Extremal Problems in 3-uniform Hypergraphs

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Chapter 1

Introduction

We consider extremal problems for hypergraphs. A *k-uniform hypergraph* is an ordered pair $H = (V, E)$ where V is a (finite) set whose elements are called *vertices* and where $E \subseteq V^{(k)} = \{e \subseteq V : |e| = k\}$ is a set of k-element subsets. The elements in E are called *edges*. A 2-uniform hypergraph will be simply called *graph* and, since we focus mainly on 3-uniform hypergraphs, unless stated otherwise, a hypergraph will always be 3-uniform. Frequently we omit the parenthesis and the commas in the set notation, e.g. we denote the edge $\{x, y, z\}$ simply by xyz. Since isolated vertices (i.e. vertices not contained in any edge) do not play an important rôle in the following problems we frequently identify a hypergraph with its set of edges, and therefore, for instance we may write $E \subseteq H$ for a set of edges E and a hypergraph H. Given a k-uniform hypergraph H and subset of vertices $S \subseteq V(H)$ of size t we define the *neighbourhood* and the *t-degree* of S by

$$N_H(S) = \{e : S \subseteq e \in E(H)\} \quad \text{and} \quad d_H(S) = |N_H(S)|,$$

respectively, and if H is clear from the context we omit it from the notation. For $t = 1$ and $t = k - 1$ the t-degree are called *degree* and *codegree* respectively. Moreover, we denote the minimum t-degree among all sets of vertices of size t by $\delta_t(H)$.

Given a vertex set V of size n, a k-uniform hypergraph whose edge set is exactly $V^{(k)}$ is called *complete k-uniform hypergraph* and will be denoted by $K_n^{(k)}$ (for $k = 2$ we omit the superindex and simply write K_n). Given a set of n ordered vertices $v_1, v_2, \ldots, v_n$, the *(tight)*[1] *path* $P_n^{(3)}$, is a hypergraph whose edges consist of all sets of three consecutive vertices. In other words, the edge set is given by

$$E(P_n^{(3)}) = \{v_i v_{i+1} v_{i+2} \colon 1 \leqslant i \leqslant n-2\} .$$

We frequently say that $P_n^{(3)}$ is a (v_1, v_2)-(v_{n-1}, v_n)-path, and that (v_1, v_2) and (v_{n-1}, v_n) are respectively the *starting pair* and *ending pair* of the path, and they are both called *ends*. For simplicity we denote a path by listing its vertices. A *(tight)*[1] *cycle* $C_n^{(3)}$ is a path with the two additional edges $v_1 v_{n-1} v_n$ and $v_1 v_2 v_n$. We define paths and cycles analogously for graphs and denote them by P_n and C_n respectively.

We consider three classical extremal problems in which the general question consists in determining conditions for the existence of a substructure in a host graph or hypergraph. For *Turán-type problems* this substructure consist in a hypergraph of fixed size, and we normally look for a condition in the number of edges or density of the host hypergraph. In contrast, in *Dirac-type problems* the structure is spanning, that is, it contains as many vertices as the host hypergraph. Here a minimum degree condition is a common parameter to consider. Finally, we consider *decomposition problems* in which the main goal is to find a partition of all edges into parts with certain structure.

On all these three problems we obtained results for 3-uniform hypergraphs. In particular, we determined the Turán density of complete hypergraphs of size five in hypergraphs with quasirandom links (see Subsection 2.1 and in particular Theorem 2.1.3 and Corollary 2.1.5). Moreover, we obtained asymptotically optimal uniform density conditions that enforce the existence of a Hamilton cycle in hypergraphs with mild minimum degree conditions (see Subsection 2.2 and Theorem 2.2.2). Finally, we found an asymptotically optimal minimum codegree condition that enforces the existence of a decomposition into cycles of fixed length (see Subsection 2.3 and Theorem 2.3.1).

In the following three sections we present some of the basic concepts and prototypical examples of Turán-type, Dirac-type, and decompositions problems. While for graphs,

[1]In the literature there are other definitions of 'path' or 'cycle' in hypergraphs (non necessarily tight). In this work, unless stated otherwise, we assume all paths and cycles are tight.

several results on these problems are already obtained, extensions of these results to k-uniform hypergraphs are in general very difficult. We studied restrictions and variations of these problems in hypergraphs with certain quasirandom properties. These properties are discuss in detailed in Section 1.4.

1.1 Turán-type problems

Given a positive integer n and a hypergraph F, we define the *extremal number* $\mathrm{ex}(n, F)$ as the maximum number of edges that a hypergraph on n vertices can have without containing a copy of F as a subhypergraph. This is

$$\mathrm{ex}(n, F) = \max\{|E(H)| \colon H \text{ is a hypergraph with } |V(H)| = n \text{ and } F \nsubseteq H\}\,,$$

and moreover we define the *Turán density of* F as the limit

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{3}}\,, \tag{1.1.1}$$

which always exist since the sequence $\mathrm{ex}(n, F)/\binom{n}{3}$ is non-increasing.

For graphs, $\mathrm{ex}(n, F)$ and $\pi(F)$ are defined analogously, and due to the work of Turán [63], Erdős and Stone [21], and many others we are able to determine the value of the Turán density of any graph in terms of its chromatic number.[2] In particular the following beautiful formula (that first appeared in [59]) holds for every graph F

$$\pi(F) = \frac{\chi(F) - 2}{\chi(F) - 1}\,.$$

While that and many other related results were obtained for graphs, for hypergraphs our knowledge about $\pi(F)$ is very restricted. In fact, determining the Turán density of $K_4^{(3)}$ remains a major open problem in the area and was already posed by Turán eighty years ago [63]. Even for the hypergraph on four vertices and three edges, denoted by $K_4^{(3)-}$, the value its Turán density is still unknown. The best lower and upper bounds

[2]As usual, the chromatic number of a graph G is the minimum number of colours needed to colour the vertices of G in such a way that all edges contain vertices of two different colours.

obtained so far for these hypergraphs are given by

$$\frac{2}{7} \leqslant \pi(K_4^{(3)-}) \leqslant 0.2871 \qquad \text{and} \qquad \frac{5}{9} \leqslant \pi(K_4^{(3)}) \leqslant 0.5616\,.$$

The first lower bound comes from a construction by Frankl and Füredi [23] and the second is attributed to Turán (see for example [20]). The upper bounds were obtained through computer assisted proofs in [6, 49] based on the so called *flag algebra method* introduced by Razborov in [48].

Since this problem turns out to be so difficult, several variations were studied. Here we study a variant introduced by Erdős and Sós [19]. They suggested a version of this problem in which the host hypergraphs have the restriction of being 'uniformly dense' among linear sized sets of vertices (see Definition 3.2.2). Our first contribution concerns a variation of this original problem and we will describe it in detail in Section 2.1.

1.2 Dirac-type problems

For Turán-type problems we study conditions in hypergraphs that force the existence of a subgraph of fixed size. In contrast, one can study necessary conditions for the existence of a *spanning* subgraph. For example, Dirac [16] proved that for $n \geqslant 3$ every graph G with $\delta_1(G) \geqslant n/2$ contains a cycle covering all vertices, or *Hamilton cycle*. This result is best possible in terms of the minimum degree since a graph with its vertex set partitioned into two classes of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ and containing all edges with both vertices in the same class does not contain a Hamilton cycle. In the context of graphs many optimal results of this kind were obtain. For instance, minimum degree conditions forcing the existence of clique factors [30], F-factors [3], and powers of Hamilton cycles [39].

In contrast with Turán-type problems, there are some optimal results for hypergraphs extending Dirac's theorem. In particular, Rödl, Ruciński, and Szemerédi [58] proved an asymptotically optimal version of Dirac's result. They proved that every n-vertex hypergraph H with $\delta_2(H) \geqslant (\frac{1}{2} + o(1))n$ yields a Hamilton cycle. For their proof they introduced the so-called 'Absorption Method' (see a detailed description in Subsection 4.1). Later, in [52] Reiher, Rödl, Schacht, and Szemerédi proved the same conclusion for every n-vertex hypergraph H with $\delta_1(H) \geqslant (\frac{5}{9} + o(1))\binom{n}{2}$. This last result is also asymptotically optimal.

As in the graph case the optimal constructions contain large 'holes', meaning, large sets of vertices (or pairs of vertices) containing no edge. Lenz, Mubayi, and Mycroft [41] studied conditions for the existence of spanning structures in 'uniformly dense' hypergraphs (see Definition 3.2.2) in which these kind of holes are forbidden. We follow this lead and obtained asymptotically optimal conditions for the existence of Hamilton cycles in different kind of 'uniformly dense' hypergraphs. We present our results in Section 2.2.

1.3 Decomposition problems

Given a k-uniform hypergraph H, a *decomposition of* H is a collection of subhypergraphs such that every edge of H is covered exactly once. When these subhypergraphs are all isomorphic copies of a single hypergraph F we say that it is an *F-decomposition*, and that H is *F-decomposable*. Finding conditions for the existence of decompositions of hypergraphs is one of the oldest problems in combinatorics. In general, there are divisibility conditions which are obviously needed. For example, it is easy to see that for H to contain an F-decomposition, the number of edges of H has to be divisible by $|F|$. Further, if all vertices in F have degree d, it is clear that all degrees in H have to be divisible by d. We refer to these conditions as *trivial divisibility conditions*. In recent years several new decompositions results have been proven and several major open problems were resolved.

Kirkman [38] in 1847 proved that the complete graph K_n can be decomposed into K_3 for every odd n with $\binom{n}{2}$ being a multiple of 3. Wilson [64, 65] extended this result by proving that for every fixed graph F, the complete graph K_n contains an F-decomposition whenever the trivial divisibility conditions hold. More recently, Keevash [36,37] generalised these results for k-uniform hypergraphs when n is sufficiently large. Later, Glock, Kühn, Lo, and Osthus [28] proved the same theorem using a different method. With these results we have a very good understanding of F-decomposition problems for complete k-uniform hypergraphs.

We study decomposition problems for hypergraphs with large codegree. We say that a hypergraph is d-vertex-divisible when all vertices have degrees divisible by d. A 3-vertex-divisible hypergraph H whose number of edges is divisible by ℓ satisfies all trivial divisibility conditions for finding a $C_\ell^{(3)}$-decomposition, and in such a case we say H is $C_\ell^{(3)}$*-divisible*. Moreover, given n, $\ell \in \mathbb{N}$ the $C_\ell^{(3)}$*-decomposition threshold* $\delta_{C_\ell}^{(3)}(n)$ is the minimum d such that every $C_\ell^{(3)}$-divisible hypergraph H on n vertices with $\delta_2(H) \geqslant d$ contains a $C_\ell^{(3)}$-decomposition. Further, we define the following related parameter

$$\delta_{C_\ell}^{(3)} = \limsup_{n \longrightarrow \infty} \frac{\delta_{C_\ell}^{(3)}(n)}{n}. \tag{1.3.1}$$

For graphs, the parameters $\delta_{C_\ell}(n)$ and δ_{C_ℓ} can be defined analogously. Nash-Williams [42] showed that $\delta_{C_3}(n) \geqslant \frac{3}{4}n$, and proving that this inequality is optimal is one the most famous conjectures in the area. For longer odd cycles very recently Joos and Kühn [34] proved that $\delta_{C_\ell} \longrightarrow \frac{1}{2}$ as $\ell \longrightarrow \infty$, while $\delta_{C_\ell} > \frac{1}{2}$ for every odd ℓ.

