Optimized coordinates for asymptotically conical manifolds

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Abstract

The present thesis consists of two parts. The overarching theme is the notion of a Riemannian cone.

The first part is concerned with linear stability of a certain type of Ricci solitons. Ricci solitons are generalizations of Einstein metrics. We show that if (B, g_B, Z) is a Riemannian cone and (F, g_F) is an Einstein manifold with Einstein constant μ, then the product manifold $(M, g) := (B \times F, g_B \oplus g_F)$ is a gradient Ricci soliton with Ricci potential $w := \frac{\mu}{2}|Z|^2$. There is a connection between linear stability of M and that of F. We show that if (F, g_F) is linearly unstable, then the product Ricci soliton is linearly unstable, too.

In the second part, which is the main matter of this thesis, we turn our attention to Ricci-flat asymptotically conical manifolds. The main goal of this part is to show that Ricci-flatness is a strong enough assumption prove decay rates for various tensor fields, and in particular that Ricci-flatness may be used to find suitable asymptotic coordinates in which the decay rate is optimized.

After a short study of the geometry of Riemannian cone metrics g_{cone}, we show that many geometrically interesting Laplace-type operators admit a structure that allows for a decomposition similar to the standard formula expressing the Laplacian in spherical coordinates. The role of the spherical Laplacian is played by the so-called tangential operator. After an explicit calculation, we determine the spectrum and eigenfields of the tangential operators of the Laplace–Beltrami operator $\Delta_B^{g_{\mathrm{cone}}}$, the Hodge Laplacian $\Delta_H^{g_{\mathrm{cone}}}$ on 1-forms and the Einstein operator $\Delta_E^{g_{\mathrm{cone}}}$ on symmetric 2-tensor fields.

After this, we turn our attention to decaying harmonic symmetric 2-tensor fields, and show that an explicit formula may be derived for the decay rate, involving only the spectra of the tangential operators we have just dealt with.

Next, we consider asymptotically conical manifolds. These are Riemannian manifolds (M, g) for which we can find a diffeomorphism ϕ, called the asymptotic chart, which maps $M \setminus K$ to an infinite frustum of a Riemannian cone with metric g_{cone} and such that the covariant derivatives difference of the cone metric and the pushforward metric $\phi_* g$ decays with a prescribed rate τ in terms of the radial coordinate (the radial coordinate describes the position along the "axis" of the cone). The two main components of this definition are the asymptotic chart ϕ and the decay rate τ. The rest of the thesis is dedicated to showing that we can find a suitable asymptotic chart in which the decay rate is optimized (and is the same as in the kernel of the Einstein operator on a cone).

For this, we first consider decaying $\Delta_E^{g_{\mathrm{ac}}}$-harmonic 2-tensor fields and determine that they admit the same decay rate as their $\Delta_E^{g_{\mathrm{cone}}}$-harmonic analogues. The proof is similar to the conical case but, crucially, an iterative procedure needs to be introduced.

Next, as usual for partial differential equations with geometric origins, we introduce a gauging borrowing ideas from the study of the Ricci-flow. This gauging $-2\operatorname{Ric}^g + \mathcal{L}_{V(g, g_{\mathrm{ac}})} g = 0$ leads to a quasilinear partial differential equation. The iterative procedure from before may be used to determine the decay rate of the difference tensor between gauged metrics and the reference metric.

Not all metrics are gauged but it can be shown using an argument which is based on the implicit function theorem, that in a small enough neighbourhood $\mathcal{U}$ of the asymptotically conical metric g, metrics can be pulled back to a metric for which the term with the Lie

derivative disappears. Consequently, for these metrics the gaugedness condition is equivalent to the condition of Ricci flatness.

We can construct a family of metrics $(g_R)_R$ interpolating between the asymptotically conical metric g and the pullback of the cone metric such that g_R coincides with the pullback metric of g_{cone} outside an ever increasing compact set. This family converges to g_{ac}, so in particular, for large enough R, the family lies in the neighbourhood $\mathcal{U}$, therefore it may be uniquely pulled back to a metric with vanishing Lie derivative term via a map ψ. Using this ψ and the original asymptotical coordinates ϕ we can construct a new asymptotic chart in which g has the optimalized decay rate.

Zusammenfassung

Diese Dissertation besteht aus zwei Teilen, die mit durch das Konzept des Riemannschen Kegels verbunden sind.

Im ersten Teil geht es um die lineare Stabilität einiger Ricci-Solitonen, eine Klasse von Riemannmetriken, die Klasse von Einsteinmetriken verallgemeinert. Insbesondere wird gezeigt, dass wenn (M, g_M, Z) ein riemannscher Kegel ist und (F, g_F) eine Einsteinmannigfaltigkeit mit Einsteinkonstante μ, dann ist die Produktmannigfaltigkeit $(M, g) := (B \times F, g_B \oplus g_F)$ ein Riccisoliton des Gradiententyps mit Riccipotential $w := \frac{\mu}{2}|Z|^2$. Es gibt einen gewissen Zusammenhang zwischen der linearen Stabilität von M und F. Wir zeigen, dass wenn (F, g_F) linear instabil ist, so ist das Produkt-Riccisoliton auch linear instabil.

Im zweiten Teil, dem Hauptteil der Dissertation, betrachten wir Ricciflache asymptotisch konische Mannigfaltigkeiten. Der Zweck dieses Teils ist es zu zeigen, dass die Annahme der Ricciflachheit stark genug ist um die Abfallraten bestimmter Tensorfelder zu berechnen und inbesondere dass diese Annahme stark genug sei für die Konstruktion passender asymptotischer Koordinaten, bezüglich der die Abfallrate verbessert ist.

Nach einem Kapitel über die Geometrie der riemannschen Kegelmetriken g_{cone} zeigen wir, dass die Struktur vieler geometrisch interessanter laplaceartiger Operatoren eine Darstellung zulässt, die die Standardformel für "den Laplaceoperator in Polarkoordinaten" ähneln. Die Rolle des sphärischen Laplaceoperators wird in diesem allgemeineren Fall durch den sogenannten Tangentialoperator übernommen. Wir führen explizite Rechnungen vor, die das Spektrum und die Eigentensorfelder des Laplace–Beltrami-Operators $\Delta_B^{g_{\text{cone}}}$, des Hodge–Laplace-Operators $\Delta_H^{g_{\text{cone}}}$ auf 1-Formen und des Einsteinoperators $\Delta_E^{g_{\text{cone}}}$ auf symmetrischen 2-Tensorfelder bestimmen.

Nach dieser Rechnung werden wir uns an das Abfallverhalten $\Delta_E^{g_{\text{cone}}}$-harmonische symmetrische 2-Tensorfelder. Wir zeigen, dass sich eine explizite Formel für die Abfallrate herleiten lässt, die nur vom Spektrum der früher betrachteten Tangentialoperatoren abhängt.

Nach dieser Vorarbeit geht es weiter mit asymptotisch konischen Mannigfaltigkeiten. Eine riemannsche Mannigfaltigkeit (M, g) heißt asymptotisch konisch, wenn es einen Diffeomorphismus ϕ (die asymptotische Karte) gibt, die die Mannigfaltigkeit außerhalb eines Kompaktums nach einem Kegel g_{cone} ohne seine Spitze abbildet und wenn die Differenz der Pushforwardmetrik $\phi_* g$ und der Kegelmetrik g_{cone} mit einer Abfallrate τ bezüglich der Radialkoordinate abfällt (und die Ableitungen fallen entsprechend ab). Hier steht die Radialkoordinate für die Position auf der "Kegelachse". Die für uns wichtigste Komponente im vorherigen Satz sind die asymptotische Karte ϕ und die Abfallrate τ. Das Ziel im Rest dieser Dissertation ist zu zeigen, dass sich die Abfallrate durch die Wahl einer passenden Karte verbessern lässt (und dass diese Abfallrate mit der Abfallrate für $\Delta_E^{g_{\text{cone}}}$ übereinstimmt).

Als erster Schritt betrachten wir Δ_E^{g}-harmonische 2-Tensorfelder und zeigen, dass sie die gleiche Abfallrate haben, wie im $\Delta_E^{g_{\text{cone}}}$-harmonischen Fall. Der Beweis geht analog, aber ein iteratives Verfahren muss eingeführt werden.

Wie üblich im Kontext von partielle Differentialgleichungen geometrischen Ursprungs, führen wir eine Eichung ein. Hierfür entlehnen wir ein Konzept aus der Studie des Ricciflusses. Die Eichung $-2 \operatorname{Ric}^g + \mathcal{L}_{V(g, g_{\text{ac}})} g = 0$ führt zu einer quasilinearen partiellen Differentialgleichung. Das frühere iterative Verfahren kann hier wieder eingesetzt werden um die Abfallrate der Differenz zwischen einer geeichten Metrik und der Referenzmetrik zu

bestimmen.

Es kann mit einem Implizitfuntionssatzargument gezeigt werden, dass jede Metrik in einer genügend kleiner Umgebung der Referenzmetrik g_{ac} durch einen Diffeomorphismus zurückgezogen werden kann, sodass der Term mit der Lieableitung verschwindet. Für diese zurückgezogene Metrik ist also die Eichung gleich der Bedingung der Ricciflachheit.

Es lässt sich eine Familie von Metriken $(g_R)_R$ konstruieren, die zwischen der exakten Kegelmetrik und der asymptotisch konischen Metrik g_{ac} interpoliert, sodass g_R mit g_{ac} innerhalb eines mit R größer werdenden Kompaktums übereinstimmt und sodass g_R mit dem exakten Kegelmetrik außerhalb eines mit R immer größer werdenden Komapktums übereinstimmt. Ferner konvergiert diese Familie nach g_{ac}, also für genügend großes R liegen die Metriken g_R in der Umgebung $\mathcal{U}$. Nehmen wir ein solches R_1. Dann kann die Metrik g_{R_1} durch einen Diffeomorphismus ψ zurückgezogen werden, sodass der Lieableitungsterm verschwindet. Man kann mithilfe von ψ und der ursprünglichen asymptotischen Karte ϕ eine neue asymptotische Karte konstruieren bezüglich dessen die verbesserte Abfallrate erreicht werden kann.

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Chapter 1

Introduction and overview

This thesis aims to make a contribution to the field of global analysis. This subfield of mathematics, also known as analysis on manifolds, may be loosely defined that applies methods from (partial) differential equations to solve differential geometric problems on manifolds or bundles of manifolds. Several applications exist in physics.

The overarching theme of this thesis is the notion of a Riemannian cone.

1.1 A type of Ricci-solitons coming from cones

The first topic is so-called gradient Ricci solitons. These are Riemannian manifolds (M, g) satisfying

$$\mathrm{Ric}^g + \frac{1}{2}\mathcal{L}_{\mathrm{grad}_g w} g = \lambda g$$

for some $\lambda \in \mathbb{R}$ and some function $w \in \mathcal{C}^\infty(M)$. (This name comes from the fact that these metrics evolve by a homothetic rescaling and a pullback under the Ricci flow [CK04]). Note that Einstein metrics are gradient Ricci solitons where $w = \text{const}$.

We call a gradient Ricci soliton (linearly) stable if the Laplace-type operator $\Delta_E^{g,w} := \Delta^g + \nabla^g_{\mathrm{grad}_g w} - 2\mathring{R}^g$ on symmetric 2-tensor fields has positive spectrum, otherwise call the gradient Ricci-soliton (linearly) unstable [Krö15b].

I have calculated that if (B, g_B) is a Riemannian cone with Euler vector field Z and (F, g_F) is an Einstein manifold with Einstein constant μ, then the Riemannian product $B \times F$ is a Ricci-soliton with Ricci potential $f := \frac{\mu}{2}|Z|^2$ (cf. Lemma 4.2). This generalizes the so-called Gaussian Ricci-soliton, cf. e.g. [Zha18]. Moreover, I have shown that if the Einstein manifold (F, g_F) is linearly unstable, then $M \times F$ is also linearly unstable (Theorem 4.10).

1.2 Optimizing the decay rate of Ricci-flat asymptotically conical manifolds

The second part of the thesis is dedicated to the study of Ricci-flat asymptotically conical manifolds, and in particular how certain tensor fields decay on such manifolds.

The Einstein operator Consider a Ricci-flat metric g. The follow so-called Einstein operator will be of central importance to us.

$$\Delta_E \colon \Gamma^\infty\left(S^2T^*M\right) \to \Gamma^\infty\left(S^2T^*M\right), \quad h \mapsto \Delta_g h - 2\overset{\circ}{R}{}^g h, \tag{1.1}$$

where Δ_g is the analysts' raw g-Laplacian and $\overset{\circ}{R}{}^g$ is a linear operator depending on the curvature of g (see Chapter 2 for precise definitions).

Riemannian cones Before we can define what asymptotically conical manifolds are, we first need to describe their asymptotics: cones. A Riemannian cone with compact link (L, g_L) is the Riemannian manifold

$$\mathrm{Cone}(L) := (\mathbb{R}^+ \times L, g_{\mathrm{cone}} := dr \otimes dr + r^2 g_L), \tag{1.2}$$

where r is the canonical coordinate on $\mathbb{R}^+$. The vector field $Z := r\partial_r$ is called the Euler vector field. (A more detailed exposition of Riemannian cones can be found in Chapter 6.1.) One easy example of a cone is the Euclidean space without the origin, which has a round sphere as its link. In general, the cone with link L is Ricci-flat if and only if the link is Einstein with Einstein constant $\dim L - 1$. In this thesis, we are interested in Ricci-flat cones only.

The Einstein operator on cones The geometry of a cone and the geometry of its link are closely related. One prominent example of this interplay is the fact that that the Einstein operator of a cone can be written in the following form

$$\Delta_E = -\nabla^2_{\partial_r, \partial_r} - \frac{\dim L}{r} \nabla_{\partial_r} + \frac{1}{r^2} \square_E$$

where $\square_E$ is the so-called tangential operator. (This formula is reminiscent of the formula for the Laplace–Beltrami operator of $\mathbb{R}^{n+1}$ in spherical coordinates and, in fact, other operators also have a similar expression, cf. Section 6.4.)

I have determined the spectrum and the eigenvectors of the operator $\square_E$ (Theorem 6.55). The spectrum consists entirely of eigenvalues and they are given by explicit functions of the spectra of certain geometrical Laplace type operators on the link L.

The eigenvectors of the tangential operator $\square_E$ are, with the appropriate scaling, covariantly constant in the r direction (cf. Corollary 6.24). Moreover, any symmetric 2-tensor field h can be expressed as an infinite linear combination of eigenvectors of $\square_E$ with r-dependent coefficients. Consequently, the equation $\Delta_E h = 0$ reduces to a system of ordinary differential equations (Lemma 9.5). This equation can be explicitly solved (Lemma 9.5), and the growth rates (the so-called exceptional values) can be determined (again Lemma 9.5) be $m_\pm(\xi)$ where ξ is an eigenvalue of $\square_E$ and $m_\pm$ are explicitly given functions (cf. Definition 9.4)

Based on the description of the spectrum of $\square_E$, I have determined the decay rate of elements in the kernel that are decaying in the first place (Proposition 9.14). The kernel of the Einstein operator on an exact cone is an important model calculation that can be mimicked with other operators as well.

Asymptotically conical manifolds Cones have a singularity at their apices and their are not complete as Riemannian manifolds. One way to remedy this fact is to consider complete Riemannian manifolds that have the cone as asymptotic behaviour. In the literature, there are several concepts for describing the asymptotic behaviour of a Riemannian metric, cf. e.g. [PT01]. These are not all equivalent to each other. The notion that is used in this dissertation is based on the concept of an asymptotic chart.

We say (cf. Definition 7.1) that a complete Riemannian manifold (M, g_{ac}) is asymptotically conical with rate $\tau \in \mathbb{R}$ if there is a compact set $K \subset M$, a number $R > 0$ and a diffeomorphism (the so-called asymptotic chart) $\phi\colon M \setminus K \to (R, \infty) \times L$ such that the covariant derivatives of the metrics satisfy

$$|\nabla^{g_{\mathrm{cone}},k}(\phi_* g_{\mathrm{ac}} - g_{\mathrm{cone}})|_{g_{\mathrm{cone}}} = O\left(r^{-\tau-k}\right) \tag{1.3}$$

for any natural number $k \in \mathbb{N}$ (with $\nabla^0 := \mathrm{id}$, of course). Here, g_{cone} is a cone metric on $\mathbb{R}^+ \times L$ (cf. Equation (1.2)), $\nabla^{g_{\mathrm{cone}}}$ denotes the Levi-Civita connection of g_{cone}, $|\cdot|$ denotes the pointwise norm induced by g_{cone} and r denotes the standard coordinate on $\mathbb{R}^+$.

Let us introduce the core $\mathrm{Core}(R)$ which is, morally speaking, the compact set where $r \leq R$. (For a precise definition, see Definition 7.4.)

The two most important ingredients are the decay rate τ and the asymptotic chart ϕ. In fact, as Deruelle and Kröncke have shown in [DK20] that, for an important class of asymptotically conical manifolds, good stability properties can be proved under the Ricci flow if one assumes a good enough decay rate. Motivated by this, one of the goals of this thesis is to show that an a priori given decay rate of a Ricci-flat asymptotically conical manifold can be optimized by choosing a different asymptotic chart, cf. also the classical result in [BKN89]).

Before attacking this problem, we consider elements in the kernel of the Einstein operator of g_{ac} under the assumption that g_{ac} is Ricci-flat. Based on Equation (1.3), the equation $\Delta_E^{g_{\mathrm{ac}}} h = 0$ may be reformulated as $\Delta_E^{g_{\mathrm{cone}}} h = F(h)$ where $F(h)$ depends linearly on h and has appropriate decay (cf. Equation (9.5) on page 113). This leads to a system of ordinary differential equations that can be solved explicitly (Lemma 9.20). Based on this, the the decay rate of elements in $\ker \Delta_E^{g_{\mathrm{ac}}}$ can be determined by an iterative procedure (Theorem 9.24).

A family of metrics Using an interpolating function, we construct a family of metrics $(g_R)_{R \geq R_1}$ which agree with g_{ac} on $\mathrm{Core}(R)$ and agree with $\phi^* g_{\mathrm{cone}}$ outside $\mathrm{Core}(2R)$. Because of this, the metric is Ricci-flat outside $\mathrm{Core}(2R) \setminus \mathrm{Core}(R)$. This family converges to g_{ac} in a weighted Sobolev sense (Proposition 8.30).

Gauging We are interested in Ricci-flat asymptotically conical manifolds. The condition of Ricci-flatness can be expressed as a second-order quasilinear partial equation in terms of the metric. As a partial differential equation of geometric origin, it is diffeomorphism invariant. This leads to the need for the choice of a gauge if we want to solve the corresponding partial differential equation. We borrow the DeTurck vector field from the study of the Ricci flow (Section 8.1). We call a Riemannian metric g gauged with respect to g_{ac} if it comes from the set

$$\mathcal{F}_{g_0} := \left\{ g \in \mathrm{Met}\,(M) \;\middle|\; -2\,\mathrm{Ric}^g + \mathcal{L}_{V(g,g_0)} g = 0 \right\}. \tag{1.4}$$

(Note that these metrics are the fixed points of the corresponding Ricci–DeTurck flow.)

The condition of gaugedness of $g_0 + h$ can be expressed in the form $\Delta_E^{g_{\mathrm{cone}}} h = F(h)$ where $F(h)$ depends nonlinearly on h and has appropriate decay (Lemma 10.5). Using an iterative argument similar to the one used in the study of the kernel of the Einstein operator of g_{ac}, one sees that there is a neighbourhood of g_{ac} in the $L^p(S^2T^*M) \cap L^\infty(S^2T^*M)$-topology in which all gauged metrics have the same decay rate we observed for elements in the kernel of the Einstein operator (Theorem 10.6).

Optimalized decay rate with a different asymptotic chart A study of the properties of the DeTurck vector field yields that there is a neighbourhood of g_0 (in a weighted Sobolev sense) in which any metric neighbourhood can be uniquely pulled back to a metric for which the DeTurck vector field vanishes (Proposition 8.23). In particular, the family (g_R), will eventually run in this neighbourhood. This way, we obtain a diffeomorphism ψ, with the help of which we can construct a new asymptotic chart where the decay rate agrees with the decay rate in the kernel of the Einstein operator (Theorem 10.8).

Chapter 2

Some differential operators on Riemannian manifolds

The goal of this chapter is twofold. First, we introduce some general notation; secondly, we collect the most important differential operators that are used in this thesis and state some of their most important properties. This is especially crucial since there are several competing signs conventions used in the literature. We will stick to the what is sometimes called the analyst's sign convention.

While the smooth structure is enough to define many differential operators on differential forms, this thesis is mainly interested in symmetric tensor fields. Here, the smooth structure is not powerful enough to induce natural operators but a Riemannian metric is. The proofs for these statements, where not indicated otherwise, can be found in [Bes87] or [Lee18].

In the following, M is assumed to be a smooth manifold and we fix a Riemannian metric $g \in \operatorname{Met}(M)$. Note that g induces pointwise inner products on tensor fields (which we will denote by $\langle \cdot, \cdot \rangle_g$) and in particular we have the following global L^2-scalar product

$$(S_1, S_2)_g := \int_M \langle S_1, S_2 \rangle_g \operatorname{vol}_g,$$

where S_1 and S_2 are compactly supported smooth tensor fields of the same type, and vol_g is the volume form induced by the metric g. Given a differential operator D, we define its formal adjoint D^* with respect to g via the condition $(S_1, DS_1)_g = (D^* S_1, S_1)_g$ for any compactly supported smooth tensor fields S_1 and S_2 of the appropriate type.

Sometimes, a weighted volume form will be used for the norms. If $\rho\colon M \to \mathbb{R}^+$ is a smooth function, then we introduce

$$(T, T')_{g,\rho} := \int_M \langle T, T' \rangle_g \, \rho \operatorname{vol}_g .$$

The set of smooth sections of a bundle $E \to M$ is denoted by $\Gamma^\infty(E)$; the set of smooth sections of the same bundle with compact support is denoted by $\Gamma^\infty_c(E)$. The trivial bundle with fibre V over the manifold M is denoted by $\underline{V}_M$, or, if the base manifold is clear from context, by $\underline{V}$. In particular, smooth functions on the smooth manifold M can be identified with $\Gamma^\infty(\underline{\mathbb{R}}_M)$. We will denote the bundle of symmetric p tensors by S^pT^*M, and use the following symmetric tensor product

$$\odot\colon (T^*M)^{\otimes p} \otimes (T^*M)^{\otimes q} \to S^{p+q}T^*M, h \otimes k \mapsto h \odot h,$$

where

$$(h \odot k)(X_1, \ldots, X_{p+q}) := \sum_{\pi \in S_{p+q}} h(X_{\pi(1)}, \ldots, X_{\pi(p)})k(X_{\pi(p+1)}, \ldots, X_{\pi(p+q)}),$$

where S_{p+q} denotes the permutation group of $(p+q)$ symbols In particular, for $p = q = 1$ and $\alpha, \beta \in T^*M$, we obtain $(\alpha \odot \beta)(X, Y) = \alpha(X)\beta(Y) + \alpha(Y)\beta(X)$ for any $X, Y \in TM$.

2.1 Zeroth-order operators

We define the Riemannian curvature tensor with the sign convention

$$R^g(X, Y)Z := \nabla^{g,2}_{X,Y} Z - \nabla^{g,2}_{X,Y} Z.$$

The curvature tensor induces an action of symmetric 2-tensor fields via the formula

$$\overset{\circ}{R}{}^g\colon S^2T^*M \to S^2T^*M, h \mapsto \overset{\circ}{R}{}^g h \text{ with } (\overset{\circ}{R}{}^g h)(X, Y) := \sum_{i=1}^{\dim M} h(R^g(e_i, X)Y, e_i),$$

where $\{e_i \mid i = 1, \ldots, \dim M\}$ is a g-orthonormal frame. We call

$$\mathrm{Ric}^g := \overset{\circ}{R}{}^g g$$

the Ricci tensor of g, and

$$\mathrm{scal}_g := \mathrm{Tr}_g \mathrm{Ric}^g$$

the scalar curvature of g.

2.2 First-order operators

The divergence operator Consider now the cotangent bundle T^*M. Since the Levi-Civita connection of g maps functions to 1-forms, $\nabla^g\colon \Gamma^\infty_c(\mathbb{R}_M) \to \Gamma^\infty_c(T^*M)$, we may define an operator δ^g, called the divergence (or codifferential) of the metric g as the formal adjoint of the Levi-Civita connection on functions:

$$\delta^g := (\nabla^g)^*\colon \Gamma^\infty_c(T^*M) \to \Gamma^\infty_c(\mathbb{R}_M).$$

One can show that at a point $p \in M$, we have

$$(\delta^g\omega)(p) = -\left(\sum_{i=1}^{\dim M} (\nabla^g_{e_i}\omega)(e_i)\right)(p),$$

where $\{e_i \mid i = 1, \ldots, \dim M\}$ is a g-orthonormal frame around p and g^{ij} is the matrix representing the induced metric g^{-1} on T^*M.

In fact, the divergence operator may be defined on higher-rank tensor fields, too. For instance, we define the divergence if $h \in \Gamma_c^\infty\left(S^2T^*M\right)$ is a symmetric 2-tensor field, then we define

$$(\delta^g h)(X) := -\sum_{i=1}^{\dim M} (\nabla^g_{e_i} h)(e_i, X),$$

where $\{e_i \,| i = 1, \ldots, \dim M\}$ is a g-orthonormal frame and $X \in TM$.

Lemma 2.1. *Let* $f \in \Gamma^\infty(\mathbb{R}_M)$*,* $\omega \in \Gamma^\infty(T^*M)$ *and* $h \in \Gamma^\infty\left(S^2T^*M\right)$*. Then* $\delta^g(f\omega) = f\delta^g\omega - \omega(\operatorname{grad}_g f)$ *and* $\delta^g(fh) = f\delta^g h - h(\operatorname{grad}_g f, \cdot)$*.* □

The codivergence operator The formal adjoint of the divergence operator on symmetric 2-tensor fields is called the covidergence operator, and it is given by

$$(\delta^{g,\star}\omega)(X,Y) := \tfrac{1}{2}\left((\nabla^g_X\omega)(Y) + (\nabla^g_Y\omega)(X)\right),$$

where $\omega \in \Gamma^\infty(T^*M)$ and $X, Y \in TM$.

The following elementary lemma is useful for calculations.

Lemma 2.2. *For any* $\omega \in \Gamma^\infty(T^*M)$*, one has* $\operatorname{Tr}_g \delta^{g,\star}\omega = -\omega$*.* □

2.3 Second-order operators

In general, we will call the operator $\Delta_g := (\nabla^g)^*\nabla^g$ the raw Laplacian. The name is motivated by the fact that many geometrically interesting Laplace-type operators have a zeroth-order contribution (usually depending on the curvature of g) to the raw Laplacian.

The Laplace–Beltrami operator The Laplace–Beltrami operator Δ^g_B is the Laplacian type operator defined on smooth functions via $\Delta^g_B := (\nabla^g)^*\nabla^g$. One can easily show that at a point $p \in M$, one has $(\Delta^g_B f)(p) := -\left(\sum_{i,j=1}^{\dim M} g^{ij}\nabla^{g,2}_{e_i,e_j} f\right)(p)$ where $\{e_i \,| i = 1, \ldots, \dim M\}$ is a g-orthonormal frame around p and g^{ij} is the matrix representing the induced metric g^{-1} on T^*M. More succinctly, we may write this as

$$\Delta^g_B = -g^{-1} \circ \nabla^{g,2}.$$

Note that this is the opposite sign convention to the usual differential geometric Laplacian.

The Hodge Laplacian Since symmetric 1-tensor fields coincide with differential 1-forms, it makes sense to consider the Hodge Laplacian Δ^g_H on symmetric 1-tensor fields. The Hodge Laplacian is related to the raw Laplacian by the following well-known formula.

Lemma 2.3 (Weitzenböck)**.** $\Delta^g_H = (\nabla^g)^*(\nabla^g) + \operatorname{Ric}^g$ *where* Ric^g *denotes the Weitzenböck curvature operator [Pet12].*

The well-known formula $\Delta_H = d \circ \delta + \delta \circ d$, where $\delta\colon \Gamma^\infty\left(\wedge^k T^*M\right) \to \Gamma^\infty\left(\wedge^{k-1}T^*M\right)$ is the g-codifferential, has the following analogue for symmetric tensor fields.

Lemma 2.4. $\delta^{g_L}\delta^{g_L,\star}\omega = \frac{1}{2}\Delta_H g_L \omega + d\delta^{g_L}\omega - \operatorname{Ric}^{g_L}\omega$

Proof. Let $\omega \in T^*M$ and $X \in TM$. Let $p \in M$ and let $\{e_i \,|i = 1, \dots, \dim M\}$ be a g-orthonormal normal frame around p (i.e. $g_p(e_i(p), e_j(p) = \delta_{ij}$ and $(\nabla^g e_i)(p) = 0$). We will perform the calculations at p.

One checks by straightforward calculations that

$$(\delta^g\delta^{g,\star}\omega)(X) = \frac{1}{2}(\Delta_g\omega)(X) - \frac{1}{2}\sum_{i=1}^{\dim M}(\nabla^{g,2}_{e_i,X}\omega)(e_i).$$

On the other hand

$$(d\delta^g\omega)(X) = -\sum_{i=1}^{\dim M}(\nabla^g_X\nabla^g_{e_i}\omega)(e_i) - (\nabla^g_{e_i}\omega)(\nabla^g_X e_i) = -\sum_{i=1}^{\dim M}(\nabla^{g,2}_{X,e_i}\omega)(e_i).$$

Now

$$\begin{aligned}\left(\delta^g\delta^{g,\star}\omega - \frac{1}{2}d\delta^g\omega\right)(X) &= \frac{1}{2}(\Delta_g\omega)(X) - \frac{1}{2}\sum_{i=1}^{\dim M}(\nabla^{g,2}_{e_i,X}\omega - \nabla^{g,2}_{X,e_i}\omega)(e_i)\\ &= \frac{1}{2}(\Delta_g\omega)(X) - \frac{1}{2}\sum_{i=1}^{\dim M}(R(e_i,X)\omega)(e_i)\\ &= \frac{1}{2}(\Delta_g\omega)(X) - \frac{1}{2}\operatorname{Ric}^g(\omega)(X)\\ &= \frac{1}{2}(\Delta^g_H\omega)(X) - \operatorname{Ric}^g(\omega)(X).\end{aligned}$$

□

The Lichnerowicz Laplacian and the Einstein operator The geometrically natural Laplace operator for symmetric 2-tensor fields is the so-called Lichnerowicz Laplacian

$$\Delta^g_L\colon \Gamma^\infty\left(S^2T^*M\right) \to \mathcal{C}^\infty(S^2T^*M), h \mapsto (\nabla^g)^*(\nabla^g)h - 2\overset{\circ}{R}{}^g h + \operatorname{Ric}^g \circ h + h \circ \operatorname{Ric}^g.$$

If the metric g is Einstein, then we define the Einstein operator as

$$\Delta^g_E\colon \Gamma^\infty\left(S^2T^*M\right) \to \mathcal{C}^\infty(S^2T^*M), h \mapsto (\nabla^g)^*(\nabla^g)h - 2\overset{\circ}{R}{}^g h.$$

Note that for an Einstein metric with Einstein constant μ, we have $\operatorname{Ric}^g = \mu\,\mathrm{id}_{TM}$ and thus $\Delta^g_L = \Delta^g_E + 2\mu\,\mathrm{id}_{S^2T^*M}$.

The Laplace–Beltrami operator (on (symmetric) 0-tensor fields, i.e. functions), the Hodge Laplacian on (symmetric) 1-tensor fields and the Lichnerowicz Laplacian on symmetric 2-tensor fields are intimately connected.

Proposition 2.5 ([Krö15a, Lemma 4.2])**.** *$\Delta^g_L(fg) = (\Delta^g_B f)g$, $\operatorname{Tr}_g\Delta^g_L h = \Delta_B(\operatorname{Tr}_g h)$ for any $f \in \Gamma^\infty(\mathbb{R}_M)$ and any $h \in \Gamma^\infty\left(S^2T^*M\right)$. Moreover, if Ric^g is parallel (e.g. if g is Einstein), then $\Delta^g_L \circ \delta^{g,\star} = \delta^{g,\star}\circ\Delta^g_H$, $\delta^g\circ\Delta^g_L = \Delta^g_H\circ\delta^g$, $\Delta^g_L\circ\nabla^{g,2} = \nabla^{g,2}\circ\Delta^g_B$. Moreover, $d\circ\Delta^g_B = \Delta^g_H\circ d$ on functions.*

2.4 The difference of second covariant derivatives

The goal of this section is to characterise the difference of two second covariant derivatives using the difference of the first covariant derivatives.

Lemma 2.6. *Let (M, ∇^M) be a manifold with connection and let $E \to M$ be a vector bundle over M. Moreover, let $\nabla, \bar{\nabla}$ be two connections on E and set $T := \bar{\nabla} - \nabla \in \Gamma^\infty(T^*M \otimes \operatorname{End} E)$. Then, for any $h \in \Gamma^\infty(E)$, one has*

$$\bar{\nabla}^2 h - \nabla^2 h = T \star (\nabla h) + (\nabla T) \star h + T \star T \star h,$$

where $\star$ denotes various linear combinations of tensorial contractions (with coefficients ∇-covariantly constant).

Proof. For vector fields $X, Y \in \Gamma^\infty(TM)$, we have

$$\begin{aligned}
\bar{\nabla}^2_{X,Y} h &= \bar{\nabla}_X(\bar{\nabla}_Y h) - \bar{\nabla}_{\nabla^M_X Y} h \\
&= \bar{\nabla}_X(\nabla_Y h + T(Y, h))\nabla_{\nabla^M_X Y} h - T(\nabla^M_X Y, h) \\
&= \nabla_X(\nabla_Y h) + T(X, \nabla_Y h) + \nabla_X(T(Y, h)) + T(X, T(Y, h)) - \nabla_{\nabla^M_X Y} h - T(\nabla^M_X Y, h) \\
&= \nabla_X(\nabla_Y h) + T(X, \nabla_Y h) + (\nabla_X T)(Y, h) + T(\nabla^M_X Y, h) + T(Y, \nabla_X h) \\
&\quad\; + T(X, T(Y, h)) - \nabla_{\nabla^M_X Y} h - T(\nabla^M_X Y, h) \\
&= \nabla^2_{X,Y} h + T(Y, \nabla_X h) + T(X, \nabla_Y h) + (\nabla_X T)(Y, h) + T(X, T(Y, h)) \\
&= (\nabla^2 h + T \star (\nabla h) + (\nabla T) \star h + T \star T \star h)(X, Y).
\end{aligned}$$

Rearranging yields the claim. ☐

Remark 2.7. *Note that the tensor T is locally given by the difference of the Christoffel symbols ${T_{i\alpha}}^\beta = {\Gamma\bar{\nabla}_{i\alpha}}^\beta - {\Gamma\nabla_{i\alpha}}^\beta$. Accordingly, we may write the statement of Lemma 2.6 also as*

$$\bar{\nabla}^2 h - \nabla^2 h = (\Gamma\bar{\nabla} - \Gamma\nabla) \star h + \nabla(\Gamma\bar{\nabla} - \Gamma\nabla) \star h + (\Gamma\bar{\nabla} - \Gamma\nabla) \star (\Gamma\bar{\nabla} - \Gamma\nabla).$$

2.5 A few words about dual metrics

In our calculations, we will face the situation often, when the difference of two dual metrics appears in a formula. The next elementary lemma explains how this is related to the difference of the original metrics.

Lemma 2.8. *Let g, g_0 be inner products on a vector space V. Then*

$$\sharp^g - \sharp^{g_0} = -\sharp^{g_0} \circ (\flat^g - \flat^{g_0}) \circ \sharp^g,$$

where we used self-explanatory notation for the musical isomorphisms induced by the metrics. In local coordinates, this equality reads $g^{kl} - (g_0)^{kl} = -(g_0)^{ka}(g_{ab} - (g_0)_{ab})g^{bl}$. ☐

Another useful formula is how the covariant derivative of a dual metric relates to the covariant derivative of the original metric.

Lemma 2.9. *Let $g \in \mathrm{Met}(M)$ be a metric on the smooth manifold M and let ∇ be a connection on M (not necessarily the Levi-Civita connection of g!). Then*

$$\nabla(g^{-1}) = -(\nabla g) \circ (\sharp^g \otimes \sharp^g).$$

In local coordinates, $\nabla_a g^{ij} = -g^{ik}(\nabla_a g_{kp})g^{pj}$. In particular, the dual metric g^{-1} is parallel with respect to the connection induced by Levi-Civita connection ∇^g of the metric g.

Proof. Let $\lambda, \mu \in \Gamma^\infty(T^*M)$ and $X \in \Gamma^\infty(TM)$. Then

$$\begin{aligned}\mu(\nabla_X(\lambda^{\sharp^g})) &= X^a \nabla_a(\lambda_b g^{bc})\mu_c dx^c \\ &= X^a((\nabla_a \lambda_b)g^{bc} + \lambda_b(\nabla_a g^{bc}))\mu_c dx^c \\ &= \mu((\nabla_X \lambda)^{\sharp^g}) + (\nabla_X(g^{-1}))(\lambda, \mu),\end{aligned}$$

thus

$$\mu(\nabla_X(\lambda^{\sharp^g}) - (\nabla_X \lambda)^{\sharp^g}) = (\nabla_X(g^{-1}))(\lambda, \mu),$$

and evidently, this holds also with the roles of λ and μ exchanged. Therefore,

$$\begin{aligned}(\nabla_X(g^{-1}))(\lambda, \mu) &= X((g^{-1})(\lambda, \mu)) - (g^{-1})(\nabla_X \lambda, \mu) - (g^{-1})(\lambda, \nabla_X \mu) \\ &= X(g(\lambda^{\sharp^g}, \mu^{\sharp^g})) - g((\nabla_X \lambda)^{\sharp^g}, \mu^{\sharp^g}) - g(\lambda^{\sharp^g}, (\nabla_X \mu)^{\sharp^g}) \\ &= (\nabla_X g)(\lambda^{\sharp^g}, \mu^{\sharp^g}) + g(\nabla_X(\lambda^{\sharp^g}), \mu^{\sharp^g}) + g(\lambda^{\sharp^g}, \nabla_X(\mu^{\sharp^g})) \\ &\quad - g((\nabla_X \lambda)^{\sharp^g}, \mu) - g(\lambda^{\sharp^g}, (\nabla_X \mu)^{\sharp^g}) \\ &= (\nabla_X g)(\lambda^{\sharp^g}, \mu^{\sharp^g}) \\ &\quad + g(\nabla_X(\lambda^{\sharp^g}) - (\nabla_X \lambda)^{\sharp^g}, \mu^{\sharp^g}) \\ &\quad + g(\lambda^{\sharp^g}, \nabla_X(\mu^{\sharp^g}) - (\nabla_X \mu)^{\sharp^g}) \\ &= (\nabla_X g)(\lambda^{\sharp^g}, \mu^{\sharp^g}) \\ &\quad + \mu((\nabla_X(\lambda^{\sharp^g}) - (\nabla_X \lambda)^{\sharp^g}) \\ &\quad + \lambda(\nabla_X(\mu^{\sharp^g}) - (\nabla_X \mu)^{\sharp^g}) \\ &= (\nabla_X g)(\lambda^{\sharp^g}, \mu^{\sharp^g}) + 2(\nabla_X(g^{-1}))(\lambda, \mu).\end{aligned}$$

Rearranging gives the claim. □

Part I

Linear stability of certain product Ricci solitons

Chapter 3

The Einstein operator on warped products

Here we calculate quantities related to the Einstein operator on warped products. These are of independent interest as well but in this thesis we will apply them only in the product case.

3.1 Setup, notations

We will use the notations and terminology of Chapters 2 and 4 of [Che11].

Let (B, g_B) and (F, g_F) be pseudo-Riemannian manifolds and let $f\colon B \to \mathbb{R}^+$ be a smooth function. Let $(M, g) := B \times_f F$ be the warped product of B and F with the warping factor f, i.e. $g = \pi^* g_B + (f \circ \pi)\eta^* g_F$ where $\pi\colon M \to B$ and $\eta\colon M \to F$ denote the projections. These projections will be suppressed later. Introduce the vector field $F \in \mathfrak{X}(M)$ by $F := \operatorname{grad}_g \ln f$ for the sake of brevity.

We will denote the horizontal lifts of vectors on B and horizontal components of vectors in M by a superscript ^{H}. Similarly, the vertical lifts of vectors on F and vertical components of vectors on M will be denoted by a superscript ^{V}.

Note that $B \times \{f\}$ as a submanifold of M is totally geodesic for any $f \in F$ and $\{b\} \times F$ is totally umbilical in M for all $b \in B$. Moreover it is easy to see by [Che11, Proposition 4.1] that the mean curvature vector of F in M is $H = -\operatorname{grad}_g \ln f = -F$. In the direct product case, $f = 1$ and consequently $F = 0$.

In the following, we will be interested in symmetric 2-tensors on M. For tensor fields that are horizontal or vertical lifts of tensor fields on B or F, a subscript will be used. In particular: $\alpha_B \in \mathcal{C}^\infty(B)$, $\alpha_F \in \mathcal{C}^\infty(F)$, $h_B \in S^2 B$, $h_F \in S^2 F$ and $h_B \odot h_F \in \Omega^1(B) \odot \Omega^1(F)$ will be used where $\odot$ denotes the symmetric tensor product. We will use the adjectives basic and fibrous. As for vectors, $X, Y, Z, Z' \in \mathfrak{X}(B)$ and $U, V, W, W' \in \mathfrak{X}(F)$.

In some calculations it is useful to have a g-orthonormal frame in M. Let $\{e_i \,|i \in I_B\}$ be a g-orthonormal basis for B and let $\{e_a \,|a \in I_F\}$ be a g-orthonormal basis for F. Then $\{e_i \,|i \in I := I_B \cup I_F\}$ is a g-orthonormal basis for M. Also note that in this case $\{f e_a \,|a \in I_F\}$ is a g_F-orthonormal frame in F. Recall that the induced scalar product of tensors can be calculated as

$$\langle T, T' \rangle_g = \sum_{i_1,\dots,i_k \in I} T(e_{i_1}, \dots, e_{i_k}) T'(e_{i_1}, \dots, e_{i_k})$$

for any k-tensors T, T'. We set $|T|_g^2 := \langle T, T \rangle_g$ for any k-tensor T.

We collect a few things that we will use routinely.

Lemma 3.1. *1. $\langle T_B, T'_B \rangle_g = \langle T_B, T'_B \rangle_{g_B}$ for all k-tensor fields T_B, T'_B on B*

2. $\langle T_F, T'_F \rangle_g = f^{2k} \langle T_F, T'_F \rangle_{g_F}$ for all k-tensor fields T_F, T'_F on F.

3. $|g_B|_{g_B}^2 = \dim B$, $|g_F|_{g_F}^2 = \dim F$ and $|g|_g^2 = \dim M$.

4. $|d \ln f|_g^2 = |F|_g^2$.

5. For the volume forms $\mathrm{vol}_g = f^{2 \dim F} \mathrm{vol}_{g_B} \wedge \mathrm{vol}_{g_F}$.

6. Let $\alpha \in \mathcal{C}^\infty(M)$ and let T and T' be k-tensor fields on M. Then

$$\langle \nabla^g T, d\alpha \otimes T' \rangle_g = \left\langle \nabla^g_{\mathrm{grad}_g \alpha} T, T \right\rangle_g .$$

Proof. The first four claims follow by straightforward calculations. The fifth claim can be proved in local coordinates.

As for the sixth claim, first note that $g(\sum_{i \in I} d\alpha(e_i) e_i, e_j) = d\alpha(e_j) = g(\mathrm{grad}_g \alpha, e_j)$ for all $j \in I$, therefore $\sum_{i \in I} d\alpha(e_i) e_i = \mathrm{grad}_g \alpha$. As a consequence

$$\begin{aligned}
\langle \nabla^g T, d\alpha \otimes T' \rangle_g &= \sum_{i, j_1, \dots, j_k \in I} (\nabla^g_{e_i} T)(e_{j_1}, \dots, e_{j_k}) d\alpha(e_i) T(e_{j_1}, \dots, e_{j_k}) \\
&= \sum_{j_1, \dots, j_k \in I} (\nabla^g_{\sum_{i \in I} d\alpha(e_i) e_i} T)(e_{j_1}, \dots, e_{j_k}) T(e_{j_1}, \dots, e_{j_k}) \\
&= \sum_{j_1, \dots, j_k \in I} (\nabla^g_{\mathrm{grad}_g \alpha} T)(e_{j_1}, \dots, e_{j_k}) T(e_{j_1}, \dots, e_{j_k}) \\
&= \left\langle \nabla^g_{\mathrm{grad}_g \alpha} T, T \right\rangle_g .
\end{aligned}$$

□

We introduce partial traces of 2-tensors as

$$\mathrm{PTr}^g_B h := \sum_{i \in I_B} (h(e_i, e_i)) \quad \text{and} \quad \mathrm{PTr}^g_F h := \sum_{i \in I_F} (h(e_i, e_i)).$$

3.2 The curvature part pointwise: $\mathring{R}h$ and $\left\langle \mathring{R}h, h \right\rangle_g$

The zeroth order part of the Einstein operator is related to the curvature tensor. The curvature tensor acts on the set of symmetric 2-forms by the formula

$$\mathring{R}h(X, Y) := \mathrm{Tr}_g\, h(R(\cdot, X)Y, \cdot) = \sum_{i \in I} h(R(e_i, X)Y, e_i),$$

where $\{e_i \,| i \in I\}$ is a g-orthonormal frame in TM. This action is clearly $\mathcal{C}^\infty(M)$-linear.

Lemma 3.2. *For $h \in S^2M$ and $Y, Z \in \mathfrak{X}(B)$, $V, W \in \mathfrak{X}(F)$, we have*

$$\begin{aligned}
\overset{\circ}{R}h(Y,Z) &= \sum_{i \in I_B} h(({}^B R(e_i, Y)Z)^H, e_i) - \frac{1}{f} \operatorname{PTr}^F_g h \cdot \operatorname{Hess}_g f(Y,Z), \\
\overset{\circ}{R}h(Y,V) &= h(\nabla^g_Y F, V), \\
\overset{\circ}{R}h(V,W) &= \sum_{a \in I_F} h(({}^F R(e_a, V)W)^V, e_a) \\
&\quad - |F|^2_{g_B} \operatorname{PTr}^F_g h \cdot g(V,W) + |F|^2_{g_B} h(V,W) - \operatorname{PTr}^B_{g_B} h(\nabla^g_\cdot F, \cdot) g(V,W)
\end{aligned}$$

Proof. Let $\{e_i \,|i \in I_B\}$ be a g-orthonormal basis for B and let $\{e_a \,|a \in I_F\}$ be a g-orthonormal basis for F. Then $\{e_i \,|i \in I := I_B \cup I_F\}$ is a g-orthonormal basis for M.

$$\begin{aligned}
\overset{\circ}{R}h(Y,Z) &= \sum_{i \in I_B} h(R(e_i, Y)Z, e_i) + \sum_{a \in I_F} h(R(e_a, Y)Z, e_a) \\
&= \sum_{i \in I_B} h(({}^B R(e_i, Y)Z)^H, e_i) - \sum_{a \in I_F} h(\frac{1}{f} \operatorname{Hess}_g f(Y,Z) e_a, e_a) \\
&= \sum_{i \in I_B} h(({}^B R(e_i, Y)Z)^H, e_i) - \frac{1}{f} \operatorname{Hess}_g f(Y,Z) \operatorname{PTr}^F_g h \\
\overset{\circ}{R}h(Y,V) &= \sum_{i \in I_B} h(R(e_i, Y)V, e_i) + \sum_{a \in I_F} h(R(e_a, Y)V, e_a) \\
&= 0 + \sum_{a \in I_F} h(g(e_a, V) \nabla^g_Y F, e_a) \\
&= h(\nabla^g_Y F, \sum_{a \in I_F} g(e_a, V) e_a) \\
&= h(\nabla^g_Y F, V) \\
\overset{\circ}{R}h(V,W) &= \sum_{i \in I_B} h(R(e_i, V)W, e_i) + \sum_{a \in I_F} h(R(e_a, V)W, e_a) \\
&= -\sum_{i \in I_B} g(V,W) h(\nabla^g_{e_i} F, e_i) + \sum_{a \in I_F} h(({}^F R(V,W)e_a)^V, e_a) \\
&\quad + \|F\|^2_g \left[\sum_{a \in I_F} h(g(e_a, W)V - g(V,W)e_a, e_a) \right] \\
&= -g(V,W) \sum_{i \in I_B} h(\nabla^g_{e_i} F, e_i) + \sum_{a \in I_F} h(({}^F R(V,W)e_a)^V, e_a) \\
&\quad + \|F\|^2_g \left[h(V,W) - \operatorname{PTr}^F_g h \cdot g(V,W) \right]
\end{aligned}$$

□

Corollary 3.3. *Taking into account that $f \in \mathcal{C}^\infty(B)$ and therefore $F \in \mathfrak{X}(B)$, the only nonzero combinations are the following. For $h_B \in S^2B$*

$$\overset{\circ}{R}h_B(Y,Z) = {}^B\overset{\circ}{R}h_B(Y,Z)$$

$$\overset{\circ}{R}h_B(V,W) = -\operatorname{PTr}_g^B(h_B(\nabla^g_{\cdot}F,\cdot))\cdot g(V,W),$$

for $h_F \in S^2F$

$$\overset{\circ}{R}h_F(Y,Z) = -f^{-3}\operatorname{PTr}_{g_F}^F h\cdot \operatorname{Hess}_{g_B} f(Y,Z)$$
$$\overset{\circ}{R}h_F(V,W) = {}^F\overset{\circ}{R}h_F(V,W) + |F|^2_{g_B}(h(V,W) - \operatorname{PTr}_g^F h\cdot g(V,W)),$$

and for $h_B \odot h_F \in \Omega^1(B)\odot\Omega^1(F)$

$$\overset{\circ}{R}(h_B\odot h_F)(Y,V) = h_B(\nabla_Y^{g_B}F)h_F(V)$$
$$\overset{\circ}{R}(h_B\odot h_F)(V,W) = -g(V,W)\sum_{i\in I_B} h_B(e_i)h_F((\nabla^g_{e_i}F)^V).$$

Corollary 3.4. *For the pointwise inner product of symmetric 2-tensor fields, we have*

$$\left\langle \overset{\circ}{R}h_B, h_B\right\rangle_g = \left\langle {}^B\overset{\circ}{R}h_B, h_B\right\rangle_{g_B},$$
$$\left\langle \overset{\circ}{R}h_F, h_F\right\rangle_g = f^2\left\langle {}^F\overset{\circ}{R}h_F, h_F\right\rangle_{g_F},$$
$$\left\langle \overset{\circ}{R}(h_B\odot h_F), h_B\odot h_F\right\rangle_g = f^{-2}\left\langle h_B, h_B(\nabla^{g_B}_{\cdot}F)\right\rangle_{g_B}.$$

3.3 The covariant derivative: $\nabla^g h$

The main part of the Einstein–Laplace operator involves the covariant derivative of the symmetric 2-form $h \in S^2M$. In this subsection, we develop formulae to deal with this easily.

Lemma 3.5. *With our notations,*

$$(\nabla^g_Y h_B)(Z,Z') = (\nabla^{g_B}_Y h_B)(Z,Z')$$
$$(\nabla^g_U h_B)(Z,V) = f^2 g_F(U,V)h_B(Z,F)$$
$$(\nabla^g_Y h_F)(V,W) = -2Y(\ln f)h_F(V,W)$$
$$(\nabla^g_U h_F)(Z,V) = -Z(\ln f)h_F(U,V)$$
$$(\nabla^g_U h_F)(V,W) = (\nabla^{g_F}_U h_F)(V,W)$$
$$(\nabla^g_U(h_B\odot h_F))(Z,Z') = -(h_B(Z')Z(\ln f) + h_B(Z)Z'(\ln f))h_F(U)$$
$$(\nabla^g_Y(h_B\odot h_F))(Z,V) = (\nabla^{g_B}_Y h_B)(Z)h_F(V) - Y(\ln f)h_B(Z)h_F(V)$$
$$(\nabla^g_U(h_B\odot h_F))(Z,V) = h_B(Z)(\nabla^{g_F}_U h_F)(V)$$
$$(\nabla^g_U(h_B\odot h_F))(V,W) = f^2h_B(F)(g_F((U,V)h_F(W) + g_F(U,W)h_F(V)),$$

and the other combinations are zero.

Proof. We use [Che11, Proposition 4.1]. For the horizontal covariant derivatives of h_B, we have

$$\begin{aligned}
(\nabla^g_Y h_B)(Z,Z') &= Y(h_B(Z,Z')) - h_B(\nabla^g_Y Z, Z') - h_B(Z, \nabla^g_Y Z') \\
&= Y(h_B(Z,Z')) - h_B(\nabla^{g_B}_Y Z, Z') - h_B(Z, \nabla^{g_B}_Y Z') \\
&= (\nabla^{g_B}_Y h_B)(Z,Z'), \\
(\nabla^g_Y h_B)(Z,V) &= Y(h_B(Z,V)) - h_B(\nabla^g_Y Z, V) - h_B(Z, \nabla^g_Y V) \\
&= 0 - 0 - Y(\ln f) h_B(Z,V) \\
&= 0, \\
(\nabla^g_Y h_B)(V,W) &= Y(h_B(V,W)) - h_B(\nabla^g_Y V, W) - h_B(V, \nabla^g_Y W) \\
&= 0 - 0 - 0 \\
&= 0,
\end{aligned}$$

and similarly for $\nabla^g_Y h_F$

$$\begin{aligned}
(\nabla^g_Y h_F)(Z,Z') &= Y(h_F(Z,Z')) - h_F(\nabla^g_Y Z, Z') - h_F(Z, \nabla^g_Y Z') \\
&= 0 - 0 - 0 \\
&= 0, \\
(\nabla^g_Y h_F)(Z,V) &= Y(h_F(Z,V)) - h_F(\nabla^g_Y Z, V) - h_F(Z, \nabla^g_Y V) \\
&= 0 - 0 - Y(\ln f) h_F(Z,V) \\
&= 0, \\
(\nabla^g_Y h_F)(V,W) &= Y(h_F(V,W)) - h_F(\nabla^g_Y V, W) - h_F(V, \nabla^g_Y W) \\
&= Y(h_F(V,W)) - Y(\ln f) h_F(V,W) - Y(\ln f) h_F(V,W) \\
&= -2Y(\ln f) h_F(V,W).
\end{aligned}$$

For the horizontal covariant derivative of a "mixed" tensor $h_B \odot h_F$, we have

$$\begin{aligned}
(\nabla^g_Y (h_B \odot h_F))(Z,Z') &= Y((h_B \odot h_F)(Z,Z')) - (h_B \odot h_F)(\nabla^g_Y Z, Z') - (h_B \odot h_F)(Z, \nabla^g_Y Z') \\
&= 0 - h_F(\nabla^g_Y Z) h_B(Z') - h_F(\nabla^g_Y Z') h_B(Z) \\
&= 0, \\
(\nabla^g_Y (h_B \odot h_F))(Z,V) &= Y((h_B \odot h_F)(Z,V)) - (h_B \odot h_F)(\nabla^g_Y Z, V) - (h_B \odot h_F)(Z, \nabla^g_Y V) \\
&= Y(h_B(Z) h_F(V)) - h_B(\nabla^g_Y Z) h_F(V) - h_B(Z) h_F(\nabla^g_Y V) \\
&= Y(h_B(Z)) h_F(V) + h_B(Z) Y(h_F(V)) - h_B(\nabla^{g_B}_Y Z) h_F(V) \\
&\quad - h_B(Z) h_F(Y(\ln f) V) \\
&= (\nabla^{g_B}_Y h_B)(Z) h_F(V) + h_B(Z)(\nabla^g_Y h_F)(V) \\
&= (\nabla^{g_B}_Y h_B)(Z) h_F(V) - Y(\ln f) h_B(Z) h_F(V), \\
(\nabla^g_Y (h_B \odot h_F))(V,W) &= Y((h_B \odot h_F)(V,W)) - (h_B \odot h_F)(\nabla^g_Y V, W) - (h_B \odot h_F)(V, \nabla^g_Y W) \\
&= 0 - Y(\ln f)(h_B \odot h_F)(V,W) - Y(\ln f)(h_B \odot h_F)(V,W) \\
&= 0.
\end{aligned}$$

For the vertical covariant derivatives of h_B, we have

$$(\nabla^g_U h_B)(Z,Z') = U(h_B(Z,Z')) - h_B(\nabla^g_U Z, Z') - h_B(Z, \nabla^g_U Z')$$

$$
\begin{aligned}
&= 0 - Z(\ln f)h_B(U, Z') + Z'(\ln f)h_B(Z, U) \\
&= 0, \\
(\nabla^g_U h_B)(Z, V) &= U(h_B(Z, V)) - h_B(\nabla^g_U Z, V) - h_B(Z, \nabla^g_U V) \\
&= 0 - 0 + g(U, V)h_B(Z, F) \\
&= f^2 g_F(U, V)h_B(Z, F), \\
(\nabla^g_U h_B)(V, W) &= U(h_B(V, W)) - h_B(\nabla^g_U V, W) - h_B(V, \nabla^g_U W) \\
&= 0 - 0 - 0 \\
&= 0,
\end{aligned}
$$

and similarly for h_F:

$$
\begin{aligned}
(\nabla^g_U h_F)(Z, Z') &= U(h_F(Z, Z')) - h_F(\nabla^g_U Z, Z') - h_F(Z, \nabla^g_U Z') \\
&= 0 - 0 - 0 \\
&= 0, \\
(\nabla^g_U h_F)(Z, V) &= U(h_F(Z, V)) - h_F(\nabla^g_U Z, V) - h_F(Z, \nabla^g_U V) \\
&= 0 - Z(\ln f)h_F(U, V) - 0 \\
&= -Z(\ln f)h_F(U, V), \\
(\nabla^g_U h_F)(V, W) &= U(h_F(V, W)) - h_F(\nabla^g_U V, W) - h_F(V, \nabla^g_U W) \\
&= U(h_F(V, W)) - h_F(\nabla^{g_F}_U V, W) - h_F(V, \nabla^{g_F}_U W) \\
&= (\nabla^{g_F}_U h_F)(V, W).
\end{aligned}
$$

Lastly, for the vertical covariant derivatives of a "mixed" tensor $h_B \odot h_F$, we have

$$
\begin{aligned}
(\nabla^g_U(h_B \odot h_F))(Z, Z') &= U((h_B \odot h_F)(Z, Z')) - (h_B \odot h_F)(\nabla^g_U Z, Z') - (h_B \odot h_F)(Z, \nabla^g_U Z') \\
&= 0 - h_B(Z')h_F(U)Z(\ln f) - h_B(Z)h_F(U)Z'(\ln f) \\
&= -(h_B(Z')Z(\ln f) + h_B(Z)Z'(\ln f))h_F(U), \\
(\nabla^g_U(h_B \odot h_F))(Z, V) &= U((h_B \odot h_F)(Z, V)) - (h_B \odot h_F)(\nabla^g_U Z, V) - (h_B \odot h_F)(Z, \nabla^g_U V) \\
&= U(h_B(Z)h_F(V)) - h_B(\nabla^g_U Z)h_F(V) - h_B(Z)h_F(\nabla^g_U V) \\
&= U(h_B(Z))h_F(V) + h_B(Z)U(h_F(V)) - Z(\ln f)h_B(U)h_F(V) \\
&\quad - h_B(Z)h_F(\nabla^{g_F}_U V) \\
&= h_B(Z)(\nabla^{g_F}_U h_Z)(V), \\
(\nabla^g_U(h_B \odot h_F))(V, W) &= U((h_B \odot h_F)(V, W)) - (h_B \odot h_F)(\nabla^g_U V, W) - (h_B \odot h_F)(V, \nabla^g_U W) \\
&= 0 - h_B(\nabla^g_U V)h_F(W) - h_B(\nabla^g_U W)h_F(V) \\
&= g(U, V)h_B(F)h_F(W) + g(U, W)h_B(F)h_F(V) \\
&= h_B(F)(g(U, V)h_F(W) + g(U, W)h_F(V)) \\
&= f^2 h_B(F)(g_F(U, V)h_F(W) + g_F(U, W)h_F(V)).
\end{aligned}
$$

□

Corollary 3.6. *In the product case where $f = 1$, we have the following formulae.*

$$
\begin{aligned}
(\nabla^g_Y h_B)(Z, Z') &= (\nabla^{g_B}_Y h_B)(Z, Z') \\
(\nabla^g_U h_F)(V, W) &= (\nabla^{g_F}_U h_F)(V, W)
\end{aligned}
$$

$$(\nabla^g_Y(h_B \odot h_F))(Z,V) = (\nabla^{g_B}_Y h_B)(Z)h_F(V)$$
$$(\nabla^g_U(h_B \odot h_F))(Z,V) = h_B(Z)(\nabla^{g_F}_U h_F)(V)$$

The rest is zero. Note that the last two lines can be uniformly written as a Leibniz type formula:

$$(\nabla^g_K(h_B \odot h_F))(Z,V) = (((\pi^*\nabla^{g_B})_K h_B) \odot h_F)(Z,V) + (h_B \odot ((\eta^*\nabla^{g_F})_K h_Z))(Z,V),$$

where $K \in \mathfrak{X}(B \times F)$ is any vector field.

