1.5. An *(integral) polyhedral complex* is a set $\mathscr{P}$ of (integral) polyhedra along with a set of (integral) affine maps $\omega \to \tau$ identifying ω with a face of τ, making $\mathscr{P}$ into a category. We require that any proper face of any $\tau \in \mathscr{P}$ occurs as the domain of a morphism in $\mathscr{P}$ with target τ. Let B the direct limit in the category of topological spaces

$$B := \varinjlim_{\tau \in \mathscr{P}} \tau$$

By abuse of notation we view the elements of $\mathscr{P}$ as subsets of B, called *cells* of $\mathscr{P}$. We assume the following conditions:

(1) For each $\tau \in \mathscr{P}$ the map $\tau \to B$ is injective, i.e., no cells self-intersect.
(2) B is pure dimension n, that is, every cell of $\mathscr{P}$ is contained in at least one n-dimensional cell.
(3) Every $(n-1)$-dimensional cell of $\mathscr{P}$ is contained in one or two n-dimensional cells, so that B is an affine manifold away from codimension ≥ 2 cells.
(4) *The S_2 condition.* If $\tau \in \mathscr{P}$ is of dimension $\leq n-2$, then any $x \in \mathrm{Int}(\tau)$ has a neighborhood basis on B consisting of open sets V with $V \setminus \tau$ connected.

Denote by $\mathscr{P}^{[k]}$ the set of k-dimensional cells of $\mathscr{P}$. Cells of dimension 0, 1 and n are called *vertices*, *edges* and *maximal cells*. The *boundary* ∂B of B is the union of all $(n-1)$-dimensional cells contained in only one maximal cell. Cells not contained in ∂B are called *interior*.

Construction 1.6 ([GHS], Construction 1.1)**.** Let $\mathscr{P}$ be an integral polyhedral complex. From this we construct an affine manifold with singularities B. First we construct the discriminant locus Δ as follows. For each bounded cell $\tau \in \mathscr{P}$ let $a_\tau \in \mathrm{Int}(\tau)$. For each unbounded cell $\tau \in \mathscr{P}$ let $a_\tau \in \Lambda_{\tau,\mathbb{R}}$ be an element of the relative interior of the asymptotic cone of τ, i.e. the Hausdorff limit $\lim_{\epsilon\to 0} \epsilon\tau$. In the latter case we require that if $\tau' \subseteq \tau$ is a face with the same asymptotic cone, then $a_{\tau'} = a_\tau$. Then a_τ can be viewed as a point at infinity. For any chain $\tau_1 \subseteq \tau_2 \subseteq \ldots \subseteq \tau_{n-1}$ with $\dim \tau_i = i$ and τ_i bounded iff $i \leq r$, define

$$\Delta_{\tau_1,\ldots,\tau_{n-1}} := \mathrm{Conv}\{a_{\tau_i} \mid 1 \leq i \leq r\} + \sum_{i=r+1}^{n-1} \mathbb{R}_{\geq 0} \cdot a_{\tau_i} \subseteq \tau_{n-1}.$$

Here summation means taking the Minkowski sum. Then define Δ as the union of all such polyhedra. For bounded cells a canonical choice of a_τ is the barycenter of τ. However, to run the reconstruction of [GS3] (see §1.6) we need that any proper rational affine subspace of an $n-1$-cell intersects Δ transversely, i.e., that Δ does not contain any rational point. By [GS3] Lemma 1.3, this can be achieved by a sufficiently general choice of a_τ, e.g. by choosing a_τ irrational.

B has a well-defined integral affine structure on each maximal cell by embedding to $M_{\mathbb{R}} \simeq \mathbb{R}^n$. To obtain an integral affine structure away from Δ we need to provide, for each connected component $\underline{\rho}$ of $\rho \setminus \Delta$ for $\rho \in \mathscr{P}^{[n-1]}$, an identification of tangent spaces of the adjacent maximal cells σ, σ' inducing the identity of Λ_ρ.

Definition 1.7. A *polyhedral affine manifold* is a pair $(B, \mathscr{P})$ consisting of an integral polyhedral complex $\mathscr{P}$ and an affine manifold with singularities B obtained from $\mathscr{P}$ by Construction 1.6.

For a connected component $\underline{\rho}$ of $\rho \setminus \Delta$ for $\rho \in \mathscr{P}^{[n-1]}$, an identification of tangent spaces of the adjacent maximal cells inducing the identity of Λ_ρ, as required to extend the affine structure across $\underline{\rho}$ in Construction 1.6 can be given by a *fan structure* at $\underline{\rho}$, or at the unique vertex v contained in $\underline{\rho}$:

Definition 1.8. Let $\mathscr{P}$ be an integral polyhedral complex. The *open star* of a cell $\tau \in \mathscr{P}$ is the open neighbourhood of $\mathrm{Int}(\tau)$ given by

$$U_\tau = \bigcup_{\substack{\sigma \in \mathscr{P} \\ \mathrm{Hom}(\tau,\sigma) \neq \emptyset}} \mathrm{Int}(\sigma).$$

A *fan structure* along $\tau \in \mathscr{P}$ is a continuous map $S_\tau : U_\tau \to \mathbb{R}^k$ with

(i) $S_\tau^{-1}(0) = \mathrm{Int}(\tau)$;

(ii) if $e : \tau \to \sigma$ is a morphism, then $S_\tau|_{\mathrm{Int}(\sigma)}$ is an integral affine submersion onto its image, that is, is induced by an epimorphism $\Lambda_\sigma \to W \cup \mathbb{Z}^k$ for some vector subspace $W \subseteq \mathbb{R}^k$;

(iii) the collection of cones $K_e := \mathbb{R}_{\geq 0} \cdot S_\tau(\sigma \cap U_\tau)$ for all $e : \tau \to \sigma$ defines a finite fan Σ_τ in $\mathbb{R}^k$.

Two fan structures $S_\tau, S'_\tau : U_\tau \to \mathbb{R}^k$ are *equivalent* if they differ by an integral linear transformation of $\mathbb{R}^k$. If $S_\tau : U_\tau \to \mathbb{R}^k$ is a fan structure along $\tau \in \mathscr{P}$ and $\tau \subseteq \sigma$, then $U_\sigma \subseteq U_\tau$. The *fan structure along σ induced by S_τ* is the composition

$$U_\sigma \longrightarrow U_\tau \xrightarrow{S_\tau} \mathbb{R}^k \longrightarrow \mathbb{R}^k / L_\sigma \simeq \mathbb{R}^l,$$

where $L_\sigma \subseteq \mathbb{R}^k$ is the linear span of $S_\tau(\mathrm{Int}(\sigma))$. This is well-defined up to equivalence.

Proposition 1.9. *Let $\mathscr{P}$ be an integral polyhedral complex, together with a fan structure $S_v : U_v \to \mathbb{R}^n$ at each vertex $v \in \mathscr{P}^{[0]}$. Then the underlying topological space B carries a well-defined integral affine structure outside a closed subset Δ of codimension two.*

Proof. For a vertex v of $\mathscr{P}$ let W_v denote a choice of open neighborhood of v with $W_v \subset U_v$ satisfying the condition that if $v \in \rho$ with ρ a codimension 1 cell, then $W_v \cap \rho$ is the connected component of $\rho \setminus \Delta$ containing v. Then

$$\{\mathrm{Int}(\sigma) \mid \sigma \in \mathscr{P}^{[n]}\} \cup \{W_v \mid v \in \mathscr{P}^{[0]}\}$$

is an open cover of $B_0 = B \setminus \Delta$. We obtain an affine structure on B_0 by the charts

$$\psi_\sigma : \mathrm{Int}(\sigma) \hookrightarrow M_{\mathbb{R}}, \qquad \text{and} \qquad \psi_v : W_v \hookrightarrow U_v \xrightarrow{S_v} \mathbb{R}^n.$$

The left hand charts give an integral affine structure on the maximal cells of $\mathscr{P}$ via embedding into $M_{\mathbb{R}}$. The right hand charts extend this affine structure across the connected components of $\rho \setminus \Delta$ for any codimension 1 cell ρ. □

Definition 1.10 (Local monodromy)**.** Let $\omega \in \mathscr{P}^{[1]}$, $\rho \in \mathscr{P}^{[n-1]}$ with $\omega \subseteq \rho$, $\rho \nsubseteq \partial B$ and ω bounded. Then ρ is contained in two n-cells $\sigma^\pm$, and ω contains two vertices $v^\pm$. Following the change of affine charts given by (i) the fan structure at v^+ (ii) the polyhedral structure of σ^+ (iii) the fan structure at v^- (iv) the polyhedral structure of σ^- and back to (v) the fan structure at v^+ defines a transformation $T_{\omega\rho} \in \mathrm{SL}(\Lambda_{v^+})$. By [GS1] §1.5, it takes the form

$$T_{\omega\rho}(m) = m + \kappa_{\omega\rho} \langle m, \check{d}_\rho \rangle\, d_\omega.$$

Here $d_\omega \in \Lambda_\omega \subseteq \Lambda_{v^+}$ and $\check{d}_\rho \in \Omega_\rho^\perp \subseteq \Lambda_{v^+}^\star$ are the primitive integral vectors pointing from v^+ to v^- and, in the chart at v^+, evaluating positively on σ^+, respectively. The constant $\kappa_{\omega\rho} \in \mathbb{Z}$ is independent of the choices of $v^\pm$ and $\sigma^\pm$.

Definition 1.11. An integral polyhedral affine manifold is *positive* if $\kappa_{\omega\rho} \geq 0$ for all $\omega \subseteq \rho$ with ω bounded and $\rho \nsubseteq \partial B$.

Example 1.12. Let $(B, \mathscr{P})$ be a 2-dimensional polyhedral affine manifold and let $\rho \in \mathscr{P}^{[1]}$ be a 1-dimensional cell. Then, in suitable coordinates, the monodromy transformation $T_{\rho\rho}$ takes the form

$$T_{\rho\rho} = \begin{pmatrix} 1 & n_\rho \\ 0 & 1 \end{pmatrix},$$

for some $n_\rho \in \mathbb{Z}$. Here $n_\rho = 0$ exactly if ρ does not contain an affine singularity. Now $(B, \mathscr{P})$ is positive if and only if $n_\rho > 0$ for all $\rho \in \mathscr{P}^{[1]}$.

Definition 1.13. A 2-dimensional polyhedral affine manifold $(B, \mathscr{P})$ is *simple* if $n_\rho = 1$ for all $\rho \in \mathscr{P}^{[1]}$, with n_ρ as in Example 1.12. There is a notion of simplicity for arbitrary dimension ([GS1], Definition 1.60). In the case of dimension 2 it restricts to the definition above. We will only be concerned with 2-dimensional polyhedral affine manifolds in the main part of the thesis.

Definition 1.14. Let $(B, \mathscr{P})$ be an integral polyhedral affine manifold.

1. An *integral affine function* on an open set $U \subseteq B$ is a continuous map $U \to \mathbb{R}$ that is integral affine on $U \setminus \Delta$. These functions form a sheaf $\mathcal{Aff}(B, \mathbb{Z})$ on B.

2. An *integral piecewise affine function* on U is a continuous map $\varphi : U \to \mathbb{R}$ such that if $S_\tau : U_\tau \to \mathbb{R}^k$ is the fan structure along $\tau \in \mathcal{P}$, then $\varphi|_{U \cap U_\tau} = \lambda + S_\tau^\star(\varphi_\tau)$ for an integral affine function $\lambda : U_\tau \to \mathbb{R}$ and a function $\varphi_\tau : \mathbb{R}^k \to \mathbb{R}$ that is integral piecewise linear with respect to the fan Σ_τ, i.e., linear on each maximal cone of Σ_τ. By abuse of notation we write φ_τ instead of $S_\tau^\star(\varphi_\tau)$. Again, these functions form a sheaf $\mathcal{PL}_{\mathcal{P}}(B, \mathbb{Z})$ on B.
3. A *multi-valued integral piecewise affine function* on B is a section of the quotient sheaf $\mathcal{PA}_{\mathscr{P}}(B, \mathbb{Z})/\mathcal{Aff}(B, \mathbb{Z})$. It is called a *polarization* of the integral polyhedral affine manifold $(B, \mathscr{P})$ and we write $(B, \mathscr{P}, \varphi)$ for a polarized integral polyhedral affine manifold.

Construction 1.15 (The discrete Legendre transform, [GS1] §1.4)**.** Let $(B, \mathscr{P}, \varphi)$ be a polarized integral polyhedral affine manifold. For $\tau \in \mathscr{P}$ let

$$\check{\tau} = \{n \in N_{\mathbb{R}} \mid \langle n, m \rangle \geq -\varphi_\tau(m) \ \forall m \in B\}$$

be the Newton polyhedron of φ_τ. For $\omega \to \tau$ a morphism in $\mathscr{P}$ let $\check{\tau} \to \check{\omega}$ be the identification of $\check{\tau}$ with the face of $\check{\omega}$ given by $\{n \in N_{\mathbb{R}} \mid \langle n, m \rangle = -\varphi_\omega(m) \ \forall m \in \tau\}$. If $\sigma \in \mathscr{P}$ is maximal cell, then $\check{\sigma} \in \check{\mathscr{P}}$ is a vertex. In this case, the boundary of the dual of the cone over $\sigma \times \{1\}$ defines the graph of $\check{\varphi}$ on the open star $U_{\check{\sigma}}$, hence also the fan structure at $\check{\sigma}$. By [GS1], Proposition 1.51, the descriptions of $\check{\varphi}$ on the open stars $U_{\check{\sigma}}$ of the vertices $\check{\sigma} \in \check{\mathscr{P}}$ glue to form a multi-valued integral piecewise affine function. Hence $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is a polarized integral polyhedral affine manifold. We call it the *discrete Legendre transform* of $(B, \mathscr{P}, \varphi)$.

Proposition 1.16 ([GS1], Proposition 1.51)**.** *Let $(B, \mathscr{P}, \varphi)$ be a polarized integral polyhedral affine manifold. Then its discrete Legendre transform $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is again a polarized integral polyhedral affine manifold, and the discrete Legendre transform of $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is $(B, \mathscr{P}, \varphi)$.*

Proposition 1.17 ([GS1], Proposition 1.55)**.** *The discrete Legendre transform preserves positivity (Definition 1.11).*

1.2 Toric varieties

Toric varieties are a special class of algebraic varieties that can be described via combinatorial objects like polyhedra and polyhedral fans. The material of this section can be found in [Ful1] and [CLS].

Definition 1.18. A *toric variety* is an algebraic variety X containing an algebraic torus an a dense open subset such that the action of the torus on itself extends to the whole variety. A *toric stratum* of X is a subvariety of X that is the closure of a torus orbit. A *toric divisor* is a codimension 1 toric stratum. The *toric boundary* ∂X of X is the union of its toric divisors.

Definition 1.19. A (convex rational polyhedral) *cone* $\sigma \subset M_{\mathbb{R}}$ is a polyhedron of the form

$$\sigma = \{r_1 m_1 + \ldots + r_s m_s \mid r_i \in \mathbb{R}_{\geq 0}\}$$

for some $m_1, \ldots, m_s \in M$, the *generators* of σ.

Definition 1.20. Let Q be an integral polyhedron. The *cone over* Q is defined as

$$C(Q) := \mathrm{cl}\left(\{(rm, r) \mid m \in Q, r \in \mathbb{R}\}\right) \subset M_{\mathbb{R}} \times \mathbb{R},$$

where cl means taking the closure. The *asymptotic cone* of Q is the Hausdorff limit of rQ as $r \to 0$, i.e.,

$$\mathrm{Asym}(Q) = C(Q) \cap (M_{\mathbb{R}} \oplus \{0\}).$$

Definition 1.21. Let Q be an integral polyhedron. The *toric variety defined by* Q is

$$X_Q := \mathrm{Proj}\ \mathbb{C}[C(Q) \cap (M \times \mathbb{Z})].$$

Here $\mathbb{C}[C(Q) \cap (M \times \mathbb{Z})]$ is graded by the projection $M \times \mathbb{Z} \to \mathbb{Z}$, i.e., deg $z^{(m,d)} = d$. Its degree 0 piece is $\mathbb{C}[\mathrm{Asym}(Q) \cap M]$, so X_Q is projective over Spec $\mathbb{C}[\mathrm{Asym}(Q) \cap M]$. In particular, if Q is bounded, i.e., a polytope, then X_Q is projective.

Remark 1.22. There is an inclusion-preserving correspondence between the faces of Q and the toric strata of X_Q. In particular, the boundary ∂Q of Q corresponds to the toric boundary ∂X of X.

Definition 1.23. Let $\sigma \subset M_{\mathbb{R}}$ be a cone. The *dual cone* of σ is the cone $\sigma^\vee \subset N_{\mathbb{R}}$ defined by

$$\sigma^\vee = \{n \in N_{\mathbb{R}} \mid \langle n, m \rangle \geq 0\ \forall m \in \sigma\}.$$

Definition 1.24. Let $\sigma \subset M_{\mathbb{R}}$ be a cone. The *affine toric variety defined by* σ is

$$X_\sigma = \mathrm{Spec}\ \mathbb{C}[\sigma^\vee \cap N].$$

Definition 1.25. A (convex rational polyhedral) *fan* in $M_{\mathbb{R}}$ is a collection Σ of cones $\sigma \subset M_{\mathbb{R}}$ such that

(1) Each face of a cone in Σ is also a cone in Σ;

(2) The intersection of two cones in Σ is a face of each.

The 1-dimensional cones in Σ are called *rays*.

Definition 1.26. Let Σ be a fan. The *toric variety defined by* Σ is

$$X_\Sigma = \varinjlim_{\sigma \in \Sigma} X_\sigma,$$

where Σ is ordered by inclusion of faces. This means we are gluing the affine toric varieties X_σ for n-dimensional cones $\sigma \in \Sigma$ along the divisors $X_\tau \subset X_\sigma$ defined by their common faces τ.

Remark 1.27. There is an inclusion-reversing correspondence between the cones in Σ and the toric strata of X_Σ. In particular, the rays in Σ correspond to the toric divisors of X_Σ.

Definition 1.28. Let $Q \subset M_\mathbb{R}$ be a polyhedron. The (inner) *normal fan* of Q is the fan Σ_Q in $N_\mathbb{R}$ formed by the (inner) *normal cones*, for all $\sigma \in \Sigma$,

$$C_\sigma = \left\{ n \in N_\mathbb{R} \,\middle|\, \langle n, m_1 - m_2 \rangle \geq 0 \ \forall m_1 \in Q, m_2 \in \sigma \right\}.$$

Proposition 1.29 ([CLS], Proposition 3.1.6)**.** *If $Q \subset M_\mathbb{R} \simeq \mathbb{R}^n$ is an n-dimensional polyhedron, then*

$$X_Q \cong X_{\Sigma_Q}.$$

Example 1.30. Figure 1.2 shows a polytope $Q \subset \mathbb{R}^2$ defining the projective plane $\mathbb{P}^2$ and its normal fan Σ_Q. Indeed, $C(Q) \cap \mathbb{Z}^3$ is a monoid generated by three independent elements, all of degree 1 with respect to the grading.

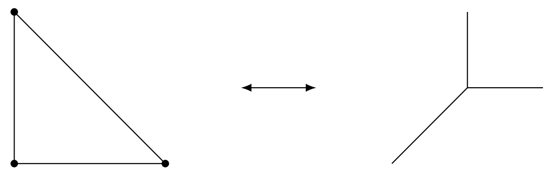


Figure 1.1: A polytope Q defining $\mathbb{P}^2$ and its normal fan.

Example 1.31. Figure 1.2 shows a polytope $Q \subset \mathbb{R}^2$ defining the weighted projective plane $\mathbb{P}(1,1,2)$ and its normal fan Σ_Q. Indeed,

$$\text{Proj } \mathbb{C}[C(Q) \cap \mathbb{Z}^3] = \text{Proj } \mathbb{C}[X,Y,Z,W]/(XY - Z^2),$$

and this is isomorphic to $\mathbb{P}(1,1,2)$ via

$$\mathbb{P}(1,1,2) \xrightarrow{\sim} V(XY - Z^2) \subset \mathbb{P}^3, (X_0, X_1, X_2) \mapsto (X_0^2, X_1^2, X_0X_1, X_2),$$

with inverse $(X,Y,Z,W) \mapsto (X, Z, XW)$.

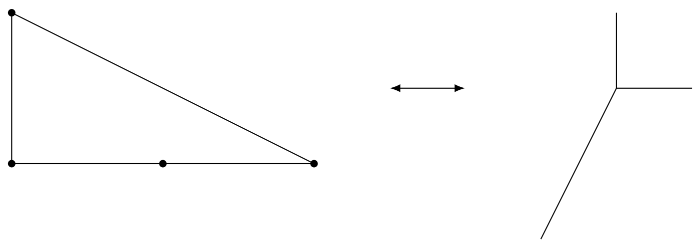


Figure 1.2: A polytope Q defining $\mathbb{P}(1,1,2)$ and its normal fan.

Remark 1.32. By construction, the toric variety X_Q defined by a polytope $Q \subset M_{\mathbb{R}}$ comes with an embedding into $\mathbb{P}^{N-1}$, where N is the number of integral points of Q. This induces an ample line bundle $\mathscr{L}_Q$ (*polarization*) on X_Q by pullback of $\mathcal{O}_{\mathbb{P}^{N-1}}(1)$. Scaling the polytope Q amounts to changing the polarization of X_Q. Passing to the normal fan we are losing this information. We can define a polarization on a toric manifold X_Σ defined by a fan Σ as follows. Let φ be a strictly convex piecewise linear function on Σ, i.e., φ is continuous and linear precisely on the maximal cones of Σ. This defines a polarization $\mathscr{L}_\varphi$ on X_Σ as the line bundle corresponding to the divisor

$$D_\varphi = \sum_{\rho \in \Sigma^{[1]}} \varphi(m_\rho) D_\rho.$$

The sum is over the rays (1-dimensional cones) ρ of Σ, $m_\rho \in M$ is the primitive generator of ρ and D_ρ is the divisor of X_Σ corresponding to ρ. In fact, there is a bijective correspondence between polarizations on X_Σ and strictly convex piecewise linear functions on Σ modulo linear functions.

1.3 Log structures

Log structures are a way to treat mildly singular schemes as being smooth, for example by remembering information about an embedding into a larger scheme. The material of this section is taken from [Gro2], §3.2.

Definition 1.33. A pre-log structure on a scheme X is a sheaf of monoids $\mathcal{M}_X$ on X along with a homomorphism of sheaves of monoids

$$\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$$

where the monoid structure on $\mathcal{O}_X$ is given by multiplication. It is a *log structure* if $\alpha_X : \alpha_X^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. Here $\mathcal{O}_X^\times$ denotes the sheaf of invertible elements of $\mathcal{O}_X$.

Definition 1.34. A *log scheme* is a scheme X equipped with a log structure $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$. A *morphism of log schemes* is a morphism of schemes $f : X \to Y$ along with a morphism of sheaves of monoids $f^\# : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ such that

$\alpha_X \circ f^\# = f^\star \circ \alpha_Y$, where $f^\star$ is the usual pullback of regular functions defined by f. We will usually write X for the log scheme and $\underline{X}$ for the underlying scheme.

Example 1.35. Let X be a scheme and $D \subset X$ a closed subset of pure codimension 1. Let $j : X \setminus D \hookrightarrow X$ be the inclusion and consider

$$\mathcal{M}_{(X,D)} := (j_\star \mathcal{O}^\times_{X\setminus D}) \cap \mathcal{O}_X.$$

This is the sheaf of regular functions on X which are invertible on $X \setminus D$. We take $\alpha_X : \mathcal{M}_{(X,D)} \to \mathcal{O}_X$ to be the inclusion. This is called the *divisorial log structure induced by* D.

Definition 1.36. Let $\alpha : \mathcal{P}_X \to \mathcal{O}_X$ be a pre-log structure on X. Then the *log structure associated to* α is given by the sheaf of monoids

$$\mathcal{M}_X := \mathcal{P}_X \oplus \mathcal{O}^\times_X \Big/ \left\{ (p, \alpha(p)^{-1}) \mid p \in \alpha^{-1}(\mathcal{O}^\times_X) \right\}$$

and $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ is given by $\alpha_X(p, f) := \alpha(p) \cdot f$.

Definition 1.37. Let $f : X \to Y$ be a morphism of schemes and let Y carry a log structure $\alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y$. The *pullback log structure* of the log structure α_Y along f is the log structure associated to the pre-log structure

$$\alpha : f^{-1}(\mathcal{M}_Y) \xrightarrow{\alpha_Y} f^{-1}(\mathcal{O}_Y) \xrightarrow{f^\star} \mathcal{O}_X.$$

Example 1.38. Consider an affine toric variety $X = \operatorname{Spec} \mathbb{C}[P]$ with the divisorial log structure induced by $\{0\} \subset X$. Let $\operatorname{Spec} \mathbb{C}$ be a point with equipped with the pullback log structure along the map to the origin $\operatorname{Spec} \mathbb{C} \hookrightarrow \mathbb{A}^1$. We denote this points by pt_P. The log structure can be given explicitly by

$$\alpha_X : \mathcal{M}_X = \mathbb{C}^\times \oplus P, (x, p) \mapsto \begin{cases} x & p = 0 \\ 0 & p \neq 0 \end{cases}.$$

In the special case $P = \mathbb{N}$ with $X = \mathbb{A}^1$ we call $\mathrm{pt}_\mathbb{N}$ the *standard log point.*

Definition 1.39. Let X be a scheme and let $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ be a log structure on X. Define the *ghost sheaf* of α_X to be the sheaf of monoids $\overline{\mathcal{M}}_X$ defined as the quotient

$$1 \to \mathcal{O}^\times_X \xrightarrow{\alpha_X^{-1}} \mathcal{M}_X \to \overline{\mathcal{M}}_X \longrightarrow 0.$$

Example 1.40. Consider $X = \operatorname{Spec} \mathbb{C}[x, y]$ with divisorial log structure induced by $D = V(xy)$. Then $\overline{\mathcal{M}}_{(X,D)} = i_{1\star}\underline{\mathbb{N}} \oplus i_{2\star}\underline{\mathbb{N}}$, where $i_1 : V(x) \hookrightarrow X$ and $i_2 : V(y) \hookrightarrow X$ are the inclusions and $\underline{\mathbb{N}}$ is the constant sheaf of $\mathbb{N}$. The map $\mathcal{M}_{(X,D)} \to i_{1\star}\underline{\mathbb{N}} \oplus i_{2\star}\underline{\mathbb{N}}$

takes a regular function f on an open set U invertible on $U \setminus D$ to the orders of vanishing of f on $U \cap V(x)$ and $U \cap V(y)$. If we pullback $\mathcal{M}_{(X,D)}$ via the inclusion $D \hookrightarrow X$, we obtain a log structure on D with $\overline{\mathcal{M}}_D = i_{1\star}\mathbb{N} \oplus i_{2\star}\mathbb{N}$. One should think of this log structure as remembering some information of how D sits inside X.

Definition 1.41. A monoid P is *integral* if the cancellation law holds, that is, for each $p, p', p'' \in P$ we have $p + p' = p + p'' \Rightarrow p' = p''$. It is *saturated* if it is integral and whenever $p \in P^{\mathrm{gp}}$ with $mp \in P$ for some $m \in \mathbb{N}_{>0}$, then $p \in P$.

Definition 1.42. Let X be a scheme. A log structure $\mathcal{M}_X$ on X is *fine* if there is an étale open cover $\{U_i\}$ of X such that on U_i there is a finitely generated integral monoid P_i and a pre-log structure $\underline{P_i} \to \mathcal{O}_{U_i}$ (*chart*) whose associated log structure is isomorphic to the pullback log structure of $\mathcal{M}_X$. Here $\underline{P_i}$ is the constant sheaf of P_i. A fine log structure $\mathcal{M}_X$ is *saturated* if $\overline{\mathcal{M}}_{X,x}$ is saturated for all geometric points $x \in X$.

Example 1.43. The log structure on the point pt_P (Example 1.38) is fine, with chart given by a homomorphism $P \to \mathbb{C}$ to the multiplicative monoid of $\mathbb{C}$.

Example 1.44. Let P be a toric monoid, i.e., a monoid of the form $P = \sigma^\vee \cap N$ for some strictly convex rational polyhedral cone $\sigma \subset M_{\mathbb{R}}$, and let $X = X_\sigma = \mathrm{Spec}\ \mathbb{C}[P]$ be the affine toric variety defined by σ. Let ∂X be the toric boundary of X. Then the divisorial log structure $\mathcal{M}_{(X,\partial X)}$ induced by $\partial X \subset X$ has a chart $\underline{P} \to \mathcal{O}_X$ globally defined by $P \to \Gamma(X, \mathcal{O}_X) = \mathbb{C}[P], p \mapsto z^p$. We call this the *toric log structure* of X. Indeed, the map $\underline{P} \to \mathcal{O}_X$ factors as

$$\underline{P} \longrightarrow \mathcal{M}_{(X,\partial X)} \xrightarrow{\alpha_X} \mathcal{O}_X$$

via $P \to \Gamma(X, \mathcal{M}_{(X,\partial X)}), p \mapsto z^p$, since z^p is a regular function on X invertible on the big torus orbit $X \setminus \partial X$. This gives a map $\underline{P} \oplus \mathcal{O}_X^\times \to \mathcal{M}_{(X,\partial X)}$ whose kernel on an open set $U \subset X$ consists precisely of pairs (p, z^{-p}) such that z^{-p} is invertible on U. Furthermore, locally at $x \in X$, a function defined in a neighborhood U of x which is invertible on $U \setminus \partial X$ is always of the form $\varphi \cdot z^p$ for φ on U invertible and $p \in P$. So $\underline{P} \oplus \mathcal{O}_X^\times \to \mathcal{M}_{(X,\partial X)}$ is surjective.

Definition 1.45. A morphism of log schemes $f : X \to Y$ is called *strict* if the map $f^\# : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ induces an isomorphism between the pullback log structure of $\mathcal{M}_Y$ to X and the log structure $\mathcal{M}_X$.

Remark 1.46. A chart of $\mathscr{M}_X$ is equivalent to a strict morphism of log schemes $X \to \mathrm{Spec}\ \mathbb{C}[P]$, where $\mathrm{Spec}\ \mathbb{C}[P]$ is an affine toric variety equipped with the toric log structure (Example 1.44).

Example 1.47. Consider $X = \operatorname{Spec} \mathbb{C}[x, y, w, t]/(xy - wt)$ with the divisorial log structure induced by $D = V(t) \subset X$. Then $\mathcal{M}_X := \mathcal{M}_{(X,D)}$ is not fine at the point $0 = (0, 0, 0, 0) \in X$ as we will see now.

Suppose there is an étale neighborhood U of 0 and a chart $\underline{P} \to \mathcal{O}_U$ for $\mathcal{M}_{(X,D)}$. Then there is a surjective map $\underline{P} \to \overline{\mathcal{M}}_{(X,D)}$ on U. This is surjective on stalks and for any geometry point $x \in U$ we have a commutative diagram

$$\begin{tikzcd} P = \Gamma(U, \underline{P}) \arrow[r] \arrow[d, "\cong"] & \Gamma(U, \overline{\mathcal{M}}_X) \arrow[d] \\ P = \underline{P}_x \arrow[r] & \overline{\mathcal{M}}_{X,x} \end{tikzcd}$$

so $\Gamma(U, \overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_{X,x}$ must be surjective.

An element $\varphi \in \mathcal{M}_{X,0}$ is of the form $\varphi = t^n \cdot \psi$ with $\psi \in \mathcal{O}_{X,0}^\times$. Indeed, in an étale neighborhood of 0, the only Cartier divisor with support in D is a multiple of D. Thus there is some non-negative integer n such that $t^{-n} \cdot \varphi$ does not vanish along D and hence, by normality, $t^{-n} \cdot \varphi$ is an invertible regular function. This shows $\overline{\mathcal{M}}_{X,0} = \mathbb{N}$. On the other hand, if $x \in X$ is a point with coordinates $x = y = 0$ and $w \neq 0$, then both of the irreducible components of D are Cartier and $\overline{\mathcal{M}}_{X,p} = \mathbb{N}^2$. Now there is some étale neighborhood of 0 on which $\Gamma(U, \overline{\mathcal{M}}_X) = \mathbb{N}$ and this cannot surject to $\overline{\mathcal{M}}_{X,p} = \mathbb{N}^2$. Thus the log structure is not fine.

Definition 1.48. A morphism $f : X \to Y$ of fine log schemes is *log smooth* if étale locally on X and Y it fits into a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}[P] \arrow[d] \\ Y \arrow[r] & \operatorname{Spec} \mathbb{Z}[P'] \end{tikzcd}$$

with the following properties:

(1) The horizontal maps induce charts $\underline{P} \to \mathcal{O}_X$ and $\underline{P'} \to \mathcal{O}_Y$ for the log structures on X and Y, respectively;

(2) The induced morphism $X \to Y \times_{\operatorname{Spec} \mathbb{Z}[P']} \operatorname{Spec} \mathbb{Z}[P]$ is a smooth morphism of schemes;

(3) The right-hand vertical arrow is induced by a monoid homomorphism $P' \to P$ with $\ker(P'^{\mathrm{gp}} \to P^{\mathrm{gp}})$ and the torsion part of $\operatorname{coker}(P'^{\mathrm{gp}} \to P^{\mathrm{gp}})$ finite groups. Note that if P and P' are toric monoids, then P^{gp} and P'^{gp} are torsion free.

Example 1.49. A toric variety X equipped with the toric log structure $\mathcal{M}_{(X,\partial X)}$ is log smooth over a point $\underline{\mathrm{pt}}$ with trivial log structure, while of course not all toric varieties are smooth. Indeed, locally $X = \operatorname{Spec} \mathbb{C}[P]$ with toric monoid $P = \sigma^\vee \cap N$, and $X \cong \operatorname{Spec} \mathbb{C} \times_{\mathbb{Z}[P']} \operatorname{Spec} \mathbb{Z}[P]$ for $P' = 0$.

Example 1.50. Consider an affine toric variety $X = \text{Spec } \mathbb{C}[P]$ with toric log structure and $\mathbb{A}^1 = \text{Spec } \mathbb{C}[\mathbb{N}]$ with divisorial log structure by $\{0\}$. Let $X \to \mathbb{A}^1$ be given by $\mathbb{N} \to P, 1 \mapsto \rho \neq 0$. This is log smooth, since it is obtained via base change from $\text{Spec } \mathbb{Z}[P] \to \text{Spec } \mathbb{Z}[\mathbb{N}]$. Restriction to the fiber over $0 \in \mathbb{A}^1$ gives a morphism $X_0 \to \text{pt}_{\mathbb{N}}$, where $\text{pt}_{\mathbb{N}}$ is the standard log point. This is again log smooth, since

$$X_0 \cong \text{Spec } \mathbb{C} \times_{\text{Spec } \mathbb{Z}[\mathbb{N}]} \text{Spec } \mathbb{Z}[P].$$

Example 1.51. Let $\pi : \mathfrak{X} \to B$ be a semistable degeneration, i.e., a proper map from a smooth variety $\mathfrak{X}$ to a smooth curve B with $X_0 = \pi^{-1}(0)$ a divisor with normal crossings and $\pi|_{\mathfrak{X} \setminus X_0}$ smooth. Then $\mathfrak{X} \to \mathbb{A}^1$ is log smooth when $\mathfrak{X}$ is equipped with the divisorial log structure by X_0 and B is equipped with the divisorial log structure by $\{0\}$. Indeed, locally π is given by projection to the t-coordinate

$$\text{Spec } \mathbb{C}[t, x_1, \ldots, x_n]/(x_1 \cdot \ldots \cdot x_n - t^l) \to \mathbb{A}^1,$$

for some $l > 0$ and with divisorial log structures by $V(t)$ and $\{0\}$, respectively. The log structure on $\text{Spec } \mathbb{C}[t, x_1, \ldots, x_n]/(x_1 \cdot \ldots \cdot x_n - t^l)$ is the toric one, so π is log smooth by Example 1.50. Again, restricting to the central fiber gives a log smooth morphism $X_0 \to \text{pt}_{\mathbb{N}}$ to the standard log point $\text{pt}_{\mathbb{N}}$.

1.4 Toric degenerations

In this section we will introduce the notion of toric degenerations of log Calabi-Yau pairs. This can be found in [GS3]. Then we show how a piecewise affine function on a polyhedron leads to a toric degeneration of a toric variety and give some examples of toric degenerations.

Definition 1.52. For us a *log Calabi-Yau pair* is a pair (X, D) consisting of a normal projective variety X together with an anticanonical divisor D. A log Calabi-Yau pair is called *smooth* if both X and D are smooth. It is called *toric* if X is a toric variety and $D = \partial X$ is the toric boundary.

Definition 1.53. A *totally degenerate Calabi-Yau pair* is a reduced variety X together with a reduced divisor $D \subseteq X$ fulfilling the following condition. Let $\nu : \tilde{X} \to X$ be the normalization and $C \subseteq \tilde{X}$ its conductor locus. Then

(1) $\tilde{X}$ is a disjoint union of toric varieties that are algebraically convex, i.e., there exists a proper map to an affine variety;
(2) C is a reduced divisor such that $[C] + \nu^\star[D]$ is the sum of all toric prime divisors;
(3) $\nu|_C : C \to \nu(C)$ is unramified and generically two-to-one;

(4) The square

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "\nu"] \\ \nu(C) \arrow[r] & X \end{tikzcd}$$

is cartesian and cocartesian.

In other words, if (X, D) is a totally degenerate CY-pair, then X is built from a collection of toric varieties by identifying pairs of toric prime divisors torically. The remaining toric prime divisors define D.

Let T be $\mathbb{A}^1$ or the spectrum of a discrete valuation $\mathbb{C}$-algebra with uniformizing parameter $t \in \mathcal{O}(T)$, e.g. $T = \text{Spec } \mathbb{C}[\![t]\!]$. Let 0 be the origin or the unique closed point, respectively.

Definition 1.54. A *toric degeneration of log Calabi-Yau pairs* over T is a flat morphism $\pi : \mathfrak{X} \to T$ of $\mathbb{C}$-schemes together with a reduced divisor $\mathfrak{D} \subseteq \mathfrak{X}$, with the following properties:

(i) $\mathfrak{X}$ is normal;

(ii) The central fiber $X_0 := \pi^{-1}(0)$ together with $D_0 = \mathfrak{D} \cap X_0$ is a totally degenerate CY-pair;

(iii) Away from a closed subset $\mathfrak{Z} \subseteq \mathfrak{X}$ of relative codimension two not containing any toric stratum of X_0, the map $\pi : \mathfrak{X} \to T$ is log smooth, where the log structures are the divisorial ones defined by $X_0 \cup \mathfrak{D} \subseteq \mathfrak{X}$ and $0 \in T$, respectively;

The generic/general fiber $(\mathfrak{X}_\eta, \mathfrak{D}_\eta)$ is isomorphic to a log Calabi-Yau pair (X, D), since the toric boundary of a toric variety is an anticanonical divisor. We say $(\mathfrak{X} \to T, \mathfrak{D})$ is a toric degeneration of (X, D).

Construction 1.55 ([Mum]; [GS4], §1.2)**.** Let $Q \subset M_\mathbb{R}$ be a polyhedron containing the origin $0 \in M_\mathbb{R}$ and let $\check{\varphi}$ be a piecewise affine function with $\check{\varphi}(0) = 0$. Let $\check{\mathscr{P}}$ be the corner locus of $\check{\varphi}$, i.e., the locus where $\check{\varphi}$ is not affine. Then $(Q, \check{\mathscr{P}}, \check{\varphi})$ is a polarized integral polyhedral affine manifold without affine singularities. From this we construct a toric degeneration of a toric log Calabi-Yau pair as follows.

$$Q_{\check{\varphi}} = \{(m, h) \in M_\mathbb{R} \times \mathbb{R} \mid h \geq \varphi(m), m \in Q\}$$

be the convex upper hull of $\check{\varphi}$ and let

$$\mathfrak{X} = \text{Proj}\left(\mathbb{C}[C(Q_{\check{\varphi}}) \cap (M \times \mathbb{Z} \times \mathbb{Z})]\right).$$

be the toric variety defined by $C(Q_{\check{\varphi}})$, the cone over $Q_{\check{\varphi}}$. The projection to the $n+1$-th coordinate $C(Q_{\check{\varphi}}) \to \mathbb{R} \geq 0$ induces a projection $\mathfrak{X} \to \mathbb{A}^1$ to the coordinate

$t := z^{(0,1,0)}$. Define $\mathfrak{D} = V(z^{(0,0,1)})$. Then $(\mathfrak{X} \to \mathbb{A}^1, \mathfrak{D})$ is a toric degeneration of $(X_Q, \partial X_Q)$. Here Q is the toric variety defined by Q and ∂X_Q is its toric boundary. The central fiber X_0 is obtained by gluing the toric varieties defined by the polytopes in $\check{\mathscr{P}}$ along their common toric divisors, as given by the combinatorics of $\mathscr{P}$. Note that each irreducible component of X_0 comes with a polarization (ample line bundle) via the definition of toric varieties from polytopes (see Remark 1.32). This yields a polarization of X_0.