For even cycles much more is known, and in fact Barber, Khün, Lo, and Osthus [8] proved that $\delta_{C_{2\ell}} = \frac{1}{2}$ for $\ell \geqslant 3$, and $\delta_{C_4} = \frac{3}{4}$. Remarkably, Taylor [60] determined the exact values of $\delta_{C_{2\ell}}(n)$ for every $\ell \neq 3$ and n sufficiently large.

For hypergraphs not much is known about codegree conditions for F-decompositions. From the general results obtained in [28] one can deduce that $\delta_{C_\ell}^{(3)} < 1$. In this thesis we determine $\delta_{C_\ell}^{(3)}$ for all but finitely many values of ℓ (see Theorem 2.3.1 in Section 2.3).

1.4 Restrictions on the host hypergraphs

Chung, Graham, and Wilson [14] studied several equivalent quasirandom properties for graphs. Based on one of those we introduce the following definition. We say a graph is (ϱ, d)-*quasirandom* if for every two sets of vertices X and Y satisfy that

$$e(X,Y) = |\{(x,y) \in X \times Y : ex \in E(G)\}| = d|X||Y| \pm \varrho|V(G)|^2 . \tag{1.4.1}$$

For extremal problems it is natural to study densities that force certain graph properties, and therefore, take graphs in which only the lower bound in (1.4.1) is consider. More precisely, given ϱ, $d \in (0, 1]$, an n-vertex graph G is (ϱ, d)-*bidense* if for every two sets of vertices X, $Y \subseteq V(G)$ we have

$$e(X, Y) \geqslant d|X||Y| \pm \varrho n^2 . \tag{1.4.2}$$

There are several notions that extend the previous definitions to k-uniform hypergraphs. We refer the reader to [1, 62] for a more general and detailed discussion on quasirandomness in k-uniform hypergraphs. For 3-uniform hypergraphs the following is a natural extension of (1.4.2).

Definition 1.4.1. Let ϱ, $d \in (0, 1]$ and let H be a hypergraph on n vertices. We say that H is $(\varrho, d, \therefore)$-*dense* if for every three sets of vertices X, Y, Z we have

$$e(X, Y, Z) = |\{(x, y, z) \in X \times Y \times Z : xyz \in E(H)\}| \geqslant d|X||Y||Z| - \varrho n^3.$$

It is easy to see that sufficiently large (ϱ, d)-bidense graphs contain a copy of every fixed graph F. This can be done by picking a vertex of average degree and since, the neighbourhood of that vertex is d-bidense as well, we can continue picking vertices inductively in successive neighbourhoods. The following construction due to Rödl [57] shows that this property does not hold for $(\varrho, d, \therefore)$-dense hypergraphs (and it also does not hold for the further extensions of (1.4.2) considered in Definition 1.4.3 below). In particular, it proves the existence of arbitrarily large $\therefore$-dense hypergraphs not containing $K_4^{(3)}$.

Example 1.4.2. Given a sufficiently large n, let $V = \{1, 2, \dots, n\}$ and for every pair $ij \in V^{(2)}$ assign the colour *red* or *blue* uniformly at random. We construct the hypergraph H whose edges are all triplets $i < j < k$ for which the colours ij and ik are different. Since this happen with probability $\frac{1}{2}$ a standard application of Azuma's inequality yields that for every $\varrho > 0$, H is asymptotically almost surely $(\varrho, 1/2, \therefore)$-dense.

Moreover, observe that given four vertices $i < j < k < \ell$, two of the three pairs ij, ik, and $i\ell$ have the same colour, and therefore one of the edges ijk, $ij\ell$, or $ik\ell$ is not present in H. This means that H does not contain $K_4^{(3)}$.

In this thesis we study following two further extensions of (1.4.2) for hypergraphs (also considered in [2, 50, 53, 55] among others).

Definition 1.4.3. Let ϱ, $d \in (0,1]$ and let H be a hypergraph on n vertices. We say that H is $(\varrho, d, \dot{\smile})$-*dense* if for every set of vertices X and every collection of ordered pairs of vertices $P \subseteq V \times V$ we have

$$e(X,P) = |\{(x,(y,z)) \in X \times P \colon xyz \in E(H)\}| \geqslant d|X||P| - \varrho n^3\,.$$

We say that H is $(\varrho, d, \wedge)$-*dense* if for every two collections of ordered pairs of vertices $P, Q \subseteq V \times V$ we have

$$e(P,Q) = |\{((x,y),(y,z)) \in P \times Q \colon xyz \in E(H)\}| \geqslant d|K_{\wedge}(P,Q)| - \varrho n^3\,, \qquad (1.4.3)$$

where $K_{\wedge}(P,Q) = \{((x,y),(y,z)) \in P \times Q\}$.

Observe that $\therefore$ is the weakest notion and $\wedge$ is the strongest. In these definitions, the symbols $\therefore$, $\dot{\smile}$, and $\wedge$ refer to the different choices for the vertex sets X, Y, or Z and the sets of pairs of vertices P or Q.

We are now ready to state the main result of this thesis.

Chapter 2

Main results

2.1 Turán densities in uniformly dense hypergraphs

As mentioned in Section 1.1 the problem of determining the Turán density $\pi(F)$ (see definition in (1.1.1)) is in general a difficult problem and, therefore, Erdős and Sós [19] consider a restricted version for uniformly dense hypergraphs. In light of Definitions 1.4.1 and 1.4.3 we consider the corresponding notions of Turán densities.

Definition 2.1.1. Given a hypergraph F and $\star \in \{\therefore, \dot{-}, \wedge\}$ let

$$\begin{aligned}\pi_\star(F) = \sup\{d \in [0,1] \colon & \text{ for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists an } F\text{-free,} \\ & (\eta, d, \star)\text{-dense hypergraph with at least } n \text{ vertices}\}\,.\end{aligned}$$

For a thorough discussion on Turán problems for uniformly dense hypergraphs we refer the reader to [50].

The original question from Erdős and Sós [19] asks for determining $\pi_{\therefore}(K_4^{(3)-})$. By a computer assisted proof Glebov, Král', and Volec [25] answered this question by showing

$$\pi_{\therefore}(K_4^{(3)-}) = \frac{1}{4}\,.$$

The same result was obtained later by Reiher, Rödl, and Schacht [54] with a different proof based on the regularity method for hypergraphs. For $K_4^{(3)}$ Example 1.4.2 shows

$$\frac{1}{2} \leqslant \pi_{\therefore}(K_4^{(3)})$$

and proving that this construction is optimal is a well known open problem (see for example [50]).

For $\pmb{\wedge}$-density much more is known. In fact, Reiher, Rödl, and Schacht [55] obtained a general upper bound for $\pi_{\wedge}(K_t^{(3)})$, which turned out to be best possible for all $t \leqslant 16$ except for $t = 5$, 9, and 10.

Theorem 2.1.2 (Reiher, Rödl, and Schacht [55])**.** *For every integer $r \geqslant 2$ we have*

$$\pi_{\wedge}(K_{2^r}^{(3)}) \leqslant \frac{r-2}{r-1}\,.$$

Moreover, we have

$$\begin{aligned} 0 &= \pi_{\wedge}(K_4^{(3)})\,, \\ \tfrac{1}{3} \leqslant \pi_{\wedge}(K_5^{(3)}) \leqslant \tfrac{1}{2} &= \pi_{\wedge}(K_6^{(3)}) = \cdots = \pi_{\wedge}(K_8^{(3)})\,, \\ \textit{and} \quad \tfrac{1}{2} \leqslant \pi_{\wedge}(K_9^{(3)}) \leqslant \pi_{\wedge}(K_{10}^{(3)}) \leqslant \tfrac{2}{3} &= \pi_{\wedge}(K_{11}^{(3)}) = \cdots = \pi_{\wedge}(K_{16}^{(3)})\,. \qquad \square \end{aligned}$$

We closed the gap for $\pi_{\wedge}(K_5^{(3)})$ and showed that the lower bound is best possible.

Theorem 2.1.3 (Berger, Piga, Reiher, Rödl, and Schacht [10])**.** *We have that*

$$\pi_{\wedge}(K_5^{(3)}) = \frac{1}{3}\,.$$

Theorem 2.1.3 has a consequence for hypergraphs with quasirandom links. For a hypergraph H and a vertex x, define the *link graph of* x, by the edges

$$H(x) = \{yz \in V^{(2)} \colon xyz \in E(H)\}. \tag{2.1.1}$$

One can check that if all the vertices of a hypergraph H have a (δ, d)-quasirandom link graph (see (1.4.1)), then H is $(f(\delta), d, \pmb{\wedge})$-dense, where $f(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. In fact, such hypergraphs would even satisfy in addition a matching upper bound for $e_{\wedge}(P, Q)$ in (1.4.3) and, hence, having quasirandom links is a stronger property. However, the lower bound construction for $\pi_{\wedge}(K_5^{(3)})$ (see below) has quasirandom links with density 1/3 and, therefore, Theorem 2.1.3 yields an asymptotically optimal result for such hypergraphs.

Example 2.1.4. *For a map* $\psi\colon V^{(2)} \longrightarrow \mathbb{Z}/3\mathbb{Z}$*, let* $H_\psi = (V, E)$ *be the hypergraph defined by*

$$xyz \in E \qquad \Longleftrightarrow \qquad \psi(xy) + \psi(xz) + \psi(zy) \equiv 1 \pmod 3\,. \tag{2.1.2}$$

Observe that for any set of five different vertices $U = \{u_1, u_2, u_3, u_4, u_5\}$ *the following equality follows by double counting*

$$\sum_{u_iu_ju_k \in U^{(3)}} \psi(u_iu_j) + \psi(u_iu_k) + \psi(u_ju_k) = 3 \sum_{u_iu_j \in U^{(2)}} \psi(u_iu_j)\,.$$

Since the second sum is zero modulo 3 *at least one of the ten triplets in the first sum fails to satisfy* (2.1.2)*. Consequently,* H_ψ *is* $K_5^{(3)}$*-free for every map* ψ*.*

Moreover, if ψ *is chosen uniformly at random, then following the lines of the proof of* [55, Proposition 13.1] *shows that for every fixed* $\delta > 0$ *and sufficiently large* $|V|$ *with high probability the hypergraph* H_ψ *has the property that all link graphs are* $(\delta, 1/3)$*-quasirandom.*

Summarising the discussion above we arrive at the following corollary, which in light of Example 2.1.4 is asymptotically best possible.

Corollary 2.1.5. *For every* $\varepsilon > 0$ *there exist* $\delta > 0$ *and an integer* n_0 *such that every hypergraph on at least* n_0 *vertices with all link graphs being* $(\delta, 1/3 + \varepsilon)$*-quasirandom contains a copy of* $K_5^{(3)}$*.* □

The proof of Theorem 2.1.3 is based on the regularity method for hypergraphs and we explain the details of the proof in Chapter 3.

2.2 Hamilton cycles in uniformly dense hypergraphs

In this section we study conditions in uniformly dense hypergraph for the existence of a Hamilton cycle. Observe that the notions of uniform density given in Definitions 1.4.1 and 1.4.3 cannot prevent the existence of an isolated vertex (which immediately forbids the existence of a Hamilton cycle). Therefore, minimum degree conditions have to be considered as well.

