Algebraic Structures on Moduli Spaces of Mirror Geometries

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Martin Vogrin
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Abstract

This thesis is concerned with algebraic structures appearing on the moduli spaces of mirror geometries enhanced with differential forms. It was shown in [Mov12] that in the case of elliptic curves, the local structure of such moduli spaces can be constructed from a variation of polarized Hodge structure, giving rise to finitely generated graded differential rings, generalizing the ring of quasi-modular forms. An algebra, named the Gauss-Manin or AMSY Lie algebra, of derivations on these rings was constructed from a suitable combination of vector fields on the moduli space. The program of investigating moduli spaces of Landau-Ginzburg models enhanced with differential forms was named the Gauss-Manin Connection in Disguise (GMCD).

The thesis is split in three independent parts:

In the first part the GMCD construction is carried out for several families of lattice polarized elliptic K3 surfaces. The local structure of moduli spaces of K3 surfaces enhanced with differential forms is identified and an algebra of derivations on the rings of regular functions is found. We show that the ring of regular functions can be identified with the ring of quasi-modular forms in two variables.

In the second part of the thesis we extend the GMCD program to families of non-compact Calabi-Yau threefolds. A definition of families enhanced with differential forms is proposed and the local structure of the moduli spaces of such families is investigated. We show that in the case of mirrors of local $\mathbb{P}^2$ and local $\mathbb{F}^2$ the rings of regular functions are closely related to the rings of quasi-modular forms, arising from the associated mirror curves. We construct the Gauss-Manin Lie algebra in both cases and identify an $\mathrm{sl}_2(\mathbb{C})$ Lie subalgebra.

The third part of the thesis is concerned with extending the GMCD program to families of toric Landau-Ginzburg models. We propose a definition of Landau-

Ginzburg models, enhanced with a GKZ local system of solutions to differential equations, generalizing both of the previous constructions. We apply the constructions to Landau-Ginzburg mirrors of $\mathbb{CP}^n$, constructing differential rings associated to the families and giving a first example of a GMCD construction for non-Calabi-Yau Landau-Ginzburg models.

Zusammenfassung

Die vorliegende Arbeit befasst sich mit algebraischen Strukturen auf Modulräumen von, mit Differentialformen ausgestatteten, Spiegelgeometrien. In [Mov12] wurde gezeigt, dass die lokale Struktur solcher Modulräume, im Fall von elliptischen Kurven, durch die Variation einer polarisierten Hodge-Struktur konstruiert werden kann. Dies gibt Anlass zur Betrachtung von endlich erzeugten graduierten Differentialringen, welche den Ring der Quasi-Modulformen verallgemeinern. Durch eine passende Kombination von Vektorfeldern auf dem Modulraum kann auf diesen Ringen eine Algebra der Derivationen - genannt die Gauss-Manin oder AMSY Lie-Algebra - konsturiert werden. Das Verfahren zur Untersuchung von Modulräumen des Landau-Ginzburg Modells mit Differentialformen wird als Gauss-Manin Connection in Disguise (GMCD) bezeichnet.

Die Arbeit ist unterteilt in drei unabhängige Teile:

Im ersten Teil wird die GMCD für diverse Familien von gitterpolarisierten K3-Flächen konstruiert. Es wird die lokale Struktur von Modulräumen der K3-Flächen mit Differentialformen identifiziert und eine Algebra der Derivationen auf dem Ring der regulären Funktionen gefunden. Wir zeigen, dass der Ring der regulären Funktionen mit den Ring der Quasi-Modulformen in zwei Variablen identifiziert werden kann.

Im zweiten Teil der Arbeit erweitern wir das GMCD Verfahren auf Familien von nicht kompakten dreidimensionalen Calabi-Yau Mannigfaltigkeiten. Eine Definition von Familien mit darauf definierten Differentialformen wird unterbreitet und die lokale Struktur auf den Modulräumen solcher Familien untersucht. Wir zeigen, dass im Fall von Spiegelungen des lokalen $\mathbb{P}^2$ und lokalen $\mathbb{F}^2$ die Ringe von regulären Funktionen genau mit den Ringen der Quasi-Modulformen, welche aus den damit verbundenen Spiegelkurven entstehen, zusammenhängen. Wir konstruieren die Gauss-Manin Lie-Algebra für beide Fälle und identifizeren eine $\mathrm{sl}_2(\mathbb{C})$ Lie-Unteralgebra.

Der dritte Teil der Arbeit beschäftigt sich mit der Erweiterung des GMCD Verfahrens auf Familien von torischen Landau-Ginzburg-Modellen. Wir führen eine Definition von Landau-Ginzburg Modellen, ausgestattet mit einem GKZ Lokalsystem von Lösungen von Differentialgleichungen ein, wodurch beide der vorhergehenden Konstruktionen verallgemeinert werden. Wir wenden die Konstruktionen auf Landau-Ginzburg Spiegelgeometrien der $\mathbb{CP}^n$ an, konstruieren zu den Familien assoziierte Differentialringe und geben ein erstes Beispiel für eine GMCD-Konstruktion eines Nicht-Calabi-Yau Landau-Ginzburg Modells.

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Contents

Chapter 1

Introduction

The main object of study of this work are moduli spaces of Landau-Ginzburg models. In the recent years the wide-spread interest in the study of their properties was ignited by the discovery of a duality between different supersymmetric Landau-Ginzburg models, called mirror symmetry [LVW89]. Mirror symmetry was initially formulated for Calabi-Yau manifolds of dimension three and has famously lead to the computation of rational curves of an arbitrary degree for the quintic threefold in [CdlOGP91]. The construction was extended beyond compact Calabi-Yau threefolds to compact Calabi-Yau manifolds in any dimension, as well as non-compact Calabi-Yau manifolds and Fano manifolds, see [CK99, HKK+03, KKP08] and references therein.

Mirror symmetry at higher genus was formulated in [BCOV94], providing a recursive procedure to obtain the generating functions of higher genus curve counts using the lower genus ones. In [YY04] it was shown that the higher genus Gromov-Witten generating functions for the quintic threefold are elements in special polynomial rings, which are fixed by the special Kähler geometry. The result was later generalised to Calabi-Yau families with an arbitrary number of moduli [AL07] and has since been extended to a number of Calabi-Yau geometries, see e.g. [Ali13] and references therein. The results found applications in solving the holomorphic anomaly equations and establishing rigorously mirror symmetry for elliptic curves [Li11] conjectured in [Dij95], as well as higher genus Gromov-Witten theory of $\mathrm{K3} \times T$ in [OP16], proving the Igusa cusp form conjecture of [KKV99].

In the case of elliptic curves, it was shown in [Li11], that the higher genus generating functions are elements of the ring of almost holomorphic modular forms, confirming the prediction of [Dij95]. At the point of maximally unipotent monodromy, the generating functions were shown to be elements in the ring of quasi-modular forms of [KZ95]. These are graded differential rings, finitely generated by the Eisenstein series

$$E_{2k} = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n}, \quad \text{for} \quad k = 1, 2, 3, \tag{1.1}$$

where ζ is the Riemann ζ-function. The Eisenstein series give, for example, the normal forms for elliptic curves and are closely related to the Weierstrass $\wp$-function and thus famously provide an interface between number theory, geometry and analysis. The ring of quasi-modular forms is equipped with three distinguished differential structures. It is common (see e.g. [Zag08]), to represent elements $f \in \widetilde{\mathcal{M}} \cong \mathbb{C}[E_2, E_4, E_6]$ as formal sums

$$P_f = \sum_{i,j,k \in \mathbb{N}_0} f_{ijk} E_2^i E_4^j E_6^k, \tag{1.2}$$

where $f_{ijk} \in \mathbb{C}$, and the algebra of derivations on $\widetilde{\mathcal{M}}$ can be defined via

$$\begin{aligned} \partial_\tau f &= \left(\frac{E_2^2 - E_4}{12} \frac{\partial}{\partial E_2} + \frac{E_4 E_2 - E_6}{3} \frac{\partial}{\partial E_4} + \frac{E_6 E_2 - E_4^2}{2} \frac{\partial}{\partial E_6} \right) P_f, \\ W f &= \left(2 E_2 \frac{\partial}{\partial E_2} + 4 E_4 \frac{\partial}{\partial E_4} + 6 E_6 \frac{\partial}{\partial E_6} \right) P_f, \\ F f &= -12 \frac{\partial}{\partial E_2} P_f. \end{aligned} \tag{1.3}$$

It can be checked that the algebra of derivations is isomorphic to $\mathrm{sl}_2(\mathbb{C})$.

While the geometric interpretation of the Eisenstein series in terms of line bundles on elliptic curves is well-understood, more general q-expansions associated to Calabi-Yau varieties are still largely unexplored. The project of constructing similar differential rings was initiated in [Mov12] (based on the related work [Mov08]), where it was found that the ring of quasi-modular forms $\widetilde{\mathcal{M}} \cong \mathbb{C}[E_2, E_4, E_6]$ appears naturally as the ring of regular functions on the moduli space of elliptic curves, enhanced with differential one forms. This discovery prompted the study of moduli spaces of varieties, enhanced with differential forms and their rings of regular functions. In [Mov12] a set of techniques for describing moduli spaces (denoted throughout this work by T) of varieties enhanced with differential forms was developed and soon after in a sequence of works [Mov13, Mov11, AMSY16] several properties of such moduli spaces were established. The program was named the Gauss-Manin Connection in Disguise (GMCD) [Mov17b], due to its close relationship with the variation of Hodge structure and the important role the Gauss-Manin connection plays in finding the algebraic structure on the rings of regular functions. The program aims to describe the structure of moduli spaces of enhanced varieties, and determine their algebraic structure. It was applied to a number of families of varieties, such as families of elliptic curves [Mov12], K3 surfaces [DHMW16, Ali17, Nik20, AV18], Calabi-Yau threefolds [Mov15, AMSY16, Ali17] and Dwork families [MN16, Nik20]. The success of the program shows in reproducing the classical quasi-modular forms in the elliptic curve case [Mov12], as well as giving rise to more general rings of automorphic forms, for example Siegel modular forms in the case of genus 2 hyperelliptic curves [CMY19], rings of automorphic forms for Calabi-Yau threefolds [Mov15, AMSY16], and abelian varieties [Fon18]. The approach found applications in the study of modularity of elliptic Calabi-Yau manifolds [Hag17], the study of Noether Lefschetz loci in [Mov17c] and Hitchin systems in [ABF19]. In [AMSY16, Ali17], a synthesis of Movasati's approach and the differential rings of the special geometry of Calabi-Yau threefolds [YY04, AL07] was made, see also [Zho13].

The algebra of derivations on the rings of regular functions $\mathcal{O}_{\mathsf{T}}$ was introduced in [AMSY16] and was named the Gauss-Manin or AMSY Lie algebra in the literature. The algebra of derivations on $\mathcal{O}_{\mathsf{T}}$ was related to the existence of special (modular) vector fields on T. It was observed that this algebra is closely related to the Lie algebra of the group of automorphisms of the Hodge filtration, preserving compatibility with the Hodge filtration in the case of Calabi-Yau manifolds. The Gauss-Manin Lie algebra for elliptic curves was characterized in the framework of GMCD in [Mov12] and was computed for the mirror quintic in [Mov15], for elliptic K3 surfaces in [AV18] and for Dwork families in [MN16, Nik20].

Main Results

The contribution of this thesis is the following:

Gauss-Manin Lie algebra for elliptic K3 surfaces

In the first part of this work, we construct the moduli space T from the data of the holomorphic Gauss-Manin connection on the middle dimensional cohomology of the mirrors of the elliptically fibered K3 manifolds. We show that T is 6-dimensional in accordance with its general construction based on special geometry of Ref. [Ali17]. Away from the discriminant locus T is a locally ringed space with the ring of regular functions $\mathcal{O}_{\mathsf{T}}$.

Theorem 1.1. *There is an isomorphism*

$$\mathcal{O}_{\mathsf{T}} \cong \widetilde{\mathcal{M}}(\Gamma_0(N) \times \Gamma_0(N)), \tag{1.4}$$

between the local ring $\mathcal{O}_{\mathsf{T}}$ and the graded ring of quasi-modular forms of the modular subgroup $\Gamma_0(N)$ in two variables. The level N of the congruence subgroup is determined by the type of elliptic fiber of the mirror.

We construct the Gauss-Manin Lie algebra $\mathfrak{G}$ attached to T.

Theorem 1.2. *There is an isomorphism*

$$\mathfrak{G} \cong \mathrm{sl}_2(\mathbb{C}) \oplus \mathrm{sl}_2(\mathbb{C}). \tag{1.5}$$

The algebra of derivations of quasi modular forms from mirror symmetry

In the second part of this work we develop the GMCD construction for families of non-compact Calabi-Yau threefolds. The program is applied to two cases of interest, the mirror families to local $\mathbb{P}^2$ and local $\mathbb{F}^2$, as an illustration. The developed techniques are universal and can be generalised to other non-compact Calabi-Yau varieties, as well as variations of mixed Hodge structure associated to different families. For the mirrors of local $\mathbb{P}^2$ and $\mathbb{F}^2$, the local structure of the moduli spaces can partially be characterised by the inclusion of differential rings of the mirror curves into the differential ring of the non-compact Calabi-Yau threefold family. The following structural theorem gives the isomorphism of differential rings in the case of the mirror of local $\mathbb{P}^2$:

Theorem 1.3. *There is an isomorphism between the ring of regular functions $\mathcal{O}_{\mathsf{T}}$ for the mirror of local $\mathbb{P}^2$ equipped with the differential structure coming from a modular vector field R on T and the ring of quasi-modular forms on $\Gamma_0(3)$ with the derivation ∂_τ.*

For the mirror of local $\mathbb{F}^2$ we prove the following theorem, relating the differential ring of the family of non-compact Calabi-Yau threefold to the differential ring of the associated mirror curve:

Theorem 1.4. *The ring of regular functions $\mathcal{O}_{\mathsf{T}}$ for the mirror of local $\mathbb{F}^2$ equipped with the differential structures coming from modular vector fields R_1 and R_2 contains as a differential sub-ring the ring of quasi-modular forms on $\Gamma_0(2)$ with the derivation ∂_τ.*

Moreover, the Gauss-Manin Lie algebra is computed in both cases and the following theorems characterise its relation to the Gauss-Manin Lie algebra of the associated mirror curve:

Theorem 1.5. *There is an* $\mathrm{sl}_2(\mathbb{C})$ *Lie subalgebra of the Gauss-Manin Lie algebra* $\mathfrak{G}$ *for mirrors of local* $\mathbb{P}^2$ *and local* $\mathbb{F}^2$*.*

Gauss-Manin Connection in Disguise: $\mathbb{CP}^n$

The third part of this work is concerned with extending the GMCD program to the case of Landau-Ginzburg models. In particular, we consider the case of complex projective spaces, whose mirrors are given by the triples (Y_q, f_q, ω_q), where $Y_q : \{y_1 \cdots y_{n+1} = q\} \subset \mathbb{C}^{n+1}$ is a family of affine surfaces over the moduli space B spanned by the formal parameter $q = e^t$, $f_q = y_1 + \ldots + y_{n+1}$ restricted to Y_q is called the superpotential and ω_q is a symplectic form on Y_q. We consider these Landau-Ginzburg models, enhanced with solutions of the extended GKZ system (3.37). For the mirror families of $\mathbb{CP}^n$, we construct a (modular) vector field on the moduli space T of weakly enhanced Landau-Ginzburg models mirror to $\mathbb{CP}^n$. We further determine the induced differential structure on the local ring and compute the Gauss-Manin Lie algebra.

Theorem 1.6. *There is an* $\mathrm{sl}_2(\mathbb{C})$ *Lie subalgebra of the Gauss-Manin Lie algebra* $\mathfrak{G}$ *for mirrors of complex projective spaces. For* $\mathbb{CP}^1$*,* $\mathfrak{G} \cong \mathrm{sl}_2(\mathbb{C})$*.*

Publications

The following manuscripts have resulted from the work of this thesis:

- Murad Alim and Martin Vogrin. Gauss-Manin Lie algebra of mirror elliptic K3 surfaces. *arXiv preprint: arXiv:1812.03185*, 2018 (Accepted for publication in Math. Res. Lett.).
- Murad Alim, Vadym Kurylenko, and Martin Vogrin. The algebra of derivations of quasi modular forms from mirror symmetry. *arXiv preprint: arXiv:2008.06523*, 2020.

Organisation of the thesis

This thesis is organized as follows:

Chapter 2

In chapter 2 we put forward the basic definitions. We define graded differential rings of modular forms and introduce certain algebraic operations on it. We further introduce the concept of Hodge structures and their variation and the related concept of Frobenius structure.

Chapter 3

In chapter 3 we review the basics of toric geometry and mirror symmetry. We construct variation of Hodge structure for Calabi-Yau manifolds in large generality and comment on the extension to Landau-Ginzburg models, which is treated later in chapter 7. We relate the variation of Hodge structure on middle-dimensional cohomology to a variation of Hodge structure on the space of solution of a certain differential operator. In the last part we show that the constructed variations of Hodge structure are naturally polarized.

Chapter 4

Chapter 4 is designed as a self-contained introduction to the GMCD program. We introduce enhanced varieties and construct their moduli spaces. We further introduce the Gauss-Manin connection for them and an algebraic group acting on enhanced varieties. From its Lie algebra, we give an algebraic construction of the Gauss-Manin Lie algebra and show that it is isomorphic to a certain subalgebra of vector fields on the moduli space of enhanced varieties. We apply the construction to the family of elliptic curves.

Chapter 5

In chapter 5 we study families of elliptic K3 surfaces given by lattice polarized K3 surfaces as introduced by Dolgachev [Dol96]. We construct the moduli space of elliptic K3 surfaces enhanced with differential forms T and show that, away from the discriminant locus T is a locally ringed space with the local ring $\mathcal{O}_{\mathsf{T}} \cong \widetilde{\mathsf{M}}(\Gamma_0(N)\times\Gamma_0(N))$, where the level N of the congruence subgroup is determined by the type of elliptic fiber of the mirror. Moreover, we construct the Gauss-Manin Lie algebra $\mathfrak{G}$ attached to T and prove in Theorem 5.13 that there is an isomorphism $\mathfrak{G} \cong \mathrm{sl}_2(\mathbb{C}) \oplus \mathrm{sl}_2(\mathbb{C})$.

Chapter 6

After defining a suitable generalization of the intersection pairing for non-compact Calabi-Yau families, we define moduli spaces of non-compact varieties enhanced with differential forms and give their local construction. We introduce certain algebraic structures on the ring of regular functions, such as Ramanujan-Serre differential structure and the Gauss-Manin Lie algebra and show how they arise naturally in the geometric framework of the GMCD. In the second part of the chapter, we apply the construction to two families, mirror to non-compact Calabi-Yau threefolds and prove the main Theorems 1.3, 1.4 and 1.5 for these families.

Chapter 7

In chapter 7 we introduce the notion of (weakly) enhanced toric Landau-Ginzburg models, by attaching to toric Landau-Ginzburg models a local system of solutions to the associated GKZ system. We investigate moduli spaces of such pairs and construct the rings of regular functions. We treat in detail the case of Landau-Ginzburg models mirror to complex projective spaces.

Chapter 2

Preliminaries

Cohomology groups of compact Kähler manifolds admit a Hodge decomposition and thus carry a natural (polarized) Hodge structure. Similarly, families of compact Kähler manifolds are naturally equipped with a variation of (polarized) Hodge structure on their cohomology groups. One might adapt the notion of frame bundles (see e.g. [KN96]) to this setting, by imposing compatibility of the frame with the variation of Hodge structure. The moduli space of families of compact Kähler manifolds, together with a frame of its middle cohomology compatible with the variation of polarized Hodge structure on it, is a locally ringed space with the ring of regular functions being a graded differential ring, generalizing the ring of quasi-modular forms. In this sense the GMCD program serves as a bridge between geometry of (enhanced) varieties, and the theory of quasi-modular forms.

Moduli spaces of Landau-Ginzburg models are naturally equipped with a Frobenius algebra on their holomorphic tangent space. Determining the Frobenius structure is equivalent to solving the WDVV equations [DVV91, Wit90] for these Landau-Ginzburg models and in this sense provides a geometric version of the problem [Dub93, Dub96]. The Frobenius structure plays an important role in mirror symmetry, as one version of the correspondence identifies the Frobenius structures on the moduli spaces of mirror geometries. The first geometric construction of a Frobenius structure for Landau-Ginzburg models was given by K. Saito [Sai83] in the case of isolated singularities, by introducing primitive forms (see also [Her03]). For Landau-Ginzburg models mirror to Calabi-Yau manifolds in any dimension Frobenius structure on the moduli spaces was constructed in [Bar00, Bar02] and for mirrors of projective spaces in [Bar01], by introducing the notion of variation of semi-infinite Hodge structure, later investigated in the context of homological mirror symmetry in [KKP08] and twistorial structures [Her03, HS08]. For Calabi-Yau families, Frobenius structure on their moduli space is typically constructed from variation of Hodge structure on the middle cohomology of the deformation family by introducing a special set of (flat) coordinates.

This chapter is intended as an introduction to the geometry of moduli spaces of Landau-Ginzburg models, variation of Hodge structure, Frobenius manifolds and differential rings of quasi-modular forms.

2.1 Modular forms

The exposition in this section follows to a large extent [Zag08], see also [Zho13, Ali14] and references therein.

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ be the upper half plane and $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ be the extended upper half plane. Consider the natural action of $\mathrm{SL}_2(\mathbb{Z})$ on $\overline{\mathbb{H}}$

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.1}$$

Definition 2.1. A (meromorphic) modular form of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$ is a function $f : \overline{\mathbb{H}} \to \mathbb{P}^1$ such that the following conditions hold:

1. $$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{2.2}$$
   where $(c\tau + d)^k$ is called the automorphy factor,
2. f is meromorphic on $\mathbb{H}$,
3. f is meromorphic at the cusps in the sense that function $(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$ is meromorphic at $\tau = i\infty$ for any $\gamma \in \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$.

We denote by $\mathcal{M}_k$ the space of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of a fixed weight k. It is non-trivial only for even k and finite dimensional for any k, see e.g. [Zag08]. The direct sum

$$\mathcal{M} = \bigoplus_{k \in 2\mathbb{Z}} \mathcal{M}_k, \tag{2.3}$$

will denote the graded ring of modular forms. Define the Eisenstein series

$$
\begin{aligned}
E_2(\tau) &= 1 - 24\sum_{k=1}^{\infty}\sigma_1(k)e^{2\pi ik\tau}, \quad \sigma_1(k) = \sum_{d:d|k} d,\\
E_4(\tau) &= 1 + 240\sum_{k=1}^{\infty}\sigma_3(k)e^{2\pi ik\tau}, \quad \sigma_3(k) = \sum_{d:d|k} d^3,\\
E_6(\tau) &= 1 - 504\sum_{k=1}^{\infty}\sigma_5(k)e^{2\pi ik\tau}, \quad \sigma_5(k) = \sum_{d:d|k} d^5.
\end{aligned}
\tag{2.4}
$$

The ring of modular forms $\mathcal{M}$ is isomorphic to $\mathbb{C}[E_4, E_6]$.

Definition 2.2. A (meromorphic) quasi-modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight k and depth p is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ and meromorphic at the cusps, such that there exist $p+1$ meromorphic functions $f_0, \ldots, f_p$, such that

$$
(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{i=0}^{p} f_i(\tau)\left(\frac{c}{c\tau + d}\right)^i. \tag{2.5}
$$

Example 2.3. The Eisenstein series E_2 is a quasi-modular form of depth 1 and weight 2

$$
E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) + 12c(c\tau + d), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.6}
$$

It was shown in [KZ95], that there is a ring isomorphism $\widetilde{\mathcal{M}} \cong \mathcal{M} \otimes \mathbb{C}[E_2] \cong \mathbb{C}[E_2, E_4, E_6]$. The second Eisenstein series E_2 is manifestly non-holomorphic, however we its holomorphic completion can be introduced

$$
\widehat{E_2} = E_2 - \frac{3}{\pi \mathrm{Im}\tau}. \tag{2.7}
$$

The ring $\widehat{\mathcal{M}} = \mathbb{C}[\widehat{E_2}, E_4, E_6]$ is called the ring of almost-holomorphic modular forms.

Remark 2.4. An element $\widehat{f} \in \widehat{\mathcal{M}}$ can be written as a polynomial

$$
\widehat{f} = \sum_{i \in \mathbb{N}_0} f_i Y^i, \qquad Y = \frac{1}{\mathrm{Im}\tau}, \tag{2.8}
$$

with f_i holomorphic functions on $\mathbb{H}$. The constant term map $\widehat{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is given by $\widehat{f} \mapsto f_0$. In other words, treating $Y = \frac{1}{\mathrm{Im}\tau}$ as an independent parameter, it is given by the $Y \to 0$ limit.

Rings of quasi and almost holomorphic modular forms carry a natural differential structure given by differentiation by τ. Explicitly, the differential structure on the ring of quasi-modular forms is given by the following action on the generators

$$\begin{aligned}\partial_\tau E_2 &= \frac{1}{12}(E_2^2 - E_4),\\ \partial_\tau E_4 &= \frac{1}{3}(E_2 E_4 - E_6),\\ \partial_\tau E_6 &= \frac{1}{2}(E_2 E_6 - E_4^2),\end{aligned} \tag{2.9}$$

where $\partial_\tau = \frac{1}{2\pi i}\frac{\partial}{\partial \tau}$. A similar relation holds for the ring of almost-holomorphic modular forms with E_2 replaced by $\widehat{E}_2$.

Remark 2.5. The differential ∂_τ gives a map $\mathcal{M} \to \widetilde{\mathcal{M}}$. The ring of modular forms $\mathcal{M}$ is closed under the Ramanujan-Serre differential $D = \partial_\tau - \frac{k}{12}E_2$, where k is the weight of the modular form.

In addition to the differential structure ∂_τ, there exist two other differential structures on $\widetilde{\mathcal{M}}$ (see e.g. [Zag08, Zag16]), which we denote by W and F. Due to the fact that $\widetilde{\mathcal{M}}$ is finitely generated, we can write any quasi-modular form f of weight k as a polynomial in E_2

$$f = P_f(E_2) = \sum_{i\in\mathbb{N}_0} f_i E_2^i, \tag{2.10}$$

with $f_i \in \mathcal{M}$ of weight $k - 2i$. Define

$$Ff = -12 P_f'(E_2). \tag{2.11}$$

The second operator operator simply returns the weight of the quasi-modular form

$$Wf = kf. \tag{2.12}$$

Remark 2.6. The same differentiations can be defined for the ring of almost-holomorphic modular forms by replacing E_2 with $\widehat{E}_2$.

Proposition 2.7. *The Lie algebra of derivations on $\widetilde{\mathcal{M}}$ is isomorphic to* $\mathrm{sl}_2(\mathbb{C})$. *Explicitly*

$$[W, \partial_\tau] = 2\partial_\tau, \quad [W, F] = -2F, \quad [\partial_\tau, F] = W. \tag{2.13}$$

Proof. Let $f \in \widetilde{\mathcal{M}}$ be a quasi-modular form of weight k. It can be expanded as (2.10) for $f_i \in \mathcal{M}$ of weight $k-2i$. It then holds

$$(W \circ F - F \circ W)(f) = \sum_{i \in \mathbb{N}_0} (-12i(k-2) f_i E_2^{i-1}) + 12ik f_i E_2^{i-1})) = -2F(f). \tag{2.14}$$

Other commutators follow by a similar argument. □

Remark 2.8. There exists a similar Lie algebra of differentiations in the case of Jacobi modular forms, see [OP19].