Example 1.56. Figure 1.3 shows a polarized polyhedral affine manifold $(Q, \check{\mathscr{P}}, \check{\varphi})$ defining a toric degeneration $(\mathfrak{X} \to \mathbb{A}^1, \mathfrak{D})$ of $(\mathbb{P}^2, \partial\mathbb{P}^2)$, where $\partial\mathbb{P}^2$ is the toric boundary. The polytope $Q \subset M_{\mathbb{R}}$ defines $\mathbb{P}^2$ via a degree 3 embedding into $\mathbb{P}^9$, since Q has 10 integral points. Pulling back $\mathcal{O}_{\mathbb{P}^9}(1)$ to $\mathbb{P}^2$ gives the anticanonical line bundle $\mathcal{O}_{\mathbb{P}^2}(3)$. The total space $\mathfrak{X}$ is defined as a subvariety of $\mathbb{P}^9 \times \mathbb{A}^1$. However, the vertices of Q form a sublattice M' of M of index 3 and $Q \cap M'$ consists of 4 points, so we can write $\mathfrak{X}$ as

$$\mathfrak{X} = V(XYZ - t^3W) \subset \mathbb{P}(1,1,1,3) \times \mathbb{A}^1.$$

Here X, Y, Z, W are the coordinates of $\mathbb{P}(1,1,1,3)$ and t is the coordinate of $\mathbb{A}^1$. The map $\pi : \mathfrak{X} \to \mathbb{A}^1$ is given by projection to t and $\mathfrak{D} = V(W) \subset \mathfrak{X}$.

For $t \in \mathbb{A}^1$ write $X_t = \pi^{-1}(t)$. For $t \neq 0$ the fiber X_t is isomorphic to $\mathbb{P}^2$ via elimination of W by $W = t^{-3}XYZ$ and $\mathfrak{D} = \partial\mathbb{P}^2$ is the toric boundary. The central fiber X_0 is a union of three $\mathbb{P}(1,1,3)$ glued along common toric divisors as described by the combinatorics of Figure 1.3 and D_0 is still a cycle of three lines.

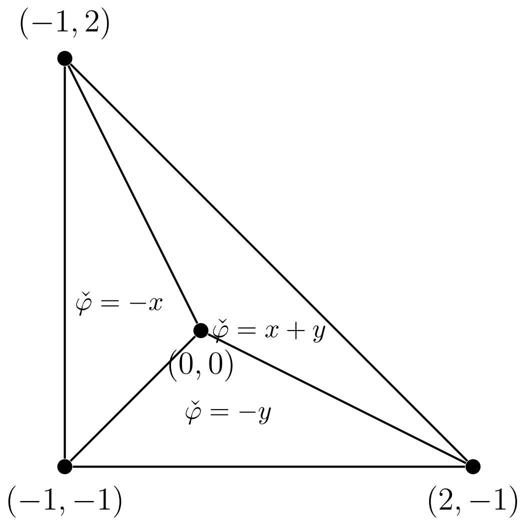


Figure 1.3: A polarized polyhedral affine manifold $(Q, \check{\mathscr{P}}, \check{\varphi})$ defining a toric degeneration of $(\mathbb{P}^2, \partial\mathbb{P}^2)$. The piecewise affine function $\check{\varphi}$ takes values 0 at the origin and 1 on the boundary of Q.

Example 1.57. We can deform the general fiber of the above example to make the divisor D_t smooth for $t \neq 0$. This will lead to affine singularities on the interior eges of $\check{\mathscr{P}}$ (see Example 1.59 below).

Define

$$\mathfrak{X} = V\left(XYZ - t^3(W + f_3)\right) \subset \mathbb{P}(1,1,1,3) \times \mathbb{A}^1,$$

where X, Y, Z, W are the coordinates of $\mathbb{P}(1,1,1,3)$ and f_3 is a general homogeneous degree 3 polynomial in X, Y, Z. Define $\mathfrak{D} = V(W) \subset \mathfrak{X}$ and let $\pi : \mathfrak{X} \to \mathbb{A}^1$ be the projection to the coordinate t of $\mathbb{A}^1$. Then $(\mathfrak{X} \to \mathbb{A}^1, \mathfrak{D})$ is a toric degeneration of $(\mathbb{P}^2, E)$, where $E \subset \mathbb{P}^2$ is an elliptic curve:

For $t \neq 0$ we have $X_t = \mathbb{P}^2$, since we can eliminate W by $W = t^{-3}XYZ - f_3$. The divisor $D_t \subset \mathbb{P}^2$ is defined by a general degree 3 polynomial, so is an elliptic curve E. For $t = 0$ we have $X_0^s = V(XYZ) \subset \mathbb{P}(1,1,1,3)$. This is a union of three $\mathbb{P}(1,1,3)$ glued along common toric divisors. D_0 is a cycle of three lines. For a picture consider the intersection complex of $(\mathfrak{X} \to \mathbb{A}^1, \mathfrak{D})$ in Figure 1.3 below.

The family $\mathfrak{X} \to \mathbb{A}^1$ is log smooth away from three points on the central fiber X_0. These are given by $t = X = Y = W + f_3 = 0$ and its permutations of X, Y, Z and are locally of the form Spec $\mathbb{C}[x, y, \tilde{w}, t]/(xy - \tilde{w}t)$ as in Example 1.47, where we dehomogenized by setting $Z = 1$ and wrote $\tilde{w} = w + f_3$. At other points $\mathfrak{X}$ is semistable, so log smooth by Example 1.51.

1.5 The intersection complex and its dual

Let $(\mathfrak{X} \to T, \mathfrak{D})$ be a toric degeneration of log Calabi-Yau pairs. Consider the divisorial log structure on $\mathfrak{X}$ defined by $X_0 \cup \mathfrak{D} \subseteq \mathfrak{X}$ and pull this log structure back to the central fiber X_0. There are two natrual polarized integral polyhedral affine manifolds induced by the pair $(X_0, \mathcal{M}_{X_0})$ or, equivalently, by $(\mathfrak{X} \to T, \mathfrak{D})$. The first one is the *intersection complex* or the *cone picture*. The second one is the *dual intersection complex* or the *fan picture*.

Construction 1.58 (The intersection complex, [GS3], Example 1.13)**.** Let $(\mathfrak{X} \to T, \mathfrak{D})$ a toric degeneration of log Calabi-Yau pairs, together with a polarization $\mathscr{L}$ of the central fiber X_0. The *intersection complex* of $(\mathfrak{X} \to T, \mathfrak{D})$ is constructed as follows. Any toric stratum of X_0 (meaning a toric stratum of some irreducible component of X_0) together with the restriction of $\mathscr{L}$ is a polarized toric variety. The sections of powers of the line bundle define the integral points of a cone $C(\sigma)$ over some integral polytope $\sigma \subset M_{\mathbb{R}}$. This is the converse of the construction of toric varieties from σ and σ is determined uniquely up to integral affine transformations. Inclusions of toric strata define integral affine inclusions of polytopes as faces. Hence, we have an integral polyhedral complex $\check{\mathscr{P}}$ (Definition 1.5).

To obtain a polyhedral affine manifold we need fan structures (Definition 1.8) at the vertices of $\check{\mathscr{P}}$. At a 0-dimensional toric stratum the degeneration is étale locally described by a toric morphism Spec $\mathbb{C}[\sigma \cap (M \times \mathbb{Z})] \to$ Spec $\mathbb{C}[\mathbb{N}]$ for some rational polyhedral cone $\sigma \subset M_{\mathbb{R}} \cap \mathbb{R}$. Denote by $\rho \in \sigma \cap (M \times \mathbb{Z})$ the image of $1 \in \mathbb{N}$. Then the images of the faces of σ not containing ρ under the projection $M_{\mathbb{R}} \times \mathbb{R} \to (M_{\mathbb{R}} \times \mathbb{R})/\mathbb{R}\rho \simeq \mathbb{R}^n$ define an n-dimensional fan. This defines the fan structure at the vertex v corresponding to the 0-dimensional stratum. Thus we have a polyhedral affine manifold $(\check{B}, \check{\mathscr{P}})$ via Construction 1.6 and Proposition 1.9

For each vertex, the total space of the local model Spec $\mathbb{C}[\sigma \cap (M \times \mathbb{Z})] \to \mathbb{A}^1$ is the toric variety defined by the graph of a strictly convex integral piecewise affine function $\check{\varphi}_v$ on Σ_v (Definition 1.8, (iii)), unique up to linear functions. The $\check{\varphi}_v$ only differ by linear functions on intersections, so they define a polarization $\check{\varphi}$ of $(\check{B}, \check{\mathscr{P}})$. The triple $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ is called the *intersection complex* of $(\mathfrak{X} \to T, \mathfrak{D})$.

Example 1.59. The intersection complex $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ of the toric degeneration of $(\mathbb{P}^2, E)$ from Example 1.57 (with polarization $\mathcal{O}_{\mathbb{P}^2}(3)$) is shown in Figure 1.4. Indeed, locally at 0-dimensional strata the toric degeneration is given by Spec $\mathbb{C}[x, y, \tilde{w}, t]/(xy - t^l)$ or some permutation of x, y, z, where $\tilde{w} = w + f_3$. This is the union of two $\mathbb{A}^2$ glued along a common $\mathbb{A}^1$, so it gives the fan structures as shown. The boundary of $\check{B}$ is a straight line. $\check{B}$ has three affine singularities, lying on the interiors of bounded edges of $\check{\mathscr{P}}$. Their monodromy $T_{\rho\rho}$ (Definition 1.10) is given locally by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in suitable coordinates, so $(\check{B}, \check{\mathscr{P}})$ is simple (Definition 1.13).

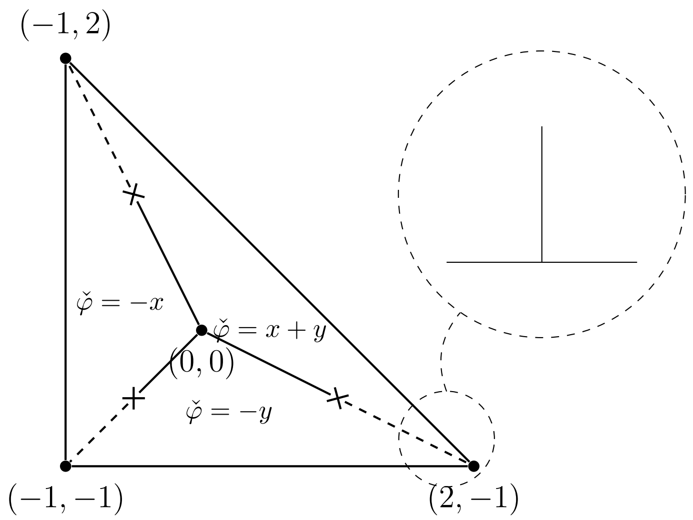


Figure 1.4: The intersection complex $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ of a toric degeneration of $(\mathbb{P}^2, E)$, where E is an elliptic curve. The fan structure at a vertex is shown inside the circle.

Construction 1.60 (The dual intersection complex, [GS3], Example 1.11)**.** For a toric stratum S of X_0 let $\eta_S \in \mathfrak{X}$ be the generic point and let $Y_1, \ldots, Y_r$ be the irreducible components of $X_0 \cup \mathfrak{D}$ containing S. Choose the order in such a way that $Y_i \subseteq X_0$ iff $i \leq s$. Define the monoid

$$P_S := \left\{ (m_1, \ldots, m_r) \in \mathbb{N}^r \mid \sum_{i=1}^{r} m_i [Y_i] \text{ is a Cartier divisor at } \eta_S \in \mathfrak{X} \right\}.$$

This is a toric monoid, i.e., the set of integral points of a cone in $\mathbb{R}^r$. Define

$$\sigma_S := \{\lambda \in \mathrm{Hom}(P_S, \mathbb{R}_{\geq 0}) \mid \lambda(\rho_S) = 1\},$$

where $\mathbb{R}_{\geq 0}$ is viewed as an additive monoid and $\rho_S = (1, \ldots, 1, 0, \ldots, 0)$ is the the vector with entry 1 at the first s places. If $S_1 \subseteq S_2$, then generization maps P_{S_1} surjectively to P_{S_2}, and this induces an inclusion of σ_{S_2} as a face of σ_{S_1}. This shows that the σ_S form an integral polyhedral complex $\mathscr{P}$. The toric irreducible components of X_0 define fan structures at the vertices of $\mathscr{P}$, yielding a integral polyhedral affine manifold $(B, \mathscr{P})$ by Construction 1.6 and Proposition 1.9

For each vertex v of $\mathscr{P}$ there is a polarization of the corresponding irreducible component $X_v = X_{\Sigma_v}$ by restriction of $\mathscr{L}$. This gives a strictly convex piecewise affine functions φ_v on Σ_v as in Remark 1.32. The $\check{\varphi}_v$ only differ by linear functions on intersections, so they define a polarization φ of $(B, \mathscr{P})$. The triple $(B, \mathscr{P}, \varphi)$ is called the *dual intersection complex* of $(\mathfrak{X} \to T, \mathfrak{D})$.

Example 1.61. Figure 1.5 shows the dual intersection complex of the toric degeneration from Example 1.59, i.e., the discrete Legendre transform of Figure 1.4. For each vertex v of $\mathscr{P}$, the fan Σ_v describing $\mathscr{P}$ locally at v defines $X_{\Sigma_v} = \mathbb{P}(1, 1, 3)$.

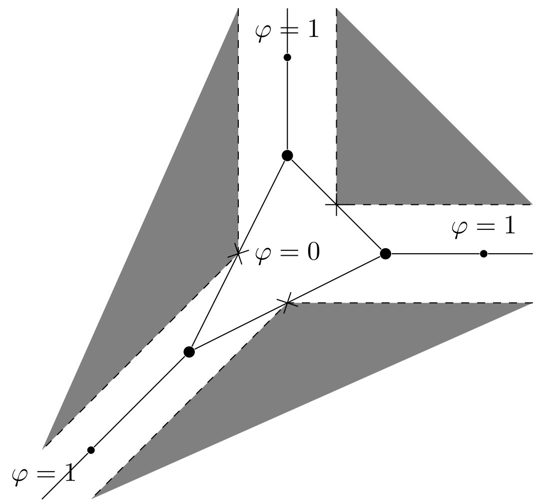


Figure 1.5: The dual intersection complex $(B, \mathscr{P}, \varphi)$ of a toric degeneration of $(\mathbb{P}^2, E)$. The shaded regions are cut out and the dashed lines are mutually identified, so in fact the unbounded edges are all parallel. The piecewise affine function φ is zero on the bounded maximal cell and has slope 1 along the unbounded edges.

Remark 1.62. By construction the intersection complex and its dual are exchanged by the discrete Legendre transform (Construction 1.15).

1.6 The reconstruction algorithm

The procedures of taking the intersection complex (cone picture) or the dual intersection complex (fan picture) can be reversed to construct a totally degenerate log Calabi-Yau pair (X_0, D_0) from a polarized polyhedral affine manifold. Moreover, both pictures give local models for a toric degeneration at the 0-dimensional strata of X_0. However, these do not necessarily fit together when there are affine singularities. In other words, the local models for the toric degeneration may depend on the affine chart in which the construction is performed.

This problem was solved in [GS3] by using *scattering diagrams* and *wall structures* to refine X_0 into a larger number of components and apply non-toric automorphisms of the toric divisors along which these components are glued. We will first review scattering diagrams and wall structures and then give an idea of how to obtain toric degenerations.

The material of this section is taken from the original paper [GS3] and its introductory accompanist [GS4].

1.6.1 Scattering diagrams

Let $M \simeq \mathbb{Z}^2$ be a lattice and write $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let Σ be a fan in $M_{\mathbb{R}}$ and let φ be an integral strictly convex piecewise affine function on Σ with $\varphi(0) = 0$. Note that the rays of Σ form the corner locus of φ. Let P_φ be the monoid of integral points in the upper convex hull of φ,

$$P_\varphi = \{p = (\overline{p}, h) \in M \oplus \mathbb{Z} \mid h \geq \varphi(\overline{p})\}.$$

Write $t := z^{(0,1)}$ and let R_φ be the $\mathbb{C}[\![t]\!]$-algebra obtained by completion of $\mathbb{C}[P_\varphi]$ with respect to (t),

$$R_\varphi = \varprojlim \mathbb{C}[P_\varphi]/(t^k).$$

Definition 1.63. A *ray* for φ is a half-line $\mathfrak{d} = \mathbb{R}_{\geq 0} \cdot m_{\mathfrak{d}} \subseteq M_{\mathbb{R}}$, with $m_{\mathfrak{d}} \in M$ primitive, together with an element $f_{\mathfrak{d}} \in R_\varphi$ such that

(1) each exponent $p = (\overline{p}, h)$ in $f_{\mathfrak{d}}$ satisfies $\overline{p} \in \mathfrak{d}$ or $-\overline{p} \in \mathfrak{d}$. In the first case the ray is called *incoming*, in the latter it is called *outgoing*;

(2) if $m_{\mathfrak{d}}$, is not a ray generator of Σ, then $f_{\mathfrak{d}} \equiv 1 \bmod (z^{m_{\mathfrak{d}}} t)$;

A *scattering diagram* for φ is a set $\mathfrak{D}$ of rays for φ such that for every power $k > 0$ there are only a finite number of rays $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$ with $f_{\mathfrak{d}} \not\equiv 1 \bmod (t^k)$.

Remark 1.64. Note that [GPS] deals with the case $\varphi \equiv 0$, where the t-order of an element $z^{(\overline{p},h)}$ is simply given by h. In our case, the t-order is $\varphi(-\overline{p}) + h \geq 0$:

$$z^{(\overline{p},h)} = \left(z^{(-\overline{p},\varphi(-\overline{p}))}\right)^{-1} t^{\varphi(-\overline{p})+h}.$$

Let $\mathfrak{D}$ be a scattering diagram for φ. Let $\gamma : [0,1] \to M_{\mathbb{R}}$ be a closed immersion not meeting the origin and with endpoints not contained in any ray of $\mathfrak{D}$. Then, for each power $k > 0$, we can find numbers $0 < r_1 \leq r_2 \leq \ldots \leq r_s < 1$ and rays $\mathfrak{d}_i = (\mathbb{R}_{\geq 0} m_i, f_i) \in \mathfrak{D}$ with $f_i \not\equiv 1 \bmod (t^k)$ such that (1) $\gamma(r_i) \in \mathfrak{d}_i$, (2) $\mathfrak{d}_i \neq \mathfrak{d}_j$ if $r_i = r_j$ and $i \neq j$, and (3) s is taken as large as possible.

For each ray $\rho \in \Sigma^{[1]}$ write $f_\rho = 1 + z^{(m_\rho,\varphi(m_\rho))}$ for $m_\rho \in M$ the primitive generator of ρ, and define

$$\tilde{R}^k_\varphi = \left(R_\varphi/(t^{k+1})\right)_{\prod_\rho f_\rho}.$$

For each i, define a $\mathbb{C}[\![t]\!]$-algebra automorphism of $\tilde{R}^k_\varphi$ by $\theta^k_{\mathfrak{d}_i} := \exp(\log(f_i)\partial_{n_i})$ for $\partial_n(z^p) := \langle n, \overline{p}\rangle\, z^p$, i.e.,

$$\theta^k_{\mathfrak{d}_i}(z^p) = f_i^{-\langle n_i, \overline{p}\rangle} z^p,$$

where $n_i \in N = \mathrm{Hom}(M,\mathbb{Z})$ is the unique primitive vector satisfying $\langle n_i, m_i\rangle = 0$ and $\langle n_i, \gamma'(r_i)\rangle > 0$. Define

$$\theta^k_{\gamma,\mathfrak{D}} = \theta^k_{\mathfrak{d}_s} \circ \ldots \circ \theta^k_{\mathfrak{d}_1}.$$

If $r_i = r_j$, then $\theta_{\mathfrak{d}_i}$ and $\theta_{\mathfrak{d}_j}$ commute. Hence, $\theta^k_{\gamma,\mathfrak{D}}$ is well-defined. Moreover, define

$$\theta_{\gamma,\mathfrak{D}} = \lim_{k\to\infty} \theta^k_{\gamma,\mathfrak{D}}.$$

Definition 1.65. A scattering diagram $\mathfrak{D}$ is *consistent to order* k if, for all γ such that $\theta^k_{\gamma,\mathfrak{D}}$ is defined,

$$\theta^k_{\gamma,\mathfrak{D}} \equiv 1 \bmod (t^{k+1}).$$

It is *consistent to any order*, or simply *consistent*, if $\theta_{\gamma,\mathfrak{D}} = 1$.

Proposition 1.66 ([GPS], Theorem 1.4)**.** *Let $\mathfrak{D}$ be a scattering diagram such that $f_{\mathfrak{d}} \equiv 1$ mod (t) for all $\mathfrak{d} \in \mathfrak{D}$. Then there exists a consistent scattering diagram $\mathfrak{D}_\infty$ containing $\mathfrak{D}$ such that $\mathfrak{D}_\infty \setminus \mathfrak{D}$ consists only of outgoing rays.*

Proof. The proof is constructive, so we will give it here.

Take $\mathfrak{D}_0 = \mathfrak{D}$. We will show inductively that there exists a scattering diagram $\mathfrak{D}_k$ containing $\mathfrak{D}_{k-1}$ that is consistent to order k. Let $\mathfrak{D}'_{k-1}$ consist of those rays $\mathfrak{d}$ in $\mathfrak{D}_{k-1}$ with $f_{\mathfrak{d}} \not\equiv 1 \bmod (t^{k+1})$. Let γ be a closed simple loop around the origin. Then $\theta_{\gamma,\mathfrak{D}_{k-1}} \equiv \theta_{\gamma,\mathfrak{D}'_{k-1}} \bmod (t^{k+1})$. By the induction hypothesis this can be uniquely written as

$$\theta_{\gamma,\mathfrak{D}'_{k-1}} = \exp\left(\sum_{i=1}^{s} c_i z^{m_i} \partial_{n_i}\right)$$

with $m_i \in M \setminus \{0\}$m $n_i \in m_i^\perp$ primitive and $c_i \in (t^k)$. Define

$$\mathfrak{D}_k = \mathfrak{D}_{k-1} \cup \{(\mathbb{R}_{\geq 0} m_i, 1 \pm c_i z^{m_i}) \mid i = 1, \ldots, s\},$$

with sign chosen in each ray such that its contribution to $\theta_{\gamma,\mathfrak{D}_k}$ is $\exp(c_i z^{m_i} \partial_{n_i})$ modulo (t^{k+1}). These contributions exactly cancel the contributions to $\theta_{\gamma,\mathfrak{D}_k}$ coming from $\mathfrak{D}_{k-1}$, so $\theta_{\gamma,\mathfrak{D}_k} \equiv 1 \bmod (t^{k+1})$.

Then take $\mathfrak{D}_\infty$ to be the non-disjoint union of the $\mathfrak{D}_k$ for all $k \in \mathbb{N}$. The diagram $\mathfrak{D}_\infty$ will usually have infinitely many rays. □

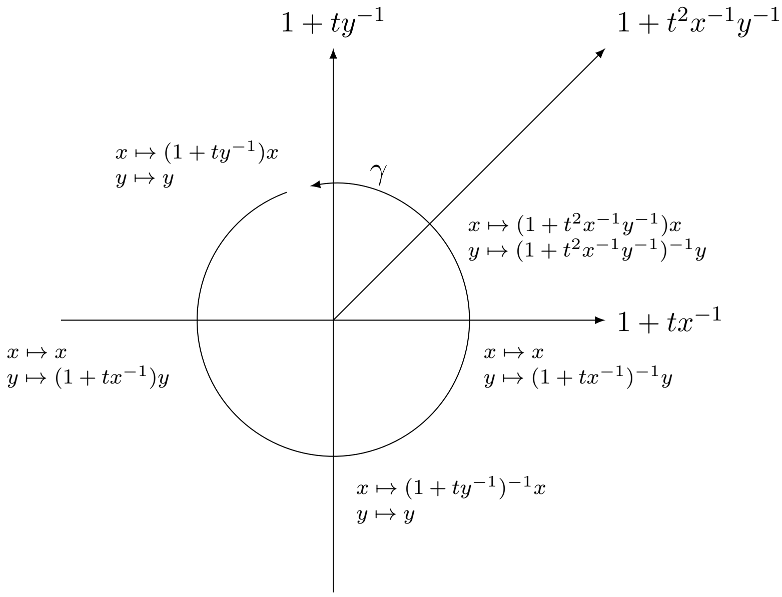


Figure 1.6: The consistent scattering diagram $\mathfrak{D}_\infty$ obtained from the scattering diagram $\mathfrak{D} = \{(\mathbb{R}(1,0), 1 + tx^{-1}), (\mathbb{R}(0,1), 1 + ty^{-1})\}$. The automorphisms corresponding to the rays of $\mathfrak{D}$ are given.

Definition 1.67. A *line* in a scattering diagram $\mathfrak{D}$ is a pair of two rays in opposite directions with the same attached functions, necessarily one incoming and one outgoing. We write a line as

$$\mathfrak{d} = (\mathbb{R} m_{\mathfrak{d}}, f_{\mathfrak{d}}),$$

where $m_{\mathfrak{d}}$ is the direction of the outgoing ray.

Example 1.68. Consider the scattering diagram $\mathfrak{D}$ for $\varphi \equiv 0$ consisting of the lines $(\mathbb{R}(1,0), 1 + tx^{-1})$ and $(\mathbb{R}(0,1), 1 + ty^{-1})$. Let γ be a simple loop around the origin. Then

$$\theta_{\gamma,\mathfrak{D}} : \quad x \mapsto (1 + t^2x^{-1}y^{-1})^{-1}x, \quad y \mapsto (1 + t^2x^{-1}y^{-1})y.$$

Inserting the ray $(\mathbb{R}_{\geq 0}(1,1), 1 + t^2x^{-1}y^{-1})$ cancels $\theta_{\gamma,\mathfrak{D}}$, hence

$$\mathfrak{D}_\infty = \mathfrak{D} \cup \{(\mathbb{R}_{\geq 0}(1,1), 1 + t^2x^{-1}y^{-1})\}.$$

This is shown in Figure 1.6. In this special example we only needed to add one ray to obtain a consistent scattering diagram. In general this will not be the case.

Example 1.69. Consider the scattering diagram for $\varphi \equiv 0$ consisting of the lines $(\mathbb{R}(1,0),(1+tx^{-1})^2)$ and $(\mathbb{R}(0,1),(1+ty^{-1})^2)$. By [GPS], Example 1.6,

$$\begin{aligned}\mathfrak{D}_\infty = \mathfrak{D} \quad &\cup \quad \left\{(\mathbb{R}_{\geq 0}(n+1,n),(1+t^{2n+1}x^{-n-1}y^{-n})^2 \mid n \in \mathbb{N}\setminus\{0\}\right\}\\ &\cup \quad \left\{(\mathbb{R}_{\geq 0}(n,n+1),(1+t^{2n+1}x^{-n}y^{-n-1})^2 \mid n \in \mathbb{N}\setminus\{0\}\right\}\\ &\cup \quad \left\{(\mathbb{R}_{\geq 0}(1,1),(1-t^2x^{-1}y^{-1})^{-4}\right\}\end{aligned}$$

A small part of $\mathfrak{D}_\infty$ is shown in Figure 1.7.

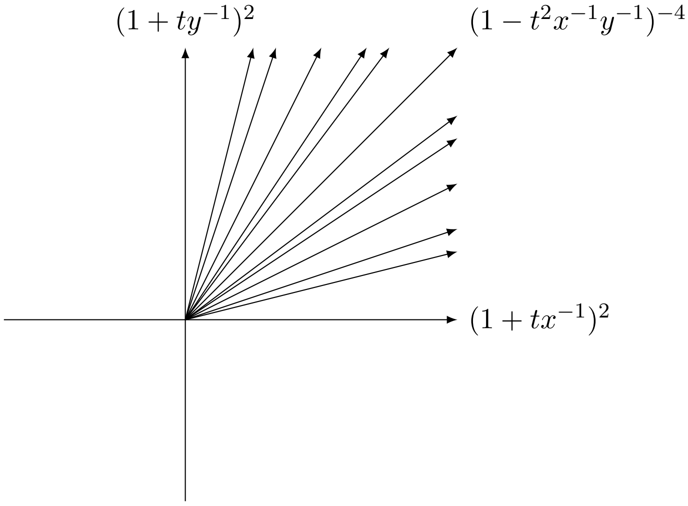


Figure 1.7: A small part of the consistent scattering diagram $\mathfrak{D}_\infty$ with infinitely many rays defined by $\mathfrak{D} = \{(\mathbb{R}(1,0),(1+tx^{-1})^2),(\mathbb{R}(0,1),(1+ty^{-1})^2)\}$.

Definition 1.70. Two scattering diagrams $\mathfrak{D}$, $\mathfrak{D}'$ are *equivalent* if $\theta_{\gamma,\mathfrak{D}} = \theta_{\gamma,\mathfrak{D}'}$ for any closed immersion γ for which both sides are defined.

Definition 1.71. A scattering diagram $\mathfrak{D}$ is called *minimal* if
(1) any two rays $\mathfrak{d},\mathfrak{d}'$ in $\mathfrak{D}$ have distinct support , i.e., $m_\mathfrak{d} \neq m_{\mathfrak{d}'}$;
(2) it contains no *trivial* ray, i.e., with $f_\mathfrak{d} = 1$.

Remark 1.72. Every scattering diagram $\mathfrak{D}$ is equivalent to a unique minimal scattering diagram. In fact, if $\mathfrak{d},\mathfrak{d}'$ have the same support, then we can replace these two rays with a single ray with the same support and attached function $f_\mathfrak{d} \cdot f_{\mathfrak{d}'}$. Moreover, we can remove any trivial ray without affecting $\theta_{\gamma,\mathfrak{D}}$.

1.6.2 Wall structures

Let $(B,\mathscr{P},\varphi)$ be a 2-dimensional polarized polyhedral affine manifold with $(B,\mathscr{P})$ simple (Definition 1.13). Note that for each $x \in B\setminus\Delta$ (where Δ is the discriminant locus of B), φ defines an integral strictly convex piecewise affine function

$$\varphi_x : \Lambda_{B,x} \simeq M \to \mathbb{R}$$

on the fan Σ_x describing $(B, \mathscr{P})$ locally at x. If $\tau_x \in \mathscr{P}$ is the smallest cell containing x, this is given by

$$\Sigma_x = \{K_{\tau_x}\sigma \mid \tau_x \subseteq \sigma \in \mathscr{P}\},$$

where $K_\tau\sigma$ is the cone generated by σ relative to τ, i.e.,

$$K_\tau\sigma = \mathbb{R}_{\geq 0}(\sigma - \tau) = \{m \in M_{\mathbb{R}} \mid \exists m_0 \in \tau, m_1 \in \sigma, \lambda \in \mathbb{R}_{\geq 0} : m = \lambda(m_1 - m_0)\}.$$

As in §1.6.1 this defines a monoid by the integral points in the upper convex hull of φ_x,

$$P_x := P_{\varphi_x} = \{p = (\overline{p}, h) \in \Lambda_{B,x} \oplus \mathbb{Z} \mid h \geq \varphi_x(\overline{p})\}. \tag{1.1}$$

Note that Spec $\mathbb{C}[P_x]$ gives a local toric model for the toric degeneration defined by $(B, \mathscr{P}, \varphi)$ at a point on the interior of the toric stratum corresponding to τ_x.

Definition 1.73. For $x, x' \in B$ integral points, let $m_{xx'} \in \Lambda_B$ denote the primitive vector pointing from x to x'. For a 1-cell ρ and $x \in \rho \setminus \Delta$ let $v[x]$ be the vertex of the irreducible component of $\rho \setminus \Delta$ containing x. This is unique by the construction of the discriminant locus Δ for polyhedral affine manifolds (Construction 1.6).

(1) A *slab* $\mathfrak{b}$ on $(B, \mathscr{P}, \varphi)$ is a 1-dimensional rational polyhedral subset of a 1-cell $\rho_{\mathfrak{b}} \in \mathscr{P}^{[1]}$ together with elements $f_{\mathfrak{b},x} \in \mathbb{C}[P_x]$, one for each $x \in \mathfrak{b} \setminus \Delta$, satisfying the following conditions.
   (1) $f_{\mathfrak{b},x} \equiv 1 \mod (t)$ if $\rho_{\mathfrak{b}}$ does not contain an affine singularity;
   (2) $f_{\mathfrak{b},x} \equiv 1 + z^{(m_{v[x]\delta}, \varphi(m_{v[x]\delta}))} \mod (t)$ if $\rho_{\mathfrak{b}}$ contains an affine singularity δ;
   (3) $f_{\mathfrak{b},x} = z^{(m_{v[x]v[x']}, \varphi(m_{v[x]v[x']}))} f_{\mathfrak{b},x'}$ for all $x, x' \in \mathfrak{b} \setminus \Delta$.
   Note that conditions (1) and (2) are compatible with (3).

(2) A *wall* $\mathfrak{p}$ on $(B, \mathscr{P}, \varphi)$ is a 1-dimensional rational polyhedral subset of a maximal cell $\sigma_{\mathfrak{p}} \in \mathscr{P}^{[2]}$ with $\mathfrak{p} \cap \mathrm{Int}(\sigma_{\mathfrak{p}}) \neq \emptyset$ together with (i) a *base point* $\mathrm{Base}(\mathfrak{p}) \in \mathfrak{p} \setminus \partial\mathfrak{p}$, (ii) an *exponent* $p_{\mathfrak{p}} \in \Gamma(\sigma_{\mathfrak{p}}, \Lambda \oplus \underline{\mathbb{Z}})$ such that $p_{\mathfrak{p},x} = (\overline{p}_{\mathfrak{p},x}, h_{\mathfrak{p},x}) \in P_x$ for all $\mathfrak{p} \setminus \Delta$ with $h_{\mathfrak{p},x} > \varphi(\overline{p}_{\mathfrak{p},x})$ for $x \neq \mathrm{Base}(\mathfrak{p})$, and (iii) $c_{\mathfrak{p}} \in \mathbb{C}$, such that

$$\mathfrak{p} = (\mathrm{Base}(\mathfrak{p}) - \mathbb{R}_{\geq 0}\overline{p}_{\mathfrak{p}}) \cap \sigma_{\mathfrak{p}}.$$

   For each $x \in \mathfrak{p} \setminus \Delta$ this defines a function

$$f_{\mathfrak{p},x} = 1 + c_{\mathfrak{p}} z^{p_{\mathfrak{p}}} \in \mathbb{C}[P_x].$$

(3) A *wall structure* $\mathscr{S}$ on $(B, \mathscr{P}, \varphi)$ is a locally finite set of slabs and walls with a polyhedral decomposition $\mathscr{P}_{\mathscr{S}}$ of its support $|\mathscr{S}| = \cup_{\mathfrak{b} \in \mathscr{S}} \mathfrak{b}$ such that
   (1) The map sending a slab $\mathfrak{b} \in \mathscr{S}$ to its underlying 1-cell of $\mathscr{P}$ is injective;
   (2) Each closure of a connected component of $B \setminus |\mathscr{S}|$ (*chamber*) is convex and its interior is disjoint from any wall;
   (3) Any wall in $\mathscr{S}$ is a union of elements of $\mathscr{P}_{\mathscr{S}}$;
   (4) Any maximal cell of $\mathscr{P}$ contains only finitely many slabs or walls in $\mathscr{S}$.

(4) A *joint* $\mathfrak{j}$ of a wall structure $\mathscr{S}$ on $(B, \mathscr{P}, \varphi)$ is a vertex of $\mathscr{P}_{\mathscr{S}}$. At each joint $\mathfrak{j}$, the wall structure defines a scattering diagram $\mathfrak{D}_{\mathfrak{j}}$ for $\varphi_{\mathfrak{j}}$.

Definition 1.74. A polarized polyhedral affine manifold $(B, \mathscr{P}, \varphi)$ induces an *initial wall structure* $\mathscr{S}_0$ consisting only of slabs as follows. For each 1-cell ρ containing an affine singularity δ there is a slab $\mathfrak{b}$ with underlying polyhedral subset ρ and with

$$f_{\mathfrak{b},v} = 1 + z^{(m_{v\delta}, \varphi(m_{v\delta}))},$$

where $m_{v\delta} \in \Lambda_{B,v}$ is the primitive vector pointing from v to δ.

Definition 1.75. A wall structure $\mathscr{S}$ is *consistent (to order k) at a joint* $\mathfrak{j}$ if the associated scattering diagram $\mathfrak{D}_{\mathfrak{j}}$ is consistent (to order k). It is *consistent (to order k)* if it is consistent (to order k) at any joint.

Definition 1.76. Two wall structures $\mathscr{S}$, $\mathscr{S}'$ are *compatible to order k* if the following conditions hold.

(1) If $\mathfrak{p} \in \mathscr{S}$ is a wall with $c_{\mathfrak{p}} \neq 0$ and $f_{\mathfrak{p},x} \not\equiv 1 \bmod (t^{k+1})$ for some $x \in \mathfrak{p} \setminus \Delta$, then $\mathfrak{p} \in \mathscr{S}'$, and vice versa.

(2) If $x \in \mathrm{Int}(\mathfrak{b}) \cap \mathrm{Int}(\mathfrak{b}')$ for slabs $\mathfrak{b} \in \mathscr{S}, \mathfrak{b}' \in \mathscr{S}'$, then $f_{\mathfrak{b},x} \equiv f_{\mathfrak{b}',x} \bmod (t^{k+1})$.

Proposition 1.77. *If $(B, \mathscr{P}, \varphi)$ is a polarized polyhedral affine manifold with $(B, \mathscr{P})$ simple, then there exists as sequence of wall structures $(\mathscr{S}_k)_{k \in \mathbb{N}}$ such that*

(1) $\mathscr{S}_0$ is the initial wall structure defined by $(B, \mathscr{P}, \varphi)$;

(2) $\mathscr{S}_k$ is consistent to order k;

(3) $\mathscr{S}_k$ and $\mathscr{S}_{k+1}$ are compatible to order k.

Proof. By [GS3], Remark 1.29, if $(B, \mathscr{P})$ is simple, then the central fiber of the corresponding toric degeneration is locally rigid for any choice of open gluing data. In this case, the existence of a sequence of wall structures as claimed is the main part (§3, §4) of [GS3]. Roughly speaking, the proof goes by induction as follows. To obtain $\mathscr{S}_k$ from $\mathscr{S}_{k-1}$, for each joint $\mathfrak{j}$ of $\mathscr{S}_{k-1}$ we calculate the scattering diagram $\mathfrak{D}_{\mathfrak{j},k}$ consistent to order k from $\mathfrak{D}_{\mathfrak{j},k-1}$ as in Proposition 1.66. Then we add walls corresponding to these rays to the scattering diagram $\mathscr{S}_{k-1}$. This will possibly produce some new joint or complicate the scattering diagrams at other joints. However, it is shown in [GS3] that after finitely many steps this procedure gives a wall structure $\mathscr{S}_k$ consistent to order k. $\square$

1.6.3 Reconstruction of toric degenerations

Although we will not need this construction in the sequel, we will present an upshot of it to emphasize the importance of the scattering procedure and to interpret the main result of the thesis in the context of mirror symmetry.