In the case of graphs, this problem is degenerate in the sense that arbitrarily small local density and a minimum degree at least $\Omega(n)$ is enough to force the existence of a Hamilton cycle. More precisely, using a result from Chvátal and Erdős [15], it is not hard to prove that for every $\alpha, d > 0$ there is an $\varrho > 0$ for which every large (ϱ, d)-bidense n-vertex graph with minimum degree at least αn contains a Hamilton cycle.

For hypergraphs, this line of research can be traced back to the work of Lenz, Mubayi and Mycroft [41] who studied conditions in uniformly dense hypergraphs for *loose Hamilton cycles*. For an even $n \in \mathbb{N}$ and an n-vertex hypergraph H, a loose Hamilton cycle is an ordering of the vertex set $V(H) = \{v_1, \dots, v_n\}$ such that $v_{2i-1}v_{2i}v_{2i+1} \in E(H)$ for every $1 \leqslant i \leqslant \frac{n}{2}$, where the indices are taken in $\mathbb{Z}/n\mathbb{Z}$. In [41] they proved that for arbitrarily small $d, \alpha > 0$ there is an $\varrho > 0$ such that every sufficiently large $(\varrho, d, \therefore)$-dense n-vertex hypergraph with minimum degree αn^2 contains a loose Hamilton cycle. As this density condition is the weakest one, this theorem implies the same result for the stronger notions $\dot{-}$ and $\wedge$.

Aigner-Horev and Levy [2] proved the same conclusion for tight cycles instead of loose cycles, but considering minimum codegree conditions instead of vertex degrees and assuming the strongest density notion $\wedge$. More precisely, they proved that for every $d, \alpha > 0$ there is a $\varrho > 0$ such that every sufficiently large $(\varrho, d, \wedge)$-dense hypergraph with minimum codegree αn contains a tight Hamilton cycle. It turns out that for the $\dot{-}$-density an analogous result is not possible due the following counterexample.

Example 2.2.1. Let G be a random graph[1] $G_{n-2,1/2}$ and define a hypergraph on the same set of vertices for which a triple of vertices is a edge if it forms a triangle in G or in $\overline{G}$. Observe that every cycle in H uses edges such that either all of them induce triangles in G or all of them induce triangles en $\overline{G}$. Finally, add two new vertices x and y in such a way that $N_H(x) = E(G)$ and $N_H(y) = E(\overline{G})$. Then x is covered only by cycles induced by triangles in G and y is covered only by cycles induced by triangles in $\overline{G}$. Hence H contains no tight Hamilton cycle. Moreover, if we add all the edges containing

[1] As usual $G_{n,p}$ represents a n-vertex random graph for which every edge is taken independently at random with probability p.

the pair $\{x, y\}$ then the hypergraph H only yields a Hamilton path, but not a Hamilton cycle. One can show that for every $\varrho > 0$ with high probability H is $(\varrho, 1/4, \therefore)$-dense and it has minimum degree $(1/4 - \varrho)\binom{n}{2}$ and even minimum codegree $(1/4 - \varrho)n$.

We proved that the previous example is essentially best possible.

Theorem 2.2.2 (Araújo, Piga, and Schacht [5])**.** *For every $\varepsilon > 0$ there exist $\varrho > 0$ and n_0 such that every $(\varrho, 1/4 + \varepsilon, \therefore)$-dense hypergraph H on $n \geqslant n_0$ vertices with $\delta_1(H) \geqslant \varepsilon\binom{n}{2}$ contains a Hamilton cycle.*

We also strengthen a result of Aigner-Horev and Levy [2] by showing that their codegree assumption for tight Hamilton cycles in Λ-dense hypergraphs can be relaxed to a minimum vertex degree assumption.

Theorem 2.2.3 (Araújo, Piga, and Schacht [5])**.** *For every d, $\alpha > 0$ there exist $\varrho > 0$ and n_0 such that every (ϱ, d, Λ)-dense hypergraph H on $n \geqslant n_0$ vertices with $\delta_1(H) \geqslant \alpha\binom{n}{2}$ contains a Hamilton cycle.*

Theorem 2.2.3 was conjectured in [2] and was obtained independently in [24]. We mainly focus on the proof of Theorem 2.2.2, but the proof of Theorem 2.2.3 is based on similar ideas. The details of the proofs of Theorems 2.2.2 and 2.2.3 are presented in Chapter 4.

2.3 Minimum codegree conditions for cycle decompositions

We study minimum codegree conditions for decompositions of hypergraphs into cycles. First, we determined $\delta^{(3)}_{C_\ell}$ for all but finitely many values of $\ell \in \mathbb{N}$.

Theorem 2.3.1 (Piga and Sanhueza-Matamala [44])**.** *Suppose ℓ satisfies one of the following:*

(i) ℓ is divisible by 3 and it is at least 9, or

(ii) $\ell \geqslant 10^7$.

Then $\delta^{(3)}_{C_\ell} = 2/3$.

As seen in Section 1.3, the C_ℓ-decomposition thresholds for graphs depend on the parity of ℓ. In contrast, Theorem 2.3.1 implies that $\delta^{(3)}_{C_\ell} = \frac{2}{3}$ for all sufficiently large ℓ, regardless of whether the cycle is tripartite or not.

We also study conditions for *cycle decompositions* – not necessarily of the same length–. The following is a simple corollary of Theorem 2.3.1.

Corollary 2.3.2. *Any* 3*-vertex-divisible hypergraph* H *with* $\delta_2(H) \geqslant (2/3 + o(1))|H|$ *has a cycle decomposition.*

Corollary 2.3.2 turned out to be best possible (see Theorem 2.3.4 below) and was conjectured[2] by Glock, Kühn, and Osthus [29, Conjecture 5.6].

A *tour* is a sequence of non-necessarily distinct vertices $v_1, \ldots, v_\ell$ such that, for every $1 \leqslant i \leqslant \ell$ the three consecutive vertices $v_i v_{i+1} v_{i+2}$ induce an edge (understanding the indices modulo ℓ) and moreover all of these edges are distinct. If a hypergraph H contains a tour that covers each edge exactly once, we call it *Euler tour* and we say that H is *Eulerian*.

With analogous definitions for graphs, Euler [22] famously proved that every Eulerian graph must be 2-vertex-divisible, and he stated (later proved by Hierholzer and Wiener [31]) that connected and 2-vertex-divisible graphs are Eulerian. Analogously, it is an easy observation that every Eulerian hypergraph must be 3-vertex-divisible. However, the characterisation of Eulerian hypergraphs is not as simple as for graphs. In [32] Jackson proved that $K_n^{(3)}$ is Eulerian for every n such that the degrees are divisible by 3. This was conjectured before by Chung, Diaconis, and Graham [13] and they even believe it should be true for complete k-uniform hypergraphs for every $k \geqslant 2$. Recently, Glock, Joos, Kühn, and Osthus [26] proved this conjectured for sufficiently large n.

[2]In a previous version of their paper their conjectured that cycle decompositions should already exists in hypergraphs with $\delta_2(H) \geqslant (1/2 + o(1))|H|$.

In fact, from a more general result in [26] one can deduced a 'minimum codegree' version of their theorem: there exists $c > 0$ such that any sufficiently large 3-vertex-divisible hypergraph H with $\delta_2(H) \geqslant (1-c)|H|$ is Eulerian. The constant c which they obtained is fairly small and therefore improving the minimum codegree condition becomes a natural problem. Their proof is based fundamentally on a reduction to the problem of finding a cycle decomposition. In the same fashion, we can use Theorem 2.3.1 to improve the minimum codegree condition.

Corollary 2.3.3. *Any 3-vertex-divisible hypergraph H with $\delta_2(H) \geqslant (2/3 + o(1))|H|$ is Eulerian.*

Glock, Joos, Kühn, and Osthus [26] conjectured that a minimum codegree condition of $(1/2 + o(1))|H|$ should be enough to guarantee the existence of Euler tours. However, Corollary 2.3.3 turned out to be asymptotically best possible (see Theorem 2.3.4 below).

We use the same construction to prove that Theorem 2.3.1 and Corollaries 2.3.2 and 2.3.3 are asymptotically best possible. Note that $C_\ell^{(3)}$-decompositions, cycles decompositions, and Eulerian tours are particular instances of decompositions into tours. Hence, the following theorem imply a lower bound construction for all the aforementioned results.

Theorem 2.3.4 (Piga and Sanhueza-Matamala [44])**.** *Let $\ell \geqslant 4$ and $n \geqslant 3(\ell + 3)$ be divisible by 18. Then there exists a C_ℓ-divisible hypergraph H on n vertices which satisfies $\delta_2(H) \geqslant (2n - 15)/3$, but does not admit a tour decomposition.*

The proof of Theorem 2.3.1 is based in the so called *iterative absorption method* and we present the details in Chapter 5. The proofs of Corollaries 2.3.2 and 2.3.3 and Theorem 2.3.4 are included in Chapter 5 as well.

Chapter 3

Turán density of $K_5^{(3)}$ in $\wedge$-dense hypergraphs

The main goal of this chapter is proving Theorem 2.1.3. The proof is based on the regularity method for hypergraphs and in the next section we recall the relevant concepts. We follow the ideas in [50] to transfer Theorem 2.1.3 to a statement for reduced hypergraphs $\mathcal{A}$ (see Proposition 3.1.3). The proof of Proposition 3.1.3 is based on a further reduction to the case in which there exists an underlying bicolouring of the pairs $V^{(2)}$, which corresponds to a bicolouring of the vertices in the reduced hypergraph $\mathcal{A}$ (see Proposition 3.1.5). We proved this proposition by analysing 'holes' in the hypergraph (see Section 3.3). Finally, we show that in the context of Theorem 2.1.3 such bicoloured reduced hypergraphs yield a $K_5^{(3)}$ (see Proposition 3.1.6). Sections 3.3 and 3.4 are devoted to the proofs of Propositions 3.1.5 and 3.1.6. Finally in Section 3.5 we discuss related open problems and variations of the main problem.

The work corresponding to this chapter was done in collaboration with Berger, Reiher, Rödl, and Schacht [10].

3.1 Hypergraph regularity and bicoloured reduced hypergraphs

Given a large hypergraph $H = (V, E)$, the regularity lemma for hypergraph provides a vertex partition $V_1 \dot\cup V_2 \dot\cup \ldots \dot\cup V_t = V$ together with partitions $\mathcal{P}^{ij}$ of the edges of the complete bipartite graphs between all $\binom{t}{2}$ pairs of classes V_i, V_j. Each class $P^{ij} \in \mathcal{P}^{ij}$ is ε-regular in the sense of Szemerédi's regularity lemma for graphs. Moreover, the hypergraph H is "regular" among most *triads*, i.e., among most of the tripartite graphs

$$P^{ijk}_{\alpha\beta\gamma} = P^{ij}_{\alpha} \cup P^{ik}_{\beta} \cup P^{jk}_{\gamma}$$

with $P^{ij}_{\alpha} \in \mathcal{P}^{ij}$, $P^{ik}_{\beta} \in \mathcal{P}^{ik}$, and $P^{jk}_{\gamma} \in \mathcal{P}^{jk}$. Roughly speaking, here "regular" means, that the hyperedges of H match the same proportion of triangles for every tripartite subgraph of such a triad.