Quasi-modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$

Let N be a positive integer, then the principal congruence sugbroup of level N is

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}. \tag{2.15}$$

A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$. For example,

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}. \tag{2.16}$$

The ring of (quasi-)modular forms for Γ is defined analogously to definitions 2.1, respectively 2.2 with $\mathrm{SL}_2(\mathbb{Z})$ replaced by Γ. We define weight one modular forms associated to the congruence subgroups $\Gamma_0(N)$:

N	A	B	C
1	$E_4(\tau)^{1/4}$	$\left(\frac{E_4(\tau)^{3/2}+E_6(\tau)}{2}\right)^{1/6}$	$\left(\frac{E_4(\tau)^{3/2}-E_6(\tau)}{2}\right)^{1/6}$
2	$\frac{(64\eta(2\tau)^{24}+\eta(\tau)^{24})^{1/4}}{\eta(\tau)^2\eta(2\tau)^2}$	$\frac{\eta(\tau)^4}{\eta(2\tau)^2}$	$2^{3/2}\frac{\eta(2\tau)^4}{\eta(\tau)^2}$
3	$\frac{(27\eta(3\tau)^{12}+\eta(\tau)^{12})^{1/3}}{\eta(\tau)\eta(3\tau)}$	$\frac{\eta(\tau)^3}{\eta(3\tau)}$	$3\frac{\eta(3\tau)^3}{\eta(\tau)}$

where $\eta(\tau)$ denotes the Dedekind η-function

$$\eta(\tau) = \left(\frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} \right)^{\frac{1}{24}}. \tag{2.17}$$

Define also the analogue of the Eisenstein series E_2 as

$$E = \partial_\tau \log B^r C^r, \tag{2.18}$$

where $r = 6$ for $N = 1$, $r = 4$ for $N = 2$ and $r = 3$ for $N = 3$. From the Ramanujan-Serre differential ring of Eisenstein series (2.9) we deduce the following differential relations

$$\begin{aligned}
\partial_\tau A &= \frac{1}{2r} A\left(E + \frac{C^r - B^r}{A^{r-2}}\right),\\
\partial_\tau B &= \frac{1}{2r} B(E - A^2),\\
\partial_\tau C &= \frac{1}{2r} C(E + A^2),\\
\partial_\tau E &= \frac{1}{2r}(E^2 - A^4).
\end{aligned} \tag{2.19}$$

The other derivations on the ring are given by

$$W : f \mapsto kf, \tag{2.20}$$

and

$$F : f \mapsto -2r\partial_E P_f(E), \tag{2.21}$$

for a quasi modular form $f \in \widetilde{\mathcal{M}}(\Gamma_0(N))$ for a congruence subgroup $\Gamma_0(N)$, where $P_f(E)$ is again viewed as a formal expansion of f in E. The Lie algebra of derivations is $\mathrm{sl}_2(\mathbb{C})$.

2.2 Hodge theory

The motivating examples for defining (variations of) Hodge structures are the cohomology groups of compact Kähler manifolds. For a pedagogical exposition to the construction of Hodge structure we refer for example to [Voi07, Huy05] and to [Mov17a] for a recent treatment, with an emphasis on the connection to the theory of enhanced varieties, which appear later in the text.

Definition 2.9. A Hodge structure $(H_\mathbb{Z}, \{H^{p,q}\})$ of weight $n \in \mathbb{Z}$ is a finitely generated free abelian group $H_\mathbb{Z}$ and a decomposition $H_\mathbb{C} = \oplus_{p+q=n} H^{p,q}$ of the complexification $H_\mathbb{C} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$, which satisfies $\overline{H^{p,q}} = H^{q,p}$.

Definition 2.10. A polarized Hodge structure $(H_\mathbb{Z}, H^{p,q}, Q)$ is a Hodge structure $(H_\mathbb{Z}, H^{p,q})$ together with an integral non-degenerate bilinear form Q on $H_\mathbb{Z}$, which extends to $H_\mathbb{C}$ by linearity and satisfies

1. Q is symmetric if n is even and skew-symmetric if n is odd,

2. $Q(\zeta, \eta) = 0$ for $\zeta \in H^{p,q}$ and $\eta \in H^{p',q'}$ with $p \neq q'$,

3. $(-1)^{\frac{n(n-1)}{2}} i^{p-q} Q(\zeta, \overline{\zeta}) > 0$ for $\zeta \in H^{p,q}$ non-zero.

A Hodge structure is said to be polarizable if it admits a polarization.

Equivalently one can define a (polarized) Hodge structure in terms of a decreasing filtration F^p on $H_{\mathbb{C}}$

$$H_{\mathbb{C}} = F^0 \supset F^1 \supset \ldots \supset F^n \supset \{0\}. \tag{2.22}$$

The two definitions are equivalent, by defining

$$F^p = \oplus_{i \geq p} H^{i,n-i}, \tag{2.23}$$

or

$$H^{p,q} = F^p \cap \overline{F^q}. \tag{2.24}$$

Definition 2.11. A variation of Hodge structure on a complex variety B is a pair $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ such that

- $\mathcal{H}_{\mathbb{Z}}$ is a locally constant sheaf of finitely generated $\mathbb{Z}$-modules on B,
- $\mathcal{F}^\bullet$ is a finite decreasing filtration on $\mathcal{H} = \mathcal{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathsf{B}}$ by holomorphic subbundles,

such that

- For each $b \in \mathsf{B}$ the stalks $\mathcal{F}_b^\bullet$ form a decreasing filtration on $(\mathcal{H}_{\mathbb{Z}})_b \otimes \mathbb{C}$,
- The Gauss-Manin conection $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{\mathsf{B}}$ defined by

  $$\nabla(s \otimes f) = s \otimes df, \tag{2.25}$$

  for $s \in \mathcal{H}_{\mathbb{Z}}$ and $f \in \mathcal{O}_{\mathsf{B}}$, satisfies the Griffiths' transversality condition

  $$\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes_{\mathcal{O}_{\mathsf{B}}} \Omega^1_{\mathsf{B}}. \tag{2.26}$$

A variation of Hodge structure is said to be polarized if there exists a ∇-flat bilinear pairing

$$\mathsf{Q} : \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{H}_{\mathbb{Z}} \to \mathbb{Z}, \tag{2.27}$$

such that it polarizes $(\mathcal{H}_{\mathbb{Z}})_b$ for each $b \in \mathsf{B}$.

Example 2.12. Let $\pi : \mathcal{X} \to \mathsf{B}$ be a family of compact Calabi-Yau varieties of dimension n. The middle cohomology of a general fibre $X_b = \pi^{-1}(b),\ b \in \mathsf{B}$ carries a Hodge structure $(H^n(X_b, \mathbb{Z}), F^\bullet(X_b))$, naturally polarized by

$$Q(\alpha, \beta) = (-1)^{\frac{n(n-1)}{2}} \int_{X_b} \alpha \wedge \beta, \qquad \alpha, \beta \in H^n(X_b, \mathbb{Z}), \tag{2.28}$$

and there is a polarized variation of Hodge structure $(\mathcal{H}^n_{\mathbb{Z}}(\mathcal{X}/\mathsf{B}), \mathcal{F}^\bullet)$, where $\mathcal{H}^n_{\mathbb{Z}}(\mathcal{X}/\mathsf{B}) = R^n\pi_*\mathbb{Z}$, and $\mathcal{F}^\bullet$ are unique vector subbundles of $\mathcal{H}^n(\mathcal{X}/\mathsf{B}) = \mathcal{H}^n_{\mathbb{Z}}(\mathcal{X}/\mathsf{B}) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathsf{B}}$ such that $\mathcal{F}^\bullet_b = F^\bullet(X_b)$. The variation of Hodge structure is naturally polarized by (2.28).

In contrast to compact Calabi-Yau varieties, cohomology groups of non-compact Calabi-Yau varieties are naturally endowed with a mixed Hodge structure introduced by [Del71].

Definition 2.13. A mixed Hodge structure $(H_{\mathbb{Z}}, W_\bullet, F^\bullet)$ of weight $n \in \mathbb{Z}$ is a $\mathbb{Z}$-module $H_{\mathbb{Z}}$ together with an increasing (weight) filtration $W_\bullet$ on $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and a decreasing (Hodge) filtration $F^\bullet$, which defines a pure $\mathbb{Q}$-Hodge structure of weight i on the graded piece $\mathrm{Gr}^W_i = W_i/W_{i-1}$.

A mixed Hodge structure $(H_{\mathbb{Z}}, W_\bullet, F^\bullet)$ is called graded-polarized if the induced Hodge structure on Gr^W_i is polarized for all i.

Definition 2.14. A variation of mixed Hodge structure on a complex variety B is a triple $(\mathcal{H}_{\mathbb{Z}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$, where

1. $\mathcal{H}_{\mathbb{Z}}$ is a locally constant sheaf of finitely generated $\mathbb{Z}$-modules on B,
2. $\mathcal{W}_\bullet$ is an increasing filtration of $\mathcal{H}_{\mathbb{Q}} = \mathcal{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ by locally constant subsheaves,
3. $\mathcal{F}^\bullet$ is a decreasing filtration on $\mathcal{H} = \mathcal{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathsf{B}}$ by holomorphic subbundles, satisfying Griffiths' transversality.

A variation of mixed Hodge structure is called graded-polarized if there exists a pairing Q, flat with respect to the Gauss-Manin connection ∇ on each $\mathrm{Gr}^{\mathcal{W}}_i = \mathcal{W}_i/\mathcal{W}_{i-1}$.

2.3 Frobenius structure

An important set of manifolds which play a role in mirror symmetry for Landau-Ginzburg models are Frobenius manifolds, investigated in [Dub93], based on the tt^* geometry of [CV91]. A detailed investigation of the relation between the two and the construction for singularities was given in [Her03].

Let B be a complex manifold. We will denote by $\mathcal{T}_{\mathsf{B}}$ the holomorphic tangent bundle of B.

Definition 2.15. A Frobenius manifold of rank $n \in \mathbb{Z}$ is a tuple $(\mathsf{B}, \eta, \circ, E, e)$ where

- B is a complex manifold of dimension n,
- $\eta : \mathcal{T}_{\mathsf{B}} \otimes \mathcal{T}_{\mathsf{B}} \to \mathcal{O}_{\mathsf{B}}$ is a non-degenerate $\mathcal{O}_{\mathsf{B}}$-bilinear form,
- $\circ$ is an associative commutative η-invariant multiplication on $\mathcal{T}_{\mathsf{B}}$, i.e.

$$\eta(u \circ v, w) = \eta(u, v \circ w), \quad u, v, w \in \mathcal{T}_{\mathsf{B}}, \tag{2.29}$$

- The Levi-Civita connection $\nabla : \mathcal{T}_{\mathsf{B}} \to \mathcal{T}_{\mathsf{B}} \otimes \Omega^1_{\mathsf{B}}$ of η is flat

$$\begin{aligned} [\nabla_u, \nabla_v] &= \nabla_{[u,v]}, \quad u, v \in \mathcal{T}_{\mathsf{B}}, \\ \nabla_u v - \nabla_v u &= [u, v], \quad u, v \in \mathcal{T}_{\mathsf{B}}, \\ u\eta(v, w) &= \eta(\nabla_u v, w) + \eta(v, \nabla_u w), \quad u, v, w \in \mathcal{T}_{\mathsf{B}}, \end{aligned} \tag{2.30}$$

  and satisfies the potentiality condition $\nabla C = 0$, where C is the Higgs field, i.e. an $\mathcal{O}_{\mathsf{B}}$-linear map $C : \mathcal{T}_{\mathsf{B}} \to \mathcal{T}_{\mathsf{B}} \otimes \Omega^1(\mathsf{B})$, s.t. $C_u(v) = -u \circ v$,

- E is a holomorphic vector field called the Euler vector field, satisfying

$$Lie_E(\circ) = \circ, \quad Lie_E(\eta) = (2 - d)\eta, \quad \text{for some } d \in \mathbb{C}, \tag{2.31}$$

- e is a ∇-parallel holomorphic vector field, which is an identity for the multiplication, i.e. $e \circ u = u, \ \forall u \in \mathcal{T}_{\mathsf{B}}$.

Remark 2.16. Let us consider the space of parallel sections of ∇

$$\mathcal{T}_{\mathsf{B}}^f = \{u \in \mathcal{T}_{\mathsf{B}} \mid \nabla_u v = 0 \ \forall u \in \mathcal{T}_{\mathsf{B}}\}, \tag{2.32}$$

which is a local system of rank n on B, such that η takes constant values on $\mathcal{T}_{\mathsf{B}}^f$. Since $\mathcal{T}_{\mathsf{B}}^f$ is abelian with respect to the commutator, it holds that at each point in B, there exists a local coordinate system $(t_0, \ldots, t_{n-1})$, called flat coordinates, such that

$$\text{i) } e = \partial_0, \quad \text{and} \quad \text{ii) } \partial_i = \frac{\partial}{\partial t_i} \ (i = 1, \ldots, n-1) \text{ span } \mathcal{T}_{\mathsf{B}}^f \text{ over } \mathbb{C}. \tag{2.33}$$

e is sometimes called a primitive vector field, or a primitive derivation.

Remark 2.17. The potentiality axiom $\nabla C = 0$ implies the existence of a prepotential $\mathcal{F}$ for the flat metric. At each point of B there exists a holomorphic function $\mathcal{F}$ satisfying

$$\eta(\partial_i \circ \partial_j, \partial_k) = \eta(\partial_i, \partial_j \circ \partial_k) = \partial_i \partial_j \partial_k \mathcal{F}, \quad \forall i,j,k = 0,\dots,n-1, \tag{2.34}$$

where ∂_i correspond to flat coordinates. In particular, one has

$$\eta_{ij} = \eta(\partial_i, \partial_j) = \partial_i \partial_j \mathcal{F}. \tag{2.35}$$

A notion closely related to Frobenius structure is the notion of a Saito structure, which we introduce next.

Definition 2.18. A Saito structure of weight $w \in \mathbb{Z}$ on B is a tuple $(\mathsf{B}, \pi : V \to \mathsf{B}, \nabla, \eta, C, R_0, R_\infty)$, where

- $\pi : V \to \mathsf{B}$ is a complex vector bundle on B equipped with a metric η, a flat connection ∇ and two endomorphisms R_0 and R_∞ satisfying the following conditions:

$$R^\nabla = 0, \quad d^\nabla C = 0 \quad \nabla\eta = 0, \quad C \wedge C = 0, \quad C^* = C, \tag{2.36}$$

  and

$$\begin{aligned} &\nabla R_0 + C = [C, R_\infty], \quad [R_0, C] = 0, \quad R_0^* = R_0, \\ &\nabla R_\infty = 0, \quad R_\infty^* + R_\infty = -w\mathrm{Id}_V. \end{aligned} \tag{2.37}$$

  Above $d^\nabla C$ and $C \wedge C$ are $\mathrm{End}(V)$-valued 2-forms, defined by

$$\begin{aligned} (d^\nabla C)_{u,v} &= \nabla_u(C_v) - \nabla_v(C_u), \\ (C \wedge C)_{u,v} &= C_u C_v - C_v C_u, \end{aligned} \tag{2.38}$$

  for any $u, v \in \mathcal{T}_\mathsf{B}$, and * denotes the adjoint with respect to η.

A Frobenius manifold $(\mathsf{B}, \circ, e, \eta, E)$ defines a Saito structure $(\pi : \mathcal{T}_\mathsf{B} \to \mathsf{B}, \nabla, \eta, C, R_0, R_\infty)$, where ∇ is the Levi-Civita connection of η, $C_u v = u \circ v$, $R_0 = C_E$ and $R_\infty = \nabla E$. Conversely a Saito bundle, whose rank is equal to the dimension of the base, together with a suitably chosen parallel section of ∇ (usually called primitive homogeneous), gives rise to a Frobenius structure on the base of the bundle.

Chapter 3

Mirror Geometries

In this chapter we introduce basics of toric geometry following to a large extent [CR18] and [HKT95]. Toric geometry gives a very important set of examples of varieties, for which mirror constructions are well understood. Two of the most commonly used mirror constructions are the Batyrev-Borisov [Bat94, BB96] and Hori-Vafa [HV00, HIV00] mirror constructions. For the families mirror to hypersurfaces in toric varieties a variation of polarized Hodge structure can be constructed from the toric data. We define the classical periods of Landau-Ginzburg models and introduce a set of (GKZ) differential operators annihilating them.

3.1 Landau-Ginzburg models

Definition 3.1. A Landau–Ginzburg model is a tuple (Y_q, f_q, ω_q), where

- Y_q is a family of $\mathcal{C}^\infty$-manifold parametrized by q, with a symplectic form ω_q,
- $f_q : Y_q \to \mathbb{C}$ is a proper $\mathcal{C}^\infty$-map, such that there exists an $R > 0$ so that over $\{q \in \mathbb{C} | |q| \geq R\}$ the map f_q is a smooth fibration with fibers symplectic submanifolds in (Y_q, ω_q).

Definition 3.2. A Laurent polynomial $f_q \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is called toric if there is an embedded degeneration Y_q to a toric variety T whose fan polytope (the convex hull of generators of its rays) coincides with the Newton polytope (the convex hull of non-zero coefficients) of f_q. A Laurent polynomial without the toric condition is called a weak Landau–Ginzburg model.

Definition 3.3. A Calabi-Yau manifold X is a Kähler manifold with $c_1(X) = 0$.

Remark 3.4. The condition $c_1(X) = 0$ implies the existence of a unique nowhere vanishing holomorphic form of degree n on X [Yau78].

Definition 3.5. A Landau-Ginzburg model is said to be of Calabi-Yau type if any fiber of $f_q : Y_q \to \mathbb{C}$ after some fiberwise compactification is a Calabi-Yau manifold.

3.2 Toric geometry

Definition 3.6. An affine toric variety of dimension n is an irreducible affine variety V containing an algebraic torus $T = (\mathbb{C}^*)^n$ as a Zariski open subset, such that the action of T on itself extends to an algebraic action on all V.

Let N be a lattice, i.e. a free abelian group of finite rank, and set $N_\mathbb{R} = N \otimes \mathbb{R}$.

Definition 3.7. A convex rational polyhedral cone in $N_\mathbb{R}$ is a set

$$\sigma = \left\{ \sum_i a_i v_i \mid \mathbb{C} \ni a_i \geq 0 \right\} \subseteq N_\mathbb{R}, \tag{3.1}$$

for $v_1, \ldots, v_n \in N$.

A polyhedral cone is said to be strongly convex if $\sigma \cap (-\sigma) = \{0\}$.

Definition 3.8. Let M be the lattice dual to N and denote $\langle \, , \, \rangle : M \times N \to \mathbb{Z}$. The dual cone to σ is a cone

$$\sigma^\vee = \{m \in M_\mathbb{R} \mid \langle m, v \rangle \geq 0 \; \forall v \in \sigma\}. \tag{3.2}$$

Let $\sigma \subseteq N_\mathbb{R}$ be a polyhedral cone, then $\sigma^\vee$ is also a polyhedral cone in $M_\mathbb{R}$ and $(\sigma^\vee)^\vee = \sigma$.

Definition 3.9. Let σ be a cone and let $\lambda \in \sigma^\vee \cap M$. Then $\tau = \sigma \cap \lambda^\perp$ is called a face of σ.

The construction of toric varieties from a cone is as follows. For a cone σ consider

$$S_\sigma = \sigma^\vee \cap M, \tag{3.3}$$

finitely generated by vectors $n_1, \ldots, n_p, \geq \dim M$. The non-trivial linear relations between the generators of S_σ are denoted by

$$\sum_{n_i \in S_\sigma} \mu_i n_i = \sum_{n_i \in S_\sigma} \nu_i n_i, \tag{3.4}$$

with μ_i, ν_i non-negative integers. The cone σ defines an affine toric variety

$$U_\sigma = \Big\{(Z_1, \dots, Z_p) \in \mathbb{C}^p \mid Z^\mu = Z^\nu\Big\}, \tag{3.5}$$

where $Z^\mu = Z_1^{\mu_1} \cdots Z_p^{\mu_p}$.

Definition 3.10. We call a collection Σ of strongly convex rational polyhedral cones in $N_\mathbb{R}$ a fan, if

1. Each face of a cone in Σ is also a cone in Σ,
2. The intersection of any two cones is a face of each of them.

A toric variety $\mathbb{P}_\Sigma$ for a fan Σ is constructed by gluing $U_\sigma, \sigma \in \Sigma$ along the faces

$$\mathbb{P}_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma. \tag{3.6}$$

The gluing works because $U_{\sigma \cap \sigma'}$ is an open subset of both U_σ and $U_{\sigma'}$.

Definition 3.11. Let $\{v_1, \dots, v_n\}$ be the generators of the one-dimensional cones $\Sigma(1) \subset \Sigma$. We call the elements $Q_{ij} \in \mathbb{Z}$ satisfying

$$\sum_{i=1}^n Q_{ij} v_i = 0, \tag{3.7}$$

the toric charges and the matrix $Q = (Q_{ij})$ the charge matrix.

An equivalent description of toric varieties, which will turn out to be more applicable to mirror constructions, is the description in terms of polytopes.

Definition 3.12. An integral polytope in $M_\mathbb{R}$ is the convex hull of finitely many points in M.

Definition 3.13. A rational polytope $\Delta \subset M_\mathbb{R}$, containing the origin, is called reflexive if the dual polytope

$$\Delta^\circ = \{n \in N_\mathbb{R} \mid \langle m, n \rangle \geq -1, \ \forall m \in \Delta\}, \tag{3.8}$$

is again a rational polytope.

Remark 3.14. To a rational polytope Δ one can associate a fan $\Sigma(\Delta)$, whose cones are the cones over the faces of Δ with apex at the origin. In this way one can construct a toric variety associated to Δ.

Definition 3.15. Let $f = \sum a_m x^m \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial. Its Newton polyhedron is the convex hull of $\{m \in \mathbb{Z}^n \mid a_m \neq 0\}$ in $\mathbb{R}^n$. We denote the space of Laurent polynomials, whose Newton polyhedra are Δ by $\mathbb{L}(\Delta)$.

Definition 3.16. A Laurent polynomial $f \in \mathbb{L}(\Delta)$ is said to be Δ-regular if for every l-dimensional face $\Delta' \subset \Delta$, the equations

$$F^{\Delta'} = \sum_{m \in \Delta' \cap \mathbb{Z}^n} a_m x^m, \quad \frac{\partial F^{\Delta'}}{\partial x_1} = \ldots = \frac{\partial F^{\Delta'}}{\partial x_n} = 0, \tag{3.9}$$

have no common solutions in T. We will denote the space of Δ-regular Laurent polynomials by $\mathbb{L}_{\mathrm{reg}}(\Delta)$.

3.3 Mirror symmetry and mirror families

Mirror symmetry originates in physics [LVW89] as a duality between superconformal field theories. The construction has been adapted in various ways in the mathematics literature. The simplest version is the so-called topological mirror symmetry, which states that two Calabi-Yau manifolds $(X, \check{X})$ of dimension n are mirror dual to each other if

$$h^{p,q}(X) = h^{n-q,p}(\check{X}), \quad \forall p, q \in \mathbb{N}_0, \tag{3.10}$$

where $h^{p,q} = \dim(H^{p,q})$ are the Hodge numbers of the variety. Mirror symmetry has been extended to non-Calabi-Yau manifolds, notably to Fano manifolds, by [Giv95, Giv97, Giv98], [BK98] and [HV00], for which the mirror pairs are known to be non-Calabi-Yau Landau-Ginzburg models. The mirror duality was reinterpreted in [Kon95] as the isomorphism

$$\mathrm{Fuk}(X) \cong D^b(\mathrm{coh}\check{X}), \tag{3.11}$$

of the Fukaya category associated to a Calabi-Yau manifold X and the derived category of coherent sheaves on the mirror $\check{X}$. This formulation goes under the name of homological mirror symmetry and can be used as the definition of the mirror pairs. The conjecture implies the existence of the Frobenius manifold structure on the moduli space of A_∞-deformations of the derived category of coherent sheaves on $\check{X}$. This structure coincides with the Frobenius structure on the formal neighbourhood of zero in the even-degree cohomology $H^{\mathrm{even}}(X)$, which can be constructed via Gromov-Witten classes, as reviewed for example in [Man99].

Remark 3.17. A notion of topological mirror symmetry can be formulated for non-Calabi-Yau Landau-Ginzburg models by a suitable definition of Hodge numbers [KKP08], see also [PS15].

There is a number of mirror constructions in the literature. For toric Calabi-Yau varieties the mirror construction was proposed in [Bat94, BB96], and generalized later to toric Landau-Ginzburg models [HV00], see also [HKK$^+$03, CR18] for a review. More recently a general mirror construction for Landau-Ginzburg models was proposed in [SYZ96] and later formalised and significantly extended in [GS11]. The SYZ mirror construction was applied to toric Fano [CL10] and Calabi-Yau manifolds [CLL$^+$12].

3.3.1 Batyrev-Borisov mirrors

In case $X(\Delta)$ is a compact Calabi-Yau, a construction of a mirror pair due to Batyrev and Borisov [Bat94, BB96] can be implemented.

Definition 3.18. Let Δ be a reflexive polyhedron and $X(\Delta)$ the associated family of Calabi-Yau hypersurfaces. The Batyrev-Borisov mirror to $X(\Delta)$ is the family $X(\Delta^\circ)$ for Δ° a dual to Δ.