Let $\mathscr{S}$ be a wall structure consistent to order k on a polarized polyhedral affine manifold $(B,\mathscr{P},\varphi)$. Let $\sigma \in \mathscr{P}_{\mathscr{S}}^{[n]}$ be a maximal cell and let $\omega,\tau \in \mathscr{P}_{\mathscr{S}}$ be cells with $\omega \subseteq \tau$. Define

$$P_{\omega,\sigma} = \{(\overline{p},h) \in \Lambda_{B,\omega} \oplus \mathbb{Z} \mid h \geq \varphi_v(\overline{p}) \ \forall v \in \omega\}$$

and let $I^{>k}_{\omega\to\tau,\sigma}$ be the ideal in $\mathbb{C}[P_{\omega,\sigma}]$ generated by the monoid ideal

$$P^{>k}_{\omega\to\tau,\sigma} = \{(\overline{p},h) \in P_{\omega,\sigma} \mid h - \varphi_v(\overline{p}) > k \ \forall v \in \tau\} \subset P_{\omega,\sigma}.$$

Moreover, define

$$R^k_{\omega\to\tau,\sigma} = \left(\mathbb{C}[P_{\omega,\sigma}]/I^{>k}_{\omega\to\tau,\sigma}\right)_{\prod_{\substack{\rho\in\mathscr{P}_{\mathscr{S}}^{[n-1]}\\ \rho\supseteq\tau}} f_{\rho,v}},$$

where $f_{\rho,v} := f_{\mathfrak{b},v}$ for $\mathfrak{b}$ the slab or wall with support ρ. Then Spec $R^k_{\omega\to\tau,\sigma}$ is a k-th order non-reduced version (*thickening*) of $(X_\tau \cap U_\omega)\setminus \cup_{\rho\supseteq\tau} V(f_{\rho,v})$ for $v \in \omega$ arbitrary and

$$U_\omega = X_0 \setminus \bigcup_{\omega\not\subseteq\tau} X_\tau.$$

Note that for $v,v' \in \tau$ the gluing functions $f_{\rho,v}$ and $f_{\rho,v'}$ differ by a monomial that is invertible on U_ω, so $V(f_{\rho,v'}) \cap U_\omega = V(f_{\rho,v}) \cap U_\omega$.

By construction of the initial wall structure (Definition 1.74) and the scattering calculations, the gluing of the affine pieces Spec $R^k_{\omega\to\tau,\sigma}$ along common divisors is well-defined, leading to a k-th order family $\check{\mathfrak{X}}_k \to \text{Spec } \mathbb{C}[t]/(t^{k+1})$. For more details see [GS3] and [GS4]. This gives the following corollary of Proposition 1.77.

Corollary 1.78 ([GS3], Proposition 2.42)**.** *Let $(B,\mathscr{P},\varphi)$ be a simple polarized integral polyhedral affine manifold. Then there exists a formal toric degeneration of log Calabi-Yau pairs with intersection complex $(B,\mathscr{P},\varphi)$.*

1.7 Stable log maps

In this section we review the basics of logarithmic Gromov-Witten theory. This is a powerful extension of Gromov-Witten theory designed to be well-behaved under degenerations. The material of this section can be found in [GS5] and [ACGS1].

Let $X \to S$ be a morphism of log schemes, with the log structure on X being defined in the Zariski topology.

Definition 1.79. A *stable log map* $(C/W, \mathbf{p}, f)$ to $X \to S$ is a commutative diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\gamma"] & X \arrow[d] \\ W \arrow[r] & S \end{tikzcd} \tag{1.2}$$

such that

(1) $\gamma : C \to W$ is a proper, log smooth and integral morphism of fine saturated log schemes together with a tuple of section $\mathbf{p} = (p_1, \ldots, p_k)$ of γ such that every geometric fiber of γ is a reduced and connected curve, and if $U \subset \underline{C}$ is the non-critical locus of γ then $\overline{\mathcal{M}}_C|_U \simeq \gamma^\star \overline{\mathcal{M}}_W \oplus \bigoplus_{i=1}^k p_{i\star}\underline{\mathbb{N}}_W$.

(2) For every geometric point $w \in W$, the restriction of f to $\underline{C}_w$ together with $\mathbf{p}$ is an ordinary stable map.

We will later use the different notation $f : C/W \to X/S$ or just $f : C \to X$ for a stable log map, when $\mathbf{p}$ is clear from the context.

Proposition 1.80 ([Kat2], p.222)**.** *Let $\gamma : C \to W$ be a log smooth and integral morphism of fine saturated log schemes such that every geometric fiber is a reduced curve and $W = pt_P$ is a point with log structure given by a chart $\sigma : P \to \mathbb{C}$. Then étale locally C is isomorphic to one of the following log schemes over W.*

(I) $C = Spec\ \mathbb{C}[z]$ with the log structure induced from the chart

$$P \to \mathcal{O}_C, p \mapsto \sigma(p).$$

(II) $C = Spec\ \mathbb{C}[z]$ with the log structure induced from the chart

$$P \oplus \mathbb{N} \to \mathcal{O}_C, (p, a) \mapsto z^a \sigma(p).$$

(III) $C = Spec\ \mathbb{C}[z, w]/(zw)$ with the log structure induced from the chart

$$P \oplus_{\mathbb{N}} \mathbb{N}^2 \to \mathcal{O}_C, (p, (a, b)) \mapsto z^a z^b \sigma(p).$$

Here $\mathbb{N} \to \mathbb{N}^2$ is the diagonal embedding and $\mathbb{N} \to P, 1 \mapsto \rho \neq 0$ is some monoid homomorphism uniquely defined by $C \to pt_P$.

We call these cases (I) general points, (II) marked points and (III) nodes, respectively.

Discussion 1.81 ([GS5], Discussion 1.8)**.** Let $(C/\mathrm{pt}_P, \mathbf{p}, f)$ be a stable log map to $X \to S$. As above, pt_P is a point with log structure given by a chart $P \to \mathbb{C}$. For $x \in C$ a geometric point with underlying scheme theoretic point $\underline{x}$, the map $\overline{f}^{\#} : f^{-1}\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$ induces maps $\varphi_x : P_x \to \overline{\mathcal{M}}_{C,x}$ for $P_x := \overline{\mathcal{M}}_{X,f(x)}$. Note that $\overline{\mathcal{M}}_{X,f(x)}$ is independent of the choice of $x \to \underline{x}$, since the log structure on X is Zariski. For a monoid P write $P^\vee := \mathrm{Hom}(P, \mathbb{N})$. We have the following three types of points on C:

(I) $x = \eta$ is a generic point or a general closed point. Then $\overline{\mathcal{M}}_{C,\eta} = P$ and φ defines a monoid homomorphism

$$\varphi_\eta : P_\eta \to P.$$

(II) $x = p$ is a marked point. Then $\overline{\mathcal{M}}_{C,p} = P \oplus \mathbb{N}$ and φ_p is determined by φ_η for η the generic point of the irreducible component containing p together with

$$u_p : P_p \xrightarrow{\varphi_p} P \oplus \mathbb{N} \xrightarrow{\mathrm{pr}_2} \mathbb{N}.$$

The element $u_p \in P_p^\vee$ is called the *contact order* at p.

(III) $x = q$ is a node. Then $\overline{\mathcal{M}}_{C,q} \simeq P \oplus_{\mathbb{N}} \mathbb{N}^2$. If q is contained in the closures of η_1 and η_2 and $\chi_i : P_q \to P_{\eta_i}$ are the generization maps, there exists an element $u_q \in \mathrm{Hom}(P_q, \mathbb{Z})$ called *contact order* at q, such that

$$\varphi_{\eta_2}(\chi_2(m)) - \varphi_{\eta_1}(\chi_1(m)) = u_q(m)\rho_q,$$

with $\rho_q = \rho$ as in Proposition 1.80 above. The maps $\varphi_{\eta_i} \circ \chi_i$ and u_q are equivalent to providing the local homomorphism $\varphi_q : P_q \to P \oplus_{\mathbb{N}} \mathbb{N}^2$.

Definition 1.82. Let $(C/\mathrm{pt}_P, \mathbf{p}, f)$ be a stable log map to $X \to S$ over pt_P as above. For a monoid P we write $P^\vee := \mathrm{Hom}(P, \mathbb{N})$. Define a monoid P_{basic} by defining its dual

$$P_{\mathrm{basic}}^\vee = \left\{ ((V_\eta)_\eta, (e_q)_q) \in \bigoplus_\eta P_\eta^\vee \oplus \bigoplus_q \mathbb{N} \;\middle|\; V_{\eta_2} - V_{\eta_1} = e_q u_q \;\forall q \right\}.$$

Here the sum is over generic points η and nodes q of C. The duals of the maps $\varphi_\eta : P_\eta \to \mathcal{M}_{C,\eta} = P$ together with the maps $\rho_q \in P \cong \mathrm{Hom}(P^\vee, \mathbb{N})$ define a map

$$P^\vee \to P_{\mathrm{basic}}^\vee \subset \bigoplus_\eta P_\eta^\vee \oplus \bigoplus_q \mathbb{N}, \quad m \mapsto \left((m \circ \varphi_\eta)_\eta, (m(\rho_q))_q \right).$$

Let $P_{\text{basic}} \to P$ be the dual morphism of monoids.

The stable log map $(C/\text{pt}_P, \mathbf{p}, f)$ is called *basic* if $P_{\text{basic}} \to P$ is an isomorphism. More generally, a stable log map over some base W is called basic, if its restriction to any geometric point is basic.

Definition 1.83. A *class* β of stable log maps to $X \to S$ consists of the following:

(i) The data of an underlying ordinary stable map, i.e., the genus g, an effective curve class $\underline{\beta} \in H_2^+(X, \mathbb{Z})$, and the number of marked points k;

(ii) Integral elements $u_{p_1}, \ldots, u_{p_k} \in |\Sigma(X)|$.

We say a stable log map $(C/W, \mathbf{p}, f)$ is of class β if

(1) The underlying ordinary stable map is of type $(g, \underline{\beta}, k)$;

(2) Define the closed subset $Z_i \subset \underline{X}$ to be the union of strata with generic points η such that u_{p_i} lies in the image of $\sigma_\eta \to |\Sigma(X)|$. Then for any i we have $\text{im}(f \circ p_i) \subset Z_i$ and for any geometric point $w \in W$ such that $p_i(w)$ lies in the stratum of X with generic point η, there exists $u \in \sigma_\eta = \text{Hom}(\overline{\mathcal{M}}_{X,\eta}, \mathbb{N})$ mapping to $u_{p_i} \in |\Sigma(X)|$, making the following diagram commute:

$$\begin{tikzcd} \overline{\mathcal{M}}_{X,f(p_i(w))} \arrow[r, "\overline{f}^\#"] \arrow[d, "\chi"'] & \overline{\mathcal{M}}_{C,p_i(w)} \arrow[r, equal] & \overline{\mathcal{M}}_{W,w} \oplus \mathbb{N} \arrow[d, "\text{pr}_2"] \\ \overline{\mathcal{M}}_{X,\eta} \arrow[rr, "u"] & & \mathbb{N} \end{tikzcd}$$

Here χ is the generization map. In particular this specifies the contact orders u_{p_i} at the marked points $p_i(w)$.

Definition 1.84. Let $\mathscr{M}(X/S, \beta)$ denote the stack of basic stable log maps to $X \to S$ of class β.

Proposition 1.85 ([GS5], Theorems 0.2 and 0.3)**.** *If $X \to S$ is proper, then $\mathscr{M}(X/S, \beta)$ is a proper Deligne-Mumford stack. If furthermore $X \to S$ is log smooth, then $\mathscr{M}(X/S, \beta)$ carries a perfect obstruction theory, defining a virtual fundamental class $[\![\mathscr{M}(X/S, \beta)]\!]$ in the rational Chow group of $\mathscr{M}(X/S, \beta)$.*

1.8 Tropicalization

In this section we review the notion of tropicalization of log schemes and families of tropical curves. We closely follow [ACGS1].

Definition 1.86. Let **Poly** be the category whose objects are pairs $\sigma = (\sigma_\mathbb{R}, N_\sigma)$ where $N_\sigma \cong \mathbb{Z}^n$ is a lattice and $\sigma_\mathbb{R} \subseteq N_{\sigma,\mathbb{R}} = N_\sigma \otimes_\mathbb{Z} \mathbb{R}$ is an n-dimensional convex integral polyhedron. A morphism of polyhedra $\varphi : \sigma_1 \to \sigma_2$ is a homomorphism $\varphi : N_{\sigma_1} \to N_{\sigma_2}$ which takes $\sigma_{1,\mathbb{R}}$ into $\sigma_{2,\mathbb{R}}$. It is a *face morphism* if it identifies

$\sigma_{1,\mathbb{R}}$ with a face of $\sigma_{2,\mathbb{R}}$ and N_{σ_1} with a saturated sublattice of N_{σ_2}. A *generalized polyhedral complex* Σ is a topological space with a presentation as the colimit of an arbitrary finite diagram in the category **Poly** with all morphisms being face morphisms. Write $|\Sigma|$ for the underlying topological space.

Let **Cone** be the subcategory of **Poly** of objects $(\sigma_{\mathbb{R}}, N_\sigma)$ such that $\sigma_{\mathbb{R}} \subseteq N_{\sigma,\mathbb{R}}$ is a strictly convex rational polyhedral cone. A *generalized cone complex* is a topological space with a presentation as the colimit of an arbitrary finite diagram in the category **Cone** with all morphisms being face morphisms.

Example 1.87. A polyhedral affine manifold (Definition 1.7) is in particular a generalized polyhedral complex.

Definition 1.88. Let X be a fine saturated log scheme with log structure defined in the Zariski topology. For the generic point η of a stratum of X, its characteristic monoid $\overline{\mathcal{M}}_{X,\eta}$ defines a dual monoid $(\overline{\mathcal{M}}_{X,\eta})^\vee := \mathrm{Hom}(\overline{\mathcal{M}}_{X,\eta}, \mathbb{N})$ lying in the group $(\overline{\mathcal{M}}_{X,\eta})^\star := \mathrm{Hom}(\overline{\mathcal{M}}_{X,\eta}, \mathbb{Z})$, hence a dual cone

$$\sigma_\eta := \left((\overline{\mathcal{M}}_{X,\eta})^\vee_{\mathbb{R}}, (\overline{\mathcal{M}}_{X,\eta})^\star\right).$$

If η is specialization of η', there is a well-defined generization map $\overline{\mathcal{M}}_{X,\eta} \to \overline{\mathcal{M}}_{X,\eta'}$. Dualizing, we obtain a face morphism $\sigma_{\eta'} \to \sigma_\eta$. This gives a diagram of cones indexed by strata of X with face morphisms, hence gives a generalized cone complex $\Sigma(X)$, the *tropicalization* of X. This construction is functorial: given a morphism of fine saturated log schemes $f : X \to Y$, the map $\overline{f}^\# : f^{-1}\overline{\mathcal{M}}_Y \to \mathcal{M}_X$ induces a morphism if generalized cone complexes $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$.

Example 1.89. Let pt_P be a point with log structure given by a chart $P \to \mathbb{C}$. Then $\Sigma(\mathrm{pt}_P)$ consists of a single cone $\mathrm{Hom}(P, \mathbb{R}_{\geq 0})$. In particular, for the standard log point $\mathrm{pt}_{\mathbb{N}}$ (Example 1.38) we have $\Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathbb{R}_{\geq 0}$.

Example 1.90. If X is a toric variety with the toric log structure $\mathcal{M}_{(X,\partial X)}$, then $\Sigma(X)$ is abstractly the fan defining X. It is missing the embedding of $|\Sigma|$ as a fan in a vector space $N_{\mathbb{R}}$.

Example 1.91. Let $\pi : \mathfrak{X} \to T$ be a toric degeneration with divisorial log structures given by $X_0 \cup \mathfrak{D} \subset \mathfrak{X}$ and $0 \in T$. Let $\pi_0 : X_0 \to \mathrm{pt}_{\mathbb{N}}$ be the central fiber with the pullback log structures. Tropicalization gives a morphism of generalized cone complexes

$$\Sigma(\pi_0) : \Sigma(X_0) \to \mathbb{R}_{\geq 0}.$$

By construction the fiber $\Sigma(\pi_0)^{-1}(1)$ is homeomorphic to the dual intersection complex of $\mathfrak{X} \to T$.

Definition 1.92. For us a *graph* Γ is a set of vertices $V(\Gamma)$, a set of edges $E(\Gamma)$ and a set of legs or half-edges $L(\Gamma)$, with appropriate incidence relations between them. We admit multiple edges and loops. Every edge $E \in E(\Gamma)$ is a pair of orientations of E, so that the automorphism group of a graph with single loop is $\mathbb{Z}/2\mathbb{Z}$.

Example 1.93. Let $\gamma : C \to \mathrm{pt}_P$ be a log smooth curve over the point with log structure given by a chart $P \to \mathbb{C}$. Tropicalization gives a morphism of generalized cone complexes

$$\Sigma(\gamma) : \Sigma(C) \to P^\vee_{\mathbb{R}} = \mathrm{Hom}(Q, \mathbb{R}).$$

By Proposition 1.80, for any nontrivial $b \in \mathrm{Hom}(Q, \mathbb{R})$ the fiber $\Sigma(\gamma)^{-1}(b)$ is homeomorphic to (the topological realization of) the dual intersection graph of C. This is the graph Γ_C given by

(i) a vertex V for each irreducible component C_V of C;
(ii) a bounded edge E for each node q of C, with vertices V, V' corresponding to the irreducible components $C_V, C_{V'}$ containing q;
(iii) a leg L for each marked point p of C, with vertex V corresponding to the irreducible component C_V containing p.

Definition 1.94. A *family of tropical curves* over a cone $\omega \in$ **Cones** is a connected graph Γ together with a bijection $L(\Gamma) \to \{1, \ldots, k\}$ (*leg ordering*) and two maps

$$g : V(G) \to \mathbb{N}, \quad \ell : E(\Gamma) \to \mathrm{Hom}(\omega \cap N_\omega, \mathbb{N}) \setminus \{0\}.$$

For $V \in V(\Gamma)$ and $E \in E(\Gamma)$ we call $g(V)$ the *genus* of V and $\ell(E)$ the *length function* of E.

Construction 1.95 ([ACGS1], Construction 2.20)**.** To a family of tropical curve Γ over ω we can associate a morphism of generalized cone complexes

$$\pi_\Gamma : \Sigma_\Gamma \to \omega$$

as follows:

(i) For each vertex $V \in V(\Gamma)$ take a copy ω_V of ω;
(ii) For each edge $E \in E(\Gamma)$ take the cone

$$\omega_E = \left\{(s, \lambda) \in \omega \times \mathbb{R}_{\geq 0} \mid \lambda \leq L(E)(s)\right\}.$$

This has two facets, each isomorphic to ω via projection the first factor. The corresponding inclusions define face morphisms $\omega_V \to \omega_E$ and $\omega_{V'} \to \omega_E$ for the vertices V, V' of E.

(iii) For each leg $L \in L(\Gamma)$ take the cone $\omega_L = \omega \times \mathbb{R}$ with face morphism $\omega_V \to \omega_L$ defined by the facet $\omega \times \{0\}$.

The morphism π_Γ is defined on each cone by projection to the first factor. Each vertex $V \in V(\Gamma)$ defines a section of π_Γ denoted by $\omega \to \Sigma_\Gamma, s \mapsto V(s) \in \omega_V$. Note that if $s \in \omega$ is not contained in a proper face, then $\pi_\Gamma^{-1}(s)$ is homeomorphic to Γ.

Example 1.96. Let $\gamma : C \to \mathrm{pt}_P$ be a log smooth curve over pt_P for some monoid P. Then the dual intersection graph Γ_C defines a family of tropical curves over the cone $P_{\mathbb{R}}^\vee$, with associated morphism of generalized cone complexes given by the tropicalization $\Sigma(C) \to P_{\mathbb{R}}^\vee$ as follows:

(1) If η is a generic point of C, then $\omega_\eta = P_{\mathbb{R}}^\vee$ and $\Sigma(\gamma)|_{\omega_\eta}$ is the identity, so each fiber $\Sigma(\gamma)|_{\omega_\eta}$ is a point V. We define the genus $g(V)$ of a vertex $V \in \Gamma_C$ to be the geometric genus $g(C_V)$ of the corresponding irreducible component C_V of C.

(2) The cone of $\Sigma(C)$ defined by a node q of C is

$$\omega_q = \mathrm{Hom}(P \oplus_{\mathbb{N}} \mathbb{N}^2, \mathbb{R}_{\geq 0}) = P_{\mathbb{R}}^\vee \times_{\mathbb{R}_{\geq 0}} \mathbb{R}_{\geq 0}^2,$$

with maps $P_{\mathbb{R}}^\vee \to \mathbb{R}_{\geq 0}$ given by evaluation at $\rho_q \in P \setminus \{0\}$ (Discussion 1.81, (III)) and $\mathbb{R}_{\geq 0}^2 \to \mathbb{R}_{\geq 0}, (a,b) \mapsto a+b$. The projection $\mathbb{R}_{\geq 0}^2 \to \mathbb{R}_{\geq 0}$ to the first factor defines an isomorphism

$$P_{\mathbb{R}}^\vee \times_{\mathbb{R}_{\geq 0}} \mathbb{R}_{\geq 0}^2 \xrightarrow{\sim} \left\{(m, \lambda) \in P_{\mathbb{R}}^\vee \times \mathbb{R}_{\geq 0} \mid \lambda \leq m(\rho_q)\right\}.$$

Thus defining $\ell(E_q) = \rho_q$ we have a canonical isomorphism $\omega_q \simeq \omega_{E_q}$.

(3) For a marked point $p \in C$ we have $\omega_p = P_{\mathbb{R}}^\vee \times \mathbb{R}_{\geq 0}$ and $\Sigma(\pi)|_{\omega_p}$ is the projection onto the first component, again compatible with the definition of Σ_Γ above.

Definition 1.97. A *family of tropical maps* to a generalized cone complex Σ over $\omega \in$ **Cones** is a family of tropical curves Γ over ω together with a morphism of cone complexes $h : \Sigma_\Gamma \to \Sigma$.

Definition 1.98. From a family of tropical maps to a generalized cone complex Σ we can extract the following discrete data:

(1) *Image cones:* Define

$$\sigma : V(\Gamma) \cup E(\Gamma) \cup L(\Gamma) \to \Sigma$$

by mapping x to the minimal cone in Σ containing the image of ω_x, the cone associated to x. Note that this respects incidence relations.

(2) *Contact orders at nodes:* For an edge $E \in E(\Gamma)$ with a chosen oder of vertices V, V' the image of $(0,1) \in N_{\omega_E} = N_\omega \times \mathbb{R}$ under h defines $u_q \in N_{\sigma(E)}$ such that for all $s \in \omega_E$

$$h(v(s)) - h(v'(s)) = \ell(E_q)(s) \cdot u_q.$$

(3) *Contract orders at marked points:* For a leg $L \in L(\Gamma)$ the image of $(0,1) \in N_{\omega_L} = N_\omega \times \mathbb{R}$ defines $u_p \in N_{\sigma(L)} \cap \sigma(L)$ with $h(\mathrm{Int}(\omega_L)) \subset \mathrm{Int}(\sigma(L))$.

Example 1.99. The tropicalization of a stable log map $(C/\mathrm{pt}_P, \mathbf{p}, f)$ to X defines a family of stable log maps to $\Sigma(X)$ over $P^\vee_{\mathbb{R}}$.

In this case the dual of the basic monoid $P^\vee_{\mathrm{basic}}$ of $(C/\mathrm{pt}_P, \mathbf{p}, f)$ can be interpreted as the moduli space of families of tropical maps with the same genus decorated graph $(\Gamma, (g_V)_V)$ and discrete data $(\sigma, (u_q)_q, (u_p)_p)$. The boundary of $P^\vee_{\mathrm{basic}}$ corresponds to degenerate tropical curves, i.e., with some edges contracted to a point.

Discussion 1.100 ([ACGS1], 2.5.3)**.** Let $\pi : X \to \mathrm{pt}_{\mathbb{N}}$ be a log scheme over the standard log point $\mathrm{pt}_{\mathbb{N}}$. Tropicalization gives a morphism of generalized cone complexes $\Sigma(X) \to \mathbb{R}_{\geq 0}$. The fiber $\pi^{-1}(1)$ is a generalized polyhedral complex. Note that the morphism $\pi : \Sigma(X) \to \mathbb{R}_{\geq 0}$ can be recovered from $\pi^{-1}(1)$ by replacing each $\sigma = (\sigma_{\mathbb{R}}, N)$ by the closure of $\mathbb{R}_{\geq 0}(\sigma \times \{1\})$ in $N_{\mathbb{R}} \times \mathbb{R}$.

Let $(C/\mathrm{pt}_P, \mathbf{p}, f)$ be a stable log map to $X \to \mathrm{pt}_{\mathbb{N}}$. Write the morphism of bases as $\alpha : \mathrm{pt}_P \to \mathrm{pt}_{\mathbb{N}}$. Then the family of tropical maps $\Sigma(C) \to \Sigma(X)$ over $P^\vee$ carries the same information as the restriction to the fiber over $1 \in \mathbb{R}_{\geq 0}$, a family of maps from metric graphs to $\Delta(X)$ parametrized by the polyhedron $\Sigma(\alpha)^{-1}(1) \subset P^\vee$.

We have seen that in some cases, e.g. when $X \to \mathrm{pt}_{\mathbb{N}}$ is the central fiber of a toric degeneration, the generalized polyhedral complex $\Sigma(\pi)^{-1}(1)$ carries the structure of an affine manifold with singularities. This leads to a definition of tropical curves on an affine manifolds with singularities (Definition 4.1).

1.9 Artin fans and logarithmic modifications

An *Artin fan* is a logarithmic algebraic stack that is logarithmically étale over a point. Artin fans were introduced in [AW] to prove the invariance of logarithmic Gromov-Witten invariants under *logarithmic modifications*, that is, proper birational logarithmically étale morphisms. We will briefly summarize this subject.

To any fine saturated log smooth scheme X one can define an associated Artin fan $\mathcal{A}_X$. It has an étale cover by finitely many *Artin cones* – stacks of the form $[V/T]$, where V is a toric variety and T its dense torus. In this way, $\mathcal{A}_X$ encodes the combinatorial structure of X. A *subdivision* of the Artin fan $\mathcal{A}_X$ induces a logarithmic modification of X via pull-back. Moreover, all logarithmic modifications of X arise this way. This ultimately leads to a proof of the birational invariance in logarithmic Gromov-Witten theory [AW].

Olsson [Ols] showed that a logarithmic structure on a given underlying scheme $\underline{X}$ is equivalent to a morphism $\underline{X} \to \underline{\mathbf{Log}}$, where $\underline{\mathbf{Log}}$ is a zero-dimensional algebraic stack – the moduli stack of logarithmic structures. It carries a universal logarithmic structure whose associated logarithmic algebraic stack we denote by $\mathbf{Log}$ – providing a universal family of logarithmic structures $\mathbf{Log} \to \underline{\mathbf{Log}}$. As shown in [AW], if X is a fine saturated log smooth scheme, then the morphism $X \to \mathbf{Log}$ factors through an initial morphism $X \to \mathcal{A}_X$, where $\mathcal{A}_X$ is an Artin fan and $\mathcal{A}_X \to \mathbf{Log}$ is étale and representable. While this serves as a definition of the associated Artin fan $\mathcal{A}_X$, there is a more explicit description of $\mathcal{A}_X$ in terms of the *tropicalization* of X, given below. Let S be a fine saturated log scheme.

Definition 1.101. The *moduli stack of log structures over* S is the category $\underline{\mathbf{Log}_S}$ fibered over the category of $\underline{S}$-schemes defined as follows. The objects over a scheme morphism $\underline{X} \to \underline{S}$ are the log morphisms $X \to S$ over $\underline{X} \to \underline{S}$. The morphisms from $X \to S$ to $X' \to S$ are the log morphisms $h : X \to X'$ over S for which $h^\star \mathcal{M}_{X'} \to \mathcal{M}_X$ is an isomorphism.

Proposition 1.102 ([Ols] Theorem 1.1)**.** $\underline{\mathbf{Log}_S}$ *is an algebraic stack locally of finite presentation over* $\underline{S}$.

Definition 1.103. An *Artin fan* is a logarithmic algebraic stack that is logarithmically étale over a point. An *Artin cone* is a logarithmic algebraic stack isomorphic to $[V/T]$, where V is a toric variety and T its dense torus.

Remark 1.104. If a logarithmic algebraic stack has a strict representable étale cover by Artin cones, then it is an Artin fan. In fact, in [AW] Artin fans were defined this way. Later the definition was generalized to the one above.

Lemma 1.105 ([AW], Lemma 2.3.1)**.** *An algebraic stack that is representable and étale over* **Log** *has a strict étale cover by Artin cones.*

Proposition 1.106 ([ACMW], Proposition 3.1.1)**.** *Let X be a logarithmic algebraic stack that is locally connected in the smooth topology. Then there is an initial factorization of $X \to$ **Log** through a strict étale morphism $\mathcal{A}_X \to$ **Log** which is representable by algebraic spaces.*

Definition 1.107. Let X be a fine saturated log smooth scheme. The *Artin fan of* X is the stack $\mathcal{A}_X$ from Proposition 1.106. Indeed, this is an Artin fan by Lemma 1.105 and Remark 1.104.

We now give a more explicit description of the Artin fan $\mathcal{A}_X$ of a fine saturated log smooth scheme X. By Lemma 1.105 and Proposition 1.106, $\mathcal{A}_X$ has a strict étale cover by Artin cones. In fact, $\mathcal{A}_X$ is a colimit of Artin cones $\mathcal{A}_\sigma$ corresponding to the cones σ in the tropicalization $\Sigma(X)$ of X.

Definition 1.108. For a cone $\sigma \subseteq N_{\mathbb{R}}$, let $P = \sigma^\vee \cap M$ be the corresponding monoid. The *Artin cone defined by* σ is the logarithmic algebraic stack

$$\mathcal{A}_\sigma = \left[\mathrm{Spec}\ \mathbb{C}[P] \middle/ \mathrm{Spec}\ \mathbb{C}[P^{\mathrm{gp}}]\right]$$

with the toric log structure coming from the global chart $P \to \mathbb{C}[P]$.

Definition 1.109. Let Σ be a generalized cone complex (Definition 1.86) that is a colimit of a diagram of cones $s : I \to \mathbf{Cones}$. Then define $\mathcal{A}_\Sigma$ to be the colimit as sheaves over Log of the corresponding diagram of sheaves given by $I \ni i \mapsto \mathcal{A}_{s(i)}$.

Proposition 1.110 ([ACGS1], Proposition 2.2.2)**.** *Let X be a fine saturated log smooth scheme with tropicalization (Definition 1.88) a generalized cone complex $\Sigma(X)$. Then*

$$\mathcal{A}_X \cong \mathcal{A}_{\Sigma(X)}.$$

Definition 1.111. A *subdivision* of an Artin fan $\mathcal{X}$ is a morphism of Artin fans $\mathcal{Y} \to \mathcal{X}$ whose base change via any map $\mathcal{A}_\sigma \to \mathcal{X}$ is isomorphic to $\mathcal{A}_\Sigma$ for some subdivision Σ of σ.

Definition 1.112. A *logarithmic modification* of fine saturated log smooth schemes is a proper surjective logarithmically étale morphism.

Let X be a fine saturated log smooth scheme with tropicalization $\Sigma(X)$. Then a subdivision $\tilde{\Sigma}(X)$ of $\Sigma(X)$ gives a subdivision $\mathcal{A}_{\tilde{\Sigma}(X)} \to \mathcal{A}_X$ of the Artin fan of X. The pull back $\tilde{X} := \mathcal{A}_{\tilde{\Sigma}(X)} \times_{\mathcal{A}_X} X \to X$ is a logarithmic modification. Moreover, all logarithmic modifications of X arise this way:

Proposition 1.113 ([AW], Corollary 2.6.7)**.** *If $Y \to X$ is a logarithmic modification of fine saturated log smooth schemes, then $Y \to \mathcal{A}_Y \times_{\mathcal{A}_X} X$ is an isomorphism.*

Theorem 1.114 ([AW], Theorem 1.1)**.** *Let $h : Y \to X$ be a logarithmic modification of log smooth schemes. This induces a projection $\pi : \bar{\mathscr{M}}(Y) \to \bar{\mathscr{M}}(X)$ with*

$$\pi_\star [\![\bar{\mathscr{M}}(Y)]\!] = [\![\bar{\mathscr{M}}(X)]\!],$$

where $\bar{\mathscr{M}}(X)$ is the stack of stable log maps to X.

2 Deforming toric degenerations

Definition 2.1. A *smooth very ample log Calabi-Yau pair* is a log Calabi-Yau pair (X, D) consisting of a smooth del Pezzo surface X of degree $d \geq 3$ and a smooth very ample anticanonical divisor D.

2.1 The cone picture

Construction 2.2. Let $M \simeq \mathbb{Z}^2$ be a lattice and let $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ be the corresponding vector space. Let $Q \subset M_\mathbb{R}$ be a Fano polytope, i.e., a convex lattice polytope containing the origin and with all vertices being primitive integral vectors. The polytope Q can be seen as an affine manifold via its embedding into $M_{\mathbb{R}} \simeq \mathbb{R}^2$. Let $\check{\mathscr{P}}$ be the polyhedral decomposition of Q obtained by inserting edges connecting the vertices of Q to the origin. Let $\check{\varphi} : Q \to \mathbb{R}$ be the strictly convex piecewise affine function on $(Q, \check{\mathscr{P}})$ defined by $\check{\varphi}(0) = 0$ and $\check{\varphi}(v) = 1$ for all vertices v of Q. This means $\check{\varphi}$ is affine on the maximal cells of $\check{\mathscr{P}}$ and locally at each vertex v of $\check{\mathscr{P}}$ gives a strictly convex piecewise affine function on the fan Σ_v describing $\check{\mathscr{P}}$ locally. The triple $(Q, \check{\mathscr{P}}, \check{\varphi})$ is a *polarized polyhedral affine manifold* ([GHS], Construction 1.1). From this one obtains a toric degeneration of a toric del Pezzo surfaces with cyclic quotient singularities via the construction of Mumford [Mum] (see Construction 1.55) as

$$\mathfrak{X}^0 := \operatorname{Proj}\left(\mathbb{C}[C(Q_{\check{\varphi}}) \cap (M \times \mathbb{Z} \times \mathbb{Z})]\right).$$

By construction $\mathfrak{X}^0$ comes with an embedding into $\mathbb{P}^{N-1} \times \operatorname{Spec} \mathbb{C}[t]$, where N is the number of lattice points of Q and $t = z^{(0,1,0)}$. Projection to the last coordinate gives a toric degeneration $\mathfrak{X}^0 \to \mathbb{A}^1$ of a toric del Pezzo surface X_0 with quotient singularities. The polytope Q is the momentum polytope of X_0 and the Fano condition on Q corresponds to the condition on X having very ample anticanonical bundle. The divisor

$$\mathfrak{D}^0 = \{z^{(0,0,0,1)} = 0\} \subset \mathfrak{X}^0$$

defined by setting the coordinate corresponding to the origin in Q to zero is a very ample anticanonical divisor, since it corresponds to the pullback of the line bundle $\mathcal{O}_{\mathbb{P}^{N-1}}(1)$, which is the anticanonical bundle on the general fiber of $\mathfrak{X}^0$. By construction, the polarized polyhedral affine manifold $(Q, \check{\mathscr{P}}, \check{\varphi})$ is the *intersection complex* of the toric degeneration $\mathfrak{X}^0$.

Remark 2.3. Let $M' \subset M$ be the sublattice generated by the vertices of $\check{\mathscr{P}}$. We naturally have an embedding $\mathfrak{X}^0 \subset \mathbb{P}_{\check{B},\check{\mathscr{P}}} \times \mathbb{A}^1$, where $\mathbb{P}_{\check{B},\check{\mathscr{P}}}$ is the weighted projective space of dimension $|Q \cap M' - 1|$ and weights $(1, \ldots, 1, d)$ for d the index of M' in M.

One can deform $\mathfrak{X}^0$ by perturbing its defining equations. This means we add a term $t^l sf$ to each equation, where l is the lowest non-trivial t-order, $s \in \mathbb{A}^1$ is the deformation parameter and f is a general polynomial defining a section of the anticanonical bundle of the general fiber of $\mathfrak{X}^0$. We give some examples below.

This leads to a 2-parameter family

$$(\mathfrak{X}_Q \to \mathbb{A}^2, \mathfrak{D}_Q)$$

such that

(1) for $s=0$ we have a toric degeneration $(\mathfrak{X}^0 \to \mathbb{A}^1, \mathfrak{D}^0)$ of a log Calabi-Yau pair (X^0, D^0) consisting of a toric del Pezzo surface with quotient singularities X^0 and its toric boundary $D^0 = \partial X^0$;
(2) for $s \neq 0$ we have a toric degeneration $(\mathfrak{X} \to \mathbb{A}^1, \mathfrak{D})$ of a smooth log Calabi-Yau pair (X, D) consisting of a $\mathbb{Q}$-Gorenstein smoothing X of X^0, i.e., a smooth del Pezzo surface of the same degree, and a smooth anticanonical divisor D. For different choices of $s \neq 0$ these toric degenerations are related via smooth deformation. We only care about (X, D) up to smooth deformation, since log Gromov-Witten invariants are invariant under such deformations ([MR], Appendix A).

Notation 2.4. We write the fibers of $(\mathfrak{X}_Q \to \mathbb{A}^2, \mathfrak{D}_Q)$ as (X_t^s, D_t^s), where s is the deformation parameter and t is the parameter for the toric degeneration. We denote the 1-parameter families defined by fixing one parameter by $(\mathfrak{X}_t \to \mathbb{A}^1, \mathfrak{D}_t)$ and $(\mathfrak{X}^s \to \mathbb{A}^1, \mathfrak{D}^s)$, respectively. When we fix a parameter different from zero, we sometimes omit the index, e.g. $X = X_t^s$ for $s, t \neq 0$. When writing $(\mathfrak{X} \to \mathbb{A}^1, \mathfrak{D})$ we will always mean the toric degeneration $(\mathfrak{X}^s \to \mathbb{A}^1, \mathfrak{D}^s)$ for some $s \neq 0$. By (2) above this notation makes sense. Moreover, we often supress the divisor in the notation.

Let $\check{B}$ be the affine manifold with singularities obtained from Q by introducing affine singularities on the interior edges of $\check{\mathscr{P}}$ such that the boundary of $\check{B}$ is a straight line, and let $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ be the corresponding polarized polyhedral affine manifold. Of course, there is a choice of the exact position of the affine singularities along the interior edges. In fact, one could form families of affine manifolds as in [Pri1]. However, we don't care about the exact position, as we only care about the degeneration $(\mathfrak{X} \to \mathbb{A}^1, \mathfrak{D})$ up to deformation. So we may place the affine singularities in the middle of the interior edges. Note that in [GS3] the affine singularities are required to have irrational coordinates, since otherwise some walls in the induced wall structure may cross them. However, this doesn't happen in our special case, so the middle of the edges will be a valid choice.

Proposition 2.5. *The intersection complex of $\mathfrak{X} \to \mathbb{A}^1$ is $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$, while the intersection complex of $\mathfrak{X}^0 \to \mathbb{A}^1$ is $(Q, \check{\mathscr{P}}, \check{\varphi})$.*

Proof. First note that $\check{B}$ as above exists, since by reflexivity for any vertex v the integral tangent vectors of any adjacent vertex together with $v - v_0$ generate the full lattice (see [CPS], Construction 6.2). Here v_0 is the unique interior vertex.

The t-constant term in the defining equation for X^0 is independent of the variable $z^{(0,0,0,1)}$, since $(0,0,0,1)$ is the only lattice point at which $\varphi = 0$. So the central fiber of $\mathfrak{X}^s \to \mathbb{A}^1$ is independent of s. As a consequence, the maximal cells of the intersection complex of $\mathfrak{X}^s \to \mathbb{A}^1$ are the same for each s. So the parameter s only changes the affine structure, given by the fan structures at vertices of the intersection complex. These fan structures are defined by local models for the family at zero-dimensional toric strata of the central fiber.