Important structural properties of a hypergraph H after an application of the hypergraph regularity lemma can be captured by the *reduced hypergraph*, which can be viewed as a generalisation of the reduced graph in the context of Szemerédi's regularity lemma for graphs. Given a set of indices I and pairwise disjoint, non-empty sets of vertices $\mathcal{P}^{ij}$ for every pair of indices $ij \in I^{(2)}$, let for every triple of distinct indices $ijk \in I^{(3)}$ a tripartite hypergraph $\mathcal{A}^{ijk}$ with vertex classes $\mathcal{P}^{ij}$, $\mathcal{P}^{ik}$, and $\mathcal{P}^{jk}$ be given. We consider the disjoint union of all those hyperedges and, hence, we obtain a $\binom{|I|}{2}$-partite hypergraph $\mathcal{A}$ with

$$V(\mathcal{A}) = \dot\bigcup_{ij \in I^{(2)}} \mathcal{P}^{ij} \qquad \text{and} \qquad E(\mathcal{A}) = \dot\bigcup_{ijk \in I^{(3)}} E(\mathcal{A}^{ijk})\,.$$

We say $\mathcal{A}$ is a *reduced* hypergraph with *index set* I, *vertex classes* $\mathcal{P}^{ij}$, and *constituents* $\mathcal{A}^{ijk}$. In this work the index set I will often be an ordered set and we may assume $I \subseteq \mathbb{N}$.

An application of the hypergraph regularity lemma to a given hypergraph H naturally defines a reduced hypergraph $\mathcal{A}$ in which the vertices $P^{ij} \in \mathcal{P}^{ij}$ represent a set of pairs between the vertex classes V_i and V_j. Moreover, a hyperedge $P^{ij}_{\alpha} P^{ik}_{\beta} P^{jk}_{\gamma}$ in the reduced hypergraph signifies that H is regular and dense on the triad $P^{ijk}_{\alpha\beta\gamma}$.

As mentioned above the properties of the hypergraph H are often transferred to the reduced hypergraph. We consider $\boldsymbol{\wedge}$-dense and $K_5^{(3)}$-free hypergraphs H and below we discuss the corresponding properties for the reduced hypergraph $\mathcal{A}$ after an appropriate application of the hypergraph regularity lemma.

Roughly speaking, the $\boldsymbol{\wedge}$-density condition translates into a minimal codegree condition for almost all pairs of vertices from different vertex classes in almost all constituents of the reduced graphs. However, one can always move to a large reduced hypergraph in which *all* pairs of vertices from different vertex classes in the same constituent have large codegree (see [50, Lemma 4.2] for details). This inspires the following definition of $(d, \boldsymbol{\wedge})$-density for reduced hypergraphs.

Definition 3.1.1. *For $d \in [0, 1]$, we say that a reduced hypergraph $\mathcal{A}$ with index set I is $(d, \boldsymbol{\wedge})$-*dense*, if for every $ijk \in I^{(3)}$ and all vertices $P^{ij} \in \mathcal{P}^{ij}$ and $P^{ik} \in \mathcal{P}^{ik}$ we have*

$$d\big(P^{ij}, P^{ik}\big) = \big|\{P^{jk} \in \mathcal{P}^{jk} \colon P^{ij} P^{ik} P^{jk} \in E(\mathcal{A}^{ijk})\}\big| \geqslant d\,|\mathcal{P}^{jk}|\,.$$

As discussed above (see [50, Section 5] for details), an appropriate application of the hypergraph regularity lemma to a $(\eta, d + \varepsilon, \boldsymbol{\wedge})$-dense hypergraph H yields a $(d + \varepsilon/2, \boldsymbol{\wedge})$-dense reduced hypergraphs $\mathcal{A}$. The following definition allows us to transfer $K_5^{(3)}$-freeness of H to the reduced hypergraph $\mathcal{A}$.

Definition 3.1.2. *We say a reduced hypergraph $\mathcal{A}$ with index set I* supports *a clique $K_\ell^{(3)}$ if there is a ℓ-element subset $J \subseteq I$ and vertices $P^{ij} \in \mathcal{P}^{ij}$ for every $ij \in J^{(2)}$ such that*

$$P^{ij} P^{ik} P^{jk} \in E(\mathcal{A}^{ijk})$$

for all $ijk \in J^{(3)}$.

Note that, if the reduced hypergraph $\mathcal{A}$ defined from a hypergraph H through an appropriate application of the regularity lemma supports a $K_5^{(3)}$, then the embedding/counting lemma yields a $K_5^{(3)} \subseteq H$, Hence, $K_5^{(3)}$-free hypergraphs H have reduced hypergraphs that do not support $K_5^{(3)}$.

The discussion above reduces the proof of Theorem 2.1.3 to the following statement for reduced hypergraphs.

Proposition 3.1.3. *For every $\varepsilon > 0$ every sufficiently large $\left(\frac{1}{3} + \varepsilon, \boldsymbol{\wedge}\right)$-dense reduced hypergraph $\mathcal{A}$ supports a $K_5^{(3)}$.*

For the proof of Proposition 3.1.3 we proceed by contradiction and assume that for some $\varepsilon > 0$ there are $(\frac{1}{3} + \varepsilon, \boldsymbol{\wedge})$-dense reduced hypergraphs of unbounded size that do not support $K_5^{(3)}$. This motivates the following notion.

Definition 3.1.4. *For $\varepsilon > 0$ we say a reduced hypergraph $\mathcal{A}$ is* ε-wicked *if it is $(\frac{1}{3}+\varepsilon, \boldsymbol{\wedge})$-dense and fails to support a $K_5^{(3)}$. In case ε is clear from the context or irrelevant, we may sometimes suppress it and call ε-wicked reduced hypergraphs simply* wicked.

Proposition 3.1.3 asserts that wicked reduced hypergraphs do not exist and the proof is divided in two main parts. First we reduce the problem to the case in which the reduced hypergraph $\mathcal{A}$ on some index set I can be *bicoloured*. By this we mean that there is a colouring $\varphi\colon V(\mathcal{A}) \longrightarrow \{\text{red}, \text{blue}\}$ of the vertices such that for every $ij \in I^{(2)}$ we have

$$\varphi^{-1}(\text{red}) \cap \mathcal{P}^{ij} \neq \varnothing \qquad \text{and} \qquad \varphi^{-1}(\text{blue}) \cap \mathcal{P}^{ij} \neq \varnothing \tag{3.1.1}$$

and there are no hyperedges in $\mathcal{A}$ with all three vertices of the same colour. Given such a colouring φ, we define the *minimum monochromatic codegree density of $\mathcal{A}$ and φ* by

$$\tau_2(\mathcal{A}, \varphi) = \min_{ijk \in I^{(3)}} \min \left\{ \frac{d(P^{ij}, P^{ik})}{|\mathcal{P}^{jk}|} \colon P^{ij} \in \mathcal{P}^{ij},\ P^{ik} \in \mathcal{P}^{ik},\ \text{and } \varphi(P^{ij}) = \varphi(P^{ik}) \right\}. \tag{3.1.2}$$

The following proposition reduceds Proposition 3.1.3 to bicoloured reduced hypergraphs.

Proposition 3.1.5. *Given $\varepsilon > 0$ and $t \in \mathbb{N}$, let $\mathcal{A}$ be a sufficiently large ε-wicked reduced hypergraph. There exists a reduced hypergraph $\mathcal{A}_\star$ with index set of size at least t not supporting a $K_5^{(3)}$ and a bicolouring φ of $\mathcal{A}_\star$ such that $\tau_2(\mathcal{A}_\star, \varphi) \geqslant \frac{1}{3} + \frac{\varepsilon}{8}$.*

For the proof of Proposition 3.1.5 we mainly analyse holes in wicked reduced hypergraphs, i.e., subsets of vertices with very low density. It turns out that two essentially disjoint holes can be used to define an appropriate colouring on a subhypergraph of $\mathcal{A}$ (see Section 3.3). The next proposition completes the proof of Proposition 3.1.3

by contradicting the conclusion of Proposition 3.1.5, which shows that large wicked hypergraphs indeed do not exist.

Proposition 3.1.6. *For every $\varepsilon > 0$ every sufficiently large bicoloured reduced hypergraph $\mathcal{A}$ with $\tau_2(\mathcal{A}, \varphi) \geqslant \frac{1}{3} + \varepsilon$ supports a $K_5^{(3)}$.*

The proof of Proposition 3.1.6 is deferred to Section 3.4.

3.2 Preliminaries

In this section we introduce some necessary definitions and properties for reduced hypergraphs.

3.2.1 Transversals and cherries

We start with the following notion for reduced hypergraphs $\mathcal{A}$ with index set I. For $J \subseteq I$ we refer to a sequence of vertices $\mathcal{Q}(J) = (Q^{ij})_{ij \in J^{(2)}}$ with $Q^{ij} \in \mathcal{P}^{ij}$ as a *J-transversal*. Similarly, for two disjoint subsets of indices $K, L \subseteq I$ we say that $\mathcal{Q}(K, L) = (Q^{k\ell})_{(k,\ell) \in K \times L}$ is a *(K, L)-transversal* when $Q^{k\ell} \in \mathcal{P}^{k\ell}$. For subsets $J_\star \subseteq J$, $K_\star \subseteq K$, and $L_\star \subseteq L$ we refer to the transversals $\mathcal{Q}(J_\star) \subseteq \mathcal{Q}(J)$ and $\mathcal{Q}(K_\star, L_\star) \subseteq \mathcal{Q}(K, L)$ (defined in the obvious way) as *restricted transversal*. Whenever the sets J, K, $L \subseteq I$ are clear from the context we may omit them and write *transversal* to refer to J-transversals or to (K, L)-transversals.

Since we are working with $\boldsymbol{\wedge}$-dense reduced hypergraphs (see Definition 3.1.1) pairs of vertices sharing one index will play an important rôle. More precisely, given indices $ijk \in I^{(3)}$ with $i < j < k$ and given vertices $P^{ij} \in \mathcal{P}^{ij}$, $P^{ik} \in \mathcal{P}^{ik}$, and $P^{jk} \in \mathcal{P}^{jk}$ we say that the ordered pair (P^{ij}, P^{ik}) is a *left cherry*, the ordered pair (P^{ik}, P^{jk}) is a *right cherry*, and the ordered pair (P^{ij}, P^{jk}) is a *middle cherry*. Often we refer to them simply as *cherries*.

For indices $ijk \in I^{(3)}$ and a set of cherries $\mathscr{C}^{ijk} \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{ik}$ we say a transversal $\mathcal{Q}$ *avoids* $\mathscr{C}^{ijk}$ if the pair $(Q^{ij}, Q^{ik}) \notin \mathscr{C}$ for $Q^{ij}, Q^{ik} \in \mathcal{Q}$. Furthermore, for $J \subset I$ we say $\mathcal{Q}$

avoids a set of cherries $\mathscr{C} = \bigcup_{ijk\in J^{(3)}} \mathscr{C}^{ijk}$, if it avoids $\mathscr{C}^{ijk}$ for every $ijk \in J^{(3)}$. We extend this definition to (K,L)-transversals in an analogous way.

Lemma 3.2.1. *For every $t \in \mathbb{N}$ and $\delta > 0$ there is a $\mu > 0$ such that the following holds. Suppose that $\mathcal{A}$ is a reduced hypergraph with an index set I of size $|I| = t$ and given*

(a) sets $\mathcal{Q}^{ij} \subseteq \mathcal{P}^{ij}$ of size at least $\delta|\mathcal{P}^{ij}|$ for every $ij \in I^{(2)}$,

(b) sets of left cherries $\mathscr{L}^{ijk} \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{ik}$ of size at most $\mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$ for every $ijk \in I^{(3)}$,

(c) and sets of right cherries $\mathscr{R}^{ijk} \subseteq \mathcal{P}^{ik} \times \mathcal{P}^{jk}$ of size at most $\mu|\mathcal{P}^{ik}||\mathcal{P}^{jk}|$ for every $ijk \in I^{(3)}$.