Variation of Hodge structure for the Batyrev-Borisov mirrors

Let

$$f(a_i) = \sum a_i y^{\mu_i} = 0, \tag{3.12}$$

where y_i denote coordinates in a weighted projective space $\mathbb{P}(\Delta)$ and y^{μ_i} denotes the monomials of the form $\prod {y_j}^{(\mu_i)_j}$. The Newton polynomial of Δ corresponds to the reflexive polytope defining the toric variety $X(\Delta)$. Note that by rescaling the coordinates y_i and adjusting an overall normalization we can set the number of independent parameters to $|\Delta|$. The invariant coordinates on $\mathbb{L}_{\text{reg}}(\Delta)$ can be chosen as $z_k = (-1)^{Q_{0k}} \prod_i a_i^{Q_{ik}}$. Let $\pi : \mathcal{X} \to \mathsf{B}$ be a family of compact Calabi-Yau manifolds of dimension n, given by a (possible desingularization of) Laurent polynomial $f_z = f(a_i)$, using the subscript z to emphasize the dependence of $f(a_i)$ on z_k. There exists a unique holomorphic section ω_0 of the middle cohomology bundle $\mathcal{H}^n(\mathcal{X}/\mathsf{B})$ and a Gauss-Manin connection ∇ on it. The two can be used to construct a global frame ω of $\mathcal{H}^n(\mathcal{X}/\mathsf{B})$ by successive application of the Gauss-Manin connection to ω_0. In the case of (compact) Calabi-Yau manifolds the holomorphic n-form ω_0 as well as the Gauss-Manin connection can be constructed with the Griffiths-Dwork method, as reviewed below. Define a holomorphic $(n+1)$-form on the weighted projective space $\mathbb{P}(w_1,\ldots,w_{n+2})$ by

$$\Omega_{\mathbb{P}(w_1,\ldots,w_{n+2})} = \sum_{k=1}^{n+2}(-1)^k w_k y_k dy_1 \wedge \ldots \wedge \widehat{dy_k} \wedge \ldots \wedge dy_{n+2}, \tag{3.13}$$

where the hat denotes the omission of the k-th factor. For hypersurfaces defined by the vanishing locus of f_z, $\Omega_{\mathbb{P}(w_1,\ldots,w_{n+2})}$ can be used to construct a basis of $H^n(\mathbb{P}(w_1,\ldots,w_{n+2})-X_b)$ by

$$\Xi_i = \frac{P_i\Omega_{\mathbb{P}(w_1,\ldots,w_{n+2})}}{f_z^k}, \tag{3.14}$$

where P_i are homogeneous polynomials of degree $kd-(\sum w_j+1)$. Ξ_i restrict to the hypersurface X_b via the residue map $\mathrm{Res} : H^{n+1}(\mathbb{P}(w_1,\ldots,w_{n+2})-X_b) \to PH^n(X_b)$, where $PH^n(X_b)$ denotes the primitive cohomology of X_b. Let $\omega_j = \mathrm{Res}_{f_z(y)=0}(\Xi_j) \in PH^n(X_b)$. We define the Gauss-Manin connection by

$$\nabla_{\theta_i}\omega_j = \mathrm{Res}_{f_z(y)=0}(\theta_i\Xi_j), \tag{3.15}$$

where $\theta_i = z_i\frac{\partial}{\partial z_i}$. The Gauss-Manin connection ∇ satisfies Griffiths transversality [Gri69] and consequently a basis ω constructed from a fixed holomorphic n form ω_0 by successive application of the Gauss-Manin connection is compatible with the Hodge filtration. The holomorphic n form on X_b can be chosen to be

$$\omega_0 = \mathrm{Res}_{f_z(y)=0}\left(\sum_{k=1}^{n+2}(-1)^k w_k y_k \frac{dy_1 \wedge \ldots \wedge \widehat{dy_k} \wedge \ldots \wedge dy_{n+2}}{f_z(y)}\right). \tag{3.16}$$

The Gauss-Manin connection $\nabla : \mathcal{H}^n(\mathcal{X}/\mathsf{B}) \to \mathcal{H}^n(\mathcal{X}/\mathsf{B}) \otimes_{\mathcal{O}_\mathsf{B}} \Omega^1_\mathsf{B}$ in the frame ω can be expressed as

$$\nabla\omega = B\omega, \tag{3.17}$$

where B is a $b_n \times b_n$ matrix of meromorphic 1-forms, $b_n = \dim(H^n(X_z))$. Let z_i, $i = 1,\ldots,\dim(\mathsf{B})$ be a set of coordinates on B. We can expand B as

$$B = B_1 d\log(z_1) + B_2 d\log(z_2) + \ldots + B_h d\log(z_h), \tag{3.18}$$

where $B_i = B(\theta_i)$, in the basis ω and $h = \dim(\mathsf{B})$, with $(B_i)_{j,k} = (B_k)_{j,i}$.

Remark 3.19. The connection (3.17) can be written as a set of $\dim(\mathsf{B})$ differential equations for ω of the form

$$\nabla_{\theta_i}\omega = B_i\omega, \qquad \theta_i = z_i\frac{d}{dz_i}, \tag{3.19}$$

which due to the fact that ω is constructed by successive application of ∇ gives a differential operator annihilating the holomorphic n-form ω_0.

Polarization

A polarization on the middle-dimensional cohomology of compact Kähler manifolds X_b of dimension n is given by the standard Poincaré pairing

$$Q : H^n(X_b) \times H^n(X_b) \to \mathbb{C}, \qquad Q(\alpha, \beta) = (-1)^{\frac{n(n-1)}{2}} \int_{X_b} \alpha \wedge \beta. \tag{3.20}$$

The pairing between any two cohomology elements for compact Calabi-Yau families of dimension n can be computed from the Griffiths-Yukawa couplings

$$C_{i_1 \ldots i_{b_n}} := -\int_{X_b} \omega_0 \wedge \nabla_{i_1} \ldots \nabla_{i_{b_n}} \omega_0, \tag{3.21}$$

where $\nabla_i = \nabla_{\theta_i}$ and b_n is the n-th Betti number of the general fibre X_b of the family. They satisfy a set of differential equations, which follow from the constraints on ω_0:

$$\int \omega_0 \wedge \mathcal{L}_i^{\nabla} \omega_0 = 0 \quad \text{and} \quad \int \omega_0 \wedge \nabla_{\theta_i} \omega_0 = 0, \qquad i = 1, \ldots, \dim(\mathsf{B}). \tag{3.22}$$

Here $\mathcal{L}_i^{\nabla}$ denotes the operator corresponding to (3.19).

3.3.2 Hori-Vafa mirrors

Definition 3.20. Let Δ be a not necessarily reflexive polyhedron and $X(\Delta)$ the associated toric variety, with a charge matrix $Q = (Q_{ij})$. The Hori-Vafa mirror to $X(\Delta)$ is a pair (Y, f), where

$$Y : \left\{ y \in \mathbb{C}^n \mid y_1^{Q_{1j}} \cdots y_n^{Q_{nj}} = e^{\tau_j}, \ \forall j = 1, \ldots, r \right\}, \tag{3.23}$$

with r the dimension of the algebraic torus acting on the toric variety, τ_j are complex parameters, and

$$f = y_1 + \ldots + y_n, \tag{3.24}$$

restricted to Y.

Variation of (mixed) Hodge structure for Hori-Vafa mirrors

Let Σ be a fan corresponding to the reflexive polyhedron Δ and denote by $A(\Delta)$ the set of integral points of Δ. The 1-dimensional cones of the fan Σ correspond to the integral points of the polytope Δ, thus the coordinates y_i can be parametrized by

$$y_k = \prod_i t_i^{(m_k)_i}, \tag{3.25}$$

with $t_1, \dots, t_a \in (\mathbb{C}^*)$ and $m = (m_1, \dots, m_a) \in A(\Delta)$, $\dim(A(\Delta)) = a$. Define the following rings

$$S_\Delta = \bigoplus_{k \geq 0} S_\Delta^k, \qquad S_\Delta^k = \bigoplus_{m \in \Delta(k)} \mathbb{C}\, t_0^k t^m, \tag{3.26}$$

where

$$\Delta(k) := \left\{ m \in \mathbb{R}^{n-1} \mid \frac{m}{k} \in \Delta \right\} \ (k \geq 1), \quad \Delta(0) = \{0\} \in \mathbb{R}^{n-1}, \tag{3.27}$$

and t_0 is an additional parameter. The grading is given by $\deg(t_0^k t^m) = k$.

Definition 3.21. Let J_f be the ideal in S_Δ generated by $t_0 f, t_0 \theta_{t_1} f, \dots, t_0 \theta_{t_a} f$. The Jacobian ring is defined as $R_f = S_\Delta / J_f$.

Denote by R_f^i the i-th homogeneous piece of R_f. It is possible to define a decreasing filtration using R_f^i. Let

$$E^{-i} = \bigoplus_{k \leq i} R_f^k, \tag{3.28}$$

then we have a decreasing filtration

$$0 \subset E^0 \subset E^{-1} \subset E^{-2} \subset \dots. \tag{3.29}$$

There is also an increasing filtration. Define $I^{(j)}$ to be the homogeneous ideals in S_Δ generated as $\mathbb{C}$-subspaces by all monomials $t_0^k t^m$, $k \geq 1$ with $m \in \Delta(k)$ which does not belong to any face of codimension j. Since everything belongs to codimension 0 face, we get $I^{(0)} = 0$. Everything that belongs to the face of codimension 2 also belongs to some face of codimension 1, therefore $I^{(1)} \subset I^{(2)}$, etc.. Hence $I^{(j)}$ form an increasing filtration

$$0 = I_\Delta^{(0)} \subset I_\Delta^{(1)} \subset \dots \subset I_\Delta^{(a)} = S_\Delta, \tag{3.30}$$

which defines under the quotient an increasing filtration on R_f:

$$0 = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \dots \subset \mathcal{I}_a = R_f. \tag{3.31}$$

Polarization

A polarization on the Jacobian ring R_f is given by the residue pairing

$$Q : R_f \times R_f \to \mathcal{O}_{\mathrm{B}}, \qquad Q(\alpha, \beta) = \operatorname{Res}\left(\frac{\alpha\beta}{\prod_{i=1}^a \partial_{\tau_i} f} dy_1 \wedge \dots \wedge dy_n \right). \tag{3.32}$$

Theorem 3.22. *[KM10] Let $\mathcal{X} \to \mathsf{B}$ be a family of (compact) Calabi-Yau manifolds of dimension n defined by $\{f(\tau) = 0\}$ for the restriction of $f(\tau) = y_1 + \ldots + y_n$ to Y. There is an isomorphism*

$$r : \mathcal{R}_{f(\tau)} \cong H^n(X_\tau, \mathbb{C}), \tag{3.33}$$

for every $\tau = (\tau_1, \ldots, \tau_r)$. Therefore, as a consequence the filtrations $E_\bullet$ and $\mathcal{I}_\bullet$ define a mixed Hodge structure on $H^n(X_\tau)$, namely $F^{6-k} = r(E^{-k})$ and $W_3 = r(\mathcal{I}_1)$, $W_4 = W_5 = r(\mathcal{I}_3)$, $W_6 = r(\mathcal{I}_4)$. Moreover, one can define the Gauss-Manin connection by differentiating elements of the Jacobian ring with respect to parameters of f. Furthermore, the polarization Q on $\mathcal{R}_{f(\tau)}$ induces a polarization on $H^n(X_\tau, \mathbb{C})$.

3.4 Periods and Picard-Fuchs equations

Let X be a projective compact Calabi-Yau variety of dimension n, given as the zero locus of an irreducible homogeneous polynomial f and let ω_0 be the unique holomorphic n-form on X, as in (3.16). For any real n-dimensional cycle $\gamma \subset X$, we call the integral

$$\int_\gamma \omega_0, \tag{3.34}$$

a period of X. Define also the period vector of X via

$$I = \left(\int_{\gamma_1} \omega_0, \ldots, \int_{\gamma_n} \omega_0 \right), \tag{3.35}$$

where $\gamma_1, \ldots, \gamma_n$ freely generate $H_n(X, \mathbb{Z})$.

Theorem 3.23. *[GKZ89] Let X be a compact Calabi-Yau manifold defined by a polytope Δ and let $Q = (Q_{ij})$ denote the associated charge matrix. Denote by $m_i = (1, \mu_i)$, for $\mu_i \in \Delta, i = 1, \ldots, |\Delta|$. The period vector I satisfies the GKZ system of differential equations*

$$\mathcal{L}_i I = 0, \quad \text{and} \quad \mathcal{Z} I = 0, \tag{3.36}$$

with

$$\mathcal{L}_i = \prod_{Q_{ij}>0} \left(\frac{\partial}{\partial a_j} \right)^{Q_{ij}} - \prod_{Q_{ij}<0} \left(\frac{\partial}{\partial a_j} \right)^{-Q_{ij}}, \quad \text{and} \quad \mathcal{Z} = \sum_{i=1}^{\Delta} m_i \theta_{a_i} - c, \tag{3.37}$$

where $\theta_{a_i} = a_i \frac{\partial}{\partial a_i}$ and $c = (-1, 0, \ldots, 0)$.

Remark 3.24. The GKZ system is equivalent to the system

$$\theta_i I = B_i I, \qquad \theta_i = z_i \frac{d}{dz_i}, \tag{3.38}$$

where B_i are given by (3.19), due to (3.15).

Remark 3.25. Analogously one can define periods for non-compact Calabi-Yau varieties by an appropriate choice of cycles. In the toric case they are annihilated by the system (3.37).

Definition 3.26. Let (Y_q, f_q, ω_q) be a toric Landau-Ginzburg model, and let Δ be the Newton polytope of f_q. The classical period is the integral

$$I_\Gamma = \int_\Gamma \frac{1}{1 - t f_q(x_1, \ldots, x_n)} \frac{dx_1 \wedge \ldots \wedge dx_n}{x_1 \cdots x_n}, \tag{3.39}$$

for real cycles $\Gamma \subset Y_q$.

For $f_q = \sum_i a_i \prod_j x_j^{(\mu_i)_j}$ toric, with a Newton polytope $\Delta = (\mu_1, \ldots, \mu_n)$, denote by $Q = (Q_{ij})$ the associated charge matrix. The classical period satisfies the set of differential equations (3.37). More precisely, the system (3.37) is referred to as the extended or better behaved GKZ system [HKTY95, BH13]. When Δ is a reflexive polytope, the better behaved GKZ system is the same as the standard GKZ system and admits $\mathrm{vol}(\Delta)$ solutions.

Remark 3.27. For Calabi-Yau manifolds of dimension n given by the vanishing locus of a homogeneous polynomial f with a Newton polyhedron Δ, the periods of the holomorphic n-form and classical periods of f satisfy the same set of (GKZ) differential equations and are thus related by a simple transformation (see e.g. [AB18] for the computation of the transformation matrices in the case of varieties of Fermat type).

Chapter 4

Gauss-Manin Connection in Disguise

This chapter is concerned with describing the local structure of the moduli spaces of compact Calabi-Yau varieties, equipped with differential forms. An extension of the program to non-compact Calabi-Yau varieties and non-Calabi-Yau Landau-Ginzburg models is treated later in chapters 6 and 7. The investigation of moduli spaces of varieties enhanced with differential forms was initiated in [Mov12]. The starting point of the program was the identification of the ring of regular functions $\mathcal{O}_{\mathsf{T}}$ of the moduli space T of elliptic curves, together with a basis of their first cohomology group, with the ring $\widetilde{\mathcal{M}}$ of quasi-modular forms. It has since been extended to a number of compact Calabi-Yau varieties (see e.g. [Mov20] for a recent account), as well as hyperelliptic curves of genus 2 [CMY19] and examples of abelian varieties [Fon18]. The ring of regular functions $\mathcal{O}_{\mathsf{T}}$ associated to compact Calabi-Yau varieties can be equipped with a distinguished set of differential structures R_a, analogous to the Ramanujan differential structure on the ring of quasi-modular forms $\widetilde{\mathcal{M}}$. The existence of the Ramanujan differential structures on $\mathcal{O}_{\mathsf{T}}$ is equivalent to existence of special (Ramanujan/modular) vector fields on T, given in (4.14). For an n-dimensional compact Calabi-Yau varieties, both the ring of regular functions $\mathcal{O}_{\mathsf{T}}$ and the (Ramanujan) differential structures R_a on it can be constructed from a variation of polarized Hodge structure on the middle-dimensional cohomology bundle $\mathcal{H}^n(\mathcal{X})$ by a suitable choice of a global frame. Ramanujan differential relations were first observed in the case of polarized variation of Hodge structure of weight 1 for families of elliptic curves in [Mov12] and were extended to variations of Hodge structures of higher weight in a number of works, e.g. [Mov15, Mov13, Ali17]. For compact Calabi-Yau varieties, there exists a subgroup G of the group of automorphisms of the Hodge filtration, satisfying certain compatibility condition. Its Lie algebra is isomorphic to a subalgebra of vector fields on T and hence gives rise to a set of derivations on the rings $\mathcal{O}_\mathsf{T}$. The algebra of derivations on $\mathcal{O}_\mathsf{T}$ is the Gauss-Manin or AMSY Lie algebra, introduced in [AMSY16].

4.1 Enhanced families

It was proposed in references [Mov12, Mov13] that the variation of Hodge structure can be cast into an algebraic form by enlarging the complex structure moduli space by a choice of the elements of filtration spaces. For a detailed account on enhanced varieties and their moduli spaces, we refer to [Mov20].

Definition 4.1. Fix a finite dimensional vector space V_0, such that $V_0 = V_\mathbb{Z} \otimes \mathbb{C}$, for some lattice $V_\mathbb{Z}$. Fix also a non-degenerate pairing $Q_0 : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{Z}$ on $V_\mathbb{Z}$, extending bilinearly to V_0. Let further $F_0^\bullet$ be a filtration on V_0. We call the tuple $(X, F^\bullet, Q, \phi)$, where X is a compact Calabi-Yau variety of dimension n and $F^\bullet$ is the Hodge filtration on the middle dimensional cohomology $H^n(X,\mathbb{C})$ of X, $Q : H^n(X,\mathbb{C}) \times H^n(X,\mathbb{C}) \to \mathbb{C}$ is the polarization on it and

$$\phi : (H^n(X,\mathbb{C}), F^\bullet, Q) \to (V_0, F_0^\bullet, Q_0), \tag{4.1}$$

is an isomorphism, an enhanced variety. $\phi : H^n(X,\mathbb{C}) \to V_0$ in (4.1) is an isomorphism in a sense that

- it respects the Hodge filtration

$$\phi(F^p(X)) = F_0^p, \quad \forall p, \tag{4.2}$$

- it respects the compatibility with the pairing

$$\phi(Q(\alpha,\beta)) = Q_0(\phi(\alpha), \phi(\beta)), \quad \alpha, \beta \in H^n(X,\mathbb{C}). \tag{4.3}$$

Definition 4.2. We call the tuple $(X, F^\bullet, \phi)$ a weakly enhanced variety if

$$\phi : (H^n(X,\mathbb{C}), F^\bullet) \to (V_0, F_0^\bullet), \tag{4.4}$$

is an isomorphism. We require for ϕ the condition (4.2), but not (4.3).

Definition 4.3. Let B be a complex manifold and $\pi : \mathcal{X} \to \mathsf{B}$ a family of projective varieties of dimension n over B. We call $\mathsf{X} = (\mathcal{H}, \mathcal{F}^\bullet, Q_{\mathcal{O}_\mathsf{B}})$ a family of enhanced varieties if there exists an isomorphism

$$\phi : (\mathcal{H}, \mathcal{F}^\bullet, Q_{\mathcal{O}_\mathsf{B}}) \to (V_0 \otimes \mathcal{O}_\mathsf{B}, F_0^\bullet \otimes \mathcal{O}_\mathsf{B}, Q_0 \otimes \mathcal{O}_\mathsf{B}), \tag{4.5}$$

where $\mathcal{H} = R^n\pi_*\mathbb{C} \otimes \mathcal{O}_\mathsf{B}$ and the bilinear pairing $Q_{\mathcal{O}_\mathsf{B}} : \mathcal{H} \times \mathcal{H} \to \mathcal{O}_\mathsf{B}$ is given by Q on each fibre.

4.2 Moduli space

Definition 4.4. A moduli space T of enhanced varieties is the set of enhanced varieties $(X, F^\bullet(X), Q, \phi)$, where two enhanced varieties are considered equivalent if there exists an isomorphism β, such that

$$\beta : \phi_1^{-1}(V_0, F_0^\bullet, Q_0) \mapsto \phi_2^{-1}(V_0, F_0^\bullet, Q_0). \tag{4.6}$$

For our discussion, a more useful way of defining the moduli space of enhanced varieties will be the geometric construction due to [Mov20]. Let T denote the moduli space of enhanced varieties, and let X/T be the associated family over it. Consider a family $\mathcal{X}/\mathsf{B}$ of Calabi-Yau varieties of dimension n over B. Around any point $b \in \mathsf{B}$ we can choose a Zariski open neighbourhood $V_b \subset \mathsf{B}$ and a global frame ω_b of the vector bundle $\mathcal{H}^n(\mathcal{X}/\mathsf{B})$, such that at every point b' of V_b the basis $\omega_b(b')$ is compatible. At any $b' \in V_b$, any compatible basis can be constructed by a transformation of $\omega_b(b')$ by a non-singular lower block-triangular matrix S,

$$\phi : \omega_b \mapsto S\omega_b, \tag{4.7}$$

such that

$$Q(S\omega_b, S\omega_b) = Q_0. \tag{4.8}$$

Equation (4.8) gives algebraic constraints on the entries of S. Define

$$U = \operatorname{Spec}\left(\mathbb{C}\left[s_{ij}^{ind}, \frac{1}{\det S}\right]\right), \tag{4.9}$$

where s_{ij}^{ind} are the algebraically independent entries of the matrix S, then a patch $\tilde{\mathsf{T}}$ of the moduli space T can be constructed as the fiber product

$$\tilde{\mathsf{T}} = V_b \times_{\mathbb{C}} U. \tag{4.10}$$

Remark 4.5. To construct the moduli space T globally one needs to ensure that the map ϕ is indeed an isomorphism, by considering in detail its singular behaviour. The construction was carried out for example in [Mov12, MN16].

4.3 Algebraic Gauss-Manin connection and modular vector fields

Let $\pi : (\mathcal{X}, \phi) \to \mathsf{T}$ be a family of enhanced compact Calabi-Yau varieties over the moduli space T. One can extend the Gauss-Manin connection to a connection on T by composition, giving rise to the algebraic Gauss-Manin connection (see e.g. [KO68])

$$\nabla : \mathcal{H}(\mathcal{X}/\mathsf{T}) \to \mathcal{H}(\mathcal{X}/\mathsf{T}) \otimes \Omega^1_{\mathsf{T}}. \tag{4.11}$$

The action of the algebraic Gauss-Manin connection on a frame $\phi(\omega)$ of $V_0 \otimes \mathcal{O}_{\mathsf{B}}$ is given by

$$\nabla_{\mathsf{t}}\phi(\omega) = \left((\mathsf{t}S)S^{-1} + \sum z_i^{-1}(\mathsf{t}z_i)S(B_i)S^{-1}\right)\phi(\omega), \tag{4.12}$$

where z_i are coordinates on B and $\nabla_{z_i \frac{\partial}{\partial z_i}} \omega = B_i \omega$, with B_i given in (3.19).

Remark 4.6. An algebraic Gauss-Manin connection can be defined analogously for non-compact Calabi-Yau varieties. We propose a modified definition for Landau-Ginzburg models in chapter 7, by considering the local system of solutions to (3.37), rather than the Gauss-Manin local system.

Definition 4.7. Let $\pi : \mathcal{X} \to \mathsf{B}$ be a family of compact Calabi-Yau varieties of dimension n and let $\nabla : \mathcal{H}(\mathcal{X}/\mathsf{T}) \to \mathcal{H}(\mathcal{X}/\mathsf{T}) \otimes \Omega^1_{\mathsf{T}}$ be the algebraic Gauss-Manin connection on $\mathcal{H}(\mathcal{X}/\mathsf{T}) = R^n\pi_*\mathbb{Z} \otimes \mathcal{O}_{\mathsf{T}}$. A modular vector field R is a rational vector field on T, such that

$$\nabla_{\mathsf{R}} : \mathsf{F}^k/\mathsf{F}^{k+1} \to \mathsf{F}^{k-1}/\mathsf{F}^k, \tag{4.13}$$

where $\mathcal{F}^\bullet$ is a filtration on $\mathcal{H}(\mathsf{X}/\mathsf{T})$ induced by $\mathcal{F}^\bullet$. Hence R is of the form

$$\nabla_{\mathsf{R}}\phi(\omega) = \begin{pmatrix} 0_{f_n,f_n} & *_{f_n,f_{n-1}} & 0_{f_n,f_{n-2}} & \cdots & 0_{f_n,f_0} \\ 0_{f_{n-1},f_n} & 0_{f_{n-1},f_{n-1}} & *_{f_{n-1},f_{n-2}} & \cdots & 0_{f_{n-1},f_0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{f_1,f_n} & 0_{f_1,f_{n-1}} & 0_{f_1,f_{n-2}} & \cdots & *_{f_1,f_0} \\ 0_{f_0,f_n} & 0_{f_0,f_{n-1}} & 0_{f_0,f_{n-2}} & \cdots & 0_{f_0,f_0} \end{pmatrix} \phi(\omega) = A_{\mathsf{R}}\phi(\omega), \tag{4.14}$$

where $f_k := \mathrm{rk}\left(\mathcal{F}^k/\mathcal{F}^{k+1}\right)$ and $*_{a,b}$ is an $a \times b$ matrix with entries in $\mathcal{O}_{\mathsf{T}}$.

Remark 4.8. The existence of the Ramanujan vector field is equivalent to the existence of differential structure on the ring $\mathcal{O}_{\mathsf{T}}$, due to (4.14), which we rewrite as

$$(\mathsf{R}S)S^{-1} + \sum z_i^{-1}(\mathsf{R}z_i)S(B_i)S^{-1} = A_{\mathsf{R}}. \tag{4.15}$$

Writing out the relations gives a system of first order differential equations for z_i and s_{ij}.

4.4 Gauss-Manin Lie algebra

Definition 4.9. A (weakly) enhanced variety $(X, F^\bullet, Q, \phi)$ is called full, if there exists an algebraic group G, acting on $\mathcal{X}$ and T from the left, which commutes both with the morphism $\mathcal{X} \to \mathsf{T}$ and ϕ.

For enhanced Calabi-Yau varieties of dimension n the condition above is equivalent to G being of the form

$$\mathsf{G} = \left\{ g \in \mathrm{GL}(b_n, \mathbb{C}) | g \text{ block lower triangular and } gQ_0 g^{\mathrm{tr}} = Q_0 \right\}, \tag{4.16}$$

where n is the n-th Betti number. The Lie algebra of G is given by

$$\mathrm{Lie}(\mathsf{G}) = \left\{ g \in \mathrm{GL}(b_n, \mathbb{C}) | \mathfrak{g} \text{ block lower triangular and } \mathfrak{g} Q_0 + Q_0 \mathfrak{g}^{\mathrm{tr}} = 0 \right\}. \tag{4.17}$$

Remark 4.10. Note that for weakly enhanced Calabi-Yau varieties the last condition is not required and G is of the form

$$\mathsf{G} = \{ g \in \mathrm{GL}(b_n, \mathbb{C}) | g \text{ block lower triangular} \}. \tag{4.18}$$

Theorem 4.11. *[AMSY16] For any $\mathfrak{g} \in \mathrm{Lie}(\mathsf{G})$, there exists a unique vector field $\mathsf{R}_\mathfrak{g} \in \mathfrak{X}(\mathsf{T})$, such that*

$$\nabla_{\mathsf{R}_\mathfrak{g}} \phi(\omega) = \mathfrak{g} \phi(\omega), \tag{4.19}$$

i.e. $\nabla_{\mathsf{R}_\mathfrak{g}} (S\omega) = \mathfrak{g}(S\omega)$.

Definition 4.12. The Gauss-Manin Lie algebra $\mathfrak{G}$ is the $\mathcal{O}_\mathsf{T}$-module generated by $\mathrm{Lie}(\mathsf{G})$ and the modular vector fields $\mathsf{R}_a \in \mathfrak{X}(\mathsf{T})$.

4.5 Example: Elliptic curves

The purpose of this section is to reproduce the algebraic construction of the ring of quasi-modular forms due to Movasati [Mov12] and establish a relation between variation of Hodge structure and rings of quasi-modular forms in the case of the Legendre family of elliptic curves.