For $s \neq 0$, locally at the zero-dimensional toric stratum of X_0^s corresponding to a vertex v on the boundary of B, the family $\mathfrak{X}^s$ is given by $\{xy = t^l\} \subset \mathbb{A}^4$ for some $l > 0$. So the fan structure at v is given by the fan of $\mathbb{P}^1 \times \mathbb{A}^1$. This shows that the boundary is a straight line.

Note that for $s = 0$ locally at a 0-dimensional stratum corresponding to $v \in \partial B$ the family $\mathfrak{X}^0$ is given by $\{xy = t^l w\} \subset \mathbb{A}^4$ for some $l > 0$. The fan structure at v is given by the fan with ray generators $(1, 0)$, $(0, 1)$ and $(1, 1)$. So the affine charts are compatible and there are no affine singularities. $\square$

Example 2.6. Figure 2.1 shows the intersection complex $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ of a toric degeneration of the log Calabi-Yau pair $(\mathbb{P}^2, E)$, where $E \subset \mathbb{P}^2$ is a smooth anticanonical divisor, i.e., an elliptic curve. This is obtained by smoothing a toric degeneration of $(\mathbb{P}^2, \partial\mathbb{P}^2)$, where $\partial\mathbb{P}^2$ is the toric boundary of $\mathbb{P}^2$. One can write down such a smoothing explicitly as in Example 1.57:

$$\begin{aligned} \mathfrak{X}_Q &= V\left(XYZ - t^3(W + sf_3)\right) \subset \mathbb{P}(1,1,1,3) \times \mathbb{A}^2 \\ \mathfrak{D}_Q &= V(W) \subset \mathfrak{X}_Q \end{aligned}$$

Here X, Y, Z, W are the coordinates of $\mathbb{P}(1,1,1,3)$, as shown in Figure 2.1, and f_3 is a general homogeneous degree 3 polynomial in X, Y, Z.

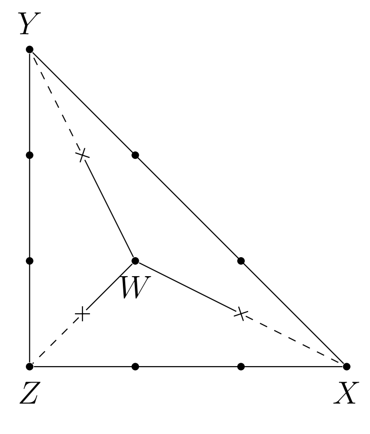


Figure 2.1: The intersection complex $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ of $(\mathbb{P}^2, E)$. The piecewise affine function $\check{\varphi}$ is 0 at the interior lattice point and 1 on the boundary.

Example 2.7. Figure 2.2 shows the intersection complex of a toric degeneration of a smooth cubic surface X (del Pezzo surface of degree 3) obtained by smoothing the Fano polytope of the toric Gorenstein del Pezzo surface $X^0 = \mathbb{P}^2/\mathbb{Z}_3$, where $\mathbb{Z}_3$ acts by $(x, y, z) \mapsto (x, \zeta y, \zeta^{-1} z)$ for ζ a nontrivial third root of unity. This can be given explicitly as follows.

$$
\begin{aligned}
\mathfrak{X}_Q &= V\left(XYZ - t^3(W^3 + sf_3)\right) \subset \mathbb{P}^3 \times \mathbb{A}^2 \\
\mathfrak{D}_Q &= V(W) \subset \mathfrak{X}_Q
\end{aligned}
$$

Again, X, Y, Z, W are the projective coordinates and f_3 is a general homogeneous degree 3 polynomial in X, Y, Z. For $t, s \neq 0$, X_t^s is a smooth cubic surface, and D_t^s is a hyperplane section. For $t \neq 0, s = 0$, X_t^0 is given by $V(XYZ - tW^3) \subset \mathbb{P}^3$, thus is a $\mathbb{Z}_3$-quotient of $\mathbb{P}^2$, and D_t^0 is a cycle of three lines. For $t = 0$ we have $X_0^s = V(XYZ) \subset \mathbb{P}^3$. This is a union of three $\mathbb{P}^2$ glued as described by the combinatorics of Figure 2.2, and again D_0^s is a cycle of three lines.

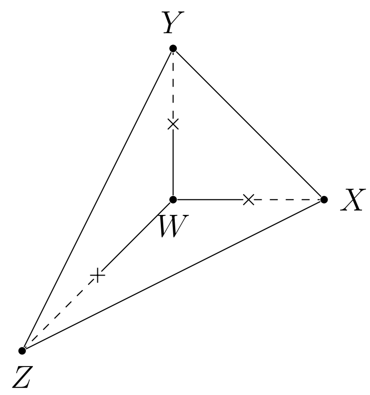


Figure 2.2: The intersection complex $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ of a smooth cubic surface, obtained by smoothing the Fano polytope of $\mathbb{P}^2/\mathbb{Z}_3$.

Example 2.8. Figure 2.3 shows the intersection complex of a toric degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ obtained by smoothing the Fano polytope of $\mathbb{P}(1,1,2)$. This can be given explicitly as follows, with f_2 a general homogeneous degree 2 polynomial in X, Y, Z, U and W the degree 2 coordinate,

$$
\begin{aligned}
\mathfrak{X}_Q &= V\left(XY - U^2 + t^2 sf_2, ZU - t^2(W + sf_2)\right) \subset \mathbb{P}(1,1,1,1,2) \times \mathbb{A}^2 \\
\mathfrak{D}_Q &= V(W) \subset \mathfrak{X}_Q
\end{aligned}
$$

Indeed, $t = 0$ implies $Z = 0$ or $U = 0$ which in turn implies $X = 0$ or $Y = 0$. We have $V(X) = V(Y) = \mathbb{P}(1,1,2)$ and $V(Z) = \{XY = U^2\} \subset \mathbb{P}(1,1,1,2)$ which is isomorphic to $\mathbb{P}(1,1,4)$. For $t \neq 0$ we have $X_t^0 = \{XY = U^2 + t^2 sf_2\} \subset \mathbb{P}^3$ by elimination of W. For $s = 0$ this is a singular quadric, so $X_t^0 = \mathbb{P}(1,1,2)$. For $s \neq 0$ it is a smooth quadric $X_t^s \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Again, D_t^s is smooth if and only if $t, s \neq 0$.

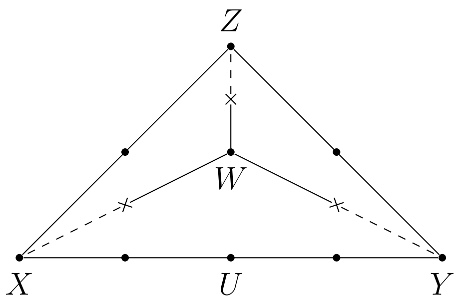


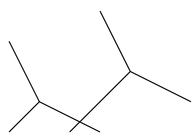
Figure 2.3: The intersection complex $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ of $\mathbb{P}^1 \times \mathbb{P}^1$, obtained by smoothing the Fano polytope of $\mathbb{P}(1,1,2)$.

Definition 2.9. Let X be a smooth del Pezzo surface. A *toric model* for X is a toric del Pezzo surface with cyclic quotient singularities X^0 that admits a $\mathbb{Q}$-Gorenstein deformation to X.

Remark 2.10. Note that there may be different toric models X^0 for the same smooth del Pezzo surface X. In fact, the Fano polytopes Q of such X^0 are related via *combinatorial mutations* ([ACC+], Theorem 3, see also [CGG+]).

Proposition 2.11. *For each smooth del Pezzo surface X with very ample anti-canonical class there exists a toric model X^0 with at most Gorenstein singularities.*

Proof. For any $\mathbb{Q}$-Gorenstein deformation $\mathfrak{X} \to \mathbb{A}^1$, the relative canonical class $K_{\mathfrak{X}/\mathbb{A}^1}$ is $\mathbb{Q}$-Cartier. By definition, the degree of X^0 is the self-intersection of its (anti)canonical class. Hence, the degree of X equals the degree of any of its toric models X^0. We need to show that the degrees of the given toric models are the ones shown in Figure 2.4. The Fano polytope Q of X^0 is exactly the Newton polytope of its anticanonical class. By the duality between subdivisions of Newton polytopes and tropical curves, the self intersection can easily be computed by intersecting two tropical curves dual to the Fano polytope Q. For (3a), i.e., $X^0 = \mathbb{P}^2/\mathbb{Z}_3$ as in Example 2.7, the intersection of tropical curves is the following:



The determinant of primitive tangent vectors at the intersection point is $|\det\left(\begin{smallmatrix} 1 & 2 \\ -1 & 1 \end{smallmatrix}\right)| = 3$. Indeed, this is the degree of $\mathbb{P}^2/\mathbb{Z}_3$. Similarly one computes the degrees of the other cases in Figure 2.4. Alternatively, one can use the fact that the degree of a del Pezzo surface equals $|\check{B} \cap M| + 1$ (see [CPS], §6).

There are two smooth del Pezzo surfaces of degree 8, the blow up of $\mathbb{P}^2$ at a point and $\mathbb{P}^1 \times \mathbb{P}^1$. The del Pezzo surface X^0 in case (8'a) is a toric model for $\mathbb{P}^1 \times \mathbb{P}^1$. The other cases are determined, up to smooth deformations, by the degree, since del Pezzos of degree $\neq 8$ have a connected moduli space. $\square$

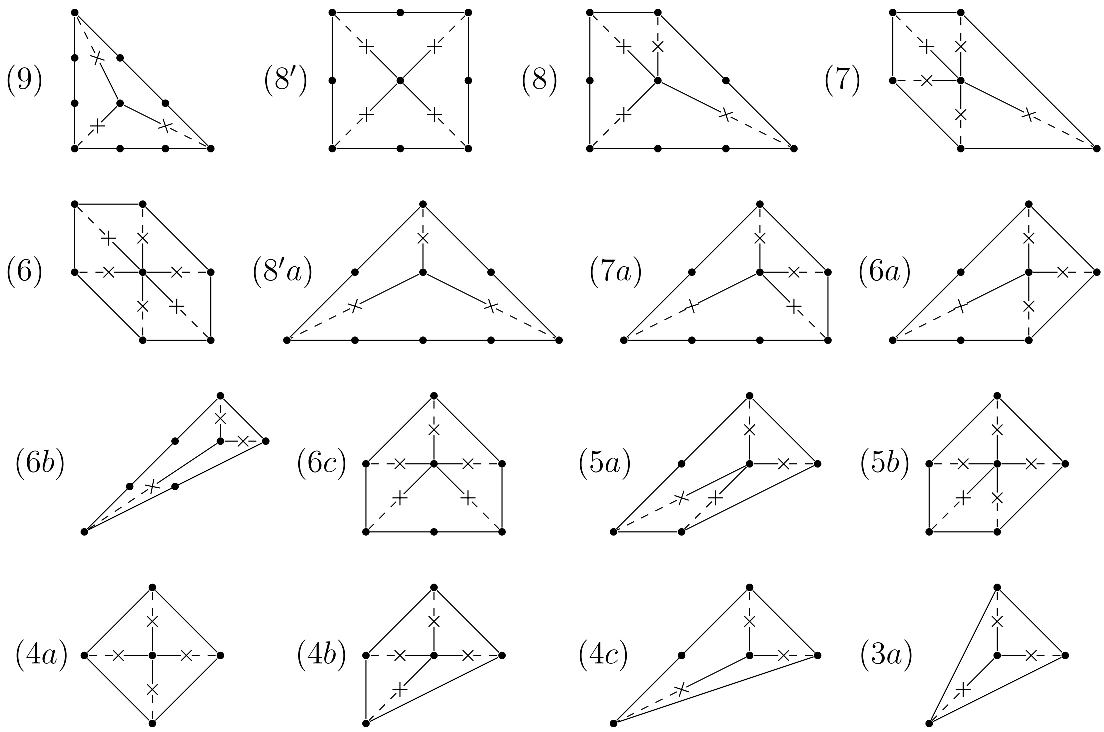


Figure 2.4: Intersection complexes $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ of smoothings X of toric Gorenstein del Pezzo surfaces X^0. The number in the labelling is the degree of X and X^0. In the first five cases X^0 is smooth.

2.2 The fan picture and refinement

Let Q be a Fano polytope and let $\mathfrak{X}_Q \to \mathbb{A}^2$ be the family from Construction 2.2. Let $(\check{B}, \check{\mathscr{P}}, \check{\varphi})$ be the intersection complex of the toric degeneration $\mathfrak{X} := \mathfrak{X}^{s\neq 0} \to \mathbb{A}^1$, i.e., one of the polarized polyhedral affine manifolds in Figure 2.4. Performing the discrete Legendre transform ([GS1], §1.4) we obtain another polarized polyhedral affine manifold that is the dual intersection complex ([GS1], §4.1) of $\mathfrak{X} \to \mathbb{A}^1$.

Definition 2.12. Write σ_0 for the unique bounded maximal cell of the dual intersection complex of $\mathfrak{X} \to \mathbb{A}^1$.

Construction 2.13. Refine the dual intersection complex of $\mathfrak{X} \to \mathbb{A}^1$ by introducing rays starting at the origin and pointing to the integral points of σ_0. This yields another polarized polyhedral affine manifold $(B, \mathscr{P}, \varphi)$, as shown in Figure 2.5. A refinement of the dual intersection complex gives a logarithmic modification of $\mathfrak{X}_Q \to \mathbb{A}^2$ (see §1.9). Since the deformation parameter s is not part of the logarithmic data, the logarithmic modification does not change the general fiber $X = X^{s\neq 0}_{t\neq 0}$. It can be constructed explicitly as follows.

(1) Blow up $\mathfrak{X}_Q \subset \mathbb{P}_{\check{\mathscr{P}}} \times \mathbb{A}^2$ at $X_{\sigma_0} \times \{(0,0)\}$, where X_{σ_0} is the point corresponding to σ_0. This corresponds to a refinement of σ_0 by edges from the origin to the corners of σ_0.

(2) Introducing the ray starting at the origin and pointing in the direction of an integral vector on the interior of a bounded edge ω of $\mathscr{P}$ corresponds to a blow up at $X_\omega \times \mathbb{A}^1 \times \{s = 0\}$, where $X_\omega = \mathbb{P}^1$ is the component corresponding to ω, i.e., the line through the points corresponding to the bounded maximal cell and the unbounded maximal cell containing ω, respectively.

In cases where X^0 is not smooth we refine the asymptotic fan of $\mathscr{P}$. This corresponds to a toric blow up of the toric model X^0. This blow up is nef but not necessarily ample. By [KMi], Proposition A.2, the deformation of such a nef toric model still is (X, D) and has Picard group isomorphic to $\mathrm{Pic}(X)$. Note that $\mathrm{Pic}(X)$ is isomorphic to $H_2(X, \mathbb{Z})$ for the del Pezzo surface X by the Kodaira vanishing theorem and Poincaré duality.

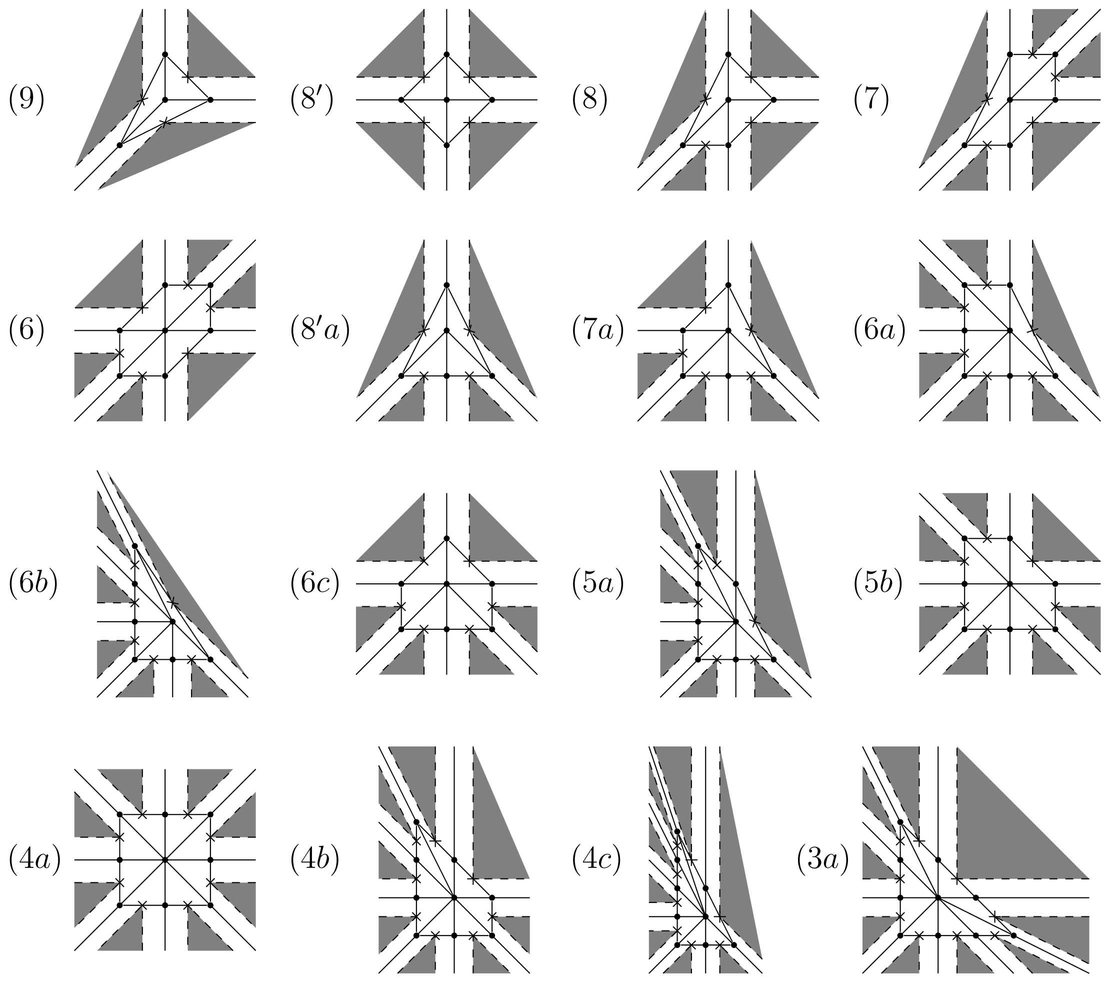


Figure 2.5: Dual intersection complexes $(B, \mathscr{P}, \varphi)$ of smooth very ample log Calabi-Yau pairs. The shaded regions are cut out and the dashed lines are mutually identified. Compare this with [KMi], Figure 2, and [Pum], Figure 5.15.

Definition 2.14. By abuse of notation, from now on $\mathfrak{X}_Q \to \mathbb{A}^2$ will denote the logarithmic modification from Construction 2.13. Note that X^0 is smooth, toric and nef, but not necessarily ample. We call it a *smooth toric model* of X. If X^0 is ample it coincides with the toric model of X constructed from Q (Definition 2.9).

Example 2.15. Consider the smoothing of $\mathbb{P}(1,1,2)$ to $\mathbb{P}^1 \times \mathbb{P}^1$ (case (8'a)) from Example 2.8. The logarithmic modification from Construction 2.13 is a 2-parameter family $\mathfrak{X}_Q \rightarrow \mathbb{A}^2$ such that $\mathfrak{X} \rightarrow \mathbb{A}^1$ is a toric degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathfrak{X}^0 \rightarrow \mathbb{A}^1$ is a toric degeneration of the Hirzebruch surface $\mathbb{F}_2$, the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ given by $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. This is the smooth surface obtained by blowing up the singular point on $\mathbb{P}(1,1,2)$, corresponding the the subdivision of the asymptotic fan given in Figure 2.6.

The Picard group $\mathrm{Pic}(\mathbb{F}_2) \simeq H_2(\mathbb{F}_2,\mathbb{Z})$ is generated by the class of a fiber F and the class of a section, e.g., the exceptional divisor E of the blow up. The intersection numbers are $F^2 = 0$, $E^2 = -2$ and $E \cdot F = 1$. The anticanonical bundle is $-K_{\mathbb{F}_2} = 2F + S + E = 4F + 2E$, where $S = 2F + E$ is the class of a section different from the exceptional divisor. The classes of curves corresponding to the rays in the fan of $\mathbb{F}_2$ are given in Figure 2.6.

The Picard group of $\mathbb{P}^1 \times \mathbb{P}^1$ is generated by the class of a bidegree $(1,0)$ curve L_1 and a bidegree $(0,1)$ curve L_2, with intersection numbers $L_1^2 = 0$, $L_2^2 = 0$ and $L_1 \cdot L_2 = 1$. There is an isomorphism

$$\mathrm{Pic}(\mathbb{F}_2) \xrightarrow{\sim} \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1),\ F \mapsto L_2,\ E \mapsto L_1 - L_2.$$

Note that there is another isomorphism by the symmetry $L_1 \leftrightarrow L_2$ and we made a choice here, fixed by the deformation $\mathfrak{X}^0 \hookrightarrow \mathfrak{X}_Q$. We will use this isomorphism in §8.2.1 to calculate the logarithmic Gromov-Witten invariants of $\mathbb{P}^1 \times \mathbb{P}^1$ in an alternative way.

Remark 2.16. Note that $(B,\mathscr{P})$ is simple (Definition 1.13), for all cases in Figure 2.5, since all affine singularities have monodromy $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ in suitable coordinates. Thus we can apply the reconstruction theorem (Corollary 1.78) together with the construction of a tropical superpotential from [CPS] to obtain the mirror Landau-Ginzburg model to (X,D).

2.3 Affine charts

Figure 2.5 shows the dual intersection complexes $(B,\mathscr{P},\varphi)$ in the chart of σ_0 (Definition 2.12). The shaded regions are cut out and the dashed lines are mutually identified, so in fact all unbounded edges are parallel.

Definition 2.17. Let $m_{\mathrm{out}} \in \Lambda_B$ denote the primitive integral tangent vector pointing in the unique unbounded direction on B.

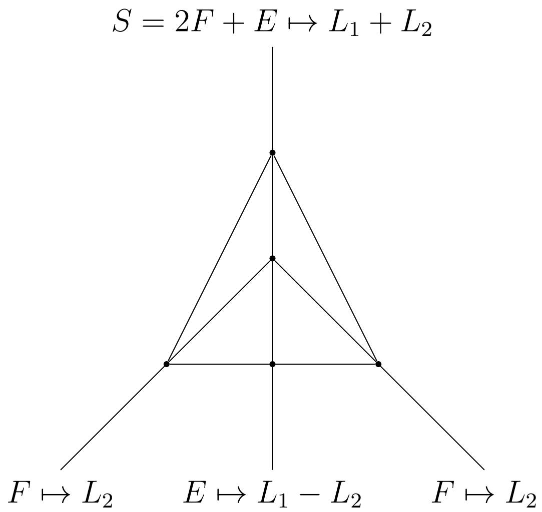


Figure 2.6: The dual intersection complex of a toric degeneration of $\mathbb{F}_2$. The classes of curves corresponding to the rays and their images under $\mathrm{Pic}(\mathbb{F}_2) \xrightarrow{\sim} \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ are given.

Example 2.18. Figure 2.7 shows the dual intersection complex $(B, \mathscr{P}, \varphi)$ of $(\mathbb{P}^2, E)$ in the chart of an unbounded maximal cell. Intuitively, this picture can be obtained by mutually gluing the dashed lines in Figure 2.5, (9). The two horizontal dashed lines are identified. The monodromy transformation by passing across the upper horizontal dashed line is given by $\Lambda_B \to \Lambda_B, m \mapsto \left(\begin{smallmatrix} 1 & 9 \\ 0 & 1 \end{smallmatrix}\right) \cdot m$.

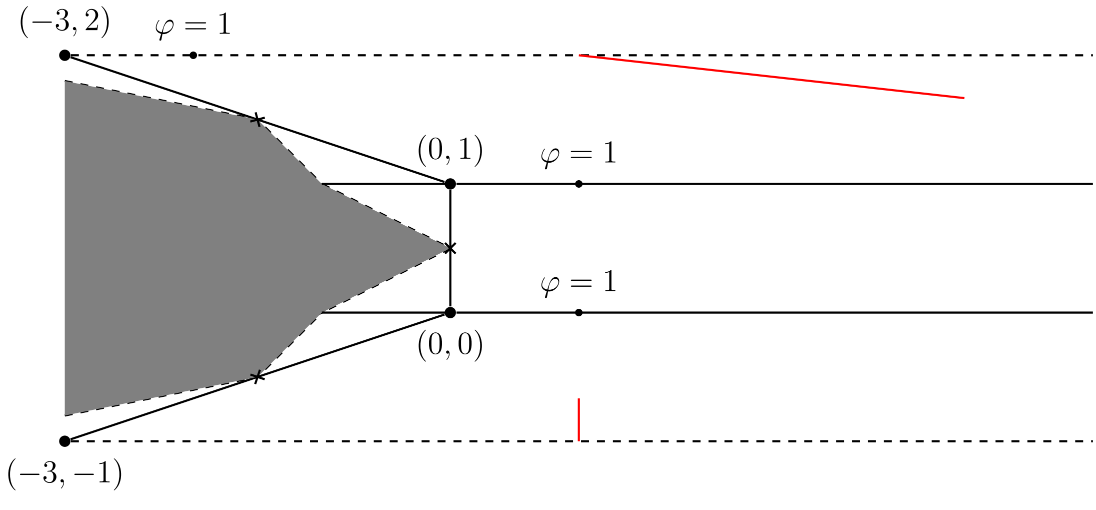


Figure 2.7: $(B, \mathscr{P}, \varphi)$ for $(\mathbb{P}^2, E)$ in the chart of an unbounded maximal cell. The dark region is cut out and the dashed lines are mutually identified. A straight line is shown in red.

We can extend the description of the affine structure across the horizontal dashed line by giving a chart of a discrete covering space $\bar{B}$ of B (Figure 2.8). Passing from one fundamental domain to an adjacent one amounts to applying the monodromy transformation by crossing the horizontal dashed line in B.

This gives a trivialization $\Lambda_{\bar{B}} \simeq M = \mathbb{Z}^2$ on $\bar{B} \setminus (\mathrm{Int}(\bar{\sigma_0}) \cup \bar{\Delta})$, where $\bar{\sigma_0}$ and $\bar{\Delta}$ are the preimages of the bounded maximal cell σ_0 and the discriminant locus Δ, respectively. We will see in Lemma 6.3 that the consistent wall structure $\mathscr{S}_\infty$ defined by $(B, \mathscr{P}, \varphi)$ has support disjoint from $\mathrm{Int}(\sigma_0)$. Hence, the whole scattering procedure can be described in this affine chart of the covering space $\bar{B}$. This allows for a simple implementation of the scattering algorithm (see §8).

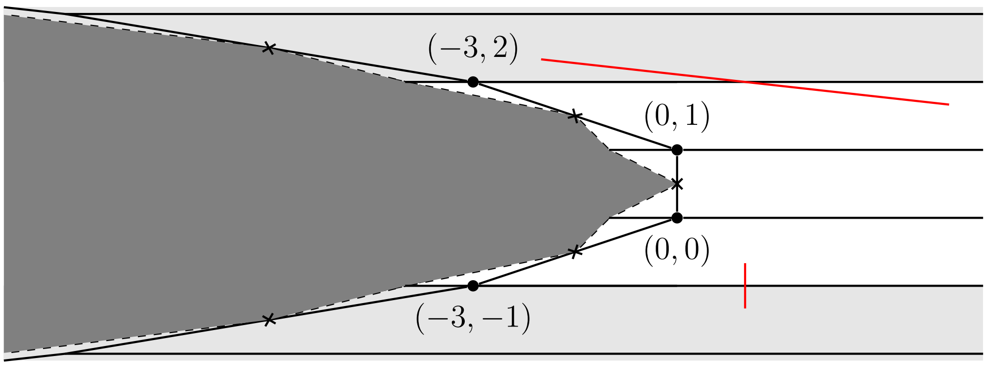


Figure 2.8: A chart of a covering space $\bar{B}$ of B with fundamental domain the white region (including one of the rays on its border). The preimage of the straight line from Figure 2.7 is shown in red.

3 Resolution of log singularities

Let Q be a Fano polytope and consider the family $\mathfrak{X}_Q \to \mathbb{A}^2$ from Construction 2.13. Equip $\mathbb{A}^2$ with the divisorial log structure defined by $V(t) \subset \mathbb{A}^2$ and $\mathfrak{X}_Q$ with the divisorial log structure defined by $\mathfrak{X}_0 \cup \mathfrak{D}_Q \subset \mathfrak{X}_Q$, that is, the sheaf of monoids

$$\mathcal{M}_{(\mathfrak{X}_Q, \mathfrak{X}_0 \cup \mathfrak{D}_Q)} := (j_\star \mathcal{O}^\times_{\mathfrak{X}_Q \setminus (\mathfrak{X}_0 \cup \mathfrak{D}_Q)}) \cap \mathcal{O}_{\mathfrak{X}_Q}, \quad j : \mathfrak{X}_Q \setminus (\mathfrak{X}_0 \cup \mathfrak{D}_Q) \hookrightarrow \mathfrak{X}_Q.$$

If we consider the fibers X_t^s or the families $\mathfrak{X}_t \to \mathbb{A}^1$ or $\mathfrak{X}^s \to \mathbb{A}^1$ as log schemes, we always mean equipped with the log structure by restriction of the above log structure. Now $\mathfrak{X}^0 \to \mathbb{A}^1$ is log smooth, and $\mathfrak{X} \to \mathbb{A}^1$ is log smooth away from finitely many points on the central fiber, corresponding to the affine singularities of the dual intersection complex $(B, \mathscr{P}, \varphi)$. At these points $\mathfrak{X}^s$ is locally given by $\mathrm{Spec}\, \mathbb{C}[x, y, w, t]/(xy - t^l(w + s))$ with log structure given by $V(t) \cup V(w)$. This is isomorphic to $\mathrm{Spec}\, \mathbb{C}[x, y, \tilde{w}, t]/(xy-t^l\tilde{w})$ with $\tilde{w} = w+s$. The log structure is given by $V(t) \cup V(\tilde{w})$ for $s = 0$ and by $V(t)$ for $s \neq 0$. Arguments as in [Gro2], Example 3.20, show that for $s \neq 0$ this is not fine at the point given by $x = y = w = t = 0$. Following [GS2], Lemma 2.12, we describe a small log resolution $\tilde{\mathfrak{X}}^s \to \mathfrak{X}^s$ such that $\tilde{\mathfrak{X}}^s$ is fine and log smooth over $\mathbb{A}^1$.

3.1 The local picture

Spec $\mathbb{C}[x,y,\tilde{w},t]/(xy-t^l\tilde{w})$ is the affine toric variety defined by the cone σ generated by $(0,0,1)$, $(0,1,0)$, $(1,0,1)$ and $(l,1,0)$. In fact,

$$\begin{aligned} \text{Spec } \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^3] &= \text{Spec } \mathbb{C}[z^{(1,0,0)}, z^{(-1,l,1)}, z^{(0,0,1)}, z^{(0,1,0)}] \\ &= \text{Spec } \mathbb{C}[x,y,\tilde{w},t] \Big/ (xy-\tilde{w}t^l). \end{aligned}$$

We obtain a toric blow up by subdividing the fan consisting of the single cone σ. There are two ways of doing this and they are related by a *flop*. We choose the subdivision Σ as in [GS2], Lemma 2.12, with maximal cones σ_1 generated by $(0,0,1)$, $(1,0,1)$ and $(0,1,0)$, and σ_2 generated by $(l,1,0)$, $(1,0,1)$ and $(0,1,0)$.

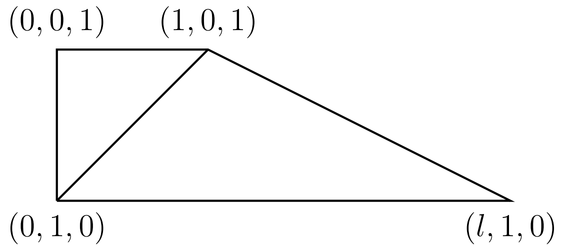


Figure 3.1: Generators of the cone defining a toric model of a log singularity and a choice of subdivision.

These cones define affine toric varieties

$$\begin{aligned} X_{\sigma_1} &= \text{Spec } \mathbb{C}[z^{(1,0,0)}, z^{(0,1,0)}, z^{(-1,0,1)}] = \text{Spec } \mathbb{C}[x,t,u] = \mathbb{A}^3, \\ X_{\sigma_2} &= \text{Spec } \mathbb{C}[z^{(-1,l,1)}, z^{(0,0,1)}, z^{(0,1,0)}, z^{(1,0,-1)}] = \text{Spec } \mathbb{C}[y,\tilde{w},t,v]/(yv-t^l), \\ X_{\sigma_{12}} &= \text{Spec } \mathbb{C}[z^{(1,0,0)}, z^{(0,1,0)}, z^{\pm(-1,0,1)}] = \text{Spec } \mathbb{C}[x,t,u^{\pm 1}] = \mathbb{A}^2 \times \mathbb{G}_m. \end{aligned}$$

The toric variety X_Σ defined by Σ is obtained by gluing X_{σ_1} and X_{σ_2} along $X_{\sigma_{12}}$. This is the fibered coproduct (with $u = U/V$ and $v = V/U$)

$$X_\Sigma = X_{\sigma_1} \amalg_{X_{\sigma_{12}}} X_{\sigma_2} = \text{Proj } \mathbb{C}[x,y,\tilde{w},t][U,V] \Big/ (\tilde{w}V - xU, yV - t^lU).$$

Note that we take Proj of the polynomial ring with variables U,V over the ring $\mathbb{C}[x,y,\tilde{w},t]$, so only U,V are homogeneous coordinates, of degree 1. The grading is given by degree in U and V. The exceptional set of the resolution $X_\Sigma \to X_\sigma$ is a line contained in the irreducible component of the central fiber $X_{\Sigma,0}$ given by $y = 0$.

Equip X_Σ with the divisorial log structure by its central fiber $X_{\Sigma,0}$ and pull this log structure back to $X_{\Sigma,0}$. Then X_{σ_1} and X_{σ_2} are log smooth with respect to the restriction of this log structure, since they are simple normal crossings. They form an affine cover of $X_{\Sigma,0}$, so $X_{\Sigma,0}$ is log smooth.

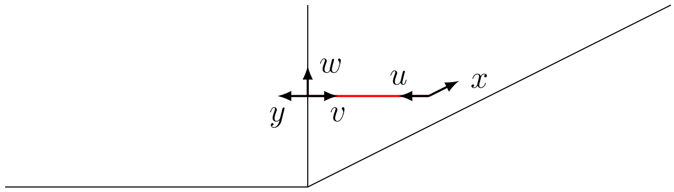


Figure 3.2: Local picture of the central fiber of the resolution, with exceptional line shown in red.

Similarly, if we make the opposite choice of subdivision, the exceptional line is contained in the irreducible component of the central fiber given by $x = 0$.

3.2 The global picture

There are two geometric descriptions of the toric blow up $X_\Sigma \to X_\sigma$ considered in §3.1 above:

(1) $X_\Sigma \to X_\sigma$ is given by blowing up X_σ along $\{y = t = 0\}$. Indeed, this corresponds to subdividing σ by cones connecting the face of σ corresponding to $\{y = t = 0\}$, in our case the ray generated by $(1, 0, 1)$, with all other faces of σ, leading to the fan Σ.

(2) Let $X_{\Sigma'}$ be the blow up of X_σ along the origin. This corresponds to inserting a ray in the center of σ and connecting all faces of σ with this ray, leading to a fan Σ'. The exceptional set of this blow up is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Choose one of the $\mathbb{P}^1$-factors and partially contract the exceptional set in $X_{\Sigma'}$ by projecting to this factor. This corresponds to one of the two ways to pair off the four maximal cones in Σ' into two cones. One choice leads to Σ, so we obtain $X_{\Sigma'} \to X_\Sigma$ by a partial contraction of the exceptional set in $X_{\Sigma'}$.

These constructions can also be performed globally on $\mathfrak{X}_Q$.

In (1) we blow up $\mathfrak{X}_Q$ along one of the irreducible components of $\mathfrak{X}_0$. We can do this for all irreducible components of $\mathfrak{X}_0$ and obtain a log smooth family over $\mathbb{A}^2$. However, this family will depend on the order of the blow ups and the irreducible components of its central fiber will contain different numbers of exceptional lines.

In (2) we blow up $\mathfrak{X}_Q$ along curves on $\mathfrak{X}_0$ and then partially contract the exceptional sets. In each step we have two ways to choose the contraction. Making the right choices we obtain a more symmetric resolution.

Construction 3.1 (The log smooth degeneration)**.** For each log singularity on $\mathfrak{X}_Q$ we have two choices of a small resolution as in (2) above, fixed by choosing which irreducible component of $\tilde{X}_0^{s\neq0}$ contains the exceptional line. We make a symmetric choice such that we have one exceptional line on each irreducible component of $\tilde{X}_0$ (see Figure 3.4). The only reason for doing so is to avoid distinction of cases. We obtain a log smooth family $(\tilde{\mathfrak{X}}_Q, \tilde{\mathfrak{D}}_Q) \to \mathbb{A}^2$. Since we only change the fibers X_0^s, we still have that $(\tilde{\mathfrak{X}}^0, \mathfrak{D}^0) \to \mathbb{A}^1$ is a degeneration of X^0 and $\mathfrak{X} \to \mathbb{A}^1$ is a degeneration of X, but these are not toric degenerations.

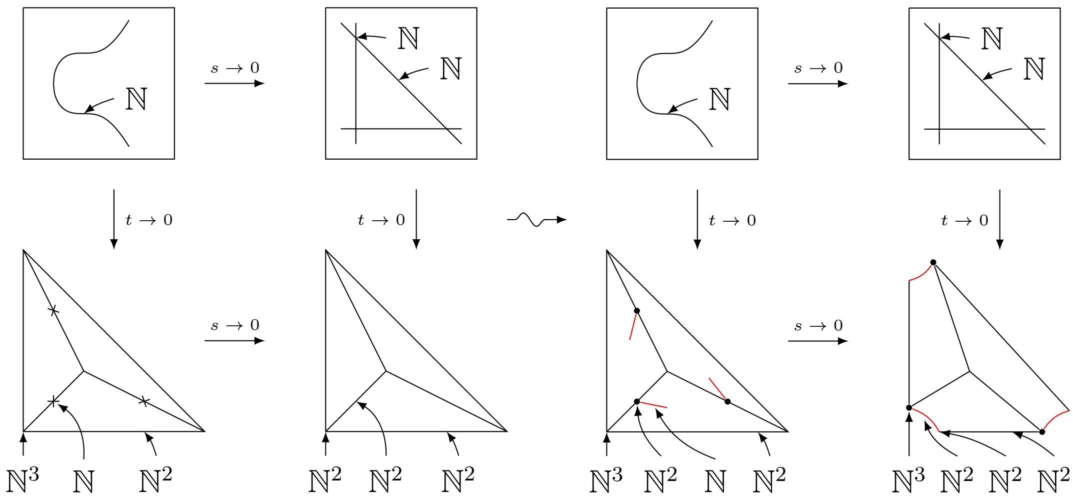


Figure 3.3: Fibers of the families $\mathfrak{X}_Q \to \mathbb{A}^2$ (left) and $\tilde{\mathfrak{X}}_Q \to \mathbb{A}^2$ (right) for $(\mathbb{P}^2, E)$. The exceptional lines are red. Some stalks of the ghost sheaves are given.

The small resolution does not change the local toric models at generic points of toric strata. As a consequence, the dual intersection complex $\tilde{B}$ of $\tilde{\mathfrak{X}} \to \mathbb{A}^1$ is homeomorphic to the dual intersection complex B of $\mathfrak{X} \to \mathbb{A}^1$. But there is one difference here. The irreducible components of $\tilde{X}_0$ are non-toric, so there is no natural fan structure at the vertices. Further, $\tilde{X}_0$ has no log singularities. Hence, there is no focus-focus singularity on bounded edges in the dual intersection complex. However, the gluing is still in such a way that the unbounded edges are parallel, leading to affine singularities at the vertices of $\tilde{B}$, coming from the gluing. This gives a triple $(\tilde{B}, \mathscr{P}, \varphi)$ as in Figure 3.4. Affine manifolds with singularities at vertices have been considered by Gross-Hacking-Keel [GHK] to construct mirrors to log Calabi-Yau surfaces.