3.4 Pointwise norm of the covariant derivative: $\langle \nabla^g h, \nabla^g h \rangle_g$

Next we study the pointwise norm of the exterior derivative of the symmetric 2-form $h \in S^2(B \times_f F)$.

First we calculate the norm of pure 2-tensors.

Lemma 3.7.

$$\begin{aligned}
|\nabla^g h_B|^2_g &= |\nabla^{g_B} h_B|^2_{g_B} + \dim F |h_B(\cdot, F)|^2_{g_B} \\
|\nabla^g h_F|^2_g &= f^{-6}|\nabla^{g_F} h_F|^2_{g_F} + 5f^{-4}|F|^2_{g_B}|h_F|^2_{g_F} \\
|\nabla^g (h_B \odot h_F)|^2_g &= f^{-2}|\nabla^{g_B} h_B|^2_{g_B}|h_F|^2_{g_F} + f^{-2}|h_B|^2_{g_B}|\nabla^{g_F} h_F|^2_{g_F} \\
&\quad + f^{-2}|h_F|^2_{g_F}\Big(3|h_B|^2_{g_B}|F|^2_{g_B} + 2h_B(F)^2(\dim F + 2) - 2\left\langle \nabla^{g_B}_F h_B, h_B\right\rangle_{g_B}\Big)
\end{aligned}$$

Proof. We can use the formulae from Lemma 3.5.

For a basic 2-tensor h_B we have

$$\begin{aligned}
|\nabla^g h_B|^2_g &= \sum_{i,j,k \in I_B} (\nabla^g_{e_i} h_B)(e_j, e_k)^2 + \sum_{\substack{i \in I_B \\ a,b \in I_F}} (\nabla^g_{e_a} h_B)(e_i, e_b)^2 \\
&= \sum_{i,j,k \in I_B} (\nabla^{g_B}_{e_i} h_B)(e_j, e_k)^2 + \sum_{\substack{i \in I_B \\ a,b \in I_F}} \Big(f^2 g_F(e_a, e_b) h_B(e_i, F)\Big)^2 \\
&= |\nabla^{g_B} h_B|^2_{g_B} + \dim F |h_B(\cdot, F)|^2_{g_B}.
\end{aligned}$$

For a fibrous 2-tensor h_F, we have

$$\begin{aligned}
|\nabla^g h_F|^2_g &= \sum_{\substack{i \in I_B \\ a,b \in I_F}} (\nabla^g_{e_i} h_F)(e_a, e_b)^2 + \sum_{\substack{i \in I_B \\ a,b \in I_F}} (\nabla^g_{e_a} h_F)(e_i, e_b)^2 \\
&\quad + \sum_{a,b,c \in I_F} (\nabla^g_{e_a} h_F)(e_b, e_c)^2 \\
&= \sum_{\substack{i \in I_B \\ a,b \in I_F}} (-2e_i(\ln f) h_F(e_a, e_b))^2 + \sum_{\substack{i \in I_B \\ a,b \in I_F}} (-e_i(\ln f) h_F(e_a, e_b))^2 \\
&\quad + \sum_{a,b,c \in I_F} (\nabla^g_{e_a} h_F)(e_b, e_c)^2
\end{aligned}$$

$$
\begin{aligned}
&= 5|F|_g^2|h_F|_g^2 + f^{-6}\sum_{a,b,c\in I_F}(\nabla^g_{fe_a}h_F)(fe_b, fe_c)^2\\
&= f^{-6}|\nabla^{g_F}h_F|^2_{g_F} + 5f^{-4}|F|^2_{g_B}|h_F|^2_{g_F}.
\end{aligned}
$$

Finally, for a mixed 2-tensor $h_B \odot h_F$, we proceed in several steps.

$$
\begin{aligned}
A &:= \sum_{\substack{i,j\in I_B\\ a\in I_F}} \nabla^g_{e_a}(h_B\odot h_F)(e_i,e_j)^2\\
&= \sum_{\substack{i,j\in I_B\\ a\in I_F}} (-(h_B(e_j)e_i(\ln f) + h_B(e_i)e_j(\ln f))h_F(e_a))^2\\
&= |h_F|_g^2 \sum_{i,j\in I_B} (h_B(e_j)e_i(\ln f) + h_B(e_i)e_j(\ln f))^2\\
&= |h_F|_g^2 \sum_{i,j\in I_B} \Big((h_B(e_j)e_i(\ln f))^2 + 2h_B(e_j)e_i(\ln f)h_B(e_i)e_j(\ln f) + (h_B(e_i)e_j(\ln f))^2\Big)\\
&= |h_F|_g^2(|h_B|_g^2|F|_g^2 + 2(h_B(F))^2 + |h_B|_g^2|F|_g^2)\\
&= f^{-2}|h_F|^2_{g_F}(|h_B|^2_{g_B}|F|^2_{g_B} + 2(h_B(F))^2 + |h_B|^2_{g_B}|F|^2_{g_B})\\
&= 2f^{-2}(|h_B|^2_{g_B}|F|^2_{g_B} + (h_B(F))^2)|h_F|^2_{g_F}\\
B &:= \sum_{\substack{i,j\in I_B\\ a\in I_F}} (\nabla^g_{e_i}(h_B\odot h_F))(e_j,e_a)^2\\
&= \sum_{\substack{i,j\in I_B\\ a\in I_F}} (\nabla^{g_B}_{e_i}h_B(e_j)h_F(e_a) - e_i(\ln f)h_B(e_j)h_F(e_a))^2\\
&= \sum_{\substack{i,j\in I_B\\ a\in I_F}} (\nabla^{g_B}_{e_i}h_B(e_j)h_F(e_a))^2\\
&\quad - 2\sum_{\substack{i,j\in I_B\\ a\in I_F}} \nabla^{g_B}_{e_i}h_B(e_j)h_F(e_a)e_i(\ln f)h_B(e_j)h_F(e_a)\\
&\quad \sum_{\substack{i,j\in I_B\\ a\in I_F}} (e_i(\ln f)h_B(e_j)h_F(e_a))^2\\
&= |\nabla^{g_B}h_B|_g^2|h_F|_g^2 - 2\left\langle\nabla^{g_B}_F h_B, h_B\right\rangle_g |h_F|_g^2 + |F|_g^2|h_B|_g^2|h_F|_g^2\\
&= |\nabla^{g_B}h_B|_g^2|h_F|_g^2 - 2\left\langle\nabla^{g_B}_F h_B, h_B\right\rangle_g |h_F|_g^2 + |F|_g^2|h_B|_g^2|h_F|_g^2\\
&= f^{-2}|\nabla^{g_B}h_B|^2_{g_B}|h_F|^2_{g_F} + f^{-2}\Big(-2\left\langle\nabla^{g_B}_F h_B, h_B\right\rangle_{g_B} + |F|^2_{g_B}|h_B|^2_{g_B}\Big)|h_F|^2_{g_F}\\
C &:= \sum_{\substack{i\in I_B\\ a,b\in I_F}} ((\nabla^g_{e_a}(h_B\odot h_F))(e_i,e_b)^2\\
&= \sum_{\substack{i\in I_B\\ a,b\in I_F}} (h_B(e_i)(\nabla^{g_F}_{e_a}h_F)(e_b))^2\\
&= |h_B|_g^2|\nabla^{g_F}h_F|_g^2\\
&= f^{-2}|h_B|^2_{g_B}|\nabla^{g_F}h_F|^2_{g_F}
\end{aligned}
$$

$$
\begin{aligned}
D &:= \sum_{a,b,c\in I_F} ((\nabla^g_{e_a}(h_B \odot h_F))(e_b,e_c)^2 \\
&= \sum_{a,b,c\in I_F} f^4 h_B(F)^2 \left(g_F(e_a,e_b)h_F(e_c) + g_F(e_a,e_c)h_F(e_b)\right)^2 \\
&= (h_B(F))^2 f^4 \cdot \left(f^{-6}\dim F|h_F|^2_{g_F} + f^{-6}\dim F|h_F|^2_{g_F} + 2f^{-6}|h_F|^2_{g_F}\right) \\
&= 2f^{-2}h_B(F)^2(\dim F+1)|h_F|^2_{g_F}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|\nabla^g(h_B\odot h_F)|^2_g &= A+B+C+D \\
&= 2f^{-2}(|h_B|^2_{g_B}|F|^2_{g_B} + (h_B(F))^2)|h_F|^2_{g_F} \\
&\quad + f^{-2}|\nabla^{g_B}h_B|^2_{g_B}|h_F|^2_{g_F} + f^{-2}\left(-2\left\langle\nabla^{g_B}_F h_B, h_B\right\rangle_{g_B} + |F|^2_{g_B}|h_B|^2_{g_B}\right)|h_F|^2_{g_F} \\
&\quad + f^{-2}|h_B|^2_{g_B}|\nabla^{g_F}h_F|^2_{g_F} \\
&\quad + 2f^{-2}h_B(F)^2(\dim F+1)|h_F|^2_{g_F} \\
&= f^{-2}|\nabla^{g_B}h_B|^2_{g_B}|h_F|^2_{g_F} + f^{-2}|h_B|^2_{g_B}|\nabla^{g_F}h_F|^2_{g_F} \\
&\quad + f^{-2}|h_F|^2_{g_F}\Big(2|h_B|^2_{g_B}|F|^2_{g_B} + 2h_B(F)^2 + 2h_B(F)^2(\dim F+1) \\
&\qquad\qquad - 2\left\langle\nabla^{g_B}_F h_B, h_B\right\rangle_{g_B} + |F|^2_{g_B}|h_B|^2_{g_B}\Big) \\
&= f^{-2}|\nabla^{g_B}h_B|^2_{g_B}|h_F|^2_{g_F} + f^{-2}|h_B|^2_{g_B}|\nabla^{g_F}h_F|^2_{g_F} \\
&\quad + f^{-2}|h_F|^2_{g_F}\left(3|h_B|^2_{g_B}|F|^2_{g_B} + 2h_B(F)^2(\dim F+2) - 2\left\langle\nabla^{g_B}_F h_B, h_B\right\rangle_{g_B}\right).
\end{aligned}
$$

□

Corollary 3.8. *If $f = 1$, we have as a special case the following.*

$$
\begin{aligned}
|\nabla^g h_B|^2_g &= |\nabla^{g_B}h_B|^2_{g_B} \\
|\nabla^g h_F|^2_g &= |\nabla^{g_F}h_F|^2_{g_F} \\
|\nabla^g(h_B\odot h_F)|^2_g &= |\nabla^{g_B}h_B|^2_{g_B}|h_F|^2_{g_F} + |h_B|^2_{g_B}|\nabla^{g_F}h_F|^2_{g_F}
\end{aligned}
$$

"Pure" tensors are not all there is but we know from [AM11, Lemma 3.1] that in the product case $\mathcal{C}^\infty(F)\cdot S^2B \oplus \Omega^1(B)\odot\Omega^1(F)\oplus\mathcal{C}^\infty(F)\cdot S^2F$ is dense in $H^{2,2}(B\times F)$.

Lemma 3.9. *For $\alpha\in\mathcal{C}^\infty(M)$ and $h\in S^2M$ we have*

$$
|\nabla^g(\alpha h)|^2_g = \alpha^2|\nabla^g h|^2_g + 2\alpha\left\langle\nabla^g_{\mathrm{grad}_g\alpha}h, h\right\rangle_g + |d\alpha|^2_g|h|^2_g.
$$

Proof. Let $\alpha\in\mathcal{C}^\infty(M)$ be a smooth function. Then

$$
\begin{aligned}
|\nabla^g(\alpha h)|^2_g &= |\alpha\nabla^g h + d\alpha\otimes h|^2_g \\
&= \alpha^2|\nabla^g h|^2_g + 2\alpha\left\langle\nabla^g h, d\alpha\otimes h\right\rangle_g + |d\alpha\otimes h|^2_g.
\end{aligned}
$$

Let us tackle this expression term by term. For the second term, we know from Lemma 3.1 that

$$\langle \nabla^g h, d\alpha \otimes h \rangle_g = \left\langle \nabla^g_{\mathrm{grad}_g \alpha} h, h \right\rangle_g .$$

Finally, for the third term, we calculate

$$|d\alpha \otimes h|_g^2 = \sum_{i,j,k \in I} (d\alpha(e_i) h(e_j, e_k))^2 = |d\alpha|_g^2 |h|_g^2. \qquad \square$$

Corollary 3.10. *For $\alpha_F \in \mathcal{C}^\infty(F)$ and $h_B \in S^2 B$, we have*

$$|\nabla^g(\alpha_F h_B)|_g^2 = \alpha_F^2 |\nabla^{g_B} h_B|_{g_B}^2 + f^2 |d\alpha|_{g_F}^2 |h_B|_{g_B}^2 + \alpha_F^2 \dim F |h_B(\cdot, F)|_{g_B}^2$$

For $\alpha_B \in \mathcal{C}^\infty(B)$ and $h_F \in S^2 F$, we have

$$|\nabla^g(\alpha_B h_F)|_g^2 = \alpha_B^2 |\nabla^g h_F|_g^2 + 2\alpha_B \left\langle \nabla^g_{\mathrm{grad}_g \alpha_B} h_F, h_F \right\rangle_g + |d\alpha_B|_g^2 |h_F|_g^2.$$

Corollary 3.11. *In the product case, we have*

$$\begin{aligned} |\nabla^g(\alpha_F h_B)|_g^2 &= \alpha_F^2 |\nabla^{g_B} h_B|_{g_B}^2 + |d\alpha_F|_{g_F}^2 |h_B|_{g_B}^2, \\ |\nabla^g(\alpha_B h_F)|_g^2 &= \alpha_B^2 |\nabla^{g_F} h_F|_{g_F}^2 + |d\alpha_B|_{g_B}^2 |h_F|_{g_F}^2. \end{aligned}$$

Chapter 4

Riemannian cones and Ricci solitons

4.1 A little motivation for the definition of Ricci solitons

The motivation of studying Ricci solitons comes from the Ricci flow. We call a family of metrics $(g_t)_{t\in[0,T)}$ for some $T>0$ a Ricci flow if $\partial_t g_t = -2\operatorname{Ric}^{g_t}$. The corresponding initial metric is g_0.

For a Ricci-flat metric g_0, the constant family $(g_0)_{t\in[0,\infty)}$ is a Ricci flow since $\partial_t g_0 = 0$ and $-2\operatorname{Ric}^{g_0} = 0$. With other words, Ricci-flat metrics are fixed points of the Ricci flow.

More generally, if g_0 is an Einstein metric with $\operatorname{Ric}^g = \lambda g$, then it is easy to see that

$$g_t = (1-2\lambda t)g_0 \qquad \text{for } t\in[0,T),$$

where $T := \frac{1}{2\lambda}$ if $\lambda > 0$ and $T := \infty$ otherwise, is a Ricci flow with initial metric g_0. With other words, Einstein metrics are fixed points of the Ricci flow up to homothetic rescaling.

Ricci-solitons offer an further generalization.

Definition 4.1. *We call a Riemmanian manifold (M,g) a Ricci-solition if there is a vector field $X\in\Gamma^\infty(TM)$ and a real number $\mu\in\mathbb{R}$ such that $\operatorname{Ric}^g + \frac{1}{2}\mathcal{L}_X g = \mu g$. If $X = \operatorname{grad}_g w$ with some function $w\in\Gamma^\infty(\mathbb{R}_M)$, then we call (M,g) a gradient Ricci soliton and the function w a Ricci potential of (M,g).*

It is easy to check that the defining equation for a gradient Ricci soliton may be rewritten as

$$\operatorname{Ric}^g + \operatorname{Hess}_g w = \mu g.$$

Evidently, Einstein manifolds are a special case of a gradient Ricci soliton where the Ricci potential is constant. Einstein manifolds are also called trivial Ricci solitons.

The name "Ricci soliton" itself comes from the fact that Ricci flows starting at Ricci solitons are fixed points up to homothetic rescaling and pullback [CK04, Lemma 2.4].

4.2 A product Ricci soliton from a Riemannian cone and an Einstein manifold

There is a way to construct a Ricci soliton from a Riemannian cone and an Einstein manifold. Riemannian cones will be treated more thoroughly in Section 6.1, here we just anticipate a few of their most important features needed for the present construction. A Riemannian manifold (B, g_B) is called a Riemannian cone if there is a vector field $X \in \Gamma^\infty(TM)$ such that $\nabla^g X = \mathrm{id}_{TM}$. In this case, $X = \mathrm{grad}_g \frac{1}{2}|X|_g^2$.

Lemma 4.2. *If (B, g_B, X) is a Ricci-flat conical manifold and (F, g_F) is an Einstein manifold with Einstein constant μ, then the Riemannian product $(M, g) := (B \times F, g_B + g_F)$ is a gradient Ricci soliton with Ricci potential $w = \frac{\mu}{2}|X|^2$.*

Proof. The Ricci tensor of a product manifold is the direct sum of the Ricci tensors, therefore

$$\mathrm{Ric}^g = \begin{pmatrix} \mathrm{Ric}^{g_B} & 0 \\ 0 & \mathrm{Ric}^{g_F} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mu g_F \end{pmatrix}.$$

An easy calculation convinces us that the gradient of w is given by $\mathrm{grad}_g w = \mu \hat{X}$, where $\hat{X} := \frac{X}{|X|_g}$. We easily convince ourselves that the Hessian of w is given by

$$\mathrm{Hess}_g w = \begin{pmatrix} \mu g_B & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore the equation $\mathrm{Ric}^g + \mathrm{Hess}_g w = \lambda g$ reads

$$\begin{pmatrix} 0 & 0 \\ 0 & \mu g_F \end{pmatrix} + \begin{pmatrix} \mu g_B & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda g_B & 0 \\ 0 & \lambda g_F \end{pmatrix},$$

which is obviously satisfied for $\lambda = \mu$. □

Remark 4.3. *Lemma 4.2 generalizes the so-called Gaussian Ricci soliton $(\mathbb{R}^n, g_{flat}, \frac{1}{2}|x|^2)$ where $|\cdot|$ denotes the Euclidean distance from the origin. In fact, if F is a one-point set and B is a Riemannian cone with link S^{n-1}, then $B \times F = \mathbb{R}^n \setminus \{0\}$ (cf. Example 6.6).*

4.3 Linear stability of Ricci solitons

Definition 4.4 (cf. e.g. [Krö15b, Definition 3.7])**.** *Let (M, g) be a gradient Ricci soliton (M, g) with Ricci potential w. Define*

$$V := \left\{ h \in \Gamma^\infty\left(S^2T^*M\right) \,\middle|\, \delta^g h + h(\mathrm{grad}_g w, \cdot) = 0 \text{ and } \int_M \langle \mathrm{Ric}^g, h \rangle_g e^{-w} \mathrm{vol}_g = 0 \right\}.$$

Definition 4.5 (cf. e.g. [Krö15b, Definition 3.7])**.** *We call a gradient Ricci soliton (M, g) with Ricci potential w linearly stable if the spectrum operator*

$$\Delta_E^{g,w} := \Delta_E^g + \nabla^g_{\mathrm{grad}_g w} \colon V \to \Gamma^\infty\left(S^2T^*M\right)$$

lies entirely in $[0, \infty)$. A gradient Ricci soliton which is not linearly stable is called linearly unstable.

Remark 4.6. *The operator $\Delta_E^{g,w}$ is related to the Einstein operator in the same way the so-called $\Delta_g + \nabla^g_{\mathrm{grad}_g w}$ is called weighted or Bakry–Émery Laplace operator [BÉ85] is related to the raw Laplacian.*

Evidently, for Einstein manifolds (where $w = \mathrm{const}$), linear stability is decided by the spectrum of the Einstein operator.

The first-order term that we introduced in the operator $\Delta_E^{M,w}$ interacts nicely with the weighted L^2-norm with weight e^{-w}.

Lemma 4.7. *The operator $\Delta_E^{M,w}$ is formally self adjoint with respect to the inner product $(\cdot,\cdot)_{g,\exp(-w)}$. Moreover, for any symmetric 2-tensor field h, we have*

$$\left(\Delta_E^{M,w} h, h\right)_{g,\exp(-w)} = \int_M \left(|\nabla^g h|_g^2 - 2\left\langle \mathring{R}h, h\right\rangle_g \right) e^{-w}\,\mathrm{vol}_g\,.$$

Proof. Recall that the second-order part of the Einstein operator is given by $(\nabla^g)^*\nabla^g$, where the star denotes the adjoint operator with respect to the pointwise inner product $\langle\cdot,\cdot\rangle_g$ induced by the metric g. Therefore by Lemma 3.1, we have for $h \in \Gamma_c^\infty\left(S^2T^*M\right)$, up to surface terms, that

$$\begin{aligned}
\langle(\nabla^g)^*\nabla^g h, h\rangle_g\, e^{-w} &= \left\langle(\nabla^g)^*\nabla^g h, e^{-w}h\right\rangle_g \\
&= \left\langle\nabla^g h, \nabla^g(e^{-w}h)\right\rangle_g \\
&= \left\langle\nabla^g h, d(e^{-w})\otimes h + e^{-w}\nabla^g h\right\rangle_g \\
&= \left\langle\nabla^g h, -e^{-w}dw\otimes h + e^{-w}\nabla^g h\right\rangle_g \\
&= -\left\langle\nabla^g h, dw\otimes h\right\rangle_g e^{-w} + \left\langle\nabla^g h, \nabla^g h\right\rangle_g e^{-w} \\
&= -\left\langle\nabla^g_{\mathrm{grad}_g w} h, h\right\rangle_g e^{-w} + |\nabla^g h|_g^2 e^{-w}.
\end{aligned}$$

Now

$$\begin{aligned}
\left(\Delta_E^{M,w} h, h\right)_{g,\exp(-w)} &= \int_M \left\langle (\nabla^g)^*\nabla^g h + \nabla^g_{\mathrm{grad}_g w} h - 2\mathring{R}h, h\right\rangle_g e^{-w}\,\mathrm{vol}_g \\
&= \int_M \left(|\nabla^g h|_g^2 - 2\left\langle \mathring{R}h, h\right\rangle_g \right) e^{-w}\,\mathrm{vol}_g,
\end{aligned}$$

as desired. □

4.4 Instability of product Ricci solitons with unstable fibre

The goal of this subsection is to prove that the Ricci soliton $M = B \times F$ is linearly unstable if F is linearly unstable as an Einstein manifold. We will proceed as before: we will construct a (sequence of) destabilising perturbation(s) if F is linearly unstable.

We start with a preparatory lemma.

Lemma 4.8. *For $n \geq 2$ and $\mu > 0$, we have*

$$\inf_{f\in C_c^\infty((0,\infty))} \frac{\int_0^\infty f'(r)^2 r^{n-1} e^{-\frac{\mu}{2}r^2}\,dr}{\int_0^\infty f(r)^2 r^{n-1} e^{-\frac{\mu}{2}r^2}\,dr} = 0.$$

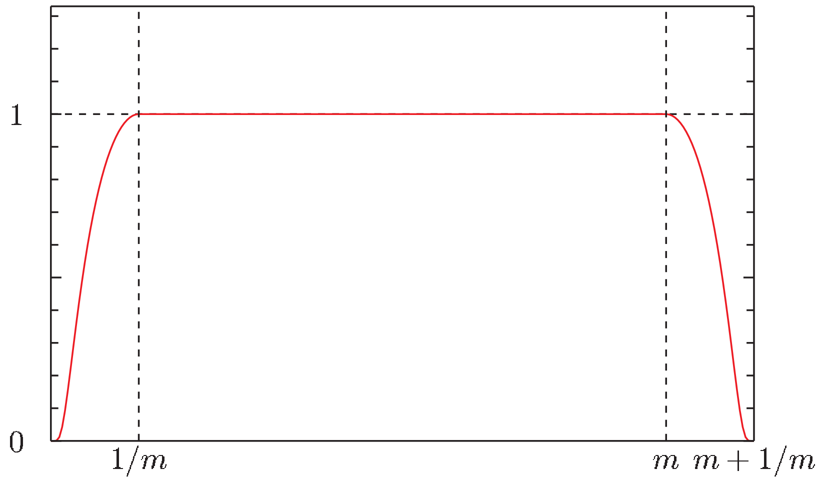


Figure 4.1: The "smooth" step function f_m used in the proof of Lemma 4.8.

Proof. The main idea is to notice that for the constant 1 function, the nominator is

$$\int_0^\infty r^{n-1}e^{-\frac{\mu}{2}r^2}dr = \frac{1}{2}\left(\frac{2}{\mu}\right)^{n/2}\Gamma\frac{n}{2} < \infty,$$

in particular it is finite and therefore the value of the quotient is zero. Based on this insight, we construct a sequence of compactly supported smooth functions such that the quotient converges to zero. Since the quotient is manifestly nonnegative, this shows that the infimum must be zero.

Let $\phi \in \mathcal{C}_c^\infty((0,\infty))$ be a nonnegative "smooth step function", i.e. a function for which[1] $\phi(x) \in [0,1]$, for which $\phi(x) = 1$ whenever $x > 1$ and for which there is a positive $\delta < 1$ such that $\phi(x) = 0$ whenever $x < \delta$. Consider the function $f_m \in \mathcal{C}_c^\infty((0,\infty))$ be the function given by

$$f_m(r) := \begin{cases} \phi(mr) & \text{if } r \leq \frac{1}{m}, \\ 1 & \text{if } \frac{1}{m} < r \leq m, \\ \phi(m^2+1-mr) & \text{if } m < r. \end{cases}$$

The function f_m is depicted in Figure 4.1. The support of f_m is the interval $[\frac{\delta}{m}, m+\frac{1}{m}-\frac{\delta}{m}]$. The support of f'_m is the compact set $[\frac{\delta}{m},\frac{1}{m}] \cup [m, m+\frac{1}{m}-\frac{\delta}{m}]$. Therefore there is a finite positive constant C such that $f'^2 \leq C$. Now it is a matter of calculation to obtain the result.

For the denominator, note that $\chi_{(\frac{1}{m},m)} \leq f_m^2 \leq 1$. Consequently,

$$\int_{\frac{1}{m}}^m r^{n-1}e^{-\frac{\mu}{2}r^2}dr \leq \int_0^\infty f_m(r)^2 r^{n-1}e^{-\frac{\mu}{2}r^2}dr \leq \frac{1}{2}\left(\frac{2}{\mu}\right)^{n/2}\Gamma\frac{n}{2},$$

and by the squeeze theorem, we obtain that $\lim_{m\to\infty}\int_0^\infty f_m(r)^2 r^{n-1}e^{-\frac{\mu}{2}r^2}dr = \frac{1}{2}\left(\frac{2}{\mu}\right)^{n/2}\Gamma\frac{n}{2}$.

[1] This assumption makes the proof easier but it is not needed since f^2, being a compactly supported continuous function, is bounded, and the quotient in question is invariant under rescaling f.

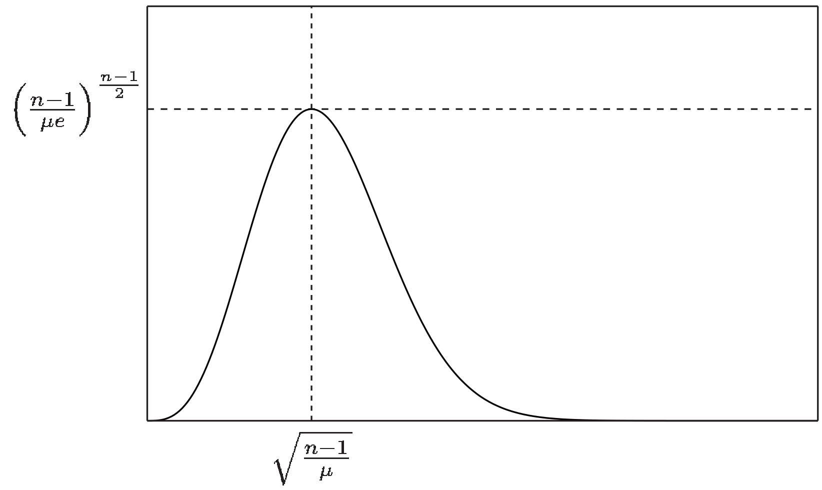


Figure 4.2: The auxiliary function $r^{n-1}e^{-\frac{\mu}{2}r^2}$ used in the proof of Lemma 4.8.

By elementary calculus, the function $r^{n-1}e^{-\frac{\mu}{2}r^2}$ is increasing if $r \in (0, \sqrt{\frac{n-1}{\mu}})$ and decreasing if $r \in (\sqrt{\frac{n-1}{\mu}}, \infty)$, cf. Figure 4.2. If m is big enough (concretely $m > \max\left\{\sqrt{\frac{n-1}{\mu}}, \sqrt{\frac{\mu}{n-1}}\right\}$), then we have the following inequalities:

$$\begin{aligned}
\int_0^\infty f_m'(r)^2 r^{k-1} e^{-\frac{\mu}{2}r^2}\,dr &= \int_0^{\frac{1}{m}} f_m'(r)^2 r^{n-1} e^{-\frac{\mu}{2}r^2}\,dr + \int_m^{m+\frac{1}{m}} f_m'(r)^2 r^{n-1} e^{-\frac{\mu}{2}r^2}\,dr \\
&\le C \int_0^{\frac{1}{m}} r^{n-1} e^{-\frac{\mu}{2}r^2}\,dr + C \int_m^{m+\frac{1}{m}} r^{n-1} e^{-\frac{\mu}{2}r^2}\,dr \\
&\le C \left(\frac{1}{m}\right)^{n-2} \int_0^{\frac{1}{m}} r e^{-\frac{\mu}{2}r^2}\,dr + C m^{n-1} e^{-\frac{\mu}{2}m^2} \int_m^{m+\frac{1}{m}} dr \\
&= C \left(\frac{1}{m}\right)^{n-2} \frac{1 - e^{-\frac{\mu}{2}\frac{1}{m^2}}}{\mu} + C m^{n-2} e^{-\frac{\mu}{2}m^2} \\
&\to 0
\end{aligned}$$

as $m \to \infty$.

Therefore $\lim_{m\to\infty} \frac{\int_0^\infty f_m'(r)^2 r^{n-1} e^{-\frac{\mu}{2}r^2}\,dr}{\int_0^\infty f_m(r)^2 r^{n-1} e^{-\frac{\mu}{2}r^2}\,dr} = 0$, as advertised. □

Lemma 4.9. *Let $(M, g) = (B, g_B, X) \times (F, g_F)$ be the product gradient Ricci soliton with Ricci potential $w := \frac{\mu}{2}|X|_g^2$. Let $h := f_B h_F$ where $f_B \in \mathcal{C}_c^\infty(B)$ and $h_F \in TT(F, g_F)$, i.e. $\operatorname{Tr}_{g_F} h_F = 0$ and $\delta^{g_F} h_F = 0$. Then $h \in V$.*

Proof. From the proof of Lemma 4.2, we already know that $\nabla^g w = \mu g(X, \cdot)$ and $\operatorname{Hess}_g w := \nabla^{g,2} w = \mu g_B$. This implies that $\operatorname{Ric}^g = \mu g - \operatorname{Hess}_g w = \mu g_F$. Consequently,

$$\begin{aligned}
\langle \operatorname{Ric}^g, h \rangle_g &= \langle \mu g_F, h \rangle_g = \operatorname{Tr}_{g_F} h_F = 0 \\
\delta^g h + h(\operatorname{grad}_g w, \cdot) &= f_B \delta^{g_F} h_F - h_F(\operatorname{grad}_g f_B, \cdot) + \mu f_B h_F(X, \cdot) = 0,
\end{aligned}$$

where we used that $\operatorname{grad}_g f_B$ and $\operatorname{grad}_g w$ are both vector fields in TB. This shows the claim. □

Theorem 4.10. *Let F be an Einstein manifold with constant $\mu > 0$, let B be a cone. If F is linearly unstable, then the Ricci soliton $M = B \times F$ is also linearly unstable.*

Proof. Let us consider the test perturbation $h := fh_F$ where $f \in \mathcal{C}_c^\infty(B)$ and $h_F \in S^2F$ is a compactly supported TT-tensor on F. Lemma 4.9 implies that $h \in V$.

It is easy to check that $(h,h)_{M,e^{-w}} = (h_F, h_F)_{g_F} \int_B f^2 e^{-w} \operatorname{vol}_{g_B}$. Moreover, we have pointwise that $|\nabla^g h|_g^2 = |df|_{g_B}^2 |h_F|_{g_F}^2 + f^2 |\nabla^{g_F} h_F|_{g_F}^2$ (the third term is missing since f depends only on the B factor and h_F is a tensor field on the F factor). We also have pointwise $\left\langle \mathring{R}^g h, h \right\rangle_g = f^2 \left\langle \mathring{R}^{g_F} h_F, h_F \right\rangle_{g_F}$ so using Fubini's theorem we obtain the following relation for the global Rayleigh quotients:

$$\frac{\left(\Delta_E^M h, h\right)_{M,e^{-w}}}{(h,h)_{M,e^{-w}}} = \frac{\int_B |df|_{g_B}^2 e^{-w} \operatorname{vol}_{g_B}}{\int_B f^2 e^{-w} \operatorname{vol}_{g_B}} + \frac{(\Delta_E^F h_F, h_F)_{g_F}}{(h_F, h_F)_{g_F}}.$$

From this we read off that h is a destabilizing perturbation if

$$\frac{(\Delta_E^F h_F, h_F)_{g_F}}{(h_F, h_F)_{g_F}} < -\frac{\int_B |df|_{g_B}^2 e^{-w} \operatorname{vol}_{g_B}}{\int_B f^2 e^{-w} \operatorname{vol}_{g_B}}.$$

This means that we can find a destabilizing perturbation of the form $h = fh_F$ if the infimum of the left-hand side is smaller than the supremum of the right-hand side, i.e. if

$$\inf_{h_F \in TT(g_F)} \frac{(\Delta_E^F h_F, h_F)_{g_F}}{(h_F, h_F)_{g_F}} < \sup_{f \in \mathcal{C}_c^\infty(B)} \left(-\frac{\int_B |df|_{g_B}^2 e^{-w} \operatorname{vol}_{g_B}}{\int_B f^2 e^{-w} \operatorname{vol}_{g_B}} \right).$$

The smallest value of the left-hand side of this equation is the smallest eigenvalue λ_1 of the Einstein operator on F. If we assume that f depends only on the radial coordinate in $B = \operatorname{Cone}(S, g_S)$, then the right-hand side becomes

$$-\frac{\int_B |df|_{g_B}^2 e^{-w} \operatorname{vol}_{g_B}}{\int_B f^2 e^{-w} \operatorname{vol}_{g_B}} = -\frac{\operatorname{Vol}_{g_S}(S) \int_0^\infty f'(r)^2 r^{n-1} e^{-\frac{\mu}{2}r^2} dr}{\operatorname{Vol}_{g_S}(S) \int_0^\infty f(r)^2 r^{n-1} e^{-\frac{\mu}{2}r^2} dr},$$

the supremum of which is 0 by Lemma 4.8. This means that we can find a destabilizing perturbation of the form $h = fh_F$ if

$$\lambda_1 < 0.$$

This concludes the proof. □

Part II

Optimizing the decay rate of Ricci-flat asymptotically conical manifolds

Chapter 5

Goals and strategy

In this part, unless otherwise noted, M denotes an $n+1$ dimensional smooth manifold.

Asympotically conical manifolds are among the most manageable noncompact manifolds. They are asymptotic to a cone, the geometry of which is largely determined by its link.

As proven for the important special case of asymptotically locally Euclidean (ALE) manifolds by Deruelle and Kröncke [DK20], if we have a Ricci-flat asymptotically locally Euclidean manifold with *some* decay rate, then in fact its decay rate may be improved based on the assumption of Ricci flatness. The goal of this part of the thesis is to establish the analogue of this theorem to the more general class of asymptotically conical metrics, and on the way prove some other decay results.

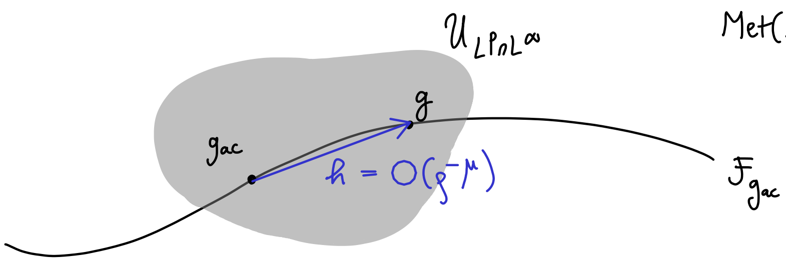
If we have a Ricci-flat asymptotically conical manifold of some decay rate, then we can find a different asymptotic chart where the decay rate is optimalized. This decay rate depends only on the spectrums of the Laplace–Beltrami operator on functions, Hodge Laplacian on 1-forms and Lichnerowicz Laplacian on symmetric 2-tensor fields corresponding to the asymptotic cone.

Our strategy is the following.

- First, we discuss cones and asymptotically conical manifolds and recall Banach spaces that are especially suitable to work with on asymptotically conical manifolds.

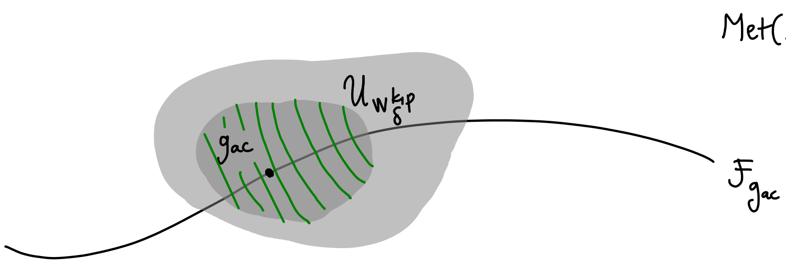
- We compute the spectrum of the so-called tangential operators to the Laplace–Beltrami operator, and the Hodge and Lichnerowicz Laplacians on a Ricci-flat cone. This data will be used later to determine decay rates.

- We fix an asymptotically conical manifold g_{ac}, introduce a g_{ac}-gauging $\mathcal{F}_{g_{\mathrm{ac}}}$ with the condition
$$-2\operatorname{Ric}^g + \mathcal{L}_{V(g,g_{\mathrm{ac}})} g_{\mathrm{ac}} = 0,$$
and show that there is a neighbour or metrics of g_{ac} in the $L^p \cap L^\infty$-sense in which the difference of gauged metrics to g_{ac} decays to order k with rate $-\mu - k$ for any $k \in \mathbb{N}$ where this μ depends on the spectrum of the tangential operators mentioned in the theorem.

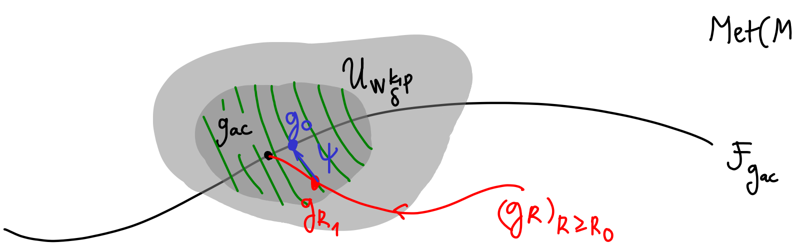


In fact, if an initial decay for h is known, then the gauging condition may even fail on a compact subset.

- Based on a careful study of the DeTurck map, we show that there is a neighbourhood of g_{ac} in the weighted Sobolev topology in which all metrics can be uniquely pulled back to a metric that has vanishing DeTurck vector field with respect to g_{ac}, i.e. $V(\phi^* g, g_{\mathrm{ac}}) = 0$ for some diffeomorphism ψ.



- We construct a family $(g_R)_R$ of metrics that coincide with g_{ac} on an ever increasing compact set K_R and coincide with $\phi^* g_{\mathrm{cone}}$ outside a bigger ever increasing subset K_R'. We show that this family converges to g_{ac} in the $W^{k,p}_\delta(S^2T^*M)$ topology. By construction, all the metrics in this family are Ricci-flat outside the compact set $\overline{K_R' \setminus K_R}$.



- The family of metrics constructed in the previous step will eventually enter the weighted Sobolev neighbourhood, thus it will be able to be pulled back to a metric under a map ψ such that $V(\psi^* g_{R_1}, g_{\mathrm{ac}}) = 0$. Now the theorem follows by changing the asymptotic chart to $\phi \circ \psi$.

Chapter 6

Riemannian cones

6.1 Definition and elementary properties

(Material similar to parts of Section 6.1 has appeared in my master's thesis [Sza16].) Following [ACM13], we define a Riemannian cone (a conical manifold in their terminology) as follows.

Definition 6.1. *A* Riemannian cone *is a triple* (M, g, Z) *where* (M, g) *is a pseudo-Riemannian manifold and* $Z \in \mathfrak{X}(M)$ *is nowhere zero, complete vector field with* $\nabla^g Z = \mathrm{id}_{TM}$*, where* ∇^g *denotes the Levi-Civita connection of* g*. The vector field* Z *is called the* Euler field*.*

Definition 6.2. *The link of the Riemannian cone* (M, g, Z) *is the set* $L := \{p \in P \,|r(p) = 1\}$ *with the induced metric. The frustum of the Riemannian cone* (M, g, Z) *at radius* R *is the set* $\mathrm{Frustum}_{(M,g)}(R) := \{p \in M \,|r(p) > R\}$*.*

Remark 6.3. *We will use the notation* g_{cone} *for a cone metric and* g_L *for the metric on the link. In some calculations, to ease notation, we introduce* $g_{\text{cone}} =: \bar{g}$ *and* $g_L =: g$*. Unless otherwise indicated,* $n := \dim L$ *and consequently* $\dim M = n + 1$*.*

The following function will play an important role in the study of conical manifolds.

Definition 6.4. *The length of the Euler vector field is called the radial coordinate, and it is denoted by*

$$r\colon M \to \mathbb{R}^+, p \mapsto \sqrt{g_p(Z_p, Z_p)}.$$

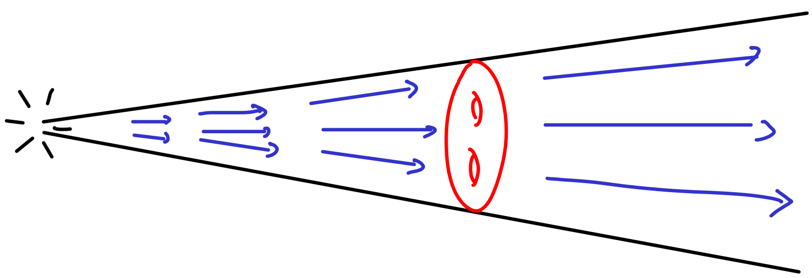


Figure 6.1: Riemannian cone with Euler field. Note that the tip is not part of the picture. The link is depicted schematically as a double torus.

Proposition 6.5. *Let (M, g, Z) be a conical pseudo-Riemannian manifold. Then*

1. *the vector field Z is the gradient of the function $\frac{1}{2}r^2$,*
2. *the vector field Z is a homothety, more precisely $\mathcal{L}_Z g = 2g$,*
3. *the orthogonal complement $\mathcal{D}$ of Z is an integrable distribution, the leaves of which are connected components of the level sets of r,*
4. *the leaf space of $\mathcal{D}$ has a natural not-necessarily-Hausdorff smooth structure such that the projection map is smooth,*
5. *the flow Fl^Z maps leaves of $\mathcal{D}$ to leaves of $\mathcal{D}$.*

Proof.

1. For $X \in \mathfrak{X}(M)$, one has

$$\begin{aligned} d(g(Z,Z))(X) &= X(g(Z,Z)) = \nabla^g_X(g(Z,Z)) \\ &= (\nabla^g_X g)(Z,Z) + 2g(\nabla^g_X Z, Z) = 2g(X,Z) \end{aligned}$$

since $\nabla^g g = 0$ for the Levi-Civita connection ∇^g.

2. Recall that $X, Y \in \mathfrak{X}(M)$, we have $\mathcal{L}_Z(g(X,Y)) = (\mathcal{L}_Z g)(X,Y) + g(\mathcal{L}_Z X, Y) + g(X, \mathcal{L}_Z Y)$, hence

$$\begin{aligned} (\mathcal{L}_Z g)(X,Y) &= \mathcal{L}_Z(g(X,Y)) - g(\mathcal{L}_Z X, Y) - g(X, \mathcal{L}_Z Y) \\ &= Z(g(X,Y)) - g([Z,X],Y) - g(X,[Z,Y]) \\ &= g(\nabla^g_Z X, Y) + g(X, \nabla^g_Z Y) - g(\nabla^g_Z X - \nabla^g_X Z, Y) - g(X, \nabla^g_Z Y - \nabla^g_Y Z) \\ &= g(\nabla^g_X Z, Y) + g(X, \nabla^g_Y Z) \\ &= g(X,Y) + g(X,Y) = 2g(X,Y), \end{aligned}$$

where we used the metricity and the torsion freeness of the Levi-Civita connection ∇^g.

3. Let us denote the distribution in question by $\mathcal{D} \subset TM$, i.e. $\mathcal{D}_p := (Z_p)^\perp$ for all $p \in M_0$. Since Z is nowhere zero on M_0, $\mathcal{D}$ is a codimension-one distribution, which is obviously transverse to the one-dimensional foliation $\mathcal{Z}$ generated by[1] Z.

   We will show now that $\mathcal{D}$ is involutive. Let $X, Y \in \mathcal{D}$. Then by torsion freeness of the Levi-Civita connection, we have

$$\begin{aligned} g([X,Y],Z) &= g(\nabla^g_X Y - \nabla^g_Y X, Z) = g(\nabla^g_X Y, Z) - g(\nabla^g_Y X, Z) \\ &= g(X,Z) - g(Y,Z) = 0, \end{aligned}$$

   thus $[X,Y] \in \mathcal{D}$. Integrability of $\mathcal{D}$ now follows from Frobenius' theorem.

   The fact that the leaves of $\mathcal{D}$ are connected components of level sets of r follows from the facts that Z is the gradient of $\frac{1}{2}r^2$ and that the level sets of $\frac{1}{2}r^2$ and r are the same since $r \geq 0$.

[1]The foliation $\mathcal{Z}$ consists of maximal integral curves of Z.

4. We show that a leaf of $\mathcal{D}$ and a leaf of $\mathcal{Z}$ intersect each other in at most one point. For this, consider a maximal integral curve $\gamma\colon I \to M_0$ of Z where $I \subset \mathbb{R}$ is an open interval around $0 \in \mathbb{R}$. Introduce the temporary notation $\rho := \frac{1}{2}r^2$. Then the function $\rho \circ \gamma$ is smooth and its derivative at $s \in I$ is

$$(\rho \circ \gamma)'_s = d\rho_{\gamma(s)}\gamma'_s = d\rho_{\gamma(s)}Z_{\gamma(s)} = g(Z_{\gamma(s)}, Z_{\gamma(s)}) = 2(\rho \circ \gamma)(s).$$

   This ordinary differential equation has the solution $(\rho \circ \gamma)(s) = (\rho \circ \gamma)(0)e^{2s}$. Consequently, we have $(r\circ\gamma)(s) = (r\circ\gamma)(0)e^s$, which is a monotonously increasing function since $\rho > 0$ on M_0. Thus it cannot happen that γ intersects a level set of r twice.

   To summarize, we see that $\mathcal{D}$ and $\mathcal{Z}$ are transverse foliations the leaves of which intersect at most in a single point. The claim now follows from the discussion in [Mat84, Section VII.7] but cf. also [Sza16, Section 6.1].

5. We have already seen in the previous point that if $r(p) = r(q)$, then $r(\mathrm{Fl}^Z_t\, p) = r(\mathrm{Fl}^Z_t\, q)$ whenever both sides make sense. Let $L_\alpha := \{p \in M\ |r(p) = \alpha\}$. Then our result can be formulated as $\mathrm{Fl}^Z_t(L_\alpha) \subset L_{\alpha e^t}$ whenever the left-hand side makes sense. Since the flow is a continuous map, it maps connected subsets (like leaves) to connected subsets (subsets of leaves). □

Example 6.6 (Punctured flat space)**.** *The easiest example of a Riemannian cone is* $(\mathbb{R}^n \setminus \{0\}, g_{std}, r\partial_r)$ *where* $r := \sqrt{(x^1)^2 + \cdots + (x^n)^2}$ *is the radial coordinate.*

Example 6.7. *Given a Riemannian manifold* (L, g_L)*, we can define a Riemannian cone, which we call the Riemannian cone over* (L, g_L)*, as follows:*

$$\mathrm{Cone}(L, g_L) := (\mathbb{R}^{>0} \times L, g := dr^2 + r^2 g_L),$$

where r *denotes the coordinate on* $\mathbb{R}^{>0}$*. One can easily see that* $(C_\pm(M, g_M), Z := r\partial_r)$ *is a Riemannian cone. The vector field* Z *is complete, its flow is* $\mathrm{Fl}^Z_t(r, p) = (re^t, p)$*. Note that* $\mathrm{Cone}(L, g_L)$ *is conformal to product metric* $(\mathbb{R}, dt \otimes dt) \times (L, g_L)$ *via the diffeomorhism*

$$\phi\colon \mathbb{R} \times L \to \mathrm{Cone}(L, g_L), (t, x) := (e^t, x),$$

where t *is the standard coordinate on* $\mathbb{R}$*. In fact, one calculates easily that* $\phi^*(dr \otimes dr + r^2 g_L) = e^{2t}(dt \otimes dt + g_L)$*.*

Remark 6.8. *If* $(L, g_L) = (B_R(0), g_{std}|_{B_R(0)})$ *is the ball in* $\mathbb{R}^2$ *of radius* R *endowed with the induced metric of the flat metric of* $\mathbb{R}^2$*, then* $\mathrm{Cone}(L, g_L)$ *may be embedded isometrically into* $\mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$*. Restricting this to the subset where* $r < R$ *for some* $R \in \mathbb{R}$*, we recover the classical definition of a right-angled cone (κῶνός) without its apex. [Euc08, Book X, Definition 18].*

In fact, it turns out that all Riemannian cones are of the form given in Example 6.7.

Proposition 6.9. *If* (M, g, Z) *is a Riemannian cone and* (L, g_L) *is its link, then*

$$\phi\colon\ \mathrm{Cone}(L, g_L) \to (M, g), (t, x) \mapsto \mathrm{Fl}^Z_t\, x$$

is a Riemannian isometry which maps $r\partial_r$ *to* Z*.*

Proof. This follows by an easy calculation based on Proposition 6.5. □

Many local calculations based on local frames can be made considerably simpler by using a frame that is "well suited" to the conical geometry.

Corollary 6.10. *Let (M, g, Z) be a Riemannian cone with link (L, g_L) and let $p = \phi(r, x) \in M$. If $\{e_i \mid i = 1, \ldots, \dim L\}$ is a g_L-orthonormal frame around $x \in L$, then $\{E_i \mid i = 0, \ldots, \dim L\}$ where $E_0 := \partial_r$ and $E_j := \frac{1}{r} d\phi e_j$ for $j > 0$, is a g-orthonormal frame around p.*

Proof. This follows from the facts that $g(\partial_r, V) = 0$ for any $V \in TL$ and that

$$g(E_i, E_j) = g_L\left(\frac{1}{r}d\phi e_i, \frac{1}{r}d\phi e_j\right) = r^2 g_L\left(\frac{1}{r}e_i, \frac{1}{r}e_j\right) = \delta_{ij}$$

since $E_i \in TL$ for $i > 0$. □

Definition 6.11. *A standard (or adapted) frame on TM is a frame constructed like in Corollary 6.10.*

From now on, we will suppress the isomorphism ϕ from the notation.

Remark 6.12. *Proposition 6.9 shows that the leaf space of $\mathcal{D}$ is isomorphic to $\mathbb{R}^+$. Moreover, each leaf is isomorphic to the link.*

Remark 6.13. *Proposition 6.9 shows that Riemannian cones are a special type of warped product:* $\mathrm{Cone}(L, g_L) = (\mathbb{R}_+, dr \otimes dr) \times_r (L, g_L)$. *Therefore, all the terminology of warped products gets inherited to cones.*

Note that for Riemannian manifolds (L_1, g_1) and (L_2, g_2) with $L_1 \cap L_2 = \emptyset$, we have $\mathrm{Cone}((L_1, g_1) \cup (L_2, g_2)) \simeq \mathrm{Cone}(L_1, g_1) \cup \mathrm{Cone}(L_2, g_2)$, thus we may restrict our attention to cones with connected link.

Definition 6.14. *A regular Riemannian cone is a Riemannian cone the link of which is compact and connected.*

Proposition 6.15 (Metric completion of a cone, [BM16, Theorem 1.5])**.** *A Riemannian cone (M, g, Z) can be completed as a metric space by adjoining an ideal point $\star$ with $r(\star) = 0$.*

Definition 6.16. *The apex (or tip) of the Riemannian cone is the unique point in $\overline{M} \setminus M$.*

Corollary 6.17. *The distance function from the apex coincides with the function r on M.*

Proof. It is a general fact for Riemannian warped products that the the projection $\pi\colon B \times_f F \to B$ to the base (in case of a cone: $B = (0, \infty)$) does not increase the length of tangent vectors, since

$$\begin{aligned} g(X, X) &= g(d\pi X + (X - X d\pi X), d\pi X + (X - X d\pi X)) \\ &= |d\pi X|_g^2 + 2g(d\pi X, X - d\pi X) + |X - d\pi X|_g^2 \\ &= |X|_{g_B}^2 + f^2 \circ \pi \cdot |X - d\pi X|_{g_F}^2 \\ &\geq |X|_{g_B}^2, \end{aligned}$$

hence we have for any curve $\gamma\colon I \to M$ that $L(\gamma) \geq L(\pi \circ \gamma)$ (lengths), and in particular $\mathrm{dist}_g(p, p') \geq \mathrm{dist}_{g_B}(\pi(p), \pi(p'))$. This means that $\mathrm{dist}_g(\{r_1\} \times L, \{r_2\} \times L) \geq \mathrm{dist}_{dr \otimes dr}(r_1, r_2)$. On the other hand, equality can be achieved via the curve $\gamma\colon [r_1, r_2] \to M, t \mapsto (t, x)$ for any fixed $x \in L$. The claim now follows from a density argument. □

From now on, we will be interested solely in regular Riemannian cones, and therefore we will leave the adjective "regular" to ease terminology. Their appeal stems from the hope[2] that they may offer a good balance between noncompactness and tracability. More concretely, it is reasonable to make the working hypothesis that the geometry of the link determines a good portion of the geometry of the cone. Moreover, regular cones can be thought of as being only "mildly" noncompact (in the sense that they are noncompact only "in one direction").

Remark 6.18. *The question arises whether the Euler vector field is unique on a Riemannian cone. It is easy to see that the set of Euler vector fields forms an affine space over the vector space of parallel vector fields. If* $\dim M \geq 2$*, then Gallot's dichotomy [Gal79] states that if* (M, g, Z) *is a Riemannian cone with compact (or, more generally, complete) link* (L, g_L)*, then either* (L, g_L) *is the round sphere or the restricted holonomy group of* g *splits. We have as a consequence, since parallel vector fields split the restricted holonomy, that the Euler vector field is unique, except on the punctured flat space (Example 6.6) where it is unique up to an additive constant.*

6.2 The stretching map

Definition 6.19. *The* stretching map of the Riemannian cone $(\bar{M}, \bar{g})$ with stretching factor $\alpha > 0$ *is the diffeomorphism* $\Phi_\alpha := \mathrm{Fl}^Z_{\ln \alpha}$ *where* $Z \in \mathfrak{X}(\bar{M})$ *is the Euler vector field.*

Remark 6.20. *Note that in the model, the stretching map with stretching factor* α *corresponds to the map* $(r, x) \mapsto (\alpha r, x)$*.*

The stretching map will be useful in the investigation of "behaviour at infinity", and therefore it is a good idea to collect some of its useful properties in the next lemma, the proof of which is straightforward calculation.