Theorem 4.13. *[Mov12] Let $\pi : \mathcal{E} \to \mathsf{B}$ be the family of elliptic curves given by (possible desingularization of) the vanishing locus*

$$\left\{ f_z = x_1^3 + x_2^3 + x_3^3 - z^{1/3} x_1 x_2 x_3 = 0 \right\} \subset \mathbb{P}^2, \quad z \in \mathbb{P}^1 \setminus \left\{ 0, \frac{1}{27}, \infty \right\} =: \mathsf{B}, \tag{4.20}$$

Let further ω denote a global frame of the middle cohomology bundle $\mathcal{H}^1(\mathcal{E},\mathbb{C}) \to \mathsf{B}$. There exists a basis $\alpha = S\omega$, S lower diagonal and filtration preserving, and a Ramanujan vector field R, such that

$$\nabla_{\mathsf{R}}\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\alpha. \tag{4.21}$$

The latter is equivalent to the existence of a differential structure R on the ring $\mathbb{C}[s_0,s_1,z]$

$$\mathsf{R}: \begin{cases} \mathsf{R}s_0 = s_1, \\ \mathsf{R}s_1 = -6z(1-27z)s_0^{-3}, \\ \mathsf{R}z = (1-27z)zs_0^{-2}. \end{cases} \tag{4.22}$$

Variation of Hodge structure

Let $\mathcal{E}$ be a family of elliptic curves in (4.20). The variation of Hodge structure can be constructed on the bundle $\mathcal{H} := R^1\pi_*\mathbb{C} \to \mathsf{B}$. We identify $\mathcal{F}^0 = \mathcal{H}$ and $\mathcal{F}^1$ as the unique vector bundle over B given by $\mathcal{F}^1\big|_z = H^{0,1}(X_z)$, for $z \in \mathsf{B}$, $X_z = \pi^{-1}(z)$. A global frame of $\mathcal{H}$ is given by

$$\omega_0 = \mathrm{Res}_{f_z=0}\left(\frac{-x_1dx_2 \wedge dx_3 + x_2dx_1 \wedge dx_3 - x_3dx_1 \wedge dx_2}{f_z}\right), \tag{4.23}$$

and the Gauss-Manin connection can be constructed as

$$\omega_k = \nabla_\theta^k\omega_0 := \mathrm{Res}_{f_z=0}\left(\theta^k\frac{-x_1dx_2 \wedge dx_3 + x_2dx_1 \wedge dx_3 - x_3dx_1 \wedge dx_2}{f_z}\right), \quad \theta = z\frac{d}{dz}. \tag{4.24}$$

Proposition 4.14. *The basis $\omega = (\omega_0,\omega_1)$ is a global frame of $\mathcal{H}$ and satisfies*

$$\nabla_\theta\omega = \begin{pmatrix} 0 & 1 \\ \frac{1}{1-27z} & \frac{27z}{1-27z} \end{pmatrix}\omega = B_z\omega. \tag{4.25}$$

Proof. The proposition follows from the explicit forms of ω_0 and ω_1. □

The polarization in the basis ω is given by the pairing (3.20)

$$Q(\omega,\omega) = \begin{pmatrix} 0 & \frac{1}{1-27z} \\ -\frac{1}{1-27z} & 0 \end{pmatrix}. \tag{4.26}$$

In order to construct the Ramanujan vector field, we only need to construct a vector field R, such that in the global frame $\alpha = S\omega$,

$$\nabla_{\mathsf{R}}\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\alpha = A_{\mathsf{R}}\alpha, \tag{4.27}$$

where S is of the form

$$S = \begin{pmatrix} s_0 & 0 \\ s_1 & s_0^{-1} \end{pmatrix}. \tag{4.28}$$

From

$$\nabla_{\mathsf{R}}\alpha = (\mathsf{R}SS^{-1} + z^{-1}\mathsf{R}(z)SB_zS^{-1})\alpha = A_{\mathsf{R}}\alpha, \tag{4.29}$$

we find $\mathsf{R} = \partial_\tau$, with

$$\tau = \frac{\pi_1}{\pi_0}, \tag{4.30}$$

where π_1 and π_0 are solutions to

$$(\theta^2 - 3z(3\theta+2)(3\theta+1))\pi_i = 0, \tag{4.31}$$

and s_i satisfy

$$\partial_\tau s_0 = s_1, \quad \partial_\tau s_1 = -6z(1-27z)s_0^{-3}, \quad \partial_\tau z = (1-27z)zs_0^{-2}. \tag{4.32}$$

The family of elliptic curves enhanced with one-forms is full with the algebraic group G given by

$$\mathsf{G} = \left\{ S = \begin{pmatrix} s_0 & 0 \\ s_1 & s_0^{-1} \end{pmatrix} \mid s_0 \in \mathbb{C}^*, s_1 \in \mathbb{C} \right\}. \tag{4.33}$$

Its Lie algebra is easily computable and has two generators

$$\mathfrak{g}_0 = \left(\frac{\partial}{\partial s_0} S\right) S^{-1}\Big|_{S=\mathrm{Id}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.34}$$

$$\mathfrak{g}_1 = \left(\frac{\partial}{\partial s_1} S\right) S^{-1}\Big|_{S=\mathrm{Id}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.35}$$

The corresponding vector fields are given by

$$\mathsf{R}_{\mathfrak{g}_0} = s_0\frac{\partial}{\partial s_0} + s_1\frac{\partial}{\partial s_1}, \qquad \mathsf{R}_{\mathfrak{g}_1} = s_0\frac{\partial}{\partial s_1}. \tag{4.36}$$

Proposition 4.15. *The algebra generated by the vector fields* R, $\mathsf{R}_{\mathfrak{g}_0}$, $\mathsf{R}_{\mathfrak{g}_1}$ *is isomorphic to the* $\mathrm{sl}_2(\mathbb{C})$ *Lie algebra.*

Proof. We can easily compute the commutators

$$[\mathsf{R}, \mathsf{R}_{\mathfrak{g}_1}] = \mathsf{R}_{\mathfrak{g}_0}, \quad [\mathsf{R}, \mathsf{R}_{\mathfrak{g}_0}] = -2\mathsf{R}, \quad [\mathsf{R}_{\mathfrak{g}_1}, \mathsf{R}_{\mathfrak{g}_0}] = 2\mathsf{R}_{\mathfrak{g}_1}. \tag{4.37}$$

□

Theorem 4.16. *[Mov12] The ring of regular functions $\mathcal{O}_{\mathsf{T}}$ equipped with differential structures coming from the vector fields R, $\mathsf{R}_{\mathfrak{g}_0}$, $\mathsf{R}_{\mathfrak{g}_1}$ is isomorphic as differential ring to the ring of quasi-modular forms for $\Gamma(3)$ with differential structures described in section 2.1.*

Proof. The isomorphism is given by

$$A = s_0^4, \qquad E = (1-54z)s_0^2 - 12 s_1 s_0^{-1}, \qquad B = (1-54z)s_0^4, \tag{4.38}$$

and the modular vector field R is equivalent to the Ramanujan relations (2.9). Moreover, the action of $\mathsf{R}_{\mathfrak{g}_0}$ and $\mathsf{R}_{\mathfrak{g}_1}$ is equivalent to the action of W and F respectively. $\square$

Chapter 5

Mirror Elliptic K3 Surfaces

The work of Movasati on the enhanced moduli space of elliptic curves [Mov12] provided an algebro-geometric context for a construction of rings quasi-modular forms with the corresponding $\mathrm{sl}_2(\mathbb{C})$ Lie algebra of derivations on it. The general automorphic or modular properties for moduli spaces of Calabi-Yau threefolds such as the quintic are less clear (see [Mov13]), although the analogous enhanced moduli spaces and Gauss-Manin Lie algebras have been put forward [AMSY16]. Lattice polarized K3 manifolds therefore provide the middle grounds between the classical theory of quasi modular forms and new structures appearing in the moduli spaces of generic threefolds. We should note, that in various limiting constructions both of elliptically fibered Calabi-Yau manifolds as well as non-compact Calabi-Yau manifolds, connections to classical quasi modular forms have been worked out, see e.g. [Hag17]. Correspondingly $\mathrm{sl}_2(\mathbb{C})$ Lie-subalgebras of the full Gauss-Manin Lie algebras have been put forward in the context of elliptic fibrations [HMY17]. A different kind of universal $\mathrm{sl}_2(\mathbb{C})$ Lie sub-algebra of the Gauss-Manin Lie algebra stemming from the rescaling of the holomorphic top form of the Griffiths-Dwork family of Calabi-Yau n-folds has been studied in [Nik20]. Our result provides on the other hand the full Gauss-Manin Lie algebra which is beyond, yet very close to the classical one, reducing to a direct sum of two copies the classical one.

5.1 Elliptic K3 surfaces

An elliptic K3 surface is a K3 surface X, together with a surjective morphism $\pi : X \to \mathbb{P}^1$, such that the general fiber is an elliptic curve. Elliptic K3 surfaces were classified in [Kod63] by classifying the singular fibers of π. A particularly interesting subset of these, which are known to exhibit modular properties are projective K3 surfaces with singular fibers of type E_6, E_7 and E_8, corresponding to elliptic singularities of the same type. In Refs. [LY95, LY96b] Lian and Yau proved that a fundamental solution of the Picard-Fuchs equations for the mirrors of these K3 surfaces factorizes as a product of two modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. The modular subgroup is given by the respective monodromy group of an elliptic fiber of the mirror. Moreover, the authors discovered an intricate relationship between the mirror map for these families and the McKay-Thompson series [LY96a] (see also Ref. [Dor00]). An orthogonal approach to connect K3 periods to quasi modular forms was taken in Ref. [YY07], where the authors started with certain quasi-modular forms in two parameters and constructed K3 surfaces for which these modular forms are realized as classical periods.

Lattice polarized K3 surfaces

In this section we give a short review of mirror symmetry for lattice polarized projective K3 surfaces following to a large extent [Dol96, Hos00]. A lattice polarized K3 surface is defined by an even, non-degenerate lattice M of signature $(1, 19-m)$ that admits a primitive embedding into the K3 lattice $\Lambda_{\mathrm{K3}} = E_8(-1) \oplus E_8(-1) \oplus H^{\oplus 3}$ where H represents the rank two hyperbolic lattice. An M-polarized K3 surface is a K3 surface X whose Picard lattice Pic(X) is given by M. The orthogonal complement of M in Λ_{K3} gives the transcendental lattice of the K3 surface and, up to a factor of H, the lattice of the mirror K3. Consider a mirror pair $X_\Delta, X_{\Delta^\circ}$ of projective K3 surfaces described by dual three-dimensional reflexive, integral polytopes (Δ, Δ°), as introduced in [Bat94, BB96], and denote by $\mathbb{P}_\Delta$ and $\mathbb{P}_{\Delta^\circ}$ the ambient toric varieties of X_Δ and X_{Δ° respectively. The polarization of the two surfaces is given by the pull-back of toric divisors, together with the divisors that arise from possible splitting of the simple divisors intersected with the hypersurfaces into several irreducible components. A toric Picard lattice is defined as $\mathrm{Pic}_{tor}(\Delta) = \iota^* A^1(X_\Delta)$, where $\iota : X_\Delta \to \mathbb{P}_\Delta$ denotes the embedding of the K3 surface into the toric variety and $A^1(\mathbb{P}_\Delta)$ is the first Chow group of the toric variety. A mirror of a lattice polarized K3 surface $(X_\Delta, \mathrm{Pic}_{tor}(\Delta))$ is a lattice polarized K3 surface $(X_{\Delta^\circ}, \mathrm{Pic}_{cor}(\Delta^\circ))$, where $\mathrm{Pic}_{cor}(\Delta^\circ)$ is the orthogonal complement of $\mathrm{Pic}_{tor}(\Delta^\circ)$ in $H^2(X, \mathbb{Z})$, see e. g. [Roh04].

Realization of elliptic K3 surfaces as hypersurfaces in weighted projective spaces

Elliptic K3 surfaces with singular fibres of types E_6, E_7 and E_8 can be obtained as hypersurfaces of degrees $d_6 = 6$, $d_7 = 8$ and $d_8 = 12$ in weighted projective spaces $\mathbb{P}(2, 2, 1, 1)$,

					$l^{(1)}$	$l^{(2)}$
D_0	1	0	0	0	$-(w_1+w_2)/2-1$	0
D_1	1	0	0	1	$w_1/2$	0
D_2	1	0	1	0	$w_2/2$	0
D_3	1	1	$w_2/2$	$w_1/2$	0	1
D_4	1	-1	$w_2/2$	$w_1/2$	0	1
D_5	1	0	$w_2/2$	$w_1/2$	1	-2

Table 5.1: Toric data of the K3 surfaces.

$\mathbb{P}(4,2,1,1)$ and $\mathbb{P}(6,4,1,1)$. Explicitly, they are given as the zero loci of Fermat polynomials of the form

$$\left\{f(x) = x_1^{d/w_1} + x_2^{d/w_2} + x_3^d + x_4^d = 0\right\} \subset \mathbb{P}(w_1, w_2, 1, 1), \tag{5.1}$$

where x_i denote the homogeneous coordinates in $\mathbb{P}(w_1, w_2, 1, 1)$. In all three cases there is a singular locus along $x_3 = x_4 = 0$ of the torus action resulting in the singular curve $\mathcal{C} : x_1^{d/w_1} + x_2^{d/w_2} = 0$. The singularity in the ambient space can be resolved by introducing a linear relation $x_4 = \lambda x_3$. This defines an exceptional divisor E, which is a ruled surface over the curve $\mathcal{C}$. The resulting geometry is a K3 surface which is a double cover of the associated elliptic curves by the map $(x_1, x_2, x_3, x_4) \mapsto (\lambda = x_3/x_4; x_1, x_2, y_3 = x_3^2)$, branched over the elliptic curve. Elliptic fibers of type E_6, E_7 and E_8 at a general point are given by hypersurfaces $\mathbb{P}(1,1,1)[3]$, $\mathbb{P}(2,1,1)[4]$ and $\mathbb{P}(3,2,1)[6]$, where the degree of the hypersurface is indicated in the square brackets. The monodromy groups of these elliptic curves are genus zero congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We define them in chapter 2 and introduce quasi-modular forms for them. The congruence subgroups are $\Gamma_0(3)$ for elliptic curve of type E_6, $\Gamma_0(2)$ for elliptic curve of type E_7 and $\Gamma_0(1)$ for elliptic curve of type E_8. The polytopes $\Delta = \{D_i\}, i \in \{0,1,2,3,4,5\}$ for elliptic K3 surfaces in (5.1) are given in Table 5.1. The vectors $l^{(i)}$ of linear relations between the polytopes generate the Mori cone in the secondary fan of $\mathbb{P}_{\Delta^\circ}$. The intersection numbers can be computed from the toric data. Let L be the linear system generated by degree one polynomials x_3 and x_4, and let H be the linear system generated by degree 2 polynomials x_3^2, x_3x_4, x_4^2, x_1 and x_2. It is straightforward to compute the intersection numbers $C_{LL} = L \cdot L = 0$, $C_{LH} = C_{HL} = H \cdot L = 2d/(w_1w_2)$, $C_{HH} = H \cdot H = 4d/(w_1w_2)$.

The mirror variety X_{Δ° is constructed by the Batyrev-Borisov construction [BB96] as a resolution of the quotient

$$\left\{f_{\phi,\psi}(y) = y_1^{d/w_1} + y_2^{d/w_2} + y_3^d + y_4^d - d\psi y_1y_2y_3y_4 - 2\phi y_3^{d/2}y_4^{d/2} = 0\right\}/G, \tag{5.2}$$

in $\mathbb{P}(w_1, w_2, 1, 1)$ with coordinates y_1, y_2, y_3, y_4 and d is the degree of $f_{\phi,\psi}(y)$. The discrete groups G are $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ and $(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ for the three K3 surfaces in the usual order. The locus $f_{\psi,\phi}(y) = 0$ defines a family $\mathcal{X}$ of lattice polarized projective K3 surfaces over the moduli space B of dimension 2 for which ψ and ϕ provide a local coordinate chart. The polarization is given, up to a a rank two hyperbolic lattice, by the orthogonal complement of the toric divisors defined by $\{D_i\}$ in the K3 lattice Λ_{K3}. For later purposes it will be useful to write down a general polynomial of the form (5.2)

$$f_{a_1,\dots,a_5}(y) = a_1 y_1^{d/w_1} + a_2 y_2^{d/w_2} + a_3 y_3^d + a_4 y_4^d + a_0 y_1 y_2 y_3 y_4 + a_5 y_3^{d/2} y_4^{d/2}. \tag{5.3}$$

It is equivalent to the form (5.2) by a projective transformation of y_i. We define the GKZ coordinates

$$z_1 = \frac{a_1^{w_1/2} a_2^{w_2/2} a_5}{a_0^{d/2}}, \qquad z_2 = \frac{a_3 a_4}{a_5^2}. \tag{5.4}$$

The relations between (ψ, ϕ) and (z_1, z_2) are $-d\psi = z_1^{-2/d} z_2$ and $-2\phi = z_2^{-1/2}$. We will denote the homogeneous polynomial $f_{\psi,\phi}$ in (5.2) by f_{z_1,z_2} to highlight the dependence on z_1 and z_2.

Variation of Hodge structure and Picard-Fuchs equations

Let B be a complex manifold and let $\pi : \mathcal{X} \to \mathsf{B}$ be a family of K3 surfaces. The vector bundle $\mathcal{H}^2_{\mathrm{dR}}(\mathcal{X}) = R^2\pi_*\mathbb{C} \otimes \mathcal{O}_{\mathsf{B}}$ carries the Gauss-Manin connection $\nabla : \mathcal{H}^2_{\mathrm{dR}}(\mathcal{X}) \to \mathcal{H}^2_{\mathrm{dR}}(\mathcal{X}) \otimes_{\mathcal{O}_{\mathsf{B}}} \Omega^1_{\mathsf{B}}$ defined by the action on the locally constant subsheaf $R^2\pi_*\mathbb{C}$ by

$$\nabla(s \otimes f) = s \otimes df, \tag{5.5}$$

for $s \in R^2\pi_*\mathbb{C}$, $f \in \mathcal{O}_{\mathsf{B}}$, where $\mathcal{O}_{\mathsf{B}}$ denotes the $\mathbb{C}$-algebra of regular functions on B and by Ω^1_{B} we denote the $\mathcal{O}_{\mathsf{B}}$-module of differential 1-forms on B. The Hodge filtration $F^\bullet(X_b) = \{F^p(X_b)\}_{p=0,1,2} = \bigoplus_{a \geq p} H^{a,2-a}(X_b)$ for each fiber specifies the Hodge bundle $\mathcal{F}^\bullet$ of the family $\mathcal{X}$. The Hodge filtration $F^\bullet(X_b)$ varies holomorphically over the base B and ∇ satisfies Griffiths' transversality

$$\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes_{\mathcal{O}_{\mathsf{B}}} \Omega^1_{\mathsf{B}}. \tag{5.6}$$

We say that a family $\mathcal{X}$ of K3 surfaces is polarized by a lattice M if each fiber of $\mathcal{X}$ is polarized by M. The image of the polarization $\iota : M \to \mathcal{H}^2_{\mathrm{dR}}(\mathcal{X})$ consists of constant sections of the Gauss-Manin connection. We will denote by $\nabla : \mathcal{H}^2_{\mathrm{dR}}(\mathcal{X})_\iota \to \mathcal{H}^2_{\mathrm{dR}}(\mathcal{X})_\iota \otimes_{\mathcal{O}_{\mathsf{B}}} \Omega^1_{\mathsf{B}}$ the induced connection on the quotient $\mathcal{H}^2_{\mathrm{dR}}(\mathcal{X})_\iota = \mathcal{H}^2_{\mathrm{dR}}(\mathcal{X})/\iota(M)$. Furthermore, we will denote by $\mathcal{F}_\iota^\bullet$ a filtration on $\mathcal{H}^2_{\mathrm{dR}}(\mathcal{X})_\iota$ induced by $F^\bullet$. We say that a basis $\omega = (\omega_1, \ldots, \omega_{m+2})$ of $H^2_{\mathrm{dR}}(X_b)_\iota$ is compatible with its Hodge filtration if $\omega_1 \in F^2$, $\omega_2, \ldots, \omega_{m+1} \in F^1 \setminus F^2$ and $\omega_{m+2} \in F^0 \setminus F^1$.

For families of projective K3 surfaces in (5.2) the variation of Hodge structure can be constructed with the Griffiths-Dwork method, as reviewed below. Define a holomorphic 3-form on $\mathbb{P}(w_1, w_2, w_3, w_4)$ by

$$\Omega_{\mathbb{P}(w_1,w_2,w_3,w_4)} = \sum_{k=1}^{4} (-1)^k w_k x_k dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_4, \tag{5.7}$$

where hat denotes the omission of the k-th factor. For hypersurfaces (5.2), $\Omega_{\mathbb{P}(w_1,w_2,w_3,w_4)}$ can be used to construct a basis of $H^3(\mathbb{P}(w_1, w_2, w_3, w_4) - X_b)$ by

$$\Xi_i = \frac{P_i \Omega_{\mathbb{P}(w_1,w_2,w_3,w_4)}}{f^k_{z_1,z_2}}, \tag{5.8}$$

where P_i are homogeneous polynomials of degree $kd - (\sum w_j + 1)$. Ξ_i restrict to the hypersurface X_b via the residue map $\mathrm{Res} : H^3(\mathbb{P}(w_1, w_2, w_3, w_4) - X_b) \to PH^2(X_b)$, where $PH^2(X_b)$ denotes the primitive cohomology of X_b. Let $\omega_j = \mathrm{Res}_{f_{z_1,z_2}(y)=0}(\Xi_j) \in PH^2(X_b)$. We define the Gauss-Manin connection by

$$\nabla_{\theta_i} \omega_j = \mathrm{Res}_{f_{z_1,z_2}(y)=0}(\theta_i \Xi_j), \tag{5.9}$$

where $\theta_i = z_i \frac{\partial}{\partial z_i}$. It is straightforward to check that Griffiths transversality is satisfied and consequently a basis constructed from a fixed $(2, 0)$ form ω_1 by successive application of the Gauss-Manin connection is compatible with the Hodge filtration. The holomorphic $(2, 0)$ form on X_b can be chosen to be

$$\omega_1 = \mathrm{Res}_{f_{z_1,z_2}(y)=0} \left(\sum_{k=1}^{4} (-1)^k w_k y_k \frac{dy_1 \wedge \ldots \wedge \hat{dy_k} \wedge \ldots \wedge dy_4}{f_{z_1,z_2}(y)} \right). \tag{5.10}$$

We furthermore define 2-forms $\omega_i, i = 2, 3, 4$ as

$$\omega_2 = \nabla_{\theta_1} \omega_1, \quad \omega_3 = \nabla_{\theta_2} \omega_1, \quad \text{and} \quad \omega_4 = \nabla_{\theta_1} \nabla_{\theta_1} \omega_1. \tag{5.11}$$

For dimensional reasons $(\omega_1, \omega_2, \omega_3, \omega_4)$ provide a basis for $H^2_{\mathrm{dR}}(X_b)_\iota$ and Griffiths transversality ensures its compatibility with the Hodge filtration.

Proposition 5.1. *The Gauss-Manin connection in the basis $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ is*

$$\nabla_{\theta_i} \omega^{\mathrm{tr}} = G_i \omega^{\mathrm{tr}}, \tag{5.12}$$

with

$$G_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2}\mu(1-\nu)z_1 & \frac{1}{2}(\Delta_1 - 1) & 0 & \frac{1}{2}\Delta_1 \\ (G_1)_{41} & (G_1)_{42} & (G_1)_{43} & (G_1)_{44} \end{pmatrix}, \tag{5.13}$$

$$G_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{1}{2}\mu(1-\nu)z_1 & \frac{1}{2}(\Delta_1 - 1) & 0 & \frac{1}{2}\Delta_1 \\ \frac{2\nu(1-\nu)z_1 z_2}{\Delta_2} & \frac{(1-2\Delta_1)z_2}{\Delta_2} & \frac{2z_2}{\Delta_2} & \frac{(1-2\Delta_1)z_2}{\Delta_2} \\ (G_2)_{41} & (G_2)_{42} & (G_2)_{43} & (G_2)_{44} \end{pmatrix}, \tag{5.14}$$

where we defined

$$\Delta_1 = 1 - \mu\nu^2 z_1, \qquad \Delta_2 = 1 - 4z_2, \tag{5.15}$$

with (μ, ν) given by $(3,3)$, $(4,4)$ and $(12,6)$ for K3 surfaces of elliptic type E_6, E_7 and E_8 respectively. The entries $(G_i)_{4j}$ of the Gauss-Manin connection matrices are

$$\begin{aligned}
(G_1)_{41} &= \frac{\mu(1-\nu)z_1(2(1-\Delta_1)-1)}{\Delta_1^2 + (\Delta_2 - 1)(\Delta_1 - 1)^2}, \\
(G_1)_{42} &= \frac{\mu z_1((1-\nu)((2-\Delta_1)\Delta_2 - 2) + \nu^2(2\Delta_1\Delta_2 - 1))}{\Delta_1^2 + (\Delta_2 - 1)(\Delta_1 - 1)^2}, \\
(G_1)_{43} &= \frac{2\mu(1-\nu)z_1\Delta_2}{\Delta_1^2 + (\Delta_2 - 1)(\Delta_1 - 1)^2}, \\
(G_1)_{44} &= \frac{3\mu\nu^2 z_1(1-(1-\Delta_1)\Delta_2)}{\Delta_1^2 + (\Delta_2 - 1)(\Delta_1 - 1)^2},
\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}
(G_2)_{41} &= \frac{\mu(1-\nu)(1-\Delta_1)z_1(1-(1+\Delta_1)\Delta_2)}{2(\Delta_1^2 + (\Delta_2 - 1)(\Delta_1 - 1)^2)}, \\
(G_2)_{42} &= \frac{\mu z_1((1-\nu)(1-\Delta_2) - \nu^2(1-\Delta_1)(1-(1+\Delta_1)\Delta_2)}{2(\Delta_1^2 + (\Delta_2 - 1)(\Delta_1 - 1)^2)}, \\
(G_2)_{43} &= -\frac{\mu(1-\nu)z_1\Delta_1\Delta_2}{\Delta_1^2 + (\Delta_2 - 1)(\Delta_1 - 1)^2}, \\
(G_2)_{44} &= \frac{(1-\Delta_1)(-1+(1-\Delta_1)(3\Delta_2 - 1) - (1-\Delta_1^2)\Delta_2)}{2(\Delta_1^2 + (\Delta_2 - 1)(\Delta_1 - 1)^2)}.
\end{aligned} \tag{5.17}$$

Parameters μ and ν can be computed from the toric data as

$$\mu = \frac{2d}{w_1 w_2} \left(\frac{d}{w_1}\right)^{w_1/2-1} \left(\frac{d}{w_2}\right)^{w_2/2-1}, \qquad \nu = \frac{d}{2}. \tag{5.18}$$

Proof. Picard-Fuchs system for hypersurfaces in weighted projective varieties is equivalent to the GKZ hypergeometric system [GKZ89] and can thus be determined from the generators of the Mori cone $l^{(i)}$ in Table 5.1. $\square$

Let W_b be the Poincaré dual of $\iota(M)$ in $H^2_{\mathrm{dR}}(X_b)$. We define

$$H_2(X_b,\mathbb{Z})_\iota = H_2(X_b,\mathbb{Z})/W_b. \tag{5.19}$$

Denote the basis of $H_2(X_b,\mathbb{Z})_\iota$ by γ_b^j, $j = 1,\dots,m+2$, and define the period matrix $\mathbf{\Pi}$ by integrating the basis ω of $H^2_{\mathrm{dR}}(X_b)_\iota$ over the integral cycles $\gamma_b^j \in H_2(X_b,\mathbb{Z})_\iota$

$$\mathbf{\Pi}_{ij} = \int_{\gamma_b^j} \omega_i, \quad \gamma_b^j \in H_2(X_b,\mathbb{Z})_\iota, \quad i,j = 1,\dots,m+2. \tag{5.20}$$

The first row of $\mathbf{\Pi}$ corresponds to the periods (X^0,X^1,X^2,X^3) of ω_1. They satisfy the Picard-Fuchs differential equations

$$\begin{aligned}
&(\theta_1(\theta_1-2\theta_2) - \mu z_1(\nu\theta_1+\nu-1)(\nu\theta_1+1))\cdot X^j = 0,\\
&\left(\theta_2^2 - z_2(\theta_1-2\theta_2)(\theta_1-2\theta_2-1)\right)\cdot X^j = 0, \quad j = 0,1,2,3,
\end{aligned} \tag{5.21}$$

where (μ,ν) are $(3,3)$, $(4,4)$ and $(12,6)$ as in Proposition 5.1. The system (5.21) admits a holomorphic solution

$$X^0 = \sum_{n\ge 2m\ge 0} \frac{\left(\frac{d}{2}n\right)!}{\left(\frac{w_1}{2}n\right)!\left(\frac{w_2}{2}n\right)!(m!)^2(n-2m)!} z_1^n z_2^m, \tag{5.22}$$

with leading term 1. There are unique solutions X^a, $a = 1,2$ of (5.21) of the form

$$X^a = (2\pi i)^{-1} X^0 \log(z_a) + S^a, \qquad a = 1,2, \tag{5.23}$$

where S^a is a convergent power series in z_1 and z_2, with $S^a \to 0$ as $|z_i| \to 0$. The series S^a are fixed uniquely as a solution to (5.21).