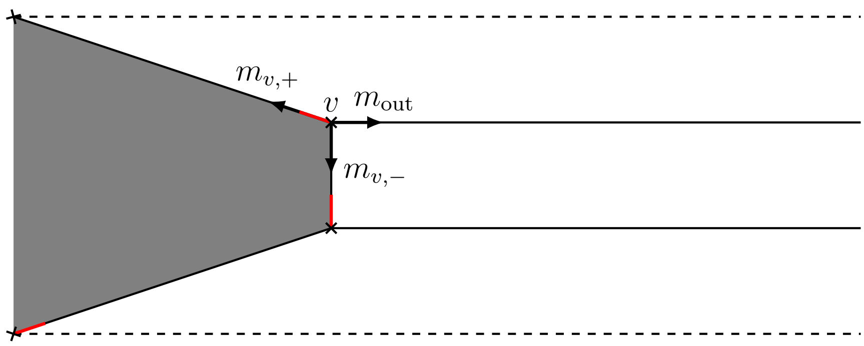


Figure 3.4: The dual intersection complex $(\tilde{B}, \mathscr{P}, \varphi)$ of $\tilde{X}_0$ for $(\mathbb{P}^2, E)$ away from σ_0, with choices of resolutions indicated.

Definition 3.2. For a vertex v of $\mathscr{P}$, let L_v^{exc} be the unique exceptional line contained in the irreducible component $\tilde{X}_v$ of $\tilde{X}_0$ corresponding to v.

For later convenience we indicate the choices of small resolutions by red stubs attached to the vertices of $\mathscr{P}$. The stub at a vertex v points in the direction corresponding to the toric divisor of X_v intersecting L_v^{exc}. Denote the primitive vector in the direction of the red stub adjacent to v by $m_{v,+} \in \Lambda_{\tilde{B},v}$. Denote the primitive vector in the direction of the other edge of σ_0 adjacent to v by $m_{v,-} \in \Lambda_{\tilde{B},v}$. Further, m_{out} is the unique unbounded direction (Definition 2.17).

3.3 Logarithmic Gromov-Witten invariants

Logarithmic Gromov-Witten invariants have been defined in [Che1][AC] and [GS5] as counts of stable log maps. A stable log map is a stable map defined in the category of log schemes with additional logarithmic data at the marked points, allowing for specification of contact orders. This leads to a generalization of Gromov-Witten theory in log smooth situations. For example, Gromov-Witten invariants relative to a (log) smooth divisor can be defined in this context, avoiding the target expansion of relative Gromov-Witten theory [Li1][Li2]. This is the case of interest to us.

Let $\tilde{\mathfrak{X}}_Q \to \mathbb{A}^2$ be the log smooth family from Construction 3.1. Note that $\tilde{X}_{t\neq 0} = X$. For the definition of stable log maps and their classes see §1.7.

The group of curve (= divisor) classes on X is isomorphic to the singular homology group $H_2(X,\mathbb{Z})$ by Poincaré duality and since $H^1(X,\mathcal{O}_X) = 0$ for del Pezzo surfaces by the Kodaira vanishing theorem. We write $H_2^+(X,\mathbb{Z})$ for the monoid of effective curve classes.

Definition 3.3. For an effective curve class $\underline{\beta} \in H_2^+(X,\mathbb{Z}) \simeq H_2^+(X^0,\mathbb{Z})$ define a class β of stable log maps to $\tilde{\mathfrak{X}}_Q \to \mathbb{A}^2$ as follows:

(1) genus $g = 0$;

(2) $k = 1$ marked point p;

(3) fibers have curve class $\underline{\beta}$;

(4) contact data $u_p = (D \cdot \underline{\beta})m_{\mathrm{out}}$, that is, full tangency with D at the marked point. Here $m_{\mathrm{out}} \in \Lambda_{\tilde{B}}$ is the primitive integral tangent vector pointing in the unbounded direction on $\tilde{B}$ (Definition 2.17).

Remark 3.4. One comment is in order about the space in which u_p lives. By definition ([GS5], Discussion 1.8(ii)), u_p is an element of $P_p^\vee := \mathrm{Hom}(f^\star\overline{\mathcal{M}}_{\tilde{\mathfrak{X}}_Q}|_p,\mathbb{N})$ and m_{out} is an element of $\Lambda_{\tilde{B}}$, the sheaf of integral tangent vectors on the dual intersection complex $\tilde{B}$. Let ω be the cell of $\mathscr{P}$ corresponding to the minimal stratum of $\tilde{X}_0^s$ to which the marked point is mapped. Then ω is an unbounded 1- or 2-dimensional cell and m_{out} defines an element of $\Lambda_{\tilde{B},\omega}$. Both, $P_p^\vee$ and $\Lambda_{\tilde{B},\omega}$ are submonoids of

$\Lambda_{\Sigma(\tilde{X}_0^s),\omega}$ and their intersection is $\mathbb{N}\cdot m_{\text{out}} \subseteq \Lambda_{\tilde{B},\omega}$. Thus $\mathbb{N}\cdot m_{\text{out}}$ can be viewed as a submonoid of $P_p^\vee$, so the definition above makes sense.

Definition 3.5. Define $w_{\text{out}} = \min\{D\cdot\underline{\beta} \mid \underline{\beta}\in H_2^+(X,\mathbb{Z})\}$.

Example 3.6. For $(X,D)=(\mathbb{P}^2,E)$ we have $w_{\text{out}}=3$, since E has degree 3. For $X=\mathbb{P}^1\times\mathbb{P}^1$ we have $w_{\text{out}}=2$ and for the smooth cubic surface we have $w_{\text{out}}=1$.

Definition 3.7. Let $\mathscr{M}(\tilde{\mathfrak{X}},\beta)$ be the moduli space of basic stable log maps to $\tilde{\mathfrak{X}} := \mathfrak{X}^{s\neq 0}\to\mathbb{A}^1$ of class β.

By [GS5], Theorems 0.2 and 0.3, $\mathscr{M}(\tilde{\mathfrak{X}},\beta)$ is a proper Deligne-Mumford stack and admits a virtual fundamental class $[\![\mathscr{M}(\tilde{\mathfrak{X}},\beta)]\!]$. Since (X,D) is a log Calabi-Yau pair, the class β is combinatorially finite ([GS5], Definition 3.3). Hence, the virtual dimension of $\mathscr{M}(\tilde{\mathfrak{X}},\beta)$ is zero and the following definition makes sense.

Definition 3.8. For β as in Definition 3.3 define the logarithmic Gromov-Witten invariant

$$N_\beta = \int_{[\![\mathscr{M}(\tilde{\mathfrak{X}},\beta)]\!]} 1.$$

Definition 3.9. Define $w_{\text{out}} = \min\{D\cdot\underline{\beta} \mid \underline{\beta}\in H_2^+(X,\mathbb{Z})\}$ so e.g. for $(X,D)=(\mathbb{P}^2,E)$ we have $w_{\text{out}}=3$, since E has degree 3. For $d>0$ define

$$N_d = \sum_{\substack{\underline{\beta}\in H_2^+(X,\mathbb{Z})\\ D\cdot\underline{\beta}=dw_{\text{out}}}} N_\beta.$$

Remark 3.10. Logarithmic Gromov-Witten invariants are constant in log smooth families ([MR], Appendix A). This means the following. Let $\gamma:\{\text{pt}\}\to\mathbb{A}^1$ be a point and let $\gamma^!$ be the corresponding refined Gysin homomorphism . Then $\gamma^!\beta$ defines a class of stable log maps to the fiber $\tilde{X}_t$ that, by abuse of notation, we also write as β. We get a moduli space and a virtual fundamental class $[\![\mathscr{M}(\tilde{X}_t,\beta)]\!]$. Then, for all $t\in\mathbb{A}^1$,

$$N_\beta = \int_{[\![\mathscr{M}(\tilde{X}_t,\beta)]\!]} 1.$$

This shows that N_β equals the logarithmic Gromov-Witten invariant N_β defined in the introduction. Moreover, as shown in [AMW], N_β equals the relative Gromov-Witten invariant of the smooth pair (X,D) as defined in [Li1].

3.4 Log BPS numbers

The logarithmic Gromov-Witten invariants N_β are not integers but rather rational numbers. The fractional part comes from multiple cover contributions of curves with class β' such that $\underline{\beta}=k\cdot\underline{\beta}'$.

Proposition 3.11 ([GPS], Proposition 6.1)**.** *The k-fold cover of an irreducible curve of class β' contributes the following factor to $N_{k\cdot\beta'}$:*

$$M_{\beta'}[k] = \frac{1}{k^2}\binom{k(D\cdot\underline{\beta}'-1)-1}{k-1}$$

We use the same formula for reducible curves, though it may be unclear how to interpret this as a multiple cover contribution.

Definition 3.12. Define numbers n_β by subtracting multiple cover contributions:

$$N_\beta = \sum_{\beta':\underline{\beta}=k\cdot\underline{\beta}'} M_{\beta'}[k]\cdot n_{\beta'}.$$

They are called *Gopakumar-Vafa invariants* or *log BPS numbers* as they are related to BPS state counts in string theory [GV].

Remark 3.13. The logarithmic Gromov-Witten invariants N_β are related to local Gromov-Witten invariants N_β^{loc} of the total space of the canonical bundle K_X of X by the formula $N_\beta = (-1)^{D\cdot\underline{\beta}-1}(D\cdot\underline{\beta})N_\beta^{\text{loc}}$. This was conjectured by Takahashi ([Tak2], Remark 1.11) and proved by Gathmann ([Gat], Example 2.2) and more generally by van Garrel, Graber and Ruddat [GGR]. The log BPS numbers n_d were shown to be integers in [GWZ], using integrality of local BPS numbers.

4 Tropical curves and refinement

In this section we analyze what tropicalizations of stable log maps contributing to N_d look like. We prove that for each d there are only finitely many such tropical curves (Corollary 4.19). Choosing a subdivision of the dual intersection complex $(\tilde{B},\mathscr{P},\varphi)$ such that tropicalizations contributing to N_d are contained in the 1-skeleton of the polyhedral decomposition leads to a logarithmic modification $\tilde{\mathfrak{X}}_d$ of $\tilde{\mathfrak{X}}$ (Construction 4.29) with the property that stable log maps to the central fiber Y of $\tilde{\mathfrak{X}}_d$ contributing to N_d are torically transverse.

4.1 Tropicalization of stable log maps

Let Q be a Fano polytope and let $\tilde{\mathfrak{X}}\to\mathbb{A}^1$ be the log smooth degeneration of the corresponding smooth very ample log Calabi-Yau pair (X,D) from Construction 3.1. Let $\underline{\beta}\in H_2^+(X,\mathbb{Z})$ be an effective curve class and consider a basic stable log map $f:C/\text{pt}_{Q_{\text{basic}}}\to\tilde{X}_0/\text{pt}_{\mathbb{N}}$ of class β (Definition 3.3). Here Q_{basic} is the basic monoid of f (Definition 1.82). We will see in Corollary 4.19 that in our situation we have $Q_{\text{basic}} = \mathbb{N}$.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\gamma"] & \tilde{X}_0 \arrow[d, "\tilde{\pi}_0"] \\ \mathrm{pt}_{Q_{\text{basic}}} \arrow[r, "g"] & \mathrm{pt}_{\mathbb{N}} \end{tikzcd}$$

Tropicalization (Definition 1.88) gives a diagram of generalized cone complexes. Note that $\Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathbb{R}_{\geq 0}$. The fiber $\Sigma(\tilde{\pi}_0)^{-1}(1)$ is homeomorphic to the dual intersection complex $\tilde{B}$ of $\tilde{X}_0$. Similarly, for a general element b of the cone $\Sigma(\mathrm{pt}_{Q_{\text{basic}}})$ the fiber $\Sigma(\gamma)^{-1}(b)$ is homeomorphic to the dual intersection graph Γ_C of C. Hence, tropicalization of the above diagram and restriction to the fiber over $1 \in \Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathbb{R}_{\geq 0}$ gives a map

$$\tilde{h} : \Gamma_C \to \tilde{B}. \tag{4.1}$$

There is additional data on Γ_C making $\tilde{h} : \Gamma_C \to \tilde{B}$ into a tropical curve in the sense of [ACGS1]. Note that such a tropical curve only fulfills a modified version of the balancing condition ([GS5], Proposition 1.15). In §4.2 we will see what this means in our case. To make the connection with scattering diagrams in §6 it is useful to consider tropical curves on B (not $\tilde{B}$) that are balanced in the usual sense but may have some bounded legs.

Definition 4.1. Let B be a 2-dimensional integral affine manifold with singularities. Let $\Delta \subset B$ be the discriminant locus and write $B_0 := B \setminus \Delta$. A *(parametrized) tropical curve on* B, written $h : \Gamma \to B$, is a homogeneous map $h : \Gamma \to B_0$, where Γ is the topological realization of a graph[3], possibly with some non-compact edges (*legs*), together with

(1) a non-negative integer g_V (*genus*) for each vertex V;
(2) a non-negative integer ℓ_E (*length*) for each compact edge E;
(3) an element $u_{(V,E)} \in i_\star \Lambda_{B_0,h(V)}$ (*weight vector*) for every vertex V and edge or leg E adjacent to V. Here Λ_{B_0} is the sheaf of integral affine tangent vectors on B and $i : B_0 \hookrightarrow B$ is the inclusion. The index of $u_{(V,E)}$ in the lattice $\Lambda_{B,h(V)}$ is called the *weight* w_E of E;

such that

(i) if E is a compact edge with vertices V_1, V_2, then h maps E affine linearly[4] to the line segment connecting $h(V_1)$ and $h(V_2)$, and $h(V_2) - h(V_1) = \ell_E u_{(V_1,E)}$. In particular, $u_{(V_1,E)} = -u_{(V_2,E)}$;
(ii) if E is a leg with vertex V, then h maps E affine linearly either to the ray

[3]The topological realization of a graph Γ is a topological space which is the union of line segments corresponding to the edges. By abuse of notation, we also denote this by Γ. Whenever we talk about a map from a graph we mean a homogeneous map from its topological realization.

[4]The affine structure on Γ is given by the lengths ℓ_E of its edges.

$h(V) + \mathbb{R}_{\geq 0} u_{(V,E)}$ or to the line segment $[h(V), \delta)$ for δ an affine singularity of B such that $\delta - h(V) \in \mathbb{R}_{>0} u_{(V,E)}$, i.e., $u_{(V,E)}$ points from $h(V)$ to δ.

We write the set of compact edges of Γ as $E(\Gamma)$, the set of legs as $L(\Gamma)$, the set of legs mapped to a ray (*unbounded legs*) as $L_\infty(\Gamma)$ and the set of legs mapped to an open line segment (*bounded legs*) as $L_\Delta(\Gamma)$ (since such edges end at the singular locus Δ of B).

The *genus* of a parametrized tropical curve $h : \Gamma \to B$ is defined by

$$g_h := g_\Gamma + \sum_{V \in V(\Gamma)} g(V),$$

where g_Γ is the genus (first Betti number) of the graph Γ.

Remark 4.2. Note that if E is a leg of Γ, then $h(E)$ must be parallel to the edge of $\mathscr{P}$ containing δ, since there is only one tangent direction at δ, i.e., $\Lambda_{B,\delta} \simeq \mathbb{Z}$.

Definition 4.3. An *isomorphism* of tropical curves $h : \Gamma \to B$ and $h' : \Gamma' \to B$ is a homeomorphism $\phi : \Gamma \to \Gamma'$ such that $h = h' \circ \phi$, $g_{\phi(V)} = g_V$ and $u_{(\phi(V),\phi(E))} = u_{(V,E)}$. An *automorphism* of a tropical curve h is an isomorphism of h with itself. Here we use the convention that an edge E is a pair of orientations of E, so that the automorphism group of a graph with a single loop is $\mathbb{Z}/2\mathbb{Z}$.

Remark 4.4. We will only consider tropical curves of genus 0. In particular, our tropical curves will have no loops.

Definition 4.5. Let $(B, \mathscr{P})$ be a 2-dimensional polyhedral affine manifold. A tropical curve $h : \Gamma \to B$ is *compatible with* $\mathscr{P}$ if

(1) the edges of Γ do not extend across several maximal cells of $\mathscr{P}$. In other words, we have a well-defined map $E(\Gamma) \cup L(\Gamma) \to \mathscr{P}$ associating to an edge or leg E the minimal cell of $\mathscr{P}$ containing it.

(2) there are no bivalent vertices in Γ mapped to a maximal cell of $\mathscr{P}$.

Construction 4.6. Let $\tilde{h} : \Gamma_C \to \tilde{B}$ be the continuous map from (4.1). We describe additional data making $\tilde{h}$ a tropical curve compatible with $\mathscr{P}$.

(1) For each vertex V, the genus is $g_V = 0$.

(2) Let $E \in E(\Gamma_C)$ be a compact edge with vertices V_1, V_2, corresponding to a node $q \in C$. Then $\overline{\mathcal{M}}_{C,q}$ is isomorphic to the submonoid S_{e_q} of $\mathbb{N}^2$ generated by $(e_q, 0)$, $(0, e_q)$ and $(1,1)$ for some $e_q \in \mathbb{N}_{>0}$ ([Kat2], 1.8). Moreover, there is an equation $\tilde{h}(V_2) - \tilde{h}(V_1) = \pm e_q u_q$ for some $u_q \in \Lambda_{\tilde{B}}$ (see [GS5], Discussions 1.8, 1.13). Then the length of E is $\ell_E = e_q$ and the weight vectors are $u_{(V_i,E)} = \pm u_q$, with sign chosen such that $u_{(V_i,E)}$ points away from $\tilde{h}(V_i)$.

(3) Let $E \in L_\infty(\Gamma_C)$ be an unbounded leg with vertex V, corresponding to a marked point $p \in C$. Then $\overline{\mathcal{M}}_{C,p}$ is isomorphic to $\mathbb{N} \oplus \mathbb{N}$ and

$$f^\star \overline{\mathcal{M}}_X|_p \to \overline{\mathcal{M}}_{C,p} \overset{\mathrm{pr}_2}{\to} \mathbb{N}$$

is determined by an element of $P_p^\vee = \mathrm{Hom}(f^\star \overline{\mathcal{M}}_X|_p, \mathbb{N})$, inducing an element $u_p \in \Lambda_{\tilde{B},\tilde{h}(V)}$. The weight vector is $u_{(V,E)} = u_p$.

The properties for $\tilde{h} : \Gamma_C \to \tilde{B}$ to be compatible with $\mathscr{P}$ can be achieved by (1) inserting vertices at points mapping to vertices or edges of $\mathscr{P}$ and (2) removing bivalent vertices mapping to a maximal cell of $\mathscr{P}$, by replacing a chain of edges connected via bivalent vertices with a single edge. The latter is possible, since vertices of Γ_C not mapping to vertices of $\mathscr{P}$ are balanced by Proposition 4.10, (I), below.

Definition 4.7. Let B be a 2-dimensional integral affine manifold with singularities and $m \in \Lambda_B$ an integral tangent vector. A tropical curve $h : \Gamma \to B$ is called *of degree d relative to m* if there is exactly one unbounded leg $E_{\mathrm{out}} \in L_\infty(\Gamma)$, and its weight vector is $u_{(V_{\mathrm{out}},E_{\mathrm{out}})} = d \cdot m$. Here V_{out} is the unique vertex of E_{out}.

Proposition 4.8. *Given a stable log map $f : C/\mathrm{pt}_{Q_{basic}} \to \tilde{X}_0/\mathrm{pt}_{\mathbb{N}}$ of class β as in Definition 3.3, the continuous map $\tilde{h} : \Gamma_C \to \tilde{B}$ from* (4.1) *together with the additional data defined in Construction 4.6 is a tropical curve without bounded legs, of genus* 0*, degree $D \cdot \underline{\beta}$ relative to $m_{\mathrm{out}} \in \Lambda_{\tilde{B}}$ (Definition 2.17) and compatible with the dual intersection complex $\mathscr{P}$. By abuse of notation, we call this tropical curve $\tilde{h} : \Gamma_C \to \tilde{B}$ the tropicalization of f.*

Proof. The properties (i) and (ii) of Definition 4.1 follow by the structure of $f : C \to \tilde{X}_0$ on the level of ghost sheaves (see [GS5], Discussions 1.8, 1.13). Hence, $\tilde{h}$ is a tropical curve. Moreover, these discussions show that $\tilde{h}$ has no bounded legs. $\tilde{h}$ is of degree dw_{out} relative to m_{out} by Definition 3.3, (4). □

Remark 4.9. There is one issue here, since we lost some information by smoothing (X^0, D^0). An effective curve class $\beta \in H_2^+(X^0, \mathbb{Z})$ is determined by its intersection numbers $d_1, \ldots, d_k$ with the components of $D^0 = D_1, \ldots, D_k$. After smoothing D^0 we only see the sum $d = d_1 + \ldots + d_k$. In particular, if X^0 is a smooth toric del Pezzo surface with Picard number > 1, i.e., different from $(\mathbb{P}^2, E)$, then we only see the total degree, not the multi-degree. One could solve this problem using non-trivial gluing data capturing information of the divisor D^0. We will give a more geometric solution in §4.4 by looking at the limit of curves under $s \to 0$, where $s \in \mathbb{A}^2$ is the deformation parameter of the family $\mathfrak{X}_Q \to \mathbb{A}^2$ from Construction 2.13.

4.2 Types of vertices

Let $f : C/\mathrm{pt}_{Q_{\mathrm{basic}}} \to \tilde{X}_0/\mathrm{pt}_{\mathbb{N}}$ be a stable log map of class β as in Definition 3.3 and let $\tilde{h} : \Gamma_C \to \tilde{B}$ be the corresponding tropical curve.

Proposition 4.10. *Let C_V be an irreducible component of C, corresponding to a vertex V of Γ_C. Then the following three cases can occur:*

(I) If C_V is mapped to a 0- or 1-dimensional toric stratum of $\tilde{X}_0$, i.e., if V is not mapped to a vertex of $\mathscr{P}$, then the ordinary balancing condition holds:

$$\sum_{E \ni V} u_{(V,E)} = 0.$$

The sum is over all edges or legs $E \in E(\Gamma_C) \cup L(\Gamma_C)$ containing V.

(II) If C_V is mapped onto an exceptional divisor L_v^{exc} on some component $\tilde{X}_v$ of $\tilde{X}_0$ (Definition 3.2), then C_V is a k-fold multiple cover of $L_v^{\mathrm{exc}} \simeq \mathbb{P}^1$ for some $k > 0$. It is fully ramified at the point $p = L_v^{\mathrm{exc}} \cap \partial\tilde{X}_v$, where $\partial\tilde{X}_v$ is the proper transform of the toric boundary ∂X_v under the resolution from §3. The vertex V is mapped to the vertex v of $\mathscr{P}$. It is 1-valent with adjacent edge E mapped onto the edge of $\mathscr{P}$ containing the red stub adjacent to v. The balancing condition reads (with $m_{v,+}$ as in Figure 3.4)

$$u_{(V,E)} = k m_{v,+}.$$

(III) Otherwise, V is mapped to a vertex v of $\mathscr{P}$ and has exactly one adjacent edge or leg $E_{V,\mathrm{out}}$ that is not mapped onto a compact edge of $\mathscr{P}$. All other edges (possibly none) are compact with other vertex of type (II) above. In this case, for some $k \geq 0$, the following balancing condition holds:

$$\sum_{E \ni V} u_{(V,E)} + k m_{v,+} = 0.$$

Proof. If C_V does not intersect an exceptional line, the log structure on $\tilde{X}_0$ along the image of C_V is the toric one. Then by [ACGS2], Remark 2.26, the ordinary balancing condition holds. This proves (I).

If C_V is mapped onto an exceptional line $L_v^{\mathrm{exc}} \simeq \mathbb{P}^1$ on some component $\tilde{X}_v$, it is a k-fold multiple cover for some $k > 0$. Suppose it is not fully ramified at the point where L_v^{exc} meets $\partial\tilde{X}_v$, i.e., V has valency > 1. Let E_1, E_2 be two distinct edges adjacent to V. We have $V \neq V_{\mathrm{out}}$, since C_V does not meet the toric divisor of $\tilde{X}_v$ belonging to $\tilde{D}_0$. By Proposition 4.8, Γ_C has only one leg, and this leg is attached to V_{out}. Thus E_1 and E_2 are bounded. Let V_1, V_2 be the vertices of E_1, E_2 different from V, respectively. There is a chain of vertices and edges (possibly the trivial one) connecting V_1 to V_{out} and similarly for V_2. These two chains form a cycle of the graph Γ_C, so $g(\Gamma_C) > 0$ in contradiction with rationality of C. Hence, there is a unique bounded edge E adjacent to V. Let V' be its other vertex. Then $h(V') - h(V)$ points in the direction of $m_{v,+}$, since the only special point (node) of C_V is mapped to $L_v^{\mathrm{exc}} \cap \partial\tilde{X}_v$. It follows by Definition 4.1 that $u_{(V,E)}$ points in the direction of $m_{v,+}$. Its affine length, the weight w_E, is the multiplicity of the node which is the ramification order k. This proves (II).

Let C_V be a component of C that intersects an exceptional divisor L_v^{exc} on some component $\tilde{X}_v$ but is not mapped onto it. We first prove the balancing condition. Let $m \in \Gamma(C_V, f^\star\overline{\mathcal{M}}_X|_{C_V})$ be the generator of the submonoid $\mathbb{N}$ of $\Gamma(C_V, f^\star\overline{\mathcal{M}}_X|_{C_V})$ corresponding to the exponent of the degeneration parameter t. Then by [GS5], Lemma 1.14, the line bundle associated to m is $f^\star\nu^\star\mathcal{O}_{X_v}(-kD_{v,+})$, where $\nu : \tilde{X}_v \to X_v$ is the resolution and $D_{v,+}$ is the toric divisor of X_v whose proper transform $\tilde{X}_v$ intersects L_v^{exc}. This is because the condition that C_V intersects L_v^{exc} is equivalent to the condition that $\nu(C_V)$ intersects $D_{v,+}$ in the point $\nu(L_v^{\mathrm{exc}})$. The corresponding contact data is $km_{v,+}$. This shows $\sum_{E\ni V} u_{(V,E)} + km_{v,+} = 0$, where k is the sum of the affine lengths of the additional u_p. Now we show uniqueness of an edge or leg $E_{V,\mathrm{out}}$ as claimed. If all legs and edges adjacent to V are mapped onto compact edges of $\mathscr{P}$, this balancing condition can not be achieved. So there is at least one such edge or leg. We show by contradiction that there is at most one. Assume that there are two edges or legs E, E' adjacent to V that are not mapped onto compact edges of $\mathscr{P}$. Then E, E' are either unbounded legs or bounded edges with other vertex of type (I). If E or E' is unbounded, then, since vertices of type (I) fulfill the ordinary balancing condition, we would have at least two unbounded legs, contradicting the assumptions. If E and E' are bounded edges with other vertices of type (I), their paths to V_{out} form a cycle, contradicting $g = 0$. So there is a unique edge not mapped onto a compat edge of $\mathscr{P}$. This proves (III). $\square$

Definition 4.11. Denote the set of vertices of the given types in Proposition 4.10 by $V_I(\Gamma_C)$, $V_{II}(\Gamma_C)$ and $V_{III}(\Gamma_C)$, respectively.

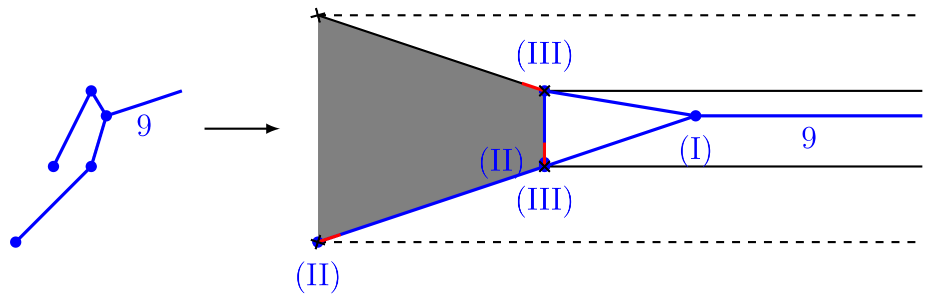


Figure 4.1: A tropical curve $\tilde{h} : \tilde{\Gamma} \to \tilde{B}$ in $\tilde{\mathfrak{H}}_3$ for $(\mathbb{P}^2, E)$, with types of vertices shown. The weight of the outgoing edge is 9. All other edges have weight 1. Two vertices are mapped to the same vertex of $\mathscr{P}$ but not connected by an edge.

Definition 4.12. Let $\tilde{\mathfrak{H}}_d$ be the set of isomorphism classes of tropical curves $\tilde{h} : \tilde{\Gamma} \to \tilde{B}$ compatible with $\mathscr{P}$ of genus 0 and degree $dw_{\rm out}$ relative to $m_{\rm out} \in \Lambda_{\tilde{B}}$, without bounded legs and with vertices of one of the types (I)-(III) above.

Lemma 4.13. *Let $\tilde{h} : \tilde{\Gamma} \to \tilde{B}$ be a tropical curve in $\tilde{\mathfrak{H}}_d$ for some $d > 0$. Then $\tilde{h}(\tilde{\Gamma})$ is disjoint from the interior of σ_0 (Definition 2.12).*

Proof. Let $\tilde{h} : \tilde{\Gamma} \to \tilde{B}$ be a tropical curve in $\tilde{\mathfrak{H}}_d$. Give $\tilde{\Gamma}$ the structure of a rooted tree by defining the root vertex to be the vertex $V_{\rm out}$ of the unique unbounded leg $E_{\rm out}$. Let V be a vertex of $\tilde{\Gamma}$ and let $E_{V,\rm out}$ be the edge connecting V with its parent, or $E_{V,\rm out} = E_{\rm out}$ if V is the root vertex $V_{\rm out}$. By Proposition 4.10, if V is mapped to a vertex v of $\mathscr{P}$, hence of type (II) or (III), then $E_{V,\rm out}$ is mapped to the conical subset $\mathbb{R}_{\leq 0} m_{v,+} + \mathbb{R}_{\leq 0} m_{v,-}$ of $\tilde{B}$, and if V is of type (I), then by induction $E_{V,\rm out}$ is mapped to the subset $\bigcup_{V'} \mathbb{R}_{\leq 0} m_{\tilde{h}(V'),+} + \mathbb{R}_{\leq 0} m_{\tilde{h}(V'),-}$ of $\tilde{B}$, where the union is over all vertices of type (II) and (III) in the subgraph of $\tilde{\Gamma}$ with root V. In particular, $\tilde{h}(\tilde{\Gamma})$ is contained in

$$\bigcup_{V' \in V_{II}(\tilde{\Gamma}) \cup V_{III}(\tilde{\Gamma})} \mathbb{R}_{\leq 0} m_{\tilde{h}(V'),+} + \mathbb{R}_{\leq 0} m_{\tilde{h}(V'),-}$$

This is disjoint from the interior of σ_0. ∎

4.3 Balanced tropical curves

We describe a procedure to obtain tropical curves in $\tilde{\mathfrak{H}}_d$ from tropical curves to B (not $\tilde{B}$!) that are balanced in the usual sense. This makes the connection to scattering diagrams in §6 more transparent. Moreover, the degeneration formula gets more symmetric when expressed in invariants labeled by balanced tropical curves (see Theorem 5.17).

Definition 4.14. Let $\mathfrak{H}_d$ be the set of isomorphism classes of tropical curves $h : \Gamma \to B$ compatible with $\mathscr{P}$, possibly with bounded legs, of genus 0 and degree $dw_{\rm out}$ relative to $m_{\rm out}$, satisfying the ordinary balancing condition at each vertex V of Γ:

$$\sum_{E \ni V} u_{(V,E)} = 0,$$

Construction 4.15. We construct a surjective map $\mathfrak{H}_d \to \tilde{\mathfrak{H}}_d$ as follows.

Let $h : \Gamma \to B$ be a tropical curve in $\mathfrak{H}_d$. Let $E \in L_\Delta(\Gamma)$ be a bounded leg with vertex V. Then E is mapped to the line segment $[h(V), \delta)$ for δ an affine singularity on an edge ω of $\mathscr{P}$. Since $\Lambda_{B,\delta}$ is one-dimensional, $h(E)$ is parallel to ω. Since h is compatible with $\mathscr{P}$ and by the balancing condition, $h(V)$ must be a vertex v of

$\mathscr{P}$. Let $m_{v,\delta}$ be the primitive integral tangent vector pointing from v to δ and let $m_{v,+}, m_{v,-}$ be as in Figure 3.4. Two cases can occur.

(1) If $m_{v,\delta} = m_{v,+}$, i.e., if E is mapped in the direction of the red stub attached to v, then remove E from Γ.

(2) Otherwise, $m_{v,\delta} = m_{v,-}$. Then add a vertex V' to E to obtain a compact edge $\tilde{E}$. Define $u_{(V',E)} = -u_{(V,E)}$ and $\tilde{h}(\tilde{E}) = \omega$, such that $\tilde{h}(V') = v'$ is a vertex of $\mathscr{P}$. This determines the length $\ell_{\tilde{E}}$ by Definition 4.1, (i).

We show that the map $\mathfrak{H}_d \to \tilde{\mathfrak{H}}_d$ constructed this way is surjective. Let $\tilde{h} : \tilde{\Gamma} \to \tilde{B}$ be a tropical curve in $\tilde{\mathfrak{H}}_d$. We can construct a preimage of $\tilde{h}$ as follows. (1) For each vertex $V \in V_{III}(\tilde{\Gamma})$, add a bounded leg E with vertex V and weight vector $u_{(V,E)} = -\sum_{E' \ni V} u_{(V,E')}$. The image of E is specified by Definition 4.1, (ii). (2) For each vertex $V \in V_{II}(\tilde{\Gamma})$, let E be the unique adjacent edge. It is a bounded edge and we remove the vertex V from E to obtain a bounded leg. This shows that the map $\mathfrak{H}_d \to \tilde{\mathfrak{H}}_d$ is surjective. Note that in step (1) we could also add several bounded legs with weights a partition of $\sum_{E' \ni V} u_{(V,E')}$, so the number of preimages of $\tilde{h}$ is the number of such partitions.

Definition 4.16. Let $(\bar{B}, \bar{\mathscr{P}})$ be the covering space of $(B, \mathscr{P})$ described in §2.3. Let $\bar{\mathfrak{H}}_d$ be the set of isomorphism classes of balanced tropical curves $\bar{h} : \bar{\Gamma} \to \bar{B}$ compatible with $\bar{\mathscr{P}}$ of genus 0 and degree dw_{out} relative to m_{out} satisfying the ordinary balancing condition and such that the image of E_{out} lies in a fixed fundamental domain.

Construction 4.17. Define a map $\bar{\mathfrak{H}}_d \to \mathfrak{H}_d$ by sending $\bar{h} : \Gamma \to \bar{B}$ to $h : \Gamma \to B$, where h is the composition of $\bar{h}$ with the covering map $\bar{B} \to B$. This map is bijective. The inverse map is given as follows. Let $h : B \to \Gamma$ be a tropical curve in $\mathfrak{H}_d$ and choose an unbounded maximal cell of $\mathscr{P}$. Choose a fundamental domain of $\bar{B} \to B$ and let $\bar{h}(E_{\text{out}})$ be the preimage of $h(E_{\text{out}})$ in that fundamental domain. Whenever the image $h(V)$ of an edge V lies on the horizontal dashed line in Figure 2.7 with respect to the chart on the unbounded maximal cell chosen, we change the fundamental domain and apply the monodromy transformation.

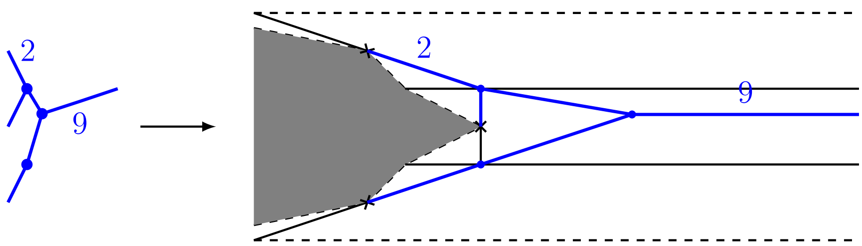


Figure 4.2: A balanced tropical curve $h : \Gamma \to B$ in $\mathfrak{H}_3$ for $(\mathbb{P}^2, E)$ giving the tropical curve in Figure 4.1 under the map from Construction 4.15.

Lemma 4.18. *The set $\bar{\mathfrak{H}}_d$ is finite.*

Proof. Let $\bar{h} : \bar{\Gamma} \to \bar{B}$ be a tropical curve in $\bar{\mathfrak{H}}_d$. The graph $\bar{\Gamma}$ together with the set of weight vectors of bounded legs $\{u_{(V,E)} \mid V \in E \in L_\Delta(\bar{\Gamma})\}$ determines the image of $\bar{h}$. Indeed, for a bounded leg E, the weight vector $u_{(V,E)}$ determines its image, since all edges containing affine singularities have different direction. The images of all other edges are determined by the balancing condition. If we only know the set $\{u_{(V,E)} \mid V \in E \in L_\Delta(\bar{\Gamma})\}$, there are finitely many possibilities for $\bar{\Gamma}$, since the number of leaves is specified. So we need to show that there are only finitely many possible sets $\{u_{(V,E)} \mid V \in E \in L_\Delta(\bar{\Gamma})\}$ for a tropical curve $\bar{h} : \bar{\Gamma} \to \bar{B}$ in $\bar{\mathfrak{H}}_d$.

Let V_{out} be the vertex of the unique unbounded edge E_{out} and let σ_{out} be an unbounded maximal cell containing V_{out}. Let φ_{out} be a representative of the piecewise affine function φ on σ_{out}. Note that via an affine transformation of $\bar{B}$ we can achieve that $\varphi_{\text{out}}(m) = \langle m, m_{\text{out}} \rangle$. Let E be a bounded leg of $\bar{\Gamma}$ with vertex V and write $u_{(V,E)} = w_E m_E$ with $w_E \in \mathbb{Z}_{>0}$ and $m_E \in \Lambda_{\bar{h}(V)} \simeq \mathbb{Z}^2$ primitive. Let E' be an edge on the path from V to V_{out} that is *pointing towards* V_{out}, i.e., such that the ray $\bar{h}(V) + \mathbb{R}_{>0} m_E$ intersects the interior of the bounded maximal cell σ_0. Then $\varphi_{\text{out}}(m_{E'}) > 0$ and by convexity of the bounded maximal cell and of φ_{out} we have $w_{E'}\varphi_{\text{out}}(m_{E'}) \geq w_E|\varphi_{\text{out}}(m_E)|$. In particular $w_{E_{\text{out}}}\varphi_{\text{out}}(m_{\text{out}}) = dw_{\text{out}} \geq w_E|\varphi(m_E)|$. This gives a bound $w_E \leq \lfloor \frac{dw_{\text{out}}}{|\varphi_{\text{out}}(m_E)|} \rfloor$ and there are only finitely many bounded legs E with $|\varphi_{\text{out}}(m_E)| \leq dw_{\text{out}}$, i.e., such that this bound is nonzero. $\square$

Corollary 4.19. *The sets $\mathfrak{H}_d$ and $\tilde{\mathfrak{H}}_d$ are finite. In particular, tropical curves in $\tilde{\mathfrak{H}}_d$ are rigid and the basic monoid of stable log maps in $\tilde{\mathfrak{H}}_d$ is $Q_{basic} = \mathbb{N}$.*

4.4 The limit $s \to 0$

Here we consider the 2-parameter family $\tilde{\mathfrak{X}}_Q \to \mathbb{A}^2$ and describe limits of stable log maps to $\mathfrak{X} := \mathfrak{X}^{s\neq 0}$ under $s \to 0$. This will enable us to read off the curve class $\underline{\beta} \in H_2^+(X,\mathbb{Z})$ of a stable log map from its tropicalization.