Then there is a transversal $\mathcal{Q}(I) = (Q^{ij})_{ij\in I^{(2)}}$ with $Q^{ij} \in \mathcal{Q}^{ij}$ avoiding $\mathscr{L} = \bigcup_{ijk\in I^{(3)}} \mathscr{L}^{ijk}$ and $\mathscr{R} = \bigcup_{ijk\in I^{(3)}} \mathscr{R}^{ijk}$.

Lemma 3.2.1 follows from a simple counting argument.

Proof. Obviously by assumption (a) there are $\delta^{\binom{t}{2}} \prod_{ij\in I^{(2)}} |\mathcal{P}^{ij}|$ transversals with all vertices in $\bigcup_{ij\in I^{(2)}} \mathcal{Q}^{ij}$. On the other hand, it follows from assumptions (b) and (c) that at most $2\binom{t}{3}\mu \prod_{ij\in I^{(2)}} |\mathcal{P}^{ij}|$ of these transversals may contain a left or a right cherry from $\mathscr{L} \cup \mathscr{R}$. Consequently, the lemma holds for sufficiently small $\mu = \mu(t,\delta)$. $\square$

3.2.2 Inhabited transversals in ∴-dense reduced hypergraphs

We shall utilise the main result from [56] for ∴-dense hypergraphs. As discussed in Section 3.1 uniform density conditions translates to reduced hypergraphs through an appropriate application of the regularity lemma for hypergraphs. The following correspond to the notion of ∴-density in the context of reduced hypergraphs (see, e.g., [50, 56] for more details).

Definition 3.2.2. *Let $\mu > 0$ and $\mathcal{A}$ be a reduced hypergraph on an index set I. We say that $\mathcal{A}$ is $(\mu, ∴)$-*dense*, if for every $ijk \in I^{(3)}$ we have*

$$e(\mathcal{A}^{ijk}) \geqslant \mu\,|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,. \tag{3.2.1}$$

Further, for disjoint subsets of indices $K, L, M \subseteq I$ we say that $\mathcal{A}$ is $(\mu, \therefore)$-tridense on K, L, M, *if* (3.2.1) *holds for every triple (i, j, k) in $K \times L \times M$.*

Note that by definition every $(d, \wedge)$-dense reduced hypergraph is also $(d, \therefore)$-dense. The following result from [56, Lemma 3.1], states the existence of transversals containing edges in $\therefore$-dense reduced hypergraphs.

Theorem 3.2.3. *Let $t \in \mathbb{N}$, $\mu > 0$, and let $\mathcal{A}$ be a $(\mu, \therefore)$-dense reduced hypergraph on a sufficiently large index set I. There exist a set $I_\star \subseteq I$ of size t and three transversals $\mathcal{Q}(I_\star)$, $\mathcal{R}(I_\star)$, and $\mathcal{S}(I_\star)$ such that $Q^{ij}R^{ik}S^{jk} \in E(\mathcal{A})$ for every $i < j < k$ in $I_\star$.* $\square$

Triples of transversals satisfying the conclusion of Theorem 3.2.3 will play an important rôle here and we motivates the following definition.

Definition 3.2.4 (inhabited triple of transversals)**.** *Given a reduced hypergraph $\mathcal{A}$ with index set I. We say a triple of transversals $\mathcal{Q}(J)\mathcal{R}(J)\mathcal{S}(J)$ for some $J \subseteq I$ is* inhabited *if for every $i < j < k$ in J we have $Q^{ij}R^{ik}S^{jk} \in E(\mathcal{A})$.*

Similarly, for pairwise disjoint sets of indices K, L, $M \subseteq I$, we say a triple of transversals $\mathcal{Q}(K, L)\mathcal{R}(K, M)\mathcal{S}(L, M)$ is inhabited *if for every $k \in K$, $\ell \in L$, and $m \in M$ we have $Q^{k\ell}R^{km}S^{\ell m} \in E(\mathcal{A})$.*

Here we will also need a version of Theorem 3.2.3 in which the resulting transversals avoid given sets of forbidden cherries.

Lemma 3.2.5. *For $t \in \mathbb{N}$ and $\mu > 0$ there is $\mu' > 0$ such that the following holds. Let $\mathcal{A}$ be a $(\mu, \therefore)$-dense reduced hypergraph on sufficiently large index set I and for all $i < j < k$ in I let $\mathscr{L}^{ijk} \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{ik}$ and $\mathscr{R}^{ijk} \subseteq \mathcal{P}^{ik} \times \mathcal{P}^{jk}$ be sets of left and right cherries satisfying*

$$|\mathscr{L}^{ijk}| \leqslant \mu' |\mathcal{P}^{ij}||\mathcal{P}^{ik}| \qquad \text{and} \qquad |\mathscr{R}^{ijk}| \leqslant \mu' |\mathcal{P}^{ik}||\mathcal{P}^{jk}| \,.$$

There exist a set $I_\star \subseteq I$ of size t and a triple of inhabited transversals $\mathcal{Q}(I_\star)$, $\mathcal{R}(I_\star)$, and $\mathcal{S}(I_\star)$ avoiding the cherries $\mathscr{L}^{ijk}$ and $\mathscr{R}^{ijk}$, for every $ijk \in I_\star^{(3)}$.

For the proof of Lemma 3.2.5 we will consider random preimages of reduced hypergraphs.

Definition 3.2.6 (random preimages of reduced hypergraphs). *Given a reduced hypergraph $\mathcal{A}$ with index set I and vertex classes $\mathcal{P}^{ij}$ for $ij \in I^{(2)}$ and given an integer $\ell \geqslant 1$, we fix $\binom{|I|}{2}$ disjoint sets $\mathcal{P}^{ij}_\bullet$ of size ℓ and consider the uniform probability space $\mathfrak{A}(\mathcal{A}, \ell)$ of all mappings h from $\bigcup_{ij \in I^{(2)}} \mathcal{P}^{ij}_\bullet$ to $\bigcup_{ij \in I^{(2)}} \mathcal{P}^{ij}$ satisfying*

$$h(\mathcal{P}^{ij}_\bullet) \subseteq \mathcal{P}^{ij}$$

for every $ij \in I^{(2)}$.

With each such map h we associate a reduced hypergraph $\mathcal{A}_h$ with index set I and vertex classes $\mathcal{P}^{ij}_\bullet$ for $ij \in I^{(2)}$, where edges are defined by

$$P^{ij}_\bullet P^{ik}_\bullet P^{jk}_\bullet \in E(\mathcal{A}^{ijk}_h)\,, \text{ whenever } \quad h(P^{ij}_\bullet)h(P^{ik}_\bullet)h(P^{jk}_\bullet) \in E(\mathcal{A}^{ijk})$$

for all $ijk \in I^{(3)}$ and all $P^{ij}_\bullet \in \mathcal{P}^{ij}_\bullet$, $P^{ik}_\bullet \in \mathcal{P}^{ik}_\bullet$, and $P^{jk}_\bullet \in \mathcal{P}^{jk}_\bullet$. In particular, h signifies a homomorphism $\mathcal{A}_h \longrightarrow \mathcal{A}$.

Below we pass to such a random preimage $\mathcal{A}_h$ of $\mathcal{A}$ for sufficiently large ℓ, which will allow us to deduce Lemma 3.2.5 for $\mathcal{A}$ by applying Theorem 3.2.3 to $\mathcal{A}_h$.

Proof of Lemma 3.2.5. Given $t \in \mathbb{N}$ and $\mu > 0$, let t_1 be sufficiently large for an application of Theorem 3.2.3 with t and $\frac{\mu}{2}$ in place of t and μ. Further, we fix an integer ℓ and $\mu' > 0$ to satisfy the hierarchy

$$\mu, t_1^{-1} \gg \ell^{-1} \gg \mu'.$$

Finally, let $\mathcal{A}$ be a reduced hypergraph as in the statement of Lemma 3.2.5 and we may assume that its index set I is of size t_1.

Similar as in the proof of [50, Lemma 4.2] we consider the probability space $\mathfrak{A}(\mathcal{A}, \ell)$ from Definition 3.2.6 and we shall prove that with high probability the associated reduced hypergraph $\mathcal{A}_h$ is $(\frac{\mu}{2}, \therefore)$-dense and no cherry has its image in the sets $\mathscr{L}^{ijk}$ or $\mathscr{R}^{ijk}$.

For every constituent $\mathcal{A}_h^{ijk}$ the random variable $e(\mathcal{A}_h^{ijk})$ satisfies $\mathbb{E}[e(\mathcal{A}_h^{ijk})] = \mu\ell^3$ and by Azuma's inequality (see, e.g. [33, Corollary 2.27]) we obtain

$$\mathbb{P}\big(\mathcal{A}_h \text{ is not } (\tfrac{\mu}{2},\therefore)\text{-dense}\big) \leqslant \sum_{ijk\in I^{(3)}} \mathbb{P}\big(e(\mathcal{A}_h^{ijk}) < \tfrac{\mu}{2}\ell^3\big) \leqslant \binom{t_1}{3}\exp(-\tfrac{\mu^2\ell}{24})\,.$$

Moreover, since $\mathscr{L}^{ijk} \leqslant \mu'|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$, the probability that the image of some cherry lies in those sets is bounded by

$$\sum_{ijk\in I^{(3)}} \mathbb{P}\Big(h(P_\bullet^{ij})h(P_\bullet^{ik}) \in \mathscr{L}^{ijk} \text{ for some } P_\bullet^{ij}P_\bullet^{ik} \in \mathcal{P}_\bullet^{ij} \times \mathcal{P}_\bullet^{ik}\Big) \leqslant \binom{t_1}{3}\mu'\ell^2\,.$$

The same inequality holds for the sets $\mathscr{R}^{ijk}$ and note that by our choice of variables

$$\binom{t_1}{3}\exp(-\tfrac{\mu^2\ell}{8}) + 2\binom{t_1}{3}\mu'\ell^2 < 1.$$

Therefore, we can fix an h such that $\mathcal{A}_h$ is $(\frac{\mu}{2},\therefore)$-dense and no cherry has its image in the sets $\mathscr{L}^{ijk}$ or $\mathscr{R}^{ijk}$.

Applying Theorem 3.2.3 to $\mathcal{A}_h$ yields a set $I_\star \subseteq I$ of size t and three transversals $\mathcal{Q}_h(I_\star)$, $\mathcal{R}_h(I_\star)$, and $\mathcal{S}_h(I_\star)$ such that $Q_h^{ij}R_h^{ik}S_h^{jk} \in E(\mathcal{A}_h)$ for every $i<j<k$ in $I_\star$. It is easy to see that the transversals

$$\mathcal{Q}(I_\star) = \big(h(Q_h^{ij})\big)_{ij\in I_\star^{(2)}}\,, \quad \mathcal{R}(I_\star) = \big(h(R_h^{ij})\big)_{ij\in I_\star^{(2)}}\,, \quad \text{and} \quad \mathcal{S}(I_\star) = \big(h(S_h^{ij})\big)_{ij\in I_\star^{(2)}}$$

satisfy the desired properties and the lemma follows. □

3.2.3 Partite versions

We will also need a slightly more technical variant of Theorem 3.2.3, which guarantees the existence of inhabited triples of transversals in the intersection of multiple $\therefore$-tridense reduced subhypergraphs.