Definition 5.2. The Griffiths-Yukawa couplings Y_{ij} are

$$\mathsf{Y}_{ij} = -\int_X \omega_1 \wedge \nabla_{\theta_i}\nabla_{\theta_j}\omega_1, \quad i,j = 1,2. \tag{5.24}$$

Proposition 5.3. *The Griffiths-Yukawa couplings for the mirrors of elliptically fibered K3 surfaces are given by*

$$\begin{aligned}
\mathsf{Y}_{11} &= \frac{2c}{\Delta_1^2 + (\Delta_2-1)(\Delta_1-1)^2},\\
\mathsf{Y}_{12} = \mathsf{Y}_{21} &= \frac{c\Delta_1}{\Delta_1^2 + (\Delta_2-1)(\Delta_1-1)^2},\\
\mathsf{Y}_{22} &= \frac{2c(2\Delta_1-1)z_2}{\Delta_1^2 + (\Delta_2-1)(\Delta_1-1)^2}.
\end{aligned} \tag{5.25}$$

Proof. We compute

$$\theta_k \mathsf{Y}_{ij} = -\int_X \nabla_{\theta_k}\omega_1 \wedge \nabla_{\theta_i}\nabla_{\theta_j}\omega_1 - \int_X \omega_1 \wedge \nabla_{\theta_i}\nabla_{\theta_j}\nabla_{\theta_k}\omega_1. \tag{5.26}$$

Integrating by parts the first term, we find

$$\theta_k \mathsf{Y}_{ij} = -\frac{2}{3}\int_X \omega \wedge \nabla_{\theta_i}\nabla_{\theta_j}\nabla_{\theta_k}\omega_1. \tag{5.27}$$

From the Gauss-Manin system we can express the action of $\nabla_{\theta_i}\nabla_{\theta_j}\nabla_{\theta_k}$ in terms of lower order operators. By Griffiths transversality only operators of second order will contribute and Y_{ij} are the solutions of the resulting differential equations. Moreover, we have the following relations between Griffiths-Yukawa couplings

$$\Delta_1\mathsf{Y}_{11} - 2\mathsf{Y}_{12} = 0 \qquad \text{and} \qquad \Delta_2\mathsf{Y}_{22} + 4z_2\mathsf{Y}_{12} - z_2\mathsf{Y}_{11} = 0, \tag{5.28}$$

which fix $\mathsf{Y}_{12} = \mathsf{Y}_{21}$ and Y_{22} in terms of Y_{11}. □

The intersection pairing

$$Q : H^2_{\mathrm{dR}}(X) \times H^2_{\mathrm{dR}}(X) \to \mathbb{C}, \quad Q(\omega_i, \omega_j) = \int_X \omega_i \wedge \omega_j, \tag{5.29}$$

in the basis ω will be denoted by Q_ω. It is given by

$$Q_\omega = \begin{pmatrix} 0 & 0 & 0 & -\mathsf{Y}_{11} \\ 0 & \mathsf{Y}_{11} & \mathsf{Y}_{12} & \frac{1}{2}\theta_1\mathsf{Y}_{11} \\ 0 & \mathsf{Y}_{21} & \mathsf{Y}_{22} & -\frac{1}{2}\theta_2\mathsf{Y}_{11} + \theta_1\mathsf{Y}_{12} \\ -\mathsf{Y}_{11} & \frac{1}{2}\theta_1\mathsf{Y}_{11} & -\frac{1}{2}\theta_2\mathsf{Y}_{11} + \theta_1\mathsf{Y}_{12} & \mathsf{Y}_{44} \end{pmatrix}, \tag{5.30}$$

with

$$\begin{aligned} \mathsf{Y}_{44} = -\frac{1}{\Delta_1^2 + (\Delta_2-1)(\Delta_1-1)^2}\Big(&-4\theta_1^2\mathsf{Y}_{11} + \frac{1}{2}(\Delta_1-1)(1+\Delta_2(1+\Delta_1))\theta_1\mathsf{Y}_{11} \\ &+\frac{1}{2}\left((\Delta_1-1)(2\Delta_2(1-\Delta_1)-1) + 4\mu(\nu-1)z_1(4(1-\Delta_2)+3\Delta_2)\right)\mathsf{Y}_{11}\Big). \end{aligned} \tag{5.31}$$

Note that the logarithmic derivatives of Y_{ij} can be expressed in terms of multiplication factors

$$\theta_1\mathsf{Y}_{11} = \frac{(1-\Delta_1)(1+(\Delta_1-1)\Delta_2)}{\Delta_1^2 + (\Delta_2-1)(\Delta_1-1)^2}\mathsf{Y}_{11}, \qquad \theta_2\mathsf{Y}_{11} = \frac{(1-\Delta_1)^2(\Delta_2-1)}{\Delta_1^2 + (\Delta_2-1)(\Delta_1-1)^2}\mathsf{Y}_{11}, \tag{5.32}$$

hence all the entries in Q_ω can be expressed in terms of Y_{11} and algebraic prefactors.

Monodromy of Elliptic K3 Surfaces

The monodromy group of the families (5.2) can be derived by an analytic continuation of solutions to (5.21) along closed paths around singular points in the moduli space. In the cases considered here, the monodromy group is closely related to the monodromy group of the elliptic fibre.

Proposition 5.4. *[LY96b] The Gauss-Manin system* (5.21) *is equivalent to the system*

$$\theta_{\alpha_i}^2 - \mu\alpha_i(\nu\theta_{\alpha_i} + \nu - 1)(\nu\theta_{\alpha_i} + 1), \quad i = 1, 2, \tag{5.33}$$

where α_i are related to z_i in (5.21) *by*

$$z_1 = \frac{1}{d_N}(\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2), \qquad z_2 = \frac{1}{d_N^2}\frac{\alpha_1\alpha_2(1-\alpha_1)(1-\alpha_2)}{z_1^2}. \tag{5.34}$$

In Ref. [CDM17] a similar factorization is found for the Appell function F_2, where the solution vector of a rank 4 system analogous to (5.21) is constructed as a Kronecker product of solutions to (5.33). In terms of differential forms on $H^2_{\mathrm{dR}}(X_b)_\iota$, the following corollary gives the structure of the Gauss-Manin connection matrices.

Corollary 5.5. *There exists a basis of $H^2_{\mathrm{dR}}(X_b)_\iota$, such that the Gauss-Manin connection matrices composed with $\alpha_i \frac{d}{d\alpha_i}$ are of the form*

$$\widetilde{G}_1 = G^{ell}_{\alpha_1} \boxplus \mathbb{1}_{2\times 2}, \qquad \widetilde{G}_2 = \mathbb{1}_{2\times 2} \boxplus G^{ell}_{\alpha_2}, \tag{5.35}$$

where $\boxplus$ denotes the Kronecker product and G^{ell}_α is the rank two Gauss-Manin system for an elliptic curve, given by

$$G^{ell}_\alpha = \begin{pmatrix} 0 & 1 \\ \frac{\mu(\nu-1)\alpha}{1-\mu\nu^2\alpha} & \frac{\mu\nu^2\alpha}{1-\mu\nu^2\alpha} \end{pmatrix}. \tag{5.36}$$

From the structure of the connection matrices, we infer that the monodromy of the fundamental solutions is generated by

$$\gamma_1 \boxplus \mathbb{1}_{2\times 2} \quad \text{and} \quad \mathbb{1}_{2\times 2} \boxplus \gamma_2, \quad \gamma_1, \gamma_2 \in \Gamma_0(N), \tag{5.37}$$

for a K3 family with the monodromy group of elliptic fibre $\Gamma_0(N)$.

Corollary 5.6. *The monodromy of the elliptic K3 families with the monodromy group of elliptic fibre $\Gamma_0(N)$ in* (5.2) *is isomorphic to*

$$\Gamma_0(N) \times \Gamma_0(N). \tag{5.38}$$

5.2 Moduli space of enhanced K3 surfaces

By moduli spaces T of lattice polarized K3 surfaces enhanced with differential forms we refer to the moduli spaces of pairs $(X, \{\omega_i\}_{i=1,\dots,m+2})$ where X is a K3 surface polarized by a lattice M of rank $\mathrm{rk}(M) = 20 - m$ with signature $(1, 19 - m)$ and $\{\omega_i\}_{1=1,\dots,m+2}$ is a basis of $H^2_{\mathrm{dR}}(X)/\iota(M)$, where $\iota : M \to H^2_{\mathrm{dR}}(X)$ denotes the polarization map. A basis $\{\alpha_i\}_{i=1,\dots,m+2}$ of $H^2_{\mathrm{dR}}(X)/\iota(M)$ can be fixed, such that the intersection pairing in this basis is given by the pairing

$$\Phi = \begin{pmatrix} 0 & 0 & -1 \\ 0 & C_{ab} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{5.39}$$

where C_{ab} denote the intersection numbers of the elliptic K3 surface. Mirror families introduced in the previous section are two-parameter families of hypersurfaces in weighted projective spaces (5.2). They are polarized by the pull-back of the lattice of toric divisors to the hypersurface. The rank of $\mathrm{Pic}_{cor}(X^\circ)$ is 18, $m = 2$ and the moduli space T is 6-dimensional. Away from the discriminant locus T is a locally ringed space with the local ring $\mathcal{O}_{\mathsf{T}}$. We will show that there is an isomorphism

$$\mathcal{O}_{\mathsf{T}} \cong \widetilde{\mathsf{M}}(\Gamma_0(N) \times \Gamma_0(N)), \tag{5.40}$$

between the local ring $\mathcal{O}_{\mathsf{T}}$ and the graded ring of quasi-modular forms of the modular subgroup $\Gamma_0(N)$ in two variables. The level N of the congruence subgroup is determined by the type of elliptic fiber of the elliptic K3, as explained in the next subsection. For explicit construction of the coordinates on T consider the filtration preserving transformation $\omega \mapsto \alpha = \mathsf{S}\omega,\ G_i \mapsto G_a = \sum_i \frac{1}{z_i}\frac{\partial z_i}{\partial t_a}\mathsf{S}G_i\mathsf{S}^{-1} + \partial_a\mathsf{S}\cdot\mathsf{S}^{-1},\ i, a = 1, 2$, where S is of the form

$$\mathsf{S} = \begin{pmatrix} \mathsf{s}_0 & 0 & 0 \\ \mathsf{s}_a & \mathsf{s}_{a,i} & 0 \\ \mathsf{s}_{3,0} & \mathsf{s}_{3,i} & \mathsf{s}_{3,3} \end{pmatrix}, \tag{5.41}$$

and $\partial_a = \frac{\partial}{\partial t_a}$ denotes the differentiation with respect to coordinates t_a. The moduli space T of K3 surfaces enhanced with differential forms consists of the moduli space B, together with the independent parameters of S. The condition on the pairing

$$\mathsf{S}Q_\omega\mathsf{S}^{\mathrm{tr}} = \Phi, \tag{5.42}$$

reads explicitly:

$$
\begin{aligned}
\mathsf{s}_{3,3} &= \frac{1}{\mathsf{s}_0 \mathsf{Y}_{1,1}},\\
\mathsf{C}^{alg}_{ab} &= \mathsf{s}_{a,i}\mathsf{s}_{b,j}\mathsf{Y}_{ij}, \quad i,j = 1,2,\\
s_{3,i} &= \frac{1}{\mathsf{s}_0}\mathsf{s}^{-1}_{a,j}\mathsf{Y}^{-1}_{ji}\mathsf{s}_a + \frac{1}{\mathsf{s}_0}\frac{\mathsf{Y}^{-1}_{i1}\mathsf{Y}_{24} + \mathsf{Y}^{-1}_{i2}\mathsf{Y}_{34}}{\mathsf{Y}_{11}},\\
\mathsf{s}_{3,0} &= \frac{1}{2\mathsf{s}_0}(\mathsf{C}^{alg})^{-1}_{ab}\mathsf{s}_{a,0}\mathsf{s}_{b,0} + \frac{\mathsf{Y}_{22}(\mathsf{Y}^2_{24} + \mathsf{Y}_{11}\mathsf{Y}_{44}) - \mathsf{Y}^2_{12}\mathsf{Y}_{44} - 2\mathsf{Y}_{12}\mathsf{Y}_{24}\mathsf{Y}_{34}}{2\mathsf{s}_0\mathsf{Y}^2_{11}(\mathsf{Y}_{11}\mathsf{Y}_{22} - \mathsf{Y}^2_{12})}.
\end{aligned}
\tag{5.43}
$$

For elliptic K3 surfaces we find 4 independent parameters. As coordinates on T we choose $\mathsf{s}_0, \mathsf{s}_1, \mathsf{s}_2$ and $\mathsf{s}_{1,1}$.

Algebraic variation of Hodge structure for projective elliptic K3 surfaces

Let $(\mathcal{X}, \alpha) \to \mathsf{T}$ be a family of lattice polarized projective K3 surfaces with a fixed choice of basis α of $\mathcal{H}^2_{\mathrm{dR}}(\mathcal{X})_\iota$, such that the intersection pairing in the basis α is Φ. Let furthermore R denote the function ring of T. The relative algebraic de Rham cohomology $H^2_{\mathrm{dR}}(\mathcal{X}/\mathsf{T})$ (see [Gro66]) carries the Gauss-Manin connection $\nabla : H^2_{\mathrm{dR}}(\mathcal{X}) \to H^2_{\mathrm{dR}}(\mathcal{X}/\mathsf{T}) \otimes_{\mathsf{R}} \Omega^1_{\mathsf{T}}$, where Ω^1_{T} is an R-module of differential forms in R [KO68]. As before, the Gauss-Manin connection restricts to the quotient $\mathcal{H}^2_{\mathrm{dR}}(\mathcal{X})_\iota$. Let $\mathrm{Vec}(\mathsf{T})$ be the Lie algebra of vector fields on T. The algebraic Gauss-Manin connection ∇ acts on α as

$$
\nabla_{\mathsf{E}_i}\alpha = \mathsf{A}_{\mathsf{E}_i}\alpha, \qquad \mathsf{E}_i \in \mathrm{Vec}(\mathsf{T}), \tag{5.44}
$$

where $\mathsf{A}_{\mathsf{E}_i}$ are $(m+2)\times(m+2)$ matrices with entries in $\mathcal{O}_{\mathsf{T}}$.

Theorem 5.7. *There are unique vector fields* $\mathsf{R}_a \in \mathrm{Vec}(\mathsf{T})$ *and unique* $\mathsf{C}^{alg}_{ab} \in \mathcal{O}_{\mathsf{T}}, a, b = 1,2$ *symmetric in* a, b *such that*

$$
\mathsf{A}_{\mathsf{R}_a} = \begin{pmatrix} 0 & \delta^b_a & 0 \\ 0 & 0 & \mathsf{C}^{alg}_{ac} \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.45}
$$

We call them modular vector fields. Furthermore

$$
\mathsf{R}_{a_1}\mathsf{C}^{alg}_{a_2 a_3} = 0. \tag{5.46}
$$

Theorem 5.7 amounts to finding S and t_a as above such that

$$
\mathsf{A}_{\frac{\partial}{\partial t_a}} = \sum_i \frac{1}{z_i}\frac{\partial z_i}{\partial t_a}\mathsf{S}G_i\mathsf{S}^{-1} + \partial_a\mathsf{S}\cdot\mathsf{S}^{-1}. \tag{5.47}
$$

Proposition 5.8. *The system* (5.47) *is solved by*

$$\mathsf{s}_0 = \left(X^0\right)^{-1}, \qquad t_a = \frac{X^a}{X^0}, \tag{5.48}$$

where X^0 denotes the fundamental period and X^a denote the the periods with a logarithmic pole at $z_a = 0$. Furthermore, the other independent parameters in S satisfy

$$\mathsf{s}_{a,i} = \frac{1}{z_i}\frac{\partial z_i}{\partial t_a}\mathsf{s}_0, \quad \mathsf{s}_a = \sum_{i=1,2} \mathsf{s}_{a,i}\theta_i \log \mathsf{s}_0 = \partial_{t_a} \log \mathsf{s}_0. \tag{5.49}$$

Proof. The proof is computational. For a proof using the special Kähler structure of B see [Ali17]. □

Corollary 5.9. *With this choice, we find that C^{alg}_{ab} are $\mathsf{C}^{alg}_{11} = C_{HH}, \mathsf{C}^{alg}_{12} = \mathsf{C}^{alg}_{21} = C_{HL}$ and $\mathsf{C}^{alg}_{22} = C_{LL} = 0$, which finishes the proof of Theorem 5.7.*

Theorem 5.10. *[LY95, LY96b] The fundamental period X^0 of a mirror to a projective elliptic K3 surface factorizes*

$$X^0 = A(\tau_1)A(\tau_2), \tag{5.50}$$

with $\tau_1 = t_1, \tau_2 = t_1 + t_2$ and the weight 1 modular forms A are given in 2.1. For each model, the quasi-modular form A is the quasi-modular form associated to the monodromy group of the respective elliptic fiber. This can be checked by comparison with (5.22).

Proposition 5.11. *There is an isomorphism*

$$\mathcal{O}_{\mathsf{T}} \cong \widetilde{\mathsf{M}}(\Gamma_0(N) \times \Gamma_0(N)), \tag{5.51}$$

between the local ring $\mathcal{O}_{\mathsf{T}}$ and the graded ring of quasi-modular forms of the modular subgroup $\Gamma_0(N)$ in two variables. The level N of the congruence subgroup is the same as in the monodromy group of the elliptic fibre of the elliptic K3 surface.

Proof. The independent variables $\mathsf{t} := \{z_i, \mathsf{s}_0, \mathsf{s}_a, \mathsf{s}_{1,1}\}_{i,a=1,2}$ form a local chart for T. Denote by $\mathcal{O}_{\mathsf{T}}$ the local ring at $\mathsf{t} \in \mathsf{T}$. Theorem 5.10 provides an isomorphism between the ring $\mathcal{O}_{\mathsf{T}}$ and the ring of quasi-modular forms in two variables. The isomorphism is given by the inverse mirror map $z_i = z_i(t_1, t_2)$. Fix $\alpha_a = \left(\frac{C(\tau_a)}{A(\tau_a)}\right)^r$, r as in 2.1. It satisfies

$$\partial_{\tau_a}\alpha_b = \delta^b_a \alpha_b(1-\alpha_b)A^2(\tau_b). \tag{5.52}$$

In terms of these the inverse mirror map is

$$z_1 = \frac{1}{d_N}(\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2), \qquad z_2 = \frac{1}{d_N^2}\frac{\alpha_1\alpha_2(1-\alpha_1)(1-\alpha_2)}{z_1^2}. \tag{5.53}$$

The remaining elements of the ring are found from (5.48), (5.49), (5.50) and (2.19)

$$\begin{aligned} s_a &= -\frac{1}{2r}\left(E(\tau_a) + \frac{2C^r(\tau_a) - A^r(\tau_a)}{A^{r-2}(\tau_a)}\right), \\ \mathsf{s}_{1,1} &= \frac{\alpha_1(1-\alpha_1)(1-2\alpha_2)}{\alpha_1(1-\alpha_2)+\alpha_2(1-\alpha_1)}\frac{A(\tau_1)}{A(\tau_2)}. \end{aligned} \tag{5.54}$$

□

Algebraic Group acting on T

We define a Lie group G by

$$\mathsf{G} = \{\mathsf{g} \in \mathrm{GL}(m+2,\mathbb{C}) |\ \mathsf{g} \text{ block lower triangular and } \mathsf{g}\Phi\mathsf{g}^{\mathrm{tr}} = \Phi\}. \tag{5.55}$$

It acts on T from the right as

$$(X,\alpha)\cdot\mathsf{g} = (X, \alpha^{\mathrm{tr}}\mathsf{g}), \tag{5.56}$$

where $\alpha = (\alpha_1,\ldots,\alpha_{m+2})^{\mathrm{tr}}$ is the special basis defined as above, $\mathsf{g} \in \mathsf{G}$, and $\alpha^{\mathrm{tr}}\mathsf{g}$ is the standard matrix product. The condition $\mathsf{g}\Phi\mathsf{g}^{\mathrm{tr}} = \Phi$ fixes $\dim(\mathsf{G}) = \dim(\mathsf{T}) - 2 = 4$. The group G is generated by two elements isomorphic to the multiplicative group $\mathbb{C}^*$ and two elements isomorphic to the additive group $\mathbb{C}$. The following lemma gives the generators of G:

Lemma 5.12. *For any* $\mathsf{g} \in \mathsf{G}$ *there are unique elements* $\mathsf{g}_i \in \mathsf{G}$, $i = 1,2,3,4$ *such that* g *can be written as a product of at most four* g_i. *For families* (5.2), g_i *are given by*

$$\begin{aligned} \mathsf{g}_1 &= \begin{pmatrix} \mathsf{h}_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathsf{h}_0^{-1} \end{pmatrix}, \quad \mathsf{g}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathsf{h}_{11} & \mathsf{h}_{21} & 0 \\ 0 & \mathsf{h}_{12} & \mathsf{h}_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \mathsf{g}_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathsf{C}^{alg}_{11}\mathsf{h}_1 & 1 & 0 & 0 \\ \mathsf{C}^{alg}_{12}\mathsf{h}_1 & 0 & 1 & 0 \\ 0 & \mathsf{h}_1 & 0 & 1 \end{pmatrix}, \quad \mathsf{g}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathsf{C}^{alg}_{12}\mathsf{h}_2 & 1 & 0 & 0 \\ \mathsf{C}^{alg}_{22}\mathsf{h}_2 & 0 & 1 & 0 \\ 0 & 0 & \mathsf{h}_2 & 1 \end{pmatrix}, \end{aligned} \tag{5.57}$$

where h_{ij}, $i,j = 1,2$ *satisfy the constraints*

$$\sum_{i,j=1,2} \mathsf{C}^{alg}_{ij}\mathsf{h}_{ik}\mathsf{h}_{jl} = \mathsf{C}^{alg}_{lk}. \tag{5.58}$$

The constraints can be solved by a simple algebraic manipulation, which yields only one independent parameter. We fix $\mathsf{h}_{11} = \mathsf{C}^{alg}_{12}\mathsf{h}_3$ *as the independent parameter and express* h_{ij} *in terms of* h_3.

The Lie algebra of G is given by

$$\mathrm{Lie}(\mathsf{G}) = \{\mathfrak{g} \in \mathrm{Mat}(m+2,\mathbb{C}) | \ \mathfrak{g} \text{ is block lower triangular and } \mathfrak{g}\Phi + \Phi\mathfrak{g} = 0\}. \tag{5.59}$$

The Lie algebra $\mathrm{Lie}(\mathsf{G})$ is a Lie sub-algebra of $\mathrm{Vec}(\mathsf{T})$. The basis of $\mathrm{Lie}(\mathsf{G})$ can be constructed from the elements $\mathsf{g}_i \in \mathsf{G}$. We find

$$\begin{aligned}
\mathfrak{g}_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathfrak{g}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathsf{C}^{alg}_{12} & -\mathsf{C}^{alg}_{11} & 0 \\ 0 & \mathsf{C}^{alg}_{22} & -\mathsf{C}^{alg}_{12} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\mathfrak{g}_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathsf{C}^{alg}_{11} & 0 & 0 & 0 \\ \mathsf{C}^{alg}_{12} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathsf{C}^{alg}_{12} & 0 & 0 & 0 \\ \mathsf{C}^{alg}_{22} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{5.60}$$

5.3 The Gauss-Manin Lie algebra

The Gauss-Manin Lie algebra is defined to be the $\mathcal{O}_\mathsf{T}$ module generated by $\mathrm{Lie}(\mathsf{G})$ and the modular vector fields R_a in (5.45). We write the action of the modular vector fields R_a on α explicitly

$$\mathsf{A}_{\mathsf{R}_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathsf{C}^{alg}_{11} \\ 0 & 0 & 0 & \mathsf{C}^{alg}_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{A}_{\mathsf{R}_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathsf{C}^{alg}_{12} \\ 0 & 0 & 0 & \mathsf{C}^{alg}_{22} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.61}$$

Theorem 5.13. *The Gauss-Manin Lie algebra, generated by* $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4$ *and* $\mathsf{A}_{\mathsf{R}_1}, \mathsf{A}_{\mathsf{R}_2}$, *is isomorphic to* $\mathrm{sl}_2(\mathbb{C}) \oplus \mathrm{sl}_2(\mathbb{C})$.