Definition 4.20. For an effective curve class $\underline{\beta} \in H_2^+(X,\mathbb{Z})$ let $\mathscr{M}(\tilde{\mathfrak{X}}_Q,\beta)$ be the moduli space of basic stable log maps to $\tilde{\mathfrak{X}}_Q \to \mathbb{A}^2$ of class β (Definition 3.3). Since $\tilde{\mathfrak{X}}_Q$ is projective over $\mathbb{A}^1$ by projection to s, the moduli space $\mathscr{M}(\tilde{\mathfrak{X}}_Q,\beta)$ is proper over $\mathbb{A}^1$ by [GS5], Theorem 0.2. Figure 4.3 shows the fibers of a stable log map of degree 1 in $\mathscr{M}(\tilde{\mathfrak{X}}_Q,\beta)$ for $(\mathbb{P}^2, E)$.

Lemma 4.21. *Let $f : \mathfrak{C} \to \mathfrak{X}$ be a stable log map in $\mathscr{M}(\tilde{\mathfrak{X}}_Q,\beta)$. Then the fibers $f_t^0 : C_t^0 \to X_t^0$ map entirely to the divisor D_t^0.*

Proof. Suppose there is an irreducible component of C_t^0 not mapped to D_t^0. Then, since the marked point is mapped to D_t^0, there exists an irreducible component of C_t^0 that is not mapped onto D_t^0 and is not contracted to a point. But then the tropicalization of f_t^0 must have at least two legs, since the balancing condition implies balancing of the legs. This means C_t^0 must have at least two marked points, in contradiction with the definition of β (Definition 3.3). □

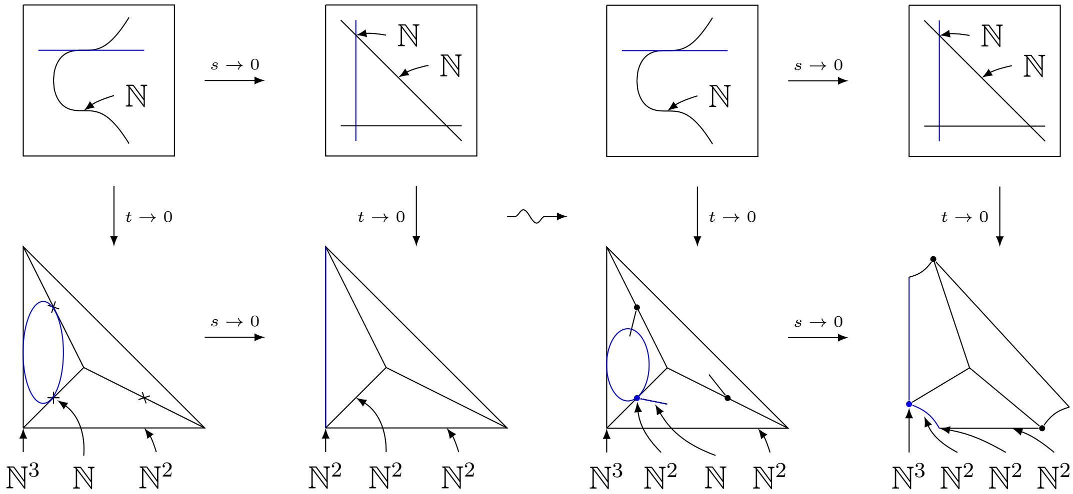


Figure 4.3: Fibers of a stable log map of degree 1 to $\tilde{\mathfrak{X}}_Q \to \mathbb{A}^2$ for $(\mathbb{P}^2, E)$ (right) and their image on $\mathfrak{X}_Q \to \mathbb{A}^2$ (left) under the resolution from §3. Some stalks of the ghost sheaves are given.

Definition 4.22. Let $\Gamma_{D_t^0}$ be the dual intersection graph of D_t^0. This is a cycle with r vertices. Let $\mathfrak{G}_d$ be the set of graph morphisms $g : \Gamma \to \Gamma_{D_t^0}$ where Γ is a tree (genus 0 graph) with vertices V decorated by $d_V \in \mathbb{N}_{>0}$ such that $\sum_V d_V = d$.

Construction 4.23. Note that projection to the unique unbounded direction defines a map $B \to \Gamma_{D_t^0}$, where vertices of $\Gamma_{D_t^0}$ correspond to unbounded edges of B. Define a surjective map

$$\mathfrak{H}_d \to \mathfrak{G}_d$$

by composing $h : \Gamma \to B$ with this projection and defining the label d_V at a vertex V as follows. Let $h : \Gamma \to B$ be a tropical curve in $\mathfrak{H}_d$. Give Γ the structure of a rooted tree by defining the root vertex to be the vertex V_{out} of the unique unbounded leg E_{out}. Let V be a vertex of Γ and let $E_{V,\text{out}}$ be the edge connecting V with its parent, or $E_{V,\text{out}} = E_{\text{out}}$ if V is the root vertex V_{out}. Let $E_1, \ldots, E_r$ be the other edges of Γ adjacent to V. Then define

$$d_V = \varphi(u_{(V,E_{V,\text{out}})}) - \sum_{i=1}^{r} \varphi(-u_{(V,E_i)})$$

Let $f : C \to \mathfrak{X} := \mathfrak{X}^{s\neq 0}$ be a stable log map in $\mathscr{M}(\mathfrak{X}, \beta)$. Since all the fibers $\mathfrak{X}^s$ are isomorphic for $s \neq 0$ this gives a family of stable log maps over $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$.

Since $\mathscr{M}(\tilde{\mathfrak{X}}_Q, \beta)$ is proper this family can be uniquely completed to a family over $\mathbb{A}^2$. In other words, the limit of a stable log map in $\mathscr{M}(\mathfrak{X}, \beta)$ under $s \to 0$ is well defined.

Proposition 4.24. *Let $f : \mathfrak{C} \to \mathfrak{X}_Q$ be a stable log map with tropicalization $\tilde{h}$ mapping to $h \in \mathfrak{H}_d$ under the map from Construction 4.15. The limit of f with respect to the family $\mathfrak{X}_t^0 \to \mathbb{A}^1$ has dual graph that is given by the image of h under the map from Construction 4.23.*

Proof. Let $f : \mathfrak{C} \to \mathfrak{X}_Q$ be a stable log map with tropicalization $\tilde{h}$ mapping to $h : \Gamma \to B$. Consider the fiber $f_0^0 : C_0^0 \to X_0^0$. If a vertex V of Γ is mapped to a vertex v of $\mathscr{P}$ or the unbounded edge adjacent to v, then the corresponding irreducible component C_V of C_0^0 is mapped to the irreducible component X_v of X_0^0 corresponding to v. But then, for $t \neq 0$, the corresponding irreducible component C_V of C_t^0 is mapped to the irreducible component D_v of D_t^0 corresponding to v. This is the image of V under the map from Construction 4.23. The map $f_t^0 : C_t^0 \to D_v = \mathbb{P}^1$ is a multiple cover of a line. Its degree is precisely the label d_V of V as above. $\square$

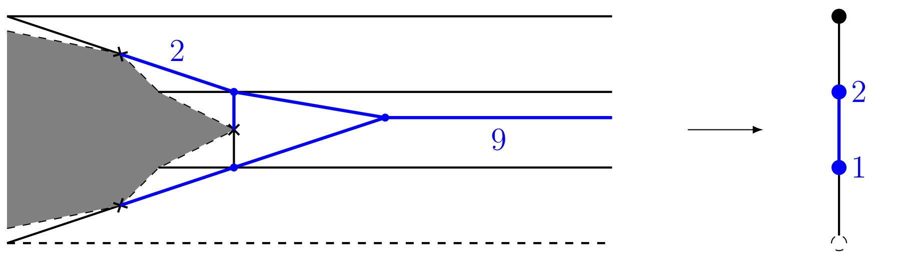


Figure 4.4: A tropical curve with degree splitting $[2, 1]$.

For $s = 0$ fix a cyclic labelling of the cycle of lines $D_t^0 = D_1 + \ldots + D_k$ and write $v_1, \ldots, v_r$ for the corresponding vertices of $\Gamma_{D_t^0}$.

Definition 4.25. Given $g : \Gamma \to \Gamma_{D_t^0}$ in $\mathfrak{G}_d$ write

$$d_i := \sum_{\substack{V \in \Gamma^{[0]} \\ g(V) = v_i}} d_V.$$

Then the collection $[d_1, \ldots, d_k]$ gives the degrees of the curve over the lines $D_i = \mathbb{P}^1$. If all components of D_t^0 are isomorphic as divisors of X_t^0, e.g. for $X = \mathbb{P}^2$ or $X = \mathbb{P}^1 \times \mathbb{P}^1$, we omit all zeros in this collection. We call this collection the *degree splitting* corresponding to g or any tropical curve h mapping to g under the map from Construction 4.23. For example, the tropical curve for $(\mathbb{P}^2, E)$ in Figure 4.4 has degree splitting $[2, 1, 0]$ for any choice of labelling of $D_t^0 = D_1 + D_2 + D_3$, and we simply write this as $[2, 1]$.

Construction 4.26. Define a surjective map

$$\mathfrak{G}_d \to H_2^+(X, \mathbb{Z})$$

by sending an element $g : \Gamma \to \Gamma_{D_t^0}$ of $\mathfrak{G}_d$ with degree splitting $[d_1, \ldots, d_k]$ to the curve class $\underline{\beta}$ defined by

$$D_i \cdot \underline{\beta} = d_i.$$

This is well-defined by the balancing condition and since $H_2^+(X, \mathbb{Z}) \simeq H_2^+(X^0, \mathbb{Z})$, where X^0 is a toric variety.

Definition 4.27. Let $\mathfrak{H}_\beta$ be the set of tropical curves mapping to $\underline{\beta}$ under the decomposition of the maps from Constructions 4.23 and 4.26. Let $\tilde{\mathfrak{H}}_\beta$ be the preimage of $\mathfrak{H}_\beta$ under the map from Construction 4.15.

A direct consequence of Proposition 4.24 is the following.

Corollary 4.28. *The tropicalization of a stable log map of class β is in $\tilde{\mathfrak{H}}_\beta$.*

4.5 Refinement and logarithmic modification

To apply the degeneration formula in §5 we need a degeneration of (X, D) such that all stable log maps to the central fiber are torically transverse. We achieve this as follows.

Construction 4.29 (The refined degeneration)**.** Let $\mathscr{P}_d$ be a refinement of $\mathscr{P}$ such that each tropical curve in $\mathfrak{H}_{\leq d} = \cup_{d' \leq d} \mathfrak{H}_{d'}$ (or equivalently in $\tilde{\mathfrak{H}}_{\leq d}$) is contained in the 1-skeleton of $\mathscr{P}_d$. This is well-defined by finiteness of $\mathfrak{H}_d$ (Corollary 4.19) and defines a refinement of the generalized cone complex $\Sigma(\tilde{X}_0)$ by taking cones over cells of $\mathscr{P}_d$. In turn, $\mathscr{P}_d$ induces a logarithmic modification $\tilde{\mathfrak{X}}_d \to \mathbb{A}^1$ of $\tilde{\mathfrak{X}} \to \mathbb{A}^1$ (see §1.9) without changing the generic fiber. By making a base change $\mathbb{A}^1 \to \mathbb{A}^1, t \mapsto t^e$ we can scale $\mathscr{P}_d$ and thus assume it has integral vertices (c.f. [NS], Proposition 6.3).

Remark 4.30. The dual intersection complex of the central fiber Y of $\tilde{\mathfrak{X}}_d$ is given by $(\tilde{B}, \mathscr{P}_d, \varphi)$. Hence, all stable log maps to $Y \to \mathrm{pt}_{\mathbb{N}}$ of class β as in Definition 3.3 are torically transverse, since their tropicalizations are contained in the 1-skeleton of $\mathscr{P}_d$, with vertices mapping to vertices of $\mathscr{P}_d$ (see [MR], Proposition 4.6).

It was shown in [AW] that Gromov-Witten invariants are invariant under logarithmic modifications. Hence,

$$N_\beta = \int_{[\![\mathscr{M}(Y,\beta)]\!]} 1.$$

In the next section we will apply the degeneration formula of logarithmic Gromov-Witten theory to get a formula for N_d in terms of logarithmic Gromov-Witten invariants of irreducible components of Y.

5 The degeneration formula

Consider a projective semi-stable degeneration $\pi : \mathfrak{X} \to T = \operatorname{Spec} R$, for R a discrete valuation ring. This is a projective surjection such that the generic fiber is smooth and the fiber $X = \pi^{-1}(0)$ over the closed point $0 \in T$ is simple normal crossings with two smooth connected (hence irreducible) components X_1, X_2 meeting in a smooth connected divisor D. In the logarithmic language this means that π is log smooth when T and $\mathfrak{X}$ carry the divisorial log structures given by $0 \in T$ and $X \subseteq \mathfrak{X}$, respectively. The degeneration formula relates invariants (relative or logarithmic Gromov-Witten invariants) on the generic fiber of $\pi : \mathfrak{X} \to T$ to invariants on the components X_1, X_2 of the special fiber.

The degeneration formula was proved for stable relative maps, in symplectic geometry [LR][IP] and in algebraic geometry [Li2][AF], as well as for stable log maps using expanded degenerations [Che2]. A pure log-geometric version avoiding the target expansions of relative Gromov-Witten theory was worked out by Kim, Lho and Ruddat [KLR] using logarithmic Gromov-Witten theory [GS5].

The formula in [KLR] is stated in the setup above, with two smooth irreducible components X_1 and X_2. In our setup we have several irreducible components, indexed by the vertices of a tropical curve. One could generalize the formula in [KLR] to the case where X_1, X_2 are only log smooth, in particular they might be reducible, and then apply it repeatedly to get a formula for several components. For the sake of self-containedness we will take a different approach and prove the formula more explicitly in our setting, only referring to [KLR] for some general statements.

Fix an integer $d > 0$ and let $\tilde{\mathfrak{X}}_d \to \mathbb{A}^1$ be the refined log smooth degeneration of (X, D) from Construction 4.29. Write the central fiber as Y and let β be a class of stable log maps as in Definition 3.9 with $D \cdot \underline{\beta} \leq dw_{\text{out}}$. Let Y° be the complement of zero-dimensional toric strata in Y and write

$$\mathscr{M}_\beta := \mathscr{M}(Y, \beta).$$

Since $\mathscr{M}(Y^\circ, \beta)$ is canonically isomorphic to the moduli space of torically transverse stable log maps to Y of class β and all such maps are torically transverse (Remark 4.30), the canonical inclusion gives an isomorphism $\mathscr{M}_\beta \cong \mathscr{M}(Y^\circ, \beta)$.

For a vertex $v \in \mathscr{P}_d^{[0]}$, let Y_v° be the complement of the 0-dimensional toric strata of the irreducible component Y_v of Y corresponding to v. Then Y° is a union of finitely many log smooth schemes Y_v° over $\mathrm{pt}_{\mathbb{N}}$, with $Y_v^\circ \cap Y_{v'}^\circ = \emptyset$ if there is no edge connecting v and v', and $D_E^\circ := Y_v^\circ \cap Y_{v'}^\circ$ log smooth and a divisor of both Y_v° and $Y_{v'}^\circ$ if there is an edge E connecting v and v'. The intersection of any triple of components is empty. Hence, we can apply the degeneration formula.

5.1 Toric invariants

We introduce logarithmic Gromov-Witten invariants of toric varieties with point conditions on the toric boundary, following [GPS].

Let $M \simeq \mathbb{Z}^2$ be a lattice and let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ be the associated vector space. Let $(m_1, \ldots, m_n)$ be an n-tuple of distinct nonzero primitive vectors in M and let $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ be an n-tuple of weight vectors $\mathbf{w}_i = (w_{i1}, \ldots, w_{il_i})$ with $l_i > 0$, $w_{ij} \in \mathbb{N}$ such that

$$\sum_{i=1}^{n} |\mathbf{w}_i| m_i = w_{\mathrm{out}} m_{\mathrm{out}}$$

for $0 \neq m_{\mathrm{out}} \in M$ primitive and $w_{\mathrm{out}} > 0$. Here $|\mathbf{w}_i| := \sum_{j=1}^{l_i} w_{ij}$. Let Σ be the complete rational fan in $M_{\mathbb{R}}$ whose rays are generated by $-m_1, \ldots, -m_n, m_{\mathrm{out}}$ and let X_Σ be the corresponding toric surface over $\mathbb{C}$. By refining Σ if necessary, we can assume that X_Σ is nonsingular. Let $D_1, \ldots, D_n, D_{\mathrm{out}} \subseteq X_\Sigma$ be the toric divisors corresponding to the given rays. Let X_Σ° be the complement of the 0-dimensional torus orbits in X_Σ, and let $D_i^\circ = D_i \cap X_\Sigma^\circ$, $D_{\mathrm{out}}^\circ = D_{\mathrm{out}} \cap X_\Sigma^\circ$. Then define a class $\beta_{\mathbf{w}}$ of stable log maps to X_Σ as follows.
(1) genus $g = 0$;
(2) $k = l_1 + \ldots + l_n + 1$ marked points $p_{ij}, i = 1, \ldots, n, j = 1, \ldots, l_i$ and p;
(3) $\underline{\beta}_{\mathbf{w}} \in H_2(X_\Sigma, \mathbb{Z})$ defined by intersection numbers with toric divisors,

$$D_i \cdot \underline{\beta}_{\mathbf{w}} = |\mathbf{w}_i|, \quad D_{\mathrm{out}} \cdot \underline{\beta}_{\mathbf{w}} = w_{\mathrm{out}};$$

(4) contact data $u_{p_{ij}} = w_{ij} m_i$ and $u_p = w_{\mathrm{out}} m_{\mathrm{out}}$.
By restriction we get a class of stable log maps to X_Σ° that we also denote by $\beta_{\mathbf{w}}$. The moduli space $\mathscr{M}(X_\Sigma^\circ, \beta_{\mathbf{w}})$ in general is not proper, since X_Σ° is not proper. However, the evaluation map

$$\mathrm{ev}^\circ : \mathscr{M}(X_\Sigma^\circ, \beta_{\mathbf{w}}) \to \prod_{i=1}^{n} (D_i^\circ)^{l_i}$$

is proper ([GPS], Proposition 4.2) and we obtain a proper moduli space via base change to a point. To be precise, let $\gamma : \mathrm{Spec}\ \mathbb{C} \to \prod_{i=1}^{n} (D_i^\circ)^{l_i}$ be a point. Then

$$\mathscr{M}_\gamma := \mathrm{Spec}\ \mathbb{C} \times_{\prod_{i=1}^{n} (D_i^\circ)^{l_i}} \mathscr{M}(X_\Sigma^\circ, \beta_{\mathbf{w}})$$

is a proper Deligne-Mumford stack admitting a virtual fundamental class, and we can define the logarithmic Gromov-Witten invariant

$$N_{\mathbf{m}}(\mathbf{w}) := \int_{\mathscr{M}_\gamma} \gamma^! [\![\mathscr{M}(X_\Sigma^\circ, \beta_{\mathbf{w}})]\!]. \tag{5.1}$$

Since the codimension of γ equals the virtual dimension of $\mathscr{M}(X_\Sigma^\circ, \beta_{\mathbf{w}})$, this definition makes sense. Note that we may add further primitive vectors m_i to $\mathbf{m}$, with weight vectors $\mathbf{w}_i = 0$. This leads to a subdivision of Σ, hence to a toric blow up of X_Σ, but the logarithmic Gromov-Witten invariants do not change.

5.2 The decomposition formula

By the decomposition formula for stable log maps ([ACGS1], Theorem 1.2), the moduli space $\mathscr{M}_\beta$ decomposes into moduli spaces indexed by certain decorated tropical curves. Here decorated means that there are classes of stable log maps β_V attached to the vertices. In this section we show that a tropical curve in $\tilde{\mathfrak{H}}_d$ automatically carries such decorations.

Proposition 5.1. *Let $f : C/\mathrm{pt}_{\mathbb{N}} \to Y/\mathrm{pt}_{\mathbb{N}}$ be a stable log map in $\mathscr{M}_\beta$ with tropicalization $\tilde{h} : \tilde{\Gamma} \to \tilde{B}$. For each vertex $V \in \tilde{\mathscr{P}}_d^{[0]}$, the class $[C_V] \in H_2^+(Y_{\tilde{h}(V)}, \mathbb{Z})$ is uniquely determined by the intersection numbers of C_V with components of $\partial Y_{\tilde{h}(V)}$, i.e., by $\tilde{h}$.*

Proof. If V is of type (I) as in Definition 4.11, then $Y_{\tilde{h}(V)}$ is a toric variety, so the statement is true. If V is of type (II), then C_V is a multiple cover of some exceptional line L_v^{exc} (Definition 3.2). Its intersection with ∂Y_v determines the degree d of the multiple cover, hence the curve class $[C_V] = d[L_v^{\mathrm{exc}}] \in H_2^+(Y_v, \mathbb{Z})$. Let V be a vertex of type (III). It is mapped to a vertex v of $\mathscr{P}$. Let X_v be the corresponding component of X_0. This is a toric variety. By Proposition 4.10, (III), we know the intersection of the image of C_V under the resolution $\nu : \tilde{\mathfrak{X}} \to \mathfrak{X}$ from §3 with the toric divisors of X_v, hence the curve class $[\nu(C_V)] \in H_2^+(X_v, \mathbb{Z})$. But this determines $[C_V] = [\nu(C_V)] - k[L_v^{\mathrm{exc}}] \in H_2^+(Y_v, \mathbb{Z})$, where k is as in Proposition 4.10, (III). □

Definition 5.2. For $\tilde{h} : \tilde{\Gamma} \to \tilde{B}$ in $\tilde{\mathfrak{H}}_d$, let $\mathscr{M}_{\tilde{h}}$ be the moduli space of stable log maps with tropicalization $\tilde{h}$. This is proper by [ACGS1], Proposition 2.34.

Remark 5.3. In fact [ACGS1] deals with moduli spaces $\mathscr{M}_{\tilde{\tau}}$ of stable log maps *marked by* $\tilde{\tau} = (\tau, \mathbf{A})$, where τ is a type of tropical maps and $\mathbf{A}$ is a vertex decoration by curve classes. Since the virtual dimension of $\mathscr{M}_\beta$ is zero and tropical curves in $\tilde{\mathfrak{H}}_d$ are rigid, such $\tilde{\tau}$ are in bijection with vertex decorated tropical curves. We showed in Proposition 5.1 that tropical curves in $\tilde{\mathfrak{H}}_d$ carry unique vertex decorations. So $\tilde{\tau}$ uniquely defines a tropical curve $\tilde{h}$ and $\mathscr{M}_{\tilde{\tau}}$ equals $\mathscr{M}_{\tilde{h}}$.

Proposition 5.4 (Decomposition formula)**.**

$$[\![\mathscr{M}_\beta]\!] = \sum_{\tilde{h}\in\tilde{\mathfrak{H}}_\beta} \frac{l_{\tilde{\Gamma}}}{|\mathrm{Aut}(\tilde{h})|} F_\star[\![\mathscr{M}_{\tilde{h}}]\!],$$

where $l_{\tilde{\Gamma}} := \mathrm{lcm}\{w_E \mid E \in E(\tilde{\Gamma})\}$ *and* $F : \mathscr{M}_{\tilde{h}} \to \mathscr{M}_\beta$ *is the forgetful map. Here* $Aut(\tilde{h})$ *is the group of automorphisms of* $\tilde{h}$ *(Definition 4.3).*

Proof. The decomposition formula ([ACGS1], Theorem 1.2) gives $[\![\mathscr{M}_\beta]\!]$ as a sum over decorated types of tropical maps $\tilde{\tau}$. By Remark 5.3 this is a summation over $\tilde{\mathfrak{H}}_\beta$. The multiplicity m_τ in [ACGS1], Theorem 1.2, is defined as the index of the image of the lattice $\Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathbb{N}$ inside the lattice $\Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathbb{N}$. Here the first $\mathrm{pt}_{\mathbb{N}}$ is the base of the curve while the second $\mathrm{pt}_{\mathbb{N}}$ is the base of Y. In other words, m_τ is the smallest integer such that scaling $\tilde{B}$ by m_τ leads to a tropical curve with integral vertices and edge lengths. By Construction 4.29 $\mathscr{P}_d$ has integral vertices (by the base change $t \mapsto t^e$) and tropical curves in $\tilde{\mathfrak{H}}_\beta$ are contained in the 1-skeleton of $\mathscr{P}_d$ with vertices mapping to vertices of $\mathscr{P}_d$. The affine length of the image of an edge $E \in E(\tilde{\Gamma})$ is $\ell_E w_E$ by Definition 4.1, (i). So the scaling necessary to obtain integral edge lengths is $m_\tau = l_{\tilde{\Gamma}} := \mathrm{lcm}\{w_E \mid E \in E(\tilde{\Gamma})\}$. Moreover $\mathrm{Aut}(\tau) = \mathrm{Aut}(\tilde{h})$ by our definition of automorphisms (see Definition 4.3). Then [ACGS1], Theorem 1.2, gives the formula above. □

5.3 Contributions of the vertices

Let $\tilde{h} : \tilde{\Gamma} \to \tilde{B}$ be a tropical curve in $\tilde{\mathfrak{H}}_d$. Define

$$\mathscr{M}_V^\circ := \mathscr{M}(Y^\circ_{\tilde{h}(V)}, \beta_V),$$

where $Y^\circ_{\tilde{h}(V)}$ is the complement of the 0-dimensional toric strata in $Y_{\tilde{h}(V)}$.

For $V \in V_{II}(\tilde{\Gamma})$ (Definition 4.11) with adjacent edge E, the moduli space $\mathscr{M}_V^\circ$ is proper, since it is isomorphic to the moduli space of w_E-fold multiple covers of $\mathbb{P}^1$ totally ramified at a point.

For $V \in V(\tilde{\Gamma}) \setminus V_{II}(\tilde{\Gamma})$ we obtain a proper moduli space as follows. Again, $\tilde{\Gamma}$ is a rooted tree with root vertex V_{out}. There is a natural orientation of the edges of $\tilde{\Gamma}$ by choosing edges to point from a vertex to its parent. For each vertex $V \in V(\tilde{\Gamma}) \setminus V_{II}(\tilde{\Gamma})$ there is an evaluation map

$$\mathrm{ev}^\circ_{V,-} : \mathscr{M}_V^\circ \to \prod_{E\to V} D_E^\circ,$$

where the product is over all edges of $\tilde{\Gamma}$ adjacent to V and pointing towards V.

Lemma 5.5. *The evaluation map* $\mathrm{ev}^\circ_{V,-}$ *is proper.*

Proof. For $V \in V_I(\tilde{\Gamma})$ this is [GPS], Proposition 4.2. For $V \in V_{III}(\tilde{\Gamma})$ it is similar to [GPS], Proposition 5.1. Let us carry this out. Let $V \in V(\tilde{\Gamma})$ be a vertex. We use the valuative criterion for properness, so let R be a valuation ring with residue field K, and suppose we are given a diagram

$$\begin{tikzcd} T = \mathrm{Spec}\ K \arrow[r] \arrow[d] & \mathscr{M}_V^\circ \arrow[r] \arrow[d, "\mathrm{ev}_{V,-}^\circ"] & \mathscr{M}_V \arrow[d, "\mathrm{ev}_{V,-}"] \\ S = \mathrm{Spec}\ R \arrow[r] & \prod_{E \to V} D_E^\circ \arrow[r] & \prod_{E \to V} D_E \end{tikzcd}$$

Since $\mathscr{M}_V$ is proper, $\mathrm{ev}_{V,-}$ is proper and we obtain a unique family of stable log maps

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & Y_{\tilde{h}(V)} \times S \arrow[d] \\ S \arrow[r, "="] & S \end{tikzcd}$$

We will show that f is a family of stable log maps to $Y^\circ_{\tilde{h}(V)}$. Let $0 \in S$ be the closed point and consider $f_0 : C_0 \to Y_{\tilde{h}(V)}$. The marked points of C_0 map to $Y^\circ_{\tilde{h}(V)}$.

Suppose $f_0(C_0)$ intersects a toric divisor $D \subset Y_{\tilde{h}(V)}$ at a point of $D \setminus Y^\circ_{\tilde{h}(V)}$. The intersection number of C_0 with D is accounted for in $Y^\circ_{\tilde{h}(V)}$. For $V \in V_I(\tilde{\Gamma})$ this is clear and for $V \in V_{III}(\tilde{\Gamma})$ this follows since the intersection number is accounted for after composing with the resolution $\tilde{\mathfrak{X}} \to \mathfrak{X}$ from §3. Hence, there must be an irreducible component C of C_0 dominating D. Let D_1 and D_2 be the two distinct toric divisors of $Y_{\tilde{h}(V)}$ intersecting D only at two distinct torus fixed points of D. It was shown in [GPS], Proposition 4.2, that there are irreducible components $C_1, C_2 \subset C_0$ intersecting C and dominating E_1 and E_2, respectively. By applying this statement repeatedly, replacing C with C_1 or C_2, we find that C_0 contains a cycle of components dominating the union of toric divisors of $Y_{\tilde{h}(V)}$. But then C_0 would have genus $g > 0$, contradicting the assumptions. We have shown by contradiction that $f : \mathcal{C} \to Y_{\tilde{h}(V)} \times S$ is a family of stable log maps to $Y^\circ_{\tilde{h}(V)}$. Hence, $\mathrm{ev}^\circ_{V,-}$ is proper by the valuative criterion for properness. □

Since properness of morphisms is stable under base change, we obtain a proper moduli space by base change to a point

$$\gamma_V : \mathrm{Spec}\ \mathbb{C} \to \prod_{E \to V} D_E^\circ,$$

that is,

$$\mathscr{M}_{\gamma_V} := \mathrm{Spec}\ \mathbb{C} \times_{\prod_{E\to V} D_E^\circ} \mathscr{M}_V^\circ$$

is a proper Deligne-Mumford stack.

Lemma 5.6. *For $V \in V_{II}(\tilde{\Gamma})$ the virtual dimension of $\mathscr{M}_V$ is zero. Otherwise the virtual dimension of $\mathscr{M}_V$ equals the codimension of γ_V.*

Proof. For $V \in V_{II}(\tilde{\Gamma})$ with adjacent edge E the moduli space $\mathscr{M}_V$ is isomorphic to the moduli space of w_E-fold multiple covers of $\mathbb{P}^1$ totally ramified at a point. However, the two moduli spaces carry obstruction theories which differ by $H^1(C, f^\star\mathcal{O}_{\mathbb{P}^1}(-1))$ at a moduli point $[f : C \to L_V^{\mathrm{exc}}]$ (c.f. [GPS], §5.3). The rank of $H^1(C, f^\star\mathcal{O}_{\mathbb{P}^1}(-1))$ is $w_E - 1$ and so is the virtual dimension of $\mathscr{M}(\mathbb{P}^1/\infty, w_E)$. Hence, the virtual dimension of $\mathscr{M}_V$ is zero.

Otherwise, the virtual dimension is easily seen to be the number of edges of $\tilde{\Gamma}$ pointing towards V, with orientation of $\tilde{\Gamma}$ as given above. By definition this is the codimension of γ_V. □

Definition 5.7. For a vertex V of $\tilde{\Gamma}$ define

$$N_V := \begin{cases} \int_{[\![\mathscr{M}_V^\circ]\!]} 1, & V \in V_{II}(\tilde{\Gamma}); \\ \int_{\mathscr{M}_{\gamma_V}} \gamma_V^! [\![\mathscr{M}_V^\circ]\!], & V \in V_I(\tilde{\Gamma}) \cup V_{III}(\tilde{\Gamma}). \end{cases}$$

This is a finite number by Lemma 5.6 and independent of γ_V by Lemma 5.5.

Proposition 5.8. *We give N_V for the different types of vertices (Definition 4.11).*

(I) For $V \in V_I(\tilde{\Gamma})$ let $e_1, \ldots, e_n$ be the edges of $\mathscr{P}_d$ adjacent to $\tilde{h}(V)$ and let $m_1, \ldots, m_n$ be the corresponding primitive vectors. Let $\boldsymbol{w}_i = (w_{i1}, \ldots, w_{il_i})$ be the weights of edges of $\tilde{\Gamma}$ mapping to e_i and write $\boldsymbol{w} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)$. Then N_V is the toric invariant from (5.1)

$$N_V = N_{\boldsymbol{m}}(\boldsymbol{w}),$$

(II) If $V \in V_{II}(\tilde{\Gamma})$, then

$$N_V = \frac{(-1)^{w_E - 1}}{w_E^2},$$

where E is the unique edge adjacent to V.

(III) If $V \in V_{III}(\tilde{\Gamma})$, then

$$N_V = \sum_{\boldsymbol{w}_{V,+}} \frac{N_{\boldsymbol{m}}(\boldsymbol{w})}{|\mathrm{Aut}(\boldsymbol{w}_{V,+})|} \prod_{i=1}^{l} \frac{(-1)^{w_{V,i}-1}}{w_{V,i}}.$$

The sum is over all weight vectors $\boldsymbol{w}_{V,+} = (w_{V,1}, \ldots, w_{V,l_V})$ such that $|\boldsymbol{w}_{V,+}| := \sum_{i=1}^{l_V} w_{V,i} = k$, with k as in Proposition 4.10, (III). Further, $N_{\boldsymbol{m}}(\boldsymbol{w})$ is as in (5.1) *with $\boldsymbol{m} = (m_{v,-}, m_{v,+})$ and $\boldsymbol{w} = ((w_E)_{E \in E_{V,-}(\tilde{\Gamma})}, \boldsymbol{w}_{V,+})$, where $E_{V,-}$ is the set of edges adjacent to V and mapped to direction $m_{v,-}$.*

Proof. (I) is by the definition of $N_{\mathbf{m}}(\mathbf{w})$ in (5.1). For (II) recall from the proof of Lemma 5.6 that $\mathscr{M}_V$ is isomorphic to $\mathscr{M}(\mathbb{P}^1/\infty, w_E)$ with obstruction theory differing by $H^1(C, f^\star\mathcal{O}_{\mathbb{P}^1}(-1))$. Hence,

$$N_V = \int_{[\![\mathscr{M}(\mathbb{P}^1/\infty, w_E)]\!]} e(H^1(C, f^\star\mathcal{O}_{\mathbb{P}^1}(-1)))$$

which is equal to $(-1)^{w_E-1}/w_E^2$ by the genus zero part of [BP], Theorem 5.1, see also [GPS], Propositions 5.2 and 6.1. For (III) we apply [GPS], Proposition 5.3. We only blow up one point on the divisor $D_{v,+}$, so in the notation of [GPS], Proposition 5.3, we have $\mathbf{P} = (P_+)$ with $P_+ = k$ as in Proposition 4.10, (III). Note that our i is called j in [GPS] and the i of [GPS] is equal to 1 here. Further, $R_{\mathbf{P}_+|\mathbf{w}_+} = \prod_{i=1}^{l_i} \frac{(-1)^{w_i-1}}{w_i^2}$ by [GPS], Proposition 5.2, and the discussion thereafter. Then [GPS], Proposition 5.3, gives (III). $\square$

5.4 Gluing

Define $\bigtimes_{V\in V(\tilde{\Gamma})} \mathscr{M}_V$ to be the moduli space of stable log maps in $\prod_V \mathscr{M}_V$ matching over the divisors D_E, $E \in E(\tilde{\Gamma})$, i.e., the fiber product

$$\begin{tikzcd}
\bigtimes_V \mathscr{M}_V \arrow[r] \arrow[d] & \prod_V \mathscr{M}_V \arrow[d, "\mathrm{ev}"] \\
\prod_{E\in E(\tilde{\Gamma})} D_E \arrow[r, "\delta"] & \prod_V \prod_{\substack{E\in E(\tilde{\Gamma})\\ V\in E}} D_E
\end{tikzcd}$$

Here ev is the product of evaluation maps to common divisors (labeled by compact edges) and δ is the diagonal map. Similarly define $\bigtimes_V \mathscr{M}_V^\circ$.

Definition 5.9. Let cut : $\mathscr{M}_{\tilde{h}} \to \bigtimes_V \mathscr{M}_V$ be the morphism defined by cutting a curve along its gluing nodes. For a precise definition see [Bou1], §7.1. Here $\mathscr{M}_{\tilde{h}}$ denotes the moduli space of stable log maps with tropicalization $\tilde{h}$ (Definition 5.2). Since every stable log map in $\mathscr{M}_{\tilde{h}}$ is torically transverse (Remark 4.30) this is in fact a morphism

$$\mathrm{cut} : \mathscr{M}_{\tilde{h}} \to \bigtimes_V \mathscr{M}_V^\circ.$$

Lemma 5.10. *The morphism cut is étale of degree*

$$\deg(\mathrm{cut}) = \frac{\prod_{E\in E(\tilde{\Gamma})} w_E}{l_{\tilde{\Gamma}}},$$

where $l_{\tilde{\Gamma}} = \mathrm{lcm}\{w_E\}$.

Proof. Since stable log maps in $\mathscr{M}_{\tilde{h}}$ are torically transverse by Construction 4.29, locally we are gluing along a smooth divisor as in [KLR]. Then the statement is [KLR], Lemma 9.2 (4), and the degree is computed by [KLR], (6.13). We will briefly explain how to arrive at this expression.

For each edge E we have a choice of w_E-th root of unity in the log structure of C at the corresponding node, contributing a factor of w_E to deg(cut). This was computed e.g. in [NS], Proposition 7.1, [Gro2], Proposition 4.23, and [Bou1], Proposition 18. In fact its is a bit more involved: In the definition of tropicalization we removed some of the bivalent vertices from the dual intersection graph Γ_C. For each compact edge E in Γ_C we have a choice of w_E-th root of unity as follows. Locally at a node q we have $C = \text{Spec } \mathbb{C}[u,v]/(uv)$ and $\tilde{\mathfrak{X}}_d = \text{Spec } \mathbb{C}[x,y,w^{\pm 1},t]/(xy-t^\ell)$ for some $\ell \in \mathbb{Z}_{>0}$, so $Y := \tilde{X}_{d,0} = \text{Spec } \mathbb{C}[x,y,w^{\pm 1}]/(xy)$. Locally at q a log structure on C is given by a commutative diagram

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r, "f^\#"] \arrow[d, "\alpha_Y"] & \mathcal{M}_C \arrow[d, "\alpha_C"] \\ f^{-1}\mathcal{O}_Y \arrow[r, "f^\star"] & \mathcal{O}_C \end{tikzcd}$$

For any w_E-th root of unity ζ there is a chart for $\mathcal{M}_C$ locally at q given by

$$S_\ell \to \mathcal{O}_C, ((a,b,c)) \mapsto \begin{cases} (\zeta^{-1}u)^a v^b & c = 0, \\ 0 & c \neq 0. \end{cases}$$

Here $S_\ell = \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}$ with $\mathbb{N} \to \mathbb{N}^2$ the diagonal embedding and $\mathbb{N} \to \mathbb{N}, 1 \mapsto \ell$ (see Construction 4.6, (2)). None of these choices are identified via a scheme theoretically trivial isomorphism and all possible extensions are of the above form (see [Gro2], Proposition 4.23, Step 2).

Now consider a chain of edges of Γ_C connected by bivalent vertices not mapping to vertices of $\mathscr{P}_d$. Then these bivalent vertices get removed by producing the tropicalization and the chain of edges is replaced by a single edge E. In this case there are some isomorphisms between the above stable log maps that are not scheme theoretically trivial. Up to such isomorphisms there are exactly w_E stable log maps (see [Gro2], Proposition 4.23, Step 3). So we really only get one factor of w_E for each edge in the tropicalization. The log structure at general points and marked points is uniquely determined (see [Gro2], Proposition 4.23, Step 1). This gives the nominator of deg(cut) as above.