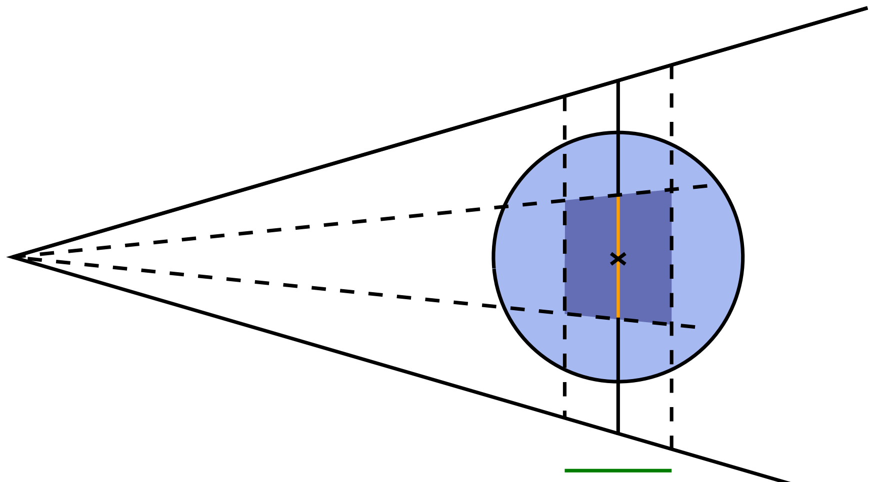
Lemma 6.21.
1. *The stretching map is a group homomorphism* $(\mathbb{R}_+, \cdot) \to \mathrm{Diff}(M)$ *and its generating vector field is* $\left.\frac{d}{d\alpha}\right|_{\alpha=1} \Phi_\alpha(p) = Z_p$ *for any* $p \in \bar{M}$*.*
2. *The stretching map is a homothety, more precisely* $(\Phi_\alpha)^* \bar{g} = \alpha^2 \bar{g}$*.*
3. *The stretching map scales the volume form as* $(\Phi_\alpha)^* \mathrm{vol}_{\bar{g}} = \alpha^{n+1} \mathrm{vol}_{\bar{g}}$*.*
4. *The Levi-Civita connections (and all induced connections) of* $\bar{g}$ *and* $(\Phi_\alpha)^* \bar{g}$ *coincide.*
5. *The stretching map intertwines between Laplace operators:* $L^{(\Phi_\alpha)^* \bar{g}} \circ (\Phi_\alpha)^* = (\Phi_\alpha)^* \circ L^{\bar{g}}$*, where* $L \in \{\Delta_B, \Delta_H, \Delta_L\}$*. In particular, the kernel of* $L^{\bar{g}}$ *gets mapped to the kernel of* $L^{(\Phi_\alpha)^* \bar{g}}$ *under pullback under* Φ_α*.*

[2] This hope is going to be realized later.

Lemma 6.22 (Non collapsing balls)**.** *On an exact cone $\bar{M}$, we have* $\inf_{p\in\bar{M}} B_p(R) > 0$ *for all* $R > 0$.

Proof. Let us work in the model, and let us fix a point $p_0 := (r_0, x_0)$ and a radius $R > 0$. Let $R_0 < \min\{r_0, R\}$. From the triangle inequality, we have the inclusion

$$(r_0 - R_0/3, r_0 + R_0/3) \times B^M_{x_0}(R_0/3) \subseteq B^{\bar{M}}_p(R).$$



To see this, let $p := (r, x)$ be an element of the set on the left-hand side, and let $\hat{p} := (r_0, x)$ be the point that is on the same Z integral curve as p and on the same level set of r as p_0. Then by the triangle inequality

$$d(p, p_0) \leq d(p, \hat{p}) + d(\hat{p}, p_0) \leq R_0/3 + R_0/3 < R,$$

so $p \in B^{\bar{M}}_{p_0}(R)$. Thus

$$\begin{aligned}
\mathrm{Vol}_{\bar{g}}\Big(B^{\bar{M}}_p(R)\Big) &\geq \mathrm{Vol}_{\bar{g}}\Big((r_0 - R_0/3, r_0 + R_0/3) \times B^M_{x_0}(R_0/3)\Big) \\
&= \int_{(r_0 - R_0/3, r_0 + R_0/3)\times B^M_{x_0}(R_0/3)} \mathrm{vol}_{\bar{g}} \\
&= \int_{r_0 - R/3}^{r_0 + R_0/3} \int_{B^M_{x_0}(R_0/3)} r^n \,\mathrm{vol}_g \, dr \\
&= \int_{r_0 - R/3}^{r_0 + R_0/3} r^n dr \cdot \int_{B^M_{x_0}(R_0/3)} \mathrm{vol}_g \\
&= \frac{1}{n+1}((\underbrace{r_0 + R_0/3}_{\geq r_0})^{n+1} - (\underbrace{r_0 - R_0/3}_{\leq \frac{2}{3} r_0})^{n+1}) \,\mathrm{Vol}_g\Big(B^M_{x_0}(R_0/3)\Big) \\
&\geq \frac{1}{n+1}(1 - (2/3)^{n+1}) r_0^{n+1} \underbrace{\inf_{x\in M} \mathrm{Vol}_g\Big(B^M_x(R_0/3)\Big)}_{>0} \\
&=: c(R) > 0,
\end{aligned}$$

uniformly in $p \in M$ for all $R > 0$. ? In the last step we have used that the volume of $R_0/3$-balls in a compact manifold do not collapse, which is due to e.g. [Heb96, Theorem 3.18(2)] since Sobolev embeddings hold on compact manifolds. □

Let us call the set $U_1 := \left\{ p \in \bar{M} \,\middle|\, 1 < |Z|_{\bar{g}}(p) < 2 \right\}$ standard ring, and let us set $U_\alpha := \Phi_\alpha(U_1)$. Note that the set U_α corresponds to the product set $(\alpha, 2\alpha) \times M$ in the model from Proposition 6.9.

6.3 The curvature of a Riemannian cone

Due to Proposition 6.9, Riemannian cones are warped products. This means we can specialize several results. The base is $(0, \infty)$, the fibre is (L, g_L), and we suppress the projections from the notation.

Lemma 6.23. *Let (M, g_{cone}) be a Riemannian cone with link (L, g_L). Let X, Y, Z be vector fields on M. Then the following hold.*

1. *The Levi-Civita connection of g_{cone} can be calculated as follows*

$$\nabla^{g_{\text{cone}}}_X Y = X(dr(Y))\partial_r + \frac{1}{r} dr(X)Y + \frac{1}{r} dr(Y)X + \nabla^{g_L}_X Y - r g_L(X, Y)\partial_r$$

2. $\nabla^{g_{\text{cone}}} r = dr$*,* $\nabla^{g_{\text{cone}},2} r = r g_L$ *and* $\Delta_B r = -\frac{\dim M - 1}{r}$.

3. *The Riemannian curvature can be calculated as follows.*

$$R^{g_{\text{cone}}}(X, Y)Z = R^{g_L}(X, Y)Z + g_L(X, Z)Y - g_L(Y, Z)X.$$

   In particular, $R^{g_{\text{cone}}}(X, Y)Z$ is always vertical, and it is always zero if at least one of the three vector fields playing a role here is horizontal.

4. *The Ricci curvature can be calculated as follows.*

$$\operatorname{Ric}^{g_{\text{cone}}} = \operatorname{Ric}^{g_L} + (2 - \dim M) g_L.$$

   In particular $\operatorname{Ric}^{g_{\text{cone}}} = 0$ *if and only if (L, g_L) is Einstein with Einstein constant* $\dim M - 2$.

Proof.

1. This is a direct consequence of [Che11, Proposition 4.1] and the flatness of $dr \otimes dr$. Note that the notation $\langle \cdot, \cdot \rangle$ is used for the warped product metric in this book.

2. This follows by direct calculation from the last claim. For the Laplace–Beltrami operator, we can use an adapted frame, cf. Definition 6.11.

3. This is a direct consequence of the flatness of $dr \otimes dr$, the previous claims and [Che11, Proposition 4.2].

4. This is either a short calculation in an adapted frame based on previous claims or a special case of [Che11, Corollary 4.1].

□

Multiplying tensor fields with an appropriate radial factor makes them radially covariantly constant. Using this fact can simplify calculations substantially.

Corollary 6.24. *If $\omega_L, \eta_L \in \Gamma^\infty\left(T^*L\right)$ and $h_L \in \Gamma^\infty\left(S^2T^*L\right)$, then*

$$\nabla^{g_{\text{cone}}}_{\partial_r}(r\omega_L) = \nabla^{g_{\text{cone}}}_{\partial_r}(dr) = 0$$

and

$$\nabla^{g_{\text{cone}}}_{\partial_r}(dr \otimes dr) = \nabla^{g_{\text{cone}}}_{\partial_r}(r\omega_L \odot dr) = \nabla^{g_{\text{cone}}}_{\partial_r}(r\omega_L \odot r\eta_L) = \nabla^{g_{\text{cone}}}_{\partial_r}(r^2 h_L) = 0.$$

Proof. This follows from direct calculation based on Lemma 6.23. □

The pointwise norm of the Riemannian curvature decays quadratically (with derivatives) on a Riemannian cone.

Lemma 6.25. *On the exact cone (M, g_{cone}), we have for the pointwise norm $|\nabla^{g_{\text{cone}},k} R^{g_{\text{cone}}}|_{g_{\text{cone}}} = O\left(r^{-2-k}\right)$.*

Proof. As usual, we calculate in an adapted frame $\{E_i \mid i = 0, \ldots, \dim M - 1\}$.

$k = 0$: We obtain by direct calculation that $|R^{g_{\text{cone}}}|_{g_{\text{cone}}} = O\left(r^{-2}\right)$.

$k = 1$: For $X, Y, Z, W \in TM$, we can use the formula for the Riemannian curvature tensor from Lemma 6.23 to obtain

$$\begin{aligned}
(\nabla^{g_{\text{cone}}}_W R^{g_{\text{cone}}})(X,Y)Z &= \nabla^{g_{\text{cone}}}_W (R^{g_{\text{cone}}}(X,Y)Z) \\
&\quad - R^{g_{\text{cone}}}(\nabla^{g_{\text{cone}}}_W X, Y)Z - R^{g_{\text{cone}}}(X, \nabla^{g_{\text{cone}}}_W Y)Z - R^{g_{\text{cone}}}(X,Y)\nabla^{g_{\text{cone}}}_W Z \\
&= \nabla^{g_{\text{cone}}}_W (R^{g_L}(X,Y)Z + g(X,Z)Y - g(Y,Z)X) \\
&\quad - R^{g_L}(\nabla^{g_{\text{cone}}}_W X, Y)Z - g(\nabla^{g_{\text{cone}}}_W X, Z)Y + g(Y,Z)\nabla^{g_{\text{cone}}}_W X \\
&\quad - R^{g_L}(X, \nabla^{g_{\text{cone}}}_W Y)Z - g(X,Z)\nabla^{g_{\text{cone}}}_W Y + g(\nabla^{g_{\text{cone}}}_W Y, Z)X \\
&\quad - R^{g_L}(X,Y)\nabla^{g_{\text{cone}}}_W Z - g(X, \nabla^{g_{\text{cone}}}_W Z)Y + g(Y, \nabla^{g_{\text{cone}}}_W Z)X \\
&= \nabla^{g_{\text{cone}}}_W (R^{g_L}(X,Y)Z) \\
&\quad - R^{g_L}(\nabla^{g_{\text{cone}}}_W X, Y)Z - R^{g_L}(X, \nabla^{g_{\text{cone}}}_W Y)Z - R^{g_L}(X,Y)\nabla^{g_{\text{cone}}}_W Z \\
&= \nabla^{g_L}_W (R^{g_L}(X,Y)Z) + \frac{1}{r}dr(Z)(R^{g_L}(X,Y)Z) + R^{g_L}(\nabla^{g_{\text{cone}}}_W X, Y)Z \\
&+ R^{g_L}(X, \nabla^{g_{\text{cone}}}_W Y)Z + R^{g_L}(X,Y)\nabla^{g_{\text{cone}}}_W Z \\
&\quad - R^{g_L}(\nabla^{g_{\text{cone}}}_W X, Y)Z - R^{g_L}(X, \nabla^{g_{\text{cone}}}_W Y)Z - R^{g_L}(X,Y)\nabla^{g_{\text{cone}}}_W Z \\
&= \nabla^{g_L}_W (R^{g_L}(X,Y)Z) + \frac{1}{r}dr(Z)(R^{g_L}(X,Y)Z) \\
&= (\nabla^{g_L} R^{g_L} + \frac{1}{r}dr \otimes R^{g_L})(W,X,Y,Z).
\end{aligned}$$

Now a direct calculation reveals that $|\nabla^{g_{\text{cone}}} R^{g_{\text{cone}}}|_{g_{\text{cone}}} = O\left(r^{-3}\right)$.

$k > 1$: Since $\nabla^{g_{\text{cone}}}(\frac{1}{r}dr) = -\frac{1}{r^2}dr \otimes dr + \frac{1}{r}rg$ and $\nabla^{g_{\text{cone}}} h = \nabla^{g_L} h$ for any r-independent vertical tensor field h, the Leibniz rules implies that $\nabla^{g_{\text{cone}},k} R^{g_{\text{cone}}}$ will be a linear combination of tensor products of $\frac{1}{r}dr$ and vertical fields independent of r. The g-norm of each of these is $O\left(r^{-1}\right)$, and so the $(k+2)$-contravariant tensor $\nabla^{g_{\text{cone}},k} R^{g_{\text{cone}}}$ has pointwise $|\nabla^{g_{\text{cone}},k} R^{g_{\text{cone}}}|_{g_{\text{cone}}} = O\left(r^{-k-2}\right)$, as advertised. □

6.4 The tangential operator

Definition 6.26. *A family of vector bundles on the manifold M parametrized by the manifold A is a vector bundle $\pi\colon E \to A \times M$.*

The reason for this definition becomes clear if we introduce the notation $E_a := \pi^{-1}(\{a\}\times M)$ for any $a \in A$. After identifying $\{a\} \times M$ with M using the canonical projection, the bundle E_a becomes an ordinary vector bundle over M (with the restriction of π as the bundle projection). We identify sections via the map $M \to \{a\} \times M, x \mapsto (a, x)$.

Note that the set $\{a\} \times M$ is a closed set in $A \times M$, and therefore we can extend any section from $\{a\} \times M$ to an open neighbourhood using coordinates patches and a partition of unity. (In fancy sheaf theoretic language: the sheaf of sections of a vector bundle is fine and consequently soft.)

Definition 6.27. *A differential operator $D\colon \Gamma(E) \to \Gamma(E)$ is restrictable to E_a if for any section $h_a \in \Gamma(E_a)$, the expression $D(\tilde{h_a})|_{\{a\}\times M}$ is independent of the extension $\tilde{h_a}$ of h_a. In this case, we define $D_a\colon \Gamma(E_a) \to \Gamma(E_a)$ via $D_a(h_a) := D(\tilde{h_a})|_{\{a\}\times M}$.*

Restrictable operators form a $\mathcal{C}^\infty(M)$-module.

Example 6.28. *Consider[3] the Riemannian cone $(\bar{M}, \bar{g})$ with link (M, g). As we have seen before in Proposition 6.9, $\bar{M} = I \times M$ as manifolds with $I = (0, \infty)$. This way, any tensor bundle E over $\bar{M}$ can be considered as a family of vector bundles on M parametrized by I.*

Given a g-orthogonal frame $e_1, \ldots, e_n$ of TM, the raw Laplacian on E can be written as follows, with $\bar{\nabla}$ denoting the connection induced by the Levi-Civita connection of the metric $\bar{g}$.

$$\begin{aligned}
\bar{\Delta}_B &= -\bar{\nabla}^2_{\partial_r,\partial_r} - \frac{1}{r^2}\sum_{i>0}\bar{\nabla}^2_{e_i,e_i} \\
&= -\bar{\nabla}_{\partial_r}\bar{\nabla}_{\partial_r} + \bar{\nabla}_{\bar{\nabla}_{\partial_r}\partial_r} - \frac{1}{r^2}\sum_{i>0}\left(\bar{\nabla}_{e_i}\bar{\nabla}_{e_i} - \bar{\nabla}_{\bar{\nabla}_{e_i}e_i}\right) \\
&= -\bar{\nabla}_{\partial_r}\bar{\nabla}_{\partial_r} - \frac{1}{r^2}\sum_{i>0}\left(\bar{\nabla}_{e_i}\bar{\nabla}_{e_i} - \bar{\nabla}_{\nabla_{e_i}e_i - rg(e_i,e_i)\partial_r}\right) \\
&= -\bar{\nabla}_{\partial_r}\bar{\nabla}_{\partial_r} - \frac{n}{r}\bar{\nabla}_{\partial_r} - \frac{1}{r^2}\sum_{i>0}\left(\bar{\nabla}_{e_i}\bar{\nabla}_{e_i} - \bar{\nabla}_{\nabla_{e_i}e_i}\right).
\end{aligned}$$

(This formula is a slight generalization of the well-known 'Laplacian in spherical coordinates' formula.)

The operator $D := -\frac{1}{r^2}\sum_{i>0}\left(\bar{\nabla}_{e_i}\bar{\nabla}_{e_i} - \bar{\nabla}_{\nabla_{e_i}e_i}\right)$ is restrictable.

Example 6.29. *In this thesis, the operators of interest differ from the raw Laplacian by a zeroth-order term. Such terms, being pointwise, are always restrictable.*

Counterexample 6.30. *The operator $-\bar{\nabla}_{\partial_r}\bar{\nabla}_{\partial_r}$ is a non-restrictable second-order operator.*

[3] Here we see the convention mentioned earlier: $(M, g) = \mathrm{Cone}(()L, g_L)$ and $(\bar{M}, \bar{g}) = \mathrm{Cone}(M, g)$. The presence/absence of the bar in the notation makes it unambiguous which convention is being used.

As we have discussed in Proposition 6.5, the flow of the Euler vector field establishes a diffeomorphism between $\{r_1\} \times M$ and $\{r_2\} \times M$ for any $r_1, r_2 \in I$. Recall the notation $\Phi_\alpha := \mathrm{Fl}^Z_{\ln \alpha}$ for the stretching map. This map maps $\{r\} \times M$ to $\{\alpha r\} \times M$. The stretching map allows us to define an even better class of operators on subbundles of the tensor bundle, for example for $E := S^2T^*(I \times M)$.

Definition 6.31. *A restrictable operator $D \colon \Gamma(E) \to \Gamma(E)$ on a finite-rank subbundle E of the tensor bundle of $I \times M$ is conical or Euler if the following diagram commutes for any $r_1, r_2 \in I$.*

$$\begin{array}{ccc} \Gamma(E_{r_1}) & \xrightarrow{D_{r_1}} & \Gamma(E_{r_1}) \\ {\scriptstyle (\Phi_{r1/r2})^*}\big\downarrow & & \big\downarrow{\scriptstyle (\Phi_{r_1/r_2})^*} \\ \Gamma(E_{r_2}) & \xrightarrow[D_{r_2}]{} & \Gamma(E_{r_2}) \end{array}$$

An easy calculation convinces us of the next statement.

Lemma 6.32. *A restrictable operator $D \colon \Gamma(E) \to \Gamma(E)$ is conical if and only if the following diagram commutes for all $r \in I$.*

$$\begin{array}{ccc} \Gamma(E_{r}) & \xrightarrow{D_{r}} & \Gamma(E_{r}) \\ {\scriptstyle (\Phi_{r})^*}\big\downarrow & & \big\downarrow{\scriptstyle (\Phi_{r})^*} \\ \Gamma(E_{1}) & \xrightarrow[D_{1}]{} & \Gamma(E_{1}) \end{array}$$

Euler operators are in a one to one correspondence with operators on any of the bundles E_r.

Example 6.33. *Consider the operator D from Example 6.28 acting on functions. We claim that $r^2 D$ is an Euler operator .*

Indeed, note first that whenever ϕ is a diffeomorphism, α is a 1-form and X is a vector field, we have

$$\phi^*(\alpha(X)) = (\phi^*\alpha)(d\phi^{-1}X).$$

Now we calculate for $f \in \mathcal{C}^\infty(\{r\} \times M)$, using that $\Phi_r|_{TM} = \mathrm{id}_{TM}$ and naturality of the pullback,

$$\begin{aligned} (\Phi_r)^*(D_r f) &= (\Phi_r)^* \left(-\frac{1}{r^2} \sum_{i>0} \left(\bar{\nabla}_{e_i} \bar{\nabla}_{e_i}(f) + \bar{\nabla}_{\nabla_{e_i} e_i}(f) \right) \right) \\ &= -\frac{1}{r^2} \sum_{i>0} (\Phi_r)^* \left(\bar{\nabla}_{e_i} \bar{\nabla}_{e_i}(f) - \bar{\nabla}_{\nabla_{e_i} e_i}(f) \right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{r^2}\sum_{i>0}(\Phi_r)^*\left(e_i(e_i(f)) - df(\nabla_{e_i}e_i)\right)\\
&= -\frac{1}{r^2}\sum_{i>0}(\Phi_r)^*\left(d(e_i(f))(e_i) - df(\nabla_{e_i}e_i)\right)\\
&= -\frac{1}{r^2}\sum_{i>0}\Big((\Phi_r)^*d(e_i(f))(d\Phi_r^{-1}e_i) - (\Phi_r)^*(df)(d\Phi_r^{-1}\nabla_{e_i}e_i)\Big)\\
&= -\frac{1}{r^2}\sum_{i>0}\left(d[(\Phi_r)^*(e_i(f))](d\Phi_r^{-1}e_i) - d(\underbrace{(\Phi_r)^*f}_{=:f_r})(d\Phi_r^{-1}\nabla_{e_i}e_i)\right)\\
&= -\frac{1}{r^2}\sum_{i>0}\Big(d[(\Phi_r)^*(df(e_i))](d\Phi_r^{-1}e_i) - d(f_r)(d\Phi_r^{-1}\nabla_{e_i}e_i)\Big)\\
&= -\frac{1}{r^2}\sum_{i>0}\Big(d[((\Phi_r)^*df)(d\Phi_r^{-1}e_i))](d\Phi_r^{-1}e_i) - d(f_r)(d\Phi_r^{-1}\nabla_{e_i}e_i)\Big)\\
&= -\frac{1}{r^2}\sum_{i>0}\Big(d[d(f_r)(d\Phi_r^{-1}e_i))](d\Phi_r^{-1}e_i) - d(f_r)(d\Phi_r^{-1}\nabla_{e_i}e_i)\Big)\\
&= -\frac{1}{r^2}\sum_{i>0}\Big((d\Phi_r^{-1}e_i)((d\Phi_r^{-1}e_i)(f_r)) - (d\Phi_r^{-1}\nabla_{e_i}e_i)(f_r)\Big)\\
&= -\frac{1}{r^2}\sum_{i>0}\left((e_i(e_i(f_r)) - (\nabla_{e_i}e_i)(f_r)\right)\\
&= \frac{1}{r^2}D_1(f_r)\\
&= \frac{1}{r^2}D_1((\Phi_r)^*(f)),
\end{aligned}
$$

thus $(\Phi_r)^* \circ (r^2 D_r) = 1^2 D_1 \circ (\Phi_r)^*$*, as claimed.*

Example 6.34. *Similarly, the operator* $r^2 D$ *(cf. Example 6.28) acting on symmetric 2-tensor fields is an Euler operator.*

Example 6.35. *Consider the potential term in the Hodge Laplacian on* $\bar{M}$*. Since* $(\bar{M}, \bar{g})$ *is Ricci-flat, the potential term is zero, we are done.*

Example 6.36. *Consider now the potential term in the Lichnerowicz Laplacian on* $\bar{M}$*, i.e.* $-2\mathring{\bar{R}}$*. We claim that* $r^2\mathring{\bar{R}}$ *is an Euler operator. This is a zeroth-order operator, so it suffices to check the Euler condition on a single frame of* E_r *at any point* $x \in M$*. Such a frame can be chosen to consist of forms of the form* $dr \otimes dr$*,* $dr \odot rh_1$ *and* $r^2 h_2$ *where* $h_1 \in \Gamma(T^*M)$ *and* $h_2 \in \Gamma\left(S^2(T^*M)\right)$*. Note that* $d(\Phi_r)_{(1,x)}(X^0\partial_r + X_M) = rX^0\partial_r + X_M$*, and therefore*

$$
\begin{aligned}
(\Phi_r)^*(dr \otimes dr) &= r^2 dr \otimes dr,\\
(\Phi_r)^*(dr \odot rh_1) &= rdr \odot r(h_1 \circ \Phi_r)\\
(\Phi_r)^*(r^2 h_2) &= r^2 h_2 \circ \Phi_r,
\end{aligned}
$$

or more concisely

$$(\Phi_r)^* h = r^2 h \circ \Phi_r \tag{6.1}$$

for any of the tensors above.

We calculate, using that $(\Phi_r)^* h \in \Gamma(E_1)$ *and therefore* $E_i = e_i$ *for* $i > 0$,

$$\begin{aligned}
\mathring{\bar{R}}_1((\Phi_r)^*(dr \otimes dr)) &= r^2 \sum_{i=0}^{n} (dr \otimes dr)(\underbrace{\bar{R}(E_i, \cdot)\cdot}_{\in TM}, E_i) \\
&= 0 \\
\mathring{\bar{R}}_1((\Phi_r)^*(dr \odot h_1)) &= \sum_{i=0}^{n} r^2 (dr \odot (h_1 \circ \Phi_r))(\underbrace{\bar{R}(E_i, \cdot)\cdot}_{\in TM}, E_i) \\
&= r^2 dr(\partial_r)(h_1 \circ \Phi_r)(\underbrace{\bar{R}(\partial_r, \cdot)\cdot}_{=0}) + \sum_{i=1}^{n} \underbrace{dr(e_i)}_{=0}(h_1 \circ \Phi_r)(\bar{R}(e_i, \cdot)\cdot) \\
&= 0 \\
\mathring{\bar{R}}_1((\Phi_r)^*(r^2 h_2)(X, Y) &= \sum_{i=0}^{n} r^2 (h_2 \circ \Phi_r)(\bar{R}(E_i, X)Y, E_i) \\
&= r^2 (h_2 \circ \Phi_r)(\bar{R}(\partial_r, X)Y, \partial_r) + r^2 \sum_{i=1}^{n} (h_2 \circ \Phi_r)(\bar{R}(e_i, X)Y, e_i) \\
&= r^2 \sum_{i=1}^{n} (h_2 \circ \Phi_r)(R(e_i, X)Y + g(e_i, Y)X - g(X, Y)e_i, e_i) \\
&= r^2 (\mathring{R}(h_2 \circ \Phi_r) + (h_2 \circ \Phi_r) - g(X, Y) \operatorname{Tr}_g h \circ \Phi_r)(X, Y)
\end{aligned}$$

On the other hand, similarly

$$\begin{aligned}
(\Phi_r)^*(\mathring{\bar{R}}_r(dr \otimes dr)) &= (\Phi_r)^* 0 = 0 \\
(\Phi_r)^*(\mathring{\bar{R}}_r(dr \odot r h_1)) &= (\Phi_r)^* 0 = 0 \\
(\Phi_r)^*(\mathring{\bar{R}}_r(r^2 h_2))(X, Y) &= r^2 \frac{1}{r^2} \sum_{i>0} h_2(\bar{R}(e_i, d(\Phi_r)X)(d(\Phi_r)Y), e_i) \\
&= \sum_{i>0} h_2(\bar{R}(e_i, d(\Phi_r)X)(d(\Phi_r)Y), e_i) \\
&= (\mathring{R}(h_2 \circ \Phi_r) + (h_2 \circ \Phi_r) - g(X, Y) \operatorname{Tr}_g h \circ \Phi_r)(X, Y).
\end{aligned}$$

Thus we have shown that $r^2 (\Phi_r)^* \circ \mathring{\bar{R}}_r = \mathring{\bar{R}}_1 \circ (\Phi_r)^*$. *This means that* $r^2 \mathring{\bar{R}}$ *is an Euler type operator.*

We have observed the same scaling in Examples 6.34 and 6.36. This motivates the following definition.

Definition 6.37. *Let* $(\bar{M}, \bar{g})$ *be a Ricci-flat Riemannian cone. The tangential operator to the operator* $\bar{\Delta}_L$ *is the Euler operator*

$$\Box_E := \Box_L := -\sum_{i>0} \left(\bar{\nabla}_{E_i} \bar{\nabla}_{E_i} - \bar{\nabla}_{\nabla_{E_i} E_i} \right) - 2r^2 \mathring{\bar{R}}.$$

The tangential cone to the Hodge Laplacian on 1-forms is

$$\square_H := -\sum_{i>0}\left(\bar{\nabla}_{E_i}\bar{\nabla}_{E_i} - \bar{\nabla}_{\nabla_{E_i}E_i}\right),$$

(where we note that the Hodge Laplacian coincides with the raw Laplacian by Ricci-flatness). The tangential cone to the Laplace–Beltrami operator is

$$\square_B := -\sum_{i>0}\left(\bar{\nabla}_{E_i}\bar{\nabla}_{E_i} - \bar{\nabla}_{\nabla_{E_i}E_i}\right).$$

With the tangential operators, we can write

$$\bar{\Delta}_L = -\bar{\nabla}_{\partial_r}\bar{\nabla}_{\partial_r} - \frac{n}{r}\bar{\nabla}_{\partial_r} + \frac{1}{r^2}\square_L \tag{6.2}$$

$$= \frac{1}{r^2}(-\bar{\nabla}_Z\bar{\nabla}_Z - (n-1)\bar{\nabla}_Z + \square_L). \tag{6.3}$$

Of course, the decomposition of the operator Δ_L into an Euler operator and a rest is not canonical: we have the freedom of an Euler type operator.

The next lemma collects some important properties of the tangential operator.

Lemma 6.38. *1. For any $r \in I$, we have the following Leibniz type rule for $(\square_L)_r$:*

$$(\square_L)_r(fh) = r^2(\Delta_B)_r(f)h - 2\bar{\nabla}_{\mathrm{grad}_g f}h + f(\square_L)_r h \tag{6.4}$$

whenever $f \in \mathcal{C}^\infty(M)$ and $h \in \Gamma(E_r)$.

2. *For any $r \in I$, the restriction $(\square_L)_r$ is an elliptic operator.*

3. *For any $r \in I$, the restriction $(\square_L)_r$ is formally self-adjoint with respect to the L^2 metric $\int_{\{r\}\times M}\langle\cdot,\cdot\rangle_{\bar{g}}\,\mathrm{vol}_g$.*

4. *For $r = 1$, the eigenfields w_i of $(\square_L)_1$ can be chosen to satisfy the normalization condition*

$$\int_{\{1\}\times M}\langle w_i, w_j\rangle_{\bar{g}|_{\{1\}\times M}}\,\mathrm{vol}_g = \delta_{ij}.$$

5. *For any $r \in I$, the symmetric 2-tensor fields $(\Phi_{1/r})^* w_i$ are eigenfields of $(\square_L)_r$ satisfying the normalization condition*

$$\int_{\{r\}\times M}\left\langle(\Phi_{1/r})^* w_i, (\Phi_{1/r})^* w_j\right\rangle_{\bar{g}}\,\mathrm{vol}_{g|_{\{r\}\times M}} = r^n\delta_{ij}. \tag{6.5}$$

Proof. 1. Let $f \in \mathcal{C}^\infty(\{r\}\times M)$ and $h \in \Gamma(E_r)$. Then

$$(\square_L)_r(f^2h) = -\sum_{i>0}\left(\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}(fh) - \bar{\nabla}_{\nabla_{e_i}e_i}(fh) - 2r^2\mathring{\bar{R}}fh\right)$$

$$= -\sum_{i>0}\left(\bar{\nabla}_{e_i}(e_i(f)h + f\bar{\nabla}_{e_i}h)\right.$$

$$- \nabla_{\nabla_{e_i} e_i}(f)h + f\bar{\nabla}_{\nabla_{e_i} e_i} h - 2r^2 f \mathring{\bar{R}} h\Big)$$

$$= -\sum_{i>0} \Big(e_i((e_i(f))h + e_i(f)\bar{\nabla}_{e_i} h + e_i(f)\bar{\nabla}_{e_i} h + f\bar{\nabla}_{e_i}\bar{\nabla}_{e_i} h)$$

$$- \nabla_{\nabla_{e_i} e_i}(f)h + f\bar{\nabla}_{\nabla_{e_i} e_i} h - 2r^2 f \mathring{\bar{R}} h\Big)$$

$$= r^2(\Delta_B)_r(f)h - 2\bar{\nabla}_{\mathrm{grad}_g f} h + f(\Box_L)_r h$$

2. It is clear that $(\Box_L)_r$ is a second-order operator. To determine its principal symbol, let $x \in M$, $\lambda \in T_x M$ and $h \in (E_r)_X$. Then we can find a function $f \in \mathcal{C}^\infty(M)$ with $f(x) = 0$ and $df_x = \lambda$. Moreover, we can find a section $\sigma \in \Gamma(E_r)$ with $\sigma(x) = h$. Now by the definition of the principal symbol and the Leibniz rule, we calculate

$$\begin{aligned}
\sigma_x((\Box_L)_r)(\lambda)(h) &= ((\Box_L)_r(f^2\sigma))(x) \\
&= \Big(r^2(\Delta_B)_r(f^2)\sigma - 2\bar{\nabla}_{\mathrm{grad}_g(f^2)}\sigma + f^2(\Box_L)_r\sigma\Big)(x) \\
&= \Big(r^2(\Delta_B)_r(f^2)\sigma - 4f\bar{\nabla}_{\mathrm{grad}_g f}\sigma + f^2(\Box_L)_r\sigma\Big)(x) \\
&= \Big(r^2(2f(\Delta_B)_r(f) - 2\langle df, df\rangle_{\bar{g}})\sigma\Big)(x) \\
&= -2\langle\lambda,\lambda\rangle_{\bar{g}}\, h \\
&= -\frac{2}{r^2}\langle\lambda,\lambda\rangle_g\, h
\end{aligned}$$

thus whenever $\lambda \neq 0$, the symbol $\sigma(D)(\lambda)$ is invertible. (Note that E_r is a bundle over M, so $\lambda \in TM$.)

3. Consider two smooth symmetric 2-tensor fields $h, k \in \Gamma(E_r)$. It is easy to show that $\delta^{\bar{g}} E_i = \delta^g e_i = 0$ and $\nabla_{E_i} E_i = 0$ for $i > 0$. By the standard technique, we see that for $X \in TM$, $(\bar{\nabla}_X)^* = -\bar{\nabla}_X - \delta^{\bar{g}} X$. Moreover, since $\mathring{\bar{R}}$ is a pointwise operation, its adjoint is also pointwise, and we know that it is formally self-adjoint in $\bar{\Delta}_L$.

   Based on these, we calculate

$$\begin{aligned}
((\Box_L)_r)^* &= \left(-\sum_{i>0}\Big(\bar{\nabla}_{E_i}\bar{\nabla}_{E_i} - \bar{\nabla}_{\nabla_{E_i} E_i}\Big) - 2r^2\mathring{\bar{R}}\right)^* \\
&= -\sum_{i>0}\Big((\bar{\nabla}_{E_i}\bar{\nabla}_{E_i})^* - (\bar{\nabla}_{\nabla_{E_i} E_i})^*\Big) - 2r^2(\mathring{\bar{R}})^* \\
&= -\sum_{i>0}\Big((\bar{\nabla}_{E_i})^*(\bar{\nabla}_{E_i})^* - (\bar{\nabla}_{\nabla_{E_i} E_i})^*\Big) - 2r^2(\mathring{\bar{R}})^* \\
&= -\sum_{i>0}\Big((-\bar{\nabla}_{E_i} - \delta^{\bar{g}} E_i)(-\bar{\nabla}_{E_i} - \delta^{\bar{g}} E_i) - 0\Big) - 2r^2(\mathring{\bar{R}})^* \\
&= -\sum_{i>0}\Big((-\bar{\nabla}_{E_i})(-\bar{\nabla}_{E_i}) - 0\Big) - 2r^2\mathring{\bar{R}}
\end{aligned}$$

$$= -\sum_{i>0}\left(\bar{\nabla}_{E_i}\bar{\nabla}_{E_i} + \bar{\nabla}_{\nabla_{E_i}E_i}\right) - 2r^2\mathring{\bar{R}}$$
$$= (\square_L)_r.$$

In the last step, we have used that $\nabla_{E_i}E_i = 0$ so we can add $\bar{\nabla}_{\nabla_{E_i}E_i}$ freely.

4. By the second claim, $(\square_L)_r$ is an elliptic operator, so elliptic regularity implies that its eigenfields are smooth. Then we can show using the standard technique and the second claim, that eigenfields corresponding to different eigenvalues are orthogonal. Lastly, we can perform a Gram–Schmidt orthogonalization process in each of the eigenspaces since they are finite dimensional.

5. This part follows from the fact that $\square_L$ is an Euler operator and the integral transformation formula. □

6.5 The spectra of the tangent operators of certain Laplacian operators

Next, we determine the spectra of some tangential operators. Similar calculations have been done in [Krö20, Section 2].

6.5.1 The general strategy

The general strategy of finding the spectrum of the tangential operator is as follows.

1. Obtain an L^2-orthogonal decomposition of the tensor fields of the appropriate on the link type using tensor fields of lower rank on the link. Show that the corresponding Laplacian on the link acts diagonally with respect to this decomposition.

2. Find orthonormal bases for the direct summands from the first step.

3. Obtain a formula relating a general tensor field on the cone to the decomposition in the second step.

4. Calculate the tangential operator on a general tensor field using the formula from the third step. (This is the most calculation intensive step.)

5. Represent the tangential operator with matrices on certain finite-dimensional subspaces and solve the eigenvalue problem explicitly (in terms of the spectra of Laplacian operators).

The higher the rank of the tensor field, the more complicated these steps are.

6.5.2 The Laplace–Beltrami operator on functions

Proposition 6.39. *Let $(M, g) := \mathrm{Cone}(L, g_L)$ be a Ricci-flat cone. Then the spectrum of the tangential operator to the Laplace–Beltrami operator on (M, g) is $\sigma(\square_B) = \sigma(\Delta_B^{g_L})$.* □

Proof. The proof of this proposition is trivial, but we follow the steps outlined for the general case methodically to illustrate them in this easy case.

1. This step is not necessary for the Laplace–Beltrami operator.
2. Let $\{v_i \,|i \in \mathbb{N}\}$ be an orthonormal basis of $L^2(\mathbb{R}_L, g_L)$ consisting of eigenvectors of $\Delta_B^{g_L}$. (This is possible by the fact that $\Delta_B^{g_L}$ is an elliptic operator acting on a compact manifold.)
3. Given any smooth function $f \in \mathcal{C}^\infty(M)$, we obtain the expansion
$$f(r,x) = \sum_{i\in\mathbb{N}} a_i(r)v_i(x),$$
where $(r,x) \in M$ and $a_i \in \mathcal{C}^\infty(\mathbb{R}^+)$.
4. Let $v \in \mathcal{C}^\infty(L)$ with $\Delta_B^{g_L} v = \lambda v$. Now $\Box_B(v) = \Delta_B^{g_L} v = \lambda v$.
5. We see that the subspace $\mathcal{C}^\infty(\mathbb{R}^+)v_i$ is invariant under $\Box_B$. This means that we may represent $\Box_B$ on this subspace as the 1-by-1 matrix
$$\begin{pmatrix}\lambda_i\end{pmatrix},$$
the eigenvalue of which is λ_i. □

6.5.3 The Hodge Laplacian on 1-forms

Decomposition of 1-forms on the link

Definition 6.40. *The vector space $D(L, g_L) := D(L) := \{h \in \Gamma^\infty(T^*L) \,|\delta^{g_L}\alpha = 0\}$ is called the vector space of divergence-free 1-forms on the manifold L with respect to the metric g_L.*

If it is clear from context, we suppress the metric from the notation.

Lemma 6.41. *Let (L, g_L) be a compact Riemannian manifold. Then*

1. *We have the L^2-orthogonal decomposition $\Gamma^\infty(T^*L) = d(\mathcal{C}^\infty(L)) \oplus D(L, g_L)$.*
2. *The Hodge Laplacian $\Delta_H^{g_L}$ acts diagonally with respect to this decomposition.*

Proof.
1. This follows from the Hodge decomposition theorem, by noting that $D(L, g_L) = \delta^{g_L}\Omega^1(L) \oplus \ker \Delta_H^{g_L}$.
2. For $f \in \mathcal{C}^\infty(L)$, we have by Proposition 2.5
$$\Delta_H^{g_L}(df) = d\delta^{g_L}df + \delta^{g_L}ddf = d\delta^{g_L}df = d\Delta_B^{g_L}f,$$
thus $\Delta_H^{g_L}(d\mathcal{C}^\infty(L)) \subset d\mathcal{C}^\infty(L)$. On the other hand, if $\omega \in D(L, g_L)$, we have for any $f \in \mathcal{C}^\infty(L)$ that
$$\begin{aligned}(\delta^{g_L}\Delta_H^{g_L}\omega, f)_{g_L} &= (\Delta_H^{g_L}\omega, df)_{g_L} = (d\delta^{g_L}\omega + \delta^{g_L}d\omega, df)_{g_L} \\ &= (\delta^{g_L}d\omega, df)_{g_L} = (d\omega, ddf)_{g_L} = 0,\end{aligned}$$
thus $\delta^{g_L}\Delta_H^{g_L}\omega \perp \mathcal{C}^\infty(L)$, a dense subset in $L^2(L, g_L)$. This means $\delta^{g_L}\Delta_H^{g_L}\omega = 0$ and thus $\Delta_H^{g_L}(D(L, g_L)) \subset D(L, g_L)$.

□

Formula for a general 1-form Note that any 1-form $\omega \in \Gamma^\infty(T^*M)$ may be written as

$$w_{(r,x)} = \omega_0(r,r)dr + r(\omega_1)_{(r,x)}, \tag{6.6}$$

where $(r,x) \in M$, and $\omega_0(r,x) \in \mathbb{R}$ and $(\omega_1)_{(r,x)} \in T^*_x L$. Moreover, if ω is smooth, so are $\omega_0(r,\cdot)$ and $\omega_1(r,\cdot)$ for any fixed r.

Orthonormal bases for the direct summands Our next goal is to find an orthonormal basis of $L^2(L, g_L)$ consisting of eigenvectors of the Hodge Laplacian $\Delta_H^{g_L}$. By Lemma 6.41, we may reduce this task to finding L^2-orthonormal bases of $L^2(d(\mathcal{C}^\infty(L)))$ and $L^2(D(L, g_L))$.

Lemma 6.42.

- *Let $\{v_i \in \mathcal{C}^\infty(L) \mid i \in \mathbb{N}\}$ be an L^2-orthonormal basis of the Laplace–Beltrami operator $\Delta_B^{g_L}$ (these are automatically smooth by elliptic regularity), and let $\lambda_i \in \mathbb{R}$ be the corresponding eigenvalues., i.e. $\Delta_B^{g_L} v_i = \lambda_i v_i$. Then $\left\{ \frac{dv_i}{\sqrt{\lambda_i}} \,\middle|\, i \in \mathbb{N}^+ \right\}$ is an orthonormal basis for $L^2(d(\mathcal{C}^\infty(L)))$ consisting of eigenvectors of the Hodge Laplacian.*
- *There is an orthonormal basis for $L^2(D(L, g_L))$ consisting of eigenvectors of the Hodge Laplacian.*

Proof.

- The fact that $\frac{dv_i}{\sqrt{\lambda_i}}$ is an eigenvector of $\Delta_H^{g_L}$ of eigenvalue λ_i follows from Proposition 2.5. Pairwise orthogonality can be shown by an easy calculation. (Note that we needed to exclude the index $i = 0$ from this new basis since compactness of L implies that $v_0 = \text{const}$ and thus $dv_0 = 0$.)
- Let $\tilde{\omega} \in \Omega^L(1)$. According to Lemma 6.41, we may write $\tilde{\omega} = d\tilde{f} + \tilde{\eta}_j$ where $\tilde{f} \in \mathcal{C}^\infty(L)$ and $\tilde{\eta} \in D(L, g_L)$. One checks that the L^2-orthogonality of the decomposition in Lemma 6.41 implies that for $\tilde{\omega} = d\tilde{f} + \tilde{\eta}$, where $\tilde{f} \in \mathcal{C}^\infty(L)$ and $\tilde{\eta} \in D(L, g_L)$, we have $\Delta_H^{g_L}\tilde{\omega} = \mu\tilde{\omega}$ if and only if $\Delta_H^{g_L} d\tilde{f} = \mu d\tilde{f}$ and $\Delta_H^{g_L}\tilde{\eta} = \mu\tilde{\eta}$. (Note that this relation does *not* mean that either $d\tilde{f}$ or $\tilde{\eta}$ are eigenvectors of the Hodge Laplacian $\Delta_H^{g_L}$ since it is not excluded that they are zero.)

  Let now $\{\tilde{\omega}_j \mid j \in \mathbb{N}\}$ be an orthonormal basis of $L^2(T^*L, g_L)$ consisting of eigenvalues of $\Delta_H^{g_L}$ and let $\mu_j \in \mathbb{R}$ be the corresponding eigenvalue, i.e. $\Delta_H^{g_L}\tilde{\omega}_j = \mu_j\tilde{\omega}_j$. According to Lemma 6.41, we may write $\tilde{\omega}_j = d\tilde{f}_j + \tilde{\eta}_j$ where $\tilde{f}_j \in \mathcal{C}^\infty(L)$ and $\tilde{\eta}_j \in D(L, g_L)$. Since $\{\tilde{\omega}_j \mid j \in \mathbb{N}\}$ be an orthonormal basis of $L^2(T^*L, g_L)$, $\{\tilde{\eta}_j \mid j \in \mathbb{N}\}$ be an orthogonal generating set of $L^2(D(L, g_L), g_L)$. Since the Hodge Laplacian $\Delta_H^{g_L}$ is elliptic, its eigenspace E_μ corresponding to a given eigenvalue μ is finite dimensional. Therefore, we may chose a linearly independent system of E_μ. Performing Gram–Schmidt orthogonalization yields an orthonormal basis of E_μ. The set of the vectors in $D(L, g_L)$ obtained as described above yields an orthonormal basis for $L^2(D(L, g_L))$, which we denote by $\{\omega_j \mid j \in \mathbb{N}\}$.

□

Lemma 6.42 and the decomposition in Equation (6.6) means that any one-form on the cone may be written as

$$\omega = \sum_{i \in \mathbb{N}} (a_i v_i dr + b_i r dv_i) + \sum_{j \in \mathbb{N}} c_j r \omega_j, \tag{6.7}$$

where $a_i, b_i, c_j \in \mathcal{C}^\infty(\mathbb{R}^+)$ and $b_0 = 0$.

Calculations and the spectrum The next task is to determine the spectrum of the tangent operator to the Hodge Laplacian.

Proposition 6.43. *Let $(M,g) := \mathrm{Cone}(L,g_L)$ be a Ricci-flat cone. Then the spectrum of the tangential operator to the Hodge Laplacian on (M,g) is determined by*

$$\sigma(\square_H) = \left\{\lambda + 1 \pm \sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda}\,\middle|\,\lambda \in \sigma(\Delta_B^{g_L})\right\} \cup \left\{\mu - n + 2 \,\middle|\, \mu \in \sigma(\Delta_H^{g_L}|_{D(L,g_L)})\right\}.$$

The corresponding eigenvectors are

- *$\lambda v dr - m_\pm(\lambda) r dv$ with eigenvalue $\lambda+1\pm\sqrt{\left(\frac{n-1}{2}\right)^2+\lambda}$ of the multiplicity[4] equal to the multiplicity of λ with respect to the Laplace–Beltrami operator $\Delta_B^{g_L}$, where $v \in \mathcal{C}^\infty(L)$ with $\Delta_B^{g_L} v = \lambda v$ for some $\lambda \in \mathbb{R}$,*
- *$r\omega$ with eigenvalue $\mu - n + 2$ where $\omega \in D(L,g_L)$ with $\Delta_H^{g_L}\omega = \mu\omega$ for some $\mu \in \mathbb{R}$ of the same multiplicity as μ with respect to $\Delta_H^{g_L}$.*

Remark 6.44. *It may happen that the eigenvalues corresponding to different values of λ coincide. In this case, we adopt the convention that we count the eigenvalues and the multiplicities separately. This is the convention we will adopt also in Theorem 6.55.*

Since the calculation is not long, we calculate the multiplicities also in the classical sense. Note that we may rewrite the expression for the eigenvalue as

$$\lambda + 1 \pm \sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda} = \left(\sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda} \pm \frac{1}{2}\right)^2 + \frac{3}{4} - \left(\frac{n-1}{2}\right)^2,$$

thus the eigenvalues corresponding to $\lambda, \lambda' \in \sigma(\Delta_B^{g_L})$ coincide if and only if

$$\left(\sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda} \pm \frac{1}{2}\right)^2 = \left(\sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda'} \pm \frac{1}{2}\right)^2,$$

where the signs are a priori independent of each other. Note that since $\sigma(\Delta_B^{g_L}) \leq 0$ and $n \geq 1$, there are nonnegative numbers under the square. Thus coincidence may happen if and only if

$$\sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda} \pm \frac{1}{2} = \sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda'} \pm \frac{1}{2},$$

still with independent signs. If the signs are the same, we get $\lambda = \lambda'$, thus no nontrivial coincidence occurs. If the signs are different, then

$$\sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda} \mp \frac{1}{2} = \sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda'} \pm \frac{1}{2},$$

[4]Cf. Remark 6.44.

(now, of course, with the signs either both the top choice or both the bottom choice) i.e.

$$\sqrt{\left(\frac{n-1}{2}\right)^2+\lambda}=\sqrt{\left(\frac{n-1}{2}\right)^2+\lambda'}\pm 1.$$

Proof. Note that for any function $f \in \mathcal{C}^\infty(\mathbb{R}^+)$ (depending only on r), we have $\Box_H(f\omega) = f\Box_H\omega$, thus we may ignore the coefficient functions for the calculations. Moreover, the assumption on Ricci-flatness means that the Hodge Laplacian and the raw Laplacian coincide. As usual, to ease notation for the calculations, we will denote objects related to g by an bar, and objects related to g_L without any marking, i.e. $\bar{\nabla} := \nabla^g$, $\nabla := \nabla^{g_L}$, $\Delta_B := \Delta_B^{g_L}$.

- Let $\omega \in \Omega^L(1)$ and let let $\{e_i \,|i \in I\}$ be a g_L-orthonormal frame of T_xL at some point $x \in L$. Now we have at x that

$$\begin{aligned}
\Box_H(r\omega) &= -\sum_{i=1}^n(\bar{\nabla}_{e_i}\circ\bar{\nabla}_{e_i}-\bar{\nabla}_{\nabla_{e_i}e_i})(r\omega)\\
&= -\sum_{i=1}^n \bar{\nabla}_{e_i}(r\nabla_{e_i}\omega-\omega(e_i)dt)-\bar{\nabla}_{\nabla_{e_i}e_i})(r\omega)\\
&= -\sum_{i=1}^n(r\nabla_{e_i}\circ\nabla_{e_i}\omega-(\nabla_{e_i}\omega)(e_i)dr-e_i(\omega(e_i))dr\\
&\qquad -\omega(e_i)rg_L(e_i,\cdot)-r\nabla_{\nabla_{e_i}e_i}\omega+\omega(\nabla_{e_i}e_i)dr)\\
&= -\sum_{i=1}^n(r\nabla^2_{e_i,e_i}\omega-2(\nabla_{e_i}\omega)(e_i)dr-r\omega(g_L(e_i,\cdot),e_i)\\
&= r\Delta_{g_L}\omega-2(\delta^{g_L}\omega)dr-r\omega\\
&= r\Delta_H^{g_L}\omega-(n-1)r\omega-2(\delta^{g_L}\omega)dr-r\omega\\
&= r\Delta_H^{g_L}\omega-(n-2)r\omega-2(\delta^{g_L}\omega)dr,
\end{aligned}\tag{6.8}$$

  where we used that the assumption of Ricci-flatness of the cone means that $\mathrm{Ric}^{g_L} = n-1$ and thus $\Delta_H^{g_L}\omega = \Delta_{g_L}\omega + (n-1)\omega$.

- In particular, if $v \in \mathcal{C}^\infty(L)$ with $\Delta_B^{g_L}v = \lambda v$ and $\omega := dv$, then $\Delta_H^{g_L}\omega = \lambda dv$ by Proposition 2.5 and $\delta^{g_L}\omega = \delta^{g_L}dv = \Delta_B^{g_L}v = \lambda v$, thus Equation (6.8) implies

$$\Box_H(rdv) = r\lambda\omega-(n+2)r\omega = (\lambda-n+2)rdv-2\lambda vdr. \tag{6.9}$$

- On the other hand, if $\omega \in D(L, g_L)$, then Equation (6.8) implies

$$\Box_H(r\omega) = r\mu\omega-(n-2)r\omega-0 = (\mu-n+2)r\omega. \tag{6.10}$$

  Equation (6.10) implies already that $\sigma(\Delta_H^{g_L}|_{D(L,g_L)}) \subset \sigma(\Box_H)$.

- Let $v \in \mathcal{C}^\infty(L)$ with $\Delta_B^{g_L}v = \lambda v$ and let $\{e_i \,|i \in I\}$ be a g_L-orthonormal frame of T_xL at some point $x \in L$. Now we have at x that

$$\Box_H(vdr) = -\sum_{i=1}^n(\bar{\nabla}_{e_i}\circ\bar{\nabla}_{e_i}-\bar{\nabla}_{\nabla_{e_i}e_i})(vdr)$$

$$
\begin{aligned}
&= -\sum_{i=1}^{n} \bar{\nabla}_{e_i}((dv(e_i)dr + vrg_L(e_i,\cdot)) - \bar{\nabla}_{\nabla_{e_i}e_i}(vdr) \\
&= -\sum_{i=1}^{n} [e_i(e_i(v))dr + dv(e_i)rg_L(e_i,\cdot) + dv(e_i)rg_L(e_i,\cdot) - vr\bar{\nabla}_{e_i}(g_L(e_i,\cdot))] \\
&\quad - dv(\nabla_{e_i}e_i) - vrg_L(\nabla_{e_i}e_i,\cdot)] \\
&\overset{(\star)}{=} -[-(\Delta_B^{g_L}v)dr + rdv + rdv - \frac{n}{r}vrdr + \sum_{i\in I} vrg_L(\nabla_{e_i}e_i,\cdot) - \sum_{i\in I} vrg_L(\nabla_{e_i}e_i,\cdot)] \\
&= -[(-\Delta_B^{g_L}v - nv)dr + 2rdv] \\
&= (\lambda+n)vdr - 2rdv, \qquad (6.11)
\end{aligned}
$$

where, at step $(\star)$, we used that $\nabla_{e_i}(g_L(e_i,\cdot)) = -\frac{n}{r}dr + g_L(\nabla_{e_i}e_i,\cdot)$.

- Equations (6.11) and (6.9) imply that for $v \in \mathcal{C}^\infty(L)$ with $\Delta_B^{g_L}v = \lambda v$ we have

$$
\begin{aligned}
\Box_H(avdr + brdv) &= a((\lambda+n)vdr - 2rdv) + b((\lambda-n+2)rdv - 2\lambda vdr) \\
&= ((\lambda+n)a - 2\lambda b)vdr + (-2a + (\lambda-n+2)b)rdv.
\end{aligned}
$$

  This formula shows that the space $\mathcal{C}^\infty(\mathbb{R}^+)vdr + \mathcal{C}^\infty(\mathbb{R}^+)vdr$ is invariant under the tangential operator $\Box_H$. In fact, the tangential operator may be represented on this subspace by the matrix

$$
\begin{pmatrix} \lambda+n & -2\lambda \\ -2 & \lambda-n+2 \end{pmatrix}.
$$

  The eigenvalues of this matrix may be computed explicitly:

$$
\lambda + 1 \pm \sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda},
$$

  with corresponding eigenvectors $\lambda vdr - m_\pm(\lambda)rdv$.

This finishes the proof. □

Lemma 6.45. *Let (N, g_N) be a Riemannian manifold and let $\flat_{g_N} : TN \to T^*N, X \mapsto g_N(X,\cdot) := X^{\flat_{g_N}}$ be the musical isomorphism induced by the metric g_N. Then $\flat_{g_N} \circ \Delta_{TM}^{g_N} = \Delta_{T^*M}^{g_N} \circ \flat_{g_N}$.*

Proof. For any vector field $X \in \Gamma^\infty(TN)$ and any tangent vector $Y \in TN$, one has $\nabla_Y^{g_N}(X_{g_N}^\flat) = (\nabla_Y^{g_N}X)_{g_N}^\flat$ since g_N is parallel. Consequently, with a g_N-orthonormal frame $\{e_i \,|i = 1,\ldots,\dim N\}$ at a point $p \in N$, we have at p that

$$
\begin{aligned}
\Delta_{T^*M}^{g_N}(X^{\flat_{g_N}}) &= -\sum_{i=1}^{\dim N} \nabla_{e_i,e_i}^{g_N,2}(X^{\flat_{g_N}}) = -\sum_{i=1}^{\dim N} \nabla_{e_i}^{g_N} \circ \nabla_{e_i}^{g_N}(X^{\flat_{g_N}}) - \nabla_{\nabla_{e_i}^{g_N}e_i}^{g_N}(X^{\flat_{g_N}}) \\
&= \left(-\sum_{i=1}^{\dim N} \nabla_{e_i}^{g_N} \circ \nabla_{e_i}^{g_N}X - \nabla_{\nabla_{e_i}^{g_N}e_i}^{g_N}X\right)^{\flat_{g_N}} = (\nabla_{TM}^{g_N}X)^{\flat_{g_N}}.
\end{aligned}
$$
□

Corollary 6.46. *In the setting of Proposition 6.43, the spectrum of the tangential operator to the raw Laplacian on TM is $\sigma(\Box_{raw}^{TM}) = \sigma(\Box_H)$.* □

6.5.4 The Einstein operator

Decomposition of symmetric 2-tensor fields on the link Recall that the operator

$$\delta^{g_L}\colon \Gamma^\infty\left(S^2T^*L\right) \to \Gamma^\infty\left(T^*L\right), h \mapsto -\sum_{i\in I}(\nabla^{g_L}_{e_i}h)(e_i,\cdot)$$

(where $\{e_i \,|i \in I\}$ is a g_L-orthonormal frame) has formal adjoint

$$\delta^{g_L,\star}\colon \Gamma^\infty\left(T^*L\right) \to \Gamma^\infty\left(S^2T^*L\right), \omega \mapsto -\frac{1}{2}\mathcal{L}_{\omega^{\sharp_{g_L}}} g_L,$$

where $\sharp_{g_L}\colon T^*L \to TL$ denotes the musical isomorphism induced by the metric g_L. Note that $(\delta^{g_L,\star}\omega)(X,Y) = \frac{1}{2}(\nabla^{g_L}_X\omega)(Y) + \frac{1}{2}(\nabla^{g_L}_Y\omega)(X)$ for any 1-form $\omega \in \Gamma^\infty\left(T^*L\right)$ and any tangent vectors $X, Y \in TL$.

Definition 6.47. *The denote the set of transverse and traceless tensors by* $TT(L, g_L) := \left\{h \in \Gamma^\infty\left(S^2T^*L\right) \,|\delta^{g_L}h = 0 \text{ and } \operatorname{Tr}_{g_L} h = 0\right\}$.

Lemma 6.48. *Let (L, g_L) be a compact Riemannian manifold different from the round sphere. Then we have the L^2-orthogonal decomposition*

$$\begin{aligned}\Gamma^\infty\left(S^2T^*L\right) = (\mathcal{C}^\infty(L))g_L \oplus \left\{n\nabla^{g_L,2}v + (\Delta^{g_L}_B v)g_L \,\middle|\, v \in \mathcal{C}^\infty(L)\right\}\\ \oplus\, \delta^{g_L,\star}(\Omega^1(L)) \oplus TT(L, g_L).\end{aligned}$$

Proof. This is the discussion after Lemma 2.2 in [Krö17]. □

The Einstein operator $\Delta^{g_L}_E$ is elliptic, whence its spectrum consists of eigenvalues with finite multiplicity and we obtain an orthonormal eigenbasis $\{r^2t_k \,|k \in \mathbb{N}\}$ on $TT(L, g_L)$ as before.