Lemma 3.2.7. *Let t, $r \in \mathbb{N}$, $\mu > 0$ there is an $s \in \mathbb{N}$ such that the following is true. Let $\mathcal{A}$ be a reduced hypergraph on index set I. Suppose that there are*

(*a*) *disjoint subsets of indices $K, L, M \subseteq I$ each of size s,* (*b*) *sets* $X_1, \ldots, X_r$ *of size* s*, and*

(*c*) *for every* r*-tuple* $\vec{x} \in \prod_{i\in[r]} X_i$ *a* $(\mu, \therefore)$*-tridense subhypergraph* $\mathcal{A}_{\vec{x}} \subseteq \mathcal{A}$ *on* K, L, M*.*

Then, there are

(*i*) *subsets* $K_\star \subseteq K, L_\star \subseteq L, M_\star \subseteq M$ *of size* t*,*

(*ii*) *subsets* $Y_i \subseteq X_i$ *of size* t *for every* $i \in [r]$*, and*

(*iii*) *there is a triple of transversals* $\mathcal{Q}(K_\star, L_\star)\mathcal{R}(K_\star, M_\star)\mathcal{S}(L_\star, M_\star)$*, which is inhabited in* $\mathcal{A}_{\vec{y}}$*, for every* $\vec{y} \in \prod_{i\in[r]} Y_i$*.*

The proof of Lemma 3.2.7 relies on repeated applications of the following auxiliary lemma.

Lemma 3.2.8. *Let* t*,* $r \in \mathbb{N}$*,* $\mu > 0$ *there is an* $s \in \mathbb{N}$ *such that the following is true. Let* $\mathcal{A}$ *be a reduced hypergraph on index set* I*. Suppose that there are*

(*a*) *disjoint subsets of indices* $K, L \subseteq I$ *each of size* s*,*

(*b*) *sets* $X_1, \ldots, X_r$ *of size* s*, and*

(*c*) *for every* r*-tuple* $\vec{x} \in \prod_{i\in[r]} X_i$*, every* $k \in K$*, and every* $\ell \in L$ *we have a subset* $\mathcal{P}^{kl}_{\vec{x}} \subset \mathcal{P}^{kl}$ *of size at least* $\mu|\mathcal{P}^{k\ell}|$*.*

Then, there are

(*i*) *subsets* $K' \subseteq K, L' \subseteq L$ *of size* t*,*

(*ii*) *subsets* $X'_i \subseteq X_i$ *of size* t *for every* $i \in [r]$*, and*

(*iii*) *a transversal* $\mathcal{Q}(K', L')$ *such that for every* $\vec{x} \in \prod_{i\in[r]} X'_i$ *and every* $k\ell \in K' \times L'$ *we have that* $Q^{k\ell} \in \mathcal{P}^{k\ell}_{\vec{x}}$*.*

Proof. Given t, $r \in \mathbb{N}, \mu > 0$ we fix an integer s such that

$$t, r, \mu^{-1} \ll s\,. \tag{3.2.2}$$

Let $\mathcal{A}$ be a reduced hypergraph as in the statement of the lemma and further let $K' \subseteq K$, and $L' \subseteq L$ be arbitrary subsets of size t.

For every (K', L')-transversal $\mathcal{Q}$ we consider the set

$$\mathfrak{x}(\mathcal{Q}) = \left\{ \vec{x} \in \prod_{i \in [r]} X_i \colon Q^{k\ell} \in \mathcal{P}^{k\ell}_{\vec{x}} \text{ for all } k\ell \in K' \times L' \right\}.$$

Summing over all (K', L')-transversal $\mathcal{Q}$ assumption (c) yields

$$\sum_{\mathcal{Q}} |\mathfrak{x}(\mathcal{Q})| = \sum_{\vec{x} \in \prod_{i \in [r]} X_i} \prod_{k\ell \in K' \times L'} \left|\mathcal{P}^{k\ell}_{\vec{x}}\right| \geqslant \mu^{t^2} \prod_{k\ell \in K' \times L'} \left|\mathcal{P}^{k\ell}\right| \prod_{i \in [r]} |X_i| .$$

Hence, we can fix a (K', L')-transversal $\mathcal{Q}$ such that

$$|\mathfrak{x}(\mathcal{Q})| \geqslant \mu^{t^2} \prod_{i \in [r]} |X_i| .$$

We may view $\mathfrak{x}(\mathcal{Q})$ as an r-partite r-uniform hypergraph of density at least μ^{t^2} on vertex classes of size s. Consequently, a result of Erdős [18] combined with the hierarchy (3.2.2) yields subsets $X_i' \subseteq X_i$ of size t for every $i \in [r]$ such that

$$\prod_{i \in [r]} X_i' \subseteq \mathfrak{x}(\mathcal{Q}) ,$$

which concludes the proof of Lemma 3.2.8. □

Next we derive Lemma 3.2.7.

Proof of Lemma 3.2.7. Given $t, r \in \mathbb{N}, \mu > 0$ we fix integers s, s', and s'' such that

$$t, r, \mu^{-1} \ll s'' \ll s' \ll s$$

and let $\mathcal{A}$ be a reduced hypergraph as in the statement of the lemma.

We will prove the lemma by applying Lemma 3.2.8 three times, once for every pair from K, L, and M.

For every $k \in K$, $\ell \in L$, $m \in M$, and every $\vec{x} \in \prod_{i \in [r]} X_i$ we consider the set

$$\mathcal{P}^{k\ell}_{(\vec{x},m)} = \left\{ P^{k\ell} \in \mathcal{P}^{k\ell} \colon \left| N_{\mathcal{A}^{k\ell m}_{\vec{x}}}(P^{k\ell}) \right| > \tfrac{\mu}{2} |\mathcal{P}^{km}||\mathcal{P}^{\ell m}| \right\}.$$

Since $A_{\vec{x}}$ is $(\mu, \therefore)$-tridense we have

$$e(A_{\vec{x}}^{k\ell m}) \geqslant \mu|\mathcal{P}^{k\ell}||\mathcal{P}^{km}||\mathcal{P}^{\ell m}|$$

and a standard averaging argument implies

$$\left|\mathcal{P}^{k\ell}_{(\vec{x},m)}\right| \geqslant \tfrac{\mu}{2}\left|\mathcal{P}^{k\ell}\right|.$$

Lemma 3.2.8 applied with s', $r+1$, and $\frac{\mu}{2}$ in place of t, r, and μ and with $X_{r+1} = M$ yields subsets $K' \subseteq K$, $L' \subseteq L$, $M' \subseteq M$, and $X'_i \subseteq X_i$ for every $i \in [r]$, of size s' and a transversal $\mathcal{Q}(K', L')$ such that for every $(\vec{x}, m) \in \prod_{i\in[r]} X'_i \times M'$ and every $k\ell \in K' \times L'$ we have that $Q^{k\ell} \in \mathcal{P}^{k\ell}_{(\vec{x},m)}$.

For the second application of Lemma 3.2.8 we consider the set

$$\mathcal{P}^{km}_{(\vec{x},\ell)} = \left\{P^{km} \in \mathcal{P}^{km} \colon \left|N_{\mathcal{A}_{\vec{x}}^{k\ell m}}(Q^{k\ell}, P^{km})\right| \geqslant \tfrac{\mu}{4}|\mathcal{P}^{\ell m}|\right\}$$

for every $k \in K'$, $\ell \in L'$, $m \in M'$, and every $\vec{x} \in \prod_{i\in[r]} X'_i$. By our choice of the transversal $\mathcal{Q}(K', L')$ we have

$$|N_{\mathcal{A}_{\vec{x}}^{k\ell m}}(Q^{k\ell})| \geqslant \tfrac{\mu}{2}|\mathcal{P}^{km}||\mathcal{P}^{\ell m}|$$

and, as before, this implies

$$|\mathcal{P}^{km}_{(\vec{x},\ell)}| \geqslant \tfrac{\mu}{4}|\mathcal{P}^{km}|.$$

Again, we apply Lemma 3.2.8, now with s'', $r+1$, and $\frac{\mu}{4}$ in place of t, r, and μ and with $X'_{r+1} = L'$, to reach subsets $K'' \subseteq K'$, $L'' \subseteq L'$, $M'' \subseteq M'$, and $X''_i \subseteq X'_i$, for every $i \in [r]$, of size s'' and a transversal $\mathcal{R}(K'', M'')$ such that for every $(\vec{x}, \ell) \in \prod_{i\in[r]} X''_i \times L''$ and every $km \in K'' \times M''$ it is $R^{km} \in \mathcal{P}^{km}_{(\vec{x},\ell)}$.

Last, we consider the set

$$\mathcal{P}^{\ell m}_{(\vec{x},k)} = N_{\mathcal{A}_{\vec{x}}^{k\ell m}}(Q^{k\ell}, R^{km})$$

for every $\ell \in L'', k \in K'', m \in M''$, and every $\vec{x} \in \prod_{i\in[r]} X''_i$. By our choice of the transversals $\mathcal{Q}(K'', L'')$ and $\mathcal{R}(K'', M'')$ we have $|\mathcal{P}^{\ell m}_{(\vec{x},k)}| \geqslant \frac{\mu}{4}|\mathcal{P}^{\ell m}|$. The final application of Lemma 3.2.8, with t, $r+1$, and $\frac{\mu}{4}$ in place of t, r, and μ, yields t-sized subsets $K_\star \subseteq K''$, $L_\star \subseteq L''$, $M_\star \subseteq M''$, and $Y_i \subseteq X''_i$, for every $i \in [r]$, and a transversal $\mathcal{S}(L_\star, M_\star)$ such that for every $\vec{y} \in \prod_{i\in[r]} Y_i$ and every $k\ell m \in K_\star \times L_\star \times M_\star$ we have that $Q^{k\ell}R^{km}S^{\ell m} \in E(\mathcal{A}_{\vec{y}})$. □

3.3 Bicolouring wicked reduced hypergraphs

In this chapter we prove Proposition 3.1.5. The proof pivots on the analysis of holes in a reduced hypergraph and develops its theory in § 3.3.1–§ 3.3.5, before we deduce Proposition 3.1.5 at the end of this section in § 3.3.6.

3.3.1 Holes and links in reduced hypergraphs

Given a reduced hypergraph $\mathcal{A}$ with index set I, a natural definition of a *hole* across a subset of indices $J \subseteq I$ and subsets of vertices $\Phi^{ij} \subseteq \mathcal{P}^{ij}$ for $ij \in J^{(2)}$ would maybe require that for every $ijk \in J^{(3)}$ the sets Φ^{ij}, Φ^{ik}, Φ^{jk} span no hyperedges in $\mathcal{A}^{ijk}$. However, this notion is too restrictive for our analysis and we shall only require that these sets induce hypergraphs of low density.

Definition 3.3.1. *Given a reduced hypergraph $\mathcal{A}$ and a subset of indices $J \subseteq I$ we say that a subset of vertices $\Phi \subseteq V(\mathcal{A})$ is a μ-*hole* on J if $\Phi^{ij} = \Phi \cap \mathcal{P}^{ij}$ is nonempty for all $ij \in J^{(2)}$ and*

$$e(\Phi^{ij}, \Phi^{ik}, \Phi^{jk}) \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|$$

for every $ijk \in J^{(3)}$.