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{w_1 w_2}{2d} & -2\frac{w_1 w_2}{2d} & 0 \\ 0 & 0 & \frac{w_1 w_2}{2d} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.62}$$

Here $\frac{w_1 w_2}{2d} = C_{HL}^{-1}$. The Lie algebra is given by the generators

$$\mathcal{J}^1 = A\cdot(\mathfrak{g}_1+\mathfrak{g}_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathcal{J}^2 = A\cdot(\mathfrak{g}_1-\mathfrak{g}_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
$$\mathcal{J}_-^1 = A\cdot\mathfrak{g}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_-^2 = A\cdot\mathfrak{g}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{5.63}$$
$$\mathcal{J}_+^1 = A\cdot\mathsf{A}_{\mathsf{R}_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_+^2 = A\cdot\mathsf{A}_{\mathsf{R}_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which form a basis of $\mathrm{sl}_2(\mathbb{C})\oplus\mathrm{sl}_2(\mathbb{C})$, with commutation relations

$$[\mathcal{J}_+^a, \mathcal{J}_-^a] = \mathcal{J}^a, \quad [\mathcal{J}_0^a, \mathcal{J}_+^a] = \mathcal{J}_+^a, \quad [\mathcal{J}_0^a, \mathcal{J}_-^a] = -\mathcal{J}_-^a, \quad [\mathcal{J}_\bullet^1, \mathcal{J}_\bullet^2] = 0, \quad a = 1,2, \tag{5.64}$$

where $\bullet$ denotes any generator. □

Chapter 6

Non-compact Calabi-Yau Manifolds

In this chapter we study moduli spaces of mirror non-compact Calabi-Yau threefolds enhanced with choices of differential forms in their middle cohomology. The Gauss-Manin Lie algebra of derivations on the ring of regular functions is constructed. We study in detail two families, appearing in mirror symmetry, mirrors to local $\mathbb{P}^2$ and local $\mathbb{F}_2$. The families share several properties with the corresponding mirror curves, a locus where the mirror families degenerate. The mirror curves in this case are of genus one and the corresponding differential ring of quasi-modular forms (for the appropriate congruence subgroup) is identified as a subring of the ring of regular functions of the moduli space of enhanced non-compact families. Moreover, we identified the $\mathrm{sl}_2(\mathbb{C})$ Lie subalgebra corresponding to the algebra of derivations of quasi-modular forms in the cases of the mirrors of local $\mathbb{P}^2$ and local $\mathbb{F}_2$. We note that the mirror geometry of local $\mathbb{F}_2$ has two parameters, although the moduli space is only one-dimensional, this was already addressed, for instance, in [ABK08] and is related to the fact that $\mathbb{F}_2$ has two distinct curve classes but only one non-trivial four-cycle. On the level of the Gauss-Manin Lie algebra we note that the additional parameter gives rise to an additional vector field, enlarging the $\mathrm{sl}_2(\mathbb{C})$ algebra of derivations of the quasi-modular forms in this case.

6.1 Local Calabi-Yau and mirror families

In this work we consider toric Calabi-Yau threefolds which are given by the total space of a bundle over a surface or a curve, we call these local Calabi-Yau threefolds. Toric varieties can be described by a fan Σ, but substantial amount of information about it is encapsulated by a charge matrix Q_k^i that encodes the linear relations between one dimensional cones $\Sigma(1)$ of the fan Σ, i.e. $\sum_{i=0}^{s-1} Q_k^i v_i = 0$ for $v_i \in \Sigma(1)$ and $s = |\Sigma(1)|$. The mirror families of such varieties are known to have the following form, see e. g. [KLM+96, KMV97, CKYZ99, Hos00]:

$$\mathcal{X} = \left\{ uv + F_{\boldsymbol{a}}(y_i) = uv + \sum_{i=0}^{s-1} a_i y_i = 0 \mid u, v \in \mathbb{C}, y_i \in \mathbb{C}^*, \prod_{i=0}^{s-1} y_i^{Q_k^i} = 1 \right\}. \tag{6.1}$$

It is also common to set as a mirror family a family of curves

$$\Sigma = \left\{ F_{\boldsymbol{a}}(y_i) = \sum_{i=0}^{s-1} a_i y_i = 0 \mid y_i \in \mathbb{C}^*, \prod_{i=0}^{s-1} y_i^{Q_k^i} = 1 \right\}. \tag{6.2}$$

We are interested in a special class of local Calabi-Yau varieties, namely canonical bundles over del Pezzo surfaces. In this case all the information can be encoded in a 2-dimensional reflexive polytope Δ. In particular, the set of the one dimensional cones v_i of the fan Σ corresponds to the set $A(\Delta)$ of the integral points m_i of the polytope Δ, thus the coordinates y_i can be parametrized by

$$y_i = t^{m_i} := t_1^{m_{i,1}} t_2^{m_{i,2}}, \quad t_1, t_2 \in \mathbb{C}^*, \tag{6.3}$$

for $m_i = (m_{i,1}, m_{i,2}) \in A(\Delta)$. Hence the information about the general fibre $X_{\boldsymbol{a}}$ of $\mathcal{X}$ is encoded in the polynomial

$$F_{\boldsymbol{a}}(t_1, t_2) = \sum_{m_i \in A(\Delta)} a_i t^{m_i}. \tag{6.4}$$

The parameters a_i redundantly describe the deformations of complex structure on $X_{\boldsymbol{a}}$. The GIT quotient of the space of parameters a_i by the natural torus action induced by the action $F(t_1, t_2) \to \lambda_0 F(\lambda_1 t_1, \lambda_2 t_2)$ gives the complex structure moduli space B, for which the torus-invariant local coordinates can be chosen as $z_k = (-1)^{Q_k^0} \prod_i a_i^{Q_k^i}$.

The holomorphic 3-form on $X_{\boldsymbol{a}}$ is given by the residue of the top form on the ambient space

$$\Omega = \operatorname{Res}\left[\frac{1}{F_{\boldsymbol{a}}(t_1, t_2) + uv} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge du \wedge dv \right], \tag{6.5}$$

where Res is the residue map

$$\operatorname{Res} : H^4(\mathbb{C}^2 \times (\mathbb{C}^*)^2, X_{\boldsymbol{a}}) \to H^3(X_{\boldsymbol{a}}), \tag{6.6}$$

which sends an $\omega \in H^4(\mathbb{C}^2 \times (\mathbb{C}^*)^2, X_{\boldsymbol{a}})$ to $\int_\gamma w$, with $\gamma \in H_4(\mathbb{C}^2 \times (\mathbb{C}^*)^2, X_{\boldsymbol{a}})$ being a cycle around $uv + F_{\boldsymbol{a}}(t_1, t_2) = 0$. The periods of the holomorphic 3-form satisfy Picard-Fuchs equations, which are obtained from a GKZ system [GKZ89] of differential equations

$$\left[\prod_{i:Q_k^i > 0} \left(\frac{\partial}{\partial a_i} \right)^{Q_k^i} - \prod_{i:Q_k^i < 0} \left(\frac{\partial}{\partial a_i} \right)^{-Q_k^i} \right] \int_\gamma \Omega = 0, \quad \gamma \in H_3(X_{\boldsymbol{a}}, \mathbb{Z}). \tag{6.7}$$

Mixed Hodge structure

This contents of this subsection follow closely section 3.4 and are meant as establishing notation and making the chapter self-contained. A mixed Hodge structure on the middle cohomology $H^3(X_{\boldsymbol{a}}, \mathbb{C})$ for the families (6.1) was described in [Bat94, KM10] (see also [Sti98]). Define the following ring:

$$\mathbf{S}_\Delta = \bigoplus_{k\geq 0} \mathbf{S}_\Delta^k, \qquad \mathbf{S}_\Delta^k = \bigoplus_{m\in A(\Delta(k))} \mathbb{C}t_0^k t^m, \tag{6.8}$$

where t_0 is an additional parameter and

$$\Delta(k) := \left\{ m \in \mathbb{Z}^2 \subseteq \mathbb{R}^2 \mid \frac{m}{k} \in \Delta \right\} \tag{6.9}$$

for $k \geq 1$, $\Delta(0) := \{0\} \subset \mathbb{R}^2$. The grading is given by $\deg(t_0^k t^m) = k$. Define also the following differential operators on $\mathbf{S}_\Delta$

$$\mathcal{D}_0 := \theta_{t_0} + t_0 F_{\boldsymbol{a}}, \quad \mathcal{D}_i := \theta_{t_i} + t_0 \theta_{t_i} F_{\boldsymbol{a}}, \quad i = 1, \dots, s-1, \tag{6.10}$$

and a $\mathbb{C}$-vector space

$$\mathcal{R}_F := \mathbf{S}_\Delta / (\sum_{i=0}^{s-1} \mathcal{D}_i \mathbf{S}_\Delta). \tag{6.11}$$

There is a decreasing filtration on $\mathcal{R}_F$

$$0 \subset \mathcal{E}^0 \subset \mathcal{E}^{-1} \subset \mathcal{E}^{-2} \subset \dots, \tag{6.12}$$

with $\mathcal{E}^{-k}$ being a subspace generated by monomials of degree $\leq k$.

There is also an increasing filtration. Define $I_\Delta^{(j)}$ to be the homogeneous ideals in $\mathbf{S}_\Delta$ generated as $\mathbb{C}$-subspaces by all monomials $t_0^k t^m$ ($k \geq 1$) with $m \in \Delta(k)$ which does not belong to any face of codimension j. Since everything belongs to the codimension 0 face, we get $I_\Delta^{(0)} = 0$. Everything that belongs to the face of codimension 2 also belongs to some face of codimension 1, therefore $I_\Delta^{(1)} \subset I_\Delta^{(2)}$. There are no faces of codimension 3, thus $I_\Delta^{(3)} = \bigoplus_{k\geq 1} \mathbf{S}_\Delta^k$, which contains the last two, and set $I_\Delta^{(4)} = \mathbf{S}_\Delta$. These form an increasing filtration on $\mathbf{S}_\Delta$:

$$0 = I_\Delta^{(0)} \subset I_\Delta^{(1)} \subset I_\Delta^{(2)} \subset I_\Delta^{(3)} \subset I_\Delta^{(4)} = \mathbf{S}_\Delta, \tag{6.13}$$

which defines under the quotient an increasing filtration on $\mathcal{R}_{F_{\boldsymbol{a}}}$:

$$0 = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \mathcal{I}_3 \subset \mathcal{I}_4 = \mathcal{R}_{F_{\boldsymbol{a}}}, \tag{6.14}$$

where $\mathcal{I}_j$ is the image of $I_\Delta^{(j)}$ in $\mathcal{R}_{F_{\boldsymbol{a}}}$.

Let us consider $\mathbf{S}_\Delta[\boldsymbol{a}] = \mathbf{S}_\Delta \otimes \mathbb{C}[\boldsymbol{a}]$ and

$$\mathcal{R}_{F_{\boldsymbol{a}}}[\boldsymbol{a}] := \mathbf{S}_\Delta[\boldsymbol{a}]/(\sum_{i=0}^{n} \mathcal{D}_i \mathbf{S}_\Delta[\boldsymbol{a}]). \tag{6.15}$$

On $\mathbf{S}_\Delta[\boldsymbol{a}]$ we can define the following differential operators:

$$\mathcal{D}_{a_i} := \frac{\partial}{\partial a_i} + t_0 t^{m_i}. \tag{6.16}$$

Since they commute with $\mathcal{D}_i$, they descend to $\mathcal{R}_{F_{\boldsymbol{a}}}[\boldsymbol{a}]$.

Theorem 6.1 ([KM10]). *There is an isomorphism*

$$r : \mathcal{R}_{F_{\boldsymbol{a}}} \cong H^3(X_{\boldsymbol{a}}, \mathbb{C}), \tag{6.17}$$

for every $\boldsymbol{a}$. Therefore, as a consequence, the filtrations $\mathcal{E}_\bullet, \mathcal{I}_\bullet$ define a mixed Hodge structure on $H^3(X_{\boldsymbol{a}}, \mathbb{C})$, namely $F^{6-k} = r(\mathcal{E}^{-k})$ and $W_3 = r(\mathcal{I}_1)$, $W_4 = W_5 = r(\mathcal{I}_3)$, $W_6 = r(\mathcal{I}_4)$. Moreover the derivations $\mathcal{D}_{a_i}$ correspond to the Gauss-Manin connection $\nabla_{\frac{\partial}{\partial a_i}}$.

Remark 6.2 ([KM10]). The weight 3 filtration space W_3 is in fact just the first cohomology of the corresponding curve $\overline{\Sigma}_{\boldsymbol{a}}$, which is the compactification of a fibre of (6.2) at $\boldsymbol{a}$:

$$W_3 H^3(X_{\boldsymbol{a}}) \cong H^1(\overline{\Sigma}_{\boldsymbol{a}}). \tag{6.18}$$

For the families (6.1) there is an inclusion $W_3 H^3(X_{\boldsymbol{a}}) \cong H^1(\overline{\Sigma}_{\boldsymbol{a}}) \subset H^3(X_a)$. We define a polarization $\langle -, - \rangle$ to be equal to the intersection pairing on $W_3 H^3(X_a)$ and 0 otherwise. It was shown in [KM10] that this pairing is flat on Gr_i^W with respect to the Gauss-Manin connection. Since in the case of local del Pezzo all the mirror curves are elliptic curves, in a suitable basis the pairing matrix is

$$\begin{pmatrix} 0_{(b_n-2)\times(b_n-2)} & 0_{(b_n-2)\times 2} \\ 0_{2\times(b_n-2)} & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix}, \tag{6.19}$$

where $0_{a\times b}$ is an $a \times b$ block of zeroes.

6.2 Enhanced non-compact Calabi-Yau varieties

An important property of the intersection pairing in the case of compact Calabi-Yau varieties is that it is flat with respect to the Gauss-Manin connection. In the case of variation of mixed Hodge structure we no longer have this property, but we have flatness of the pairing on the graded components $\mathrm{Gr}_i^{\mathcal{W}} = \mathcal{W}_i/\mathcal{W}_{i-1}$. This fact motivates the following definition.

Definition 6.3. An enhanced Calabi-Yau variety of dimension n is a pair (X, ω), where X is a Calabi-Yau variety of dimension n and

$$\omega = (\omega_1, \ldots, \omega_i, \ldots, \omega_{b_n})^{\mathrm{tr}}, \quad b_n = \dim H^n(X, \mathbb{C}), \tag{6.20}$$

is a basis of $H^n(X, \mathbb{C})$ such that:

- It respects the Hodge filtration, i.e.

  $$\omega = (\omega_1, \ldots, \omega_{\dim F^k}) \tag{6.21}$$

  spans F^k;
- it respects the weight filtration, i.e.

  $$\omega = (\omega_{b_n - \dim W_k}, \ldots, \omega_{b_n}) \tag{6.22}$$

  spans W_k;
- the pairing $\langle -, - \rangle$ on Gr_i^W takes the form of a constant matrix Φ_i in this basis for all i.

The construction of the moduli space of enhanced varieties, as well as the Gauss-Manin Lie algebra generalize to this setting. The difference is the additional constraints imposed on the entries of S by the second condition above. It sets some of the lower-diagonal blocks to zero, in a similar fashion as the Hodge filtration condition sets the upper-diagonal blocks of S to zero. Moreover, the pairing condition of the Definition 4.3 is imposed on each graded component of the weight filtration separately.

6.3 Examples from mirror symmetry

The construction of sections 6.1, 6.2 is applied to two families of non-compact Calabi-Yau threefolds, mirrors to local $\mathbb{P}^2$ and local $\mathbb{F}^2$. We construct the rings of regular functions on T and the Gauss-Manin Lie algebra and prove Theorems 1.3, 1.4 and 1.5.

Local $\mathbb{P}^2$

Setup

The first example we study is the total space of the canonical bundle over the projective plane $\mathbb{P}^2$. It is defined by the toric charge vectors $Q = (-3, 1, 1, 1)$ and the mirror family is given by

$$\mathcal{X} = \left\{ uv + a_0 y_0 + a_1 y_1 + a_2 y_2 + a_3 y_3 = 0 \mid u, v \in \mathbb{C}, y_k \in \mathbb{C}^*, \frac{y_1 y_2 y_3}{y_0^3} = 1 \right\}. \tag{6.23}$$

We have

$$F_{\boldsymbol{a}}(t_1, t_2) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1 t_2}. \tag{6.24}$$

The torus invariant coordinate reads

$$z = -\frac{a_1 a_2 a_3}{a_0^3}, \tag{6.25}$$

and the Picard-Fuchs operator is

$$\mathcal{L} = (\theta^2 - 3z(3\theta + 1)(3\theta + 2))\theta, \quad \theta = z\frac{d}{dz}. \tag{6.26}$$

The middle cohomology can be written as

$$H^3(X_z) \cong \mathcal{R}_{F_{\boldsymbol{a}}} = \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}t_0^2, \tag{6.27}$$

where we denote the general fibre of $\mathcal{X}$ by X_z to emphasize the dependence on z. The mixed Hodge structure described in the previous section is

$$W_3 = W_4 = W_5 = \mathbb{C}t_0 \oplus \mathbb{C}t_0^2 \subset W_6 = \mathcal{R}_{F_{\boldsymbol{a}}}, \tag{6.28}$$

$$F^3 = \mathbb{C}1 \subset F^2 = \mathbb{C}1 \oplus \mathbb{C}t_0 \subset F^1 = F^0 = \mathcal{R}_{F_{\boldsymbol{a}}}. \tag{6.29}$$

We can generate it with the help of $\mathcal{D}_{a_0}$:

$$\mathcal{D}_{a_0}(1) = \left(\frac{\partial}{\partial a_0} + t_0\right)(1) = t_0, \qquad \mathcal{D}_{a_0}(t_0) = t_0^2. \tag{6.30}$$

There is a patch of the moduli space B, where

$$\theta_{a_0} = -3\theta = -3z\frac{d}{dz} = -3\theta, \tag{6.31}$$

holds. Thus we can take $\left(\Omega, \nabla_\theta\Omega, \nabla_\theta^2\Omega\right)$ as a basis of $H^3(X_z)$ that satisfies first two conditions from the Definition 6.3. The only non-zero pairing is between $\nabla_\theta\Omega$ and $\nabla_\theta^2\Omega$. It is the Yukawa coupling Y_{111} for local $\mathbb{P}^2$, namely

$$\langle \nabla_\theta^2\Omega, \nabla_\theta\Omega\rangle = Y_{111} = -\frac{1}{3(1-27z)}. \tag{6.32}$$

In the basis

$$\omega = \left(\Omega, \nabla_\theta\Omega, Y_{111}^{-1}\nabla_\theta^2\Omega\right), \tag{6.33}$$

the pairing has the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.34}$$

Moduli space T

Let us fix $\Phi_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\Phi_6 = \begin{pmatrix} 0 \end{pmatrix}$. We saw in the previous subsection that ω is a basis satisfying the conditions of the Definition 6.3, therefore we can construct (locally) the moduli space T by considering the complex structure modulus z and algebraically independent entries of the matrix S. The requirement on S to preserve the Hodge filtration restricts it to the lower-diagonal form

$$S = \begin{pmatrix} s_{11} & 0 & 0 \\ s_{21} & s_{22} & 0 \\ s_{31} & s_{32} & s_{33} \end{pmatrix}. \tag{6.35}$$

The second condition – preservation of the weight filtration, sets $s_{21} = s_{31} = 0$. Next, we have to satisfy the condition that

$$\langle -,-\rangle|_{\mathrm{Gr}_3^W} = \Phi_3, \qquad \langle -,-\rangle|_{\mathrm{Gr}_6^W} = \Phi_6, \tag{6.36}$$

note that Gr_4^W and Gr_5^W are empty. The second condition is empty, and the first one implies $s_{22} = s_{33}^{-1}$, thus we have

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s_{33}^{-1} & 0 \\ 0 & s_{32} & s_{33} \end{pmatrix}. \tag{6.37}$$

The element s_{11} corresponds to the normalization of the holomorphic 3-form and can be set to 1, as it decouples from other s_{ij}.

Modular vector field and Gauss-Manin Lie algebra

Modular vector fields have the form

$$\nabla_{\mathsf{R}}\omega_{\mathsf{t}} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}\omega_{\mathsf{t}}. \tag{6.38}$$

We make a choice of normalization inspired by [ASYZ14]

$$\nabla_{\mathsf{R}}\omega_{\mathsf{t}} = \begin{pmatrix} 0 & Y^{-1}(\mathsf{t}) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\omega_{\mathsf{t}}. \tag{6.39}$$

The Gauss-Manin connection matrix in the basis ω reads

$$\nabla_{\theta}\omega = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & Y_{111} \\ 0 & 18z & 0 \end{pmatrix}\omega = A_{\theta}\omega. \tag{6.40}$$

Let

$$\mathsf{R} = i_z\frac{\partial}{\partial z} + i_{32}\frac{\partial}{\partial s_{32}} + i_{33}\frac{\partial}{\partial s_{33}}, \tag{6.41}$$

then,

$$\nabla_{\mathsf{R}}\omega_{\mathsf{t}} = \left[\frac{i_z}{z}(SA_{\theta}S^{-1}) + i_{32}\frac{\partial S}{\partial s_{32}}S^{-1} + i_{33}\frac{\partial S}{\partial s_{33}}S^{-1}\right]\omega_{\mathsf{t}}. \tag{6.42}$$

Proposition 6.4. *There exists a unique modular vector field* R *on* T*. It is given by*

$$\mathsf{R} = \frac{zs_{33}^2}{Y_{111}}\frac{\partial}{\partial z} - s_{32}s_{33}^2\frac{\partial}{\partial s_{33}} + \frac{18zs_{33}^3}{Y_{111}}\frac{\partial}{\partial s_{32}}. \tag{6.43}$$

Introduce $\dot{f} = df(\mathsf{R})$ for $f \in \mathcal{O}_{\mathsf{T}}$, then the existence of R is equivalent to the following differential structure on the ring $\mathcal{O}_{\mathsf{T}}$:

$$\begin{cases} \dot{z} = zs_{33}^2Y_{111}^{-1}, \\ \dot{s}_{32} = 18zs_{33}^3Y_{111}^{-1}, \\ \dot{s}_{33} = -s_{32}s_{33}^2. \end{cases} \tag{6.44}$$

Theorem 6.5. *There is an isomorphism between the ring of regular functions* $\mathcal{O}_{\mathsf{T}}$ *equipped with the differential structure* R *and the ring* $\widetilde{\mathcal{M}}(\Gamma_0(3))$ *of quasi-modular forms on* $\Gamma_0(3)$ *with the derivation* ∂_{τ}*.*

Proof. Define

$$A = \sqrt{-3}s_{33}, \qquad \alpha = 27z, \tag{6.45}$$

$$B = (1-27z)^{1/3}A, \quad C = 3z^{1/3}A, \quad E = d\log(B^3C^3)(\mathsf{R}), \tag{6.46}$$

Then we can rewrite the differential ring as

$$\begin{aligned}
\dot{\alpha} &= \alpha(1-\alpha)A^2,\\
\dot{A} &= \frac{1}{6}\left(AE + C^3 - B^3\right),\\
\dot{B} &= \frac{1}{6}B(E - A^2),\\
\dot{C} &= \frac{1}{6}C(E + A^2),\\
\dot{E} &= \frac{1}{6}(E^2 - A^4),
\end{aligned} \tag{6.47}$$

which is the differential ring of quasi-modular forms on $\widetilde{\mathcal{M}}(\Gamma_0(3))$ in section 2.1. □

Gauss-Manin Lie algebra

Theorem 6.6. *The Gauss-Manin Lie algebra for local* $\mathbb{P}^2$ *is spanned by*

$$\begin{aligned}
\mathsf{R} &= \frac{zs_{33}^2}{Y_{111}}\frac{\partial}{\partial z} - s_{32}s_{33}^2\frac{\partial}{\partial s_{33}} + \frac{18zs_{33}^3}{Y_{111}}\frac{\partial}{\partial s_{32}},\\
\mathsf{R}_{\mathfrak{g}_{33}} &= s_{33}\frac{\partial}{\partial s_{33}} + s_{32}\frac{\partial}{\partial s_{32}},\\
\mathsf{R}_{\mathfrak{g}_{32}} &= s_{33}\frac{\partial}{\partial s_{32}},
\end{aligned} \tag{6.48}$$

and it is isomorphic to $\mathrm{sl}_2(\mathbb{C})$.

Proof. $\mathrm{Lie}(\mathsf{G})$ is generated by

$$\mathfrak{g}_{33} = \left(\frac{\partial}{\partial s_{33}}S\right)S^{-1}\bigg|_{S=\mathrm{Id}} = \begin{pmatrix} 0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{6.49}$$

$$\mathrm{g}_{21} = \left(\frac{\partial}{\partial s_{32}}S\right)S^{-1}\bigg|_{S=\mathrm{Id}} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}. \tag{6.50}$$

The corresponding vector fields are exactly $\mathsf{R}_{\mathrm{g}_{33}}$ and $\mathsf{R}_{\mathrm{g}_{32}}$. It can be checked explicitly that commutators are

$$[\mathsf{R}, \mathsf{R}_{\mathfrak{g}_{32}}] = \mathsf{R}_{\mathfrak{g}_{33}}, \quad [\mathsf{R}, \mathsf{R}_{\mathfrak{g}_{33}}] = -2\mathsf{R}, \quad [\mathsf{R}_{\mathfrak{g}_{32}}, \mathsf{R}_{\mathfrak{g}_{33}}] = 2\mathsf{R}_{\mathfrak{g}_{32}}. \tag{6.51}$$

□

Local $\mathbb{F}_2$

The canonical bundle over the second Hirzebruch surface $K_{\mathbb{F}_2}$ is defined by the toric charge vectors $Q_1 = (-2,1,0,1,0)$ and $Q_2 = (0,0,1,-2,1)$ and the mirror family is given by

$$\mathcal{X} = \left\{ uv + \sum_{i=0}^{4} a_i y_i = 0 \mid u,v \in \mathbb{C}, y_k \in \mathbb{C}^*, \frac{y_1 y_3}{y_0^2} = 1, \frac{y_2 y_4}{y_3^2} = 1 \right\}. \tag{6.52}$$

We have

$$F_{\boldsymbol{a}}(t_1,t_2) = a_0 + a_1 t_1 + a_2 t_2 + \frac{a_3}{t_1} + \frac{a_4}{t_1^2 t_2}. \tag{6.53}$$

The torus-invariant coordinates are

$$z_1 = \frac{a_1 a_3}{a_0^2}, \quad z_2 = \frac{a_2 a_4}{a_3^2}, \tag{6.54}$$

and the Picard-Fuchs operators read

$$\begin{aligned} \mathcal{L}_1 &= \theta_1(\theta_1 - 2\theta_2) - 2z_1(2\theta_1 + 1)\theta_1\,, \\ \mathcal{L}_2 &= \theta_2^2 - z_2(\theta_1 - 2\theta_2 - 1)(\theta_1 - 2\theta_2), \quad \theta_i = z_i \frac{d}{dz_i}. \end{aligned}$$

The middle cohomology can be written as

$$H^3(X_z) \cong \mathcal{R}_{F_{\boldsymbol{a}}} = \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}\frac{t_0}{t_1} \oplus \mathbb{C}t_0^2. \tag{6.55}$$

The mixed Hodge structure is

$$W_3 = \mathbb{C}t_0 \oplus \mathbb{C}t_0^2 \subset W_4 = W_5 = W_3 \oplus \mathbb{C}\frac{t_0}{t_1} \subset W_6 = \mathcal{R}_{F_{\boldsymbol{a}}}, \tag{6.56}$$

$$F^3 = \mathbb{C}1 \subset F^2 = \mathbb{C}1 \oplus \mathbb{C}t_0 \oplus \mathbb{C}\frac{t_0}{t_1} \subset F^1 = F^0 = \mathcal{R}_{F_{\boldsymbol{a}}}. \tag{6.57}$$

Using the Gauss-Manin connection we obtain

$$\mathcal{D}_{a_0}(1) = t_0, \qquad \mathcal{D}_{a_0}(t_0) = t_0^2, \qquad \mathcal{D}_{a_3}(1) = \frac{t_0}{t_1}. \tag{6.58}$$

On the moduli space B there is a patch where

$$a_0 \frac{\partial}{\partial a_0} = -2\theta_1, \qquad a_3 \frac{\partial}{\partial a_3} = \theta_1 - 2\theta_2, \tag{6.59}$$

therefore in the basis

$$\left(\Omega, \nabla_{(\theta_1 - 2\theta_2)}\Omega, \nabla_{\theta_1}\Omega, \nabla_{\theta_1}^2\Omega\right), \tag{6.60}$$

the pairing has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Y_{111} \\ 0 & 0 & Y_{111} & 0 \end{pmatrix}, \tag{6.61}$$

where Y_{111} is the Yukawa coupling

$$Y_{111} = \frac{1}{(1-4z_1)^2 - 64z_1^2 z_2}. \tag{6.62}$$

Moduli space

Let us fix

$$\Phi_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Phi_4 = \begin{pmatrix} 0 \end{pmatrix}, \quad \Phi_6 = \begin{pmatrix} 0 \end{pmatrix}. \tag{6.63}$$

In addition to the complex structure moduli z_1 and z_2 we need to identify the independent elements of the matrix S. The Hodge filtration condition restricts it to the lower-block diagonal form

$$S = \begin{pmatrix} s_{11} & 0 & 0 & 0 \\ s_{21} & s_{22} & s_{23} & 0 \\ s_{31} & s_{32} & s_{33} & 0 \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}. \tag{6.64}$$

The weight filtration condition gives

$$S = \begin{pmatrix} s_{11} & 0 & 0 & 0 \\ 0 & s_{22} & s_{23} & 0 \\ 0 & 0 & s_{33} & 0 \\ 0 & 0 & s_{43} & s_{44} \end{pmatrix}. \tag{6.65}$$

The conditions on the pairing on Gr_4^W and Gr_6^W are empty and the corresponding condition on Gr_3^W gives us the final form of S:

$$S = \begin{pmatrix} s_{11} & 0 & 0 & 0 \\ 0 & s_{22} & s_{23} & 0 \\ 0 & 0 & s_{44}^{-1} & 0 \\ 0 & 0 & s_{43} & s_{44} \end{pmatrix}. \tag{6.66}$$

Moreover, we again set $s_{11} = 1$.