Formula for a general symmetric 2-tensor field on the cone Note that any symmetic 2-tensor field $h \in \Gamma^\infty\left(S^2T^*M\right)$ on the cone can be written at a point $(r, x) \in M$ as

$$h(r, x) = h_0(r, x)dr \otimes dr + rh_1(r, x) \odot dr + r^2h_2(r, x),$$

where $h_0(r, x) \in \mathbb{R}$, $h_1(r, x) \in T^*_xL$ and $h_2(r, x) \in S^2T^*_xL$. Based on the decomposition and the orthonormal bases before, we may write

$$\begin{aligned}h = \sum_{i\in\mathbb{N}} P_iv_i dr \otimes dr + \sum_{i\in\mathbb{N}^+} Q_i dr \odot rdv_i + \sum_{j\in\mathbb{N}} A_j dr \odot r\omega_j + \sum_{i\in\mathbb{N}} S_ivir^2g_L\\ + \sum_{i\in\mathbb{N}^+} R_i(nr^2\nabla^{g_L,2}v + (\Delta^{g_L}_B v)r^2g_L) + \sum_{j\in\mathbb{N}} B_jr\delta^{g_L,\star}(r\omega_j) + \sum_{k\in\mathbb{N}} F_kr^2t_k,\end{aligned}$$

where

- $\{v_i \,|i \in \mathbb{N}\}$ is an orthonormal eigenbase for $\Delta^{g_L}_B$ with $\Delta^{g_L}_B v_i = \lambda_iv_i$,
- $\{\omega_j \,|j \in \mathbb{N}\}$ is an orthonormal eigenbase for $\Delta^{g_L}_H|_{D(L,g_L)}$ with $\Delta^{g_L}_H\omega_j = \mu_j$,
- $\{t_k \,|k \in \mathbb{N}\}$ is an orthonormal eigenbase for $\Delta^{g_L}_E|_{TT(L,g_L)}$ with $\Delta^{g_L}_E t_k = \nu_k$,

and $P_i, Q_i, R_i, S_i, A_j, B_j, F_k \in \mathcal{C}^\infty(\mathbb{R}^+)$. It will turn out to be advantageous to regroup the expression as follows

$$\begin{aligned} h = &\sum_{i\in\mathbb{N}}(P_i v_i dr\otimes dr + Q_i dr\odot r dv_i + R_i(nr^2\nabla^{g_L,2}v + (\Delta_B^{g_L}v)r^2 g_L) + S_i vir^2 g_L) \\ &+ \sum_{j\in\mathbb{N}}(A_j dr\odot r\omega_j + B_j r\delta^{g_L,\star}(r\omega_j)) \\ &+ \sum_{k\in\mathbb{N}} F_k r^2 t_k, \end{aligned} \tag{6.12}$$

(with $Q_0 = R_0 = 0$) since, as we will see, the lines in this regrouping represent invariant subspaces.

Calculations We can calculate the tangential operator's value at the different types of tensors separately. We start with vertical (purely link-related) tensor fields.

Lemma 6.49. *For $t \in \Gamma^\infty(S^2T^*L)$, we have $\Box_E(r^2t) = r^2\Delta_E^{g_L}t - 2dr\odot r(\delta^{g_L}t) - 2(\mathrm{Tr}_{g_L} t)dr\otimes dr + 2(\mathrm{Tr}_{g_L} t)r^2 g_L$*

Proof. In the following, $t \in \Gamma^\infty(S^2T^*L)$, moreover, $X, Y \in TM$ and $U, V, W \in TL$.

- We have $r^2\mathring{\bar{R}}t = \mathring{R}t + t - (\mathrm{Tr}_{g_L} t)g_L$ since

$$\begin{aligned} (\mathring{\bar{R}}t)(\partial_r,\partial_r) &= \sum_{i=0}^n t(\bar{R}(E_i,\partial)\partial, E_i) = 0, \\ (\mathring{\bar{R}}t)(\partial_r, V) &= \sum_{i=0}^n t(\bar{R}(E_i,\partial_r)V, E_i) = 0, \\ (\mathring{\bar{R}}t)(V,W) &= \sum_{i=0}^n t(\bar{R}(E_i,V)W,E_i) = t(\bar{R}(\partial_r,V)W,\partial_r) + \frac{1}{r^2}\sum_{i=1}^n t(\bar{R}(e_i,V)W,e_i) \\ &= \frac{1}{r^2}\sum_{i=1}^t (R(e_i,V)W + g_L(e_i,W)V - g_L(V,W)e_i, e_i) \\ &= \frac{1}{r^2}((\mathring{R}t))(V,W) + t(V,W) - (\mathrm{Tr}_{g_L} t g_L(V,W)). \end{aligned}$$

- We have $\bar{\nabla}_V(r^2t) = r^2\nabla_V t - dr\odot rt(V,\cdot)$ since

$$\begin{aligned} \bar{\nabla}_V(r^2t)(\partial_r,\partial_r) &= V(r^2t(\partial_r,\partial_r)) - 2r^2t(\nabla_V\bar{\partial}_r,\partial_r) = 0, \\ \bar{\nabla}_V(r^2t)(\partial_r,W) &= V(r^2t(\partial_r,W)) - r^2t(\bar{\nabla}_V\partial_r,W) - r^2t(\partial_r,\bar{\nabla}_V W) \\ &= -rt(V,W), \\ \bar{\nabla}_V(r^2t)(W,U) &= V(r^2t(W,U)) - r^2t(\bar{\nabla}_V W,U) - r^2t(W,\bar{\nabla}_V U) \\ &= V(r^2t(W,U)) - r^2t(\nabla_V W,U) - r^2t(W,\nabla_V U) \\ &= r^2(\nabla_V t)(W,U). \end{aligned}$$

- Based on this, we calculate for the second, iterated derivative

$$\begin{aligned}
\bar\nabla_V(\bar\nabla_W(r^2t)) &= \bar\nabla_V(r^2\nabla_W t - dr\odot rt(W,\cdot))\\
&= r^2\nabla_V(\nabla_W t) - dr\odot r(\nabla_W t)(V,\cdot) - rg_L(V,\cdot)\odot rt(W,\cdot) - dr\odot r\bar\nabla_V(t(W,\cdot))\\
&\overset{(\star)}{=} r^2\nabla_V(\nabla_W t) - dr\odot r(\nabla_W t)(V,\cdot) - rg_L(V,\cdot)\odot rt(W,\cdot)\\
&\quad - dr\odot rt(\nabla_V W,\cdot) + t(V,W)dr\odot dr\\
&= r^2\nabla_V(\nabla_W t) - dr\odot r(\nabla_W t)(V,\cdot) - dr\odot r(\nabla_V t)(W,\cdot)\\
&\quad - dr\odot rt(\nabla_V W,\cdot) + 2t(V,W)dr\odot dr - rg_L(V,\cdot)\odot rt(W,\cdot),
\end{aligned}$$

where, in the marked step, we used that

$$\begin{aligned}
\bar\nabla_V(t(W,\cdot))(X) &= V(t(W,X)) - t(W,\bar\nabla_V X)\\
&= (\nabla_V t)(W,X) + t(\nabla_V W,X) + t(W,\nabla_V X)\\
&\quad - t(W,\nabla_V X - rt(V,X)\partial_r + \frac{1}{r}dr(X)V)\\
&= ((\nabla_V t)(W,\cdot) + t(\nabla_V W,\cdot) - \frac{1}{r}t(W,V)dr)(X).
\end{aligned}$$

- Now we may calculate the value of the tangential operator to the raw Laplacian on r^2t as follows.

$$\begin{aligned}
\square_{\mathrm{raw}}(r^2t) &= -\sum_{i=1}^n(\bar\nabla_{e_i}\circ\bar\nabla_{e_i} - \bar\nabla_{\nabla_{e_i}e_i})(r^2t)\\
&= -\sum_{i=1}^n r^2\nabla_{e_i}(\nabla_{e_i}t) - dr\odot r(\nabla_{e_i}t)(e_i,\cdot) - dr\odot r(\nabla_{e_i}t)(e_i,\cdot)\\
&\qquad - dr\odot rt(\nabla_{e_i}e_i,\cdot) + 2t(e_i,e_i)dr\odot dr - rg_L(e_i,\cdot)\odot rt(e_i,\cdot)\\
&\qquad - r^2\nabla_{\nabla_{e_i}e_i}Vt + dr\odot rt(\nabla_{e_i}e_i,\cdot)\\
&= -\sum_{i=1}^n r^2\nabla^2_{e_i,e_i}t - 2dr\odot r(\nabla_{e_i}t)(e_i,\cdot) + 2t(e_i,e_i)dr\odot dr\\
&\qquad - rg_L(e_i,\cdot)\odot rt(e_i,\cdot)\\
&= r^2\Delta_{g_L}t - 2dr\odot r(\delta^{g_L}t) - 2(\mathrm{Tr}_{g_L}t)dr\odot dr - 2r^2t
\end{aligned}$$

- Since $\square_E = \square_{\mathrm{raw}} - 2r^2\overset{\circ}{\bar R}$, the claim follows. □

In Equation (6.12), the corresponding part looks like

$$S_iv_ir^2g_L + R_i(nr^2\nabla^2v_i + (\Delta_B^{g_L}v))r^2g_L) + r\delta^{g_L}A_jr\omega_j + r^2F_kt_k.$$

Based on Lemma 6.49, we obtain for these tensor fields the following.

Corollary 6.50. *We have the following.*

- *If $v\in\mathcal{C}^\infty(L)$ with $\Delta_B^{g_L}v = \lambda v$ for some $\lambda\in\mathbb{R}$, then $\square_E(vr^2g_L) = -2nvdr\otimes dr + 2dt\odot rdv + (\lambda+2)vr^2g_L$.*

- *If $v \in \mathcal{C}^\infty(L)$ with $\Delta_B^{g_L} v = \lambda v$ for some $\lambda \in \mathbb{R}$, then $\square_E(nr^2\nabla^{g_L,2}v + (\Delta_B^{g_L} v)r^2 g_L) = 2(n-\lambda)(n-1)dr \odot rdv + (\lambda - 2n + 2)r^2(n\nabla^{g_L,2}v + (\Delta_B^{g_L} v)g_L)$.*
- *If $\omega \in D(L, g_L)$ with $\Delta_H^{g_L}\omega = \mu\omega$ for some $\mu \in \mathbb{R}$, then $\square_E(r\delta^{g_L} r\omega) = (-\mu + 2n + 2)dr \odot r\omega + (\mu - 2n + 2)r^2\delta^{g_L}\omega$.*
- *If $t \in TT(L, g_L)$ with $\Delta_E^{g_L} t = \nu t$ for some $\nu \in \mathbb{R}$, then $\square_E(r^2 t) = \nu r^2 t$.*

Proof.

- We calculate

$$\begin{aligned}\square_E(vr^2 g_L) &= r^2\Delta_E^{g_L}(vg_L) - 2dr \odot r\delta^{g_L}(vg_L) - 2(\mathrm{Tr}_{g_L}(vg_L))dr \odot dr + 2(\mathrm{Tr}_{g_L}(vg_L))g_L \\ &\overset{(\star)}{=} r^2(\Delta_{g_L} v - 2(n-1)v)g_L + 2dr \odot rdv - 2nvdr \odot dr + 2nvg_L \\ &= -2nvdr \odot dr + 2dr \odot rdv + (\lambda + 2)vr^2 g_L\end{aligned}$$

where, in the marked step, we used that

$$\delta^{g_L} vg_L = -\sum_{i=1}^n \nabla_{e_i}(vg_L)(e_i, \cdot) = \sum_{i=1}^n -dv(e_i)g_L(e_i, \cdot) + v(\nabla_{e_i} g_L)(e_i, \cdot) = -dv.$$

- We calculate

$$\begin{aligned}\square_E(r^2(n\nabla^2 v + (\Delta_B^{g_L} v)g_L)) &= r^2\Delta_E^{g_L}(n\nabla^2 v + (\Delta_B^{g_L} v)g_L) \\ &\quad - 2dr \odot r\delta^{g_L}(n\nabla^2 v + (\Delta_B^{g_L} v)g_L) \\ &\quad - 2\,\mathrm{Tr}_{g_L}(n\nabla^2 v + (\Delta_B^{g_L} v)g_L)(dr \otimes dr - r^2 g_L) \\ &\overset{(\star)}{=} r^2(n\nabla^2(\Delta_B^{g_L} v - 2(n-1)v) \\ &\quad + ((\Delta_B^{g_L})^2 v - 2(n-1)\Delta_B^{g_L} v)r^2 g_L \\ &\quad + 2(n-\lambda)(n-1)dr \odot rdv \\ &= 2(n-\lambda)(n-1)dr \odot rdv \\ &\quad + (\lambda - 2n + 2)r^2(n\nabla^2 v - (\Delta_B^{g_L} v)g_L),\end{aligned}$$

where, in the marked step, we used that

$$\begin{aligned}\delta^{g_L}(n\nabla^2 v + (\Delta_B^{g_L} v)g_L) &= -\sum_{i=1}^n \nabla_{e_i}(n\nabla^2 v + (\Delta_B^{g_L} v)g_L)(e_i, \cdot) \\ &= -\sum_{i=1}^n n\nabla^3_{e_i,e_i,\cdot} v + \nabla e_i(\Delta_B^{g_L} v)g_L(e_i, \cdot) \\ &= n\Delta_{g_L}(dv) - d(\Delta_B^{g_L} v) \\ &= n\Delta_H^{g_L}(dv) - n(n-1)dv - \lambda dv \\ &= -(n-\lambda)(n-1)dv.\end{aligned}$$

- We calculate

$$\square_E(r^2\delta^{g_L,\star}\omega) = r^2\Delta_E^{g_L}\delta^{g_L,\star}\omega - 2dr \odot r(\delta^{g_L}\delta^{g_L,\star}\omega) - 2(\mathrm{Tr}_{g_L}\delta^{g_L,\star}\omega)dr \odot dr$$

$$\overset{(\star)}{=} r^2\delta^{g_L,\star}(\Delta_H^{g_L}\omega - 2(n-1)\omega) - 2dr \odot r(\frac{1}{2}\Delta_H^{g_L}\omega + d\delta^{g_L}\omega - \mathrm{Ric}^{g_L}\omega)$$
$$+ 2(\delta^{g_L}\omega)dr \odot dr - 2(\delta^{g_L}\omega)r^2 g_L$$
$$= (-\mu + 2n - 2)dr \odot r\omega + (\mu - 2n + 2)r^2\delta^{g_L,\star}\omega,$$

where, in the marked step, we used Lemma 2.4 and Lemma 2.2.

- We calculate

$$\Box_E(r^2 t) = r^2\Delta_E^{g_L}t - 2dr \odot r(\delta^{g_L}t) - 2(\mathrm{Tr}_{g_L} t)dr \odot dr + 2(\mathrm{Tr}_{g_L} t)r^2 g_L$$
$$= r^2\Delta_E^{g_L}t = \nu r^2 t.$$ □

Next, we turn our attention to mixed tensor fields, i.e. tensor fields of the form $dr \odot r\omega$ where $\omega \in \Omega^1(L)$.

Lemma 6.51. *If $\omega \in \Omega^1(L)$, then $\Box_E(dr \odot r\omega) = -2(\delta^{g_L}\omega)dr \otimes dr + dr \odot r\Delta_H^{g_L}\omega + 4dr \odot r\omega + 4r^2\delta^{g_L,\star}\omega$.*

Proof. In the following, $v \in \mathcal{C}^\infty(\mathbb{R}^+)$, moreover $X, Y \in TM$ and $U, V, W, Z \in TL$, and e_i, E_i are elements of an orthonormal frame, as usual.

- $\overset{\circ}{\bar{R}}(dr \odot r\omega) = 0$ since

$$\overset{\circ}{\bar{R}}(dr \odot r\omega)(\partial_r, \partial_r) = \sum_{i=0}^n (dr \odot r\omega)(\bar{R}(E_i, \partial_r)\partial_r), E_i) = 0$$
$$\overset{\circ}{\bar{R}}(dr \odot r\omega)(\partial_r, V) = \sum_{i=0}^n (dr \odot r\omega)(\bar{R}(E_i, \partial_r)V), E_i) = 0$$
$$\overset{\circ}{\bar{R}}(dr \odot r\omega)(V, W) = \sum_{i=0}^n (dr \odot r\omega)(\bar{R}(E_i, V)W), E_i)$$
$$= (dr \odot r\omega)(\bar{r}(\partial_r, V)W, \partial_r) + \frac{1}{r^2}\sum_{i=1}^n (dr \odot r\omega)(\bar{R}(e_i, V)W, e_i) = 0,$$

where we used that $\bar{R}(\partial_r, \cdot) = 0$ and in general $\bar{R}(X, Y) \in TL$.

- We calculate

$$\bar{\nabla}_V(dr \odot r\omega) = rg_L(V, \cdot) \odot r\omega + dr \odot \bar{\nabla}_V(r\omega)$$
$$= rg_L(V, \cdot) \odot r\omega + dr \odot r\nabla_V\omega - dr \odot \omega(V)dr$$
$$= dr \odot r\nabla_V\omega + rg_L(V, \cdot) \odot r\omega - 2\omega(V)dr \otimes dr.$$

- For the iterated second derivative, we obtain

$$\bar{\nabla}_W(\bar{\nabla}_V(dr \odot r\omega)) = -2(\bar{\nabla}_W\omega)(V)dr \otimes dr - 2\omega(\bar{\nabla}_W V) - \omega(V)dr \odot rg_L(W, \cdot)$$
$$- 2(\bar{\nabla}_V\omega)(W)dr \otimes dr + dr \odot r\nabla_W\nabla_V\omega + rg_L(W, \cdot) \odot r\nabla_V\omega$$
$$+ r\bar{\nabla}_W(g_L(V, \cdot)) \odot r\omega + rg_L(V, \cdot) \odot r\bar{\nabla}_W\omega$$

$$
\begin{aligned}
&= -4\tfrac{1}{2}[(\nabla_W\omega)(V) + (\nabla_V\omega)(W)]dr \otimes dr - 2\omega(\nabla_W V)dr \otimes dr \\
&\quad - 2\omega(V)dr \odot rg_L(W,\cdot) + dr \odot r\nabla_W\nabla_V\omega + rg_L(W,\cdot) \odot r\nabla_V\omega \\
&\quad + rg_L(\nabla_W V,\cdot) \odot r\omega - g_L(V,W)dr \odot r\omega \\
&\quad + rg_L(V,\cdot)dr \otimes dr + \omega(V)dr \odot rg_L(W,\cdot) \\
&= -4(\delta^{g_L,\star}\omega)(W,V)dr \otimes dr + dr \odot r\nabla_W\nabla_V\omega \\
&\quad - 2\omega(\nabla_W V)dr \otimes dr - 2\omega(V)dr \odot rg_L(W,\cdot) \\
&\quad + rg_L(W,\cdot) \odot r\nabla_V\omega + rg_L(\nabla_W V,\cdot) \odot r\omega \\
&\quad - g_L(V,W)dr \odot r\omega + rg_L(V,\cdot) \odot r\nabla_W\omega - \omega(W)rg_L(V,\cdot) \odot dr,
\end{aligned}
$$

where we used that $\bar{\nabla}_V(t(W,\cdot)) = (\nabla_V t)(W,\cdot) + t(\nabla_V W,\cdot) - \frac{1}{r}t(W,V)dr$ for $t \in \Gamma^\infty\left(S^2T^*L\right)$.

- Now for the tangential operator to the raw Laplacian, we calculate

$$
\begin{aligned}
\square_{\mathrm{raw}}(dr \odot r\omega) &= -\sum_{i=1}^n (\bar{\nabla}_{e_i} \circ \bar{\nabla}_{e_i} - \bar{\nabla}_{\nabla_{e_i}e_i})(dr \odot r\omega) \\
&= -\sum_{i=1}^n -4(\delta^{g_L,\star}\omega)(e_i,e_i)dr \otimes dr + dr \odot r\nabla_{e_i}\nabla_{e_i}\omega \\
&\qquad - 2\omega(\nabla_{e_i}e_i)dr \otimes dr - 2\omega(e_i)dr \odot rg_L(e_i,\cdot) \\
&\qquad + rg_L(e_i,\cdot) \odot r\nabla_{e_i}\omega + rg_L(\nabla_{e_i}e_i,\cdot) \odot r\omega \\
&\qquad - g_L(e_i,e_i)dr \odot r\omega + rg_L(e_i,\cdot) \odot r\nabla_{e_i}\omega - \omega(e_i)rg_L(e_i,\cdot) \odot dr \\
&\qquad - dr \odot r\nabla_{\nabla_{e_i}e_i}\omega - rg_L(\nabla_{e_i}e_i,\cdot) \odot r\omega + 2\omega(\nabla_{e_i}e_i)dr \otimes dr \\
&= -4\delta^{g_L}\omega dr \otimes dr + dr \odot r\Delta_{g_L}\omega + 2dr \odot r\omega \\
&\quad - 4r^2\delta^{g_L,\star}\omega + ndr \odot r\omega + dr \odot r\omega \\
&= -4(\delta^{g_L}\omega)dr \otimes dr + dr \odot r\Delta_H^{g_L}\omega + 4dr \odot r\omega - 4r^2\delta^{g_L,\star}\omega.
\end{aligned}
$$

- Since $\square_E = \square_{\mathrm{raw}} - 2r^2\mathring{\bar{R}}$, we the claim follows. ☐

Based on this lemma, we can calculate the tangential operator on the following tensors.

Corollary 6.52. *We have the following.*

- *If $v \in \mathcal{C}^\infty(L)$ with $\Delta_B^{g_L}v = \lambda v$ for some $\lambda \in \mathbb{R}$, then $\square_E(dr \odot rdv) = -4\lambda vdr \otimes dr + (\lambda+4)dr \odot rdv - \frac{4}{n}(nr^2\nabla^{g_L,2}v + r^2(\Delta_B^{g_L}v)g_L) + \frac{4}{n}\lambda vr^2g_L$.*
- *If $\omega \in D(L,g_L)$ with $\Delta_H^{g_L}\omega = \mu\omega$ for some $\mu \in \mathbb{R}$, then $\square_E(dr \odot r\omega) = (\mu+4)dr \odot r\omega - 4r^2\delta^{g_L,\star}\omega$.*

Proof. We calculate the two claims separately.

- We calculate

$$
\begin{aligned}
\square_E(dr \odot rdv) &= -4(\delta^{g_L}dv)dr \otimes dr + dr \odot r\Delta_H^{g_L}dv + 4dr \odot rdv - 4r^2\delta^{g_L,\star}dv \\
&= -4\Delta_B^{g_L}vdr \otimes dr + \lambda dr \odot rdv + 4dr \odot rdv - 4r^2\nabla^{g_L,2}v
\end{aligned}
$$

$$= -4\lambda v dr \otimes dr + (\lambda+4) dr \odot r dv$$
$$- \tfrac{4}{n}(nr^2 \nabla^{g_L,2} v + r^2(\Delta_B^{g_L} v) g_L) + \tfrac{4}{n}\lambda v r^2 g_L,$$

where we used that for any $X, Y \in TM$, we have

$$\begin{aligned}(\delta^{g_L,\star} dv)(X,Y) &= \frac{1}{2}\nabla_X^{g_L}(dv)(Y) + \frac{1}{2}\nabla_Y^{g_L}(dv)(X) \\ &= \frac{1}{2}\nabla_{X,Y}^{g_L,2} v + \frac{1}{2}\nabla_{Y,X}^{g_L,2} v = \nabla_{X,Y}^{g_L,2} v.\end{aligned}$$

- For the second claim, we calculate

$$\begin{aligned}\Box_E(dr \odot r\omega) &= -4(\delta^{g_L}\omega) dr \otimes dr + dr \odot r \Delta_H^{g_L}\omega + 4 dr \otimes r\omega - 4r^2 \delta^{g_L,\star}\omega \\ &= (\mu+4) dr \odot r\omega - 4r^2\delta^{g_L,\star}\omega.\end{aligned}$$ ☐

Now only tensor fields of the form $v dr \otimes dr$, where $v \in \mathcal{C}^\infty(\mathbb{R}^+)$, are left.

Lemma 6.53. *For $v \in \mathcal{C}^\infty(\mathbb{R}^+)$, we have $\Box_E(v dr \otimes dr) = (\Delta_B^{g_L} v + 2n) dr \otimes dr - 2dr \odot r dv - 2vr^2 g_L$.*

Proof. In the following, $v \in \mathcal{C}^\infty(\mathbb{R}^+)$, moreover $X, Y \in TM$ and $U, V, W, Z \in TL$, and e_i, E_i are elements of an orthonormal frame, as usual.

- $\mathring{\bar{R}} v dr \otimes dr(X,Y) = v \sum_{i\in I} dr \otimes dr(\bar{R}(E_i, X)Y, E_i) = 0$ since $\bar{R}(E_i, X, Y) \in TL$.
- Since $\bar\nabla dr = r g_L$, we calculate $\bar\nabla_W(v dr \otimes dr) = dv(W) dr \otimes dr + v(\bar\nabla_W dr) \odot dr = dv(W) dr \otimes dr + v r g_L(W,\cdot) \odot r g_L(W,\cdot)$.
- For the iterated second derivative, we calculate

$$\begin{aligned}\bar\nabla_V(\bar\nabla_W(v dr \otimes dr)) &= \bar\nabla_V(dv(W) dr \otimes dr + v dr \odot r g_L(W,\cdot)) \\ &= V(dv(W)) dr \otimes dr + dv(W) dr \odot r g_L(V,\cdot) + dv(V) dr \odot r g_L(W,\cdot) \\ &\quad + v r g_L(V,\cdot) \odot r g_L(W,\cdot) + v dr \odot r t(\nabla_V W, \cdot) - 2t(V,W) v dr \otimes dr,\end{aligned}$$

  where we used that $\bar\nabla_V(t(W,\cdot)) = (\nabla_V t)(W,\cdot) + t(\nabla_V W,\cdot) - \frac{1}{r} t(W,V) dr$ for $t \in \Gamma^\infty(S^2T^*L)$.
- Consequently, we obtain

$$\begin{aligned}\Box_{\mathrm{raw}}(vr \otimes dr) &= -\sum_{i=1}^n (\bar\nabla_{e_i} \circ \bar\nabla_{e_i} - \bar\nabla_{\nabla_{e_i} e_i})(v dr \otimes dr) \\ &= -\sum_{i=1}^n e_i(dv(e_i)) dr \otimes dr + dv(e_i) dr \odot r g_L(e_i,\cdot) \\ &\qquad + dv(e_i) dr \odot r g_L(e_i,\cdot) + v r g_L(e_i,\cdot) \odot r g_L(e_i,\cdot) \\ &\qquad + v dr \odot r t(\nabla_{e_i} e_i, \cdot) - 2t(e_i, e_i) v dr \otimes dr \\ &\qquad - dv(\nabla_{e_i} e_i) dr \otimes dr - v r g_L(\nabla_{e_i} e_i, \cdot) \odot r g_L(\nabla_{e_i} e_i, \cdot) \\ &= (\Delta_B^{g_L} v) dr \otimes dr - 2dr \odot r dv - 2vr^2 g_L + 2nv dr \otimes dr \\ &= (\Delta_B^{g_L} v + 2n) dr \otimes dr - 2dr \odot r dv - 2vr^2 g_L.\end{aligned}$$

- Since $\Box_E = \Box_{\mathrm{raw}} - 2r^2\mathring{\bar{R}}$, the claim follows. $\Box$

In the decomposition Equation (6.12), the only term of this form if $P_i v_i dr \otimes dr$.

Corollary 6.54. *If $v \in \mathcal{C}^\infty(L)$ with $\Delta_B^{g_L} v = \lambda v$ for some $\lambda \in \mathbb{R}$, then $\Box_E(vdr \otimes dr) = (\lambda + 2n)dr \otimes dr - 2dr \odot rdv - 2vr^2 g_L$.*

Proof. $\Box_E(vdr \otimes dr) = (\Delta_B^{g_L} v + 2n)dr \otimes dr - 2dr \odot rdv - 2vr^2 g_L = (\lambda + 2n)dr \otimes dr - 2dr \odot rdv - 2vr^2 g_L$. $\Box$

Representing the tangential operator as matrices We can conclude that the following subspaces are invariant under $\Box_E$.

- $\mathcal{C}^\infty(\mathbb{R}^+)vdr \otimes dr + \mathcal{C}^\infty(\mathbb{R}^+)dr \odot rdv + \mathcal{C}^\infty(\mathbb{R}^+)(nr^2\nabla^{g_L,2}v + (\Delta_B^{g_L}v)r^2 g_L) + \mathcal{C}^\infty(\mathbb{R}^+)vr^2 g_L$ for $v \in \mathcal{C}^\infty(L)$ with $\Delta_B^{g_L} v = \lambda v$ for some $\lambda \in \mathbb{R}$.

  In fact, if we take an element (cf. Equation (6.12))

$$h = (Pvdr \otimes dr + Qdr \odot rdv + R(nr^2\nabla^{g_L,2}v + (\Delta_B^{g_L}v)r^2 g_L) + Svr^2 g_L,$$

  then

$$\begin{aligned}\Box_E h = \ & ((\lambda + 2n)P - 4\lambda Q - 2nS)vdr \otimes dr \\ & + (-2P(\lambda + 4)Q + 2(n-1)(n-\lambda)R + 2S))dr \odot rdv \\ & + (-\tfrac{4}{n}Q + (\lambda - 2n + 2)R)vr^2 g_L \\ & + (-2P + \tfrac{4\lambda}{n}Q + (\lambda + 2)S)(nr^2\nabla^{g_L,2}v + (\Delta_B^{g_L}v)r^2 g_L).\end{aligned}$$

  If $\lambda > 0$, then $v \neq \text{const}$, and the eigenvalue problem of the tangential operator $\Box_E$ may be written as the eigenvalue problem for a 4-by-4 matrix:

$$\begin{pmatrix} \lambda + 2n & -4\lambda & 0 & -2n \\ -2 & \lambda + 4 & 2(n-1)(n-\lambda) & 2 \\ 0 & -\frac{4}{n} & \lambda - 2n + 2 & 0 \\ -2 & \frac{4\lambda}{n} & 0 & \lambda + 2 \end{pmatrix},$$

  the eigenvalues of which may be computed explicitly (preferably with a computer algebra system). These turn out to be λ (with multiplicity 2), and $4m_\pm(\lambda) + \lambda + 2n + 2$ with multiplicity 1.

  If, however, $\lambda = 0$, then $v = \text{const}$ (by virtue of L being compact and without boundary), thus $dv = 0$ and $\nabla^{g_L,2}v = 0$, and the only surviving terms are $Pvdr \otimes dr + Svr^2 g_L$. Consequently, the previous matrix "collapses" to a 2-by-2 matrix

$$\begin{pmatrix} 2n & -2n \\ -2 & 2 \end{pmatrix},$$

  the eigenvalues[5] of which are $2(n+1)$ and 0, both of multiplicity 1.

[5] Note that $2(n+1) = 4m_+(\lambda) + \lambda + 2n + 2$ for $\lambda = 0$ but the corresponding "negative" branch is missing from the eigenvalues, when compared to the $\lambda > 0$ case.

- $\mathcal{C}^\infty(\mathbb{R}^+)r\omega \odot dr + \mathcal{C}^\infty(\mathbb{R}^+)r^2\delta^{g_L,\star}\omega$ for some nonzero $\omega \in D(L, g_L)$ with $\Delta_H^{g_L}\omega = \mu\omega$ for some $\mu \in \mathbb{R}$.

  In fact, for $A, B \in \mathcal{C}^\infty(\mathbb{R}^+)$, we have

  $\square_E(Ar\omega\odot dr+Br^2\delta^{g_L,\star}\omega) = ((\mu+4)A+(-\mu+2n-1)M)r\omega\odot dr+(-4A+(\mu-2n+2)B)r^2\delta^{g_L,\star}\omega.$

  This can be written as a matrix

  $$\begin{pmatrix} \mu+4 & -\mu+2n-2 \\ -4 & \mu-2n+2 \end{pmatrix},$$

  the eigenvalues of which are $2m_\pm(\mu+2-n)+\mu+2$ (each with multiplicity 1).

- $\mathcal{C}^\infty(\mathbb{R}^+)r^2t$ for some nonzero $t \in TT(L, g_L)$ with $\Delta_E^{g_L}t = \nu t$ for some $\nu \in \mathbb{R}$.

  In fact, if $F \in \mathcal{C}^\infty(\mathbb{R}^+)$, then $\square_E(Fr^2t) = \nu Fr^2t$.

The spectrum Now it is time to reap the rewards of our work.

Theorem 6.55. *Let $(M, g) = \operatorname{Cone}(L, g_L)$ be a Ricci-flat cone. Then the spectrum of the tangential operator to the Einstein operator is the following.*

$$\begin{aligned}\sigma(\square_E) = \sigma(\Delta_B^{g_L}) \\ &\cup \{4m_\pm(\lambda)+\lambda+2n+2 \,|\lambda \in \sigma(\Delta_B^{g_L}), \lambda > 0\} \\ &\cup \left\{2m_\pm(\mu+2-n)+\mu+2 \,\middle|\mu \in \sigma(\Delta_H^{g_L}|_{D(L,g_L)})\right\} \\ &\cup \sigma(\Delta_E^{g_L}|_{TT(L,g_L)}) \\ &\cup \{2n+2\}.\end{aligned}$$

The corresponding eigenvectors are

- $vdr\otimes dr + vr^2g_L$ *for* $\lambda \in \sigma(\Delta_B^{g_L})$ *with* $\Delta_Bv = \lambda v$,
- $(n-1)\lambda vdr\otimes dr + n(n-1)dr\odot rdv - 2(nr^2\nabla^{g_L,2}v + (\Delta_B^{g_L}v)g_L) - \lambda(n-1)vr^2g_L$ *for* $\lambda \in \sigma(\Delta_B^{g_L})$ *where* $\Delta_Bv = \lambda v$, *with* $\lambda > 0$,
- $-n(n-\lambda)\lambda vdr\otimes dr+n(n-\lambda)mn_\pm(\lambda)dr\odot vdr+(\lambda-nm_\pm(\lambda))(nr^2\nabla^{g_L,2}v+(\Delta_B^{g_L}v)g_L)+(n-\lambda)\lambda vr^2g_L$ *for* $4m_\pm(\lambda)+\lambda+2n+2$ *with* $\Delta_Bv = \lambda v$, $\lambda \in \sigma(\Delta_B^{g_L})$ *with* $\lambda > 0$,
- $-ndr\otimes dr + r^2g_L$ *for* $2n+2$,
- $(-\frac{\mu}{2}+n-1)dr\odot r\omega + (m_mp(\mu+2-n)-1)r^2\delta^{g_L,\star}\omega$ *for* $2m_\pm(\mu+2-n)+\mu+2$ *and* $\Delta_H^{g_L}\omega = \mu\omega$ *with* $\omega \in D(L, g_L)$,
- r^2t *for* $\nu \in \sigma(\Delta_E^{g_L}|_{TT(L,g_L)})$ *with* $\Delta_E^{g_L}t = \nu t$.

and the multiplicities are inherited from the corresponding eigenvalues of the Laplacians – except when $\lambda > 0$, when the eigenvectors of the first two type have the same eigenvalue (namely λ itself) and thus the multiplicity of λ with respect to the tangential operator $\square_E$ is twice the multiplicity of λ with respect to the Laplace–Beltrami operator $\Delta_B^{g_L}$.

Proof. This follows directly from the discussion above. Note that $0 \in \sigma(\Delta_B^{g_L})$, so it is not necessary to include 0 explicitly. Note also that $2n+2 = 4m_+(\lambda)+\lambda+2n+2$ for $\lambda = 0$. $\square$

Remark 6.56. *It may happen that the eigenvalues of the tangential operator coming from different eigenvalues of the given Laplacians coincide. In this case, we adopt the convention that the eigenvalues and their multiplicities are counted as in the theorem, cf. Remark 6.44*

Remark 6.57. *Note that the spectrum of the tangential operator $\square_E$ consists of eigenvalues and that these eigenvalues converge to infinity. The latter statement can be proved by an easy calculation based on the growth analysis of the function $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto x \pm A\sqrt{x+a^2}$ (with $A, a > 0$) and by noting that the spectra of the different Laplacians on the link are bounded from below.*

Remark 6.58. *Note that for $v \in \mathcal{C}^\infty(L)$ with $\Delta_B^{g_L} v = \lambda v$ for some $\lambda \in \mathbb{R}$, we have $\square_E(v g_{\mathrm{cone}}) = \lambda v g_{\mathrm{cone}}$.*

Chapter 7

Asymptotically conical (AC) manifolds

7.1 Definition and elementary properties

Definition 7.1. *A complete Riemannian manifold* (M, g_{ac}) *is called an asymptotically conical Riemannian manifold*[1] *(short: AC manifold) if there are*

- *a compact set* $K \subset M$,
- *a Riemannian cone* $\mathrm{Cone}(L, g_L)$ *with* L *connected,*
- *a positive number* $R > 0$,
- *a positive number* $\tau > 0$, *and*
- *a diffeomorphism* $\phi\colon M \setminus K \to \mathrm{Cone}(L, g_L) \setminus ((0, R_i] \times L_i)$

such that

$$|\nabla^{g_{\mathrm{cone}},k}(\phi_* g_{\mathrm{ac}} - g_{\mathrm{cone}})|_{g_{\mathrm{cone}}} = O\left(r^{-\tau-k}\right) \text{ for all } k \in \mathbb{N} \tag{7.1}$$

We call the set K *the core or nuculus, the set* $M \setminus K$ *the end, the number* τ *decay rate, the map* ϕ_i *the asymptotic chart and the map* ϕ^{-1} *the asymptotic parametrization. We denote the fact above by* $g \in \mathrm{AC}\,(g_{\mathrm{cone}}, \tau, \phi)$ *with* K *and* R *left implicit.*

An asymptotically conical manifold is depicted schematically in Figure 7.1.

Remark 7.2. *There are several competing definitions for the term "asymptotically conical metric" in the literature, cf. e.g. [PT01]. The basic idea of our definition can be traced back at least to Cantor [Can79], where he considers the special case* $(L, g_L) = (S^n, g_{round})$ *and calls the corresponding asymptotically conical manifolds asymptotically simple.*

Remark 7.3. *Some authors allow for multiple ends in the definition of an asymptotically conical manifold. However, asymptotically conical manifolds with Ricci curvature bounded from below have necessarily one end by the Cheeger–Gromoll splitting theorem*

[1] We will use the notation g_{ac} and g_0 for an asymptotically conical manifold.

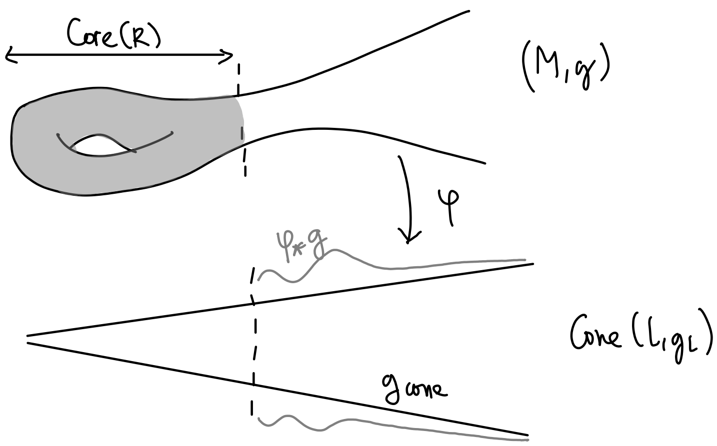


Figure 7.1: Asymptotically conical manifold

[CG71, EH84]. Since the main objects of interest in this thesis are Ricci-flat asymptotically conical manifolds, our restriction to a single end (and in particular a connected link) in the definition does not restrict the generality.

The core of an asymptotically conical manifold is not uniquely specified. In fact, there is a systematic way to generate a family of suitable cores.

Definition 7.4. *A radius function on the asymptotically conical manifold $g \in \mathrm{AC}\,(g_{\mathrm{cone}}, \tau, \phi)$ is any smooth function $\rho\colon M \to \mathbb{R}$ such that $\rho|_{M\setminus K} = r \circ \phi$. Given a radius function $\rho\colon M \to \mathbb{R}$, we define the family of manifolds*

$$\mathrm{Core}(R) := \{p \in M \mid \rho(p) \leq R\} \cup K.$$

The core is closed and bounded (since ρ is comparable to the distance function to a fixed point), thus the Hopf–Rinow theorem implies that $\mathrm{Core}(R)$ is compact. Thus, $\mathrm{Core}(R)$ is a suitable core for g and $\mathrm{Core}(R_1) \subset \mathrm{Core}(R_2)$ for $R_1 \leq R_2$.

The pointwise norm induced by the asymptotically conical metric is comparable to the pointwise norm induced by the cone metric.

Lemma 7.5 ([Pac13, Remark 6.3])**.** *Let $g \in \mathrm{AC}\,(g_{\mathrm{cone}}, \phi, \tau)$ and let S be any tensor field. Then $|S|_{\phi_* g} = |S|_{g_{\mathrm{cone}}}(1 + O\left(r^{-\tau}\right))$.*

7.2 Examples of Ricci-flat asymptotically conical manifolds

If a manifold M has the property that $M \subset K$ is diffeomorphic to $(R, \infty) \times L$ for some compact set K, then it is not complicated to construct a metric that satisfies the asymptoticity condition Equation (7.1). In fact, take any open precompact set $U \subset M$ containing K and consider the open covering $\mathcal{U} := \{U, M \setminus K\}$. If $f_U, f_{M\setminus K}$ is a partition of unity subordinate to $\mathcal{U}$ and g_U is any Riemannian metric on U, then the metric

$$g := f_U g_U + f_{M\setminus K} \phi^* g_{\mathrm{cone}}$$

(where g_{cone} is any cone metric on $(R, \infty) \times L$) satisfies the asymptoticity condition since $g - \phi^* g_{\text{cone}} = 0$ on $M \setminus \overline{U}$. (In fact, we will put this construction to use in Section 8.4.)

The situation is much more delicate if we want a Ricci-flat complete metric which satisfies the asymptoticity condition.

Example 7.6. *The easiest example for an asymptotically conical manifold is $M := \mathbb{R}^m$ with the standard flat metric g_{flat}. The link in this case is S^{m-1} with the standard round metric g_{round}. Let us fix any $R > 0$ and set $K := \{x \in \mathbb{R}^m \mid |x| < R\} \subset M$. The asymptotic chart is given, in essence, via spherical coordinates: $\phi\colon M \setminus K \to (R, \infty) \times L, x \mapsto (|x|, \frac{x}{|x|})$. Now $(\phi^{-1})^* g_{flat} = g_{\text{cone}}$, thus any positive decay rate may be chosen.*

This example is an example of what we could reasonably call an asymptotically Euclidean metric.

Example 7.7 (Eguchi–Hanson metrics). *Consider the cotangent bundle $M := T^*S^2$ of the 2-sphere S^2. We have the following chain of diffeomorphisms*

$$(T^*S^2) \setminus z(S^2) \simeq (TS^2) \setminus z(S^2) \simeq \mathbb{R}^+ \times US^2 \simeq \mathbb{R}^+ \times SO(3) \simeq \mathbb{R}^+ \times (S^3/\mathbb{Z}_2),$$

where z denotes the zero section[2], US^2 denotes the unit tangent bundle (with respect to an arbitrary metric) and where we used that US^2 is an $SO(3)$-torsor and that we have the isomorphism $SO(3) \simeq S^3/\mathbb{Z}_2$. Note that the manifold $(S^3/\mathbb{Z}_2) \times \mathbb{R}^+$ is diffeomorphic to $\text{Cone}(S^3/\mathbb{Z}_2, g_{round})$ where g_{round} denotes the metric induced by the round metric on the quotient. Note furthermore that the set $\tilde{K} := z(S^2)$ is compact in M since the zero section z is a diffeomorphism (in fact, $M \setminus \tilde{K}$ is a dense open subset of M), hence we can write the compositions of the above diffeomorphisms as

$$\tilde{\phi}\colon M \setminus \tilde{K} \to \text{Cone}(S^2/\mathbb{Z}_2, g_{round}).$$

Fixing a positive real number $R > 0$, we may thus obtain a diffeomorphism

$$\phi\colon M \setminus K \to \text{Cone}(S^2/\mathbb{Z}_2, g_{round}) \setminus (0, R] \times (S^2/\mathbb{Z}_2)$$

by restricting $\tilde{\phi}$, where $K := \phi^{-1}((0, R] \times (S^2/\mathbb{Z}_2))$.

In [EH78], Eguchi and Hanson constructed a metric on $\mathbb{R}^+ \times S^3$ which descends to a metric on $\mathbb{R}^+ \times (S^3/\mathbb{Z}_2)$ and extends to a smooth Ricci-flat metric g_{EH} on T^*S^2. The Eguchi–Hanson metric is asymptotically conical. The Eguchi–Hanson metric is hyper-Kähler, in the sense that M admits three complex structures I,J and K, satisfying the relation $IJ = K$, with respect to which g_{EH} is Kähler.

The Eguchi–Hanson metric belongs to a larger family of metrics.

Definition 7.8. *An asymptotically conical manifold is called an asymptotically localy Euclidean (ALE) manifold if the asymptotic cone is of the form $\text{Cone}(S^{\dim M - 1}/\Gamma)$ where $\Gamma \subset SO(\dim M - 1)$ is a finite subgroup of $O(n)$ acting freely on $SO(\dim M - 1)$.*

[2] In this topic, the idea to work on a bundle which has nice asymptotics outside the zero section is quite popular, cf. also Example 7.12.

The terminology comes from physics. Hyperkähler ALE spaces of dimension 4 have been classified by Kronheimer [Kro89] into classes A_k, B_k, C_k, D_k, E_6, E_7, E_8. The Eguchi–Hanson metric is A_1. Known examples of ALE metrics are either hykerkähler or arise as a finite quotient. For further examples for ALE metrics, cf. references in [HRȘ20].

Asymptotically locally Euclidean manifolds form a strict subset of asymptotically conical manifolds.

Example 7.9 (Stenzel metrics)**.** *In [Ste93], Stenzel constructed a family of compelete Ricci-flat asymptotically conical manifolds on T^*S^{n+1}, asymptotic to $\mathrm{Cone}(SO(n+2)/SO(n))$. The special case $n := 2$ recovers the Eguchi–Hanson metric.*

Example 7.10 (Ricci-flat asymptotically conical Calabi–Yau manifolds)**.** *Based on work by [CT94], Conlon and Hein gave a construction for Ricci-flat asymptotically conical Calabi–Yau manifolds [CH13, CH15].*

Example 7.11. *In [vC11], van Coevering constructed examples of Ricci-flat asymptotically conical Kähler manifolds using so-called crepant resolutions.*

Example 7.12 (G_2-manifolds)**.** *In [BS89], Bryant and Salamon constructed asymptotically conical manifolds with holonomy group contained in G_2. The underlying manifolds are the spinor bundle of S^3, the bundle of anti-self-dual 2-forms of S^4 and the bundle of anti-self-dual 2-forms of $\mathbb{CP}^2$. Their asymptotic cones are $\mathrm{Cone}(S^3 \times S^3)$, $\mathrm{Cone}(\mathbb{CP}^3)$ and $\mathrm{Cone}(SU(3)/T^2)$, respectively (cf. also [KL]). As all G_2-manifolds, the Bryant–Salomon manifolds are 7 dimensional and Ricci-flat [Bon66]. A systematic construction of asymptotically conical G_2-manifolds has been presented in [FHN].*

7.3 Functional analysis on asymptotically conical manifolds

The usual functional spaces of Sobolev and Hölder type are not well suited for asymptotically conical manifolds. However, there is a nice theory of weighted analogues of these spaces that are designed especially to deal with functional analytic issues arising on asymptotically conical manifolds. As far it could be determined, the idea of weighted function spaces originates from [Can75] who defined these spaces on $\mathbb{R}^n$, motivated by work in [NW73]. An exceptionally clearly written exposé about weighted function spaces can be found in [Pac13], here we recall only the essentials (but cf. also [McO79, CBC81, LM85, Bar86]).

Given an asymptotically conical manifold $(M, g) \in \mathrm{AC}(g_{\mathrm{cone}}, \phi, \tau)$ and a radius function $\rho \in \mathcal{C}^\infty(M)$, we define the following Banach spaces of sections of a vector bundle $E \to M$ endowed with a bundle metric and a metric connection ∇.

Definition 7.13 (cf. [Pac13, Section 5] and [DK20, page 5])**.** *The weighted Sobolev space with order $k \in \mathbb{N}$, degree $p \geq 1$, and rate $\delta \in \mathbb{R}$ is*

$$W^{k,p}_\delta(E) := \textit{Banach space completion of the space} \left\{ s \in \Gamma^\infty(E) \,\middle|\, \|s\|_{W^{k,p}_\delta(E)} \right\}$$

with respect to the norm

$$\|s\|_{W^{k,p}_\delta(E)} := \left(\sum_{j=0}^k \int_M |\rho^{-\delta-j} \nabla^j s|^p \rho^{-\dim M} \,\mathrm{vol}_g \right)^{1/p}.$$

We also introduce $H^k_\delta(E) := W^{k,2}_\delta(E)$. *This is a Hilbert space. The weighted space of* C^k *sections of rate* $\delta \in \mathbb{R}$ *is defined similarly as*

$$C^k_\delta(E) := \textit{Banach space completion of the space} \left\{ s \in \Gamma^\infty(E) \,\middle|\, \|s\|_{C^k_\delta(E)} \right\}$$

with respect to the norm

$$\|s\|_{C^k_\delta(E)} := \sum_{j=0}^{k} \sup |\rho^{-\delta-j} \nabla^j s|.$$

The weighted Hölder space with order $k \in \mathbb{N}$, *rate* $\delta \in \mathbb{R}$ *and Hölder exponent* $\alpha \in (0,1)$ *is defined as*

$$C^{k,\alpha}_\delta(E) := \left\{ s \in C^k_\delta(E) \,\middle|\, \|s\|_{C^{k,\alpha}_\delta(E)} < \infty \right\},$$

where

$$\|s\|_{C^{k,\alpha}_\delta(E)} := \|s\|_{C^k_\delta(E)} + \sup_{\substack{x,y \in M \\ 0<\mathrm{dist}_g(x,y)<\mathrm{inj}(M,g)}} \min\left\{ \rho^{-\delta+k+\alpha}(x), \rho^{-\delta+k+\alpha}(y) \right\} \frac{\left| \tau^y_x \nabla^k s(x) - \nabla^k s(y) \right|}{|\,\mathrm{dist}_g(x,y)|^\alpha},$$

where τ^y_x *denotes the parallel transport along the (unique) shortest geodesic from* x *to* y.

Evidently, if $s \in C^k_\delta(E)$, then $s = O\left(\rho^\delta\right)$.

Remark 7.14. *The notation for weighted spaces is not standardized in the literature. We follow in this thesis the convention of [Pac13] and [Bar86].*

Since different radius functions may differ only in the core, we have the following.

Lemma 7.15. *Different choices of the radius function lead to equivalent norms on the weighted Sobolev spaces.*

Weighted Sobolev and Hölder spaces are convenient to work with because analogues of several theorems for classical Sobolev and Hölder hold.

Theorem 7.16 (Weighted Sobolev embeddings)**.** *Let* $E \to M$ *be a vector bundle with metric and a metric connection.*

- *If* $k \geq l \geq 0$ *and* $k - \frac{n}{p} \geq l - \frac{n}{q}$ *and one of the following conditions hold*
  - *(a)* $p \leq q$ *and* $\beta \leq \delta$
  - *(b)* $p < q$ *and* $\beta < \delta$,

  then there is a continuous embedding $W^{p,k}_\beta(E) \leq W^{q,l}_\delta(E)$.
- *If* $\beta_1 \leq \beta_2$ *and* $k_1 + \alpha_1 \geq k_2 + \alpha_2$, *then there are continuous embeddings* $C^{k_1+1}_\beta(E) \leq C^{k_1,\alpha_1}_{\beta_1}(E) \leq C^{k_2,\alpha_2}_{\beta_2}(E) \leq C^{k_2}_{\beta_2}(E)$ *and* $C^{k_1}_{\beta_1}(E) \leq C^{k_2}_{\beta_2}(E)$.

- *If $\beta < \delta$ and $k - \frac{n}{p} \geq l + \alpha$, then there are continuous embeddings $W^{p,k}_{\beta}(E) \leq C^{l,\alpha}_{\beta}(E) \leq W^{q,l}_{\delta}(E)$.*

Proof. Cf. [Mar02, Theorem 4.17]. □

The following version of the Sobolev inequality holds on asymptotically conical manifolds.

Theorem 7.17 (Sobolev inequality)**.** *Let (M, g) be an asymptotically conical manifold of dimension m and decay rate $-\tau < 0$. Then there is a constant $C \in \mathbb{R}$ such that*

$$\|f\|_{L^{n/(n-1)}(\mathbb{R}_M, g)} \leq C \|df\|_{L^1(T^*M, g)}$$

holds for all compactly supported smooth functions $f \in \mathcal{C}^{\infty}_c(M)$.

Proof. Cf. [vC10, Theorem 2.6] or [Hei11, Theorem 1.2]. □

Proposition 7.18 (Weighted Hölder inequality, cf. e.g. [Pac13, Lemma 6.7])**.** *Let $(M, g) \in \mathrm{AC}(g_{\mathrm{cone}}, \phi, \delta)$ be an asymptotically conical manifold, and let $(E_k, (\cdot,\cdot)_k) \to M$ be Hermitian bundles, $k = 1, 2$. Let $\beta_1, \beta_2 \in \mathbb{R}$ and $q_1, q_2 \in (0, \infty]$. Then $L^{q_1}_{\beta_1}(E_1) \otimes L^{q_2}_{\beta_2}(E_2) \subset L^q_{\beta}(E)$ where $E := E_1 \otimes E_2$, $\beta := \beta_1 + \beta_2$ and $\frac{1}{q} := \frac{1}{q_1} + \frac{1}{q_2}$. If $\star\colon E_1 \otimes E_2 \to F$ is a uniformly bounded bundle map, then there is a constant $C = C(\star)$ such that (with the usual infix notation)*

$$\|u_1 \star u_2\|_{L^q_{\beta}(F)} \leq C \|u_1\|_{L^{q_1}_{\beta_1}(E_1)} \|u_2\|_{L^{q_2}_{\beta_2}(E_2)}.$$

Proof. It suffices to show the claim for decomposable tensors. Let $u_1 \in L^{q_1}_{\beta_1}(E_1)$ and $u_2 \in L^{q_2}_{\beta_2}(E_2)$. Then

$$\begin{aligned}
\|u_1 \otimes u_2\|_{L^q_{\beta}(E)} &= \left(\int_M |\rho^{-\beta} u_1 \otimes u_2|^q \rho^{-\dim M} \operatorname{vol}_g \right)^{1/q} \\
&= \left(\int_M \left(\rho^{-\beta} \rho^{-\dim M/q} \right)^q |u_1|^q |u_2|^q \operatorname{vol}_g \right)^{1/q} \\
&= \left(\int_M \left| \rho^{-\beta_1} \rho^{-\dim M/q_1} u_1 \right|^q \left| \rho^{-\beta_2} \rho^{-\dim M/q_2} u_2 \right|^q \operatorname{vol}_g \right)^{1/q} \\
&= \left\| \rho^{-\beta_1} \rho^{-\dim M/q_1} |u_1| \cdot \rho^{-\beta_2} \rho^{-\dim M/q_2} |u_2| \right\|_{L^q(E)} \\
&\overset{(\star)}{\leq} \left\| \rho^{-\beta_1} \rho^{-\dim M/q_1} u_1 \right\|_{L^{q_1}(E_1)} \cdot \left\| \rho^{-\beta_2} \rho^{-\dim M/q_2} u_2 \right\|_{L^{q_2}(E_2)} \\
&= \left(\int_M |\rho^{-\beta_1} u_1|^{q_1} \rho^{-\dim M} \operatorname{vol}_g \right)^{1/q_1} \cdot \left(\int_M |\rho^{-\beta_2} u_2|^{q_2} \rho^{-\dim M} \operatorname{vol}_g \right)^{1/q_2} \\
&= \|u_1\|_{L^{q_1}_{\beta_1}(E_1)} \|u_2\|_{L^{q_2}_{\beta_2}(E_2)},
\end{aligned}$$

where we used the ordinary Hölder inequality in the marked step. The last claim follows similarly. □

As an example of how calculations in weighted Sobolev spaces are done, consider the following lemma.

Lemma 7.19. *Let $(M, g_{\mathrm{ac}}) \in \mathrm{AC}(g_{\mathrm{cone}}, \phi, \tau)$ be an asymptotically conical manifold with asympotic link (L, g_L) and let $\rho \in \mathcal{C}^\infty(M)$ be a radius function. Then if $\alpha < \beta$ then $\rho^\alpha \in L^p_\beta(\mathbb{R}_M)$.*

Proof. Let $K \subset M$ be a compact set outside which $\rho = r \circ \phi$ (this is a suitable choice for a core). Without loss of generality, we may choose $K := \mathrm{Core}(R)$ for some $R > 0$. Then we calculate

$$\begin{aligned}
\|\rho^\alpha\|^p_{L^p_\beta(\mathbb{R}_M)} &= \int_M (\rho^{-\beta}|\rho^\alpha|)^p \rho^{-\dim M} \operatorname{vol}_{g_{\mathrm{ac}}} = \int_M (\rho^{(-\beta+\alpha)p-\dim M} \operatorname{vol}_{g_{\mathrm{ac}}} \\
&= \int_K \rho^{(-\beta+\alpha)p-\dim M} \operatorname{vol}_{g_{\mathrm{ac}}} + \int_{M\setminus K} \rho^{(-\beta+\alpha)p-\dim M} \operatorname{vol}_{g_{\mathrm{ac}}} \\
&\leq C_1 + C_2 \int_{M\setminus K} \rho^{(-\beta+\alpha)p-\dim M} \operatorname{vol}_{\phi^* g_{\mathrm{cone}}} \\
&\leq C_1 + C_2 \int_{M\setminus K} r^{(-\beta+\alpha)p-\dim M} \operatorname{vol}_{g_{\mathrm{cone}}} \\
&\leq C_1 + C_2 \int_R^\infty \int_L r^{(-\beta+\alpha)p-\dim M} r^{\dim M-1} \operatorname{vol}_{g_L} dr \\
&\leq C_1 + C_2 \operatorname{Vol}_{g_L}(L) \int_R^\infty r^{(-\beta+\alpha)p-1} dr < \infty. \qquad \square
\end{aligned}$$

7.4 Some formulas involving higher covariant derivatives

In this section we develop a few formulas that relate higher covariant derivatives with respect to different connections, with a special emphasis on Levi-Civita connections and decay rates.