There are further isomorphisms of stable log maps given by the action of l_Γ-th roots of unity on the base of the curves (see [KLR], discussion before (6.13). This gives the denominator of deg(cut) as above. □

Proposition 5.11 (Gluing formula)**.** *We have*

$$\mathrm{cut}_\star[\![\mathscr{M}_{\tilde h}]\!] = \frac{1}{\ell_{\tilde\Gamma}} \prod_{E\in E(\tilde\Gamma)} w_E \cdot \delta^! \prod_{V\in V(\tilde\Gamma)} [\![\mathscr{M}_V^\circ]\!].$$

Proof. By compatibility of obstruction theories (see [KLR], §9 and [Bou1], §7.3) we have

$$[\![\mathscr{M}_{\tilde h}]\!] = \mathrm{cut}^\star \delta^! \prod_{V\in V(\tilde\Gamma)} [\![\mathscr{M}_V^\circ]\!].$$

By the projection formula, $\mathrm{cut}_\star\mathrm{cut}^\star$ is multiplication with $\deg(\mathrm{cut})$ which is $\frac{1}{\ell_{\tilde\Gamma}}\prod_{E\in E(\tilde\Gamma)} w_E$ by Lemma 5.10. $\square$

Proposition 5.12. *We have*

$$\int_{[\![\mathscr{M}_{\tilde h}]\!]} 1 = \frac{1}{\ell_{\tilde\Gamma}} \prod_{E\in E(\tilde\Gamma)} w_E \cdot \int_{\delta^! \prod_V [\![\mathscr{M}_V]\!]} 1.$$

Proof. By Proposition 5.11 the cycles $\mathrm{cut}_\star[\![\mathscr{M}_{\tilde h}]\!]$ and $\frac{1}{\ell_{\tilde\Gamma}}\prod_{E\in E(\tilde\Gamma)} w_E \cdot \delta^! \prod_V [\![\mathscr{M}_V]\!]$ have the same restriction to the open substack $\bigtimes_V \mathscr{M}_V^\circ$ of $\bigtimes_V \mathscr{M}_V$. Hence by [Ful2], Proposition 1.8, their difference is rationally equivalent to a cycle supported on the closed substack $Z := (\bigtimes_V \mathscr{M}_V) \setminus (\bigtimes_V \mathscr{M}_V^\circ)$. Suppose there exists an element $(f_V : C_V \to Y_{\tilde h(V)})_{V\in V(\tilde\Gamma)} \in Z$. Then by the loop construction in the proof of Proposition 5.5 at least one of the source curves C_V would contain a nontrivial cycle of components, contradicting $g = 0$. So Z is empty, completing the proof. $\square$

Proposition 5.13 (Identifying the pieces)**.** *We have*

$$\int_{\delta^! \prod_V [\![\mathscr{M}_V]\!]} 1 = \prod_{V\in V(\tilde\Gamma)} N_V.$$

Proof. This is similar to the proof of [Bou1], Proposition 22. By definition of δ we have

$$\int_{\delta^! \prod_V [\![\mathscr{M}_V]\!]} 1 = \int_{\prod_V [\![\mathscr{M}_V]\!]} \mathrm{ev}^\star[\delta],$$

where $[\delta]$ is the class of the diagonal $\prod_E D_E$. Since each D_E is a projective line, we have

$$[\delta] = \prod_{E\in E(\tilde\Gamma)} (\mathrm{pt}_E \times 1 + 1 \times \mathrm{pt}_E).$$

As before we give $\tilde\Gamma$ the structure of a rooted tree by choosing the root vertex to be the vertex V_{out} of the unique unbounded leg E_{out}. For a bounded edge E let $V_{E,+}$ and $V_{E,-}$ be the vertices of E such that $V_{E,+}$ is the parent of $V_{E,-}$.

We will show by dimensional arguments that the only term of

$$\mathrm{ev}^\star[\delta] = \prod_{E\in E(\tilde{\Gamma})} \Big((\mathrm{ev}_{V_{E,-}})^\star[\mathrm{pt}_E] + (\mathrm{ev}_{V_{E,+}})^\star[\mathrm{pt}_E]\Big)$$

giving a nonzero contribution after integration over $\prod_V [\![\mathscr{M}_V]\!]$ is $\prod(\mathrm{ev}_{V_{E,+}})^\star[\mathrm{pt}_E]$. In other words:

Claim: For each compact edge E, a term of $\mathrm{ev}^\star[\delta]$ giving a nonzero contribution after integration over $\prod_V [\![\mathscr{M}_V]\!]$ does not contain a factor $(\mathrm{ev}_{V_{E,-}})^\star[\mathrm{pt}_E]$.

Let E be a compact edge with $V_{E,-}$ a vertex of type (II) as in Definition 4.11. By Proposition 5.8, (II), the virtual dimension of $\mathscr{M}_{V_{E,-}}$ is zero. So, $(\mathrm{ev}_{V_{E,-}})^\star[\mathrm{pt}_E] = 0$, since its insertion over $\mathscr{M}_{V_{E,-}}$ defines an enumerative problem of virtual dimension -1.

Now consider a compact edge E with $V_{E,-}$ of type (III). Let $E_i, i \in I$ be the edges adjacent to $V_{E,-}$ and different from E (possibly $I = \emptyset$). By Proposition 4.10, (III), the edges E_i connect $V_{E,-}$ with a vertex V_i of type (II). The terms in $\mathrm{ev}^\star[\delta]$ containing a factor $(\mathrm{ev}_{V_{E_i,-}})^\star[\mathrm{pt}_{E_i}]$ give zero after integration over $[\![\mathscr{M}_{V_i}]\!]$ by the dimensional argument above. Hence, to give a nonzero contribution, a term of $\mathrm{ev}^\star[\delta]$ must contain the factor $\prod_{i\in I}(\mathrm{ev}_{V_{E_i,+}})^\star[\mathrm{pt}_{E_i}]$. By Proposition 5.8, (III), the virtual dimension of $\mathscr{M}_V$ is $|I|$, so the insertion of $\prod_{i\in I}(\mathrm{ev}_{V_{E_i,+}})^\star[\mathrm{pt}_{E_i}]$ in $\mathscr{M}_V$ defines an enumerative problem of virtual dimension 0. Any further insertion would reduce the virtual dimension to -1, so a term of $\mathrm{ev}^\star[\delta]$ giving a nonzero contribution does not contain the factor $(\mathrm{ev}_{V_{E,-}})^\star[\mathrm{pt}_E]$.

We will show the claim for compact edges E with $V_{E,-}$ a vertex of type (I) by induction on the height of $V_{E,-}$, that is, the maximal length of chains connecting $V_{E,-}$ with a leaf of $\tilde{\Gamma}$. By Proposition 4.10, (I), a vertex of type (I) fulfills the ordinary balancing condition. In particular, it must have more than one adjacent edge, hence cannot be a leaf. This shows the set of leaves of $\tilde{\Gamma}$ is contained in $V_{II}(\tilde{\Gamma}) \cup V_{III}(\tilde{\Gamma})$. Thus we have already shown that the claim is true for compact edges E with $V_{E,-}$ of height 0. This is the base case. For the induction step assume that the claim is true for compact edges E with $V_{E,-}$ of height $\leq k$ for some $k \in \mathbb{N}$ and consider a compact edge E with $V_{E,-}$ of height $k+1$. Assume that $V_{E,+}$ is of type (I), since otherwise the claim is true by the above arguments. Let $E_i, i \in I$ be the edges connecting $V_{E,-}$ with its childs. By Proposition 5.8, (I), the virtual dimension of $\mathscr{M}_{V_{E,-}}$ is $|I|$. By the induction hypothesis, a term of $\mathrm{ev}^\star[\delta]$ giving a nonzero contribution must contain the factor $\prod_{i\in I}(\mathrm{ev}_{V_{E_i,+}})^\star[\mathrm{pt}_{E_i}]$. Inserting this factor over $\mathscr{M}_{V_{E,-}}$ gives an enumerative problem of virtual dimension 0. Again, for dimensional reasons, a term of $\mathrm{ev}^\star[\delta]$ giving a nonzero contribution cannot contain the factor $(\mathrm{ev}_{V_{E,-}})^\star[\mathrm{pt}_E]$, hence it must contain the factor $(\mathrm{ev}_{V_{E,+}})^\star[\mathrm{pt}_E]$. This proves the claim. Now

$$\int_{\prod_{V\in V(\tilde\Gamma)}[\![\mathscr{M}_V]\!]} \prod_{E\in E(\tilde\Gamma)} (\mathrm{ev}_{V_{E,+}})^\star[\mathrm{pt}_E] = \prod_{V\in V(\tilde\Gamma)} \int_{[\![\mathscr{M}_V]\!]} (\mathrm{ev}_{E\to V})^\star[\mathrm{pt}] = \prod_{V\in V(\tilde\Gamma)} N_V,$$

completing the proof. □

5.5 The degeneration formula

Proposition 5.14 (Degeneration formula)**.**

$$N_\beta = \sum_{\tilde h\in\tilde{\mathfrak{H}}_\beta} \frac{1}{|\mathrm{Aut}(\tilde h)|} \cdot \prod_{E\in E(\tilde\Gamma)} w_E \cdot \prod_{V\in V(\tilde\Gamma)} N_V.$$

Proof. Since the virtual dimension of $\mathscr{M}_\beta$ is zero, integration (i.e., proper pushforward to a point) of the decomposition formula (Proposition 5.4) gives

$$N_\beta = \sum_{\tilde h\in\tilde{\mathfrak{H}}_\beta} \frac{1}{|\mathrm{Aut}(\tilde h)|} \int_{[\![\mathscr{M}_{\tilde h}]\!]} 1.$$

Using Propositions 5.12 and 5.13 we get the above formula. □

As mentioned earlier, summation over balanced tropical curves in $\mathfrak{H}_\beta$ will give a more symmetric version of the above formula:

Definition 5.15. Let $h : \Gamma \to B$ be a tropical curve in $\mathfrak{H}_\beta$ and let V be a vertex of Γ. Then the image of V under the map from Construction 4.15 is a vertex of $\tilde\Gamma$ of type (I) or (III). Let $\mathbf{m}$ and $\mathbf{w}$ be as in the respective case of Proposition 5.8 and define

$$N_V^{\mathrm{tor}} := N_{\mathbf{m}}(\mathbf{w}).$$

Note that $N_V^{\mathrm{tor}} = N_V$ for vertices of type (I).

Definition 5.16. For a tropical curve $h : \Gamma \to B$ in $\mathfrak{H}_d$ for some d define

$$N_h := \left(\frac{1}{|\mathrm{Aut}(h)|} \cdot \prod_{E\in E(\Gamma)} w_E \cdot \prod_{E\in L_\Delta(\Gamma)} \frac{(-1)^{w_E-1}}{w_E} \cdot \prod_{V\in V(\Gamma)} N_V^{\mathrm{tor}} \right),$$

where $L_\Delta(\Gamma)$ is the set of bounded legs (see Definition 4.1).

Theorem 5.17 (Symmetric version of the degeneration formula)**.**

$$N_\beta = \sum_{h\in\mathfrak{H}_\beta} N_h.$$

Proof. Using Propositions 5.14 and 5.8, we have

$$\begin{aligned} N_d &= \sum_{\tilde{h}\in\tilde{\mathfrak{H}}_\beta} \left(\frac{1}{|\mathrm{Aut}(\tilde{h})|} \cdot \prod_{E\in E(\tilde{\Gamma})} w_E \cdot \prod_{V\in V_I(\tilde{\Gamma})} N_V^{\mathrm{tor}} \cdot \prod_{V\in V_{II}(\tilde{\Gamma})} \frac{(-1)^{w_{E_V}-1}}{w_{E_V}^2} \right. \\ &\qquad \left. \cdot \prod_{V\in V_{III}(\tilde{\Gamma})} \left(\sum_{\mathbf{w}_{V,+}} N_V^{\mathrm{tor}} \frac{1}{|\mathrm{Aut}(\mathbf{w}_{V,+})|} \prod_{i=1}^{l_V} \frac{(-1)^{w_{V,i}-1}}{w_{V,i}} \right) \right). \end{aligned}$$

Canceling the w_{E_V} for vertices of type (II) in the first product against the ones in the denominator of the third product and factoring out the second sum we get

$$\begin{aligned} N_d &= \sum_{\tilde{h}\in\tilde{\mathfrak{H}}_\beta} \sum_{(\mathbf{w}_{V,+})_{V\in V_{III}(\tilde{\Gamma})}} \left(\frac{1}{|\mathrm{Aut}(\tilde{h})||\mathrm{Aut}(\mathbf{w}_{V,+})|} \cdot \prod_{E\in E(\tilde{\Gamma})\setminus\cup_{V\in V_{II}(\tilde{\Gamma})}\{E_V\}} w_E \right. \\ &\qquad \left. \cdot \prod_{V\in V_I(\tilde{\Gamma})\cup V_{III}(\tilde{\Gamma})} N_V^{\mathrm{tor}} \cdot \prod_{V\in V_{II}(\tilde{\Gamma})} \frac{(-1)^{w_{E_V}-1}}{w_{E_V}} \cdot \prod_{V\in V_{III}(\tilde{\Gamma})} \prod_{i=1}^{l_V} \frac{(-1)^{w_{V,i}-1}}{w_{V,i}} \right). \end{aligned}$$

Now by the construction of the map $\mathfrak{H}_d \to \tilde{\mathfrak{H}}_d$ in Construction 4.15, the two summations can be replaced by a summation over $\mathfrak{H}_d$. Note that for $\tilde{h} \in \tilde{\mathfrak{H}}_d$ we have

$$\sum_{h\mapsto\tilde{h}} \frac{1}{|\mathrm{Aut}(h)|} = \frac{1}{|\mathrm{Aut}(\tilde{h})|} \sum_{(\mathbf{w}_{V,+})_{V\in V_{III}(\tilde{\Gamma})}} \frac{1}{|\mathrm{Aut}(\mathbf{w}_{V,+})|},$$

where the sum is over all $h \in \mathfrak{H}_d$ giving $\tilde{h}$ via the map from Construction 4.15. This can be seen by multiplying both sides with $|\mathrm{Aut}(\tilde{h})|$. Moreover, note that $V(\Gamma) = V_I(\tilde{\Gamma}) \cup V_{III}(\tilde{\Gamma})$ and $E(\Gamma) = E(\tilde{\Gamma}) \setminus \cup_{V\in V_{II}(\tilde{\Gamma})}\{E_V\}$, where, for a vertex V of type (II), E_V is the unique edge containing the vertex V. Then

$$\begin{aligned} N_d &= \sum_{h\in\mathfrak{H}_\beta} \left(\frac{1}{|\mathrm{Aut}(h)|} \cdot \prod_{E\in E(\Gamma)} w_E \cdot \prod_{V\in V(\Gamma)} N_V^{\mathrm{tor}} \right. \\ &\qquad \left. \cdot \prod_{V\in V_{II}(\tilde{\Gamma})} \frac{(-1)^{w_{E_V}-1}}{w_{E_V}} \cdot \prod_{V\in V_{III}(\tilde{\Gamma})} \prod_{i=1}^{l_V} \frac{(-1)^{w_{V,i}-1}}{w_{V,i}} \right). \end{aligned}$$

Using that

$$\prod_{E\in L_\Delta(\Gamma)} \frac{(-1)^{w_E-1}}{w_E} = \prod_{V\in V_{II}(\tilde{\Gamma})} \frac{(-1)^{w_{E_V}-1}}{w_{E_V}} \cdot \prod_{V\in V_{III}(\tilde{\Gamma})} \prod_{i=1}^{l_V} \frac{(-1)^{w_{V,i}-1}}{w_{V,i}}$$

completes the proof. □

Corollary 5.18.

$$N_d = \sum_{h\in\mathfrak{H}_d} N_h.$$

Proof. This follows from Theorem 5.17 and $\mathfrak{H}_d = \coprod_{\substack{\beta \in H_2^+(X,\mathbb{Z}) \\ \underline{D}\cdot\underline{\beta} = dw_{\mathrm{out}}}} \mathfrak{H}_\beta$. □

5.6 Invariants with prescribed degree splitting

Definition 5.19. As in §4.4 fix a cyclic labelling of $D_t^0 = D_1 + \ldots + D_k$. For $[d_1, \ldots, d_k] \in \mathbb{N}^k$ define

$$N_{[d_1,\ldots,d_k]} = \sum_{h \mapsto [d_1,\ldots,d_k]} N_h,$$

where the sum is over all $h \in \mathfrak{H}_d$ with degree splitting $[d_1, \ldots, d_k]$ (Definition 4.25).

Remark 5.20. Barrott and Nabijou [BN] define invariants with prescribed degree splittings by looking at the family $\mathfrak{X}_{t\neq 0} \to \mathbb{A}^1$ only degenerating the divisor and using torus localization. We conjecture that these coincide with the invariants defined above. This question will be investigated in future work.

6 Scattering calculations

In this section we prove that the consistent wall structure $\mathscr{S}_\infty$ defined by the dual intersection complex $(B, \mathscr{P}, \varphi)$ of a smooth very ample log Calabi-Yau pair (X, D) carries functions giving a formula similar to Theorem 5.17. This leads to a proof of the main theorem (Theorem 1).

For toric varieties the correspondence between scattering diagrams and logarithmic Gromov-Witten invariants was established in [GPS]. We use the relation between scattering diagrams and tropical curves from [GPS], Theorem 2.8, to extend this correspondence to our non-toric case.

6.1 Scattering diagrams and toric invariants

In [GPS], Theorem 2.4, a bijective correspondence between certain scattering diagrams and tropical curves is established, leading to an enumerative correspondence ([GPS], Theorem 2.8). Combining this result with the tropical correspondence theorem for torically transverse stable log maps with point conditions on the toric boundary ([GPS], Theorems 3.4, 4.4), we get the following.

Lemma 6.1. *Let $\boldsymbol{m} = (m_1, \ldots, m_n)$ be an n-tuple of (not necessarily distinct) primitive vectors of M. Let Σ be a fan in $M_{\mathbb{R}}$ and let φ be an integral strictly convex piecewise affine function on Σ such that $\varphi(0) = 0$. Let $\mathfrak{D}$ be a scattering diagram for φ consisting of a number of lines[5], one in each direction m_i and with attached*

[5] This means that the rays in $\mathfrak{D}$ come in pairs, one incoming and one outgoing, with the same attached function.

function $f_i \in \mathbb{C}[z^{(-m_i,0)}] \subseteq P_\varphi$. *Write the logarithm of* f_i *as*

$$\log f_i = \sum_{w=1}^{\infty} a_{iw} z^{(-wm_i,0)}, \quad a_{iw} \in \mathbb{C}.$$

Let $\mathfrak{D}_\infty$ *be the associated minimal consistent scattering diagram and let* $\mathfrak{d} \in \mathfrak{D}_\infty \setminus \mathfrak{D}$ *be a ray in direction* $m_\mathfrak{d}$ *with attached function* $f_\mathfrak{d}$. *Then*

$$\log f_\mathfrak{d} = \sum_{w=1}^{\infty} \sum_{\boldsymbol{w}} w \frac{N_{\boldsymbol{m}}(\mathbf{w})}{|\mathrm{Aut}(\mathbf{w})|} \left(\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq l_i}} a_{iw_{ij}} \right) z^{(-wm_\mathfrak{d},0)},$$

where the sum is over all n*-tuples of weight vectors* $\boldsymbol{w} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)$ *satisfying*

$$\sum_{i=1}^{n} |\boldsymbol{w}_i| m_i = w m_\mathfrak{d}.$$

Here $N_{\boldsymbol{m}}(\boldsymbol{w})$ *is the toric logarithmic Gromov-Witten invariant defined in §5.1 and* $Aut(\boldsymbol{w})$ *is the subgroup of the permutation group* S_n *stabilizing* $(w_1, \ldots, w_n)$.

Moreover, let $m \in \mathbb{Q}_{>0} m_1 + \ldots + \mathbb{Q}_{>0} m_n$ *be a primitive vector. If there is no ray* $\mathfrak{d} \in \mathfrak{D}_\infty$ *in direction* m, *then* $N_{\boldsymbol{m}}(\boldsymbol{w}) = 0$ *for all* $\boldsymbol{w}$ *satisfying* $\sum_{i=1}^n |\boldsymbol{w}_i| m_i = w m_\mathfrak{d}$.

Remark 6.2. Note that [GPS] deals with the case $\varphi \equiv 0$, where the t-order of an element $z^{(\overline{p},h)}$ is simply given by h. In our case, the t-order is $\varphi(-\overline{p}) + h \geq 0$:

$$z^{(\overline{p},h)} = \left(z^{(-\overline{p},\varphi(-\overline{p}))} \right)^{-1} t^{\varphi(-\overline{p})+h}.$$

The formula in [GPS], Theorem 2.8, contains some explicit t-factors. These are not visible in Lemma 6.1 due to this different notion of t-order.

6.2 Proof of the main theorem

Let Q be a 2-dimensional Fano polytope and let $(B, \mathscr{P}, \varphi)$ be the dual intersection complex of the corresponding toric degeneration $\mathfrak{X} \to \mathbb{A}^1$ of (X, D). Let $\mathscr{S}_\infty$ be the consistent wall structure defined by $(B, \mathscr{P}, \varphi)$, i.e., the limit of the $\mathscr{S}_k$ in Proposition 1.77. Let σ_0 be as in Definition 2.12.

Lemma 6.3. *The support* $|\mathscr{S}_\infty|$ *is disjoint from the interior of* σ_0.

Proof. $(B, \mathscr{P}, \varphi)$ defines an initial wall structure $\mathscr{S}_0$ as in Definition 1.74. The joints of $\mathscr{S}_0$ are the vertices of $\mathscr{P}$. Let v be such a vertex. By [GS3] Proposition 3.9, the walls in $\mathscr{S}_\infty$ with base point v lie in the cone $v + \mathbb{R}_{\leq 0} m_{v\delta} + \mathbb{R}_{\leq 0} m_{v\delta'} \subseteq B$. Here δ, δ' are the affine singularities on edges adjacent to v and $m_{\delta v}$ is the primitive integral tangent vector on B pointing from v to δ.

By inductively using [GS3], Proposition 3.9, all walls in $\mathscr{S}_\infty$ lie in the union of these cones, i.e.,

$$|\mathscr{S}_\infty| \subseteq \bigcup_{v\in\mathscr{P}^{[0]}} v + \mathbb{R}_{\leq 0} m_{v\delta} + \mathbb{R}_{\leq 0} m_{v\delta'}.$$

In particular, there are no walls in $\mathscr{S}_\infty$ supported on the interior of σ_0. $\square$

The unbounded walls in $\mathscr{S}_\infty$ are all parallel in the direction $m_{\text{out}} \in \Lambda_B$. Let f_{out} be the product of all functions $f_{\mathfrak{p}}$ attached to unbounded walls $\mathfrak{p}$ in $\mathscr{S}_\infty$. Then f_{out} can be regarded as an element of $\mathbb{C}[\![x]\!]$ for $x := z^{(-m_{\text{out}},0)} \in \mathbb{C}[\Lambda_B \oplus \mathbb{Z}]$.

Theorem 6.4 (Theorem 1)**.**

$$\log f_{\text{out}} = \sum_{d=1}^{\infty} (D \cdot \underline{\beta}) \cdot N_d \cdot x^{D\cdot\underline{\beta}}.$$

In fact, we will prove a more general statement, giving an enumerative meaning for the function attached to any wall in $\mathscr{S}_\infty$. For this we need the following.

Definition 6.5 ([Gro1], Definition 2.2)**.** A *tropical disk* $h : \Gamma \to B$ is a tropical curve with the choice of univalent vertex V_∞, adjacent to a unique edge E_∞, such that h is balanced for all vertices $V \neq V_\infty$.

Definition 6.6. Let $\mathfrak{p} \in \mathscr{S}_\infty$ be a wall and choose $x \in \text{Int}(\mathfrak{p})$. Define $\mathfrak{H}_{\mathfrak{p},w}$ to be the set of all tropical disks $h : \Gamma \to B$ with $h(V_\infty) = x$ and $u_{(V_\infty,E_\infty)} = -w \cdot m_{\mathfrak{p}}$.

Definition 6.7. For $h \in \mathfrak{H}_{\mathfrak{p},w}$ define, with N_V^{tor} as in Definition 5.15,

$$\frac{1}{|\text{Aut}(h)|} \prod_{E\in E(\Gamma)} w_E \cdot \prod_{V\neq V_{\text{out}}} N_V^{\text{tor}} \cdot \prod_{E\in L_\Delta(\Gamma)} \frac{(-1)^{w_E-1}}{w_E}.$$

Remark 6.8. Note that the sets $\mathfrak{H}_{\mathfrak{p},w}$ are in bijection for different choices of x. Moreover, for an unbounded wall $\mathfrak{p}$ the set $\mathfrak{H}_{\mathfrak{p},w}$ is empty for $w_{\text{out}} \nmid w$, and for each d there is an injective map $\iota : \mathfrak{H}_{\mathfrak{p},dw_{\text{out}}} \hookrightarrow \mathfrak{H}_d$ by removing V_∞ and extending E_∞ to infinity, giving E_{out}. Hence, $N_{\iota(h)} = dw_{\text{out}} N_h$.

Proposition 6.9. *For a wall* $\mathfrak{p}$ *of* $\mathscr{S}_\infty$ *we have*

$$\log f_{\mathfrak{p}} = \sum_{w=1}^{\infty} \sum_{h\in\mathfrak{H}_{\mathfrak{p},w}} N_h z^{(wm_{\mathfrak{p}},0)}.$$

Proof. We want to prove the claimed equality by induction, so we need a well-ordered set. For a wall $\mathfrak{p}$ of $\mathscr{S}_\infty$, define a set

$$\text{Parents}(\mathfrak{p}) = \{\mathfrak{p}' \in \mathscr{S}_\infty \mid \mathfrak{p} \cap \mathfrak{p}' = \text{Base}(\mathfrak{p}) \neq \text{Base}(\mathfrak{p}')\}.$$

Here $\mathrm{Base}(\mathfrak{p})$ is the base point of $\mathfrak{p}$ (see Definition 1.73, (2)). Note that $\mathrm{Base}(\mathfrak{p}')$ is only defined for walls, so we define the condition $\mathrm{Base}(\mathfrak{p}) \neq \mathrm{Base}(\mathfrak{p}')$ to be always satisfied when $\mathfrak{p}'$ is a slab. Then define inductively

$$\mathrm{Ancestors}(\mathfrak{p}) = \{\mathfrak{p}\} \cup \bigcup_{\mathfrak{p}' \in \mathrm{Parents}(\mathfrak{p})} \mathrm{Ancestors}(\mathfrak{p}').$$

For each k, the set of walls in $\mathscr{S}_k$ is finite and totally ordered by

$$\mathfrak{p}_1 \leq \mathfrak{p}_2 :\Leftrightarrow \mathfrak{p}_1 \in \mathrm{Ancestors}(\mathfrak{p}_2).$$

Hence, it is well-ordered and we can use induction. The set of smallest elements with respect to this ordering is

$$\{\mathfrak{p} \in \mathscr{S}_\infty \text{ wall} \mid \mathrm{Base}(\mathfrak{p}) \in \mathscr{P}^{[0]}_{\mathscr{S}_0} = \mathscr{P}^{[0]}\}.$$

For such $\mathfrak{p}$, the set $\mathrm{Ancestors}(\mathfrak{p})$ consists of $\mathfrak{p}$ and two slabs $\mathfrak{b}_1, \mathfrak{b}_2$. This defines a scattering diagram at the joint $\mathrm{Base}(\mathfrak{p})$ that is equivalent to the scattering diagram obtained from two lines in the directions m_1, m_2 of the slabs $\mathfrak{b}_1, \mathfrak{b}_2$ with attached functions $f_1 = 1 + z^{(-m_1,0)}$ and $f_2 = 1 + z^{(-m_2,0)}$, respectively. Note that

$$\log f_i = \sum_{w=1}^{\infty} \frac{(-1)^{w-1}}{w} z^{(wm_i,0)}.$$

Then Lemma 6.1 gives

$$\log f_{\mathfrak{p}} = \sum_{w=1}^{\infty} \sum_{\mathbf{w}} w \frac{N_{\mathbf{m}}(\mathbf{w})}{|\mathrm{Aut}(\mathbf{w})|} \left(\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq l_i}} \frac{(-1)^{w_{ij}-1}}{w_{ij}} \right) z^{(-wm_{\mathfrak{p}},0)},$$

where the sum is over all n-tuples of weight vectors $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ satisfying

$$\sum_{i=1}^{n} |\mathbf{w}_i| m_i = wm_{\mathfrak{p}}.$$

Here $N_{\mathbf{m}}(\mathbf{w})$ is the toric logarithmic Gromov-Witten invariant defined in §5.1. Tropical disks h in $\mathfrak{H}_{\mathfrak{p},w}$ have a 1-valent vertex V_∞ mapping to the interior of $\mathfrak{p}$ and another vertex V with one compact edge of weight w and several bounded legs in directions m_1 or m_2 with weights $\mathbf{w}_1 = (w_{11}, \ldots, w_{1l_1})$ and $\mathbf{w}_2 = (w_{21}, \ldots, w_{2l_2})$, respectively, such that $\sum_{i=1}^{n} |\mathbf{w}_i| m_i = wm_{\mathfrak{p}}$.

Hence, the set $\mathfrak{H}_{\mathfrak{p},w}$ is in bijection with the set of pairs $\mathbf{w}$ as above, with w_{ij} being the weights of the bounded legs of the corresponding tropical disk h. Moreover, $|\mathrm{Aut}(h)| = |\mathrm{Aut}(\mathbf{w})|$ and $N_V^{\mathrm{tor}} = N_{\mathbf{m}}(\mathbf{w})$ by definition. Hence, the summation over

$\mathbf{w}$ can be replaced by a summation over $\mathfrak{H}_{\mathfrak{p},w}$, and we obtain

$$\log f_{\mathfrak{p}} = \sum_{w=1}^{\infty} \sum_{h \in \mathfrak{H}_{\mathfrak{p},w}} w \frac{N_V^{\mathrm{tor}}}{|\mathrm{Aut}(h)|} \left(\prod_{E \in L_\Delta(\Gamma)} \frac{(-1)^{w_E - 1}}{w_E} \right) z^{(-w m_{\mathfrak{p}},0)}.$$

This is precisely the claimed formula for such $\mathfrak{p}$, completing the base case.

For the induction step, let $\mathfrak{p}$ be a wall of $\mathscr{S}_\infty$ and assume the claimed formula holds for all walls $\mathfrak{p}' \in \mathrm{Ancestors}(\mathfrak{p}) \setminus \{\mathfrak{p}\}$. By Lemma 6.1 and using the induction hypothesis,

$$\log f_{\mathfrak{p}} = \sum_{w=1}^{n} \sum_{\mathbf{w}} \left(w \frac{N_{\mathbf{m}}(\mathbf{w})}{|\mathrm{Aut}(\mathbf{w})|} \prod_{\substack{\mathfrak{p}' \in \mathrm{Parents}(\mathfrak{p}) \\ 1 \leq j \leq l_{\mathfrak{p}'}}} \sum_{h' \in \mathfrak{H}_{\mathfrak{p}',w_{\mathfrak{p}'j}}} N_{h'} \right) z^{(-w m_{\mathfrak{p}},0)},$$

where the second sum is over all tuples $\mathbf{w} = (\mathbf{w}_{\mathfrak{p}'})_{\mathfrak{p}' \in \mathrm{Parents}(\mathfrak{p})}$ of weight vectors $\mathbf{w}_{\mathfrak{p}'} = (w_{\mathfrak{p}'1}, \ldots, w_{\mathfrak{p}'l_{\mathfrak{p}'}})$ with

$$\sum_{\mathfrak{p}' \in \mathrm{Parents}(\mathfrak{p})} |\mathbf{w}_{\mathfrak{p}'}| m_{\mathfrak{p}'} = w m_{\mathfrak{p}}.$$

Factoring out, we can replace the product over $\mathrm{Parents}(\mathfrak{p})$ and $1 \leq j \leq l_{\mathfrak{p}'}$ and the sum over $h' \in \mathfrak{H}_{\mathfrak{p}',w_{\mathfrak{p}'j}}$ by a sum over tuples $(h')_{\mathfrak{p}',j} := (h' \in \mathfrak{H}_{\mathfrak{p}',w_{\mathfrak{p}'j}})_{\substack{\mathfrak{p}' \in \mathrm{Parents}(\mathfrak{p}) \\ 1 \leq j \leq l_{\mathfrak{p}'}}}$:

$$\log f_{\mathfrak{p}} = \sum_{w=1}^{n} \sum_{\mathbf{w}} \left(w \frac{N_{\mathbf{m}}(\mathbf{w})}{|\mathrm{Aut}(\mathbf{w})|} \sum_{(h')_{\mathfrak{p}',j}} \prod_{h' \in (h')_{\mathfrak{p}',j}} N_{h'} \right) z^{(-w m_{\mathfrak{p}},0)}.$$

Further, we can replace the summations over $\mathbf{w}$ and $(h')_{\mathfrak{p}',j}$ by a summation over $\mathfrak{H}_{\mathfrak{p},w}$, since this is precisely the data that determines a tropical disk in $\mathfrak{H}_{\mathfrak{p},w}$. We get the claimed formula by using that, for $h \in \mathfrak{H}_{\mathfrak{p},w}$ determined by $\mathbf{w}$ and $(h')_{\mathfrak{p}',j}$, we have

$$|\mathrm{Aut}(h)| = |\mathrm{Aut}(\mathbf{w})| \cdot \prod_{(h')_{\mathfrak{p}',j}} |\mathrm{Aut}(h')|,$$

so

$$N_h = w \frac{N_{\mathbf{m}}(\mathbf{w})}{|\mathrm{Aut}(\mathbf{w})|} \cdot \prod_{h' \in (h')_{\mathfrak{p}',j}} N_{h'}.$$

This completes the proof. □

Now Theorem 6.4 follows by summation over all outgoing walls $\mathfrak{p}$ in $\mathscr{S}_\infty$.

7 Torsion points

In this section we consider $(\mathbb{P}^2, E)$. Choose a flex point O on the elliptic curve E and consider the group law on E with O the identity. An *m-torsion point* on E is a point P such that $m \cdot P = O$. As a topological group, the elliptic curve is a torus $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. The m-torsion points form a group $\mathbb{Z}_m \times \mathbb{Z}_m$.

Lemma 7.1. *If C is a rational degree d curve intersecting E in a single point P, then P is a $3d$-torsion point.*

Proof. Let C be a rational degree d curve intersecting E in a single point P and let L be the line tangent to O. Then the cycle $C - dL$ has degree 0, so it is linearly equivalent to zero, since $\mathrm{Pic}\, \mathbb{P}^2 \cong \mathbb{Z}$ by the degree map. Moreover, it intersects E in the cycle $3d(P - O)$ which in turn is linearly equivalent to zero. $\square$

Let $T_d \simeq \mathbb{Z}_{3d} \times \mathbb{Z}_{3d}$ be the set of $3d$-torsion points on E and let β_d be the class of degree d stable log maps (Definition 3.3). Then we have a decomposition

$$\mathscr{M}(\mathbb{P}^2, \beta_d) = \coprod_{P \in T_d} \mathscr{M}(\mathbb{P}^2, \beta_d)_P,$$

where $\mathscr{M}(\mathbb{P}^2, \beta_d)_P$ is the subspace of $\mathscr{M}(\mathbb{P}^2, \beta_d)$ of maps intersecting E in P. Let $[\![\mathscr{M}(\mathbb{P}^2, \beta_d)_P]\!]$ be the restriction of $[\![\mathscr{M}(\mathbb{P}^2, \beta_d)]\!]$ to $\mathscr{M}(\mathbb{P}^2, \beta_d)_P$ and define

$$N_{d,P} := \int_{[\![\mathscr{M}(\mathbb{P}^2, \beta_d)_P]\!]} 1.$$

Definition 7.2. For $P \in \cup_{d \geq 1} T_d$ denote by $k(P)$ the smallest integer $k \geq 1$ such that P is $3k$-torsion.

Lemma 7.3. *$N_{d,P}$ only depends on P through $k(P)$.*

Proof. This was shown in [Bou4], Lemma 1.2, using ideas from [CGKT2]. The freedom of choice of O and the fact that the monodromy of the family of smooth cubics in $\mathbb{P}^2$ maps surjectively to $SL(2, \mathbb{Z})$ acting on $T_k \simeq \mathbb{Z}_{3k} \times \mathbb{Z}_{3k}$ implies that two points P, P' with $k(P) = k(P')$ are related to each other via a monodromy transformation. Then the deformation invariance of logarithmic Gromov-Witten invariants shows that $N_{d,P} = N_{d,P'}$ for all d. $\square$

Definition 7.4. Write $N_{d,k}$ for $N_{d,P}$ with P such that $k(P) = k$.

Under the toric degeneration $\mathfrak{X} \to \mathbb{A}^1$ different $3d$-torsion points may map to the same point on the central fiber X_0, and even to a 0-dimensional stratum. However, the limits of the 3-torsion points all lie on the 1-dimensional strata. The intersection points with $3d$-torsion correspond to the unbounded walls $\mathfrak{p}$ in $\mathscr{S}_\infty$ with non-zero

t^{3d}-coefficient of $\log f_{\mathfrak{p}}$. Their number is exactly $3d$ and they are distributed as the $3l$-torsion points of a circle (see Figure 0.6 and Figure 8.2).

This can be explained as follows. The 2-dimensional torus is a S^1-fibration over S^1. In the SYZ limit, the S^1-fiber shrinks to a point and we only see the S^1-base, which is the tropicalization. In this limit, the $\mathbb{Z}_{3k}$-fibers of $T_k \simeq \mathbb{Z}_{3k} \times \mathbb{Z}_{3k}$ are identified and we only see the $\mathbb{Z}_{3k}$ in the base. The toric degeneration of divisors $\mathfrak{D} \to \mathbb{A}^1$ is an elliptic fibration. It contains a singular fiber that is a cycle of three $\mathbb{P}^1$, i.e., an I_3 fiber in Kodaira's classification of singular elliptic fibers. Then the monodromy acting on the first cohomology class of the general fiber is given by $M_1 = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)$ up to conjugation. Now the action of M_1 on $\mathbb{Z}_3 \times \mathbb{Z}_3$ is trivial. This means that each 3-torsion point really defines a section of the family, which will have a limit on the special fiber. Such limit is necessarily on the smooth part of the special fiber, i.e., on a 1-dimensional toric stratum (see [SS], Theorem 6.3). The action of M_1 on $\mathbb{Z}_6 \times \mathbb{Z}_6$ has some fixed points, corresponding to 6-torsion points on E that define sections with limit on the smooth part of the special fiber, i.e., on 1-dimensional strata. The other 6-torsion points are permuted by the action of M_1, so they only define multisections with limit on the singular part of the special fiber, i.e., on 0-dimensional strata.

Now consider the refined degeneration $\tilde{\mathfrak{X}}_d \to \mathbb{A}^1$. By construction of the refinement, the limits of all $3d$-torsion points lie on 1-dimensional strata of the central fiber $X := Y$. Indeed, the central fiber of the elliptic fibration $\tilde{\mathfrak{D}} \to \mathbb{A}^1$ is a cycle of $3d$ lines, i.e., an I_{3d}-fiber. Then the monodromy acting on $\mathbb{Z}_{3k} \times \mathbb{Z}_{3k}$ is given by $\left(\begin{smallmatrix} 1 & 3d \\ 0 & 1 \end{smallmatrix}\right)$ which is the identity for all $k \mid d$.

Definition 7.5. Let $s_{k,l}$ be the number of points P on E with $k(P) = k$, fixed by the action of $M_l := \left(\begin{smallmatrix} 1 & 3l \\ 0 & 1 \end{smallmatrix}\right)$, but not fixed by the action of $M_{l'}$ for all $l' < l$. Note that $s_k := \sum_{l|k} s_{k,l}$ is the number of points P on E with $k(P) = k$ and that $\sum_{k|d} s_k = (3d)^2$ is the number of $3d$-torsion points.

$s_{1,1} = 9$					
$s_{2,1} = 9$	$s_{2,2} = 18$				
$s_{3,1} = 18$		$s_{3,3} = 54$			
$s_{4,1} = 18$	$s_{4,2} = 18$		$s_{4,4} = 72$		
$s_{5,1} = 36$				$s_{5,5} = 180$	
$s_{6,1} = 18$	$s_{6,2} = 36$	$s_{6,3} = 54$			$s_{6,6} = 108$

Table 7.1: The number $s_{k,l}$ of points P on E with $k(P) = k$, fixed by M_l, but not fixed by $M_{l'}$ for $l' < l$.