Modular vector fields

We define modular vector fields by

$$\nabla_{\mathsf{R}_1}\omega_{\mathsf{t}} = \begin{pmatrix} 0 & Y_1^{-1}(\mathsf{t}) & Y_2^{-1}(\mathsf{t}) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\omega_{\mathsf{t}}, \tag{6.67}$$

and

$$\nabla_{\mathsf{R}_2}\omega_{\mathsf{t}} = \begin{pmatrix} 0 & Y_3^{-1}(\mathsf{t}) & Y_4^{-1}(\mathsf{t}) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\omega_{\mathsf{t}}, \tag{6.68}$$

for some $Y_i(\mathsf{t}) \in \mathcal{O}_{\mathsf{T}}$.

Theorem 6.7. *There exist unique modular vector fields* $\mathsf{R}_1, \mathsf{R}_2 \in \mathfrak{X}(\mathsf{T})$.

Proof. The theorem follows from an explicit evaluation of equation (4.15) under the condition (6.66). □

The existence of the modular vector fields is equivalent to two differential structures on $\mathcal{O}_{\mathsf{T}}$. By defining

$$\partial_{\tau_1} f := df(\mathsf{R}_1) \quad \text{and} \quad \partial_{\tau_2} f := df(\mathsf{R}_2) \tag{6.69}$$

for $f \in \mathcal{O}_{\mathsf{T}}$, we can write

$$\begin{aligned}
\partial_{\tau_1} z_1 =& z_1 s_{44}^2 s_{22}^{-1} \left(s_{22} + (1-4z_1)(1-4z_2)s_{23} - 4z_1(1+4z_2)s_{22}\right), \\
\partial_{\tau_1} z_2 =& 2z_2(-1+4z_2)s_{44}^2 s_{22}^{-1} \left(s_{23} + 4z_1 s_{22}\right), \\
\partial_{\tau_1} s_{22} =& 4z_2 s_{44}^2 \left(s_{23} + 4z_1 s_{22}\right), \\
\partial_{\tau_1} s_{23} =& 2z_1 s_{44}^2 s_{22}^{-1} \Big((s_{22}+s_{23})^2 - 4z_2 s_{23}^2\Big), \\
\partial_{\tau_1} s_{43} =& -2z_1 s_{44}^2 s_{22}^{-1} \left((-1+4z_2)s_{23}(-s_{43}+4z_2 s_{44})\right. \\
& \left. + s_{22}\left(s_{43} - 4z_2 s_{43} + 4z_2(1-10z_1+8z_1 z_2)s_{44}\right)\right), \\
\partial_{\tau_1} s_{44} =& -s_{44}^2 s_{22}^{-1} \left(2z_1(-1+4z_2)s_{23}s_{44} + s_{22}\left(s_{43} + (-1+4z_2)s_{44}\right)\right)), \\
\partial_{\tau_2} z_1 =& z_1(1-4z_1)(1-4z_2)s_{44}s_{22}^{-1}, \\
\partial_{\tau_2} z_2 =& 2z_2(1-4z_2)s_{44}s_{22}^{-1}, \\
\partial_{\tau_2} s_{22} =& -4z_2 s_{44}, \\
\partial_{\tau_2} s_{23} =& s_{43} + 2z_1 s_{44} + 2z_1 s_{23}s_{44}s_{22}^{-1} + 8z_1 z_2 s_{44}(1 - s_{23}s_{22}^{-1}), \\
\partial_{\tau_1} s_{43} =& 2z_1(1-4z_2)s_{44}s_{22}^{-1}(s_{43} - 4z_2 s_{44}, \\
\partial_{\tau_2} s_{44} =& 2z_1(-1+4z_2)s_{44}^2 s_{22}^{-1}.
\end{aligned} \tag{6.70}$$

Theorem 6.8. *The differential ring* $\left(\mathcal{O}_{\mathsf{T}}, \partial_{\tau_1}, \partial_{\tau_2}\right)$ *contains as differential subring the ring of quasi-modular forms on* $\Gamma_0(2)$.

Proof. By introducing a new variable

$$u = \frac{64 z_1^2 z_2}{(1 - z_1)^2}, \tag{6.71}$$

and

$$A = \sqrt{2(1 - 4z_1)} s_{44}, \qquad B = (1 - u)^{1/4} A, \qquad C = u^{1/4} A, \tag{6.72}$$

$$E = \partial_{\tau_1} \log(B^4 C^4) = \frac{2 - 8 s_{11} s_{31} - 8 z_1 (1 + 8 z_2)}{s_{11}^2}, \tag{6.73}$$

the differential ring relations can be rewritten as

$$\begin{aligned}
\partial_{\tau_1} u &= u(1 - u) A^2, \\
\partial_{\tau_1} A &= \frac{1}{8} A\left(E + \frac{C^4 - B^4}{A^2}\right), \\
\partial_{\tau_1} B &= \frac{1}{8} B(E - A^2), \\
\partial_{\tau_1} C &= \frac{1}{8} C(E + A^2), \\
\partial_{\tau_1} E &= \frac{1}{8}(E^2 - A^4),
\end{aligned} \tag{6.74}$$

which is the differential ring of quasi-modular forms for $\Gamma_0(2)$ family of elliptic curves. Furthermore, one can check that all of the functions u, A, B, C, E are independent of τ_2. □

Gauss-Manin Lie algebra

Theorem 6.9. *The Gauss-Manin Lie algebra for local* $\mathbb{F}^2$ *is isomorphic to the semi-direct product* $\mathfrak{L}_V \rtimes \mathrm{sl}_2(\mathbb{C})$, *where* $\mathfrak{L}_V$ *denotes the Lie algebra of type V, as classified in [Bia01], corresponding to an ideal of* $\mathfrak{G}$ *generated by* R_2, $\mathsf{R}_{\mathfrak{g}_{22}}$ *and* $\mathsf{R}_{\mathfrak{g}_{23}}$.

Proof. In this case Lie(G) is generated by

$$\mathfrak{g}_{22} = \left(\frac{\partial}{\partial s_{22}} S\right) S^{-1} \bigg|_{S=\mathrm{Id}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.75}$$

$$\mathfrak{g}_{23} = \left(\frac{\partial}{\partial s_{23}} S\right) S^{-1}\Big|_{S=\mathrm{Id}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.76}$$

$$\mathfrak{g}_{43} = \left(\frac{\partial}{\partial s_{43}} S\right) S^{-1}\Big|_{S=\mathrm{Id}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{6.77}$$

$$\mathfrak{g}_{44} = \left(\frac{\partial}{\partial s_{44}} S\right) S^{-1}\Big|_{S=\mathrm{Id}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.78}$$

The associated vector fields $\mathsf{R}_{\mathfrak{g}_{ij}}$ on T defined by $\nabla_{\mathsf{R}_{\mathfrak{g}_{ij}}} \omega_{\mathsf{t}} = \mathfrak{g}_{ij} \omega_{\mathsf{t}}$ are

$$\begin{aligned} \mathsf{R}_{\mathfrak{g}_{22}} &= s_{22} \frac{\partial}{\partial s_{22}} + s_{23} \frac{\partial}{\partial s_{23}}, \\ \mathsf{R}_{\mathfrak{g}_{23}} &= s_{44}^{-1} \frac{\partial}{\partial s_{23}}, \\ \mathsf{R}_{\mathfrak{g}_{43}} &= s_{44}^{-1} \frac{\partial}{\partial s_{43}}, \\ \mathsf{R}_{\mathfrak{g}_{44}} &= s_{44} \frac{\partial}{\partial s_{44}} + s_{43} \frac{\partial}{\partial s_{43}}. \end{aligned} \tag{6.79}$$

Together with R_1 and R_2 these vector fields form the Gauss-Manin Lie algebra $\mathfrak{G}$ and the commutation relations are the following:

$$\begin{aligned} &[\mathsf{R}_1, \mathsf{R}_{\mathfrak{g}_{44}}] = -2\mathsf{R}_1, && [\mathsf{R}_1, \mathsf{R}_{\mathfrak{g}_{43}}] = \mathsf{R}_{\mathfrak{g}_{44}}, && [\mathsf{R}_{\mathfrak{g}_{43}}, \mathsf{R}_{\mathfrak{g}_{44}}] = 2\mathsf{R}_{\mathfrak{g}_{43}}, \\ &[\mathsf{R}_1, \mathsf{R}_2] = 0, && [\mathsf{R}_1, \mathsf{R}_{\mathfrak{g}_{23}}] = \mathsf{R}_2, && [\mathsf{R}_1, \mathsf{R}_{\mathfrak{g}_{22}}] = 0, \\ &[\mathsf{R}_{\mathfrak{g}_{43}}, \mathsf{R}_2] = \mathsf{R}_{\mathfrak{g}_{23}}, && [\mathsf{R}_{\mathfrak{g}_{43}}, \mathsf{R}_{\mathfrak{g}_{23}}] = 0, && [\mathsf{R}_{\mathfrak{g}_{43}}, \mathsf{R}_{\mathfrak{g}_{22}}] = 0, \\ &[\mathsf{R}_{\mathfrak{g}_{44}}, \mathsf{R}_2] = \mathsf{R}_2, && [\mathsf{R}_{\mathfrak{g}_{44}}, \mathsf{R}_{\mathfrak{g}_{23}}] = -\mathsf{R}_{\mathfrak{g}_{23}}, && [\mathsf{R}_{\mathfrak{g}_{44}}, \mathsf{R}_{\mathfrak{g}_{22}}] = 0, \\ &[\mathsf{R}_2, \mathsf{R}_{\mathfrak{g}_{23}}] = 0, && [\mathsf{R}_2, \mathsf{R}_{\mathfrak{g}_{22}}] = \mathsf{R}_2, && [\mathsf{R}_{\mathfrak{g}_{23}}, \mathsf{R}_{\mathfrak{g}_{22}}] = \mathsf{R}_{\mathfrak{g}_{23}}. \end{aligned}$$

The first line above is the $\mathrm{sl}_2(\mathbb{C})$ Lie algebra corresponding to the quasi-modular forms. The last line is the 3-dimensional algebra $\mathfrak{L}_V$ according to the Bianchi's classification and it is an ideal in the Gauss-Manin Lie algebra $\mathfrak{G}$. Therefore, we have

$$\mathfrak{G} \cong \mathfrak{L}_V \rtimes \mathrm{sl}_2(\mathbb{C}). \tag{6.80}$$

□

Chapter 7

Landau-Ginzburg Models

We adapt the techniques of the program Gauss-Manin Connection in Disguise to define and compute graded differential rings associated to Landau-Ginzburg models. In particular, we will consider the case of complex projective spaces, whose mirrors are given by the triples (Y_q, f_q, ω_q), where $Y_q : \{y_1 \cdots y_{n+1} = q\} \subset \mathbb{C}^{n+1}$ is a family of affine surfaces over the moduli space B spanned by the formal parameter $q = e^t$, $f_q = y_1 + \ldots + y_{n+1}$ restricted to Y_q is called the superpotential and ω_q is a symplectic form on Y_q. Landau-Ginzburg families of this type were investigated extensively in the literature, especially in the context of integrable systems and their higher genus Gromov-Witten theory [CV91, Giv01, OP06]. Since we will mostly be interested in toric Landau-Ginzburg models, which inherit the symplectic structure from the underlying projective space, we will denote families of Landau-Ginzburg models parametrized by q simply by (Y_q, f_q).

7.1 Mirrors of $\mathbb{CP}^n$

The mirrors of complex projective spaces are Landau-Ginzburg models (Y_q, f_q, ω_q), where $Y_q : \{y_1 \cdots y_{n+1} = q\} \subset \mathbb{C}^{n+1}$ is a family of affine surfaces over the moduli space B spanned by the formal parameter $q = e^t$, $f_q = y_1 + \ldots + y_{n+1}$ restricted to Y_q is called the superpotential and ω_q is a symplectic form on Y_q. The classical period integrals (3.39) satisfy the GKZ type differential equation (3.36) with $\Delta = \{(1,0,\ldots,0),\ldots,(0,\ldots,0,1),(1,\ldots,1)\}$ and $c = (-1,0,\ldots,0)$. In particular, the classical period integrals satisfy

$$\Box I_\Gamma = (\delta^{n+1} - t) I_\Gamma = 0, \qquad \delta = t\frac{\mathrm{d}}{\mathrm{d}t}, \tag{7.1}$$

which has $n+1$ independent solutions, given by the Meijer G functions

$$G^{m,0}_{0,n+1}\left({\emptyset \atop 0,\ldots,0} \middle| (-1)^{1+m} t \right) = \frac{1}{2\pi i} \int_L \frac{\Gamma^m(s)}{\Gamma^{n-m}(1-s)} t^{-s} \mathrm{d}s, \; m = 1,\ldots,n+1, \tag{7.2}$$

where Γ denotes the Γ-function and L is a loop in the complex plane.

The GKZ operator $\square$ induces a connection

$$\delta : \mathfrak{S}_{\mathsf{B}}(\square) \to \mathfrak{S}_{\mathsf{B}}(\square) \otimes \Omega^1_{\mathsf{B}}, \tag{7.3}$$

on the vector bundle $\mathfrak{S}_{\mathsf{B}}(\square) = \mathfrak{S}(\square) \otimes \mathcal{O}_{\mathsf{B}}$, for a local system $\mathfrak{S}(\square)$ of solutions to (7.1). Let

$$\mathsf{x} = \begin{pmatrix} I_0 & \cdots & I_n \\ \vdots & \ddots & \vdots \\ \delta^n I_0 & \cdots & \delta^n I_n \end{pmatrix}, \tag{7.4}$$

denote a global frame of $\mathfrak{S}_{\mathsf{B}}(\square)$ and denote $x_0 = (I_0, \ldots, I_n)$. The differential equation (7.1) in a frame $\mathsf{x} = (x_0, \ldots, \delta^n x_0)^{\mathrm{tr}}$ is equivalent to the system of differential equations of the form

$$\delta \mathsf{x} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ t & 0 & 0 & \cdots & 0 \end{pmatrix} \mathrm{d}\log t \ \mathsf{x} = \square_\delta \mathrm{d}\log t \ \mathsf{x}. \tag{7.5}$$

7.2 Enhanced toric Landau-Ginzburg models

The GKZ system defines a local system $\mathfrak{S}(\square)$ of solutions to (3.36). Consider the vector bundle $\mathfrak{S}_{\mathsf{B}}(\square) = \mathfrak{S}(\square) \otimes \mathcal{O}_{\mathsf{B}}$, where B is the GIT quotient of the space of parameters of f_q by the natural toric action on it. There exists a natural filtration on $\mathfrak{S}_{\mathsf{B}}(\square)$, given by the pole order. We will denote by x a global frame of $\mathfrak{S}_{\mathsf{B}}(\square)$. The GKZ operators $\square_i$ can be written as a first order differential equation for x of the form

$$\delta_i \mathsf{x} = \square_{\delta_i} \mathsf{x}, \tag{7.6}$$

where $\delta_i = a_i \frac{\partial}{\partial a_i}$ is a differentiation with respect to a parameter a_i of f_q and by $\square_{\delta_i}$ we by abuse of notation denote the matrices corresponding to the GKZ operators $\square_{\delta_i}$ adapted to the vector field δ_i on B.

Any element of $\mathfrak{S}_{\mathsf{B}}(\square)$ can be written in the form

$$\left(\frac{1}{2\pi i}\right)^n \int_{\Gamma_i} \frac{\mathfrak{i}_j}{1 - t f_q(x_1, \ldots, x_n)} \frac{dx_1 \wedge \ldots \wedge dx_n}{x_1 \cdots x_n}, \tag{7.7}$$

with $\mathfrak{i}_j \in \mathcal{O}_{\mathsf{B}}$. The pairing on $\mathfrak{S}_{\mathsf{B}}(\square)$ can be defined as the residue pairing

$$Q(I_i, I_j) = \mathrm{Res}_{f_q=0} \left[\frac{\mathfrak{i}_i \mathfrak{i}_j \frac{dx_1 \wedge \ldots \wedge dx_n}{x_1 \cdots x_n}}{x_1 \frac{df_q}{dx_1}, \cdots, x_n \frac{df_q}{dx_n}} \right]. \tag{7.8}$$

A basis of cycles Γ_i can be chosen, such that Q is constant, of the form

$$Q = Q_0 = J_{n+1}, \tag{7.9}$$

for n even, and

$$Q = Q_0 = \begin{pmatrix} 0_n & J_n \\ -J_n & 0_n \end{pmatrix}, \tag{7.10}$$

for n odd, where 0_n denotes a $n \times n$ block of zeroes and J_n is the following $n \times n$ matrix

$$J_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{7.11}$$

It induces a pairing on $\mathfrak{S}_{\mathsf{B}}(\Box)$ via

$$Q_{\mathsf{B}}(\sum \alpha_i x_i, \sum \beta_j x_j) = \sum_{i,j} \alpha_i \beta_j Q(I_i, I_j). \tag{7.12}$$

Let $\mathsf{x} = (x_0, \ldots, \delta^n x_0)^{\mathrm{tr}}$ be a frame of $\mathfrak{S}_{\mathsf{B}}(\Box)$. $Q_{\mathsf{B}}(\mathsf{x},\mathsf{x})$ satisfies the constraint

$$\delta Q_{\mathsf{B}}(\mathsf{x},\mathsf{x}) = \mathsf{A}_t Q(\mathsf{x},\mathsf{x}) + Q(\mathsf{x},\mathsf{x}) {\mathsf{A}_t}^{\mathrm{tr}}, \tag{7.13}$$

which gives a recursive relation between the elements of Q_{B}.

Remark 7.1. The mixed Hodge structure associated to a GKZ module was investigated in [RW18]. In case a weight filtration on $\mathfrak{S}_{\mathsf{B}}(\Box)$ is introduced, a construction of moduli spaces of enhanced Landau-Ginzburg models can be suitably generalized to this case, as in chapter 6 ([AKV20]).

Enhanced toric Landau-Ginzburg models

Definition of enhanced varieties generalizes to the case of Landau-Ginzburg models.

Definition 7.2. Let (Y, f) be a toric Landau-Ginzburg model and $\Box = \{\Box_i\}$ the set of associated GKZ operators (3.37). The set $\Box$ defines a local system of solutions $\mathfrak{S}(\Box)$ and we denote $h = \dim(\mathfrak{S}(\Box))$. Fix a finite dimensional vector space V_0 of dimension h, such that $V_0 = V_{\mathbb{Z}} \otimes \mathbb{C}$. Fix also a non-degenerate pairing $Q_0 : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ on $V_{\mathbb{Z}}$, extending bilinearly to V_0 and let $F_0^\bullet$ be a filtration on V_0. We call the tuple $((Y, f), F^\bullet, Q, \phi)$, where $F^\bullet$ is a filtration on $\mathfrak{S}(\Box)$, $Q : \mathfrak{S}(\Box) \times \mathfrak{S}(\Box) \to \mathbb{C}$ is the polarization on it and

$$\phi : (\mathfrak{S}(\Box), F^\bullet, Q) \to (V_0, F_0^\bullet, Q_0), \tag{7.14}$$

is an isomorphism, an enhanced Landau-Ginzburg model. As in the case of varieties, $\phi : \mathfrak{S}(\square) \to F_0^0$ in (7.14) is an isomorphism in a sense that

- It respects the Hodge filtration

$$\phi(F^p(\mathfrak{S}(\square))) = F_0^p, \quad \forall p, \tag{7.15}$$

- It respects the compatibility with the pairing

$$\phi(Q(\alpha,\beta)) = Q_0(\phi(\alpha),\phi(\beta)), \quad \alpha,\beta \in \mathfrak{S}(\square). \tag{7.16}$$

Definition 7.3. We call the tuple $((Y,f),F^\bullet,\phi)$ a weakly enhanced Landau-Ginzburg model if

$$\phi : (\mathfrak{S}(\square),F^\bullet) \to (V_0,F_0^\bullet), \tag{7.17}$$

is an isomorphism. We require for ϕ the condition (7.15), but not (7.16).

Definition 7.4. Let B be a complex manifold and $\pi : (Y_q,f_q) \to \mathsf{B}$ a family of toric Landau-Ginzburg models over B. We call $((Y_q,f_q),\phi)$ a family of enhanced Landau-Ginzburg models if there exists an isomorphism

$$\phi : (\mathfrak{S}_{\mathsf{B}}(\square),\mathcal{F}^\bullet,Q_{\mathcal{O}_{\mathsf{B}}}) \to (V_0\otimes\mathcal{O}_{\mathsf{B}},F_0^\bullet\otimes\mathcal{O}_{\mathsf{B}},Q_0\otimes\mathcal{O}_{\mathsf{B}}), \tag{7.18}$$

where the bilinear pairing $Q_{\mathcal{O}_{\mathsf{B}}} : \mathfrak{S}_{\mathsf{B}}(\square)\times\mathfrak{S}_{\mathsf{B}}(\square) \to \mathcal{O}_{\mathsf{B}}$ is given by Q on each fibre.

The same holds for families of weakly enhanced varieties with the isomorphism of pairings omitted.

Moduli space

Let $((Y_q,f_q),\phi)$ be a family of weakly enhanced toric Landau-Ginzburg models. A frame of $V_0\otimes\mathcal{O}_{\mathsf{B}}$ can be constructed from a frame x of $\mathfrak{S}_{\mathsf{B}}(\square)$ as a linear combination

$$\mathsf{x}_s = S\mathsf{x}. \tag{7.19}$$

The constraint (7.15) manifests in S being block lower triangular of the form

$$S = \begin{pmatrix} *_{f_n,f_n} & 0_{f_n,f_{n-1}} & 0_{f_n,f_{n-2}} & \cdots & 0_{f_n,f_0} \\ *_{f_{n-1},f_n} & *_{f_{n-1},f_{n-1}} & 0_{f_{n-1},f_{n-2}} & \cdots & 0_{f_{n-1},f_0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ *_{f_1,f_n} & *_{f_1,f_{n-1}} & *_{f_1,f_{n-2}} & \cdots & 0_{f_1,f_0} \\ *_{f_0,f_n} & *_{f_0,f_{n-1}} & *_{f_0,f_{n-2}} & \cdots & *_{f_0,f_0} \end{pmatrix}, \tag{7.20}$$

where $*_{f_i,f_j}$ denotes an $f_i \times f_j$ matrix with $f_\bullet = \mathrm{rk}(\mathcal{F}^\bullet/\mathcal{F}^{\bullet+1})$. We will denote the elements in S by t_{ij}, in accordance to the label T of the moduli space of enhance Landau-Ginzburg models. The moduli space T of weakly enhanced Landau-Ginzburg models $((Y_q, f_q), F, \phi)$ constructed in this way is locally a ringed space with the ring of regular functions $\mathcal{O}_\mathsf{T} \cong \mathbb{C}[t, t_{ij}]$.

To construct a moduli space of fully enhanced Landau-Ginzburg models, we further impose the compatibility condition (7.16), which reads

$$S Q_\mathsf{x} S^{\mathrm{tr}} = Q_0, \tag{7.21}$$

where Q_x is the pairing Q_B in the basis x. The moduli space T of enhanced Landau-Ginzburg models $((Y_q, f_q), F^\bullet, Q, \phi)$ constructed in this way is a locally ringed space with the local ring $\mathcal{O}_\mathsf{T} \cong \mathbb{C}[t, t_{ij}]$, t_{ij} algebraically independent.

Let $\pi : ((Y_q, f_q), \mathcal{F}^\bullet, \phi) \to \mathsf{T}$ be a family of (weakly) enhanced Landau-Ginzburg models. By writing a frame $\mathsf{x}_s = S\mathsf{x}$ of $V_0 \otimes O_\mathsf{B}$, we can define a (Gauss-Manin) connection for the family of Landau-Ginzburg models $((Y_q, f_q), \mathcal{F}^\bullet, \phi)/\mathsf{T}$ via

$$\delta_\mathsf{t} \mathsf{x}_s = \left(\mathrm{d}S(\mathsf{t})S^{-1} + \sum_i t_i^{-1} \mathrm{d}t_i(\mathsf{t}) S \square_{\delta_i} S^{-1} \right) \mathsf{x}_s. \tag{7.22}$$

Definition 7.5. A modular vector field R is a rational vector field on T such that

$$\delta_\mathsf{R} : \mathsf{F}^k/\mathsf{F}^{k+1} \to \mathsf{F}^{k-1}/\mathsf{F}^k, \tag{7.23}$$

where $\mathsf{F}^\bullet$ is a filtration on $\mathfrak{S}(\square)$ induced by $\mathcal{F}^\bullet$.