Covariant derivatives of the difference tensor We prove a few general lemmata about tensor fields related to different Riemannian metrics. Let g and g' be two Riemannian metrics on a manifold M, and set $h := g' - g$. Let the corresponding Levi-Civita connections be ∇ and ∇'. Then due to torsion freeness of both ∇ and ∇', it is known that $T := \nabla' - \nabla$ is a (2,1)-tensor field which is symmetric, i.e. $T(X, Y) = T(Y, X)$ for all $X, Y \in TM$. This tensor field can be determined by a Koszul style formula.

Lemma 7.20. *In the setting above, we have $2g'(T(X,Y), Z) = -(\nabla_Z h)(X,Y) + (\nabla_X h)(Y,Z) + (\nabla_Y h)(Z,X)$ for any $X, Y, Z \in TM$. Moreover, for the pointwise norms, we have $|T|_{g'} \leq \frac{3}{2}|\nabla h|_{g'}$.*

Proof. By g' compatibility of ∇', we obtain for $X, Y, Z \in TM$

$$\begin{aligned}
0 = (\nabla'_Z g')(X,Y) &= Z(g'(X,Y)) - g'(\nabla'_Z X, Y) - g'(X, \nabla'_Z Y) \\
&= Z(g(X,Y)) + Z(h(X,Y)) \\
&\quad - g(\nabla_Z X, Y) - g(T(Z,X), Y) - h(\nabla_Z X, Y) - h(T(Z,X), Y) \\
&\quad - g(X, \nabla_Z Y) - g(X, T(Z,Y)) - h(X, \nabla_Z Y) - h(X, T(Z,Y)) \\
&= (\nabla_Z g)(X,Y) + (\nabla_Z h)(X,Y),Y)) - g(T(Z,X),Y) - h(T(Z,X),Y) - g(X, T(Z,Y)).
\end{aligned}$$

Thus, by permutation of the arguments, we obtain

$$0 = (\nabla_Z h)(X, Y) - g'(T(Z, X), Y) - g'(X, T(Z, Y)), \tag{7.2}$$
$$0 = (\nabla_X h)(Y, Z) - g'(T(X, Y), Z) - g'(Y, T(X, Z)), \tag{7.3}$$
$$0 = (\nabla_Y h)(Z, X) - g'(T(Y, Z), X) - g'(Z, T(Y, X)). \tag{7.4}$$

Therefore, the linear combination $-(7.2) + (7.3) + (7.4)$ yields

$$\begin{aligned} 0 &= -(\nabla_Z h)(X, Y) + g'(T(X, Z), Y) + g'(T(Y, Z), X) \\ &\quad + (\nabla_X h)(Y, Z) - g'(T(Y, X), Z) - g'(T(Z, X), Y) \\ &\quad + (\nabla_Y h)(Z, X) - g'(T(Z, Y), X) - g'(T(X, Y), Z) \\ &= -(\nabla_Z h)(X, Y) + (\nabla_X h)(Y, Z) + (\nabla_Y h)(Z, X) - 2g'(T(X, Y), Z). \end{aligned}$$

Rearranging gives the first part of the claim.

For the second claim, we are going to use the fact that for any metric $B\colon V \otimes V \to \mathbb{R}$ and any vectors $v_1, \dots, v_k \in V$, we have

$$B(v_1 + \dots + v_k, v_1 + \dots + v_k) \leq k \cdot (B(v_1, v_1) + \dots + B(v_k, v_k)),$$

which follows from the Cauchy–Schwarz inequality and the inequality between geometric and arithmetic means.

For a g'-orthonormal frame $(e_i)_{i\in I}$, we have

$$\begin{aligned} |T|^2_{g'} &= \sum_{i,j,k\in I} g'(T(e_i, e_j), e_k)^2 \\ &= \sum_{i,j,k\in I} \frac{1}{4} \Big(-(\nabla_{e_k} h)(e_i, e_j) + (\nabla_{e_i} h)(e_j, e_k) + (\nabla_{e_j} h)(e_i, e_k)\Big)^2 \\ &\overset{(\star)}{\leq} \sum_{i,j,k\in I} \frac{1}{4} \Big((\nabla_{e_k} h)(e_i, e_j)^2 + (\nabla_{e_i} h)(e_j, e_k)^2 + (\nabla_{e_j} h)(e_i, e_k)^2\Big) \\ &= \frac{9}{4} |\nabla h|^2_{g'}, \end{aligned}$$

where we used the estimate given earlier to establish the inequality in the step marked by $(\star)$. □

As we will see, for the curvature, it is useful to know what the covariant derivative ∇T looks like. We can give a similar, Koszul style characterization via the metric g'.

Lemma 7.21. *In the setting above, for $X, Y, Z, W \in TM$, we have*

$$2g'((\nabla_W T)(X, Y), Z) = -(\nabla^2_{W,Z} h)(X, Y) + (\nabla^2_{W,X} h)(Y, Z) + (\nabla^2_{W,Y} h)(X, Z) - 2(\nabla_W h)(T(X, Y), Z).$$

Moreover, $|\nabla T|^2_{g'} \leq 3|\nabla^2 h|^2_{g'} + 9|\nabla h|^4_{g'}$.

Proof. The strategy is to derive the statement of Lemma 7.20 with respect to W.

$$\nabla_W(2g'(T(X, Y), Z)) = \nabla_W(-(\nabla_Z h)(X, Y) + (\nabla_X h)(Y, Z) + (\nabla_Y h)(Z, X)).$$

By the product rule we obtain

$$\begin{aligned}
2(\nabla_W g')(T(X,Y),Z) + 2g'((\nabla_W T)(X,Y),Z)&\\
&+ 2g'(T(\nabla_W X),Y),Z) + 2g'(T(X,\nabla_W Y),Z) + 2g'(T(X,Y),\nabla_W Z)\\
&= (-\nabla_W(\nabla_Z h))(X,Y) - (\nabla_Z h)(\nabla_W X,Y) - (\nabla_Z h)(X,\nabla_W Y)\\
&\quad + (\nabla_W(\nabla_X h))(Y,Z) + (\nabla_X h)(\nabla_W Y,Z) + (\nabla_X h)(Y,\nabla_W Z)\\
&\quad + (\nabla_W(\nabla_Y h))(X,Z) + (\nabla_Y h)(\nabla_W X,Z) - (\nabla_Y h)(X,\nabla_W Z).
\end{aligned}$$

Separating the term we want to express on the left-hand side, we get

$$\begin{aligned}
2g'((\nabla_W T)(X,Y),Z) &= (-\nabla_W(\nabla_Z h))(X,Y) - (\nabla_Z h)(\nabla_W X,Y) - (\nabla_Z h)(X,\nabla_Y)\\
&\quad + (\nabla_W(\nabla_X h))(Y,Z) + (\nabla_X h)(\nabla_W Y,Z) + (\nabla_Y h)(Y,\nabla_W Z)\\
&\quad + (\nabla_W(\nabla_Y h))(X,Z) + (\nabla_Y h)(\nabla_W X,Z) + (\nabla_Y h)(X,\nabla_W Z)\\
&\quad - 2(\nabla_W g')(T(X,Y),Z) - 2g'(T(\nabla_W X,Y),Z)\\
&\quad - 2g'(T(X,\nabla_W Y),Z) - 2g'(T(X,Y),\nabla_W Z).
\end{aligned}$$

Using Lemma 7.20 for $g'(T(\cdot,\cdot),\cdot)$, we obtain

$$\begin{aligned}
&= -(\nabla_W(\nabla_Z h))(X,Y) - (\nabla_Z h)(\nabla_W X,Y) - (\nabla_Z h)(X,\nabla_W Y)\\
&\quad + (\nabla_W(\nabla_X h))(Y,Z) + (\nabla_X h)(\nabla_W Y,Z) + (\nabla_X h)(Y,\nabla_W Z)\\
&\quad + (\nabla_W(\nabla_Y h))(X,Z) + (\nabla_Y h)(\nabla_W Z) + (\nabla_Y h)(X,\nabla_W Z)\\
&\quad - 2(\nabla_W g')(T(X,Y),Z)\\
&\quad + (\nabla_Z h)(\nabla_W X,Y) - (\nabla_{\nabla_W X} h)(Y,Z) - (\nabla_Y h)(\nabla_W X,Z)\\
&\quad + (\nabla_Z h)(X,\nabla_W Y) - (\nabla_X h)(\nabla_W Y,Z) + (\nabla_{\nabla_W Y} h)(X,Z)\\
&\quad + (\nabla_{\nabla_W Z} h)(X,Y) - (\nabla_X h)(Y,\nabla_W Z) - (\nabla_Y h)(X,\nabla_W Z)\\
&= -(\nabla^2_{W,Z} h)(X,Y) + (\nabla^2_{W,X} h)(Y,Z)\\
&\quad + (\nabla^2_{W,Y} h)(X,Z) - 2(\nabla_W g')(T(X,Y),Z)\\
&= -(\nabla^2_{W,Z} h)(X,Y) + (\nabla^2_{W,X} h)(Y,Z)\\
&\quad + (\nabla^2_{W,Y} h)(X,Z) - 2(\nabla_W h)(T(X,Y),Z),
\end{aligned}$$

as claimed.

For the claim about the norms, choose a g'-orthogonal frame $(e_i)_{i\in I}$ to calculate

$$\begin{aligned}
|\nabla T|^2_{g'} &= \sum_{i,j,k,l\in I} g'((\nabla_{e_i} T)(e_j,e_k),e_l)^2\\
&= \frac{1}{4}\sum_{i,j,k,l\in I} (-(\nabla^2_{e_l,e_k} h)(e_i,e_j) + (\nabla^2_{e_l,e_l} h)(e_j,e_k) + (\nabla^2_{e_l,e_j} h)(e_i,e_k)\\
&\qquad\qquad - 2(\nabla_{e_k} h)(T(e_i,e_j),e_k))^2\\
&\le \frac{4}{4}\sum_{i,j,k,l\in I} (-(\nabla^2_{e_l,e_k} h)(e_i,e_j)^2 (\nabla^2_{e_l,e_l} h)(e_j,e_k)^2 + (\nabla^2_{e_l,e_j} h)(e_i,e_k)^2\\
&\qquad\qquad + 4(\nabla_{e_k} h)(T(e_i,e_j),e_k)^2)
\end{aligned}$$

$$
\begin{aligned}
&= 3|\nabla^2 h|_g^2 + 4|\nabla h \star T|_g^2 \\
&\leq 3|\nabla^2 h|_{g'}^2 + 4|\nabla h|_{g'}^2 |T|_{g'}^2 \\
&\leq 3|\nabla^2 h|_{g'}^2 + 9|\nabla h|_{g'}^4
\end{aligned}
$$

where we used the inequality from the proof of the last lemma, the fact that the induced norm on tensors is submultiplicative with respect to compositions, and Lemma 7.20. ☐

In principle, we could continue this pattern to derive a Koszul style formula for even higher order partial derivatives in terms covariant derivatives of the difference tensor h of ever increasing order. However, for our purposes, this cumbersome road would give too much information—we need only the decay information after all. This is why we introduce the following simplifying notation. Let $T \star S$ denote any $\mathbb{R}$-linear combination of contractions of the tensor fields T and S. The meaning of $\star$ can change from line to line but it remains an associative operation. Moreover, let PT denote any $\mathbb{R}$-linear combination of the tensor T with permuted arguments.

With this notation, we can rewrite the statement of Lemma 7.21 as

$$\nabla T = P\nabla^2 h - (\nabla h) \star T = P\nabla^2 h + (\nabla h) \star (P\nabla h) = P(\nabla^2 h + (\nabla h) \star (\nabla h)).$$

From this, we can easily read off how the norm behaves:

$$
\begin{aligned}
|\nabla T|_g &= |P(\nabla^2 h + (\nabla h) \star (\nabla h))|_g \\
&= C_1 |\nabla^2 h|_g + C_2 |(\nabla h) \star (\nabla h)|_g \\
&= C_1 |\nabla^2 h|_g + C_3 |\nabla h|_g^2 \\
&= O\left(r^{-\tau-2}\right) + O\left(r^{2\cdot(-\tau-1)}\right) \\
&= O\left(r^{-\tau-2}\right).
\end{aligned}
$$

With this simplified notation, we can state and prove the following statement.

Lemma 7.22. *Let g', g be Riemannian metrics on a smooth manifold, and let $h := g' - g$, and $T := \nabla^{g'} - \nabla^g$. Then*

$$\nabla^{g,k} T = P\left(\nabla^{g,k+1} h + \sum_{p=0}^{k+1} (\nabla^{g,p} h) \star (\nabla^{g,k+1-p} h)\right),$$

where $k \in \mathbb{N}$. In particular, if $|\nabla^{g,k} h|_g = O\left(r^{-\tau-k}\right)$, then $|\nabla^{g,k} T|_g = O\left(r^{-\tau-k-1}\right)$.

Proof. We have seen the statement for $k = 0$ and $k = 1$ before in Lemmata 7.20 and 7.21. For $k > 1$, we can show the statement using induction and the Leibniz rule. ☐

Covariant derivatives of the curvature tensors Armed with these results, we can now express the curvature R' of g' in terms of the curvature R of g and various covariant derivatives of the tensors h and T.

Lemma 7.23. *In terms of the data above,* $R'(X,Y)Z = R(X,Y)Z + (\nabla_X T)(Y,Z) - (\nabla_Y T)(X,V) + T(X,T(Y,Z)) - T(Y,T(X,Z))$ *for* $X,Y,Z \in TM$*. For covariant derivatives with respect to* g*, we have* $\nabla^k R' = \nabla^k R + \sum_{l=0}^{k+1}(\nabla^l T) \star (\nabla^{k+1-l}T)$ *where* $\star$ *denotes a contraction. In particular* $|R'|_g = O\left(r^{-2}\right)$ *and* $|\nabla^k(R'-R)|_g = O\left(r^{-\tau-2-k}\right)$*.*

Proof. We will treat the cases separately, based on the value of k.

$k = 0$: Let $X, Y, Z \in TM$. The second covariant derivative with respect to the metric g' can be calculated as

$$\begin{aligned}
(\nabla')^2_{X,Y}Z &= \nabla'_X(\nabla'_Y Z) - \nabla'_{\nabla'_X Y}Z \\
&= (\nabla_X + T(X,\cdot)) \circ (\nabla_Y + T(Y,\cdot))(Z) - \nabla_{\nabla_X Y + T(X,Y)}(Z) \\
&\quad - T(\nabla_X Y + T(X,Y), Z) \\
&= \nabla^2_{X,Y}Z + (\nabla_X T)(Y,Z) + T(\nabla_X Y, Z) + T(Y, \nabla_X Z) \\
&\quad + T(X, \nabla_Y Z) + T(X, T(Y,Z)) - \nabla_{T(X,Y)}Z - T(\nabla_X Y, Z) \\
&\quad + T(T(X,Y),Z)
\end{aligned}$$

The curvature tensor is the antisymmetrization of the second covariant derivative, therefore the fact that T is symmetric yields

$$\begin{aligned}
R'(X,Y)Z &= \nabla'_{X,Y}Z - \nabla'_{Y,X}Z \\
&= R(X,Y)Z + (\nabla_X T)(Y,Z) - (\nabla_Y T)(X,Z) \qquad (7.5)\\
&\quad + T(X,T(Y,Z)) - T(Y,T(X,Z)) \\
&= (R + P\nabla T - T \star T)(X,Y,Z). \qquad (7.6)
\end{aligned}$$

Based on this equation, the triangle inequality implies that the pointwise norm satisfies

$$\begin{aligned}
|R'|_{g'} &\leq |R|_{g'} + 2|\nabla T|_{g'} + 2|T \circ (T \otimes \mathrm{id}_{TM})|_{g'} \\
&\leq |R|_g + 2|\nabla T|_g + 2|T|^2_{g'} \\
&= O\left(r^{-2}\right) + O\left(r^{-\tau-2}\right) + O\left(r^{2\cdot(-\tau-1)}\right) \\
&= O\left(r^{-2}\right).
\end{aligned}$$

Similarly, we obtain that $|R' - R|_{g'} = O\left(r^{-\tau-2}\right)$. By Lemma 7.5, we have the same decay rate in the g-norm, too.

$k > 0$: The starting point of the argument is Equation (7.5). Taking the kth covariant derivative implies, using the Leibniz rule, the statement. The decay rate follows from a similar argument as in the case $k = 0$. ☐

Higher covariant derivatives of tensor fields Next we relate higher-order covariant derivative of any tensor field to each other.

We start with a naturality statement.

Lemma 7.24. *Let M and N be smooth manifolds and let $\phi\colon M \to N$ be a diffeomorphism. Moreover, let $g \in \mathrm{Met}(N)$ and let $S \in \Gamma^\infty\left((T^*N)^{\otimes r}\right)$ be a tensor field. Then for any $k \in \mathbb{N}$, we have $|\nabla^{\phi^*g,k}(\phi^*S)|_{\phi^*g} = \phi^*\left(|\nabla^{g,k}S|_g\right)$.*

Proof. One checks that $|\phi^*S|_{\phi^*g} = \phi^*\left(|S|_g\right)$ for any tensor field T, which is incidentally the claim for $k=0$. Next, elementary calculations show that $\nabla^{\phi^*g}(\phi^*S) = \phi^*(\nabla^g S)$. The claim now follows by induction:

$$\begin{aligned}
|\nabla^{\phi^*g,k+1}(\phi^*S)|_{\phi^*g} &= |\nabla^{\phi^*g,k}(\nabla^{\phi^*g}(\phi^*S))|_{\phi^*g} \\
&= |\nabla^{\phi^*g,k}(\phi^*(\nabla^g S))|_{\phi^*g} \\
&= \phi^*\left(|\nabla^{g,k}(\nabla^g S)|_g\right) \\
&= \phi^*\left(|\nabla^{g,k+1}S|_g\right)
\end{aligned}$$

□

Lemma 7.25. *Let M be a smooth manifold and let $\nabla, \bar{\nabla}$ be two connections on M and set $T := \bar{\nabla} - \nabla$. Moreover, let $S \in \Gamma^\infty\left((T^*M)^{\otimes s}\right)$ be a tensor field. Then for any $k \in \mathbb{N}$, we have*

$$\bar{\nabla}^k S = \sum_{\substack{p,q,r=0\\ p+q+r=k}}^{k} (\nabla^p S) \star \nabla^q(\underbrace{T \star \cdots \star T}_{r \text{ times}}),$$

where $\star$ denotes a multilinear operator which factors through the tensor product via a ∇-covariantly constant map.

Proof. We can work using induction. The case $k=0$ is trivial. Note that $\nabla(A \star B) = (\nabla A) \star B + A \star (\nabla B)$ because of ∇-covariant constancy of $\star$. Thus we have by the Leibniz rule

$$\begin{aligned}
\bar{\nabla}^{k+1} S = \bar{\nabla}(\bar{\nabla}^k S) &= \bar{\nabla}\left(\sum_{\substack{p,q,r=0\\ p+q+r=k}}^{k} (\nabla^p S) \star \nabla^q(\underbrace{T \star \cdots \star T}_{r \text{ times}})\right) \\
&= (\nabla + T)\left(\sum_{\substack{p,q,r=0\\ p+q+r=k}}^{k} (\nabla^p S) \star \nabla^q(\underbrace{T \star \cdots \star T}_{r \text{ times}})\right) \\
&= \sum_{\substack{p,q,r=0\\ p+q+r=k}}^{k} \Bigg[(\nabla^{p+1} S) \star \nabla^q(\underbrace{T \star \cdots \star T}_{r \text{ times}}) + (\nabla^p S) \star \nabla^{q+1}(\underbrace{T \star \cdots \star T}_{r \text{ times}}) \\
&\qquad + T \star (\nabla^p S) \star \nabla^q(\underbrace{T \star \cdots \star T}_{r \text{ times}})\Bigg] \\
&= \sum_{\substack{p,q,r=0\\ p+q+r=k}}^{k} \Bigg[(\nabla^{p+1} S) \star \nabla^q(\underbrace{T \star \cdots \star T}_{r \text{ times}}) + (\nabla^p S) \star \nabla^{q+1}(\underbrace{T \star \cdots \star T}_{r \text{ times}}) \\
&\qquad + (\nabla^p S) \star \nabla^q(\underbrace{T \star \cdots \star T}_{r+1 \text{ times}})\Bigg]
\end{aligned}$$

$$= \sum_{\substack{p,q,r=0\\ p+q+r=k+1}}^{k+1} (\nabla^p S) \star \nabla^q (\underbrace{T \star \cdots \star T}_{r \text{ times}}). \qquad \square$$

In the context of asymptotically conical metrics, we have the following corollary.

Corollary 7.26. *Let M be a smooth manifold, let $g \in \mathrm{Met}\,(M)$ be an asymptotically conical metric and let $\bar{g} \in \mathrm{Met}\,(M)$ be another Riemannian metric with $h := \bar{g} - g$ satisfying $|\nabla^{g,k} h|_g = O\left(\rho^{-\tau-k}\right)$. Moreover, let S be a tensor field with $|\nabla^{g,k} S|_g = O\left(\rho^{-\alpha-k}\right)$. Then $|\nabla^{\bar{g},k} h|_{\bar{g}} = O\left(\rho^{-\alpha-k}\right)$ as well.*

Proof. We proceed in three steps.

- From Lemma 7.22, we know that

$$\begin{aligned}
\nabla^{k,a} T &= \nabla^{g,a+1} h + \sum_{p=0}^{a+1} (\nabla^{g,p} h) \star (\nabla^{g,a+1-p} h) \\
&= O\left(\rho^{-\tau-a-1}\right) + \sum_{p=0}^{a+1} O\left(\rho^{-\tau-p}\right) O\left(\rho^{-\tau-a-1+p}\right) \\
&= O\left(\rho^{-\tau-a-1}\right),
\end{aligned}$$

  thus we obtain from the Leibniz rule that

$$\begin{aligned}
\nabla^{g,q} (\underbrace{T \star \cdots \star T}_{k \text{ times}}) &= \sum_{\substack{a_1,\ldots,a_k= \\ a_1+\cdots+a_k=q}}^{q} (\nabla^{g,a_1} T) \star \cdots \star (\nabla^{g,a_k} T) \\
&= \sum_{\substack{a_1,\ldots,a_k= \\ a_1+\cdots+a_k=q}}^{q} O\left(\rho^{-\tau-a_1-1}\right) \ldots O\left(\rho^{-\tau-a_k-1}\right) \\
&= O\left(\rho^{-k\tau-q-k}\right) = O\left(\rho^{-q} (\rho^{-\tau-1})^k\right).
\end{aligned}$$

  (This means each derivative introduces a factor ρ^{-1} and each "power" of T introduces a factor $\rho^{-\tau-1}$.)

- By Lemma 7.25, we obtain now

$$\begin{aligned}
\nabla^{\bar{g},k} S &= \sum_{\substack{p,q,r=0\\ p+q+r=k}}^{k} (\nabla^{g,p} S) \star \nabla^{g,q} (\underbrace{T \star \cdots \star T}_{r \text{ times}}) \\
&= \sum_{\substack{p,q,r=0\\ p+q+r=k}}^{k} (\nabla^{g,p} S) \star O\left(\rho^{-r\tau-q-r}\right).
\end{aligned}$$

- Since the pointwise metrics induced by $\bar{g}$ and g are equivalent by Lemma 7.5, we obtain

$$|\nabla^{\bar{g},k} S|_{\bar{g}} \le C_1 |\nabla^{\bar{g},k} S|_g (1 + O\left(\rho^{-\tau}\right))$$

$$
\begin{aligned}
&\leq C_1 \Big| \sum_{\substack{p,q,r=0 \\ p+q+r=k}}^{k} (\nabla^{g,p} S) \star O\left(\rho^{-r\tau-q-r}\right) \Big|_g (1 + O\left(\rho^{-\tau}\right)) \\
&\leq C_2 \sum_{\substack{p,q,r=0 \\ p+q+r=k}}^{k} |(\nabla^{g,p} S)|_g O\left(\rho^{-r\tau-q-r}\right) (1 + O\left(\rho^{-\tau}\right)) \\
&\leq C_2 \sum_{\substack{p,q,r=0 \\ p+q+r=k}}^{k} O\left(\rho^{-\alpha-p}\right) O\left(\rho^{-r\tau-q-r}\right) (1 + O\left(\rho^{-\tau}\right)) \\
&= O\left(\rho^{-\alpha-k}\right) (1 + O\left(\rho^{-\tau}\right)) = O\left(\rho^{-\alpha-k}\right).
\end{aligned}
$$

□

7.5 Pointwise injectivity radius on asymptotically conical manifolds

The goal of this section is to show that on an asympototically conical manifold, far enough from the core, the injectivity radius grows at least linearly.

7.5.1 Pullback to a reference set

First, we show that the pointwise norm of the difference $g_{\mathrm{ac}} - g_{\mathrm{cone}}$ at a homothetically rescaled point can be calculated via a rescaled version of the pullback.

Lemma 7.27. *Fix $t \in \mathbb{R}$ and let $\phi_t := \mathrm{Fl}_t^Z$. Then we have for the pointwise norms that*

1. $|e^{-2t}\phi_t^* g_{\mathrm{ac}} - g_{\mathrm{cone}}|_{g_{\mathrm{cone}}} = |g_{\mathrm{ac}} - g_{\mathrm{cone}}|_{g_{\mathrm{cone}}} \circ \phi_t$,
2. $|\nabla^{g_{\mathrm{cone}}}(e^{-2t}\phi_t^* g_{\mathrm{ac}} - g_{\mathrm{cone}})|_{g_{\mathrm{cone}}} = |\nabla^{g_{\mathrm{cone}}}(g_{\mathrm{ac}} - g_{\mathrm{cone}})|_{g_{\mathrm{cone}}} \circ \phi_t$,
3. $|\nabla^{g_{\mathrm{cone}},2}(e^{-2t}\phi_t^* g_{\mathrm{ac}} - g_{\mathrm{cone}})|_{g_{\mathrm{cone}}} = |\nabla^{g_{\mathrm{cone}},2}(g_{\mathrm{ac}} - g_{\mathrm{cone}})|_{g_{\mathrm{cone}}} \circ \phi_t$.

Proof. This is an easy calculation using that $\phi_t^* g_{\mathrm{cone}} = e^{2t} g_{\mathrm{cone}}$, the properties of the pointwise norm under pushforwards and rescalings and naturality of the Levi-Civita connection.

1.

$$
\begin{aligned}
e^{2t}|e^{-2t}\phi_t^* g_{\mathrm{ac}} - g_{\mathrm{cone}}|_{g_{\mathrm{cone}}} &= |\phi_t^* g_{\mathrm{ac}} - e^{2t} g_{\mathrm{cone}}|_{g_{\mathrm{cone}}} \\
&= |\phi_t^*(g_{\mathrm{ac}} - g_{\mathrm{cone}})|_{g_{\mathrm{cone}}} \\
&= |g_{\mathrm{ac}} - g_{\mathrm{cone}}|_{(\phi_t)_* g_{\mathrm{cone}}} \circ \phi_t \\
&= e^{2t}|g_{\mathrm{ac}} - g_{\mathrm{cone}}|_{g_{\mathrm{cone}}} \circ \phi_t
\end{aligned}
$$

2.

$$
\begin{aligned}
|\nabla^{g_{\mathrm{cone}}}(e^{-2t}\phi_t^* g_{\mathrm{ac}} - g_{\mathrm{cone}})|_{g_{\mathrm{cone}}} &= e^{-2t}|\nabla^{g_{\mathrm{cone}}}(\phi_t^* g_{\mathrm{ac}})|_{g_{\mathrm{cone}}} \\
&= e^{-2t}|\nabla^{\phi_t^* g_{\mathrm{cone}}}(\phi_t^* g_{\mathrm{ac}})|_{e^{-2t}\phi_t^* g_{\mathrm{cone}}}
\end{aligned}
$$

$$\begin{aligned}
&= |(\phi_t^*\nabla^{g_{\text{cone}}})(\phi_t^* g_{\text{ac}})|_{\phi_t^* g_{\text{cone}}} \\
&= |\nabla^{g_{\text{cone}}} g_{\text{ac}}|_{g_{\text{cone}}} \circ \phi_t \\
&= |\nabla^{g_{\text{cone}}}(g_{\text{ac}} - g_{\text{cone}})|_{g_{\text{cone}}} \circ \phi_t
\end{aligned}$$

3. This is basically the same as in the last item. □

Using this lemma, we can express the C^2-norm on a homothetically rescaled compact subset.

Lemma 7.28. *Let $K \subset M$ be a compact subset and $t \in \mathbb{R}$ such that $\phi_t(K) \subset M \setminus \mathrm{Core}(R_{asy})$. Then $\|g_{\text{ac}} - g_{\text{cone}}\|_{C^2(S^2T^*M|_{\phi_t(K)}, g_{\text{cone}})} = \|e^{-2t}\phi_t^* g_{\text{ac}} - g_{\text{cone}}\|_{C^2(S^2T^*M|_K, g_{\text{cone}})} \leq \rho_K{}^{-\tau} e^{-\tau t}\left(C_0 + \frac{C_1}{\rho_K e^t} + \frac{C_2}{\rho_K^2 e^{2t}}\right)$, where $\rho_K := \min\{\rho(p)\,|p \in K\}$. In particular if $t > 0$ and $K \subset M \setminus \mathrm{Core}(R_{asy})$, we have $\|g_{\text{ac}} - g_{\text{cone}}\|_{C^2(S^2T^*M|_{\phi_t(K)}, g_{\text{cone}})} \leq Ce^{-\tau t}$ with $C := \rho_K^{-\tau}\left(C_0 + \frac{C_1}{\rho_K} + \frac{C_2}{\rho_K^2}\right)$.*

Proof.

$$\begin{aligned}
\|e^{-2t}\phi_t^* g_{\text{ac}} - g_{\text{cone}}\|^2_{C^2(S^2T^*M|_K, g_{\text{cone}})} &= \sup\left\{\begin{matrix} |e^{-2t}\phi_t^* g_{\text{ac}} - g_{\text{cone}}|^2_{g_{\text{cone}}}(p) \\ +|\nabla^{g_{\text{cone}}}(e^{-2t}\phi_t^* g_{\text{ac}} - g_{\text{cone}})|^2_{g_{\text{cone}}}(p) \\ +|\nabla^{g_{\text{cone}},2}(e^{-2t}\phi_t^* g_{\text{ac}} - g_{\text{cone}})|^2_{g_{\text{cone}}}(p) \end{matrix}\ \middle|\ p \in K\right\} \\
&= \sup\left\{\begin{matrix} |g_{\text{ac}} - g_{\text{cone}}|^2_{g_{\text{cone}}}(\phi_t(p)) \\ +|\nabla^{g_{\text{cone}}}(g_{\text{ac}} - g_{\text{cone}})|^2_{g_{\text{cone}}}(\phi_t(p)) \\ +|\nabla^{g_{\text{cone}},2}(g_{\text{ac}} - g_{\text{cone}})|^2_{g_{\text{cone}}}(\phi_t(p)) \end{matrix}\ \middle|\ p \in K\right\} \\
&= \sup\left\{\begin{matrix} |g_{\text{ac}} - g_{\text{cone}}|^2_{g_{\text{cone}}}(q) \\ +|\nabla^{g_{\text{cone}}}(g_{\text{ac}} - g_{\text{cone}})|^2_{g_{\text{cone}}}(q) \\ +|\nabla^{g_{\text{cone}},2}(g_{\text{ac}} - g_{\text{cone}})|^2_{g_{\text{cone}}}(q) \end{matrix}\ \middle|\ q \in \phi_t(K)\right\}
\end{aligned}$$

Using that $\phi_t(K) \subset M \setminus \mathrm{Core}(R_{\text{asy}})$, we obtain the estimate

$$\leq \sup\left\{\begin{matrix} C_0\rho(\phi_t(q))^{-\tau} \\ +C_1\rho(\phi_t(q))^{-\tau-1} \\ +C_2\rho(\phi_t(q))^{-\tau-2} \end{matrix}\ \middle|\ q \in K\right\}$$

Using that $\rho \circ \phi_t = e^t\rho$, we obtain

$$\begin{aligned}
&= \sup\left\{\begin{matrix} C_0\rho(q)^{-\tau}e^{-\tau t} \\ +C_1\rho(q)^{-\tau-1}e^{(-\tau-1)t} \\ +C_2\rho(q)^{-\tau-2}e^{(-\tau-2)t} \end{matrix}\ \middle|\ q \in K\right\} \\
&\leq C_0\rho_K^{-\tau}e^{-\tau t} + C_1\rho_K^{-\tau-1}e^{(-\tau-1)t} + C_2\rho_K^{-\tau-2}e^{(-\tau-2)t} \\
&= \rho_K^{-\tau}\left(C_0 + \frac{C_1}{\rho_K e^t} + \frac{C_2}{\rho_K^2 e^{2t}}\right)e^{-\tau t}.
\end{aligned}$$

In case $t > 0$, the parentheses can be estimated by its value at $t = 0$. □

7.5.2 Lower semicontinuity of the injectivity radius function

We state a few useful calculation rules for the injectivity radius.

Lemma 7.29. *Let $p \in M$, $g \in \mathrm{Met}(M)$, $\alpha > 0$, $\phi\colon M \to M$ a diffeomorphism and let $U \subset M$ be a neighbourhood of p in M. Then*

1. $\mathrm{inj}_{\alpha^2 g}(p) = \alpha\,\mathrm{inj}_g(p)$,
2. *and* $\mathrm{inj}_{\phi^* g}(p) = \mathrm{inj}_g(\phi(p))$,
3. $\mathrm{inj}_{g|_U}(p) \le \mathrm{inj}_g(p)$.

Proof. 1. Since the Koszul formula implies that the Levi-Civita connections for g and $\alpha^2 g$ are equal, the corresponding exponential maps coincide. This means in particular that the largest subset U of T_pM which gets mapped diffeomorphically onto M is also independent of α (as long as $\alpha > 0$ holds). However, the injectivity radius, i.e. the supremum of the radii of balls lying entirely in U *does* depend on the scaling.

Since for the pointwise norm of $X_p \in T_pM$, we have $|X|_{\alpha^2 g} = \sqrt{\alpha^2 g(X,X)} = \alpha |X|_g$, we have $B_r^{\alpha^2 g}(0) = B^g_{r/\alpha}(0)$ in T_pM. Consequently

$$\begin{aligned}\mathrm{inj}_p(\alpha^2 g) &= \sup\left\{r > 0 \,\middle|\, B_r^{\alpha^2 g}(0_p) \subset U\right\} = \sup\left\{r > 0 \,\middle|\, B^g_{r/\alpha}(0_p) \subset U\right\}\\ &= \sup\left\{\alpha R > 0 \,|\, B^g_R(0_p) \subset U\right\} = \alpha \sup\left\{R > 0 \,|\, B^g_R(0_p) \subset U\right\} = \alpha\,\mathrm{inj}_p(g).\end{aligned}$$

2. Since $\phi\colon M \to M$ is a diffeomorphism, the differential $d\phi_p\colon T_pM \to T_{\phi(p)}M$ is a linear isomorphism, therefore we obtain the following relation for balls in the tangent space

$$\begin{aligned}B_r^{\phi^* g}(0_p) &= \left\{X \in T_pM \,\middle|\, (\phi^*)_p(X,X) < r^2\right\}\\ &= \left\{d\phi^{-1}_{\phi(p)} Y \in T_pM \,\middle|\, (\phi^* g)_p(d\phi^{-1}_{\phi(p)} Y, d\phi^{-1}_{\phi(p)} Y) < r^2\right\}\\ &= \left\{d\phi^{-1}_{\phi(p)} Y \in T_pM \,\middle|\, g_{\phi(p)}(Y,Y) < r^2\right\}\\ &= d\phi^{-1}_{\phi(p)} \left\{Y \in T_pM \,\middle|\, g_{\phi(p)}(Y,Y) < r^2\right\}\\ &= d\phi^{-1}_{\phi(p)} B^b_r(0_{\phi(p)}).\end{aligned}$$

Now using the fact that the exponential map of a pullback metric is the pullback of the exponential map of the original metric, we obtain

$$\begin{aligned}\mathrm{inj}_p(\phi^* g) &= \sup\left\{r > 0 \,\middle|\, \exp_p^{\phi^* g}\colon B_r^{\phi^* g}(0_p) \to M \text{ is a diffeomorphism onto its image}\right\}\\ &= \sup\left\{r > 0 \,\middle|\, (\phi^* \exp^g)_p\colon B_r^{\phi^* g}(0_p) \to M \text{ is a diffeo. onto its image}\right\}\\ &= \sup\left\{r > 0 \,\middle|\, (\phi^* \exp^g)_p\colon d\phi^{-1}_{\phi(p)} B^g_r(0_{\phi(p)}) \to M \text{ is a diffeo. onto its image}\right\}\\ &= \sup\left\{r > 0 \,\middle|\, \exp_p^g\colon B^g_r(0_{\phi(p)}) \to M \text{ is a diffeo. onto its image}\right\}\\ &= \mathrm{inj}_{\phi(p)}\, g\end{aligned}$$

as claimed.

3. If $B^g_{\mathrm{inj}_p(g)}(p) \subset U$, then $\mathrm{inj}_{g|_U}(p) \leq \mathrm{inj}_g(p)$. Otherwise $\mathrm{inj}_{g|_U}(p) \leq \mathrm{inj}_g(p)$ since at least one g-geodesic must leave U before reaching the injectivity radius of g at p. □

It is a fundamental result of Riemannian geometry that, for a fixed metric, the pointwise injectivity radius function $\mathrm{inj}_g \colon M \to \mathbb{R}$ is continuous. Paul Ewing Ehrlich showed that, on compact manifolds, we can say more. Recall that a function $f \colon X \to \mathbb{R}$ from a topological space X is *lower semicontinuous* if it is continuous with respect to the left order topology on $\mathbb{R}$, i.e. if for any point $x \in X$ and any $\epsilon > 0$, there is an open neighbourhood $U = U(x, \epsilon)$ of x such that $f|_{U_x} > f(x) - \epsilon$. All continuous functions are lower semicontinuous and it is an easy exercise to show that the pointwise minimum of lower semicontinuous functions is again lower semicontinuous.

Proposition 7.30 ([Ehr74, Chapter 6, Theorem 1])**.** *Let M be a compact manifold. Then the map*

$$\mathrm{inj} \colon C^2(\mathrm{Met}\,(M)) \times M \to \mathbb{R}$$

is lower semicontinuous.

The situation is more delicate in case of noncompact manifolds but one can localize the previous result.

Corollary 7.31 ([Ehr74, Chapter 6, Remark after Theorem 1])**.** *Let (M, g_0) be a complete noncompact Riemannian manifold with $\mathrm{inj}_{g_0}(p_0) < \infty$ at some $p_0 \in M$. Moreover, let $K := \overline{B^{g_0}_{\mathrm{inj}_{g_0}(p_0)}(p_0)}$. Then the function $\mathrm{inj} \colon C^2(\mathrm{Met}\,(M)\,|_K) \times K \to \mathbb{R}$ is lower semicontinuous at (g_0, p_0).*

Remark 7.32. *If (M, g_0) is a complete connected Riemannian manifold with $\mathrm{inj}\, g_0 = \infty$, then the exponential map shows that M is diffeomorphic to $\mathbb{R}^{\dim M} = \mathrm{Cone}(S^{\dim M - 1})$. This means in turn that all asymptotically conical metrics living on M are necessarily asymptotically Euclidean. These metrics, having been thoroughly investigated elsewhere, are not our main concern here.*

Next, we prove a general lemma about lower semicontinuous functions and "restrictions" to compact subsets.

Lemma 7.33. *Let (M, g) be a Riemannian manifold and let $(E, \langle\cdot,\cdot\rangle, \nabla)$ be a vector bundle on M with inner product and compatible connection. Furthermore, let $K \subset M$ be a compact submanifold. Let $F \colon C^k(E) \times M \to \mathbb{R}$ be a lower semicontinuous function. Then $f \colon C^k(E|_K) \times K \to \mathbb{R}, (\sigma, p) \mapsto F(\sigma, p)$ is lower semicontinuous.*

Proof. Let $\iota \colon K \to M$ denote the embedding. This map induces the restriction map $\iota^* \colon C^k(E) \to C^k(E|_K)$, which is continuous by the following calculation

$$\|\iota^*\sigma\|_{C^k(E|_K)} = \sup\left\{ \sum_{a=0}^{k} |\nabla^a \sigma|_p \,\middle|\, p \in K \right\} \leq \sup\left\{ \sum_{a=0}^{k} |\nabla^a \sigma|_p \,\middle|\, p \in M \right\} = \|\sigma\|_{C^k(E)}.$$

Since $f = F \circ (\iota^*, \iota)$ all the maps on the left-hand side are lower semicontinuous, f itself is lower semicontinuous. □

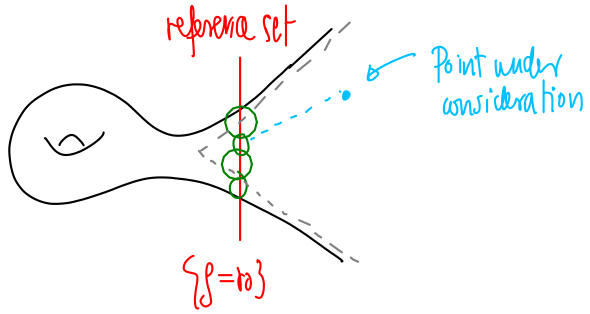


Figure 7.2: Injectivity radius in the proof of Proposition 7.34

We have arrived at the main statement of this section.

Proposition 7.34. *The injectivity radius of an asymptotically conical metric grows eventually at least linearly. More precisely, if g_{ac} is an asymptotically conical metric with $\mathrm{inj}_{g_{\mathrm{ac}}}(p) \neq \infty$ for some any $p \in M$, then there are constants $R, C > 0$ such that the pointwise injectivity radius satisfies*

$$\mathrm{inj}_{g_{\mathrm{ac}}}|_{M\setminus \mathrm{Core}(R)} \geq C\rho.$$

The constants R and C depend on g_{ac} via R_{asy} and the asymptotic constants for g_{ac}.

Proof. Let g_{cone} be the cone metric to which the metric g_{ac} is asymptotic on $\mathrm{Cone}(R_{\mathrm{asy}})$. Fix a reference radius $r_0 > R_{\mathrm{asy}}$. Now by continuity of $\mathrm{inj}_{g_{\mathrm{cone}}} : M \to \mathbb{R}$, we get a finite positive constant

$$a := \min\left\{\min\left\{ \mathrm{inj}_{g_{\mathrm{cone}}}(p) \,\middle|\, \rho(p) = r_0\right\}, R_{\mathrm{asy}} - r_0\right\} > 0.$$

Consider the restriction f of function $\min\{\mathrm{inj}, a\}$ to $C^2(\mathrm{Met}\,(M)\,|_K) \times K \to \mathbb{R}$ where $K := \{p \in M \,|r_0 - a \leq \rho(p) \leq r_0 + a\}$. Note that $K \subset M \setminus \mathrm{Core}(R_{\mathrm{asy}})$ by construction.

Let $\epsilon > 0$. By lower semicontinuity of the injectivity radius map, for all $p \in \{q \in M \,|\rho(q) = r_0\}$, there is a $\delta(\epsilon, p) > 0$ such that whenever $\|g - g_{\mathrm{cone}}\|_{C^2} < \delta$ and $d(p, q) < \delta$, we have $\mathrm{inj}_g(q) > \mathrm{inj}_{g_{\mathrm{cone}}}(p)$. In particular, we may cover the reference slice as $\{p \,|\rho(p) = r_0\} = \cup_{p\in\{p\,|\rho(p)=r_0\}} B_p(\delta(\epsilon, p))$. Since the reference slice is compact, there is a finite subcover. Let $\delta_{\min}$ and $\delta_{\max}$ denote the smallest and the biggest δ from this finite subcover, respectively.

Choosing $\epsilon := a/2$, we obtain for any metric g with $\|g - g_{\mathrm{cone}}\|_{C^2(g_{\mathrm{cone}})} < \delta_{\min}$ that

$$\mathrm{inj}_q(g) > \mathrm{inj}_p(g_{\mathrm{cone}}) \geq a - \epsilon = a/2.$$

Let $p \in M$ with $\rho(p) \geq \left(\frac{2C}{\delta_{\min}}\right)^{1/\tau} r_0 =: R$ where C denotes the constant from Lemma 7.28 for $K := \{p \in M \,|\rho(p) = r_0\}$. Moreover, let $t \in \mathbb{R}$ be such that $p_0 := \mathrm{Fl}^Z_{-t}\, p \in K$. Since we have $\rho(p) = r_0 e^t$, thus for the metric $g := e^{2t} g_{\mathrm{ac}}$, we obtain that

$$\|e^{2t} g_{\mathrm{ac}} - g_{\mathrm{cone}}\|_{C^2,K} = \|g_{\mathrm{ac}} - g_{\mathrm{cone}}\|_{C^2,\mathrm{Fl}^Z_t(K)} \leq Ce^{-\tau t}$$

$$= C \left(\frac{\rho(p)}{r_0} \right)^{-\tau} = \frac{\delta_{\min}}{2} < \delta_{\min},$$

therefore by the calculation rules for the injectivity radius derived in Lemma 7.29, we obtain

$$\begin{aligned} \frac{r_0}{\rho(p)} \operatorname{inj}_g(p) &= e^{-t} \operatorname{inj}_g(p) = \operatorname{inj}_{e^{-2t} g_{\mathrm{ac}}}(p) \geq \operatorname{inj}_{e^{-2t} g_{\mathrm{cone}}}(p) = \operatorname{inj}_{(\mathrm{Fl}^Z_{-t})^* g_{\mathrm{cone}}}(p) \\ &= \operatorname{inj}_{g_{\mathrm{cone}}}(\mathrm{Fl}^Z_{-t}\, p) = \operatorname{inj}_{g_{\mathrm{cone}}}(p_0) \geq a/2. \end{aligned}$$

Therefore

$$\operatorname{inj}_g(p) \geq \frac{a}{2r_0} \rho(p)$$

for $\rho(p) > R$, as claimed. □

Chapter 8

The DeTurck map and gauged metrics

8.1 The DeTurck map

The Ricci flow has some inconvenient analytic properties which can be remedied by adding an extra term to the evolution equation. This term – called DeTurck's term – is the Lie derivative of the metric with respect to a certain vector field. This vector field depends on the current metric and an arbitrary reference metric, and can be thought of as a first-order nonlinear differential operator

$$V\colon \operatorname{Met}(M) \times \operatorname{Met}(M) \to \mathfrak{X}(M), \quad (g_1, g_2) \mapsto \left(\delta^{g_1} g_2 - \frac{1}{2} d \operatorname{Tr}_{g_1} g_2\right)^{\sharp_{g_2}}.$$

In this thesis, we will call this operator the DeTurck map. In the evolution equation of the Ricci–DeTurck flow, the second argument if a fixed reference metric and the first argument is the actually evolving family of metrics. Nonetheless, it is worth investigating properties of the full DeTurck map.

Since the DeTurck map is defined manifestly invariantly, it is now wonder it behaves decently in the presence of diffeomorphisms.

Lemma 8.1. *The DeTurck map is equivariant under diffeomorphisms. More precisely, for any diffeomorphism $\phi\colon M \to N$ and any two metrics $g_1, g_2 \in \operatorname{Met}(N)$, we have*

$$V_M(\phi^* g_1, \phi^* g_2) = d\phi^{-1} V_N(g_1, g_2),$$

where V_N and V_M denote the DeTurck maps on N and M, respectively.

Proof. Let $\{e_i \,|i \in I\}$ be a g_1-orthonormal frame of TN. Note that $E_i := d\phi e_i$ define a $\phi^* g_1$-orthonormal frame of TM. Consequently,

$$\begin{aligned} \operatorname{Tr}_{\phi^* g_1} \phi^* g_2 &= \sum_{i \in I} (\phi^* g_2)(e_i, e_i) \\ &= \sum_{i \in I} g_2(d\phi e_i, d\phi e_i) \circ \phi \end{aligned}$$

$$\begin{aligned}
&= \sum_{i\in I} g_2(E_i, E_i)\circ\phi \\
&= \operatorname{Tr}_{g_1} g_2 \circ \phi \\
&= \phi^*(\operatorname{Tr}_{g_1} g_2),
\end{aligned}$$

and therefore $d(\operatorname{Tr}_{\phi^* g_1}\phi^* g_2) = d(\phi^*(\operatorname{Tr}_{g_1} g_2)) = \phi^*(d\operatorname{Tr}_{g_1} g_2)$. Moreover, for any vector field $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
(\delta^{\phi^* g_1}(\phi^* g_2))(X) &= -\sum_{i\in I}\left(\nabla^{\phi^* g_1}_{e_i}(\phi^* g_2)\right)(e_i, X) \\
&= -\sum_{i\in I}\left((\phi^*\nabla^{g_1})_{e_i}(\phi^* g_2)\right)(e_i, X) \\
&= -\sum_{i\in I}\left((\phi^*\nabla^{g_1})_{e_i}(\phi^* g_2)\right)(e_i, X) \\
&= -\sum_{i\in I}\left(\phi^*(\nabla^{g_1})_{d\phi e_i} g_2\right)(e_i, X) \\
&= -\sum_{i\in I}\left(\nabla^{g_1}_{d\phi e_i} g_2\right)(d\phi e_i, d\phi X)\circ\phi \\
&= -\sum_{i\in I}\left(\nabla^{g_1}_{E_i} g_2\right)(E_i, d\phi X)\circ\phi \\
&= \delta^{g_1} g_2(d\phi X)\circ\phi \\
&= \phi^*(\delta^{g_1} g_2)(X).
\end{aligned}$$

The claim now follows from the fact that for any $\lambda \in T^*N$, any metric $g \in \operatorname{Met}(N)$ and any diffeomorphism $\phi\colon M \to N$, we have $(\phi^*\lambda)^{\sharp_{\phi^* g}} = d\phi^{-1}\lambda^{\sharp_g}$. ☐

For calculations, it is useful to have a local expression of the DeTurck vector field.

Lemma 8.2. *In local coordinates, the DeTurck vector field reads*

$$V(g_1, g_2)| = (g_1)^{ij}(\Gamma(g_1)_{ij}{}^k - \Gamma(g_2)_{ij}{}^k)\partial_k.$$

Proof. Since both sides of the statement are tensorial, it suffices to check the statement in a single chart around each point. Let us take g_1-normal coordinates around a point $p \in M$ and calculate at p

$$\begin{aligned}
(g_1)^{ij}(\Gamma(g_1)_{ij}{}^k - \Gamma(g_2)_{ij}{}^k)\partial_k &= \delta^{ij}(0 - \Gamma(g_2)_{ij}{}^k)\partial_k \\
&= -\sum_{i\in I}\Gamma(g_2)_{ii}{}^k\partial_k \\
&= -\sum_{i\in I}\frac{1}{2}(\partial_i(g_2)_{ia} - \partial_a(g_2)_{ii} + \partial_i(g_2)_{ia})(g_2)^{ak}\partial_k \\
&= \sum_{i\in I}\left(-\partial_i(g_2)_{ia}(g_2)^{ak}\partial_k - \frac{1}{2}\partial_a\operatorname{Tr}_{g_1} g_2(g_2)^{ak}\partial_k\right) \\
&= \left(\delta^{g_1} g_2 - \frac{1}{2} d\operatorname{Tr}_{g_1} g_2\right)^{\sharp_{g_2}},
\end{aligned}$$

showing the claim. ☐

In order to do analysis, we will need to consider maps induced by the DeTurck map on weighted Sobolev spaces.

Proposition 8.3. *Let g_b be an asymptotically conical metric on M. Moreover, let $p > 1$ and let $k \in \mathbb{N}$ with $pk > \dim M + 1$ and $\delta \in \mathbb{R}$. Then there exists a neighbourhood U of g_b in $W^{k,p}_\delta(S^2T^*M, g_b)$ such that the map*

$$W \colon U \to W^{k-1,p}_{\delta-1}(TM, g_b), g \mapsto W(g) := V(g, g_b)$$

is well defined.

Proof. The proof is based on the weighted Hölder inequality 7.18 and the structure of the DeTurck map. □

The linearization of the DeTurck map

Lemma 8.4. *The linearization of the DeTurck map at $(g_1, g_2) \in \mathrm{Met}\,(M) \times \mathrm{Met}\,(M)$ is locally given by*

$$\begin{aligned} DV_{(g_1,g_2)}(h_1,h_2) = &-(g_1)^{ia}(h_1)_{ap}(g_1)^{pj}(\Gamma(g_1)_{ij}{}^k - \Gamma(g_2)_{ij}{}^k))\partial_k \\ &+ \tfrac{1}{2}(g_2)^{ij}((\nabla^{g_1}_{\partial_i}h_1)_{jl} - (\nabla^{g_1}_{\partial_l}h_1)_{ij} + (\nabla^{g_1}_{\partial_j}h_1)_{il})(g_1)^{kl}\partial_k \\ &- \tfrac{1}{2}(g_2)^{ij}((\nabla^{g_2}_{\partial_i}h_2)_{jl} - (\nabla^{g_2}_{\partial_l}h_2)_{ij} + (\nabla^{g_2}_{\partial_j}h_2)_{il})(g_2)(g\_2)V(g_2)^{kl}\partial_k. \end{aligned}$$

Proof. We may find the linearization componentwise.

- First consider, for fixed $g_1 \in \mathrm{Met}\,(M)$, the map $Z\colon \mathrm{Met}\,(M) \to \mathfrak{X}(M), g \mapsto V(g_1, g)$. Its Gateaux derivative at $g \in \mathrm{Met}\,(M)$ is given by

$$\begin{aligned} DZ_g(h) &= \frac{d}{d\lambda}\bigg|_{\lambda=0}\Big((g_1)^{ij}(\Gamma(g)_{ij}{}^k - \Gamma(g+\lambda h)_{ij}{}^k)\partial_k\Big) \\ &= -(g_1)^{ij}\frac{d}{d\lambda}\bigg|_{\lambda=0}\Big(\Gamma(g+\lambda h)_{ij}{}^k\Big)\,\partial_k. \end{aligned}$$

The Gateaux derivative of the Christoffel symbols can be calculated using the coordinate version of the Koszul formula.

$$\begin{aligned} \frac{d}{d\lambda}\bigg|_{\lambda=0}\Gamma(g+\lambda h)_{ij}{}^k &= \frac{d}{d\lambda}\bigg|_{\lambda=0}\tfrac{1}{2}(g+\lambda h)^{kl}(\partial_i(g+\lambda h)_{jl} - \partial_l(g+\lambda h)_{ij} + \partial_j(g+\lambda h)_{il}) \\ &= \frac{d}{d\lambda}\bigg|_{\lambda=0}\Big(\tfrac{1}{2}(g+\lambda h)^{kl}\Big)\,(\partial_i(g+\lambda h)_{jl} - \partial_l(g+\lambda h)_{ij} + \partial_j(g+\lambda h)_{il})|_{\lambda=0} \\ &\quad + \tfrac{1}{2}(g+\lambda h)^{kl}|_{\lambda=0}\frac{d}{d\lambda}\bigg|_{\lambda=0}(\partial_i(g+\lambda h)_{jl} - \partial_l(g+\lambda h)_{ij} + \partial_j(g+\lambda h)_{il}) \\ &= -\tfrac{1}{2}g^{ka}h_{ap}g^{pl}(\partial_i g_{jl} - \partial_l g_{ij} + \partial_j g_{il}) + \tfrac{1}{2}g^{kl}(\partial_i h_{jl} - \partial_l h_{ij} + \partial_j h_{il}), \end{aligned}$$

where we used the fact that $\frac{d}{d\lambda}\big|_{\lambda=0}(g+\lambda h)^{kl} = -g^{ka}h_{ap}g^{pl}$, which can be derived from the identity $(g+\lambda h)^{ka}(g+\lambda h)_{ap} = \delta^k_p$. Consequently, we have

$$DZ_g(h) = -(g_1)^{ij}\left(\frac{d}{d\lambda}\bigg|_{\lambda=0}\Gamma(g+\lambda h)_{ij}{}^k\right)\partial_k$$

$$= \tfrac{1}{2}(g_1)^{ij}\left(h^{kl}(\partial_i g_{jl} - \partial_l g_{ij} + \partial_j g_{il}) - g^{kl}(\partial_i h_{jl} - \partial_l h_{ij} + \partial_j h_{il})\right)$$

where we used the metric g to raise indices of h. This form is not manifestly invariant; however, we can rewrite the partial derivatives using the Levi-Civita connection of g to obtain

$$\begin{aligned}
DZ_g(h) &= \tfrac{1}{2}(g_1)^{ij}\left(h^{kl}(\partial_i g_j l - \partial_l g_i j + \partial_j g_{il}) - g^{kl}(\partial_i h_j l - \partial_l h_i j + \partial_j h_{il})\right)\partial_k \\
&= \tfrac{1}{2}(g_1)^{ij}(+h^{kl}(+\nabla_i g_{jl} + \Gamma(g)_{ij}{}^p g_{pl} + \Gamma(g)_{il}{}^p g_{jp} \\
&\qquad - \nabla_l g_{ij} - \Gamma(g)_{li}{}^p g_{pj} - \Gamma(g)_{lj}{}^p g_{ip} \\
&\qquad + \nabla_j g_{il} + \Gamma(g)_{ji}{}^p g_{pl} + \Gamma(g)_{jl}{}^p g_{ip}) \\
&\quad - g^{kl}(+\nabla_i h_{jl} + \Gamma(g)_{ij}{}^p h_{pl} + \Gamma(g)_{il}{}^p h_{jp} \\
&\qquad - \nabla_l h_{ij} - \Gamma(g)_{li}{}^p h_{pj} - \Gamma(g)_{lj}{}^p h_{ip} \\
&\qquad + \nabla_j h_{il} + \Gamma(g)_{ji}{}^p h_{pl} + \Gamma(g)_{jl}{}^p h_{ip}))\partial_k \\
&= -\tfrac{1}{2}(g_1)^{ij}((\nabla^g_{\partial_i} h)_{jl} - (\nabla^g_{\partial_l} h)_{ij} + (\nabla^g_{\partial_j} h)_{il})g^{kl}\partial_k,
\end{aligned}$$

where we used that expressions like $h^{kl}g_{pl}$ are symmetric under exchanging g and h.