Definition 7.6. For a wall $\mathfrak{p} \in \mathscr{S}_\infty$ let $l(\mathfrak{p})$ be the smallest number such that $\log f_{\mathfrak{p}}$ has non-trivial $t^{3l(\mathfrak{p})}$-coefficient. Let r_l be the number of walls with $l(\mathfrak{p}) = l$.

Lemma 7.7. *The number r_l can be defined recursively by*

$$r_1 = 3, \quad r_l = 3l - \sum_{l'|l} r_{l'}.$$

Proof. For a wall $\mathfrak{p} \in \mathscr{S}_\infty$, the condition $l(\mathfrak{p}) = l$ means that the corresponding toric stratum of $X := Y$ contains the limits of the points on E that are fixed by the action of M_l but not fixed by $M_{l'}$ for $l' < l$. Since in the SYZ limit we only see the base of the fibration, r_l equals the number of points on a circle S^1 that are $3l$-torsion but not $3l'$-torsion. This number can be defined as above. □

$r_1 = 3$	$r_2 = 3$	$r_3 = 6$	$r_4 = 6$	$r_5 = 12$	$r_6 = 6$

Table 7.2: The number r_l of walls $\mathfrak{p}$ with $l(\mathfrak{p}) = l$.

Note that $s_{k,l}/r_l$ is the number of points P with $k(P) = k$ and with limit on the stratum corresponding to a particular wall $\mathfrak{p}$ with $l(\mathfrak{p}) = l$. A direct consequence of Proposition 6.9 is the following.

Corollary 7.8 (Theorem 2)**.** *Let $\mathfrak{p}$ be an unbounded wall of order l in $\mathscr{S}_\infty$. Then*

$$\log f_{\mathfrak{p}} = \sum_{d=1}^{\infty} 3d \left(\sum_{k:l|k|d} \frac{s_{k,l}}{r_l} N_{d,k} \right) x^{3d}.$$

Definition 7.9. Similarly to §3.4 we can define integers $n_{d,k}$ recursively by

$$n_{d,d} = N_{d,d}, \quad n_{d,k} = N_{d,k} - \sum_{d':k|d'|d} M_{d'}[d/d'] \cdot n_{d',k}.$$

Here $M_{d'}[d/d']$ is the multiple cover contribution (see Proposition 3.11).

Some of the numbers $n_{d,k}$ have been calculated by Takahashi [Tak1]. Their relation to local BPS numbers is studied in [CGKT1][CGKT2][CGKT3].

Remark 7.10. Unfortunately we are not able to apply the methods of this section to the refined situation of §4.4 in order to calculate the contributions to $N_{[d_1,\ldots,d_m]}$ with prescribed torsion: Let $N_{[d_1,\ldots,d_m],k}$ be the logarithmic Gromov-Witten invariant of stable log maps contributing to $N_{[d_1,\ldots,d_m]}$ and meeting E in a fixed point of order $3k$, and let $n_{[d_1,\ldots,d_m],k}$ be the corresponding log BPS number. Define $l([d_1,\ldots,d_m]) := l(\mathfrak{p})$ for $\mathfrak{p}$ the unbounded wall for $[d_1,\ldots,d_m]$. Then

$$\begin{aligned} \sum_{k=1}^{d} \frac{s_{k,l}}{r_l} n_{[d_1,\ldots,d_m],k} &= n_{[d_1,\ldots,d_m]} \quad \text{for } l = l([d_1,\ldots,d_m]) \\ \sum_{l([d_1,\ldots,d_m])=l} n_{[d_1,\ldots,d_m],k} &= n_{d,k} \quad \text{for all } l|k \end{aligned}$$

This gives a system of linear equations for the indeterminates $n_{[d_1,\dots,d_m],k}$. In general the number of equations will be smaller than the number of indeterminates, so there will be no unique solution. However, for $d \leq 3$ we indeed have enough equations to determine the numbers $n_{[d_1,\dots,d_m],k}$ as we will show in §8.1.3.

8 Explicit calculations

In this section we will calculate some logarithmic Gromov-Witten invariants and log BPS numbers explicitly. To this end, I wrote a sage code for calculating scattering diagrams and wall structures. It can be found on my webpage[6], along with instructions on how to use it.

8.1 $(\mathbb{P}^2, E)$

We want to calculate the numbers $N_{d,k}$ and $n_{d,k}$ for $d \leq 6$ as well as the numbers $n_{[d_1,\dots,d_k]}$ for $d \leq 4$. Loading the code into a sage shell and typing

```
D = Diagram(case="P2",order=6)
D2 = D.scattering(order=6,case="P2")
D2.tex(initial_diagram=D,print_directions=[(1,0)])
```

one can produce a TikZ picture that under some small changes gives Figure 8.1. It shows the part of the wall structure $\bar{\mathscr{S}}_6$ on the discrete covering space $\bar{B}$ (see §2.3) that is relevant for computing the functions on the central maximal cell. The full $\bar{\mathscr{S}}_6$ would be symmetric, carrying much more walls on the outer area.

The code gives

$$\log f_{\text{out}} = 27x^3 + \frac{405}{2}x^6 + 2196x^9 + \frac{110997}{4}x^{12} + \frac{2051892}{5}x^{15} + 5527710x^{18} + \mathcal{O}(x^{20})$$

This yields the following logarithmic Gromov-Witten invariants:

$N_1 = 9$	$N_2 = \frac{135}{4}$	$n_3 = 244$	$n_4 = \frac{36999}{16}$	$n_5 = \frac{635634}{25}$	$n_6 = 307095$

Table 8.1: The invariants N_d of $(\mathbb{P}^2, E)$ for $d \leq 6$.

Subtracting multiple cover contributions we get the following log BPS numbers:

[6] `http://timgraefnitz.com/`

$n_1 = 9$	$n_2 = 27$	$n_3 = 234$	$n_4 = 2232$	$n_5 = 25380$	$n_6 = 305829$

Table 8.2: The log BPS numbers n_d of $(\mathbb{P}^2, E)$ for $d \leq 6$.

They are related to the local BPS numbers n_d^{loc}, shown in [CKYZ], Table 1, by Remark 3.13 and [CKYZ], (2.1).

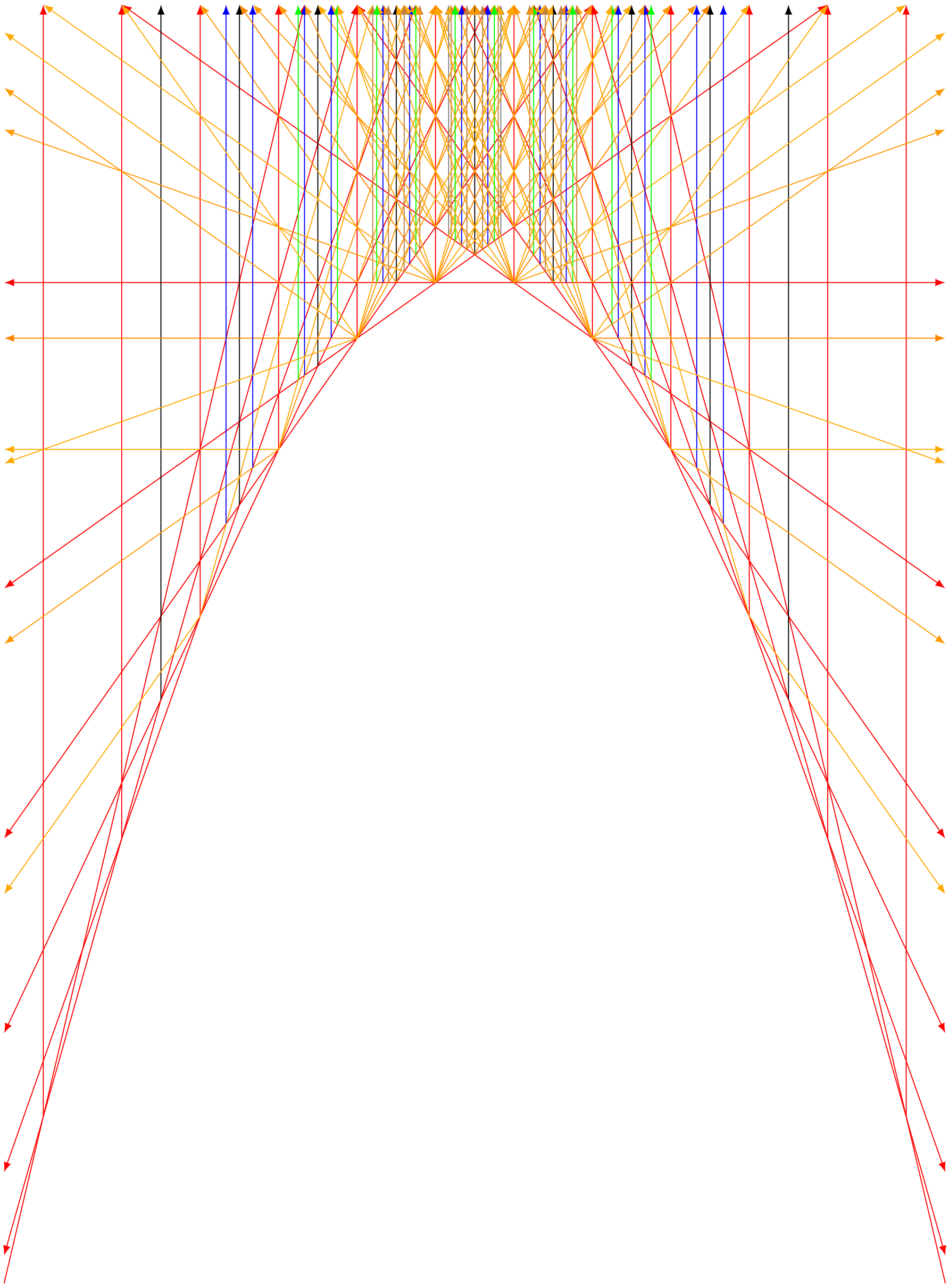


Figure 8.1: The output of the sage code, giving the relevant part of $\bar{\mathscr{S}}_6$ to compute $N_{d,k}$ for $d \leq 6$. The colors correspond to different orders.

8.1.1 Torsion points

Write f_l for the function $f_{\mathfrak{p}}$ attached to a wall $\mathfrak{p}$ with $l(\mathfrak{p}) = l$ (see Figure 8.2).

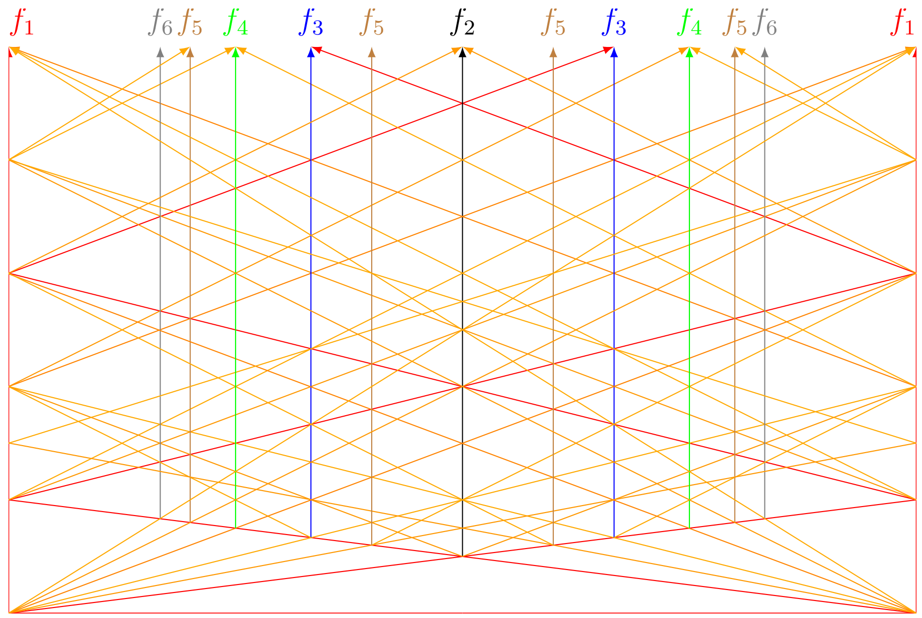


Figure 8.2: The wall structure $\mathscr{S}_6$ on one unbounded maximal cell of $\mathscr{P}$, showing the relevant attached functions f_l.

The sage code gives the following:

$$\log f_1 = 9x^3 + \frac{63}{2}x^6 + 246x^9 + \frac{9279}{4}x^{12} + \frac{175464}{5}x^{15} + 307041x^{18} + \mathcal{O}(x^{21})$$
$$\log f_2 = 36x^6 + 2322x^{12} + 307164x^{18} + \mathcal{O}(x^{21})$$
$$\log f_3 = 243x^9 + \frac{614061}{2}x^{18} + \mathcal{O}(x^{21})$$
$$\log f_4 = 2304x^{12} + \mathcal{O}(x^{21})$$
$$\log f_5 = 25425x^{15} + \mathcal{O}(x^{21})$$
$$\log f_6 = 307152x^{18} + \mathcal{O}(x^{21})$$

From this we calculate the numbers $n_{d,k}$ for $d \leq 6$ below to be the following:

$n_{1,1} = 1$					
$n_{2,1} = 0$	$n_{2,2} = 1$				
$n_{3,1} = 2$		$n_{3,3} = 3$			
$n_{4,1} = 14$	$n_{4,2} = 14$		$n_{4,4} = 16$		
$n_{5,1} = 108$				$n_{5,5} = 113$	
$n_{6,1} = 927$	$n_{6,2} = 938$	$n_{6,3} = 936$			$n_{6,6} = 948$

Table 8.3: The numbers $n_{d,k}$ calculated below.

The $n_{d,d}$ coincide with the m_d in [Tak2], Theorem 1.4. The numbers $n_{d,k}$ for $d \leq 3$ are calculated in [Tak1]. The sum $\sum_{k|d} s_k n_{d,k}$ is the log BPS number n_d of $(\mathbb{P}^2, E)$. From n_5 and $n_{5,5}$ one also obtains $n_{5,1}$. To the best of my knowledge the numbers $n_{4,1}$, $n_{4,2}$, $n_{6,1}$, $n_{6,2}$ and $n_{6,3}$ are new.

From Corollary 7.8 we get the following:

For $l = 6$ we get

$$307041 = 18 \cdot \frac{108}{6} \cdot N_{6,6},$$

hence $N_{6,6} = 948$. There are no multiple cover contributions, as there are no curves of degree < 6 meeting a point with $l = 6$. This shows $n_{6,6} = 948$.

For $l = 5$ we get

$$25425 = 15 \cdot \frac{180}{12} \cdot N_{5,5},$$

hence $N_{5,5} = 113$. Again, there are no multiple cover contributions, so $n_{5,5} = 113$.

For $l = 4$ we get

$$2304 = 12 \cdot \frac{72}{6} \cdot N_{4,4},$$

hence $N_{4,4} = n_{4,4} = 16$.

For $l = 3$ we get

$$243 = 9 \cdot \frac{54}{6} \cdot N_{3,3},$$

hence $N_{3,3} = n_{3,3} = 3$. Moreover,

$$\frac{614061}{2} = 18 \cdot \left(\frac{54}{6} \cdot N_{6,3} + \frac{54}{6} \cdot 948 \right),$$

hence $N_{6,3} = \frac{3789}{4}$. Subtracting $M_3[2] \cdot n_{3,3} = \frac{15}{4} \cdot 3$ we get $n_{6,3} = 936$.

For $l = 2$ we get

$$36 = 6 \cdot \frac{18}{3} \cdot N_{2,2},$$

hence $N_{2,2} = n_{2,2} = 1$, and

$$2322 = 12 \cdot \left(\frac{18}{3} \cdot N_{4,2} + \frac{18}{3} \cdot 16 \right),$$

hence $N_{4,2} = \frac{65}{4}$. Subtracting $M_1[4] \cdot n_{1,1} + M_2[2] n_{2,1} = \frac{35}{16} \cdot 1 + \frac{9}{4} \cdot 0$ we get $n_{4,1} = 14$. Moreover,

$$307164 = 18 \cdot \left(\frac{18}{3} \cdot N_{6,2} + \frac{36}{3} \cdot 948 \right),$$

hence $N_{6,2} = \frac{8533}{9}$. Subtracting $M_2[3] \cdot n_{2,2} = \frac{91}{9} \cdot 1$ we get $n_{6,2} = 938$.

For $l = 1$ we get

$$9 = 3 \cdot \frac{9}{3} \cdot N_{1,1},$$

hence $N_{1,1} = n_{1,1} = 1$, and

$$\frac{63}{2} = 6 \cdot \left(\frac{9}{3} \cdot N_{2,1} + \frac{9}{3} \cdot 1\right),$$

hence $N_{2,1} = \frac{3}{4}$. Subtracting $M_1[2] \cdot n_{1,1} = \frac{3}{4} \cdot 1$ we get $n_{2,1} = 0$. Moreover,

$$246 = 9 \cdot \left(\frac{9}{3} \cdot N_{3,1} + \frac{18}{3} \cdot 3\right),$$

hence $N_{3,1} = \frac{28}{9}$. Subtracting $M_1[3] \cdot n_{1,1} = \frac{10}{9} \cdot 1$ we get $n_{3,1} = 2$. Finally,

$$307041 = 18 \cdot \left(\frac{9}{3} \cdot N_{6,1} + \frac{9}{3} \cdot \frac{8533}{9} + \frac{18}{3} \cdot \frac{3789}{4} + \frac{18}{3} \cdot 948\right),$$

hence $N_{6,1} = \frac{2842}{3}$. Subtracting $M_1[6]\cdot n_{1,1} + M_2[3]\cdot n_{2,1} + M_3[2]\cdot n_{3,1} = \frac{77}{6}\cdot 1 + \frac{91}{9}\cdot 0 + \frac{15}{4}\cdot 2$ we get $n_{6,1} = 927$.

8.1.2 Degenerating the divisor

The limit $s \to 0$ corresponds to a degeneration of the elliptic curve E to a cycle of three lines $D_t^0 = D_1 + D_2 + D_3$. In this section we consider invariants of $(\mathbb{P}^2, E)$ with given degrees over the components D_i, i.e., with given degree splitting $[d_1, \ldots, d_k]$ (Definition 4.25). Since the components D_i are isomorphic as divisors of $\mathbb{P}^2$ this does not depend on the labelling of the D_i and we can omit zeros in $[d_1, \ldots, d_k]$. The tropical curves contributing to the invariants $N_{[d_1,\ldots,d_k]}$ for $d = \sum d_i \leq 4$ are shown in Figure 8.3. There is more than one tropical curve with degree splitting $[2,2]$ and $[2,1,1]$, respectively. The sage code gives the following functions:

$$\begin{aligned}
f_{[1]} &= 1 + 9x^3 + \mathcal{O}(x^6) \\
f_{[2]} &= (1+9x^3)(1+72x^6) + \mathcal{O}(t^9) \\
f_{[1,1]} &= 1 + 36x^6 + \mathcal{O}(x^9) \\
f_{[3]} &= (1+9x^3)(1+72x^6)(1-78x^9) + \mathcal{O}(x^{12}) \\
f_{[2,1]} &= 1 + 243x^9 + \mathcal{O}(x^{12}) \\
f_{[1,1,1]} &= 1 + 81x^9 + \mathcal{O}(x^{12}) \\
f_{[4]} &= (1+9x^3)(1+72x^6)(1-78x^9)(1+5256x^{12}) + \mathcal{O}(x^{15}) \\
f_{[3,1]} &= 1 + 1872x^{12} + \mathcal{O}(x^{15}) \\
f_{[2,2]} &= (1+36x^6)(1+1296x^{12})(1+1530x^{12}) + \mathcal{O}(x^{15}) \\
f_{[2,1,1]} &= (1+144x^{12})(1+432x^{12})(1+1296x^{12}) + \mathcal{O}(x^{15})
\end{aligned}$$

The invariant $N_{[d]}$ has a d-fold cover contribution from $n_{[d]}$ and $N_{[2,2]}$ has a 2-fold cover contribution from $n_{[1,1]}$. This gives the following log BPS numbers:

$n_{[1]} = 3$			
$n_{[2]} = 3$	$n_{[1,1]} = 6$		
$n_{[3]} = 15$	$n_{[2,1]} = 27$	$n_{[1,1,1]} = 9$	
$n_{[4]} = 72$	$n_{[3,1]} = 156$	$n_{[2,2]} = 168$	$n_{[2,1,1]} = 156$

Table 8.4: Log BPS numbers of $(\mathbb{P}^2, E)$ with given degrees over the D_i.

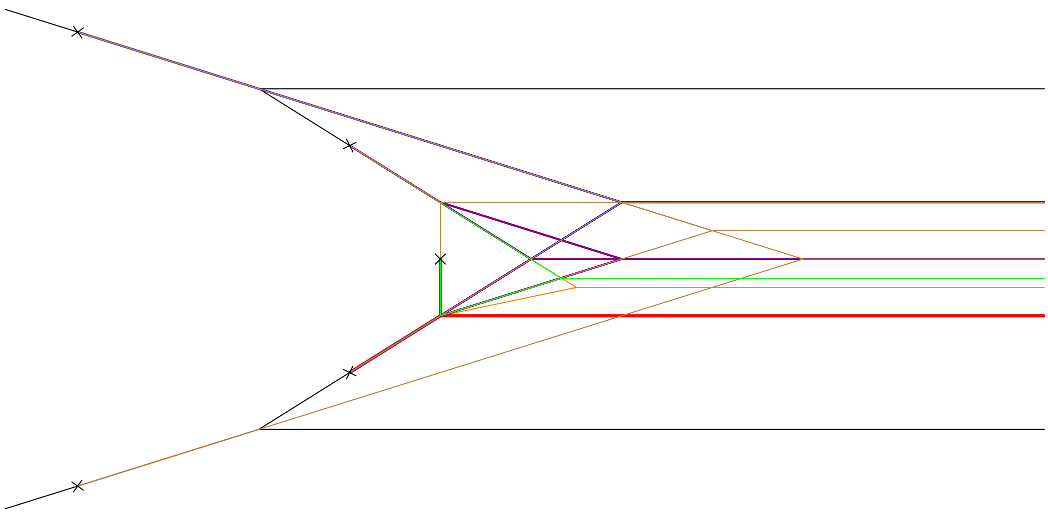


Figure 8.3: Tropical curves contributing to $n_{[d_1,\ldots,d_k]}$ for $(\mathbb{P}^2, E)$.

8.1.3 Combining the methods

As indicated in Remark 7.10 in general it is not possible to combine the above methods and calculate invariants with given degree splitting and meeting a points on E with prescribed torsion. However, for $d \leq 3$ this is indeed possible as we will show now.

For $d = 1$ we have $\mathcal{G}_1 = \{[1]\}$, so $n_{[1],1} = n_{1,1} = 1$. For $d = 2$ there are two types of tropical curves. One of them contributes to $N_{[1,1]}$ and corresponds to stable log maps meeting E in a point of order 6, so $n_{[1,1],1} = 0$ and $n_{[1,1],2} = n_{2,2} = 1$. The other one contributes to $N_{[2]}$, so $n_{[2],1} = n_{1,1} = 0$.

For $d = 3$ we have $l([3]) = 1$, $l([2,1]) = 3$ and $l([1,1,1]) = 1$. So $n_{[2,1],1} = 0$ and the equations in Remark 7.10 are the following:

$$\begin{aligned} 3n_{[3],1} + 6n_{[3],3} &= 15 & n_{[3],1} + n_{[1,1,1],1} &= 2 \\ 9n_{[2,1],3} &= 27 & n_{[3],3} + n_{[1,1,1],3} &= 3 \\ 3n_{[1,1,1],1} + 6n_{[1,1,1],3} &= 9 & n_{[2,1],3} &= 3 \end{aligned}$$

This system of linear equations has the unique solution:

$n_{[3],1} = 1$	$n_{[2,1],1} = 0$	$n_{[1,1,1],1} = 1$
$n_{[3],3} = 2$	$n_{[2,1],3} = 3$	$n_{[1,1,1],3} = 1$

Table 8.5: Log BPS numbers of $(\mathbb{P}^2, E)$ for $d = 3$ with prescribed degree splitting and torsion point.

If we try the same method for $d > 3$, we have more indeterminates than independent equations, so there is no unique solution.

8.2 $\mathbb{P}^1 \times \mathbb{P}^1$

Similarly, executing the code for $\mathbb{P}^1 \times \mathbb{P}^1$ gives the following functions:

$$\log f_{\text{out}} = 16x^2 + 72x^4 + 352x^6 + 3108x^8 + \frac{120016}{5}x^{10} + 198384x^{12} + \mathcal{O}(x^{14})$$

This gives:

$n_1 = 8$	$n_2 = 16$	$n_3 = 72$	$n_4 = 368$	$n_5 = 2400$	$n_6 = 16320$

Table 8.6: The log BPS numbers n_d of $\mathbb{P}^1 \times \mathbb{P}^1$ for $d \leq 6$.

Moreover, the functions and invariants with prescribed degree splitting are:

$$\begin{aligned}
f_{[1]} &= 1 + 4x^2 + \mathcal{O}(x^4) \\
f_{[2]} &= (1 + 4x^2)(1 + 10x^4) + \mathcal{O}(x^6) \\
f_{[1,1]} &= 1 + 16x^4 + \mathcal{O}(x^6) \\
f_{[3]} &= (1 + 4x^2)(1 + 10x^4)(1 - 20x^6) + \mathcal{O}(x^8) \\
f_{[2,1]} &= 1 + 36x^6 + \mathcal{O}(x^8) \\
f_{[1,1,1]} &= 1 + 36x^6 + \mathcal{O}(x^8) \\
f_{[4]} &= (1 + 4x^2)(1 + 10x^4)(1 - 20x^6)(1 + 115x^8) + \mathcal{O}(x^{10}) \\
f_{[3,1]} &= 1 + 64x^8 + \mathcal{O}(x^{10}) \\
f_{[2,2]} &= (1 + 16x^4)(1 + 64x^8)(1 + 264x^8) + \mathcal{O}(x^{10}) \\
f_{[2,1,1]} &= 1 + 64x^8 + \mathcal{O}(x^{10}) \\
f_{[1,2,1]} &= 1 + 256x^8 + \mathcal{O}(x^{10}) \\
f_{[1,1,1,1]} &= 1 + 64x^8 + \mathcal{O}(x^{10})
\end{aligned}$$

This gives the following log BPS numbers:

$n_{[1]} = 2$					
$n_{[2]} = 0$	$n_{[1,1]} = 4$				
$n_{[3]} = 0$	$n_{[2,1]} = 6$	$n_{[1,1,1]} = 6$			
$n_{[4]} = 0$	$n_{[3,1]} = 8$	$n_{[2,2]} = 24$	$n_{[2,1,1]} = 8$	$n_{[1,2,1]} = 32$	$n_{[1,1,1,1]} = 8$

Table 8.7: The log BPS numbers of $\mathbb{P}^1 \times \mathbb{P}^1$ with given degree splitting.

From this we compute the log BPS numbers with given curve class (bidegree) as follows. The factors are the number of unbounded walls in the wall structure contributing to $n_{[d_1,\ldots,d_m]}$ for different labellings of $D = D_1 + D_2 + D_3 + D_4$.

$$
\begin{aligned}
n_{(1,0)} &= 2 \cdot n_{[1]} && = 2 \\
n_{(2,0)} &= 2 \cdot n_{[2]} && = 0 \\
n_{(1,1)} &= 4 \cdot n_{[1,1]} && = 16 \\
n_{(3,0)} &= 2 \cdot n_{[3]} && = 0 \\
n_{(2,1)} &= 4 \cdot n_{[2,1]} + 2 \cdot n_{[1,1,1]} && = 36 \\
n_{(4,0)} &= 2 \cdot n_{[4]} && = 0 \\
n_{(3,1)} &= 4 \cdot n_{[3,1]} + 4 \cdot n_{[2,1,1]} && = 768 \\
n_{(2,2)} &= 4 \cdot n_{[2,2]} + 4 \cdot n_{[1,2,1]} + 4 \cdot n_{[1,1,1,1]} && = 1152
\end{aligned}
$$

8.2.1 Deforming $\mathbb{F}_2$

Consider the toric degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$ by deformation of the Hirzebruch surface $\mathbb{F}_2$ (case (8'a) in Figure 2.5) from Example 2.15.

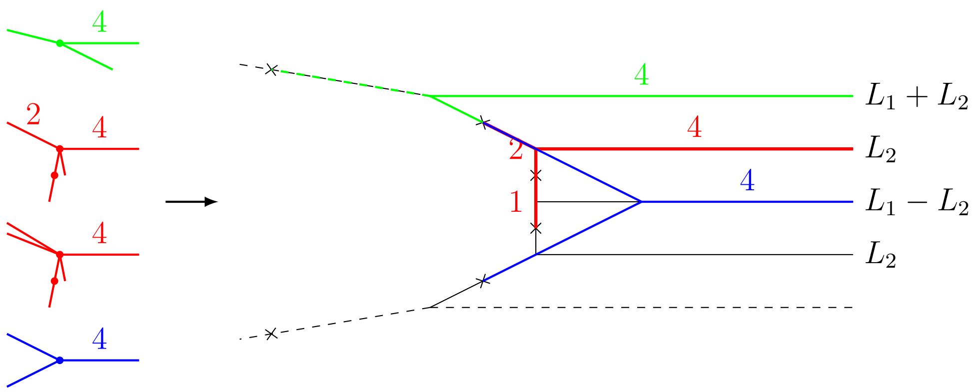


Figure 8.4: Tropical curves corresponding to $\underline{\beta} = L_1 + L_2$.

For $d = 1$ it is clear by the symmetry $L_1 \leftrightarrow L_2$ that $n_{(1,0)} = n_{L_1} = n_{L_2} = 4$. For $d = 2$ we have $n_{(2,0)} = n_{2L_1} = n_{2L_2}$ and $n_{(1,1)} = n_{L_1+L_2}$. Figure 8.4 shows the tropical curves corresponding to stable log maps of class $\underline{\beta} = L_1 + L_2$. The first one has multiplicity 4, the second one -4, the third one 8 and the last one 4. By symmetry there are two tropical curves similar to the red ones at the lower vertex, again with multiplicities -4 and 8. This gives $n_{(1,1)} = n_{L_1+L_2} = 16$ and in turn $n_{(2,0)} = 0$. One can proceed similarly for higher degrees.

8.3 Cubic surface

The dual intersection complex of the cubic surface (case (3a) in Figure 2.5) is quite similar to the one of $(\mathbb{P}^2, E)$. The only differences are that for each vertex the determinant of primitive generators of adjacent bounded edges is 1 instead of 3 and that there are three affine singularities on each bounded edge. As a consequence, by the change of lattice trick ([GHKK], Proposition C.13), the wall structure of (X, D) is in bijection with the wall structure of $(\mathbb{P}^2, E)$, and the functions attached to walls in direction m_{out}, in particular the unbounded walls, coincide. This immediately implies $N_d(X,D) = 3 \cdot N_d(\mathbb{P}^2, E)$. Subtracting multiple covers we get the following. Note that $w_{\text{out}} = 1$ and the multiple cover contributions of of degree 1 curves are $M_1[k] = \frac{1}{k^2}\binom{-1}{k-1} = \frac{(-1)^{k-1}}{k^2}$.

$n_1 = 27$	$n_2 = 108$	$n_3 = 729$	$n_4 = 6912$	$n_5 = 76275$	$n_6 = 920727$

Table 8.8: The log BPS numbers n_d of the cubic surface for $d \leq 6$.

8.3.1 Curve classes

A smooth cubic surface X can be given by blowing up six general points on $\mathbb{P}^2$. Let $e_1, \ldots, e_1$ be the classes of the exceptional divisors and let ℓ be the pullback of the class of a line in $\mathbb{P}^2$. Then $\ell, e_1, \ldots, e_6$ generate $\text{Pic}(X) \simeq H_2^+(X, \mathbb{Z}) \simeq \mathbb{Z}^7$.

The dual intersection complex of a smooth nef toric surface X^0 deforming to X is shown in Figure 8.5. Its asymptotic fan is the fan of X^0. Denote the curves corresponding to the rays of this fan by $C_1, \ldots, C_9$, labelled as in Figure 8.5.

Then (see [KMi], §4) an isomorphism $\text{Pic}(X^0) \simeq \text{Pic}(X)$ is given as follows:

$$
\begin{array}{lll}
[C_1] \mapsto e_2 - e_5 & [C_2] \mapsto \ell - e_2 - e_3 - e_6 & [C_3] \mapsto e_6 \\
[C_4] \mapsto e_3 - e_6 & [C_5] \mapsto \ell - e_1 - e_3 - e_4 & [C_6] \mapsto e_4 \\
[C_7] \mapsto e_1 - e_4 & [C_8] \mapsto \ell - e_1 - e_2 - e_5 & [C_9] \mapsto e_5
\end{array}
$$

Now we know the curve classes of the cubic surface X corresponding to the unbounded edges in the dual intersection complex $(B, \mathscr{P}, \varphi)$. In turn, we are able to associate to each tropical curve in $(B, \mathscr{P}, \varphi)$ the curve class of the corresponding stable log maps, by composition of the maps from Constructions 4.23 and 4.26.

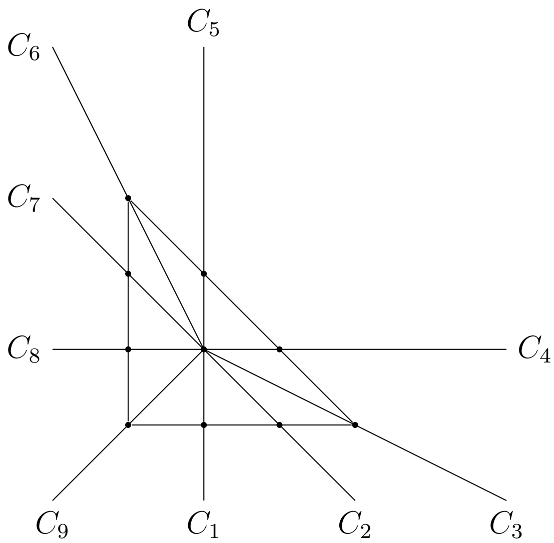


Figure 8.5: The dual intersection complexes of a smooth nef toric surface X^0 deforming to a smooth cubic surface X.

As shown in [Hos], the Weyl group W_{E_6} of type E_6 acts on $\mathrm{Pic}(X)$ as symmetries of configurations of the 27 lines and this action preserves the local Gromov-Witten invariants N_β^{loc} of X. Hence, by the log-local correspondence [GGR], it preserves the logarithmic Gromov-Witten invariants N_β of X considered here. The curve classes $\underline{\beta}$ of the cubic X giving a nonzero contribution N_β^{loc}, up to action of W_{E_6}, are given in [KMi], Table 1, along with the corresponding local BPS number n_β^{loc}.

For $d=1$ and $d=2$ there is, up to the action of W_{E_6}, only one curve class giving a nonzero contribution, so this is trivial. For $d=1$ this is $\underline{\beta}=e_6$ and the length of its orbit is 27, so $n_\beta=1$. For $d=2$ it is $\underline{\beta}=\ell-e_1$, with orbit length 27, so $n_\beta=4$. For $d=3$ there are two equivalence classes giving a nonzero contribution, with representatives ℓ and $3\ell-\sum_{i=1}^6 e_i$, respectively. Figure 8.6 shows tropical curves corresponding to stable log maps of class $\underline{\beta}=3\ell-\sum_{i=1}^6 e_i$.

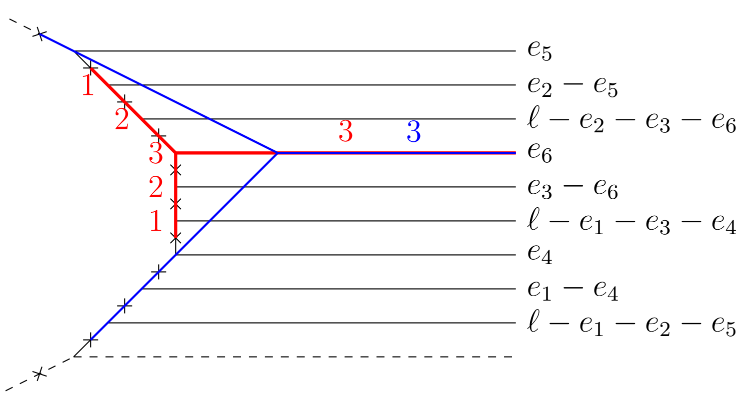


Figure 8.6: Tropical curves corresponding to $\underline{\beta}=3\ell-\sum_{i=1}^6 e_i$.

Indeed, the red tropical curve corresponds to the class

$$1\cdot(e_2-e_5)+2\cdot(\ell-e_2-e_3-e_6)+3\cdot e_6+2\cdot(e_3-e_6)+1\cdot(\ell-e_1-e_3-e_4)=3\ell-\sum_{i=1}^{6}e_i$$

and similarly for the blue tropical curve. Changing the affine singularities in which the bounded legs end may change the curve class. It turns out that for the red tropical curve any change leads to the class $\underline{\beta}=\ell$ or to a class giving a nonzero contribution. Its multiplicity is 18, so together with the choice of outgoing edge this gives a contribution of 54 to $n_{3\ell-\sum_{i=1}^6 e_i}$. For the blue tropical curve there are two changes leading again to $\underline{\beta}=3\ell-\sum_{i=1}^6 e_i$ and six changes leading to $\underline{\beta}=\ell$. The multiplicity of any of these tropical curves is 3. Together with the choice of outgoing edge this gives a contribution of $3\cdot 3\cdot 3=27$ to $n_{3\ell-\sum_{i=1}^6 e_i}$. The orbit length is 1, so $n_{3\ell-\sum_{i=1}^6 e_i}=81$. The orbit length of ℓ is 72, so $n_\ell=(729-81)/72=9$. So, in agreement with [KMi], Table 1, we have:

$n_{e_i}=1$	$n_{\ell-e_i}=4$	$n_\ell=9$	$n_{3\ell-\sum_{i=1}^6 e_i}=81$

Table 8.9: The log BPS numbers n_β of the cubic surface for $d\leq 3$.

Figure 8.7 shows tropical curves corresponding to stable log maps of class $\underline{\beta}=\ell$. In particular, any change of bounded legs of the green tropical curves still leads to class $\underline{\beta}=\ell$. For instance, the red tropical curve has class

$$1\cdot(e_2-e_5)+2\cdot(\ell-e_2-e_3-e_6)+3\cdot e_6+1\cdot(e_3-e_6)=2\ell-e_2-e_3-e_5$$

Under the action of W_{E_6} this is equivalent to $\underline{\beta}=2\ell-e_1-e_2-e_3$ and in turn to

$$2\cdot(2\ell-e_1-e_2-e_3)-(\ell-e_2-e_3)-(\ell-e_1-e_3)-(\ell-e_1-e_2)=\ell.$$

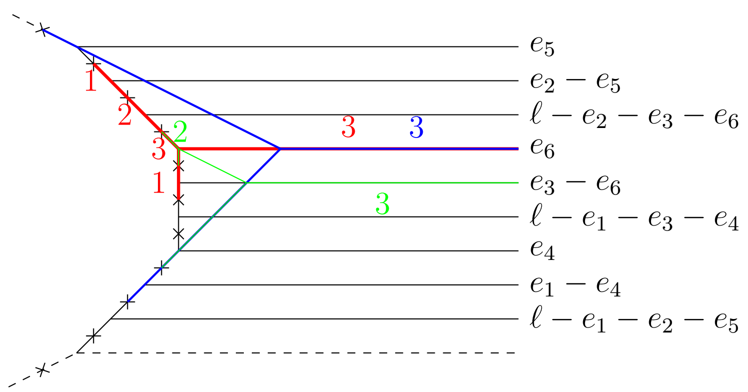


Figure 8.7: Tropical curves corresponding to $\underline{\beta}=\ell$.

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