The size *of the hole is $|J|$ and the smallest $\varsigma > 0$ such that $|\Phi^{ij}| \geqslant \varsigma|\mathcal{P}^{ij}|$ for every $ij \in J^{(2)}$ is called the* width *of the hole. We refer to μ-holes with size at least t and with width at least ς as (μ, t, ς)-holes.*

Roughly speaking, for the proof of Proposition 3.1.5 we shall find two almost disjoint holes with widths bigger than 1/3 on a large set of indices in a wicked reduced hypergraph. These holes will be used to define the desired red/blue-colouring φ for Proposition 3.1.5.

Holes may induce a few hyperedges, however, cherries that are contained in too many such hyperedges are considered to be exceptional. This leads to the following definition.

Definition 3.3.2. *Given a μ-hole Φ on J, $\varepsilon > 0$, and ijk in $J^{(3)}$ we say that a cherry $(P^{ij}, P^{ik}) \in \Phi^{ij} \times \Phi^{ik}$ is ε-*exceptional *if*

$$\left|N(P^{ij}, P^{ik}) \cap \Phi^{jk}\right| \geqslant \varepsilon\left|\Phi^{jk}\right|.$$

For indices $i < j < k$ in J we denote by

$$\mathscr{L}^{ijk}(\Phi,\varepsilon) \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{ik}\,, \quad \mathscr{R}^{ijk}(\Phi,\varepsilon) \subseteq \mathcal{P}^{ik} \times \mathcal{P}^{jk}\,, \quad \text{and} \quad \mathscr{M}^{ijk}(\Phi,\varepsilon) \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{jk}$$

the ε-exceptional left, right, and middle cherries and we set

$$\mathscr{L}(\Phi,\varepsilon) = \bigcup_{i<j<k} \mathscr{L}^{ijk}(\Phi,\varepsilon)\,, \;\; \mathscr{R}(\Phi,\varepsilon) = \bigcup_{i<j<k} \mathscr{R}^{ijk}(\Phi,\varepsilon)\,, \;\; \text{and} \;\; \mathscr{M}(\Phi,\varepsilon) = \bigcup_{i<j<k} \mathscr{M}^{ijk}(\Phi,\varepsilon)\,.$$

It is easy to see that holes can only contain few exceptional cherries. More precisely, for every μ-hole Φ on J and every $\varepsilon > 0$ we have for all $i < j < k$ in J

$$\varepsilon\,|\mathcal{P}^{jk}||\mathscr{L}^{ijk}(\Phi,\varepsilon)| \leqslant e(\Phi^{ij},\Phi^{ik},\Phi^{jk}) \leqslant \mu\,|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|$$

and the same reasoning for $\mathscr{R}$ and $\mathscr{M}$ yields

$$\big|\mathscr{L}^{ijk}(\Phi,\varepsilon)\big| \leqslant \frac{\mu}{\varepsilon}\big|\mathcal{P}^{ij}\big|\big|\mathcal{P}^{ik}\big|\,, \quad \big|\mathscr{R}^{ijk}(\Phi,\varepsilon)\big| \leqslant \frac{\mu}{\varepsilon}\big|\mathcal{P}^{ik}\big|\big|\mathcal{P}^{jk}\big|\,,$$
$$\text{and} \quad \big|\mathscr{M}^{ijk}(\Phi,\varepsilon)\big| \leqslant \frac{\mu}{\varepsilon}\big|\mathcal{P}^{ij}\big|\big|\mathcal{P}^{jk}\big|\,. \quad (3.3.1)$$

The holes Φ studied here, arise from neighbourhoods $N(P^{ik},P^{jk})$, i.e., for appropriately chosen $P^{ik} \in \mathcal{P}^{ij}$ and $P^{jk} \in \mathcal{P}^{ij}$ we set $\Phi^{ij} = N(P^{ik},P^{jk})$. Note that in $(d,\boldsymbol{\wedge})$-dense reduced hypergraphs, holes obtained this way will automatically have width at least d.

Given a (K,L)-transversal $\mathcal{Q}$, a subset $K_\star \subseteq K$, and an index $\ell \in L$ we define the *$\mathcal{Q}$-link of ℓ on $K_\star$* by

$$\Lambda(\mathcal{Q},K_\star,\ell) = \bigcup_{kk' \in K_\star^{(2)}} N(Q^{k\ell},Q^{k'\ell})\,.$$

The following lemma asserts that in $\boldsymbol{\wedge}$-dense reduced hypergraphs that do not support $K_5^{(3)}$ the $\mathcal{Q}$-links contain large holes.

Lemma 3.3.3. *Let $t \in \mathbb{N}$, μ, $d > 0$, let $\mathcal{A}$ be a $(d,\boldsymbol{\wedge})$-dense reduced hypergraph with index set I that does not support a $K_5^{(3)}$, and for sufficiently large disjoint subsets of indices $K, L \subseteq I$ let $\mathcal{Q}$ be a (K,L)-transversal.*

Then there exist $K_\star \subseteq K$ and $L_\star \subseteq L$ of size t such that $\Lambda(\mathcal{Q},K_\star,\ell)$ is a (μ,t,d)-hole for every $\ell \in L_\star$.

Proof. Let $q = \binom{\lceil \mu^{-1} \rceil}{2}$ and define an auxiliary 2-colouring of the pairs $(kk'k'', \ell) \in K^{(3)} \times L$ depending on whether

$$e\big(N(Q^{k\ell}, Q^{k'\ell}), N(Q^{k\ell}, Q^{k''\ell}), N(Q^{k'\ell}, Q^{k''\ell})\big) > \mu|\mathcal{P}^{kk'}||\mathcal{P}^{kk''}||\mathcal{P}^{k'k''}| \tag{3.3.2}$$

or not. Since K and L are sufficiently large, the product Ramsey theorem (see e.g. Proposition 9.1 in [46]) yields $K_1 \subseteq K$ with $|K_1| = \max\{3d^{-q}, t\}$ and $L_1 \subseteq L$ with $|L_1| \geqslant \max\{\mu^{-1}, t\}$ such that either (3.3.2) holds or it fails for every $kk'k'' \in K_1^{(3)}$ and $\ell \in L_1$. In fact, if (3.3.2) fails on $K_1^{(3)} \times L_1$, then $K_\star = K_1$ and $L_\star = L$ have the desired properties. Consequently, we may assume (3.3.2) holds on $K_1^{(3)} \times L_1$.

Let L_2 be a subset of L_1 of size $|L_2| = \lceil 2/\mu \rceil$ and consider some $\ell\ell' \in L_2^{(2)}$. Since we have $|N(Q^{k\ell}, Q^{k\ell'})| \geqslant d|\mathcal{P}^{\ell\ell'}|$ for every $k \in K_1$, there is a subset $K_2 \subseteq K_1$ of size at least $d|K_1|$ such that

$$\bigcap_{k \in K_2} N(Q^{k\ell}, Q^{k\ell'}) \neq \varnothing\,.$$

Repeating this argument iteratively q times for every pair in L_2 we obtain nested subsets $K_1 \supseteq K_2 \supseteq \dots \supseteq K_q$ such that

$$|K_q| \geqslant d^q|K_1| \geqslant 3 \quad \text{and} \quad \bigcap_{k \in K_q} N(Q^{k\ell}, Q^{k\ell'}) \neq \varnothing \text{ for every } \ell\ell' \in L_2^{(2)}.$$

Consequently, there is some $kk'k'' \in K_q^{(3)}$ such that for every $\ell\ell' \in L_2^{(2)}$ we can fix a vertex $P^{\ell\ell'} \in \mathcal{P}^{\ell\ell'}$ satisfying

$$P^{\ell\ell'}Q^{k\ell}Q^{k\ell'},\ P^{\ell\ell'}Q^{k'\ell}Q^{k'\ell'},\ P^{\ell\ell'}Q^{k''\ell}Q^{k''\ell'} \in E(\mathcal{A}). \tag{3.3.3}$$

We infer from (3.3.2) that

$$\sum_{\ell \in L_2} e\big(N(Q^{k\ell}, Q^{k'\ell}), N(Q^{k\ell}, Q^{k''\ell}), N(Q^{k'\ell}Q^{k''\ell})\big) > \mu|L_2||\mathcal{P}^{kk'}||\mathcal{P}^{kk''}||\mathcal{P}^{k'k''}|\,.$$

Consequently, there is an edge $R^{kk'}R^{kk''}R^{k'k''} \in E(\mathcal{A}^{kk'k''})$ such that for more than $\mu|L_2|$ indices $\ell \in L_2$ we have

$$R^{kk'}Q^{k\ell}Q^{k'\ell},\ R^{kk''}Q^{k\ell}Q^{k''\ell},\ R^{k'k''}Q^{k'\ell}Q^{k''\ell} \in E(\mathcal{A})\,. \tag{3.3.4}$$

Hence, since $\mu|L_2| \geqslant 2$ there are two indices ℓ, $\ell' \in L_2$ such that (3.3.4) holds also with ℓ replaced by ℓ'. In view of (3.3.3), we arrive at the contradiction that $P^{\ell\ell'}$, together with the six vertices $Q^{\kappa\lambda}$ for $\kappa \in \{k, k', k''\}$ and $\lambda \in \{\ell, \ell'\}$, and with the three vertices $R^{kk'}$, $R^{kk''}$, $R^{k'k''}$ support a $K_5^{(3)}$ in $\mathcal{A}$. □

Two consecutive applications of Lemma 3.3.3 yield the symmetric conclusion that both links $\Lambda(\mathcal{Q}, K_\star, \ell)$ and $\Lambda(\mathcal{Q}, L_\star, k)$ are μ-holes for every $\ell \in L_\star$ and $k \in K_\star$.

Corollary 3.3.4. *Let $t \in \mathbb{N}$, μ, $d > 0$, let $\mathcal{A}$ be a $(d, \boldsymbol{\wedge})$-dense reduced hypergraph with index set I that does not support a $K_5^{(3)}$, and for sufficiently large disjoint subsets of indices $K, L \subseteq I$ let $\mathcal{Q}$ be a (K, L)-transversal.*

Then there exist $K_\star \subseteq K$ and $L_\star \subseteq L$ of size t such that for every $\ell \in L_\star$ and for every $k \in K_\star$ the $\mathcal{Q}$-links $\Lambda(\mathcal{Q}, K_\star, \ell)$ and $\Lambda(\mathcal{Q}, L_\star, k)$ are (μ, t, d)-holes.

Proof. For sufficiently large $t' = t'(t, \mu, d)$ a first application of Lemma 3.3.3 yields subsets K' and L' of size at least t' such that $\Lambda(\mathcal{Q}, K', \ell)$ is a (μ, t', d)-hole for every $\ell \in L'$. A second application to the restricted transversal $\mathcal{Q}(K', L')$ (with the rôles of K and L exchanged) then yields subsets $L_\star \subseteq L'$ and $K_\star \subseteq K'$ of size t such that additionally $\Lambda(\mathcal{Q}, L_\star, k)$ is a (μ, t, d)-hole for every $k \in K_\star$. □

3.3.2 Intersecting and disjoint links

Next we define concepts for pairs of links having a substantial intersection and of being almost disjoint.