- Now consider, with some fixed g_2, the map $W\colon \operatorname{Met}(M) \to \mathfrak{X}(M), g \mapsto V(g, g_2)$. Its Gateaux derivative at the metric $g \in \operatorname{Met}(M)$ is given by

$$\begin{aligned}
DW_g(h) &= \frac{d}{d\lambda}\bigg|_{\lambda=0}((g+\lambda h)^{ij}(\Gamma(g+\lambda h)_{ij}{}^k - \Gamma(g_2)_{ij}{}^k))\partial_k \\
&= \frac{d}{d\lambda}\bigg|_{\lambda=0}((g+\lambda h)^{ij})(\Gamma(g+\lambda h)_{ij}{}^k - \Gamma(g_2)_{ij}{}^k))|_{\lambda=0}\partial_k \\
&\quad + ((g+\lambda h)^{ij})|_{\lambda=0}\,\frac{d}{d\lambda}\bigg|_{\lambda=0}(\Gamma(g+\lambda h)_{ij}{}^k - \Gamma(g_2)_{ij}{}^k))\partial_k \\
&= -g^{ia}h_{ap}g^{pj}(\Gamma(g)_{ij}{}^k - \Gamma(g_2)_{ij}{}^k))\partial_k + g^{ij}\,\frac{d}{d\lambda}\bigg|_{\lambda=0}\left(\Gamma(g+\lambda h)_{ij}{}^k\right)\partial_k.
\end{aligned}$$

The second term is quite like the expression $-DZ_g(h) = -(g_1)^{ij}\,\frac{d}{d\lambda}\big|_{\lambda=0}\left(\Gamma(g+\lambda h)_{ij}{}^k\right)\partial_k$, except for g appearing instead of g_1. This means that we can spare the calculation and simply substitute g for g_1 in the expression for $-DZ_g(h)$ to obtain

$$\begin{aligned}
DW_g(h) &= -g^{ia}h_{ap}g^{pj}(\Gamma(g)_{ij}{}^k - \Gamma(g_2)_{ij}{}^k))\partial_k \\
&\quad + \tfrac{1}{2}g^{ij}((\nabla^g_{\partial_i} h)_{jl} - (\nabla^g_{\partial_l} h)_{ij} + (\nabla^g_{\partial_j} h)_{il})g^{kl}\partial_k.
\end{aligned}$$

This formula is not manifestly invariant but we will not bother to rewrite it in a manifestly invariant way, cf. Remark 8.8.

- For the Gateaux derivative of the full DeTurck map at $(g_1, g_2) \in \operatorname{Met}(M) \times \operatorname{Met}(M)$, we obtain the formula in the claim by $DV_{(g_1,g_2)}(h_1, h_2) = DZ_{g_1}(h_2) + DW_{g_2}(h_1)$. □

Corollary 8.5. *The symbols of the operators in Lemma 8.4 are*

$$\sigma(DZ_g)(\lambda \otimes h) = (-\iota_{\lambda^{\#_{g_1}}} h + \tfrac{1}{2}(\operatorname{Tr}_{g_1} h)\lambda)^{\#_g},$$

$$\sigma(DW_g)(\lambda \otimes h) = (+\iota_{\lambda^{\#_g}} h - \tfrac{1}{2}(\operatorname{Tr}_g h)\lambda)^{\#_g},$$
$$\sigma(DV_{(g_1,g_2)})(\lambda \otimes (h_1, h_2)) = (+\iota_{\lambda^{\#_g}} h_1 - \tfrac{1}{2}(\operatorname{Tr}_g h_1)\lambda - \iota_{\lambda^{\#_{g_1}}} h_2 + \tfrac{1}{2}(\operatorname{Tr}_{g_1} h_2)\lambda)^{\#_g}.$$

In particular, if $\dim M > 1$*, neither the DeTurck map nor any of the maps obtained from it by fixing one of its arguments is elliptic.*

Proof. The claim about the symbols follows from the fact that covariant derivatives have the identity as symbols. To show that the operator DZ_g is not elliptic, consider a nonzero $\lambda \in T^*M$. Now, there exists a nonzero $T \in S^2(T^*M)$ with $\iota_{\lambda^{\#_{g_1}}} T = 0$. (We can take e.g. $T := \mu \otimes \mu$ where $g_1(\lambda, \mu) = 0$.) Then $h := \frac{\operatorname{Tr}_{g_1} T}{|\lambda|^2_{g_1}} \lambda \otimes \lambda + T \neq 0$ but $\sigma(DZ_g)(\lambda \otimes h) = 0$.

Showing that W is not elliptic goes along the same lines.

The above discussion shows that we can find, for each nonzero λ, a nonzero h_V in the kernel of the symbol of V and a nonzero h_W in the kernel of the symbol of W. Then (h_1, h_2) is a suitable tensor to refute ellipticity. □

Remark 8.6. *For the sake of completeness, we show that all three operators in Corollary 8.5 are, as a matter of fact, elliptic if* $\dim M = 1$*. Indeed, note that* $\sigma(DW_g)(\lambda \otimes h) = 0$ *if and only if* $\iota_{\lambda^{\#_g}} h - \frac{1}{2}(\operatorname{Tr}_g h)\lambda = 0$*. Let us choose a local coordinate* x*, and let* $g = \bar{g} dx \otimes dx$*,* $h = \bar{h} dx \otimes dx$*,* $\lambda = \bar{\lambda} dx$ *for some functions* $\bar{g}, \bar{h}$ *and* $\bar{\lambda}$*. Then* $(dx)^{\sharp_g} = \frac{1}{\bar{g}}\partial_x$ *and* $\operatorname{Tr}_g h = \frac{\bar{h}}{\bar{g}}$*, consequently*

$$\iota_{\lambda^{\#_g}} h - \frac{1}{2}(\operatorname{Tr}_g h)\lambda = \frac{1}{2}\frac{\bar{h}}{\bar{g}}\bar{\lambda} dx = \frac{1}{2}\frac{\bar{h}}{\bar{g}}\lambda.$$

If $\lambda \neq 0$*, then the map* $h \mapsto \frac{1}{2}\frac{\bar{h}}{\bar{g}}\lambda$ *is an isomorphism, showing that* W *is elliptic.*

Similarly, one can show that if $\dim M = 1$*, the operators* W *and* V *are elliptic.*

Corollary 8.7. *If* $g_1 = g_2 = g$*, we have the following formulae for the linearizations of the operators from Lemma 8.4*

$$DZ_g(h) = \delta^g h + \tfrac{1}{2}\operatorname{grad}_g \operatorname{Tr}_g h$$
$$DW_g(h) = -\delta^g h - \tfrac{1}{2}\operatorname{grad}_g \operatorname{Tr}_g h$$
$$DV_{(g,g)}(h_1, h_2) = \delta^g(h_2 - h_1) + \tfrac{1}{2}\operatorname{grad}_g \operatorname{Tr}_g(h_2 - h_1)$$

for any $h, h_1, h_2 \in S^2(T^*M)$*.* □

Remark 8.8. *Note that, for* $g = g_1 = g_2$*, we obtain* $DW_g = -DV_g$*. This means in particular that in this special case we may show that* DW_g *is an isomorphism by showing that* DZ_g *is an isomorphism, and vice versa.*

There is a nice relation between the Hodge Laplacian and the DeTurck map.

Lemma 8.9. *If* $g_1 = g_2 = g$*, then we have for any vector field* $X \in \mathfrak{X}(M)$ *any any metric* $g \in \operatorname{Met}(M)$

$$DZ_g(\mathcal{L}_X g) = \Delta_g X + \operatorname{Ric}^g = -DW_g(\mathcal{L}_X g).$$

Proof. Since DZ_g is obtained via the musical isomorphism, it is practical to calculate $(DZ_g(\mathcal{L}_X g))^{\flat_g}$ first. Let $\{e_i \,|i \in I\}$ be a g-normal frame of TM around $p \in M$ and let $V \in \mathfrak{X}(M)$ be an arbitrary vector field. Then we have

$$\begin{aligned}
\delta^g(\mathcal{L}_X g)_p(V_p) &= -\sum_{i\in I} \nabla^g_{e_i}(\mathcal{L}_X g)(e_i, V) \\
&= \sum_{i\in I} -e_i((\mathcal{L}_X g)(e_i, V)) + (\mathcal{L}_X g)(\nabla^g_{e_i} e_i, V) + (\mathcal{L}_X g)(e_i, \nabla^g_{e_i} V) \\
&= \sum_{i\in I} -e_i(g(\nabla^g_{e_i} X, V) + g(e_i, \nabla^g_V e_i)) + (\mathcal{L}_X g)(\nabla^g_{e_i} e_i, V) + (\mathcal{L}_X g)(e_i, \nabla^g_{e_i} V) \\
&= \sum_{i\in I} -e_i(g(\nabla^g_{e_i} X, V) + g(e_i, \nabla^g_V e_i)) + (\mathcal{L}_X g)(\nabla^g_{e_i} e_i, V) + (\mathcal{L}_X g)(e_i, \nabla^g_{e_i} V) \\
&= \sum_{i\in I} -g(\nabla^g_{e_i}\nabla^g_{e_i} X, V) - g(\nabla^g_{e_i} X, \nabla^g_{e_i} V) - g(\nabla^g_{e_i} e_i, \nabla^g_V e_i) - g(e_i, \nabla^g_{e_i}\nabla^g_V e_i) \\
&\qquad + g(\nabla^g_{\nabla^g_{e_i} e_i} X, V) + g(\nabla^g_{e_i} e_i, \nabla^g_V X) + g(\nabla^g_{e_i} X, \nabla^g_{e_i} V) + g(e_i, \nabla^g_{\nabla^g_{e_i} V} X) \\
&= \sum_{i\in I} g(-\nabla^{g,2}_{e_i,e_i} X, V) - g(e_i, \nabla^{g,2}_{e_i,V} e_i) \\
&= g(\Delta_g X, V) - \sum_{i\in I} g(e_i, \nabla^{g,2}_{e_i,V} e_i)
\end{aligned}$$

where we used the well-known formula $(\mathcal{L}_X g)(Y,Z) = g(\nabla^g_Y X, Z) + g(Y, \nabla^g_Z X)$ for $X, Y, Z \in \mathfrak{X}(M)$. Moreover, we have at the point p

$$\begin{aligned}
\tfrac{1}{2} d(\mathrm{Tr}_g\, \mathcal{L}_X g)(V) &= \tfrac{1}{2} d\left(\sum_{i\in I} (\mathcal{L}_X g)(e_i, e_i)\right)(V) \\
&= \tfrac{1}{2} d\left(\sum_{i\in I} g(\nabla^g_{e_i} X, e_i) + g(e_i, \nabla^g_{e_i} X)\right)(V) \\
&= d\left(g(e_i, \nabla^g_{e_i} X)\right)(V) \\
&= \sum_{i\in I} g(\nabla^g_X e_i, \nabla^g_{e_i} X) + g(e_i, \nabla^g_V \nabla^g_{e_i} X, e_i) \\
&= \sum_{i\in I} g(e_i, \nabla^{g,2}_{V,e_i} X),
\end{aligned}$$

where we used that $\{e_i \,|i \in I\}$ is a normal frame at p. Note that the order of the second covariant derivatives in the two results are different, which hints at a curvature term. In fact, we obtain by adding the previous two results that

$$\begin{aligned}
(DZ_g(\mathcal{L}_X g))^{\flat_g}(V) &= \delta^g(\mathcal{L}_X g)_p(V_p) + \tfrac{1}{2} d(\mathrm{Tr}_g\, \mathcal{L}_X g)(V) \\
&= g(\Delta_g X, V) - \sum_{i\in I} g(e_i, \nabla^{g,2}_{e_i,V} X) + \sum_{i\in I} g(e_i, \nabla^{g,2}_{V,e_i} X) \\
&= g(\Delta_g X, V) - \sum_{i\in I} g(e_i, R^g(V, e_i) X) \\
&= g(\Delta_g X, V) + \sum_{i\in I} g(R^g(e_i, V) X, e_i)
\end{aligned}$$

$$
\begin{aligned}
&= g(\Delta_g X, V) + (\overset{\circ}{R^g} g)(V, X) \\
&= g(\Delta_g X, V) + \mathrm{Ric}^g(V, X).
\end{aligned}
$$

The claim follows by applying the musical isomorphism $\flat_g$. □

8.2 Gauging

With help of the DeTurck vector field, we introduce a gauging condition.

Definition 8.10. *We call metrics in the set*

$$
\mathcal{F}_{g_0} := \left\{ g \in \mathrm{Met}\,(M) \;\middle|\; -2\,\mathrm{Ric}^g + \mathcal{L}_{V(g,g_0)} g = 0 \right\}
$$

metrics gauged with respect to the reference metric $g_0 \in \mathrm{Met}\,(M)$. Furthermore, for a compact set $K \subset M$, we introduce the notation

$$
\mathcal{F}_{g_0}^{M \setminus K} := \left\{ g \in \mathrm{Met}\,(M) \;\middle|\; -2\,\mathrm{Ric}^g + \mathcal{L}_{V(g,g_0)} g_0 = 0 \text{ on } M \setminus K \right\}.
$$

Evidently, $\mathcal{F}_{g_0} = \mathcal{F}_{g_0}^{M}$.

The gaugedness condition is equivalent to a a second-order quasilinear partial differential equation. A tedious calculation leads to the following result.

Lemma 8.11 ([Shi89, Lemma 2.1])**.** *The condition $g \in \mathcal{F}_{g_0}$ is equivalent to*

$$
\begin{aligned}
0 = {} & g^{ab} \nabla^{g_0,2}_{ab} g_{ij} \\
& + g^{ab} g^{pq} \left(\frac{1}{2} \nabla^{g_0}_i g_{pa} \nabla^{g_0}_j g_{qb} + \nabla^{g_0}_a g_{jp} \nabla^{g_0}_q g_{ib} \right) \\
& - g^{ab} g^{pq} \left(\nabla^{g_0}_a g_{jp} \nabla^{g_0}_b g_{iq} - \nabla^{g_0}_j g_{pa} \nabla^{g_0}_b g_{iq} - \nabla^{g_0}_i g_{pa} \nabla^{g_0}_b g_{jq} \right) \\
& - g^{kl} g_{ip} (g_0)^{pq} R^{g_0}_{jklq} - g^{kl} g_{jp} (g_0)^{pq} R^{g_0}_{iklq}.
\end{aligned}
$$

It is useful to know the coarse structure of the equation in Lemma 8.11 in terms of the difference tensor $h := g - g_0$.

Corollary 8.12 (cf. [DK20, Equation (5)])**.** *Fix a metric $g_0 \in \mathrm{Met}\,(M)$ and suppose $h \in \Gamma^\infty\left(S^2 T^* M\right)$ such that $g_0 + h \in \mathrm{Met}\,(M)$. Then the condition $g_0 + h \in \mathcal{F}_{g_0}$ is equivalent to a second-order quasilinear partial differential equation*

$$
\begin{aligned}
0 = {} & (g_0 + h)^{ab} \nabla^{g_0,2}_{ab} g_{ij} \\
& + (g_0 + h)^{ab} (g_0 + h)^{pq} \left(\frac{1}{2} \nabla^{g_0}_i h_{pa} \nabla^{g_0}_j h_{qb} + \nabla^{g_0}_a h_{jp} \nabla^{g_0}_q h_{ib} \right) \\
& - (g_0 + h)^{ab} (g_0 + h)^{pq} \left(\nabla^{g_0}_a h_{jp} \nabla^{g_0}_b h_{iq} - \nabla^{g_0}_j h_{pa} \nabla^{g_0}_b h_{iq} - \nabla^{g_0}_i h_{pa} \nabla^{g_0}_b h_{jq} \right) \\
& - (g_0 + h)_{ip} (g_0)^{pq} \mathrm{Ric}^{g_0}_{jq} + (g_0 + h)_{ip} (g_0)^{pq} (g_0)^{ka} h_{ab} g^{bl} (R^{g_0})_{jklq} \\
& - (g_0 + h)_{jp} (g_0)^{pq} \mathrm{Ric}^{g_0}_{iq} + (g_0 + h)_{jp} (g_0)^{pq} (g_0)^{ka} h_{ab} g^{bl} (R^{g_0})_{iklq}
\end{aligned}
$$

which can be written structurally as

$$0 = (g_0 + h)^{-1} \circ \nabla^{g_0,2} h + (\nabla^{g_0} h) \star (\nabla^{g_0} h) - h \star R^{g_0} - (g_0 + h) \star \mathrm{Ric}^{g_0},$$

where $\star$ denotes various linear combinations of tensorial contractions the coefficients of which are covariantly constant or of the same growth rate as g_0.

Proof. Note that $g := g_0 + h$ establishes the connection between the setups of Corollary 8.12 and Lemma 8.11. Rewriting the first-order and second-order terms in the equation of Lemma 8.11 is straightforward. Since

$$\begin{aligned} g^{kl}(R^{g_0})_{jklq} &= (g_0)^{kl}(R^{g_0})_{jklq} + (g^{kl} - (g_0)^{kl})(R^{g_0})_{jklq} \\ &= \mathrm{Ric}^{g_0}_{jq} - (g_0)^{ka} h_{ab} g^{bl} (R^{g_0})_{jklq} \end{aligned}$$

and $\nabla^{g_0} g = \nabla^{g_0} h$, we may rewrite zeroth-order term of the equation in Lemma 8.11 as

$$\begin{aligned} &- (g_0 + h)^{kl}(g_0 + h)_{ip}(g_0)^{pq} R^{g_0}_{jklq} + (i \leftrightarrow j) \\ &= -(g_0 + h)_{ip}(g_0)^{pq}(g_0 + h)^{kl} R^{g_0}_{jklq} + (i \leftrightarrow j) \\ &= -(g_0 + h)_{ip}(g_0)^{pq} \left[\mathrm{Ric}^{g_0}_{jq} - (g_0)^{ka} h_{ab} g^{bl}(R^{g_0})_{jklq}\right] + (i \leftrightarrow j) \\ &= -(g_0 + h)_{ip}(g_0)^{pq} \mathrm{Ric}^{g_0}_{jq} + (g_0 + h)_{ip}(g_0)^{pq}(g_0)^{ka} h_{ab} g^{bl}(R^{g_0})_{jklq} + (i \leftrightarrow j), \end{aligned}$$

which yields the claim. Here, the notation $(i \leftrightarrow j)$ means repeating the expression in the current line with the indices i and j exchanged. □

Under certain circumstances the two terms in the gaugedness equation vanish individually. One example is the next lemma from [DK20] which we reproduce from completeness' sake.

Lemma 8.13 ([DK20, Proposition 2.6])**.** *Let (M^n, g, X) be a steady Ricci soliton, i.e. $\mathrm{Ric}^g = \mathcal{L}_X g$ for some vector field $X \in \Gamma^\infty(TM)$ on M. Then $\lim_{+\infty} |X|_g = 0$ implies $X = 0$. In particular, any gauged metric $g \in \mathcal{F}_{g_0}$ that is asymptotically conical with $\lim_{+\infty} V(g, g_0) = 0$ is Ricci-flat.*

Proof. By the contracted Bianchi identity, one has:

$$\begin{aligned} \frac{1}{2}\nabla^g R^g &= \delta^g \mathrm{Ric}^g = \delta^g \mathcal{L}_X g \\ &= \frac{1}{2}\nabla^g(\mathrm{Tr}_g \mathcal{L}_X g) + \Delta_g X + \mathrm{Ric}^g(X) \\ &= \frac{1}{2}\nabla^g R^g + \Delta_g X + \mathrm{Ric}^g(X). \end{aligned}$$

Therefore, $\Delta_g X + \mathrm{Ric}^g(X) = 0$. In particular,

$$\Delta^g_B(|X|^2_g) + X(|X|^2_g) = 2|\nabla^g X|^2_{g_0} + 2\left\langle \nabla^g_X X, X\right\rangle_g - 2\,\mathrm{Ric}^g(X, X) = 2|\nabla^g X|^2_g,$$

which establishes that the nonnegative function $|X|^2_g$ is a subsolution of the operator

$$\Delta_X := \Delta^g_B + \nabla^g_X \colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \quad u \mapsto \Delta^g_B u + X(u).$$

The use of the maximum principle then implies the result in case $\lim_{+\infty} |X|_g = 0$. □

Corollary 8.14. *If (M, g_{ac}) is a Ricci-flat asymptotically conical manifold, and $X \in \Gamma^\infty(TM)$ is a Killing field that vanishes at infinity, then $X = 0$.*

Proof. Ricci-flatness implies that the Killing equation is equivalent to the steady soliton equation. □

8.3 Setting the DeTurck vector field to zero by pullback

8.3.1 The exponential of a vector field

In this section, we assume M is a manifold and g is a complete metric on M. We will use the formalism of connector maps, cf. e.g. [Pat99].

The map object of interest in this subsection is the exponential of a vector field.

Definition 8.15. *The exponential of the vector field $X\colon M \to TM$ with respect to a connection ∇ is the map $\mathcal{E}_X^\nabla\colon M \to M, p \mapsto \exp_p^\nabla(X_p)$. We will use the notation $\mathcal{E}_X^g := \mathcal{E}_X^{\nabla^g}$, and, if the connection or metric is clear from the context, simply $\mathcal{E}_X$.*

It is important that this map is not the flow of the vector field X in general.

The map $X \mapsto \mathcal{E}_X$ is injective if X is pointwise smaller than the pointwise injectivity radius.

Lemma 8.16. *The exponential map does not map points farther away than the length of the vector field at the given point. More precisely, if d_g denotes the distance induced by the metric g, then $d_g(\mathcal{E}_X^g(p), p) \leq |X_p|_g$ for all $p \in M$. Moreover, if $p_0 \in M$, then $d_g(\mathcal{E}_X^g(p), p_0) \leq |X_p|_g + d_g(p, p_0)$ for all $p \in M$. In particular, if g is an asymptotically conical metric and $|X|_g = O(\rho)$, then there exists a constant $C < \infty$ such that $d_g(\mathcal{E}_X^g(\cdot), p_0) \leq C\rho$.*

Proof. The curve $[0,1] \to M, t \mapsto \exp_p(tX_p)$ is a geodesic connecting p and $\mathcal{E}_X^g(p)$ of length $|X_p|_g$. If $|X_p|_g < \mathrm{inj}_p(g)$, then this is the distance, otherwise the distance may be smaller.

The second claim follows from the triangle inequality. The last claim follows from the fact that ρ and the distance from a given point are comparable. □

Recall [Pat99, Definition 1.1] that the geodesic flow on (M, g) is the family of diffeomorphisms

$$G_t\colon TM \to TM, v \mapsto (\gamma_v^g)'(t),$$

where $t \geq 0$ and γ_v denotes the unique geodesic with initial condition $\gamma_v(0) = \pi_{TM}(v)$ and $(\gamma_v)'(0) = v$. Jacobi fields are closely related to the differential of the geodesic flow, as the following lemma from [Pat99, Chapters 1.3.–1.5.] shows. For $v \in TM$ and $\xi \in T_v(TM)$, let $J_\xi^{\gamma_v}$ denote the unique Jacobi vector field along the geodesic γ_v with initial condition $J_\xi^{g,\gamma_v}(0) = T_v\pi_{TM}(\xi)$ and $(J_\xi^{g,\gamma_v})'(0) = K_v(\xi)$ where K is the connector map of the Levi-Civita connection of g. We will use the simplified notation $J_\xi^{g,v} := J_\xi^{g,\gamma_v}$.

Lemma 8.17 ([Pat99, Lemma 1.40])**.** *For $t > 0$ and $v \in TM$, $\xi \in T_v(TM)$, we have $T_v(G_t)(\xi) = (J_\xi^{\gamma_v})'(t)$.* □

Based on this, we can calculate the derivative of the exponential of a vector field.

Corollary 8.18. *Let $X\colon M \to TM$ be a vector field and $p \in M$.*

1. *Then for any $v \in T_pM$, we have $T_p(\mathcal{E}_{tX})v = J^{\gamma_{X_p}}_{(v,\nabla^g_v X)}(t)$.*
2. *If the curvature tensor R^g, the vector field X and the endomorphism field $\nabla^g X$ are bounded, then the derivative of $\mathcal{E}_X$ is bounded.*

Proof. 1. Let $\theta := X_p \in T_pM$ and $\xi := (T_pX)(v) \in T_\theta TM$. Let $\gamma\colon (-\epsilon,\epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = v$. Then $z := X \circ \gamma\colon (-\epsilon,\epsilon) \to TM$ is a curve with $z(0) = X_p$ and $z'(0) = (T_pX)(v)$. By construction, $\alpha := \pi_{TM} \circ z = \gamma$ and $Z = X|_\gamma$ as a vector field along γ, therefore $K^g_\theta(\xi) = (\nabla^g_{\alpha'} Z)(0) = \nabla^g_v X$. Moreover, $T_\theta \pi_{TM}(\xi) = T_{X_p}\pi_{TM}(T_pXv) = T_p(\pi_{TM} \circ X)v = v$ since X is a section of π_{TM}. Thus $T_\theta(G_t)(\xi) = (J^\theta_{v,\nabla^g_v X}(t), {J^\theta_{v,\nabla^g_v X}}'(t))$.
Since $\mathcal{E}_X = \pi_{TM} \circ G_1 \circ X$, we obtain

$$\begin{aligned} T_p(\mathcal{E}_X)v &= T_{G_1(p)}\pi_{TM} \circ T_{X_p}G_1 \circ T_pX(v) = T_{G_1(p)}\pi_{TM}(J^\theta_{v,\nabla^g_v X}(1), {J^\theta_{v,\nabla^g_v X}}'(1)) \\ &= J^\theta_{v,\nabla^g_v X}(1), \end{aligned}$$

as claimed.

The general case follows similarly.

2. Following [Cha20], let $\gamma := \gamma_{X_p}$ and $J := J^{\gamma_{X_p}}_{v,\nabla^g_v X}$ and define the function

$$f\colon [0,\infty) \to \mathbb{R}, \quad t \mapsto |J(t)|^2 + |J'(t)|^2.$$

From the Jacobi equation, we obtain

$$\begin{aligned} f' &= 2g(J,J') + 2g(J',J'') \\ &= 2g(J,J') + 2g(J',R(J,\gamma')\gamma')) \\ &\le 2|J||J'| + 2C|J||J'||\gamma'|^2 \\ &\le (C|\gamma'|^2+1)(|J|^2+|J'|^2) = (C|X_p|^2+1)f, \end{aligned}$$

where we used the Peter–Paul inequality in the last inequality. Here C is the bound on $|R|$. From Grönwall's lemma, we obtain

$$f(t) \le f(0)e^{(C|X_p|^2+1)t},$$

and this implies in particular

$$\begin{aligned} |T_p(\mathcal{E}_X)v|^2 &= |J(1)|^2 \le f(1) \le f(0)e^{C|X_p|^2+1} \\ &= (|J(0)|^2 + |J'(0)|^2)e^{C|X_p|^2+1} \\ &= (|v|^2 + |\nabla^g_v X|^2)e^{C|X_p|^2+1} \\ &= \underbrace{e^{C|X_p|^2+1}(1+|\nabla^g X|^2)}_{\le\infty}|v|^2, \end{aligned}$$

thus showing the claim. $\square$

It is well known that on a smooth manifold, the smooth structure alone is not sufficient for the definition of the exponential map. However, the following "infinitesimal pullback" along the exponential of a vector field turns out to be a famous operation that can be defined solely in terms of the smooth structure.

Lemma 8.19. *Let S be a once continuously differentiable tensor field and let X be a once continuously differentiable vector field. Then* $\frac{d}{dt}\Big|_{t=0} (\mathcal{E}_{tX}{}^*S) = \mathcal{L}_X S$

Proof. We demonstrate the proof for 2-tensor fields; other ranks can be treated analogously. Let $p \in M$ and let V, W be two vector fields around p. Then by Corollary 8.18, we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} (\mathcal{E}_{tX}{}^*S)_p(V,W) &= \lim_{t\to 0} \frac{1}{t}\Big(S_{\mathcal{E}_{tX}(p)}(T_p(\mathcal{E}_{tX})V, T_p(\mathcal{E}_{tX})W) - S_p(V,W)\Big) \\
&= \lim_{t\to 0} \frac{1}{t}\Big(S_{\gamma_{X_p}(t)}(J^{X_p}_{V,\nabla_V X}(t), J^{X_p}_{W,\nabla_W X}(t)) - S_p(V,W)\Big) \\
&= f'(0)
\end{aligned}$$

where we define $f\colon \mathbb{R} \to \mathbb{R}, t \mapsto S_{\gamma_X(t)}(J^{X_p}_{V,\nabla_V X}(t), J^{X_p}_{W,\nabla_W X}(t))$, seen as a section of $\gamma_X^*\mathbb{R}_M$. We calculate with the product rule

$$\begin{aligned}
f'(0) &= \frac{d}{dt}\Big|_{t=0} f = \nabla^{\gamma_X^*\mathbb{R}_M}_{\partial_t} f\Big|_{t=0} \\
&= (\nabla^{\gamma_X^*(T^*M\otimes T^*M)}_{\partial_t} S_{\gamma_X(t)})(J^{X_p}_{V,\nabla_V X}(t), J^{X_p}_{W,\nabla_W X}(t)) \\
&\quad + S_{\gamma_X(t)}(\nabla^{\gamma_X^*TM}_{\partial_t} J^{X_p}_{V,\nabla_V X}(t), J^{X_p}_{W,\nabla_W X}(t)) \\
&\quad + S_{\gamma_X(t)}(J^{X_p}_{V,\nabla_V X}(t), \nabla^{\gamma_X^*TM}_{\partial_t} J^{X_p}_{W,\nabla_W X}(t))\Big|_{t=0} \\
&= (\nabla_{X_p} S)_p(V,W) + S_p(\nabla_V X, W) + S_p(V, \nabla_W X) \\
&= X_p(S(V,W)) - S(\nabla_X V - \nabla_V X, W) - S(V, \nabla_X W - \nabla_W X) \\
&= X_p(S(V,W)) - S([X,V],W) - S(V,[X,V]) \\
&= (\mathcal{L}_X S)_p(V,W)
\end{aligned}$$

□

Fix a metric g_b and consider the map $\Phi(h,X) := V(\mathcal{E}_X{}^*h, g_b)$ where V is the DeTurck map. Since the exponenetial of the zero vector field is $\mathcal{E}_0 = \mathrm{id}_M$, we have that $\Phi(g_b, 0) = V(g_b, g_b) = 0$. At this point, we calculate the derivative of Φ.

Lemma 8.20. $(D_{(g_b,0)}\Phi)(h,X) = -\Delta^{g_b}X - \mathrm{Ric}^{g_b}(X) - (\delta^{g_b}h + \frac{1}{2}d\,\mathrm{Tr}_g\, h)^{\sharp_{g_b}}$*. In particular,* $(D_{(g_b,0)}\Phi)(0,X) = -\Delta^{g_b}X - \mathrm{Ric}^{g_b}(X)$.

Proof. This is a consequence of Lemma 8.4.

$$\begin{aligned}
(D_{(g_b,0)}\Phi)(0,X) &= \frac{d}{dt}\Big|_{t=0} V(\phi_{tX}^* g_b, g_b) = DW_{g_b} \frac{d}{dt}\Big|_{t=0} (\phi_{tX}^* g_b) = DW_{g_b}(\mathcal{L}_X g_b) \\
&= -\Delta^{g_b}X - \mathrm{Ric}^{g_b}(X), \\
(D_{(g_b,0)}\Phi)(h,0) &= \frac{d}{dt}\Big|_{t=0} V(\phi_0^*(g_b + th), g_b) = DW_{g_b} \frac{d}{dt}\Big|_{t=0} (g_b + th) = DW_{g_b} h \\
&= (\delta^{g_b}h + \tfrac{1}{2}d\,\mathrm{Tr}_g\, h)^{\sharp_{g_b}}.
\end{aligned}$$

□

Remark 8.21. *The fact that we used the first argument of the DeTurck map in the definition of the map Φ is inconsequential for our purposes. Indeed, if we redefined the map Φ by swapping the arguments in the DeTurck map, we would get the same result as in Lemma 8.20 up to an overall sign. This is due to Corollary 8.7.*

8.3.2 Mapping properties of the raw Laplacian on vector fields

Lemma 8.22. *Let $p > 1$, $k > \frac{\dim M}{p} + 1$ and $\beta + 1 \in (2 - \dim M, 0)$ be nonexceptional, i.e. $\beta \notin m_{\pm}(\sigma(\Box_H))$ where $\Box_H$ denotes the tangential operator of the Hodge Laplacian on 1-forms. Then the raw Laplacian induces a bounded map*

$$\Delta_{g_{\mathrm{ac}}} \colon W^{k+1,p}_{\beta+1}(TM, g_{\mathrm{ac}}) \to W^{k-1,p}_{\beta-1}(TM, g_{\mathrm{ac}}),$$

which is an isomorphism of Banach spaces.

Proof. Since β is not exceptional, the raw Laplacian induces a continuous Fredholm map (cf. [Pac13, Section 9])

$$\Delta_{g_{\mathrm{ac}}} \colon W^{k+1,p}_{\beta+1}(TM, g_{\mathrm{ac}}) \to W^{k-1,p}_{\beta-1}(TM, g_{\mathrm{ac}}).$$

Let us now consider the kernel of this operator. By elliptic regularity on weighted Sobolev spaces, the kernel of $\Delta_{g_{\mathrm{ac}}}$ consists of smooth vector fields. As in [Pac13], the kernel is independent of k and p. By the Sobolev embedding, a vector field $W^{k+1,p}_{\beta+1}(TM, g_{\mathrm{ac}}) \subset C^{1,\alpha}_{\beta+1}(TM, g_{\mathrm{ac}})$ thus vector fields in the domain of $\Delta_{g_{\mathrm{ac}}}$ decay as $O\left(\rho^{\beta+1}\right)$ (this is genuine decay since $\beta + 1 < 0$).

Let now $X \in \ker \Delta_{g_{\mathrm{ac}}}$ and set $u := |X|^2_{g_{\mathrm{ac}}}$. This is a smooth function on M with $u = O\left(\rho^{2(\beta+1)}\right)$ and we have

$$\Delta^{g_{\mathrm{ac}}}_B u = 2 \left\langle \Delta_{g_{\mathrm{ac}}} X, X \right\rangle_{g_{\mathrm{ac}}} - 2 |\nabla^{g_{\mathrm{ac}}} X|^2_{g_{\mathrm{ac}}} \leq 0,$$

thus u is a subsolution of $\Delta^{g_{\mathrm{ac}}}_B$. The weak maximum principle on the compact manifold-with-boundary $\mathrm{Core}(R)$ implies therefore that for R big enough, one has

$$\max_{\mathrm{Core}(R)} u = \max_{\partial\,\mathrm{Core}(R)} u \leq C R^{2(\beta+1)}.$$

Letting $R \to \infty$ shows that $u = 0$, thus $X = 0$ and $\ker \Delta_{g_{\mathrm{ac}}} = 0$.

Since $\Delta_{g_{\mathrm{ac}}}$ is formally self-adjoint, the same holds for the adjoint operator, too. By the way weights change under taking the adjoint [Pac13, Section 9], we see that $\operatorname{coker} \Delta_{g_{\mathrm{ac}}} = 0$ if $\beta + 1 > 2 - \dim M$. $\Box$

8.3.3 Killing the DeTurck vector field by a pullback

Given a metric g_0, introduce the notation

$$\mathcal{G}_g := \{ g \in \mathrm{Met}(M) \,|V(g, g_0) = 0 \}.$$

Some authors call this condition the Bianchi gauge.

In a neighbourhood of g_{ac}, all metrics can be pulled back to a metric in $\mathcal{G}_{g_{\mathrm{ac}}}$.

Proposition 8.23. *Let g_{ac} be a Ricci-flat asymptotically conical manifold on M. Moreover, let $p \in (1, \infty)$ and let $k > \frac{\dim M}{p} + 1$ and let $\delta \in (1 - \dim M, -1)$ be nonexceptional for the raw Laplacian of g_{ac} on vector fields. Then there is a neighbourhood $\mathcal{U}$ of g_{ac} in $W^{k,p}_{\delta}(S^2T^*M, g_{\mathrm{ac}})$ such that*

a) $\mathcal{U} \cap \mathcal{G}_{g_{\mathrm{ac}}}$ is a manifold, and

b) for any metric $g \in \mathcal{U}$, there is a unique vector field $X \in W^{k+1,p}_{\delta+1}(TM, g_{\mathrm{ac}})$ such that $(\mathcal{E}_X)^*g \in \mathcal{G}_{g_{\mathrm{ac}}}$, i.e. such that $V((\mathcal{E}_X)^*g, g_{\mathrm{ac}}) = 0$.

Proof. The proof rests on the Banach space implicit function theorem.

Note that our assumptions on k imply by the weighted Sobolev embedding theorem that $W^{k+1,p}_{\delta+1}(TM, g_{\mathrm{ac}}) \subset C^{1,\alpha}_{\delta+1}(TM, g_{\mathrm{ac}})$ for any $\alpha \in (0, 1)$.

1. Consider the map $W\colon W^{k,p}_{\delta}(S^2T^*M, g_{\mathrm{ac}}) \rightarrowtail W^{k-1,p}_{\delta-1}(TM, g_{\mathrm{ac}}), g \mapsto V(g, g_{\mathrm{ac}})$ defined on an open neighbourhood U_0 of g_b. (Such a neighbourhood exists because of the closure of weighted Sobolev spaces under tensor products, up to weights and our assumptions on k.) Evidently, $W(g_{\mathrm{ac}}) = 0$. We claim that the differential of this map at g_{ac} is surjective and its kernel is a complemented subspace.

   For surjectivity, it suffices to consider tensor fields of the form $\mathcal{L}_X g_{\mathrm{ac}}$ with $X \in W^{k+1,p}_{\delta+1}(TM, g_{\mathrm{ac}})$ since by Lemma 8.20, one has $DW_{g_b}(\mathcal{L}_X g_b) = -\Delta_{g_b}X - \mathrm{Ric}^{g_b} X = -\Delta_{g_b}X$. By our assumptions and Lemma 8.22, we have that

   $$-\Delta_{g_{\mathrm{ac}}}\colon W^{k+1,p}_{\delta+1}(TM) \to W^{k-1,p}_{\delta-1}(TM)$$

   is an isomorphism, so in particular it is surjective. Note that this also means that the operator $DW_{g_{\mathrm{ac}}}$ restricted to tensor fields of the form $\mathcal{L}_X g_b$ for some $X \in W^{k+1,p}_{\delta+1}(TM, g_{\mathrm{ac}})$ has trivial kernel, i.e. $\ker DW_{g_b} \cap \mathcal{L}_{W^{k+1,p}_{\delta+1}(TM,g_{\mathrm{ac}})} g_{\mathrm{ac}} = \{0\}$

   Next we claim that the kernel of DW_{g_b} is a complemented subspace, in fact,

   $$W^{k,p}_{\delta}(S^2T^*M) = \ker DW_{g_b} \oplus \mathcal{L}_{W^{k+1,p}_{\delta+1}(TM,g_{\mathrm{ac}})} g_{\mathrm{ac}}. \qquad (\star)$$

   The only thing left to show is that we can write any tensor field $h \in W^{k,p}_{\delta}(S^2T^*M, g_{\mathrm{ac}})$ as $h = h_0 + \mathcal{L}_{X_h} g_{\mathrm{ac}}$ for some $h_0 \in \ker DW_{g_{\mathrm{ac}}}$ and $X_h \in W^{k+1,p}_{\delta+1}(TM, g_{\mathrm{ac}})$. Let X_h be the unique solution of the equation $DW_{g_{\mathrm{ac}}}(h) = \Delta_{g_b} X_h$. Then $h_0 := h - \mathcal{L}_X g_{\mathrm{ac}}$ is a good choice.

2. Now consider the map $\Phi(h, X) := V((\mathcal{E}_X)^*h, g_{\mathrm{ac}})$ defined in a neighbourhood of $(g_{\mathrm{ac}}, 0)$. In Lemma 8.20, we have already established that

   $$(D_{(g_{\mathrm{ac}},0)}\Phi)(0, \cdot) = -\Delta_{g_{\mathrm{ac}}} - \mathrm{Ric}^{g_{\mathrm{ac}}}\colon W^{k+1,p}_{\delta+1}(TM, g_{\mathrm{ac}}) \to W^{k-1,p}_{\delta-1}(TM, g_{\mathrm{ac}})$$

   is an isomorphism.

   Note that if $X \in W^{k+1,p}_{\delta+1}(TM, g_{\mathrm{ac}}) \subset C^{1,\alpha}_{\delta+1}(TM, g_{\mathrm{ac}})$ is a Killing field, then Corollary 8.14 implies that $X = 0$. Consequently, $\mathcal{L}_{X_1} g_{\mathrm{ac}} = \mathcal{L}_{X_2} g_{\mathrm{ac}}$ can happen only if $X_1 = X_2$, and therefore the decomposition $(\star)$ can be rewritten as

   $$\ker DW_{g_b} \oplus W^{k+1,p}_{\delta+1}(TM, g_{\mathrm{ac}}) \simeq W^{k,p}_{\delta}(S^2T^*M), (h_0, X) \mapsto h_0 + \mathcal{L}_X g_b. \qquad (\star\star)$$

   Now the implicit function theorem implies the claim. □

Remark 8.24. *The proof of Proposition 8.23 follows the train of through of [DK20, Proposition 2.12]. Independently, there is an analogous result for weighted Hölder spaces [DO20, Proposition B.2]. An alternative proof for our result can be given from this proposition and a weighted Sobolev embedding.*

Remark 8.25. *Note that for* $\delta \leq -\frac{\dim M}{p}$*, one has the continuous embedding* $W^{k,p}_{\delta}(S^2T^*M, g_{\mathrm{ac}}) \subset W^{0,p}_{-\dim M/p}(S^2T^*M, g_{\mathrm{ac}}) = L^p(S^2T^*M, g_{\mathrm{ac}})$.

8.4 Eventually exactly conical metrics

Suppose M is a manifold with a single end, i.e. there exists a compact set $K \subset M$ and a diffeomorphism $\phi\colon M \setminus K \to \mathrm{Cone}(L) \setminus ((0, R_{\mathrm{end}}) \times L)$ for some $R_{\mathrm{end}} > 0$. (Note that, as manifolds, $\mathrm{Cone}(L) \simeq (0, \infty) \times L$.)

Definition 8.26. *A metric* $g_{\mathrm{eec}} \in \mathrm{Met}\,(M)$ *is eventually exactly conical if there is a threshold radius* $R_{coinc} > 0$ *such that* $\phi^* g_{\mathrm{cone}} = g_{\mathrm{eec}}$ *on* $M \setminus \mathrm{Core}(R_{coinc})$.

Next, we show that we can approximate asymptotically conical metrics with eventually exactly conical metrics arbitrarily closely in appropriate weighted Sobolev spaces. For this, we will need a bump function with controlled derivatives.

8.4.1 Interpolating function

We will need the following version of the chain rule. On $M := \mathbb{R}^n$, this lemma is known as Faà di Bruno's formula [FdB55], cf. also [Fra78].

Lemma 8.27. *Let* M *be a smooth manifold with connection* ∇*. Let* $f\colon M \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *be smooth functions and let* T *be a smooth tensor field on* M*. Then*

$$\nabla((g \circ f)T) = (g' \circ f) df \otimes T + (g \circ f)\nabla T.$$

Consequently, we have

$$\nabla^k (g \circ f) = \sum_{p=1}^{k} (g^{(p)} \circ f) \begin{pmatrix} \textit{linear combinations of terms of the form} \\ p \textit{ many factors of } f \textit{ with} \\ k \textit{ many derivatives } \nabla \textit{ acting on them} \end{pmatrix}$$

where $g^{(p)}$ *denotes the pth derivative of* g.

Proof. The first claim is easily verified in local coordinates.

For the second claim, we apply the first claim iteratively, starting with $T = 1$, and collect terms based on the order of derivative on g. This procedure works because the derivative of a real valued function can be identified with a real valued function. ☐

Lemma 8.28. *Given an asymptotically conical manifold* (M, g) *and a radius function* $\rho\colon M \to \mathbb{R}$*, there is a number* $R_0 \in \mathbb{R}$ *and a smooth family* $(f_R)_{R \geq R_0}$ *such that* $f_R|_{\mathrm{Core}(R)} = 0$*,* $f_R|_{M \setminus \mathrm{Core}(2R)} = 1$ *and for all natural numbers* $k \leq k_0$*, there is a constant* $C_k < \infty$ *with* $|\nabla^{g_{\mathrm{cone}}, k} f_R| \leq \frac{C_k}{R^k}$.

Proof. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $F|_{(-\infty,1]} = 0$ and $F|_{[2,\infty)} = 1$. (Such a function can be obtained e.g. by appropriately rescaling and shifting the indefinite integral a bump function, e.g. the bump function of Arias de Reyna [AdR82].) Since the derivative F' is compactly supported, there are constants $\tilde{C}_k < \infty$ for all $k \in \mathbb{Z}^+$ such that $|F^{(k)}(x)| \leq \tilde{C}_k$ for all $x \in \mathbb{R}$.

Let $R_{\text{prelim}} \in \mathbb{R}$ be arbitrary and let

$$R_0 := \sup \{\rho(p) \, |p \in \text{Core}(R_{\text{prelim}})\} = \max \{\rho(p) \, |p \in \text{Core}(R_{\text{prelim}})\} .$$

(We need this step because we have no control over the behaviour of ρ in the core. However, this choice of R_0 makes sure that $K \subset \rho^{-1}([-\infty, R])$, hence $\text{Core}(R) = \rho^{-1}([-\infty, R]) \cup K = \rho^{-1}([-\infty, R])$ for any $R \geq R_0$.) Define the family $(f_R)_{R \geq R_0}$ of functions via $f_R\colon M \to \mathbb{R}, p \mapsto F\left(\frac{\rho(p)}{R}\right)$. By construction, we have $f_R|_{\text{Core}(R)} = 0$, $f_R|_{M \setminus \text{Core}(2R)} = 1$.

The claim about the derivative follows by applying Lemma 8.27 and noting that $\nabla^{g,k}\rho = O\left(\rho^{-k+1}\right)$. □

8.4.2 Interpolating family of metrics

Let g_{ac} be an asymptotically conical metric with decay rate τ and let g_{cone} denote its asymptotic cone. Consider the following family of metrics:

$$g_R := f_R \phi^* g_{\text{cone}} + (1 - f_R) g_{\text{ac}}$$

for $R > R_0$. Evidently, this metric is Ricci-flat except on the set $\text{Core}(2R) \setminus \text{Core}(R)$.

Before we prove the main statement of this section, let us prove the following technical lemma.

Lemma 8.29. *Let $R_2 > R_1 > \max \{R_end, R_asy\}$, let $\alpha \in \mathbb{R}$ and let $g := \phi^* g_{\text{cone}}$. Then $\int_{\text{Core}(R_2)\setminus\text{Core}(R_1)} \rho^{\alpha - \dim M} \operatorname{vol}_g = C(\alpha, g_{\text{link}})(R_2^\alpha - R_1^\alpha)$.*

Proof. Note that $\text{Core}(R_2) \setminus \text{Core}(R_1) = \phi^{-1}((R_1, R_2] \times L)$ for the asymptotic chart ϕ and the link L. Now

$$\begin{aligned}
\int_{\text{Core}(R_2)\setminus\text{Core}(R_1)} \rho^{\alpha-\dim M} \operatorname{vol}_g &= \int_{\phi^{-1}((R_1,R_2]\times L)} \rho^{\alpha-\dim M} \operatorname{vol}_g \\
&= \int_{(R_1,R_2]\times L} (\phi^{-1})^*(\rho^{\alpha-\dim M} \operatorname{vol}_g) \\
&= \int_{(R_1,R_2]\times L} r^{\alpha-\dim M} \operatorname{vol}_{\phi^{-1*}g} \\
&= \int_{(R_1,R_2]\times L} r^{\alpha-\dim M} \operatorname{vol}_{g_{\text{cone}}} \\
&= \int_{R_1}^{R_2} r^{\alpha-\dim M} \left(\int_{\{r\}\times L} r^{\dim M} \operatorname{vol}_{g_{\text{link}}} \right) dr \\
&= \operatorname{Vol}_{g_{\text{link}}}(link) \int_{R_1}^{R_2} r^\alpha dr \\
&= \operatorname{Vol}_{g_{\text{link}}}(link) \frac{R_2^\alpha - R_1^\alpha}{\alpha} \\
&= C(\alpha, g_{\text{link}})(R_2^\alpha - R_1^\alpha)
\end{aligned}$$

□

The main statement of this section is that we can approximate any asymptotically conical metric with a family of eventually exactly conical metrics in appropriate weighted Sobolev spaces.

Proposition 8.30. *Let $\delta > -\tau$ be a real number, and let $k, p \in \mathbb{N}^+$. We have $\lim_{R\to\infty} g_R = g_{\mathrm{ac}}$ in $W^{k,p}_\delta(S^2T^*M)$. In particular, for all $\epsilon > 0$ there is an $R \in \mathbb{R}^+$ such that $\|g_{\mathrm{ac}} - g_R\|_{W^{k,p}_\delta(S^2T^*M)} < \epsilon$.*

Proof. Without loss of generality, we may assume that $R_0 > \max\{R_{\mathrm{end}}, R_{\mathrm{asy}}\}$.

We wish to estimate the following number

$$\|g_{\mathrm{ac}} - g_R\|^p_{W^{k,p}_\delta(S^2T^*M)} = \sum_{l=0}^{k} \int_M \left| \int \rho^{-\delta+l} \nabla^l (g_R - g_{\mathrm{ac}}) \right|^p \rho^{-\dim M} \operatorname{vol}_g .$$

Based on the calculations

$$g_R - g_{\mathrm{ac}} = f\phi^* g_{\mathrm{cone}} + (1-f) g_{\mathrm{ac}} - g_{\mathrm{ac}} = f(\phi^* g_{\mathrm{cone}} - g_{\mathrm{ac}}),$$

it is clear that it is advantageous to evaluate the integral using the partition

$$M = \mathrm{Core}(2R) \cup (\mathrm{Core}(2R) \setminus \mathrm{Core}(R)) \cup (M \setminus \mathrm{Core}(2R)).$$

$$I_1 := \sum_{l=0}^{k} \int_{\mathrm{Core}(R)} \left| \rho^{-\delta+l} \nabla^l (g_R - g_{\mathrm{ac}}) \right|^p \rho^{-\dim M} \operatorname{vol}_g = \sum_{l=0}^{k} \int_M 0 \operatorname{vol}_g = 0$$

Now the generalized Leibniz rule

$$\nabla^{g,l}(g_R - g_{\mathrm{ac}}) = \sum_{a=0}^{l} \binom{l}{a} \left(\nabla^{g,l-a} f\right) \otimes \nabla^{g,a}(\phi^* g_{\mathrm{cone}} - g_{\mathrm{ac}})$$

implies that

$$\begin{aligned}
I_2 &:= \sum_{l=0}^{k} \int_{\mathrm{Core}(2R)\setminus\mathrm{Core}(R)} \left| \rho^{-\delta+l} \nabla^l (g_R - g_{\mathrm{ac}}) \right|^p \rho^{-\dim M} \operatorname{vol}_g \\
&= \sum_{l=0}^{k} \int_{\mathrm{Core}(2R)\setminus\mathrm{Core}(R)} \rho^{(-\delta+l)p} \left| \sum_{a=0}^{l} \binom{l}{a} \left(\nabla^{g,l-a} f\right) \otimes \nabla^{g,a}(\phi^* g_{\mathrm{cone}} - g_{\mathrm{ac}}) \right|^p \rho^{-\dim M} \operatorname{vol}_g \\
&\leq C \sum_{l=0}^{k} \int_{\mathrm{Core}(2R)\setminus\mathrm{Core}(R)} \rho^{(-\delta+l)p} \sum_{a=0}^{l} \left(\underbrace{\left|\nabla^{g,l-a} f\right|^p}_{\leq C' R^{a-l}} \underbrace{\left|\nabla^{g,a}(\phi^* g_{\mathrm{cone}} - g_{\mathrm{ac}})\right|^p}_{\leq C'' \rho^{-\tau-a}} \right) \rho^{-\dim M} \operatorname{vol}_g \\
&\leq C''' \sum_{l=0}^{k} \int_{\mathrm{Core}(2R)\setminus\mathrm{Core}(R)} \rho^{(-\delta+l)p} \sum_{a=0}^{l} \left(R^{(a-l)p} \rho^{-(\tau+a)p} \right) \rho^{-\dim M} \operatorname{vol}_g \\
&\leq C'''' \sum_{l=0}^{k} \int_{\mathrm{Core}(2R)\setminus\mathrm{Core}(R)} \rho^{(-\delta+l)p} \sum_{a=0}^{l} \left(R^{-lp} \left(\frac{R}{\rho}\right)^{ap} \rho^{-\tau p} \right) \rho^{-\dim M} \operatorname{vol}_g
\end{aligned}$$

$$\leq C'''' \sum_{l=0}^{k} \int_{\mathrm{Core}(2R)\setminus\mathrm{Core}(R)} \rho^{(-\delta-\tau)p} \underbrace{\left(\frac{R}{\rho}\right)^{-lp} \sum_{a=0}^{l} \left(\frac{R}{\rho}\right)^{ap}}_{\leq C''''' \text{ since } R\leq\rho\leq 2R} \rho^{-\dim M} \operatorname{vol}_g$$

$$\leq C'''''' \int_{\mathrm{Core}(2R)\setminus\mathrm{Core}(R)} \rho^{-(\delta+\tau)p} \rho^{-\dim M} \operatorname{vol}_g$$

$$= C_2(\delta,\tau,p,g_{\mathrm{link}}) \frac{1}{R^{(\delta+\tau)p}}$$

The fact that $(g_R - g_{\mathrm{ac}})|_{M\setminus\mathrm{Core}(2R)} = -(g_{\mathrm{ac}} - g_{\mathrm{cone}})$ implies that

$$\begin{aligned}
I_3 &:= \sum_{l=0}^{k} \int_{M\setminus\mathrm{Core}(2R)} \left|\rho^{-\delta+l}\nabla^l(g_R - g_{\mathrm{ac}})\right|^p \rho^{-\dim M} \operatorname{vol}_g \\
&\leq C \sum_{l=0}^{k} \int_{M\setminus\mathrm{Core}(2R)} \rho^{(-\delta+l)p}\rho^{(-\tau-l)p}\rho^{-\dim M} \operatorname{vol}_g \\
&\leq C' \int_{M\setminus\mathrm{Core}(2R)} \rho^{(-\delta-\tau)p-\dim M} \operatorname{vol}_g \\
&= C_3(\delta,\tau,p,g_{\mathrm{link}}) \frac{1}{R^{(\delta+\tau)p}}
\end{aligned}$$

where we used in step $(\star)$ that $\rho = \phi^* r$ on $M \setminus \mathrm{Core}(2R)$.

Now we have

$$\|g_{\mathrm{ac}} - g_R\|^p_{W^{k,p}_\delta(S^2T^*M)} = I_1 + I_2 + I_3 = C(\delta,\tau,p,g_{\mathrm{link}}) \frac{1}{R^{(\delta+\tau)p}}$$

The claims follow as this expression converges to 0 if R tends to infinity. □

Proposition 8.31. *For large enough $R_1 > R_0$, there exists a diffeomorphism $\psi\colon M \to M$ such that*

- $V(\psi^* g_{R_1}, g_{\mathrm{ac}}) = 0$
- $\mathrm{Ric}^{\psi^* g_{R_1}} = 0$ *on $M \setminus K$ for some compact set $K \subset M$.*

In particular, the metric $g_0 := \psi^* g_{R_1}$ lies in $\mathcal{F}^{M\setminus K}_{g_{\mathrm{ac}}}$.

Proof. In particular, for large enough R, the metric g_R will lie in the neighbourhood $\mathcal{U}$ from Proposition 8.23. Let R_1 be a suitably large value. This means, there is a diffeomorphism $\psi := \mathcal{E}_X$ with some $X \in W^{k+1,p}_{\delta+1}(TM, g_{\mathrm{ac}})$ such that $g_0 := \psi^* g_{R_1}$ satisfies $V(g_0, g_{\mathrm{ac}}) = 0$.

On the other hand, $\mathrm{Ric}^{g_{R_1}}|_{\mathrm{Core}(R)} = 0$ and $\mathrm{Ric}^{g_{R_1}}|_{M\setminus\mathrm{Core}(2R)} = 0$ by construction. Since the Ricci tensor of a metric behaves naturally under pullbacks, this means that $\mathrm{Ric}^{g_0}|_{M\setminus K} = 0$ where $K := \psi^{-1}(\mathrm{Core}(2R))$.

Thus $-2\,\mathrm{Ric}^{g_0} + \mathcal{L}_{V(g_0,g_{\mathrm{ac}})} g_{\mathrm{ac}} = 0$ fails to hold on the whole of M, but it does hold on $M \setminus K$. Comparing this to Definition 8.10 shows the claim. □

Chapter 9

Decay of tensor fields and the Einstein operator

9.1 A useful expansion

Let $E := S^2T^*M$. In Theorem 6.55, we have obtained an orthonormal eigenbasis w_i of $L^2(E_1)$. By Lemma 6.38 these eigenfields extend to eigenfields on E_r as $(\Phi_{1/r})^* w_i$. Let $W \in \Gamma(E)$ be the unique section of E such that $W|_{\{r\}\times M} = (\Phi_{1/r})^* w_i$. This suggests we use these orthonormal bases to expand any section $h \in \Gamma(E)$:

$$h|_{\{r\}\times M} = \sum_{i\in\mathbb{N}} u_i(r) r^2 W_i, \tag{9.1}$$

where $u_i(r) \in \mathbb{R}$. Keeping $\mathrm{Vol}_{g_{\mathrm{cone}}}(\{r\}\times M) = r^n \,\mathrm{Vol}_{g_{\mathrm{link}}}(M)$ in mind, the coefficients $u_i(r)$ can be obtained as

$$u_i(r) = \frac{1}{r^n} \int_{\{r\}\times M} \left\langle h, r^2 W_i \right\rangle_{g_{\mathrm{cone}}} \mathrm{vol}_{g_{\mathrm{cone}}}. \tag{9.2}$$

This decomposition is useful because it “takes the burden” of the dependence along the link directions off our shoulders, and we have to deal with r-dependent functions only. The motivation of the factor r^2 is the scaling from Examples 6.34 and 6.36 but, as it turns out, $r^2 W_i$ scales just right to be covariantly constant in the radial direction:

Lemma 9.1. *The section r^2W_i is covariantly constant in the radial direction, i.e. $\bar\nabla_Z(r^2W_i) = \bar\nabla_{\partial_r}(r^2W_i) = 0$.*

Proof. For $\omega \in \Omega^1(\mathrm{link})$ and $h_2 \in S^2(T^*\mathrm{link})$, explicit calculation reveals that

$$\begin{aligned}
\bar\nabla_{\partial_r}(r^2(\Phi_{1/r})^*(dr\otimes dr)) &= \bar\nabla_{\partial_r}(dr\otimes dr) = 0,\\
\bar\nabla_{\partial_r}(r^2(\Phi_{1/r})^*(dr\odot \omega)) &= \bar\nabla_{\partial_r}(dr\odot r\omega) = 0,\\
\bar\nabla_{\partial_r}(r^2(\Phi_{1/r})^*(h_2)) &= \bar\nabla_{\partial_r}(r^2 h_2) = 0.
\end{aligned}$$

As W_i is an $\mathbb{R}$-linear combination of these fields, we obtain the statement. □

Moreover, the built-in scaling ensures that the decay rates are easy to read off the coefficient functions u_i, as the next lemma shows.

Lemma 9.2. *Let* $h = \sum_{i\in\mathbb{N}} u_i r^2 W_i \in \Gamma\left(S^2T^*\bar{M}\right)$*, and define* $S_k(\xi) := \sum_{\ell=0}^{k} \xi^{2\ell}$ *be a smooth section.*

1. *If* $h = O\left(r^{-\mu}\right)$*, then* $u_i = O\left(r^{-\mu}\right)$ *for all* $i \in \mathbb{N}$.