Remark 7.6. Such a modular vector field is of the form

$$\nabla_\mathsf{R} \mathsf{x}_s = \begin{pmatrix} 0_{f_n,f_n} & *_{f_n,f_{n-1}} & 0_{f_n,f_{n-2}} & \cdots & 0_{f_n,f_0} \\ 0_{f_{n-1},f_n} & 0_{f_{n-1},f_{n-1}} & *_{f_{n-1},f_{n-2}} & \cdots & 0_{f_{n-1},f_0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{f_1,f_n} & 0_{f_1,f_{n-1}} & 0_{f_1,f_{n-2}} & \cdots & *_{f_1,f_0} \\ 0_{f_0,f_n} & 0_{f_0,f_{n-1}} & 0_{f_0,f_{n-2}} & \cdots & 0_{f_0,f_0} \end{pmatrix} \mathsf{x}_s, \tag{7.24}$$

where $f_\bullet := \mathrm{rk}(\mathsf{F}^\bullet/\mathsf{F}^{\bullet+1})$ and $*_{f_i,f_j}$ is an $f_i \times f_j$ matrix with entries in $\mathcal{O}_\mathsf{T}$. Equation (7.22) gives a system of differential equation for the entries of S

$$\delta_\mathsf{R} \mathsf{x}_s = \left(\mathrm{d}S(\mathsf{R})S^{-1} + \sum_i t_i^{-1} \mathrm{d}t_i(\mathsf{R}) S \square_{\delta_i} S^{-1} \right) \mathsf{x}_s. \tag{7.25}$$

The existence of the modular vector field is equivalent to the existence of solutions to (7.25). Furthermore, the only elements appearing in (7.25) are the elements of the ring of regular functions $\mathcal{O}_\mathsf{T}$, thus existence of a modular vector field is equivalent to existence of a (Ramanujan) differential structure on $\mathcal{O}_\mathsf{T}$.

Gauss-Manin Lie algebra

Definition 7.7. A (weakly) enhanced Landau-Ginzburg $((Y,f),F^\bullet,Q,\phi)$ is called full, if there exists an algebraic group G, acting on $\mathcal{X}$ and T from the left, which commutes both with the morphism $\mathsf{X}\to\mathsf{T}$ and ϕ.

For enhanced Landau-Ginzburg models with $\dim(\mathfrak{S}(\square)) = n$ the condition above is equivalent to G being of the form

$$\mathsf{G} = \left\{\mathsf{g}\in \mathrm{GL}_n(\mathbb{C})|\mathsf{g} \text{ block lower triangular and } \mathsf{g}Q_0\mathsf{g}^{\mathrm{tr}} = \mathsf{Q}_0\right\}, \tag{7.26}$$

The Lie algebra of G is given by

$$\mathrm{Lie}(\mathsf{G}) = \left\{\mathfrak{g}\in \mathrm{GL}_n(\mathbb{C})|\mathfrak{g} \text{ block lower triangular and } \mathfrak{g}Q_0 + Q_0\mathfrak{g}^{\mathrm{tr}} = 0\right\}. \tag{7.27}$$

For weakly enhanced Landau-Ginzburg models the last condition is not required and G is of the form

$$\mathsf{G} = \{\mathsf{g}\in \mathrm{GL}_n(\mathbb{C})|\mathsf{g} \text{ block lower triangular}\}. \tag{7.28}$$

The following Theorem of [AMSY16] extends to the Landau-Ginzburg setting.

Theorem 7.8. *For any $\mathfrak{g}\in\mathrm{Lie}(\mathsf{G})$, there exists a unique vector field $\mathsf{R}_\mathfrak{g}\in\mathfrak{X}(\mathsf{T})$, such that*

$$\nabla_{\mathsf{R}_\mathfrak{g}}\phi(\mathsf{x}) = \mathfrak{g}\phi(\mathsf{x}), \tag{7.29}$$

i.e. $\nabla_{\mathsf{R}_\mathfrak{g}}(\mathsf{x}_s) = \mathfrak{g}(\mathsf{x}_s)$.

Definition 7.9. The Gauss-Manin Lie algebra $\mathfrak{G}$ is the $\mathcal{O}_\mathsf{T}$-module generated by $\{\mathsf{R}_\mathfrak{g}\}\cong$ Lie(G) and the modular vector fields $\mathsf{R}_a\in\mathfrak{X}(\mathsf{T})$, $a=1,\dots\dim(\mathsf{B})$.

Weakly enhanced mirrors to $\mathbb{CP}^n$

Let $(\mathfrak{S}_\mathsf{B}(\square),\mathcal{F}^\bullet,\phi)$ be a family of weakly enhanced toric Landau-Ginzburg mirrors of $\mathbb{CP}^n$. A frame of $V_0\otimes\mathcal{O}_\mathsf{B} = \mathfrak{S}_0(\square)\otimes\mathcal{O}_\mathsf{B}$ can be constructed from a frame $\mathsf{x} = (I,\delta I,\dots,\delta^n I)^{\mathrm{tr}}$ of $\mathfrak{S}_\mathsf{B}(\square)$ at $t\in\mathsf{B}$ as a linear combination

$$\mathsf{x}_s = S\mathsf{x}. \tag{7.30}$$

The constraint (7.15) manifests in S being lower triangular of the form

$$S = \begin{pmatrix} t_{00} & 0 & \cdots & 0 \\ t_{10} & t_{11} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ t_{n0} & t_{n1} & \cdots & t_{nn} \end{pmatrix}. \tag{7.31}$$

The moduli space T of weakly enhanced mirrors of $\mathbb{CP}^n$ constructed in this way is locally a ringed space of dimension $1+(n+1)(n+2)/2$, with the ring of regular functions $\mathcal{O}_\mathsf{T} \cong \mathbb{C}[t, t_{ij}]$.

Theorem 7.10. *Let $\pi : ((Y_q, f_q), \mathcal{F}^\bullet, \phi) \to \mathsf{T}$ be a family of weakly enhanced Landau-Ginzburg models mirror to $\mathbb{CP}^n$ and let $\delta : \mathfrak{S}_\mathsf{T}(\square)(\mathcal{X}/\mathsf{T}) \to \mathfrak{S}_\mathsf{T}(\square)(\mathcal{X}/\mathsf{T}) \otimes \Omega^1_\mathsf{T}$ be the algebraic Gauss-Manin connection on $\mathfrak{S}_\mathsf{T}(\square)(\mathcal{X}/\mathsf{T}) = \mathfrak{S}(\square) \otimes \mathcal{O}_\mathsf{T}$. There exists a modular vector field R on T.*

Proof. Let $\mathsf{x}_s = \phi(\mathsf{x}) = S\mathsf{x}$. The action of δ_R is given by

$$\delta_\mathsf{R}\mathsf{x}_s = \left(\mathrm{d}S(\mathsf{R})S^{-1} + \sum_i t_i^{-1}\mathrm{d}t_i(\mathsf{R})S\square_\delta S^{-1} \right)\mathsf{x}_s. \tag{7.32}$$

where $\square_\delta$ is the GKZ operator in (7.1). Definition 7.5 implies that

$$\delta_\mathsf{R}\mathsf{x}_s = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & Y_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & Y_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & Y_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \mathsf{x}_s = \square_\mathsf{R}\mathsf{x}_s, \tag{7.33}$$

for $Y_i \in \mathcal{O}_\mathsf{T}$ and x a basis of $\mathfrak{S}_\mathsf{B}(\square)$ as above. □

Denote the action $\mathsf{R} = \dot{*}$. Theorem 7.10 is equivalent to existence of a differential structure on the ring $\mathcal{O}_\mathsf{T} = \mathbb{C}[t, t_{ij}]$. For small n the differential structures are given by

- $n = 1$:

$$\mathsf{R} : \begin{cases} \dot{t} = t t_{11} t_{00}^{-1}, \\ \dot{t_{00}} = t_1, \\ \dot{t_{10}} = -t t_{11}^2 t_{00}^{-1}, \\ \dot{t_{11}} = -t_{10} t_{11} t_{00}^{-1}, \end{cases} \tag{7.34}$$

  and the vector field R is given by

$$\mathsf{R} = t t_{11} t_{00}^{-1} \frac{\partial}{\partial t} + t_{10} \frac{\partial}{\partial t_{00}} - t t_{11}^2 t_{00}^{-1} \frac{\partial}{\partial t_{10}} - t_{10} t_{11} t_{00}^{-1} \frac{\partial}{\partial t_{11}}. \tag{7.35}$$

- $n = 2$:

$$\mathsf{R}: \begin{cases} \dot{t} = tt_{11}{t_{00}}^{-1}, \\ \dot{t_{00}} = t_{10}, \\ \dot{t_{10}} = t_{11}^2 t_{20}{t_{00}}^{-1} t_{22}^{-1}, \\ \dot{t_{20}} = -tt_{11}t_{22}{t_{00}}^{-1}, \\ \dot{t_{11}} = (-t_{10}t_{11}t_{22}t_{11}^2 t_{21}){t_{00}}^{-1}{t_{22}}^{-1}, \\ \dot{t_{21}} = -t_{20}t_{11}{t_{00}}^{-1}, \\ \dot{t_{22}} = -t_{11}t_{21}{t_{00}}^{-1}, \end{cases} \tag{7.36}$$

and the vector field R is given by

$$\begin{aligned} \mathsf{R} =& tt_{11}t_{00}^{-1}\frac{\partial}{\partial t} + t_{10}\frac{\partial}{\partial t_{00}} + t_{11}^2 t_{20}t_{00}^{-1}t_{22}^{-1}\frac{\partial}{\partial t_{10}} - tt_{11}t_{22}t_{00}^{-1}\frac{\partial}{\partial t_{20}} \\ &+ (-t_{10}t_{11}t_{22}t_{11}^2 t_{21})t_{00}^{-1}t_{22}^{-1}\frac{\partial}{\partial t_{11}} - t_{20}t_{11}t_{00}^{-1}\frac{\partial}{\partial t_{21}} - t_{11}t_{21}t_{00}^{-1}\frac{\partial}{\partial t_{22}}. \end{aligned} \tag{7.37}$$

- $n = 3$:

$$\mathsf{R}: \begin{cases} \dot{t} = tt_{11}t_{00}^{-1}, \\ \dot{t_{00}} = t_{10}, \\ \dot{t_{10}} = t_{11}^2 t_{20}t_{00}^{-1}t_{22}^{-1}, \\ \dot{t_{20}} = t_{11}t_{22}t_{30}t_{00}^{-1}t_{33}^{-1}, \\ \dot{t_{30}} = -tt_{11}t_{33}t_{00}^{-1}, \\ \dot{t_{11}} = (t_{11}t_{21} - t_{10}t_{22})t_{11}t_{00}^{-1}t_{22}^{-1}, \\ \dot{t_{21}} = (t_{22}t_{31} - t_{20}t_{33})t_{11}t_{00}^{-1}t_{33}^{-1}, \\ \dot{t_{31}} = -t_{11}t_{30}t_{00}^{-1}, \\ \dot{t_{22}} = (t_{22}t_{32} - t_{21}t_{33})t_{11}t_{00}^{-1}t_{33}^{-1}, \\ \dot{t_{32}} = -t_{11}t_{31}t_{00}^{-1}, \\ \dot{t_{33}} = -t_{11}t_{32}t_{00}^{-1}, \end{cases} \tag{7.38}$$

and the vector field R is given by

$$\begin{aligned} \mathsf{R} =& tt_{11}t_{00}^{-1}\frac{\partial}{\partial t} + t_{10}\frac{\partial}{\partial t_{00}} + t_{11}^2 t_{20}t_{00}^{-1}t_{22}^{-1}\frac{\partial}{\partial t_{10}} + t_{11}t_{22}t_{30}t_{00}^{-1}t_{33}^{-1}\frac{\partial}{\partial t_{20}} - tt_{11}t_{33}t_{00}^{-1}\frac{\partial}{\partial t_{30}} \\ &+(t_{11}t_{21} - t_{10}t_{22})t_{11}t_{00}^{-1}t_{22}^{-1}\frac{\partial}{\partial t_{11}} + (t_{22}t_{31} - t_{20}t_{33})t_{11}t_{00}^{-1}t_{33}^{-1}\frac{\partial}{\partial t_{21}} - t_{11}t_{30}t_{00}^{-1}\frac{\partial}{\partial t_{31}} \\ &+(t_{22}t_{32} - t_{21}t_{33})t_{11}t_{00}^{-1}t_{33}^{-1}\frac{\partial}{\partial t_{22}} - t_{11}t_{31}t_{00}^{-1}\frac{\partial}{\partial t_{32}} - t_{11}t_{32}t_{00}^{-1}\frac{\partial}{\partial t_{33}}. \end{aligned} \tag{7.39}$$

Gauss-Manin Lie algebra

The algebraic group

$$\mathsf{G} = \{\mathrm{g} \in \mathrm{GL}(n+1,\mathbb{C}) | \mathrm{g} \text{ lower triangular}\}, \tag{7.40}$$

acts on the moduli space T of enhanced mirrors of $\mathbb{CP}^n$ from the left. Its Lie algebra is

$$\mathrm{Lie}(\mathsf{G}) = \{\mathfrak{g} \in \mathrm{GL}(n+1,\mathbb{C}) | \ \mathfrak{g} \text{ lower triangular}\}, \tag{7.41}$$

and its generators are

$$\mathfrak{g}_{ij} = \delta_{ij}, \quad j \le i, \tag{7.42}$$

where δ_{ij} is 1 for $i = j$ and 0 otherwise.

Proposition 7.11. *For small n, the generators of* $\mathrm{Lie}(\mathsf{G})$ *are given by*

- $n = 1$

$$\mathfrak{g}_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_{10} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{g}_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.43}$$

The corresponding vector fields $\mathsf{t}_{ij} \in \mathfrak{X}(\mathsf{T})$ *are given by*

$$\begin{aligned} \mathsf{t}_{00} &= t_{00}\frac{\partial}{\partial t_{00}}, \\ \mathsf{t}_{10} &= t_{00}\frac{\partial}{\partial t_{10}}, \\ \mathsf{t}_{11} &= t_{10}\frac{\partial}{\partial t_{10}} + t_{11}\frac{\partial}{\partial t_{11}}. \end{aligned} \tag{7.44}$$

- $n = 2$*:*

$$\begin{aligned} \mathfrak{g}_{00} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathfrak{g}_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{g}_{20} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ \mathfrak{g}_{11} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathfrak{g}_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathfrak{g}_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{7.45}$$

The corresponding vector fields t_{ij} *are given by*

$$\begin{aligned}
\mathsf{t}_{00} &= t_{00}\frac{\partial}{\partial t_{00}},\\
\mathsf{t}_{10} &= t_{00}\frac{\partial}{\partial t_{10}},\\
\mathsf{t}_{20} &= t_{00}\frac{\partial}{\partial t_{20}},\\
\mathsf{t}_{11} &= t_{10}\frac{\partial}{\partial t_{10}} + t_{11}\frac{\partial}{\partial t_{11}},\\
\mathsf{t}_{21} &= t_{10}\frac{\partial}{\partial t_{20}} + t_{11}\frac{\partial}{\partial t_{21}},\\
\mathsf{t}_{22} &= t_{20}\frac{\partial}{\partial t_{20}} + t_{21}\frac{\partial}{\partial t_{21}} + t_{22}\frac{\partial}{\partial t_{22}}.
\end{aligned} \tag{7.46}$$

Proof. The proof is computational and corresponds to solving the system

$$\delta_{\mathsf{t}_{ij}}\mathsf{x}_s = \mathfrak{g}_{ij}\mathsf{x}_s. \tag{7.47}$$

□

Together with R these vector fields form the Gauss-Manin Lie algebra $\mathfrak{G}$, which is increasingly large with larger n.

Enhanced mirrors to $\mathbb{CP}^n$

Let Q be the pairing defined in (7.8) and $(\mathfrak{S}_{\mathsf{B}}(\mathcal{L}), \mathcal{F}^\bullet, Q, \phi)$ be a family of enhanced toric Landau-Ginzburg models. As before frame of $V_0 \otimes \mathcal{O}_{\mathsf{B}} = \mathfrak{S}_0(\mathcal{L}) \otimes \mathcal{O}_{\mathsf{B}}$ can be constructed from a frame $\mathsf{x} = (I, \delta I, \ldots, \delta^n I)^{\mathrm{tr}}$ of $\mathfrak{S}_{\mathsf{B}}(\mathcal{L})$ at $t \in \mathsf{B}$ as a linear combination

$$\mathsf{x}_s = S\mathsf{x}. \tag{7.48}$$

The constraint (7.15) manifests in S being lower triangular of the form (7.31). Further, we impose the compatibility condition (7.16), which reads

$$SQ_{\mathsf{x}}S^{\mathrm{tr}} = Q_0, \tag{7.49}$$

where $Q_{\mathsf{x}} = Q_0$ is the pairing Q_{B} in the basis x. The moduli space T of enhanced Landau-Ginzburg models $((Y_q, f_q), F, Q, \phi)$ constructed in this way is a locally ringed space with the local ring $\mathcal{O}_{\mathsf{T}} \simeq \mathbb{C}[t, t_{ij}]$, t_{ij} algebraically independent.

Conjecture 7.12. Let $\pi : ((Y_q, f_q), \mathcal{F}^\bullet, \phi) \to \mathsf{T}$ be a family of fully enhanced Landau-Ginzburg models mirror to $\mathbb{CP}^n$ for odd n and let $\delta : \mathfrak{S}_{\mathsf{T}}(\square)(\mathcal{X}/\mathsf{T}) \to \mathfrak{S}_{\mathsf{T}}(\square)(\mathcal{X}/\mathsf{T}) \otimes \Omega^1_{\mathsf{T}}$ be the algebraic Gauss-Manin connection on $\mathfrak{S}_{\mathsf{T}}(\square)(\mathcal{X}/\mathsf{T}) = \mathfrak{S}(\square) \otimes \mathcal{O}_{\mathsf{T}}$. There exists a vector field R such that

$$\delta_{\mathsf{R}} \mathsf{x}_s = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & Y_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & Y_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & Y_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \mathsf{x}_s = \square_{\mathsf{R}} \phi(\mathsf{x}), \tag{7.50}$$

for $Y_i \in \mathcal{O}_{\mathsf{T}}$ and x a basis of $\mathfrak{S}_{\mathsf{B}}(\square)$ as above.

The existence of enhanced varieties for underlying local system of rank 2 over a one dimensional base was discussed in [ABF19], excuding the case $n = 2$. We checked the existence for some small n. As in the case of weakly enhanced Landau-Ginzburg models, the existence of the modular vector field is equivalent to the existence of differential structure on the local ring $\mathcal{O}_{\mathsf{T}}$, due to (7.50), which we rewrite as

$$\left(\mathrm{d}S(\mathsf{R})S^{-1} + \sum_i t_i^{-1} \mathrm{d}t_i(\mathsf{R}) S \square_\delta S^{-1} \right) = \square_{\mathsf{R}}. \tag{7.51}$$

Writing out the relations gives a system of first order differential equations for t and t_{ij}. We were able to construct modular vector fields for odd n, while for even n there certain algebraic constraints were not met, see e.g. [ABF19] for the constraints for local systems of rank 2 over a one-dimensional base.

Denote the action $\mathsf{R} = \dot{*}$. Theorem 7.10 is equivalent to existence of a differential structure on the ring $\mathcal{O}_{\mathsf{T}}$. For small n the differential structures are given by

- $n = 1$

$$\mathsf{R} : \begin{cases} \dot{t} = t t_0^{-2}, \\ \dot{t_0} = t_1, \\ \dot{t_1} = -t t_0^{-3}. \end{cases} \tag{7.52}$$

  and the vector field R is given by

$$\mathsf{R} = t t_0^{-2} \frac{\partial}{\partial t} + t_1 \frac{\partial}{\partial t_0} - t t_0^{-3} \frac{\partial}{\partial t_1}. \tag{7.53}$$

- $n = 3$:

$$\mathsf{R} : \begin{cases} \dot{t} = t t_{11} t_{00}^{-1}, \\ \dot{t_{00}} = t_{10}, \\ \dot{t_{10}} = t_{11}^3 t_{20} t_{00}^{-1}, \\ \dot{t_{20}} = t_{30}, \\ \dot{t_{30}} = -t t_{11} t_{00}^{-2}, \\ \dot{t_{11}} = (-t_{11} t_{10} + t_{11}^3 t_{21}) t_{00}^{-1}, \\ \dot{t_{21}} = (-2 t_{11} t_{20} + t_{21} t_{10}) t_{00}^{-1}. \end{cases} \tag{7.54}$$

and the vector field R is given by

$$\begin{aligned} \mathsf{R} = {} & t t_{11} t_{00}^{-1} \frac{\partial}{\partial t} + t_{10} \frac{\partial}{\partial t_{00}} + t_{11}^3 t_{20} t_{00}^{-1} \frac{\partial}{\partial t_{10}} + t_{30} \frac{\partial}{\partial t_{20}} - t t_{11} t_{00}^{-2} \frac{\partial}{\partial t_{30}} \\ & + (-t_{11} t_{10} + t_{11}^3 t_{21}) t_{00}^{-1} \frac{\partial}{\partial t_{11}} + (-2 t_{11} t_{20} + t_{21} t_{10}) t_{00}^{-1} \frac{\partial}{\partial t_{21}}. \end{aligned} \tag{7.55}$$

Gauss-Manin Lie algebra

In the case of enhanced Landau-Ginzburg models the families are full and there is an algebraic group

$$\mathsf{G} = \left\{ \mathrm{g} \in \mathrm{GL}(n+1, \mathbb{C}) | \mathrm{g} \text{ block lower triangular and } \mathrm{g} Q_0 \mathrm{g}^{\mathrm{tr}} = Q_0 \right\}, \tag{7.56}$$

acting on the moduli space T of enhanced mirrors of $\mathbb{CP}^n$ from the left. The condition $\mathrm{g} Q_0 \mathrm{g}^{\mathrm{tr}} = Q_0$ fixes $\dim(\mathsf{G}) = \dim(\mathsf{T}) - 1 = \frac{(n+1)(n+3)}{4}$. Its Lie algebra is

$$\mathrm{Lie}(\mathsf{G}) = \{ \mathfrak{g} \in \mathrm{GL}(n+1, \mathbb{C}) | \ \mathfrak{g} \text{ lower triangular and } \mathfrak{g} Q_0 + Q_0 \mathfrak{g}^{\mathrm{tr}} = 0 \}. \tag{7.57}$$

Proposition 7.13. *For small n, the generators of* $\mathrm{Lie}(\mathsf{G})$ *are given by*

- $n = 1$

$$\mathfrak{g}_{\mathsf{H}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathfrak{g}_{\mathsf{F}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{7.58}$$

The corresponding vector fields $\mathsf{H}, \mathsf{F} \in \mathfrak{X}(\mathsf{T})$ *are given by*

$$\begin{aligned} \mathsf{H} &= t_{00} \frac{\partial}{\partial t_{00}} - t_{10} \frac{\partial}{\partial t_{10}}, \\ \mathsf{F} &= t_{00} \frac{\partial}{\partial t_{10}}. \end{aligned} \tag{7.59}$$

- *$n = 3$:*

$$
\begin{aligned}
\mathfrak{g}_{00} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \mathfrak{g}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathfrak{g}_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\
\mathfrak{g}_{20} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ \mathfrak{g}_{30} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \mathfrak{g}_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{7.60}
$$

The corresponding vector fields t_{ij} are given by

$$
\begin{aligned}
\mathsf{t}_{00} &= t_{00}\frac{\partial}{\partial t_{00}} - t_{30}\frac{\partial}{\partial t_{30}}, \\
\mathsf{t}_{11} &= t_{10}\frac{\partial}{\partial t_{10}} - t_{20}\frac{\partial}{\partial t_{20}} + t_{11}\frac{\partial}{\partial t_{11}} - t_{21}\frac{\partial}{\partial t_{21}}, \\
\mathsf{t}_{10} &= t_{00}\frac{\partial}{\partial t_{10}} + t_{20}\frac{\partial}{\partial t_{30}}, \\
\mathsf{t}_{20} &= t_{00}\frac{\partial}{\partial t_{20}} - t_{10}\frac{\partial}{\partial t_{30}}, \\
\mathsf{t}_{30} &= t_{10}\frac{\partial}{\partial t_{20}} + t_{11}\frac{\partial}{\partial t_{21}}, \\
\mathsf{t}_{21} &= t_{00}\frac{\partial}{\partial t_{30}}.
\end{aligned}
\tag{7.61}
$$

Proof. The proof is computational and corresponds to solving the system

$$
\delta_{\mathsf{t}_{ij}} \mathsf{x}_s = \mathfrak{g}_{ij} \mathsf{x}_s. \tag{7.62}
$$

Together with R these vector fields form the Gauss-Manin Lie algebra $\mathfrak{G}$ and the commutation relations are

$$
[\mathsf{R}, \mathsf{H}] = \mathsf{R}, \quad [\mathsf{F}, \mathsf{H}] = -\mathsf{F}, \quad , [\mathsf{R}, \mathsf{F}] = 2\mathsf{H}, \tag{7.63}
$$

for $n = 1$, which is the $\mathrm{sl}_2(\mathbb{C})$ Lie algebra corresponding to the algebra of derivaitons on quasi-modular forms. □

The following structural theorem is a direct corollary of Proposition 7.13.

Theorem 7.14. *The Gauss-Manin Lie algebra of the Landau-Ginzburg family mirror to $\mathbb{CP}^1$ is isomorphic to $\mathrm{sl}_2(\mathbb{C})$.*

Inspired by these results, we identify the weight modular vector field as

Definition 7.15. The weight modular vector field H is defined as

$$\delta_{\mathsf{H}} \mathsf{x}_s = \sum \mathfrak{g}_{ij} \mathsf{x}_s, \qquad \mathfrak{g}_{ij} \text{ diagonal.} \tag{7.64}$$

The quasi-modular vector field F is defined by the condition

$$\delta_{\mathsf{F}} \mathsf{x}_s = \sum_{i \neq j} \mathfrak{g}_{ij} \mathsf{x}_s, \tag{7.65}$$

such that $[\mathsf{R}, \mathsf{F}] = \mathsf{H}$.

The weight and quasi-modular vector fields for small n are given by

- $n = 1$

$$\begin{aligned} \mathsf{H} &= t_{00} \frac{\partial}{\partial t_{00}} - t_{10} \frac{\partial}{\partial t_{10}}, \\ \mathsf{F} &= t_{00} \frac{\partial}{\partial t_{10}}. \end{aligned} \tag{7.66}$$

- $n = 3$:

$$\begin{aligned} \mathsf{H} &= t_{00} \frac{\partial}{\partial t_{00}} - t_{30} \frac{\partial}{\partial t_{30}} + t_{10} \frac{\partial}{\partial t_{10}} - t_{20} \frac{\partial}{\partial t_{20}} + t_{11} \frac{\partial}{\partial t_{11}} - t_{21} \frac{\partial}{\partial t_{21}}, \\ \mathsf{F} &= t_{00} \frac{\partial}{\partial t_{10}} + t_{20} \frac{\partial}{\partial t_{30}}. \end{aligned} \tag{7.67}$$

By definition R, H and F satisfy the $\mathrm{sl}_2(\mathbb{C})$ algebra in all cases.

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