2. *Conversely, if there is a single threshold* $R \in I$ *and a sequence* $C_i > 0$ *and such that* $u_i|_{[R,\infty)} \leq C_i r^{-\mu}$ *for all* $i \in I$ *and* $\sum_{i\in\mathbb{N}} S_k(\xi_i) C_i^2 < \infty$ *for some* $2k > (n+1)/2$*; then* $h = O\left(r^{-\mu}\right)$.

Remark 9.3. *Note that the second condition is stronger than the condition that* $u_i = O\left(r^{-\mu}\right)$*. The latter condition postulates the existence of sequences* $(R_i)_{i\in\mathbb{N}}$ *and* $(C_i)_{i\in\mathbb{N}}$ *such that* $u_i|_{[R_i,\infty)} \leq C_i r^{-\mu}$*, the former one, however, demands a single number* R *such that* $u_i|_{[R,\infty)} \leq C_i r^{-\mu}$ *for all* $i \in \mathbb{N}$*. An easy counterexample can be constructed using* $u_i(r) = r^{-\mu} + \Psi(x-i)$ *where* Ψ *is a bump function.*

Proof. 1. By assumption $|h| \leq Cr^{-\mu}$ pointwise outside some compact set K. Without loss of generality, we may take K to be of the form $[R,\infty)\times M$. Now by the Cauchy–Schwarz inequality, we obtain for any $r \geq R$ that

$$
\begin{aligned}
|u_i(r)| &= \left| \frac{1}{r^n} \int_{\{r\}\times M} \left\langle h, r^2 W_i \right\rangle_{g_{\text{cone}}} \mathrm{vol}_{g_{\text{cone}}} \right| \\
&\leq \frac{1}{r^n} |h|_{L^2(\{r\}\times M, g_{\text{cone}})} |r^2 W_i|_{L^2(\{r\}\times M, g_{\text{cone}})} \\
&= \frac{1}{r^n} |h|_{L^2(\{r\}\times M, g_{\text{cone}})} \\
&\leq \frac{1}{r^n} |h|_{L^\infty(\{r\}\times M, g_{\text{cone}})} \mathrm{Vol}_{g_{\text{cone}}}(\{r\}\times M) \\
&= \frac{1}{r^n} |h|_{L^\infty(\{r\}\times M, g_{\text{cone}})} r^n \mathrm{Vol}_{g_{\text{cone}}}(\{1\}\times M) \\
&= \frac{1}{r^n} |h|_{L^\infty(\{r\}\times M, g_{\text{cone}})} r^n \mathrm{Vol}_{g_{\text{link}}}(M) \\
&\leq C \,\mathrm{Vol}_{g_{\text{link}}}(M)\, r^{-\mu}.
\end{aligned}
$$

2. The proof of the other direction requires more subtle means. By the Sobolev embedding on E_r over the compact manifold $\{r\}\times M$ with $2k > n/2$, and by repeated application of elliptic regularity for the tangential operator $\Box$, we have

$$
\begin{aligned}
\sup_{\{r\}\times M} |h|_{\bar{g}} = \|h_r\|_{C^0(E_r)} &\leq C_0 \|h_r\|_{H^{2k}(E_r)} \\
&\leq C_0 C_{2k} (\|h_r\|_{L^2(E_r)} + \|\Box h_r\|_{H^{2k-2}(E_r)}) \\
&\leq C_0 C_{2k} (\|h_r\|_{L^2(E_r)} + C_{2k-2}(\|\Box h_r\|_{L^2(E_r)} + \|\Box^2 h_r\|_{H^{2k-4}(E_r)})) \\
&\leq \dots \\
&\leq C_0 C_{2k} (\|h_r\|_{L^2(E_r)} + C_{2k-2}(\|\Box h_r\|_{L^2(E_r)} + C_{2k-4}(\|\Box^2 h_r\|_{L^2(E_r)} \\
&\qquad + \dots + C_{2k+1}\|\Box^k h_r\|_{L^2(E_r)}) \dots)) \\
&\leq C_0 C_{\max} \sum_{l=0}^{k} \|\Box^l h_r\|_{L^2(E_r)}
\end{aligned}
$$

and, by the fact $\sum_{l=0}^k a_i \leq \sqrt{k+1}\sqrt{\sum_{l=0}^k a_i^2}$, which follows from the inequality between the arithmetic and the quadratic means, we have

$$\leq C_0 C_{\max}\sqrt{k+1}\sqrt{\sum_{l=0}^{k}\|\Box^l h_r\|^2_{L^2(E_r)}}$$

Now, let us investigate the sum under the square root further

$$\begin{aligned}
\|\Box^l h_r\|^2_{L^2(E_r)} &= \|\Box^l \sum_{i\in\mathbb{N}} u_i(r) w_i\|^2_{L^2(E_r)} \\
&= \|\sum_{i\in\mathbb{N}} u_i(r)\xi_i^l w_i\|^2_{L^2(E_r)} \\
&= \sum_{i,j\in\mathbb{N}} \int_{\{r\}\times M} \left\langle u_i(r)\xi_i^l w_i, u_j(r)\xi_j^l w_j\right\rangle_{\bar{g}} \mathrm{vol}_g \\
&= \sum_{i,j\in\mathbb{N}} u_i(r)\xi_i^l u_j(r)\xi_j^l \int_{\{r\}\times M} \langle w_i, w_j\rangle_{\bar{g}} \mathrm{vol}_g \\
&= \sum_{i,j\in\mathbb{N}} u_i(r)\xi_i^l u_j(r)\xi_j^l \delta_{ij} \\
&= \sum_{i\in\mathbb{N}} u_i(r)^2 \xi_i^{2l} \\
&= \sum_{i\in\mathbb{N}} \xi_i^{2l} u_i(r)^2,
\end{aligned}$$

therefore, we obtain

$$\begin{aligned}
\sum_{l=0}^{k}\|\Box^l h_r\|^2_{L^2(E_r)} &= \sum_{l=0}^{k}\sum_{i\in\mathbb{N}} \xi_i^{2l} u_i(r)^2 \\
&= \sum_{i\in\mathbb{N}}\sum_{l=0}^{k} \xi_i^{2l} u_i(r)^2 \\
&= \sum_{i\in\mathbb{N}} S(\xi_i) u_i(r)^2 \\
&\leq \sum_{i\in\mathbb{N}} S(\xi_i) C_i^2 r^{-2\mu} \\
&= K r^{-2\mu},
\end{aligned}$$

where $K < \infty$ by assumption. The claim now follows. □

9.2 Exceptional values and the kernel of the Einstein operator on a cone

Two particular functions of eigenvalues of the tangential operator $\Box_L$ will appear frequently in what follows so it makes sense to introduce an abbreviation for it.

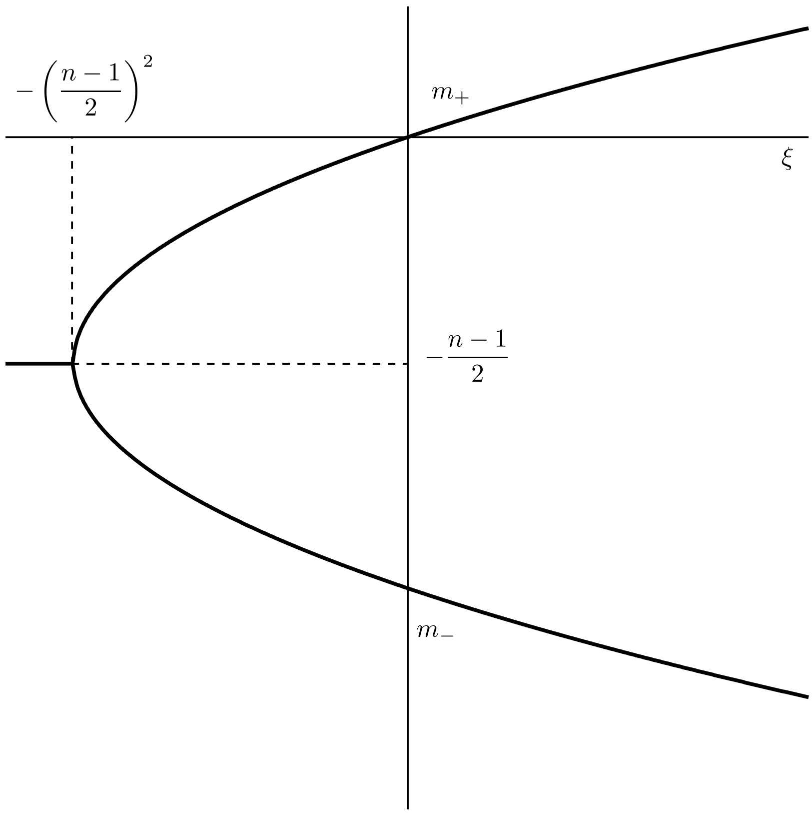


Figure 9.1: The real part of exceptional values corresponding to eigenvalues. Note that the positive-mode branch m_+ is negative if and only if $\xi < 0$, cf. Defintion 9.4.

Definition 9.4. *The positive/negative mode exceptional value (cf. Figure 9.1) corresponding to the eigenvalue ξ of the tangential operator $\square_L$ is*

$$m_\pm(\xi) := -\frac{n-1}{2} \pm \sqrt{\left(\frac{n-1}{2}\right)^2 + \xi}. \tag{9.3}$$

At this stage of exposition, $m_\pm(\xi)$ is merely a useful abbreviation. Its true significance is revealed in connection with Fredholmness properties of the Lichnerowicz Laplacian on weighted Sobolev spaces.

Note that the negative mode exceptional value is always negative but the positive mode exceptional value is negative if and only if $\xi > 0$. For the graph of the functions $m_\pm$, consider Figure 9.1.

With the help of the exceptional values, the kernel of the Lichnerowicz Laplacian on an exact cone can be described in fairly explicit terms.

Lemma 9.5. *Let $h := \sum_{i\in\mathbb{N}} u_i r^2 W_i$.*

1. $h \in \ker \Delta_L$ *if and only if*

$$-\partial_r\partial_r u_i - \frac{n}{r}\partial_r u_i + \frac{\xi_i}{r^2} u_i = 0 \qquad \textit{for all } i \in \mathbb{N}. \tag{9.4}$$

2. *A fundamental system of solutions of the differential equation* (9.4) *is* $\left\{r \mapsto r^{m_\pm(\xi_i)}\right\}$. *This fundamental system has Wrońskian determinant* $W(r) = -\frac{2}{r^n}\sqrt{\left(\frac{n-1}{2}\right)^2 + \xi}$. *Introduce* $\xi_{res} := -\left(\frac{n-1}{2}\right)^2$*, the only zero locus of the Wrońsian, (which may or may not be in the spectrum of* $\Box_E$*).*

3. *The elements in the kernel of* $\bar{\Delta}_L$ *are of the form*

$$h = \sum_{\substack{i\in\mathbb{N}\\ \xi_i\neq\xi_{res}}} \left(c_i^+ r^{m_+(\xi_i)} + c_i^- r^{m_-(\xi_i)}\right) r^2 W_i + \sum_{\substack{i\in\mathbb{N}\\ \xi_i=\xi_{res}}} \left(c_i^+ r^{-\frac{n-1}{2}} \ln r + c_i^- r^{-\frac{n-1}{2}}\right) r^2 W_i$$

$$= \sum_{\substack{i\in\mathbb{N}\\ \xi_i<\xi_{res}}} \left(\hat{c}_i^+ r^{-\frac{n-1}{2}} \sin\left(\ln r \cdot \sqrt{\left|\left(\frac{n-1}{2}\right)^2 + \xi\right|}\right)\right.$$

$$\left. + \hat{c}_i^- r^{-\frac{n-1}{2}} \cos\left(\ln r \cdot \sqrt{\left|\left(\frac{n-1}{2}\right)^2 + \xi\right|}\right)\right) r^2 W_i$$

$$+ \sum_{\substack{i\in\mathbb{N}\\ \xi_i=\xi_{res}}} \left(c_i^+ r^{-\frac{n-1}{2}} \ln r + c_i^- r^{-\frac{n-1}{2}}\right) r^2 W_i$$

$$+ \sum_{\substack{i\in\mathbb{N}\\ \xi_i>\xi_{res}}} \left(c_i^+ r^{m_+(\xi_i)} + c_i^- r^{m_-(\xi_i)}\right) r^2 W_i$$

for real constants $c_i^\pm, \hat{c}_i^\pm \in \mathbb{R}$.

Remark 9.6. *Note that the the summands for* $\xi < \xi_{res}$ *can be written in the same form as summands where* $\xi > \xi_{res}$.

Proof. 1. Note that if h satisfies $\bar{\Delta}_L h = 0$, then h must be smooth by elliptic regularity and therefore the assignment $r \mapsto u_i(r)$ can be chosen smoothly for each $i \in \mathbb{N}$. This means all the derivatives of u_i are justified.

By Equation (6.2), Lemma 6.38, Lemma 9.1 and the Leibniz rule,

$$\bar{\Delta}_L h = -\bar{\nabla}_{\partial_r}\bar{\nabla}_{\partial_r} h - \frac{n}{r}\bar{\nabla}_{\partial_r} h + \frac{1}{r^2}\Box_L h$$

$$- \bar{\nabla}_{\partial_r}\bar{\nabla}_{\partial_r}\left(\sum_{i\in\mathbb{N}} u_i r^2 W_i\right) - \frac{n}{r}\bar{\nabla}_{\partial_r}\left(\sum_{i\in\mathbb{N}} u_i r^2 W_i\right) + \frac{1}{r^2}\Box_L\left(\sum_{i\in\mathbb{N}} u_i r^2 W_i\right)$$

$$= \sum_{i\in\mathbb{N}}\left(-\partial_r\partial_r u_i - \frac{n}{r}\partial_r u_i + \frac{\xi_i}{r^2}u_i\right) r^2 W_i.$$

Now for h to be in the kernel of $\bar{\Delta}_L$, and integration on sets of the form $\{r\} \times M$ with the help of the normalization (6.5) reveals that, the expression in Equation (9.4) is zero if and only if

$$-\partial_r\partial_r u_i - \frac{n}{r}\partial_r u_i + \frac{\xi_i}{r^2}u_i = 0 \quad \text{for all } i \in \mathbb{N}.$$

2. Equation (9.4) is a Cauchy–Euler differential equation, and it can be solved with the ansatz $u_i(r) = r^a$ for some $a \in \mathbb{R}$. The corresponding index equation is

$$-a(a-1) - na + \xi_i = 0,$$

the solutions of which are $a = m_\pm(\xi_i)$. It is easy to check that the corresponding Wrońskian determinant is

$$W(r) = \det \begin{pmatrix} r^{m_+(\xi_i)} & r^{m_-(\xi_i)} \\ \partial_r(r^{m_+(\xi_i)}) & \partial_r(r^{m_-(\xi_i)}) \end{pmatrix} = -2r^{-n}\sqrt{\left(\frac{n-1}{2}\right)^2 + \xi}.$$

3. As an easy consequence of the earlier steps, the form of u_i claim follows immediately if $\xi \neq -\left(\frac{n-1}{2}\right)^2$, i.e. if the Wrońskian is not zero. If the Wrońskian is in fact zero (i.e. we have a resonance), the solution is easily checked to be $u(r) = c_1 r^{-\frac{n-1}{2}} + c_2 r^{-\frac{n-1}{2}} \ln r$. □

Based on the different behaviour of the component function u_i, it is reasonable to introduce the following concepts.

Definition 9.7. *We call an eigenvalue* $\xi \in \sigma(\square_E)$

- *subresonant if* $\xi < \xi_{res}$*,*
- *resonant if* $\xi = \xi_{res}$*,*
- *superresonant if* $\xi > \xi_{res}$*,*

Since the eigenvalues of the tangential operator $\square_E$ tend to infinity by Remark 6.57, there are only finitely many subresonant and resonant eigenvalues.

9.3 Decay rates in the kernel of the Einstein operator on a cone

The goal of this section is to describe the decay rate of decaying tensor fields in the kernel of the Einstein operator on a Ricci-flat cone. (Note that in this case, the Einstein operator and the Lichnerowicz Laplacian coincide.)

Definition 9.8. *The tensor field* T *on the noncompact Riemannian manifold* (M, g) decays at infinity *if for every positive* $\epsilon > 0$ *there is a compact set* $K_\epsilon \subset M$ *such that* $\sup\{|T|_g(p) \,|p \in M \setminus K_\epsilon\} \leq \epsilon$*.*

Evidently, we may choose the family of compact subsets to be increasing without loss of generality.

Moreover, on an AC manifold, we may choose $K_\epsilon := \mathrm{Core}(r(\epsilon))$ for some increasing function $r\colon (0,\infty) \to (0,\infty)$. Indeed, if K'_ϵ is a family of compact sets satisfying $\sup\{|T|_g(p) \,|p \in M \setminus K_\epsilon\} \leq \epsilon$, then we may choose $r(\epsilon) := \max\{\rho(p) \,|p \in K'_\epsilon\}$. The fact that r is increasing follows from the fact that the family K' may be chosen to be increasing.

Lemma 9.9. *Let T be a smooth tensor field on the Riemannian cone $(\bar{M}, \bar{g})$ decaying at infinity. Moreover, let $U_\alpha := (\alpha, 2\alpha) \times M$. Then $\lim_{\alpha\to\infty} \alpha^{-n-1}\|T\|^2_{L^2(U_\alpha,\bar{g})} = 0$.*

Proof. Let us introduce the temporary notation $f_p(\alpha) := \|T\|_{L^p(U_\alpha,\bar{g})}$.

$$\begin{aligned}
f_2(\alpha)^2 = \|T\|^2_{L^2(U_\alpha,\bar{g})} &= \int_{U_\alpha} |T|^2_{\bar{g}} \operatorname{vol}_{\bar{g}} \\
&\leq \sup_{U_\alpha} |T|^2_{\bar{g}} \int_{U_\alpha} \operatorname{vol}_{\bar{g}} \\
&= f_\infty(\alpha)^2 \int_{U_\alpha} \operatorname{vol}_{\bar{g}} = f_\infty(\alpha)^2 \int_{U_1} \Phi^*_\alpha(\operatorname{vol}_{\bar{g}}) \\
&= f_\infty(\alpha)^2 \int_{U_1} \alpha^{n+1} \operatorname{vol}_{\bar{g}} = f_\infty(\alpha)^2 \alpha^{n+1} \operatorname{Vol}_{\bar{g}}(U_1) \\
&= C(n, U_1)\alpha^{n+1} f_\infty(\alpha)^2,
\end{aligned}$$

where $C(n, U_1) := \operatorname{Vol}_{\bar{g}}(U_1)$. Thus $\alpha^{-(n+1)/2} f_2(\alpha) \leq C f_\infty(\alpha)$.

By definition of decay at infinity, we find for every $\epsilon > 0$ a compact set $K_\epsilon \subset M$ such that $|T|_{\bar{g}}(p) \leq \epsilon$ whenever $p \in \bar{M} \setminus K_\epsilon$. In particular, if we denote $\alpha_\epsilon := \max\{r(p) \mid p \in K_\epsilon\}$ (which is finite by continuity of r and compactness of K_ϵ), we have $|T|_{U_{\alpha_\epsilon}}|_{\bar{g}} \leq \epsilon$. From this it follows that $\lim_{\alpha\to\infty} f_\infty(\alpha) = 0$.

Therefore we have that $0 \leq \alpha^{-n-1} f_2(\alpha)^2 \leq C(n, U_1) f_\infty(\alpha)^2 \to 0$, and the claim follows by the squeeze theorem. □

We have established before in Lemma 9.5 that any symmetric 2-tensor field in the kernel of the Einstein operator is of the form $h = \sum_{i\in\mathbb{N}}(c_i^+ r^{m_+(\xi_i)} + c_i^- r^{m_-(\xi_i)})w_i$. With the assumption of decay at infinity, we can say more.

Corollary 9.10. *For a symmetric 2-tensor field $h \in \ker \Delta_E^{\bar{g}}$ that decays at infinity, we have $c_i^+ = 0$ in the expansion from Lemma 9.5 whenever $\xi_i \geq 0$. Moreover, there are only finitely many indices i such that $\xi_i < 0$.*

Proof. Note that $m_+(\xi_i) \geq 0$ if and only if $\xi_i \geq 0$. Moreover, note that any positive eigenvalue is necessarily superresonant. Suppose to reach contradiction that for some $i_0 \in \mathbb{N}$, we have $m_+(\xi_{i_0}) \geq 0$ and $c_{i_0}^+ \neq 0$. From the fact $\int_{\{r\}\times M} \langle w_i, w_j\rangle_{\bar{g}} \operatorname{vol}_{\bar{g}}|_{\{r\}\times M} = r^n\delta_{ij}$, we have

$$\begin{aligned}
\|h\|^2_{L^2(U_\alpha,\bar{g})} &= \int_{U_\alpha} |h|^2_{\bar{g}} \operatorname{vol}_{\bar{g}} \\
&= \int_{U_\alpha} \left|\sum_{i\in\mathbb{N}} (u_i(r) w_i\right|^2_{\bar{g}} \operatorname{vol}_{\bar{g}} \\
&= \int_\alpha^{2\alpha} \sum_{i,j\in\mathbb{N}} u_i(r)u_j(r) \int_{\{r\}\times M} \langle w_i, w_j\rangle_{\bar{g}} \operatorname{vol}_g r^n dr \\
&= \int_\alpha^{2\alpha} \sum_{i\in\mathbb{N}} u_i(r)^2 r^n dr \\
&\geq \int_\alpha^{2\alpha} u_{i_0}(r)^2 r^n dr
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\alpha}^{2\alpha} (c_{i_0}^{+} r^{m_+(\xi_{i_0})} + c_{i_0}^{-} r^{m_-(\xi_{i_0})})^2 r^n dr \\
&\geq (c_{i_0}^{+})^2 \frac{2^{2m_+(\xi_{i_0})+n+1}-1}{2m_+(\xi_{i_0})+n+1} \alpha^{2m_+(\xi_{i_0})+n+1} \\
&\quad + 2c_{i_0}^{+} c_{i_0}^{-} \frac{2^{m_+(\xi_{i_0})+m_-(\xi_{i_0})+n+1}-1}{m_+(\xi_{i_0})+m_-(\xi_{i_0})+n+1} \alpha^2 \\
&\quad + (c_{i_0}^{-})^2 \frac{2^{2m_-(\xi_{i_0})+n+1}-1}{2m_-(\xi_{i_0})+n+1} \alpha^{2m_-(\xi_{i_0})+n+1} \\
&\overset{r\gg 1}{\geq} \frac{1}{2} (c_{i_0}^{+})^2 \frac{2^{2m_+(\xi_{i_0})+n+1}-1}{2m_+(\xi_{i_0})+n+1} \alpha^{2m_+(\xi_{i_0})+n+1},
\end{aligned}
$$

where in the last step we used that $2m_+(\xi_{i_0})+n+1$ is the largest of the exponents. Thus $\alpha^{-n-1}\|h\|^2_{L^2(U_\alpha,\bar{g})} \geq \frac{1}{2}(c_{i_0}^{+})^2 \frac{2^{2m_+(\xi_{i_0})}-1}{2m_+(\xi_{i_0})} \alpha^{2m_+(\xi_{i_0})} \to \infty$, which contradicts the statement of the basic decay lemma. Thus $c_{i_0}^{+} = 0$ is necessary.

The last claim follows from the fact that the spectra of the Laplacians on the compact manifold M are discrete and the eigenvalues converge to infinity. □

Next, we determine what the decay rate in the kernel is. For this, we need to use some machinery since the different eigenvectors w are orthonormal only in the L^2 sense and not pointwise.

Lemma 9.11. *Let $f\colon (0,\infty) \to \mathbb{R}, r \mapsto c_1 r^{-a_1} + c_2 r^{-a_2}$ where $c_1, c_2 \in \mathbb{R}$ and $a_1, a_2 \in \mathbb{R}^+$ with $a_1 \neq a_2$. Then the absolute value of f is eventually decreasing, i.e. there is a constant $R = R(c_1, c_2, a_1, a_2)$ such that $f|_{[R,\infty)}$ is strictly monotonously decreasing.*

Proof. We have to consider three cases.

$c_1 \cdot c_2 > 0$. In this case, we have that

$$|f(r)| = |c_1 r^{-a_1} + c_2 r^{-a_2}| = |c_1| r^{-a_1} + |c_2| r^{-a_2},$$

and this is manifestly monotonously decreasing on $(0,\infty)$.

$c_1 \cdot c_2 = 0$. In this case, we have for some $i \in \{1,2\}$ that

$$|f(r)| = |c_i r^{-a_i}| = |c_i| r^{-a_i},$$

(with possibly $c_i = 0$) and this is manifestly monotonously decreasing on $(0,\infty)$.

$c_1 \cdot c_2 < 0$. In this case, it is technically easier to study the function f^2. It has a minimum at r_0 where $r_0 := \left|\frac{c_2}{c_1}\right|^{\frac{1}{a_2-a_1}}$ is the only zero locus of f. Moreover, the second derivative test function f^2 has a local maximum at $R := \left|\frac{c_2 a_2}{c_1 a_1}\right|^{\frac{1}{a_2-a_1}}$ is the unique zero locus of f'. Further, $\lim_{r\to\infty} f^2(r) = 0$, thus the function f^2 (and consequently $|f|$, too) is monotonously decreasing on the interval $[R,\infty)$ □

Let us use the following notation $\iota_r \colon \{r\} \times M \hookrightarrow \bar{M}$ and consider the vector bundle $E_r := (\iota_r)^*(S^2\bar{M})$ with the pullback connection $\nabla^r := (\iota_r)^*\nabla^{\bar{g}}$. Note that for the induced volume form on $\{r\} \times M$, we have $(\iota_r)^* \operatorname{vol}_{\bar{g}} = r^n \operatorname{vol}_g$ but it will be advantageous to consider the rescaled volume form vol_g instead (with this convection, we can make use of the normalization condition Equation (6.5)). For any section h of $S^2\bar{M}$, use the notation $h_r := (\iota_r)^*h$ for the induced section on E_r.

Definition 9.12. *The generic decay function is the function (cf. Figure 9.2)*

$$m\colon \mathbb{R} \to \mathbb{R}, \xi \mapsto \begin{cases} -\left(\frac{n-1}{2}\right) & \text{if } \xi \text{ is subresonant or resonant,} \\ m_+(\xi) & \text{if } \xi \text{ is nonpositive and superresonant,} \\ m_-(\xi) & \text{if } \xi \text{ is positive.} \end{cases}$$

Definition 9.13. *We call a Riemannian cone special (or resonance-dominated) if the spectrum of the tangential operator $\Box_E$ includes a resonant eigenvalue but it does not include any negative superresonant eigenvalues. A Riemannian cone is generic if it is not special. By extension, we call the cone metric and the metric on the link special or generic if the corresponding Riemannian cone is special or generic, respectively.*

Proposition 9.14. *If g_{cone} is a Ricci-flat generic cone metric, and $h \in \ker \Delta_E^{g_{\text{cone}}}$ is a tensor field in the kernel of the Einstein operator which decays at infinity, then $h = O\left(r^{-\mu}\right)$ where*

$$\mu := -\min\left\{m(\xi) \mid \xi \in \sigma(\Box_E)\right\}.$$

If g_{cone} is a Ricci-flat special cone metric, and $h \in \ker \Delta_E^{g_{\text{cone}}}$ is a tensor field in the kernel of the Einstein operator which decays at infinity, then $h = O\left(r^{-\frac{n-1}{2}} \ln r\right) = O\left(r^{-\frac{n-1}{2}+\epsilon}\right)$ for any $\epsilon > 0$.

Proof. Note that h is in the kernel of an elliptic operator, therefore it is smooth by elliptic regularity. Also note that μ is defined as a minimum, and this minimum in fact exists since the eigenvalues of the tangential operator $\Box$ approach infinity by Remark 6.57.

Suppose g_{cone} is a generic Ricci-flat cone metric.

By continuity of $\Box_L^l h_r$ for any $l \in \mathbb{N}$ and compactness of L, we have

$$\begin{aligned}
\infty &> \|\Box^l h_r\|^2_{L^2(E_r)} \\
&= \|\Box^l \sum_{i\in\mathbb{N}} u_i(r) w_i\|^2_{L^2(E_r)} \\
&= \|\sum_{i\in\mathbb{N}} u_i(r)\xi_i^l w_i\|^2_{L^2(E_r)} \\
&= \sum_{i,j\in\mathbb{N}} \int_{\{r\}\times M} \left\langle u_i(r)\xi_i^l w_i, u_j(r)\xi_j^l w_j\right\rangle_{\bar{g}} \operatorname{vol}_g \\
&= \sum_{i,j\in\mathbb{N}} u_i(r)\xi_i^l u_j(r)\xi_j^l \int_{\{r\}\times M} \langle w_i, w_j\rangle_{\bar{g}} \operatorname{vol}_g \\
&= \sum_{i,j\in\mathbb{N}} u_i(r)\xi_i^l u_j(r)\xi_j^l \delta_{ij}
\end{aligned}$$

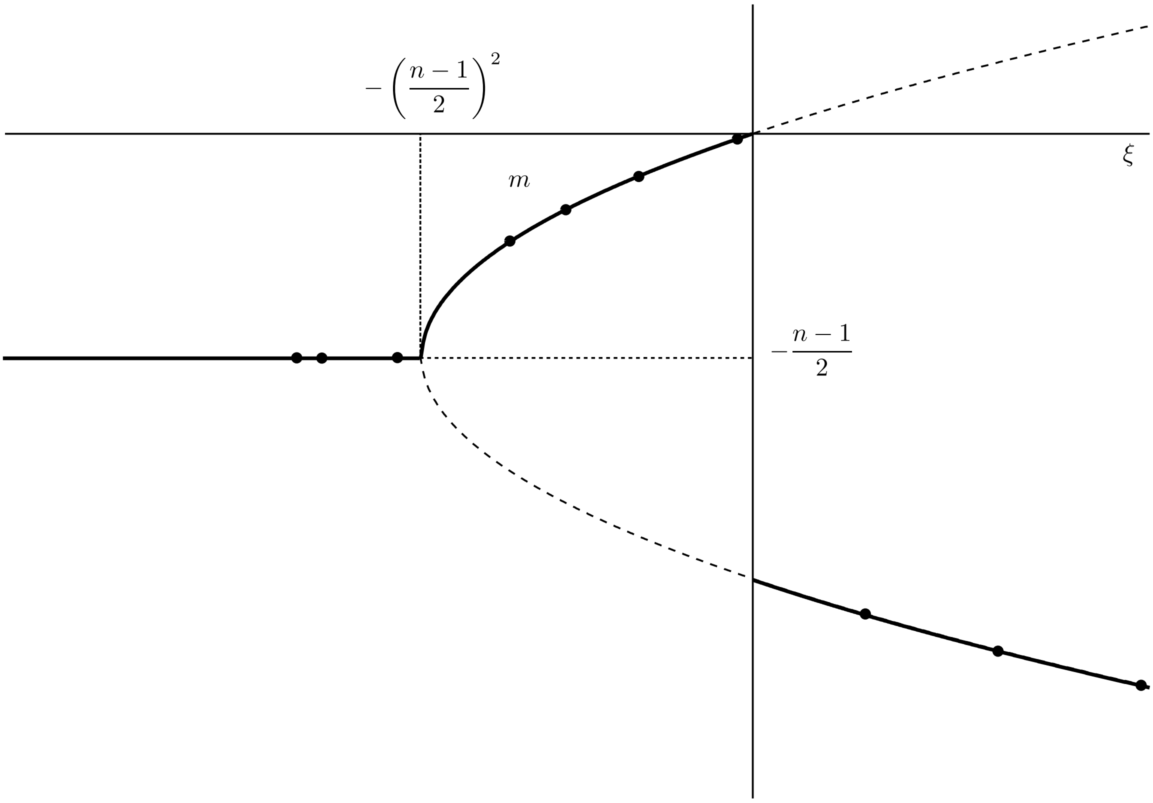


Figure 9.2: The decay rate in the exact case. The decay rate is chosen to be the highest value on the function graphs with solid line for the discrete values $\xi \in \sigma(\square_L)$, represented by dots.

$$
\begin{aligned}
&= \sum_{i\in\mathbb{N}} u_i(r)^2 \xi_i^{2l} \\
&= \sum_{i\in\mathbb{N}} \xi_i^{2l} u_i(r)^2,
\end{aligned}
$$

therefore, we obtain

$$
\begin{aligned}
\sum_{l=0}^{k} \|\Box^l h_r\|^2_{L^2(E_r)} &= \sum_{l=0}^{k}\sum_{i\in\mathbb{N}} \xi_i^{2l} u_i(r)^2 \\
&= \sum_{i\in\mathbb{N}}\sum_{l=0}^{k} \xi_i^{2l} u_i(r)^2 \\
&= \sum_{i\in\mathbb{N}} S_k(\xi_i) u_i(r)^2 < \infty.
\end{aligned}
$$

We need to show that there are constants $C, R \in \mathbb{R}$ such that the left-hand side of the equation in the claim is less than $Cr^{-\mu}$ whenever $r \geq R$.

Let us rearrange the eigenvalues ξ_i in an increasing order, and let i_0 denote the largest index such that $\xi_i < 0$.

$$
r^{2\mu}\sum_{i\in\mathbb{N}} F(\xi_i)u_i(r)^2 = \sum_{i\leq i_0} F(\xi_i) r^{2\mu} u_i(r)^2 + \sum_{i>i_0} F(\xi_i) r^{2\mu} u_i(r)^2
$$

The first sum is finite by Lemma 9.10 and each of its summands is eventually decreasing by Lemma 9.11. Let $R \in \mathbb{R}$ be a number such that all summands are decreasing on $[R, \infty)$. In the second sum, $u_i(r) = c_i^- r^{m_-(\xi_i)}$ hence each summand is a decreasing function in r. All in all, we have for $r \geq R$ that

$$
\begin{aligned}
r^{2\mu}\sum_{i\in\mathbb{N}} F(\xi_i)u_i(r)^2 &= \sum_{i\leq i_0} F(\xi_i) r^{2\mu} u_i(r)^2 + \sum_{i>i_0} F(\xi_i) r^{2\mu} u_i(r)^2 \\
&\leq \sum_{i\leq i_0} F(\xi_i) R^{2\mu} u_i(R)^2 + \sum_{i>i_0} F(\xi_i) R^{2\mu} u_i(R)^2 \\
&= \sum_{l=0}^{k} \|\Box^l h_R\|^2_{L^2(E_R)} =: C < \infty
\end{aligned}
$$

The claim now follows from Lemma 9.2. Note that the potential presence of a resonant eigenvalue causes no trouble if there is at least one negative superresonant eigenvalue which then dominates the logarithmic factor.

The case for a special Ricci-flat cone metric can be shown similarly. □

9.4 Functions fit for iteration

Definition 9.15. *A strictly monotonously increasing continuous function $F\colon (0,\infty) \to (0,\infty)$ is called a function fit for iteration if for all $x, y \in (0,\infty)$, the inequality $F(x) - F(y) \geq x - y$ holds.*

Example 9.16. *Fix a real number $\tau > 0$. Then the following functions*

- $F\colon (0,\infty) \to (0,\infty), x \mapsto x + \tau$,
- $F\colon (0,\infty) \to (0,\infty), x \mapsto \min\{2x, x+\tau\}$

are fit for iteration.

The most important property of functions fit for iteration is that we can construct a convergent sequence by

Lemma 9.17. *Let $F\colon (0,\infty) \to (0,\infty)$ be a function fit for iteration. Moreover, let $\mu, a_0 > 0$ and define a sequence $(a_n)_{n\in\mathbb{N}}$ recursively via*

$$a_{n+1} := \begin{cases} \min\{F(a_n), \mu\} & \text{if } F(a_n) \neq \mu, \\ \mu + \frac{1}{2}(a_n - \mu) & \text{if } F(a_n) = \mu \end{cases}$$

for any $n \geq 1$. Then $\lim_{n\to\infty} a_n = \mu$.

Proof. If $a_0 \geq \mu$, then $a_n = \mu$ for all $n \geq 1$ so, in particular, $\lim_{n\to\infty} a_n = \mu$. In the following, we will treat the case where $0 < a_0 < \mu$. Note that the condition on F entails in particular that F is a strictly monotonously increasing function.

1. The sequence $(a_n)_{n\in\mathbb{N}}$ is strictly monotonously increasing as the following argument shows.

   - If $F(a_n) = \mu$, then $a_{n+1} = \mu + \frac{1}{2}(a_n - F(a_n)) > a_n$.
   - If, however, $F(a_n) \neq \mu$, then $a_{n+1} = \min\{F(a_n), \mu\} > a_n$.

   Moreover, the sequence is bounded above by μ. This means that a is convergent and its limit is its supremum.

2. Next we show that the sequence $(b_n := F^n(a_0))_{n\in\mathbb{N}}$ grows without bound. Indeed, for any $n \geq 1$, we have the following telescopic sum

   $$\begin{aligned} b_n - b_0 &= (b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \cdots + (b_2 - b_1) + (b_1 - b_0) \\ &= (F(b_{n-1}) - F(b_{n-2})) + (F(b_{n-2}) - F(b_{n-3})) + \cdots + (F(b_1) - F(b_0)) + (F(b_0) - b_0) \\ &\geq (b_{n-1} - b_{n-2}) + (b_{n-2} - b_{n-3}) + \cdots + (b_1 - b_0) + (F(b_0) - b_0) \\ &= b_{n-1} - b_0 + (\underbrace{F(b_0) - b_0}_{=:\delta>0}), \end{aligned}$$

   where we used the growth property of F. In particular, we obtain $b_n = F(a_{n-1}) \geq F(a_{n-2}) + \delta \geq \cdots \geq F(a_0) + n\delta$.

3. Now we show that $\sup a_n = \mu$. It is clear that $\mu + \frac{1}{2}(a_n - \mu) \leq \mu$ if $F(a_n) = 0$ so it suffices to treat the other case, i.e. when $a_n = \min\{b_n, \mu\}$. We show that for any $\epsilon > 0$ the number $\mu - \epsilon$ is not an upper bound for the sequence b. Indeed let δ be as in the last step and let $n > \frac{\mu - \epsilon/2 - a_0}{\delta}$. From the last step we obtain that

   $$\mu - \epsilon < \mu - \frac{\epsilon}{2} \leq a_0 + n\delta \leq b_n.$$

   Thus $\mu - \epsilon$ is not an upper bound. □

Remark 9.18. *As already noted, any function fit for iteration is strictly monotonously increasing but merely demanding that the function F is strictly monotonously increasing is not enough to draw the conclusion, as the next counterexample shows. Let $F\colon (0,\infty) \to (0,\infty), x \mapsto \frac{\mu}{4} + \frac{x}{2}$, and suppose $a_0 \in (0, \frac{\mu}{2})$. Then, with the notations of the proof, we have $b_n = \frac{\mu}{4}\left(\sum_{k=0}^{n} \frac{1}{2^k}\right) + \frac{1}{2^n} a_0$. Consequently, $F(a_n) \neq \mu$ for any n, and therefore $\lim a_n = \lim b_n = \frac{\mu}{2} < \mu$.*

9.5 Decay rates in the kernel of the Einstein operator on an AC manifold

The goal of this section is to prove that the decay statement of Proposition 9.14 holds for asymptotically conical manifolds, too. The intuitive reason why this is to be expected is that the difference between the exact conical and the asymptotically conical geometries "decays faster than solutions in the kernel". The proof is based on a bootstrap argument.

Recall that we can write any symmetric 2-tensor field h on the exact cone, and consequently on the end of an asymptotically conical manifold, as $h = \sum_{i\in\mathbb{N}} u_i r^2 W_i$ where $u_i\colon I \to \mathbb{R}$ is some function. In the next lemma, we describe the decaying elements in the kernel of $\Delta_L^{g_0}$ in terms of independent ordinary differential equations for these component functions u_i.

Lemma 9.19. *Let M be a manifold with a single end and let $g_0 \in \mathrm{AC}\,(g_{\mathrm{cone}}, \tau, \phi)$ be a Ricci-flat asymptotically conical metric on M. Suppose $h \in \ker \Delta_L^{g_0}$ with $|h|_{g_{cone}} = O\left(r^{-\alpha}\right)$, $|\nabla^{g_{cone}} h|_{g_{cone}} = O\left(r^{-\alpha-1}\right)$ and $|(\nabla^{g_{cone}})^2 h|_{g_{cone}} = O\left(r^{-\alpha-2}\right)$ for some $\alpha < 0$. Then for all $i \in \mathbb{N}$, the function $u_i\colon I \to \mathbb{R}$ satisfies the ordinary differential equation*

$$-u_i'' - \frac{n}{r} u_i' - \frac{\xi_i}{r^2} u_i = f_i r^{-\alpha-\tau-2}, \tag{9.5}$$

where $f_i\colon I \to \mathbb{R}$ is a smooth function. Moreover, there is a universal constant $R \in I$ and a sequence of numbers $C_i \in \mathbb{R}_+$ such that $u_i|_{[R,\infty)} \le C_i$.

Proof. Based on , for any $h \in \ker \Delta_L^{g_0}$, we have

$$\begin{aligned}
0 = \Delta_L^{g_0} h &= \Delta_L^{g_{\mathrm{cone}}} h + (\Delta_L^{g_0} - \Delta_L^{g_{\mathrm{cone}}}) h \\
&= \Delta_L^{g_{\mathrm{cone}}} h \\
&\quad + (g_0 - g_{\mathrm{cone}}) \star (\nabla^{g_{\mathrm{cone}}})^2 h + \nabla^{g_{\mathrm{cone}}}(g_{\mathrm{cone}} - g_0) \star \nabla^{g_{\mathrm{cone}}} h \\
&\quad + (\nabla^{g_{\mathrm{cone}}})^2 (g_{\mathrm{cone}} - g_0) \star \mathring{R} h + (R^{g_0} - R^{g_{\mathrm{cone}}}) \star h \\
&= \Delta_L^{g_{\mathrm{cone}}} h \\
&\quad + O\left(r^{-\tau}\right) O\left(r^{-\alpha-2}\right) + O\left(r^{-\tau-1}\right) O\left(r^{\alpha-1}\right) \\
&\quad + O\left(r^{-\tau-2}\right) O\left(r^{-\alpha}\right) + O\left(r\right) O\left(r^{-\alpha}\right) \\
&= \Delta_L^{g_{\mathrm{cone}}} h + O\left(r^{-\alpha-\tau-2}\right).
\end{aligned}$$

This means that for any $\Delta_L^{g_0}$-harmonic h satisfying the decay properties in the statement of the lemma satisfies

$$\Delta_L^{g_{\mathrm{cone}}} h = O\left(r^{-\alpha-\tau-2}\right).$$

Without loss of generality we can introduce a symmetric 2-tensor field f with

$$\Delta_L^{g_{\mathrm{cone}}} h = f r^{-\alpha-\tau-2}. \tag{$\star$}$$

Note that $\Delta_L^{g_0}$ is an elliptic operator, therefore h must be smooth, consequently $\Delta_L^{g_{\mathrm{cone}}} h$ is also smooth, which in turn means that f must be smooth. Moreover, f is bounded outside some compact set K, which we can take without loss of generality to be $[R,\infty) \times L$ for some $R \in I$ such that the compact set in the definition of the asymptotic metric is in the complement of $[R,\infty) \times L$.

Let us introduce the function $f_i \colon I \to \mathbb{R}, r \mapsto \fint_{\{r\}\times L} \langle f, r^2 W_i \rangle_{g_{\mathrm{cone}}} \mathrm{vol}_{g_{\mathrm{link}}}$. This function is smooth by construction, and it is also bounded since the Cauchy–Schwarz inequality implies that

$$\begin{aligned} |f_i(r)| &\leq \|f\|_{L^2(\{r\}\times L, \mathrm{vol}_{g_{\mathrm{link}}})} \cdot \underbrace{\|r^2 W_i\|_{L^2(\{r\}\times L), \mathrm{vol}_{g_{\mathrm{link}}})}}_{=1} \\ &\leq \|f\|_{L^\infty(\{r\}\times L, \mathrm{vol}_{g_{\mathrm{link}}})} \cdot \mathrm{Vol}_{g_{\mathrm{link}}}(\{r\}\times L) < \infty. \end{aligned}$$

Note also that $\fint_{\{r\}\times L} \langle h, r^2 W_i \rangle_{g_{\mathrm{cone}}} \mathrm{vol}_{g_{\mathrm{link}}} = -u_i''(r) - \frac{n}{r} u_i'(r) - \frac{\xi_i}{r^2} u_i(r)$. This way, applying the operator $\fint_{\{r\}\times L} \langle \cdot, r^2 W_i \rangle_{g_{\mathrm{cone}}} \mathrm{vol}_{g_{\mathrm{link}}}$ to Equation ($\star$), we obtain

$$-u_i'' - \frac{n}{r} u_i' - \frac{1}{r^2} \xi_i u_i = f_i r^{-\alpha-\tau-2}. \qquad \square$$

Luckily, Equation (9.5) can be solved explicitly.

Lemma 9.20. *Fix an $r_0 \in I$ bigger than the asymptotic threshold and also bigger than 1. If ξ is nonresonant, then solutions of Equation (9.5) are of the form*

$$\begin{aligned} u(r) = \frac{1}{2\sqrt{\left(\frac{n-1}{2}\right)^2 + \xi}} \Bigg(& r^{m_+(\xi)} \left(\int_{r_0}^{r} f(s) s^{-m_+(\xi)-\alpha-\tau-1} ds + C_+ \right) \\ & - r^{m_-(\xi)} \left(\int_{r_0}^{r} f(s) s^{-m_-(\xi)-\alpha-\tau-1} ds + C_- \right) \Bigg), \end{aligned} \tag{9.6}$$

where $C_\pm \in \mathbb{R}$ are free constants. If ξ is resonant, i.e. if $\xi = -\left(\frac{n-1}{2}\right)^2$, then solutions of Equation (9.5) are of the form

$$\begin{aligned} u(r) = & \, r^{-\frac{n-1}{2}} \ln r \cdot \left(\int_{r_0}^{r} f(s) s^{n/2-3/2-\alpha-\tau} ds + C_+ \right) \\ & - r^{-\frac{n-1}{2}} \left(\int_{r_0}^{r} f(s) s^{n/2-3/2-\alpha-\tau} \ln s \, ds + C_- \right) \end{aligned}$$

where $C_\pm \in \mathbb{R}$ are free constants.

Proof. Equation (9.5) can be solved using the method of variation of parameters.

Suppose ξ is nonresonant, i.e. $\xi \neq -\left(\frac{n-1}{2}\right)^2$. The homogeneous equation is an Euler–Cauchy equation and it can be solved using the ansatz $u(r) = r^a$. As we have seen in the exact cone case in Lemma 9.5, a fundamental system of solutions is given by $u_\pm(r) = r^{m_\pm(\xi)}$. The corresponding Wrońskian is

$$W(r) = \det \begin{pmatrix} u_+ & u_- \\ u_+' & u_-' \end{pmatrix} = -2r^{-n}\sqrt{\left(\frac{n-1}{2}\right)^2 + \xi},$$

which is nonzero by assumption. Let us introduce $A_\pm \colon I \to \mathbb{R}$ via

$$\begin{aligned} A_\pm(r) &:= \mp \int_{r_0}^r \frac{1}{W(s)} u_\mp(s) f(s) s^{-\alpha-\tau-2} ds \\ &= \pm \frac{1}{2\sqrt{\left(\frac{n-1}{2}\right)^2+\xi}} \int_{r_0}^r f(s) s^{-n-1+m_\mp(\xi)-\alpha-\tau-1} ds \\ &= \pm \frac{1}{2\sqrt{\left(\frac{n-1}{2}\right)^2+\xi}} \int_{r_0}^r f(s) s^{-m_\pm(\xi)-\alpha-\tau-1} ds. \end{aligned}$$

Here we used the identity $-n-1+m_\mp(\xi) = -m_\pm(\xi)$. The general solution of Equation (9.5) can be written as

$$\begin{aligned} u(r) &= (A_+(r) + \tilde{C}_+)u_+(r) + (A_-(r) + \tilde{C}_-)u_-(r) \\ &= r^{m_+(\xi)} \left(\frac{1}{2\sqrt{\left(\frac{n-1}{2}\right)^2+\xi}} \int_{r_0}^r f(s) s^{-m_+(\xi)-\alpha-\tau-1} ds + \tilde{C}_+ \right) \\ &\quad - r^{m_-(\xi)} \left(\frac{1}{2\sqrt{\left(\frac{n-1}{2}\right)^2+\xi}} \int_{r_0}^r f(s) s^{-m_-(\xi)-\alpha-\tau-1} ds + \tilde{C}_- \right) \\ &= \frac{1}{2\sqrt{\left(\frac{n-1}{2}\right)^2+\xi}} \Bigg(r^{m_+(\xi)} \left(\int_{r_0}^r f(s) s^{-m_+(\xi)-\alpha-\tau-1} ds + C_+ \right) \\ &\qquad - r^{m_-(\xi)} \left(\int_{r_0}^r f(s) s^{-m_-(\xi)-\alpha-\tau-1} ds + C_- \right) \Bigg), \end{aligned}$$

where $\tilde{C}_\pm \in \mathbb{R}$ and $C_\pm := 2\tilde{C}_\pm \sqrt{\left(\frac{n-1}{2}\right)^2 + \xi}$.

Suppose now that ξ is resonant, i.e. $\xi = -\left(\frac{n-1}{2}\right)^2$. Then the functions

$$u_+(r) := r^{-\frac{n-1}{2}} \log r \quad \text{and} \quad u_-(r) := r^{-\frac{n-1}{2}}$$

form a fundamental system for the homogeneous differential equation, and the corresponding Wrońskian is

$$W(r) = \det \begin{pmatrix} u_+ & u_- \\ u_+' & u_-' \end{pmatrix} = -r^{-n},$$

which is nonzero. As before, let us introduce $A_\pm \colon I \to \mathbb{R}$ via

$$A_\pm(r) := \mp \int_{r_0}^r \frac{1}{W(s)} u_\mp(s) f(s) s^{-\alpha-\tau-2} ds$$

$$= \pm \int_{r_0}^{r} f(s)u_{\mp}(s)s^{n-\alpha-\tau-2}ds.$$

The general solution of Equation (9.5) can be written as

$$\begin{aligned} u(r) &= (A_+(r) + \tilde{C}_+)u_+(r) + (A_-(r) + \tilde{C}_-)u_(r)- \\ &= r^{-\frac{n-1}{2}} \ln r \cdot \left(\int_{r_0}^{r} f(s)s^{n/2-3/2-\alpha-\tau}ds + C_+ \right) \\ &\quad - r^{-\frac{n-1}{2}} \left(\int_{r_0}^{r} f(s)s^{n/2-3/2-\alpha-\tau} \ln s \, ds + C_- \right) \end{aligned}$$

where $C_\pm \in \mathbb{R}$ are free constants. □

The next subgoal is to establish a decay estimate for solutions (9.6) of Equation (9.5). We start this investigation with a series of computationally intensive lemmata, the results of which are summarized in Proposition 9.23 of page 120.

Lemma 9.21. *Let $\xi \geq 0$. The function $A_+ \colon I \to \mathbb{R}, r \mapsto \int_{r_0}^{r} s^{-m_+(\xi)-\alpha-\tau-1} f(s)ds$ converges to a finite value as $r \to \infty$. If the solution u of Equation9.5 tends to zero at infinity, then*

$$u_i = \begin{cases} O\left(r^{m_-(\xi)}\right) & \text{if } -\alpha-\tau < m_-(\xi) \\ O\left(r^{m_-(\xi)} \ln r\right) & \text{if } -\alpha-\tau = m_-(\xi) \\ O\left(r^{-\alpha-\tau}\right) & \text{if } m_-(\xi) < -\alpha-\tau. \end{cases}$$

Proof. For the first claim, note that, since the standard topology of $\mathbb{R}$ is metrizable, it suffices to show that the function A_+ converges sequentially. Since $\mathbb{R}$ is complete, it suffices to show Cauchy convergence. Let $(r_n)_{n\in\mathbb{N}}$ be any sequence in I converging to infinity. Define the sequence $A_n := \int_{r_0}^{r_n} s^{-m_+(\xi)-\alpha-\tau-1} f(s)ds$. For $n, m \in \mathbb{N}$, we have

$$\begin{aligned} |A_n - A_m| &= \frac{1}{2\sqrt{\left(\frac{n-1}{2}\right)^2+\xi}} \left| \int_{r_n}^{r_m} s^{-m_+(\xi)-\alpha-\tau-1} f(s)ds \right| \\ &\leq C_1 \left| \int_{r_n}^{r_m} s^{-m_+(\xi)-\alpha-\tau-1} f(s)ds \right| \\ &= C_2 \left| r_m^{-m_+(\xi)-\alpha-\tau} - r_n^{-m_+(\xi)-\alpha-\tau} \right|. \end{aligned}$$

Now we show that $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\epsilon > 0$. Then, by convergence to infinity, there is a number $N_\epsilon \in \mathbb{N}$ such that $r_p \geq \left(\frac{2C_2}{\epsilon}\right)^{\frac{1}{m_+(\xi)+\alpha+\tau}}$ whenever $p \geq N_\epsilon$. But then $r_p^{-m_+(\xi)-\alpha-\tau} \leq \frac{\epsilon}{2C_2}$, thus whenever $n, m \geq N_\epsilon$, we have

$$\begin{aligned} |A_n - A_m| &\leq C_2 \left| r_m^{-m_+(\xi)-\alpha-\tau} - r_n^{-m_+(\xi)-\alpha-\tau} \right| \\ &\leq C_2 (r_m^{-m_+(\xi)-\alpha-\tau} + r_n^{-m_+(\xi)-\alpha-\tau}) \\ &\leq C_2 \left(\frac{\epsilon}{2C_2} + \frac{\epsilon}{2C_2} \right) = \epsilon, \end{aligned}$$

showing Cauchy convergence, and thereby the first claim.

Consequently, $A_+(r)+C_+ = -\int_r^\infty s^{-m_+(\xi)-\alpha-\tau-1}f(s)ds + K_+$ for some constant $K_+ \in \mathbb{R}$, and, if $K_+ \neq 0$, then there exists a threshold $R>0$ such that $A_+(r)+C_+ > \frac{1}{2}K_+$ for all $r>R$, thus $u \to 0$ is impossible. Thus $K_+=0$ and correspondingly, by boundedness of the function f, we obtain[1]

$$\begin{aligned}
|u(r)| &= \Bigg| \frac{1}{2\sqrt{\left(\frac{n-1}{2}\right)^2+\xi}} \left(r^{m_+(\xi)} \left(\int_{r_0}^r f(s) s^{-m_+(\xi)-\alpha-\tau-1} ds + C_+ \right) \right. \\
&\qquad \left. - r^{m_-(\xi)} \left(\int_{r_0}^r f(s) s^{-m_-(\xi)-\alpha-\tau-1} ds + C_- \right) \right) \Bigg| \\
&= \Bigg| - \frac{1}{2\sqrt{\left(\frac{n-1}{2}\right)^2+\xi}} \left(r^{m_+(\xi)} \left(\int_r^\infty f(s) s^{-m_+(\xi)-\alpha-\tau-1} ds \right) \right. \\
&\qquad \left. + r^{m_-(\xi)} \left(\int_{r_0}^r f(s) s^{-m_-(\xi)-\alpha-\tau-1} ds + C_- \right) \right) \Bigg| \\
&\le C_1 \left(r^{m_+(\xi)} \int_r^\infty s^{-m_+(\xi)-\alpha-\tau-1} ds + r^{m_-(\xi)} \int_{r_0}^r s^{-m_-(\xi)-\alpha-\tau-1} ds \right) \\
&\le C_2 \left(r^{m_+(\xi)} r^{-m_+(\xi)-\alpha-\tau} + r^{m_-(\xi)} \int_{r_0}^r s^{-m_-(\xi)-\alpha-\tau-1} ds \right) \\
&\le C_2 \left(r^{-\alpha-\tau} + r^{m_-(\xi)} \int_{r_0}^r s^{-m_-(\xi)-\alpha-\tau-1} ds \right)
\end{aligned}$$

For the remaining integral, now we need to consider the following cases.

Case 1: $m_-(\xi) = -\alpha-\tau$. In this case, the integral evaluates to $\ln(r/r_0)$, thus $|u(r)| = O\left(r^{-\alpha-\tau}\right) + O\left(r^{m_-(\xi)} \ln r\right) = O\left(r^{-\alpha-\tau}\ln r\right)$.

Case 2: $m_-(\xi) \neq -\alpha-\tau$. In this case, the integral evaluates to

$$C_3(r^{-m_-(\xi)-\alpha-\tau} - (r_0)^{-m_-(\xi)-\alpha-\tau}),$$

and thus $|u(r)| = O\left(r^{-\alpha-\tau}\right) + O\left(r^{m_-(\xi)}\right)$.

The second claim follows. ☐

Lemma 9.22. *Let $\xi<0$ be superresonant. Then*

$$u_i = \begin{cases} O\left(r^{m_+(\xi)}\right) & \text{if } -\alpha-\tau < m_-(\xi) \\ O\left(r^{m_+(\xi)}\ln r\right) & \text{if } -\alpha-\tau = m_-(\xi) \\ O\left(r^{-\alpha-\tau}\right) & \text{if } m_-(\xi) < -\alpha-\tau. \end{cases}.$$

Let $\xi = -\left(\frac{n-1}{2}\right)^2 < 0$ be resonant. Then

$$u_i = \begin{cases} O\left(r^{-\frac{n-1}{2}}\ln r\right) & \text{if } -\alpha-\tau < -\frac{n-1}{2} \\ O\left(r^{-\frac{n-1}{2}}(\ln r)^2\right) & \text{if } -\alpha-\tau = -\frac{n-1}{2} \\ O\left(r^{-\alpha-\tau}\ln r\right) & \text{if } -\frac{n-1}{2} < -\alpha-\tau. \end{cases}.$$

[1]Note that $\xi \ge 0$ is alway superresonant.

Let $\xi < 0$ be subresonant. Then

$$u_i = \begin{cases} O\left(r^{-\frac{n-1}{2}}\right) & \text{if } -\alpha-\tau < -\frac{n-1}{2} \\ O\left(r^{-\frac{n-1}{2}}\ln r\right) & \text{if } -\alpha-\tau = -\frac{n-1}{2} \\ O\left(r^{-\alpha-\tau}\right) & \text{if } -\frac{n-1}{2} < -\alpha-\tau. \end{cases}.$$

Proof. • First, let us assume that $\xi_{\mathrm{res}} < \xi < 0$ is nonresonant. From now on, we use $m_\pm := m_\pm(\xi)$ in this proof in order to ease notation. Note that in this case, there is a strict inequality $m_-(\xi) < m_+(\xi)$. From Lemma 9.20, we have that

$$u(r) = \frac{1}{2\sqrt{\left(\frac{n-1}{2}\right)^2+\xi}}\left(r^{m_+(\xi)}\left(\int_{r_0}^r f(s)s^{-m_+(\xi)-\alpha-\tau-1}ds + C_+\right)\right.$$
$$\left. -r^{m_-(\xi)}\left(\int_{r_0}^r f(s)s^{-m_-(\xi)-\alpha-\tau-1}ds + C_-\right)\right),$$

thus, from boundedness of f and standard estimates, we obtain the estimate

$$|u(r)| = C_1\left(r^{m_+}\left(\int_{r_0}^r s^{-m_+-\alpha-\tau-1}ds + |C_+|\right) + r^{m_-}\left(\int_{r_0}^r s^{-m_--\alpha-\tau-1}ds + |C_-|\right)\right). \tag{9.7}$$

Based on the relative position of $-\alpha-\tau$ with respect to $m_\pm$, we have five case to consider.

Case 1 $m_- < m_+ < -\alpha-\tau$,

Case 2 $m_- < m_+ = -\alpha-\tau$,

Case 3 $m_- < -\alpha-\tau < m_+$,

Case 4 $m_- = -\alpha-\tau < m_+$,

Case 5 $-\alpha-\tau < m_- < m_+$.

Cases 1,3 and 5 may be dealt with at once (note that here $m_- \neq -\alpha-\tau \neq m_+$):

$$\begin{aligned} |u(r)| &= C_1\left(r^{m_+}\left(\int_{r_0}^r s^{-m_+-\alpha-\tau-1}ds + |C_+|\right) + r^{m_-}\left(\int_{r_0}^r s^{-m_--\alpha-\tau-1}ds + |C_-|\right)\right) \\ &= C_2\left(r^{m_+}(r^{-m_+-\alpha-\tau} + C_3) + r^{m_-}(r^{-m_--\alpha-\tau} + C_4)\right) \\ &= O\left(r^{m_+}\right) + O\left(r^{-\alpha-\tau}\right) + O\left(r^{m_-}\right) \\ &= O\left(r^{m_+}\right) + O\left(r^{-\alpha-\tau}\right). \end{aligned}$$

In case 2, we have a logarithmic integral

$$\begin{aligned} |u(r)| &= C_1\left(r^{m_+}\left(\int_{r_0}^r s^{-m_+-\alpha-\tau-1}ds + |C_+|\right) + r^{m_-}\left(\int_{r_0}^r s^{-m_--\alpha-\tau-1}ds + |C_-|\right)\right) \\ &= C_2\left(r^{m_+}(\ln(r/r_0) + C_3) + r^{m_-}(r^{-m_--\alpha-\tau} + C_4)\right) \\ &= O\left(r^{m_+}\ln r\right) + O\left(r^{m_+}\right) + O\left(r^{-\alpha-\tau}\right) + O\left(r^{m_-}\right) \\ &= O\left(r^{m_+}\ln r\right). \end{aligned}$$

Similarly, in case 4, we also have a logarithmic integral

$$\begin{aligned}|u(r)| &= C_1\left(r^{m_+}\left(\int_{r_0}^r s^{-m_+-\alpha-\tau-1}ds + |C_+|\right) + r^{m_-}\left(\int_{r_0}^r s^{-m_--\alpha-\tau-1}ds + |C_-|\right)\right)\\ &= C_2\left(r^{m_+}(r^{-m_+-\alpha-\tau} + C_3) + r^{m_-}(\ln(r/r_0) + C_4)\right.\\ &= O\left(r^{m_+}\right) + O\left(r^{-\alpha-\tau}\right) + O\left(r^{m_-}\ln r\right) + O\left(r^{m_-}\right)\\ &= O\left(r^{m_+}\right) + O\left(r^{-\alpha-\tau}\right).\end{aligned}$$

- Let us assume now that $\xi = \xi_{\mathrm{res}}$ is resonant. From Lemma 9.20, we have that

$$\begin{aligned}u(r) = r^{-\frac{n-1}{2}}\ln r\cdot&\left(\int_{r_0}^r f(s)s^{n/2-3/2-\alpha-\tau}ds + C_+\right)\\ &- r^{-\frac{n-1}{2}}\left(\int_{r_0}^r f(s)s^{n/2-3/2-\alpha-\tau}\ln s\, ds + C_-\right)\end{aligned}$$

thus, from boundedness of f and standard estimates, we obtain the estimate (valid for $r > \max\{R, 1\}$)

$$\begin{aligned}|u(r)| \le C_1 r^{-\frac{n-1}{2}}&\left(\ln r\cdot\left(\int_{r_0}^r s^{(n-1)/2-\alpha-\tau-1}ds + |C_+|\right)\right.\\ &\left. + \int_{r_0}^r f(s)s^{(n-1)/2-\alpha-\tau-1}\ln s\, ds + |C_-|\right).\end{aligned} \tag{9.8}$$

Note that for any real number $a \in \mathbb{R}$, we have the indefinite integral

$$\int r^{a-1}\ln r\, dr = \begin{cases} C_2 r^a\ln r + C_3 r^a & \text{if } a \neq 0,\\ C_4\ln r + C_5(\ln r)^2 & \text{if } a = 0\end{cases}$$

for some real numbers $C_2, C_3, C_4, C_5 \in \mathbb{R}$. Based on the relative position of $-\alpha-\tau$ with respect to $-\frac{n-1}{2}$, we have two case to consider.

If $-\alpha-\tau = -\frac{n-1}{2}$, then we obtain

$$\begin{aligned}|u(r)| &\le C_1\left(r^{-\frac{n-1}{2}}(\ln(r/r_0) + |C_+|) + C_4\ln r + C_5(\ln r)^2 + |C_-|\right)\\ &= O\left(r^{-\frac{n-1}{2}}(\ln r)^2\right).\end{aligned}$$

If $-\alpha-\tau \neq -\frac{n-1}{2}$, then we obtain

$$\begin{aligned}|u(r)| &\le C_1 r^{-\frac{n-1}{2}}\left(\ln r\cdot\left(C_6 r^{\frac{n-1}{2}-\alpha-\tau} + C_7\right) + C_1 r^{\frac{n-1}{2}-\alpha-\tau}\ln r + C_2 + |C_-|\right)\\ &= O\left(r^{-\frac{n-1}{2}}\ln r\right) + O\left(r^{-\alpha-\tau}\ln r\right) + O\left(r^{-\frac{n-1}{2}}\ln r\right) + O\left(r^{-\frac{n-1}{2}}\right)\\ &= O\left(r^{\max\left\{-\frac{n-1}{2}, -\alpha-\tau\right\}}\ln r\right).\end{aligned}$$

Hence the claim.

- Let $\xi < -\left(\frac{n-1}{2}\right)^2$ be subresonant. Then a calculation similar to the superresonant case shows the claim.

□

The decay properties of the component function can be summarized as follows. Recall the generic decay function m from Definition 9.12.

Proposition 9.23. *Let $\alpha > 0$ and $u = O(r^{-\alpha})$ is a decaying solution of Equation (9.5) with eigenvalue ξ. If ξ is nonresonant, then*

$$u = \begin{cases} O\left(r^{-\min\{-\alpha-\tau, m(\xi)\}}\right) & \text{if } -\alpha-\tau \neq m(\xi) \\ O\left(r^{m(\xi)}\ln r\right) & \text{if } -\alpha-\tau = m(\xi) \end{cases}.$$

On the other hand, if $\xi = -\left(\frac{n-1}{2}\right)^2$ is resonant, then

$$u = \begin{cases} O\left(r^{-\min\{-\alpha-\tau, m(\xi)\}}\ln r\right) & \text{if } -\alpha-\tau \neq m(\xi) \\ O\left(r^{m(\xi)}(\ln r)^2\right) & \text{if } -\alpha-\tau = m(\xi) \end{cases}.$$

As a consequence, we obtain the decay rate of decaying elements in the kernel of the Einstein operator of asymptotically conical metrics.

Theorem 9.24. *Let g_{ac} be an asymptotically conical metric with decay rate τ which is not resonance-dominated. Let $\alpha > 0$ be a real number and suppose $h \in \ker \Delta_L^{g_{\mathrm{ac}}}$ with $h = O(r^{-\alpha})$. Then $h = O(r^{-\mu})$ where $\mu := -\min\{m(\xi) \mid \xi \in \sigma(\Delta_L^{g_{\mathrm{ac}}})\}$ where m is the generic decay rate function from Definition 9.12.*

Proof. We may develop h as $h(r,x) = \sum_{i\in\mathbb{N}} u_i(r) r^2 W_i(x)$. Here, each $u_i = O(r^{-\alpha})$ by the decay assumption. Moreover, by Proposition 9.23, we have that, in fact $u_i = O(r^{-\alpha_{\mathrm{new}}})$ where α_{new} may be chosen as

$$\alpha_{\mathrm{new}} = \begin{cases} \min\{F(\alpha), \mu\} & \text{if } F(\alpha) \neq \mu \\ \mu + \frac{1}{2}(\alpha - \mu) & \text{if } F(\alpha) = \mu, \end{cases}$$

where $F\colon (0,\infty) \to (0,\infty), x \mapsto x + \tau$ since in the case where $F(\alpha) = \mu$, we have $r^{-\mu}\ln r = O\left(r^{\mu + \frac{1}{2}(\alpha-\mu)}\right)$. Note that F is a function fit for iteration. Since the decay rate of h is determined by the biggest of the decay rates of the u_i, the above iteration corresponds exactly to the situation described in Lemma 9.17. This finishes the proof. $\square$

Chapter 10

Iterative improvement of the decay rate

10.1 First-order decay of the difference tensor for gauged metrics

Our goal in this section is to establish that if we have a gauged metric $g \in \mathcal{F}_{g_0}$ with $g - g_0 = O\left(\rho^{-\alpha}\right)$, then $\nabla^{g_0} h = O\left(\rho^{-\alpha-1}\right)$

Note that the condition $g = g_0 + h \in \mathcal{F}_{g_0}$ for Ricci-flat g_0 can be expressed [Shi89, Lemma 2.1] or [DK20, Equation (5)] as a quasi-linear partial differential equation

$$\begin{aligned} &g^{ab}\nabla^{g_0,2}_{ab} h_{ij} + h_{ab} g^{ka} (g_0)^{lb} g_{ip} (g_0)^{pq} (R^{g_0})_{jklq} + h_{ab} g^{ka} (g_0)^{lb} g_{jp} (g_0)^{pq} (R^{g_0})_{iklq} \\ &= -g^{ab} g^{pq} \left(\frac{1}{2} \nabla^{g_0}_i h_{pa} \nabla^{g_0}_j h_{qb} + \nabla^{g_0}_a h_{jp} \nabla^{g_0}_q h_{ib} \right) \\ &\quad + g^{ab} g^{pq} \left(\nabla^{g_0}_a h_{jp} \nabla^{g_0}_b h_{iq} - \nabla^{g_0}_j h_{pa} \nabla^{g_0}_b h_{iq} - \nabla^{g_0}_i h_{pa} \nabla^{g_0}_b h_{jq} \right), \end{aligned}$$

Lemma 10.1. *The operator* $Q\colon \Gamma^\infty\left(S^2 T^* M\right) \rightarrowtail \Gamma^\infty\left(S^2 T^* M\right)$ *defined locally as*

$$\begin{aligned} (Qh)_{ij} := {} & (g_0 + h)^{ab} \nabla^{g_0,2}_{ab} h_{ij} + h_{ab} (g_0 + h)^{ka} (g_0)^{lb} (g_0 + h)_{ip} (g_0)^{pq} (R^{g_0})_{jklq} \\ & + h_{ab} (g_0 + h)^{ka} (g_0)^{lb} (g_0 + h)_{jp} (g_0)^{pq} (R^{g_0})_{iklq} \end{aligned}$$

induces a second-order quasilinear, uniformly elliptic differential operator

$$Q\colon W^{2,p}_{-\alpha}(S^2 T^* M, g_0) \rightarrowtail W^{0,p}_{-\alpha-2}(S^2 T^* M, g_0)$$

defined on an open neighbourhood U *of* 0*. Its symbol is* $\sigma(D_h Q) = (g_0 + h)^{-1} \otimes \mathrm{id}_{S^2 T^* M}$ *at any* $h \in U$*.*

Proof. It is clear that the operator Q is of second order and that it is quasilinear. Moreover, an easy calculation reveals that the linearization of Q at a tensor field h such that $g_0 + h$ is a metric is

$$D_h Q = (g_0 + h)^{-1} \circ \nabla^{g_0,2} + \text{lower-order terms},$$

and an equally easy calculation verifies that the symbol is indeed the map in the claim.

As for the domain of definition and the mapping properties, we work in more steps.

1. First note that $|g_0|_{g_0} = \sqrt{\dim M}$ and thus if $\|h\|_{L^\infty(S^2T^*M,g_0)} \leq \frac{1}{2}\sqrt{\dim M}$, we have from the reverse triangle inequality that

$$\begin{aligned}|g_0 + h|_{g_0} &= |g_0 - (-h)|_{g_0} \geq ||g_0|_{g_0} - |-h|_{g_0}| \\ &> \sqrt{\dim M} - \frac{1}{2}\sqrt{\dim M} = \frac{1}{2}\sqrt{\dim M}.\end{aligned}$$

   Thus $g_0 + h$ is invertible and $\frac{1}{|g_0+h|_{g_0}} < \frac{2}{\sqrt{\dim M}} =: C_1(\dim M)$. Moreover, $g_0 + h$ is a metric.
   Let us introduce $U := \left\{h \,\middle|\, \|h\|_{L^\infty(S^2T^*M)} < \frac{1}{2}\sqrt{\dim M}\right\}$.
   Now by the submultiplicative property of the induced tensor metrics with respect to the tensorial multiplication and the fact that the pointwise g_0-norm of tensors $T^*M \otimes TM$ is comparable to the pointwise g_0-operator norm on $\operatorname{End}(TM) \simeq T^*M \otimes TM$, we have

$$|g^{-1} \circ \nabla^{g_0,2}h|_{g_0} = |g^{-1} \otimes \nabla^{g_0,2}h|_{g_0} \leq C_2|g^{-1}|_{g_0}|\nabla^{g_0,2}g|_{g_0} < C_1C_2|\nabla^{g_0,2}g|_{g_0}.$$

   This estimate, together with the mapping properties of the g_0-covariant derivative on weighted Sobolev spaces imply that that the map $U \to W^{0,p}_{-\alpha-2}(S^2T^*M), h \mapsto |g^{-1} \circ \nabla^{g_0,2}h|_{g_0}$ is well-defined.

2. To establish the mapping properties of the the curvature terms in the operator Q, one uses the weighted Hölder inequality. ☐

The proof of [DK20, Lemma 2.9], originally stated for asymptotically locally Euclidean manifold, carries over to asymptotically conical manifolds. For convenience of the reader, we reproduce the proof here.

Lemma 10.2 (Weighted interpolation inequality, [DK20, Lemma 2.9])**.** *Let (M, g_{ac}) be an asymptotically conical manifold, let $\tau \in \mathbb{R}$, and let (E, ∇) be a metric bundle. If $h \in \Gamma(E)$ with $\nabla h \in W^{1,p}_{\tau+1}(T^*M \otimes E)$, then we have $|\nabla h|^2 \in L^p_\tau(\mathbb{R}_M)$ and the interpolation inequality*

$$\||\nabla h|^2_{g_{\mathrm{ac}}}\|_{L^p_\tau(\mathbb{R}_M)} \leq C\left(\|\nabla^2 h\|_{L^p_\tau(T^*M\otimes T^*M\otimes E)} + \|\nabla h\|_{L^p_{\tau+1}(T^*M\otimes E)}\right)\|h\|_{L^\infty(E)}$$

holds where $C = C(E, p, \tau)$ is independent of the section h.

Proof. By the density of compactly supported smooth sections in weighted Lebesgue spaces, we may assume that h is compactly supported and smooth. We can avoid differentiability issues at $h = 0$ if we introduce the quantity $|h|_\delta := \sqrt{|h|^2_{g_{\mathrm{ac}}} + \delta}$ for some fixed $\delta > 0$. Evidently, $|h| < |h|_\delta$.

Now we compute for any $\alpha \in \mathbb{R}$

$$\int_M |\nabla h|^{2p}\rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}} \leq \int_M \langle \nabla h, \nabla h\rangle_{g_{\mathrm{ac}}} |\nabla h|^{2p-2}_\delta \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}}$$

Integration by parts and the Leibniz rule imply

$$= -\int_M \langle h, \Delta_{g_{\mathrm{ac}}} h\rangle_{g_{\mathrm{ac}}} |\nabla h|^{2p-2}_\delta \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}}$$

$$+ \int_M \langle h, \nabla h \rangle_{g_{\mathrm{ac}}} (\nabla |\nabla h|_\delta^{2p-2}) \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}}$$
$$+ \int_M \langle h, \nabla h \rangle_{g_{\mathrm{ac}}} |\nabla h|_\delta^{2p-2} \nabla \rho \cdot \rho^{\alpha-1} \operatorname{vol}_{g_{\mathrm{ac}}}$$

The Cauchy–Schwarz inequality, the inequality $|\Delta_{g_{\mathrm{ac}}} h|_{g_{\mathrm{ac}}} \leq |\nabla^2 h|_{g_{\mathrm{ac}}}$, the well-known identity $\nabla(|\nabla h|^2_{g_{\mathrm{ac}}}) = 2 \langle \nabla^2 h, \nabla h \rangle_{g_{\mathrm{ac}}}$ and the existence of a constant C_1 such that $|\nabla \rho|_{g_{\mathrm{ac}}} \leq C_1$ imply

$$\begin{aligned} &\leq \int_M |h|_{g_{\mathrm{ac}}} |\nabla^2 h|_{g_{\mathrm{ac}}} |\nabla h|_\delta^{2p-2} \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}} \\ &\quad + \int_M |h|_{g_{\mathrm{ac}}} |\nabla h|^2_{g_{\mathrm{ac}}} |\nabla^2 h|_{g_{\mathrm{ac}}} |\nabla h|_\delta^{2p-4} \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}} \\ &\quad + C_1 \int_M |h|_{g_{\mathrm{ac}}} |\nabla h|_{g_{\mathrm{ac}}} |\nabla h|_\delta^{2p-2} \rho^{\alpha-1} \operatorname{vol}_{g_{\mathrm{ac}}} . \end{aligned}$$

After letting $\delta \to 0$, the first two terms coincide and we obtain the inequality

$$\begin{aligned} \int_M |\nabla h|^{2p}_{g_{\mathrm{ac}}} \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}} \leq 2 &\int_M |h|_{g_{\mathrm{ac}}} |\nabla^2 h|_{g_{\mathrm{ac}}} |\nabla h|^{2p-2}_{g_{\mathrm{ac}}} \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}} \\ &+ \int_M |h|_{g_{\mathrm{ac}}} |\nabla h|^{2p-1}_{g_{\mathrm{ac}}} \rho^{\alpha-1} \operatorname{vol}_{g_{\mathrm{ac}}} . \end{aligned} \qquad (\star)$$

By the Young inequality[1], we obtain, for any $\epsilon > 0$, the following inequalities:

$$\begin{aligned} |h|_{g_{\mathrm{ac}}} |\nabla^2 h|_{g_{\mathrm{ac}}} \cdot |\nabla h|^{2p-2}_{g_{\mathrm{ac}}} &\leq \epsilon |\nabla h|^{2p}_{g_{\mathrm{ac}}} + C_2(\epsilon, p) |h|^p_{g_{\mathrm{ac}}} |\nabla^2 h|^p_{g_{\mathrm{ac}}}, \\ |h|_{g_{\mathrm{ac}}} |\nabla h|_{g_{\mathrm{ac}}} \rho^{-1} \cdot |\nabla h|^{2p-2}_{g_{\mathrm{ac}}} &\leq \epsilon |\nabla h|^{2p}_{g_{\mathrm{ac}}} + C_2(\epsilon, p) |h|^p_{g_{\mathrm{ac}}} |\nabla h|^p_{g_{\mathrm{ac}}} \rho^{-p}, \end{aligned}$$

where $C_2(\epsilon, p) := -\frac{1}{p}(\epsilon p)^{-1/p^\star}$. Therefore, from Equation $(\star)$, we have

$$\begin{aligned} \int_M |\nabla h|^{2p}_{g_{\mathrm{ac}}} \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}} \leq (1 + C_1)\epsilon &\int_M |\nabla h|^{2p}_{g_{\mathrm{ac}}} \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}} \\ &+ C_2(\epsilon, p) \int_M |h|^p_{g_{\mathrm{ac}}} |\nabla^2 h|^p_{g_{\mathrm{ac}}} \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}} \\ &+ C_1 \cdot C_2(\epsilon, p) \int_M |h|^p_{g_{\mathrm{ac}}} |\nabla h|^p_{g_{\mathrm{ac}}} \rho^{\alpha-p} \operatorname{vol}_{g_{\mathrm{ac}}} . \end{aligned} \qquad (\star\star)$$

Note that the term on the left-hand side of Equation $(\star\star)$ and the first term on the right-hand side of the same equation are constant multiples of each other. Choosing $\epsilon := \epsilon_0 < \frac{1}{1+C_1}$, we obtain by subtracting the first term on the right-hand side and dividing by the constant coefficient[2] $1 - (1 + C_1)\epsilon_0$, we obtain

$$\int_M |\nabla h|^{2p}_{g_{\mathrm{ac}}} \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}} \leq C_3 \int_M |h|^p_{g_{\mathrm{ac}}} |\nabla^2 h|^p_{g_{\mathrm{ac}}} \rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}} + C_4 \int_M |h|^p_{g_{\mathrm{ac}}} |\nabla h|^p_{g_{\mathrm{ac}}} \rho^{\alpha-p} \operatorname{vol}_{g_{\mathrm{ac}}},$$

[1] The Young inequality is $ab \leq \frac{1}{p} a^p + \frac{1}{p^\star} b^{p^\star}$ where $a, b \geq 0$, $p > 1$ and $p^\star := \frac{p}{p-1}$ is the Hölder dual of p. We obtain the desired inequalities with the choices $b := (\epsilon p^\star)^{1/p^\star} |\nabla h|^{2p-2}_{g_{\mathrm{ac}}}$ and $a := (\epsilon p^\star)^{-1/p^\star} |h|_{g_{\mathrm{ac}}} T$ with $T := |\nabla^2 h|_{g_{\mathrm{ac}}}$ in the first case and $T := |\nabla h|_{g_{\mathrm{ac}}} \rho^{-1}$ in the second case.

[2] Note that this constant is positive due to our choice of ϵ_0, thus the direction of the relation sign remains invariant under the division.

where $C_3(p) := \frac{C_2(\epsilon_0,p)}{1-(1+C_1)\epsilon_0}$ and $C_4(p) := \frac{C_1 \cdot C_2(\epsilon_0,p)}{1-(1+C_1)\epsilon_0}$. Taking the pth root, we obtain a constant $C_5(p)$ which depends on the maximum of C_3 and C_4 and the constant between the p-mean and the 1-mean for which

$$\begin{aligned}\left(\int_M |\nabla h|^{2p}_{g_{\mathrm{ac}}}\rho^\alpha \operatorname{vol}_{g_{\mathrm{ac}}}\right)^{1/p} &\leq C_5\left(\int_M |h|^p_{g_{\mathrm{ac}}}|\nabla^2 h|^p_{g_{\mathrm{ac}}}\rho^\alpha\right)^{1/p} \\ &\quad + C_5\left(\int_M |h|^p_{g_{\mathrm{ac}}}|\nabla h|^p_{g_{\mathrm{ac}}}\rho^{\alpha-p}\operatorname{vol}_{g_{\mathrm{ac}}}\right)^{1/p} \\ &\leq C_5\cdot\|h\|_{L^\infty(E)}\left[\left(\int_M |\nabla^2 h|^p_{g_{\mathrm{ac}}}\rho^\alpha\right)^{1/p}\right. \\ &\quad \left. + C_5\left(\int_M |\nabla h|^p_{g_{\mathrm{ac}}}\rho^{\alpha-p}\operatorname{vol}_{g_{\mathrm{ac}}}\right)^{1/p}\right].\end{aligned}$$

Specializing to $\alpha := -\tau p - n$ yields the desired inequality (with $C := C_5$). □

Lemma 10.3. *There exists a real number $\epsilon > 0$ such that if $h \in C^0_{-\beta}(S^2T^*M, g_0)$ with $\|h\|_{L^\infty(S^2T^*M,g_0)} < \epsilon$ solves the equation $g_0 + h \in \mathcal{F}_{g_0}$, then $h \in C^1_{-\beta}(S^2T^*M, g_0)$. The same conclusion holds if the gaugedness condition is replaced with $g_0 + h \in \mathcal{F}^{M\setminus K}_{g_0}$ for some compact set $K \subset M$.*

Proof. We proceed in several steps following [DK20, Theorem 2.7].

1. To show that $h = O\left(\rho^{-\beta}\right)$ and $\nabla^{g_0} h \in O\left(\rho^{-\beta-1}\right)$ (with respect to the pointwise g_0-norm), it suffices to show that $h \in C^1_{-\beta}(S^2T^*M, g_0)$. By the definitions of the norms, it is clear that $\|h\|_{C^1_{-\beta}(S^2T^*M,g_0)} \leq \|h\|_{C^{1,\alpha}_{\beta}(S^2T^*M,g_0)}$, hence $C^{1,\alpha}_{\beta}(S^2T^*M, g_0) \subset C^{1,\alpha}_{-\beta}(S^2T^*M, g_0)$ for any $\alpha \in (0,1)$. Note that, for $p \in (n,\infty)$ and $\beta \in (0,1)$, the weighted Sobolev embedding (cf. e.g. [Mar02, Theorem 4.23]) implies
$$\|h\|_{C^{1,\alpha}_{-\beta}(S^2T^*M,g_0)} \leq C_1\|h\|_{W^{2,p}_{-\beta}(S^2T^*M,g_0)}.$$

2. This means our job has been reduced to finding an upper bound for the $W^{2,p}_{-\beta}(S^2T^*M, g_0)$-form using the equation $g_0 + h \in \mathcal{F}_{g_0}$ (and by restricting the $L^\infty(S^2T^*M, g_0)$-norm of h). By Lemma 8.11, the gaugedness relation $g_0 + h \in \mathcal{F}_{g_0}$ can be written as a quasi-linear partial differential equation
$$Q(h) = (\nabla^{g_0} h) \star (\nabla^{g_0} h).$$
Note that the g_0-norm of the right-hand side of this equation may be estimated as
$$|(\nabla^{g_0} h) \star (\nabla^{g_0}) h|_{g_0} \leq C_2|\nabla^{g_0} h|^2_{g_0}, \tag{10.1}$$
where $C_2 = C_2(g_0, \star)$. Since Q is a uniformly elliptic operator, it is natural to start with an elliptic estimate for weighted Sobolev spaces. Suppose $\|h\|_{L^\infty(S^2T^*M,g_0)} < \epsilon_1$ where $\epsilon_1 > 0$ is chosen so small such that uniform ellipticity of Q is guaranteed by Lemma 10.1.
$$\|h\|_{W^{2,p}_{-\beta}(S^2T^*M,g_0)} \leq C_3\left(\|Q(h)\|_{L^p_{-\beta-2}(S^2T^*M,g_0)} + \|h\|_{L^p_{-\beta}(S^2T^*M,g_0)}\right)$$

$$
\begin{aligned}
&\leq C_4 \left(\| |\nabla^{g_0} h|^2_{g_0} \|_{L^p_{-\beta-2}(S^2T^*M,g_0)} + \|h\|_{L^p_{-\beta}(S^2T^*M,g_0)} \right) \\
&\leq C_5 \|h\|_{L^\infty(S^2T^*M,g_0)} \\
&\quad \cdot \left(\|\nabla^{g_0,2} h\|_{L^p_{-\beta-2}(S^2T^*M,g_0)} + \|\nabla^{g_0} h\|_{L^p_{-\beta-1}(S^2T^*M,g_0)} \right) \\
&\quad\quad + C_4 \|h\|_{L^p_{-\beta}(S^2T^*M,g_0)} \\
&\leq C_5 \|\nabla^{g_0,2} h\|_{W^{2,p}_{-\beta}(S^2T^*M,g_0)} \cdot \|h\|_{L^\infty(S^2T^*M,g_0)} \\
&\quad + C_4 \|h\|_{L^p_{-\beta}(S^2T^*M,g_0)}
\end{aligned}
$$

Here we used elliptic regularity, the estimate from Equation (10.1), the weighted interpolation inequality Lemma 10.2 and the definition of the weighted Sobolev norm. Note that the far left-hand side of this chain of inequalities appears as a factor in the first term of the right-hand side of the last formula, thus

$$
(1 - C_5 \|h\|_{L^\infty(S^2T^*M,g_0)}) \|h\|_{W^{2,p}_{-\beta}(S^2T^*M,g_0)} \leq C_4 \|h\|_{L^p_{-\beta}(S^2T^*M,g_0)} \tag{10.2}
$$

Now we see a second place where it is necessary to bound the $L^\infty(S^2T^*M, g_0)$-norm of h for the argument: if $\|h\|_{L^\infty(S^2T^*M,g_0)} \leq \epsilon_2 < \frac{1}{C_5}$, then the coefficient on the left-hand side of the previous equation is positive, thus we obtain the estimate

$$
\|h\|_{W^{2,p}_{-\beta}(S^2T^*M,g_0)} \leq C_6 \|h\|_{L^p_{-\beta}(S^2T^*M,g_0)},
$$

where $C_6 := \frac{C_4}{1 - C_5 \epsilon_2}$.

3. Next, we relate the $L^p_{-\beta}(S^2T^*M, g_0)$-norm of h to the assumed decay rate of h. First of all, note that $\|h\|_{L^p_{-\beta}(S^2T^*M,g_0)} = \|\rho \cdot h\|_{L^p_{-\beta+1}(S^2T^*M,g_0)}$ since

$$
\begin{aligned}
\|h\|^p_{L^p_{-\beta}(S^2T^*M,g_0)} &= \int_M (\rho^\beta |h|_{g_0})^p \rho^{-\dim M} \operatorname{vol}_{g_0} \\
&= \int_M (\rho^{\beta-1} |\rho \cdot h|_{g_0})^p \rho^{-\dim M} \operatorname{vol}_{g_0} = \|\rho \cdot h\|^p_{L^p_{-\beta+1}(S^2T^*M,g_0)}.
\end{aligned}
$$

Now the weighted Hölder inequality Proposition 7.18 implies that

$$
\|h\|_{L^p_{-\beta}(S^2T^*M,g_0)} = \|\rho \cdot h\|_{L^p_{-\beta+1}(S^2T^*M,g_0)} \leq C_7 \|\rho\|_{L^p_{1-\eta}(\mathbb{R}_M,g_0)} \|h\|_{L^\infty_{-\beta+\eta}(S^2T^*M,g_0)},
$$

where $\eta > 0$ is a positive number. The ρ-factor is finite by Lemma 7.19. The the h-factor may be estimated, due to the fact that $h \in C^0_{-\beta}\left(S^2T^*M, g_0\right)$, as follows

$$
\begin{aligned}
\|h\|_{L^\infty_{-\beta+\eta}(S^2T^*M,g_0)} &= \operatorname*{essup}_M (\rho^{\beta-\eta} |h|_{g_0}) = \sup_M (\rho^{\beta-\eta} |h|_{g_0}) \\
&= \max \left\{ \sup_{\operatorname{Core}(R)} (\rho^{\beta-\eta} |h|_{g_0}), \sup_{M \setminus \operatorname{Core}(R)} (\rho^{\beta-\eta} |h|_{g_0}) \right\} \\
&\leq \max \left\{ C_8, \sup (C_9 \rho^{\beta-\eta} \rho^{-\beta}) \right\} < \infty,
\end{aligned}
$$

where C_8 is an upper bound in the core and C_9 and R are asymptoticity constants for h (i.e. $|h|_{g_0}|_{M \setminus \operatorname{Core}(R)} \leq C_9 \rho^{-\beta}$).

4. Thus we have, for h with $\|h\|_{L^\infty(S^2T^*M,g_0)} < \max\{\epsilon_1,\epsilon_2\}$, that

$$\|h\|_{C^{1,\alpha}_{-\beta}(S^2T^*M,g_0)} \leq C_1\|h\|_{W^{2,p}_{-\beta}(S^2T^*M,g_0)} \leq C_1C_6\|h\|_{L^p_{-\beta}(S^2T^*M,g_0)} < \infty,$$

where we used all the previous steps in order for each inequality. This shows the claim.

5. Note that the gaugedness condition has been used only in step 2. If the gaugedness condition holds only on $M \setminus K$, then we may split up the integrals to K and $M \setminus K$, and work analogously. □

Proposition 10.4. *There exists a real number $\epsilon > 0$ such that if $h \in C^0_{-\beta}\left(S^2T^*M, g_0\right)$ with $\|h\|_{L^\infty(S^2T^*M,g_0)} < \epsilon$ solves the equation $g_0 + h \in \mathcal{F}_{g_0}$, then $\nabla^{g_\text{cone},k}h = O\left(\rho^{-\beta-k}\right)$ for all $k \in \mathbb{N}$.*

Proof. The statement for $k = 0$ is true by assumption. Lemma 10.3 proves the statement for $k = 1$. For $k > 1$, we can argue using elliptic estimates [Mar02, Theorem 4.21] and the fact that $\Delta_L^{g_\text{cone}}$ is a uniformly elliptic conical operator of order 2 and rate 2. □

This proposition means that once we have established a zeroth-order decay, and the higher derivatives come for free as long as the gaugedness condition is satisfied outside of a compact set. In the next section, we will establish an iterative procedure that improves on the decay rate.

10.2 Iteration

Next, we state and prove a technical lemma.

Lemma 10.5. *Let M be a smooth manifold, and let $g_0 \in \mathrm{AC}\left(g_\text{cone}, \tau, \phi\right)$ with $\mathrm{Ric}^{g_0} = 0$. Moreover, suppose $h \in \Gamma^\infty\left(S^2T^*M\right)$ is a symmetric 2-tensor field such that $|h|_{g_\text{cone}} = O\left(\rho^{-\alpha}\right)$, $|\nabla^{g_\text{cone}}h|_{g_\text{cone}} = O\left(\rho^{-\alpha-1}\right)$ and $|\nabla^{g_\text{cone},2}h|_{g_\text{cone}} = O\left(\rho^{-\alpha-2}\right)$ for some $\alpha > 0$ and such that $g := g_0 + h \in \mathcal{F}^{M\setminus K}_{g_0}$ for some compact subset $K \subset M$. Then $\Delta_L^{g_\text{cone}}h = O\left(\rho^{-F(\alpha)-2}\right)$ where $F\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a function fit for iteration.*

Proof. The proof involves a lot of gory calculation, so it may be a good idea to summarize the strategy. We start by expressing $g^{-1} \circ \nabla^{g_0,2}h$ in two different ways. The first way uses the decay property of h and the fact that g_0 is asymptotically conical to g_cone to compare the inverse metric g^{-1} to the cone metric g_cone and the second covariant derivative $\nabla^{g_0,2}$ to $\nabla^{g_\text{cone},2}$. This leads us to a formula to the raw Laplacian $\Delta_{g_\text{cone}}h$. The second way uses the assumption that $g \in \mathcal{F}_{g_0}$ and Corollary 8.12. After adding the curvature terms, we obtain a formula for $\Delta_L^{g_\text{cone}}h$. Some tedious calculation is needed to show that the curvature terms of g_cone and g interact in such a way that their combination decays faster than any of them.

In the rest of this calculation, we will assume that $K = \emptyset$. (The behaviour inside K is anyway inconsequential for the decay properties since h is continuous.)

First, we substitute both the metric g and the Levi-Civita connection $\nabla^{g_0,2}$ for the metric g_cone and the corresponding Levi-Civita connection (at a price of certain correction terms appearing, of course). For the difference of second derivatives we use Lemma 2.6 with $T := \nabla^{g} - \nabla^{g_{\mathrm{cone}}}$.

$$\begin{aligned}
g^{-1} \circ \nabla^{g_0,2} h &= ((g_{\mathrm{cone}})^{-1} + g^{-1} - (g_{\mathrm{cone}})^{-1}) \circ (\nabla^{g_{\mathrm{cone}},2} + \nabla^{g_0,2} - \nabla^{g_{\mathrm{cone}},2}) h \\
&= (g_{\mathrm{cone}})^{-1} \circ \nabla^{g_{\mathrm{cone}},2} h + (g_{\mathrm{cone}})^{-1} \circ (\nabla^{g_0,2} - \nabla^{g_{\mathrm{cone}},2}) h \\
&\quad + (g^{-1} - (g_{\mathrm{cone}})^{-1}) \circ \nabla^{g_{\mathrm{cone}},2} h + (g^{-1} - (g_{\mathrm{cone}})^{-1}) \circ (\nabla^{g_0,2} - \nabla^{g_{\mathrm{cone}},2}) h \\
&= -\Delta_{g_{\mathrm{cone}}} h + (g_{\mathrm{cone}})^{-1} \circ (T \star \nabla^{g_{\mathrm{cone}}} h + (\nabla^{g_{\mathrm{cone}}} T) \star h + T \star T \star h) \\
&\quad + (g^{-1} - (g_{\mathrm{cone}})^{-1}) \circ \nabla^{g_{\mathrm{cone}},2} h \\
&\quad + (g^{-1} - (g_{\mathrm{cone}})^{-1}) \circ (T \star \nabla^{g_{\mathrm{cone}}} h + (\nabla^{g_{\mathrm{cone}}} T) \star h + T \star T \star h) \\
&= -\Delta_{g_{\mathrm{cone}}} h + g^{-1} \circ (T \star \nabla^{g_{\mathrm{cone}}} h + (\nabla^{g_{\mathrm{cone}}} T) \star h + T \star T \star h) \\
&\quad + (g^{-1} - (g_{\mathrm{cone}})^{-1}) \circ \nabla^{g_{\mathrm{cone}},2} h \\
&= -\Delta_{g_{\mathrm{cone}}} h \\
&\quad + g^{-1} \circ (T \star \nabla^{g_{\mathrm{cone}}} h + (\nabla^{g_{\mathrm{cone}}} T) \star h) \\
&\quad + g^{-1} \circ (T \star T \star h) \\
&\quad + (g^{-1} - (g_{\mathrm{cone}})^{-1}) \circ \nabla^{g_{\mathrm{cone}},2} h
\end{aligned}$$

Since $g^{-1} = O(1)$, $g^{-1} - (g_{\mathrm{cone}})^{-1} = g^{-1} - (g_0)^{-1} + (g_0)^{-1} - (g_{\mathrm{cone}})^{-1} = O(\rho^{-\tau}) + O(\rho^{-\alpha})$, $T = O(\rho^{-\tau-1})$ and $\nabla^{g_{\mathrm{cone}}} T = O(\rho^{-\tau-2})$, we obtain

$$\begin{aligned}
g^{-1} \circ \nabla^{g_0,2} h &= -\Delta_{g_{\mathrm{cone}}} h \\
&\quad + O(1) \circ (O\left(\rho^{-\tau-1}\right) \star O\left(\rho^{-\alpha-1}\right) + O\left(\rho^{-\tau-2}\right) \star O(\rho^{-\alpha})) \\
&\quad + O(1) \circ (O\left(\rho^{-\tau-1}\right) \star O\left(\rho^{-\tau-1}\right) \star O(\rho^{-\alpha})) \\
&\quad + (O(\rho^{-\alpha}) + O(\rho^{-\tau})) \star O\left(\rho^{-\alpha-2}\right) \\
&= -\Delta_{g_{\mathrm{cone}}} h \\
&\quad + O\left(\rho^{-\tau-\alpha-2}\right) + O\left(\rho^{-\tau-\alpha-2}\right) \\
&\quad + O\left(\rho^{-2\tau-\alpha-2}\right) \\
&\quad + O\left(\rho^{-2\alpha-2}\right) + O\left(\rho^{-\tau-\alpha-2}\right) \\
&= -\Delta_{g_{\mathrm{cone}}} h \\
&\quad + O\left(\rho^{-\alpha-2}\right) (O(\rho^{-\tau}) + O\left(\rho^{-2\tau}\right) + O(\rho^{-\alpha}) + O(\rho^{-\tau})) \\
&= -\Delta_{g_{\mathrm{cone}}} h + O\left(\rho^{-\alpha-2}\right) (O(\rho^{-\tau}) + O(\rho^{-\alpha}))
\end{aligned}$$

Note that Corollary 8.12 applies and its statement may be expressed structurally as

$$0 = g^{-1} \circ \nabla^{g_0,2} h + (\nabla^{g_0} h) \star (\nabla^{g_0} h) - h \star R^{g_0} - g \star \mathrm{Ric}^{g_0}$$

which leads us to the equality

$$\Delta_L^{g_{\mathrm{cone}}} h = \Delta_{g_{\mathrm{cone}}} h - 2 \overset{\circ}{R}{}^{g_{\mathrm{cone}}} h + \mathrm{Ric}^{g_{\mathrm{cone}}} \circ (g_{\mathrm{cone}})^{-1} \circ h + h \circ (g_{\mathrm{cone}})^{-1} \circ \mathrm{Ric}^{g_{\mathrm{cone}}}$$

$$
\begin{aligned}
&= -g^{-1} \circ \nabla^{g_0,2} h - 2\overset{\circ}{R}^{g_{\text{cone}}} h + \operatorname{Ric}^{g_{\text{cone}}} \circ (g_{\text{cone}})^{-1} \circ h + h \circ (g_{\text{cone}})^{-1} \circ \operatorname{Ric}^{g_{\text{cone}}} \\
&\quad + O\left(\rho^{-\alpha-\tau-2}\right) + O\left(\rho^{-2\alpha-2}\right) \\
&= -(\nabla^{g_0} h) \star (\nabla^{g_0} h) + h \star R^{g_0} + g \star \operatorname{Ric}^{g_0} \\
&\quad - 2\overset{\circ}{R}^{g_{\text{cone}}} h + \operatorname{Ric}^{g_{\text{cone}}} \circ (g_{\text{cone}})^{-1} \circ h + h \circ (g_{\text{cone}})^{-1} \circ \operatorname{Ric}^{g_{\text{cone}}} \\
&\quad + O\left(\rho^{-\alpha-\tau-2}\right) + O\left(\rho^{-2\alpha-2}\right) \\
&= h \star R^{g_0} + g \star \operatorname{Ric}^{g_0} - 2\overset{\circ}{R}^{g_{\text{cone}}} h + \operatorname{Ric}^{g_{\text{cone}}} \circ (g_{\text{cone}})^{-1} \circ h + h \circ (g_{\text{cone}})^{-1} \circ \operatorname{Ric}^{g_{\text{cone}}} \\
&\quad + O\left(\rho^{-\alpha-\tau-2}\right) + O\left(\rho^{-2\alpha-2}\right) \\
&= h \star R^{g_0} - 2\overset{\circ}{R}^{g_{\text{cone}}} h + O\left(\rho^{-\alpha-\tau-2}\right) + O\left(\rho^{-2\alpha-2}\right),
\end{aligned}
$$

where we used that $\operatorname{Ric}^{g_0} = O(0)$. Let us introduce the notation $(i \leftrightarrow j)$ to repeat the contents from the last equality sign with indices i and j exchanged. Now we can examine the remaining term more carefully in local coordinates

$$
\begin{aligned}
(h \star R^{g_g} - 2\overset{\circ}{R}^{g_{\text{cone}}} h)_{ij} &= -g^{kl} (g_0)^{pq} (g_{ip} (R^{g_0})_{jklq} + g_{jp} (R^{g_0})_{iklq}) \\
&\quad - 2 (R^{g_{\text{cone}}})_{iklj} h_{ab} (g_{\text{cone}})^{ak} (g_{\text{cone}})^{bl} \\
&= -g^{kl} (g_0)^{pq} g_{ip} (R^{g_0})_{jklq} - (R^{g_{\text{cone}}})_{iklj} h_{ab} (g_{\text{cone}})^{ak} (g_{\text{cone}})^{bl} + (i \leftrightarrow j)
\end{aligned}
$$

Since $g = g_0 + h$ and we have and $(g_0)^{pq} (g_0)_{ip} (R^{g_0})_{jklq} = (R^{g_0})_{jkli}$, we may continue with

$$
\begin{aligned}
&= -g^{kl} (R^{g_0})_{jkli} - g^{kl} (g_0)^{pq} (R^{g_0})_{jklq} h_{ip} \\
&\quad - (g_{\text{cone}})^{ak} (g_{\text{cone}})^{bl} (R^{g_{\text{cone}}})_{ijkl} h_{ab} + (i \leftrightarrow j)
\end{aligned}
$$

Now since

$$
\begin{aligned}
-g^{kl} (R^{g_0})_{jklt} &= -(g_0)^{kl} (R^{g_0})_{jklt} + ((g_0)^{kl} - g^{kl}) (R^{g_0})_{jklt} \\
&= -(\operatorname{Ric}^{g_0})_{jt} + (g_0)^{ka} h_{ab} g^{bl} (R^{g_0})_{jklt} \\
&= -(\operatorname{Ric}^{g_0})_{jt} + (g_0)^{ka} g^{bl} (R^{g_0})_{jklt} h_{ab},
\end{aligned}
$$

we may rewrite the expression of interest as

$$
\begin{aligned}
(h \star R^{g_g} - 2\overset{\circ}{R}^{g_{\text{cone}}} h)_{ij} &= -g^{kl} (R^{g_0})_{jkli} - g^{kl} (R^{g_0})_{jklq} (g_0)^{pq} h_{ip} \\
&\quad - (g_{\text{cone}})^{ak} (g_{\text{cone}})^{bl} (R^{g_{\text{cone}}})_{iklj} h_{ab} + (i \leftrightarrow j) \\
&= -\operatorname{Ric}^{g_0}_{ji} + (g_0)^{ka} g^{bl} (R^{g_0})_{jkli} h_{ab} \\
&\quad + (-\operatorname{Ric}^{g_0}_{jq} + (g_0)^{ka} g^{bl} (R^{g_0})_{jklq} h_{ab}) (g_0)^{pq} h_{ip} \\
&\quad - (g_{\text{cone}})^{ak} (g_{\text{cone}})^{bl} (R^{g_{\text{cone}}})_{iklj} h_{ab} + (i \leftrightarrow j) \\
&= -\operatorname{Ric}^{g_0}_{ji} + (g_0)^{ka} g^{bl} (R^{g_0})_{jkli} h_{ab} \\
&\quad - \operatorname{Ric}^{g_0}_{jq} (g_0)^{pq} h_{ip} + (g_0)^{ka} g^{bl} (R^{g_0})_{jklq} h_{ab} (g_0)^{pq} h_{ip} \\
&\quad - (g_{\text{cone}})^{ak} (g_{\text{cone}})^{bl} (R^{g_{\text{cone}}})_{iklj} h_{ab} + (i \leftrightarrow j)
\end{aligned}
$$

$$
\begin{aligned}
&= -\operatorname{Ric}^{g_0}_{ji} \\
&\quad + [(g_0)^{ka} g^{bl} (R^{g_0})_{jkli} - (g_{\text{cone}})^{ak} (g_{\text{cone}})^{bl} (R^{g_{\text{cone}}})_{iklj}] h_{ab} \\
&\quad + [-\operatorname{Ric}^{g_0}_{jq} (g_0)^{pq} + (g_0)^{ka} g^{bl} (R^{g_0})_{jklq} h_{ab} (g_0)^{pq}] h_{ip} \\
&\quad + (i \leftrightarrow j) \\
&= [(g_0)^{ka} g^{bl} (R^{g_0})_{jkli} - (g_{\text{cone}})^{ak} (g_{\text{cone}})^{bl} (R^{g_{\text{cone}}})_{iklj}] h_{ab} \\
&\quad + (i \leftrightarrow j) \\
&\quad + O\left(\rho^{-2\alpha-2}\right)
\end{aligned}
$$

Let us concentrate on the coefficient of h_{ab}. Changing the metrics to g_{cone} and using the decay rate of the difference of curvature tensors (Lemma 7.23) yields

$$
\begin{aligned}
&(g_0)^{ka} g^{bl} (R^{g_0})_{jkli} - (g_{\text{cone}})^{ak} (g_{\text{cone}})^{bl} (R^{g_{\text{cone}}})_{iklj} \\
&\quad = (g_{\text{cone}})^{ka} (g_{\text{cone}})^{bl} (R^{g_0})_{jkli} \\
&\qquad + (g_{\text{cone}})^{ka} (g^{bl} - (g_{\text{cone}})^{bl}) (R^{g_0})_{jkli} \\
&\qquad + ((g_0)^{ka} - (g_{\text{cone}})^{ka}) (g_{\text{cone}})^{bl} (R^{g_0})_{jkli} \\
&\qquad + ((g_0)^{ka} - (g_{\text{cone}})^{ka}) (g^{bl} - (g_{\text{cone}})^{bl}) (R^{g_0})_{jkli} \\
&\qquad - (g_{\text{cone}})^{ak} (g_{\text{cone}})^{bl} (R^{g_{\text{cone}}})_{iklj} \\
&\quad = (g_{\text{cone}})^{ka} (g_{\text{cone}})^{bl} ((R^{g_0})_{jkli} - R^{g_{\text{cone}}})_{iklj} \\
&\qquad + O(1) \star (O(\rho^{-\alpha}) + O(\rho^{-\tau})) \star O\left(\rho^{-2}\right) \\
&\qquad + O(\rho^{-\tau}) \star O(1) \star O\left(\rho^{-2}\right) \\
&\qquad + O(\rho^{-\tau}) \star (O(\rho^{-\alpha}) + O(\rho^{-\tau})) \star O\left(\rho^{-2}\right) \\
&\quad = (g_{\text{cone}})^{ka} (g_{\text{cone}})^{bl} ((R^{g_0})_{jkli} - R^{g_{\text{cone}}})_{iklj} \\
&\qquad + O\left(\rho^{-\alpha-2}\right) + O\left(\rho^{-\tau-2}\right) \\
&\quad = O(1) \star O(1) \star O\left(\rho^{-\tau-2}\right) + O\left(\rho^{-\alpha-2}\right) + O\left(\rho^{-\tau-2}\right) \\
&\quad = O\left(\rho^{-\alpha-2}\right) + O\left(\rho^{-\tau-2}\right).
\end{aligned}
$$

Thus the curvature term is

$$
\begin{aligned}
(h \star R^{g_0} - 2\overset{\circ}{R}{}^{g_{\text{cone}}} h)_{ij} &= \left(O\left(\rho^{-\alpha-2}\right) + O\left(\rho^{-\tau-2}\right)\right)_{ij}^{ab} h_{ab} + O\left(\rho^{-2\alpha-2}\right) \\
&= \left(O\left(\rho^{-\alpha-2}\right) + O\left(\rho^{-\tau-2}\right)\right) O(\rho^{-\alpha}) + O\left(\rho^{-2\alpha-2}\right) \\
&= O\left(\rho^{-\alpha-2}\right) (O(\rho^{-\alpha}) + O(\rho^{-\tau})).
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
\Delta_L^{g_{\text{cone}}} h &= h \star R^{g_0} - 2\overset{\circ}{R}{}^{g_{\text{cone}}} h + O\left(\rho^{-\alpha-\tau-2}\right) + O\left(\rho^{-2\alpha-2}\right) \\
&= O\left(\rho^{-F(\alpha)-2}\right)
\end{aligned}
$$

where

$$F\colon \mathbb{R}^+ \to \mathbb{R}^+, x \mapsto \begin{cases} 2x & \text{if } x \leq \tau \\ x+\tau & \text{if } x > \tau \end{cases}$$

is a function fit for iteration. □

Theorem 10.6. *Let $g_0 \in \mathrm{AC}\left(g_{\mathrm{cone}}, \phi, \tau\right)$ be a Ricci-flat asymptotically conical manifold of decay rate $\tau > 0$ and asymptotic chart ϕ which is asymptotic to* $\mathrm{Cone}(L, g_L)$. *Suppose further that g_0 is not resonance-dominated (cf. Definition 9.13). Then there exists a $p > 1$ and an $L^p \cap L^\infty$ neighbourhood of g_0 such that for any gauged metric g in this neighbourhood, we have*

$$|\nabla^{g_0,k}(g-g_0)|_{g_0} = O\left(\rho^{-\mu-k}\right) \quad \text{for any } k \in \mathbb{N}, \tag{10.3}$$

where μ is determined as in Proposition 9.14.

Proof (sketch). One can establish similarly to the first step of the proof in [DK20, Theorem 2.7] that $h = O\left(\rho^{-\alpha}\right)$ for some $\alpha > 0$ if the $L^p(S^2T^*M, g_0)$-norm and the $L^\infty(S^2T^*M, g_0)$-norm of h are small enough. By Lemma 10.5, we have that

$$\Delta_L^{g_{\mathrm{cone}}} h = O\left(\rho^{-F(\alpha)-2}\right),$$

where F is a function fit for iteration. Working as in Theorem 9.24, we establish that $h = O\left(\rho^{-\alpha_{\mathrm{new}}}\right)$, where

$$\alpha_{\mathrm{new}} = \begin{cases} \min\left\{F(\alpha)-2, \mu\right\} & \text{if } F(\alpha) \neq \mu, \\ \mu + \frac{1}{2}(\alpha-\mu) & \text{if } F(\alpha) = \mu. \end{cases}$$

where $\mu := -\min\left\{m(\xi) \mid \xi \in \sigma(\Box_E^{g_{\mathrm{cone}}})\right\}$ (recall Definition 9.12). (Again, this case distinction is necessary because of the logarithmic factor appearing in Proposition 9.23. Instead of $\mu + \frac{1}{2}(\alpha - \mu)$ any number would be admissible between α and μ.)

Now Lemma 9.17 delivers the decay rate for the zeroth derivative. Lemma 10.3 extends it to the first derivative. The higher-order decay rates follow from an inductive argument based on elliptic regularity of weighted Hölder spaces. □

Remark 10.7. *The resonance-dominated case should be treated similarly but the existence of a logarithmic factor in both when $F(\alpha) = \mu$ and when $F(\alpha) \neq 0$ means that we cannot expect an optimal decay rate, only an infimum.*

10.3 Decay improvement for asymptotically conical manifolds

Finally we are in position to show that we can find an appropriate asymptotic chart in which the optimalized decay rate is assumed.

Theorem 10.8. *Let $(M, g_{\text{ac}}) \in \text{AC}(g_{\text{cone}}, \tau, \phi)$ be an asymptotically conical manifold with asymptotic cone $(\text{Cone}(L, g_L), g_{\text{cone}})$ and $\tau > 0$. Assume that g_{ac} is not resonance-dominated*[3]*. Then there is an asymptotic chart $\tilde{\phi}$ such that $g_{\text{ac}} \in \text{AC}\left(g_{\text{cone}}, \mu, \tilde{\phi}\right)$, where*

$$\mu = -\min\{m(\xi) \mid \xi \in \sigma(\Box_E)\}$$

where m is the generic decay function from Definition 9.12. Note that the spectrum of $\Box_E$ has been computed in Theorem 6.55.

Proof. Let $g_{\text{ac}} \in \text{AC}(g_{\text{cone}}, \tau, \phi)$. Then we have $|\nabla^{g_{\text{cone}},k}((\phi^{-1})^* g_{\text{ac}} - g_{\text{cone}})|_{g_{\text{cone}}} = O\left(r^{-\tau-k}\right)$ for all $k \in \mathbb{N}$.

Recall the family of eventually exactly conical metrics $(g_R)_{R \geq R_0}$ constructed in Proposition 8.30. Metrics in this family are Ricci-flat outside of the compact set $\text{Core}(2R_1)$. Moreover, since $g_R \to g_{\text{ac}}$ in the $W^{k,p}_\delta(S^2T^*M)$ topology, we can find some $R_1 \geq R_0$ such that $g_{R_1} \in \mathcal{U}$ where $\mathcal{U}$ is a $W^{k,p}_\delta(S^2T^*M)$-neighbourhood on which every metric may be uniquely pulled back to a metric satisfying the Bianchi gauge with respect to g_{ac} (cf. Proposition 8.23). This in turn means that the metric g_{R_1} may be uniquely pulled back to a gauged metric, i.e. there is a diffeomorphism $\psi\colon M \to M$ such that $\psi^*(g_{R_1}) =: g_b \in \mathcal{G}_{g_{\text{ac}}}$, i.e. $V(g_b, g_{\text{ac}}) = 0$.

Since g_b is gauged with respect to g_{ac} outside $\psi^{-1}(\text{Core}(2R_1))$, Theorem 10.6 implies that the difference $g_b - g_{\text{ac}}$ decays optimally to all orders:

$$|\nabla^{g_{\text{ac}},k}(g_b - g_{\text{ac}})|_{g_{\text{ac}}} = O\left(\rho^{-\mu-k}\right)$$

for all $k \in \mathbb{N}$ where μ is the optimalized decay rate.

Now consider the diffeomorphism $\phi \circ \psi\colon \psi^{-1}(M \setminus K) \to (R, \infty) \times L$. Since on $M \setminus \psi^{-1}(\text{Core}(2R_1))$, we have $g_{\text{ac}} = \psi^*\phi^* g_{\text{cone}}$ by construction, we obtain

$$\begin{aligned} |\nabla^{\text{cone},k}((\phi\circ\psi)_* g_{\text{ac}} - g_{\text{cone}})|_{g_{\text{ac}}} &= |\nabla^{\text{cone},k}((\phi\circ\psi)_*(g_{\text{ac}} - \psi^*\phi^* g_{\text{cone}}))|_{g_{\text{ac}}} \\ &= |\nabla^{\text{cone},k}((\phi\circ\psi)_*(g_{\text{ac}} - g_{\text{ac}}))|_{g_{\text{ac}}}. \end{aligned}$$

The claim now follows from Corollary 7.26 and Lemma 8.16.

□

Remark 10.9. *In the resonance-dominated case, because of the presence of the logarithmic factor in both branches in Proposition 9.23, one cannot expect a definite best decay rate. It is reasonable to expect that the optimal decay is $O(\rho^{-\mu}\ln\rho)$. To accommodate this case, one would have to relax the Definition 7.1 to obtain a definite decay rate. Alternatively, one could expect a decay statement of the form $O(\rho^{-\mu+\epsilon})$ for every $\epsilon > 0$ since $\ln r = O(\rho^\epsilon)$ for any $\epsilon > 0$.*

[3] Cf. Definition 9.13

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List of publications

During my time as a doctoral student I co-authored the following papers.

- K. Kröncke, O. Lindblad Petersen, F. Lubbe, T. Marxen, W. Maurer, W. Meiser, O. C. Schnürer, Á. Szabó, B. Vertman. Mean Curvature Flow in Asymptotically Flat Product Spacetimes. To appear in: *Journal of Geometric Analysis*. https://doi.org/10.1007/s12220-020-00486-z. arXiv:1903.03502

- I. Horvath, D. Szécsi, J. Hakkila, Á. Szabó, I. I. Racz, L. V. Tóth, S. Pinter, Z. Bagoly, The clustering of gamma-ray bursts in the Hercules–Corona Borealis Great Wall: the largest structure in the Universe?, *Monthly Notices of the Royal Astronomical Society* 2020:2, http://dx.doi.org/10.1093/mnras/staa2460,

These articles are unrelated to this dissertation.

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Eidesstattliche Versicherung | *Declaration of oath*

Die vorliegende Arbeit habe ich selbständig verfasst und keine anderen als die angegebenen Hilfsmittel – insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen – benutzt. Die Arbeit habe ich vorher nicht in einem anderen Promotionsverfahren eingereicht. Die eingereichte schriftliche Fassung entspricht genau der elektronisch eingereichten Version.

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Hamburg, den 22. Dezember 2020