Definition 3.3.5. *Let $\mathcal{A}$ be a reduced hypergraph with index set I, let $K, L, M \subseteq I$ be pairwise disjoint sets of indices, and let $\mathcal{Q}(K, L)$ and $\mathcal{R}(K, M)$ be transversals.*

For $\ell \in L$ and $m \in M$ we say the links $\Lambda(\mathcal{Q}, K, \ell)$ and $\Lambda(\mathcal{R}, K, m)$ are δ-intersecting *if*

$$\left|N(Q^{k\ell}, Q^{k'\ell}) \cap N(R^{km}, R^{k'm})\right| > \delta\left|\mathcal{P}^{kk'}\right| \tag{3.3.5}$$

for all $kk' \in K^{(2)}$. If, on the other hand, (3.3.5) fails for all $kk' \in K^{(2)}$, then we say $\Lambda(\mathcal{Q}, K, \ell)$ and $\Lambda(\mathcal{R}, K, m)$ are δ-disjoint.

Moreover, we say a pair of transversals $\mathcal{Q}(K,L)\mathcal{R}(K,M)$ has δ-intersecting links *(resp.* δ-disjoint links*) if $\Lambda(\mathcal{Q},K,\ell)$ and $\Lambda(\mathcal{R},K,m)$ are δ-intersecting (resp. δ-disjoint) for every $\ell \in L$ and $m \in M$.*

We remark that the notions of δ-intersecting and δ-disjoint do not complement each other. However, by means of (the product version of) Ramsey's theorem we can always pass to subsets of K, L, and M for which one of the properties holds (see, e.g., proof of Corollary 3.3.7 below).

The next lemma shows that in reduced hypergraphs that do not support $K_5^{(3)}$ at most one pair from a triple of inhabited transversals can have an intersecting link.

Lemma 3.3.6. *Let $\delta > 0$, let $\mathcal{A}$ be a reduced hypergraph with index set I, and for sufficiently large disjoint sets $K, L, M \subseteq I$ let $\mathcal{Q}(K,L)\mathcal{R}(K,M)\mathcal{S}(L,M)$ be an inhabited triple of transversals. If both pairs of transversals $\mathcal{Q}(K,L)\mathcal{R}(K,M)$ and $\mathcal{Q}(K,L)\mathcal{S}(L,M)$ have δ-intersecting links, then $\mathcal{A}$ supports a $K_5^{(3)}$.*

Proof. Fix $m \in M$, a subset $K_\star \subseteq K$ of size at least δ^{-1}, and $q = \binom{\lceil \delta^{-1} \rceil}{2}$. Take arbitrary two distinct indices $k, k' \in K_\star$. Since $|N(Q^{k\ell}, Q^{k'\ell}) \cap N(R^{km}, R^{k'm})| \geqslant \delta |\mathcal{P}^{kk'}|$ for every $\ell \in L$ there is a subset $L_1 \subseteq L$ of size at least $\delta|L|$ such that

$$\bigcap_{\ell \in L_1} N(Q^{k\ell}, Q^{k'\ell}) \cap N(R^{km}, R^{k'm}) \neq \varnothing. \tag{3.3.6}$$

As the pair k, k' was taken arbitrarily, we can repeat the argument iteratively q times (for every pair in $K_\star$) and find nested subsets $L \supseteq L_1 \supseteq L_2 \supseteq \cdots \supseteq L_q$ such that (3.3.6) with L_1 replaced by L_q holds for every $kk' \in K_\star^{(2)}$.

Moreover, we have $|L_q| \geqslant \delta^q |L|$ and since L is sufficiently large, we have $|L_q| \geqslant 2$ and we can select $\ell\ell' \in L_q^{(2)}$. Owing to (3.3.6) with L_1 replaced by L_q, for every $kk' \in K_\star$ there is a vertex $P^{kk'} \in \mathcal{P}^{kk'}$ such that

$$P^{kk'} Q^{k\ell} Q^{k'\ell},\ P^{kk'} Q^{k\ell'} Q^{k'\ell'},\ P^{kk'} R^{km} R^{k'm} \in E(\mathcal{A})\,. \tag{3.3.7}$$

Moreover, since $\mathcal{Q}(K,L)\mathcal{S}(L,M)$ has δ-intersecting links and $K_\star$ is of size at least δ^{-1} there exists $kk' \in K_\star^{(2)}$ such that

$$N(Q^{k\ell}, Q^{k\ell'}) \cap N(Q^{k'\ell}, Q^{k'\ell'}) \cap N(S^{\ell m}, S^{\ell' m}) \neq \varnothing\,.$$

Therefore, there is a vertex $P^{\ell\ell'} \in N(Q^{k\ell}, Q^{k\ell'}) \cap N(Q^{k'\ell}, Q^{k'\ell'}) \cap N(S^{\ell m}, S^{\ell' m}) \subseteq \mathcal{P}^{\ell\ell'}$ such that

$$P^{\ell\ell'} Q^{k\ell} Q^{k\ell'},\ P^{\ell\ell'} Q^{k'\ell} Q^{k'\ell'},\ P^{\ell\ell'} S^{\ell m} S^{\ell' m} \in E(\mathcal{A})\,. \tag{3.3.8}$$

Moreover, since $\mathcal{Q}(K,L)\mathcal{R}(K,M)\mathcal{S}(L,M)$ is inhabited we have

$$Q^{k\ell} R^{km} S^{\ell m},\ Q^{k\ell'} R^{km} S^{\ell' m},\ Q^{k'\ell} R^{k'm} S^{\ell m},\ Q^{k'\ell'} R^{k'm} S^{\ell' m} \in E(\mathcal{A})\,. \tag{3.3.9}$$

Consequently, the ten hyperedges provided by (3.3.7)–(3.3.9) show that the vertices $P^{kk'}$, $P^{\ell\ell'}$, together with $Q^{k\ell}$, $Q^{k\ell'}$, $Q^{k'\ell}$, $Q^{k'\ell'}$, R^{km}, $R^{k'm}$, and $S^{\ell m}$, $S^{\ell' m}$ support a $K_5^{(3)}$ on the five indices k, k', ℓ, ℓ', and m. □

By means of the product Ramsey theorem (see e.g. Proposition 9.1 in [46]) we can move from at most one pair with intersecting links (given by Lemma 3.3.6) to at least two pairs with essentially disjoint links.

Corollary 3.3.7. *Let $t \in \mathbb{N}$, $\delta > 0$, let $\mathcal{A}$ be a reduced hypergraph with index set I that does not support $K_5^{(3)}$, and let $\mathcal{Q}(K,L)\mathcal{R}(K,M)\mathcal{S}(L,M)$ be an inhabited triple of transversals for sufficiently large disjoint sets K, L, $M \subseteq I$.*

Then there exist subsets $K_\star \subseteq K$, $L_\star \subseteq L$, and $M_\star \subseteq M$ each of size t such that at most one pair of restricted transversals $\mathcal{Q}(K_\star, L_\star)\mathcal{R}(K_\star, M_\star)$, $\mathcal{Q}(K_\star, L_\star)\mathcal{S}(L_\star, M_\star)$, $\mathcal{R}(K_\star, M_\star)\mathcal{S}(L_\star, M_\star)$ has δ-intersecting links and all other pairs have δ-disjoint links.

Proof. Define a 2-colouring on the tuples $(kk', \ell, m) \in K^{(2)} \times L \times M$ depending on whether $N(Q^{k\ell}, Q^{k'\ell}) \cap N(R^{km}, R^{k'm}) \geqslant \delta|\mathcal{P}^{kk'}|$ or not.

Since K, L, and M are large enough, we can deduce from the product Ramsey theorem that there exist large subsets $K_1 \subseteq K$, $L_1 \subseteq L$, and $M_1 \subseteq M$ for which the pair of restricted transversals $\mathcal{Q}(K_1, L_1)\mathcal{R}(K_1, M_1)$ has δ-intersecting or δ-disjoint links.

We can repeat this argument and consider the triples in $L_1^{(2)} \times K_1 \times M_1$ to obtain subsets $K_2 \subseteq K_1, L_2 \subseteq L_1$ and $M_2 \subseteq M_1$ such that the pair $\mathcal{Q}(K_2, L_2)\mathcal{S}(L_2, M_2)$ has δ-intersecting or δ-disjoint links. Observe that these properties are closed under subsets of indices and hence, we have that the pair $\mathcal{Q}(K_2, L_2)\mathcal{R}(K_2, M_2)$ has δ-intersecting or δ-disjoint links.

Using the Ramsey argument again yields subsets $K_\star \subseteq K_2$, $L_\star \subseteq L_2$, and $M_\star \subseteq M_2$ such that all pairs of restricted transversals $\mathcal{Q}(K_\star, L_\star)$, $\mathcal{R}(K_\star, M_\star)$, and $\mathcal{S}(L_\star, M_\star)$ have δ-intersecting or δ-disjoint links. Since the initial sets K, L, and M are large enough, we argue that $K_\star$, $L_\star$, and $M_\star$ can be taken of size at least t.

Finally, applying Lemma 3.3.6 we observe that at most one of those pairs of transversals has a δ-intersecting link, and hence, at least two of them have δ-disjoint links. $\square$

Finally, we may combine Corollaries 3.3.4 and 3.3.7. More precisely, after an application of Corollary 3.3.7 and three consecutive applications of Corollary 3.3.4 we arrive at the following statement.

Corollary 3.3.8. *Let $t \in \mathbb{N}$, δ, μ, $d > 0$, let $\mathcal{A}$ be a $(d, \mathord{\cdot\!\!\wedge\!\!\cdot})$-dense reduced hypergraph with index set I that does not support a $K_5^{(3)}$, and for sufficiently large disjoint sets K, L, $M \subseteq I$ let $\mathcal{Q}(K,L)\mathcal{R}(K,M)\mathcal{S}(L,M)$ be an inhabited triple of transversals.*

There exist subsets $K_\star \subseteq K$, $L_\star \subseteq L$, and $M_\star \subseteq M$ of size at least t such that

(i) at most one pair $\mathcal{Q}(K_\star, L_\star)\mathcal{R}(K_\star, M_\star)$, $\mathcal{Q}(K_\star, L_\star)\mathcal{S}(L_\star, M_\star)$, $\mathcal{R}(K_\star, M_\star)\mathcal{S}(L_\star, M_\star)$ of restricted transversals has δ-intersecting links and all other pairs have δ-disjoint links

(ii) and for every $k \in K_\star$, $\ell \in L_\star$, and $m \in M_\star$ the links $\Lambda(\mathcal{Q}, K_\star, \ell)$, $\Lambda(\mathcal{Q}, L_\star, k)$, $\Lambda(\mathcal{R}, K_\star, m)$, $\Lambda(\mathcal{R}, M_\star, k)$, $\Lambda(\mathcal{S}, L_\star, m)$, and $\Lambda(\mathcal{S}, M_\star, \ell)$ are (μ, t, d)-holes. $\square$

3.3.3 Equivalent holes

Roughly speaking, in the next step for the proof of Proposition 3.1.5 we show that for wicked reduced hypergraphs (see Definition 3.1.4), the set of holes with width bigger than 1/3 splits into only two classes defined by δ-intersections. For that we transfer the notion of δ-intersecting from links to holes.

Definition 3.3.9. *Given a reduced hypergraph $\mathcal{A}$ with index set I, a subset $J \subseteq I$, and μ, $\delta > 0$, we say two μ-holes Φ and Ψ on J are* δ-intersecting *if*

$$\left|\Phi^{ij} \cap \Psi^{ij}\right| > \delta\left|\mathcal{P}^{ij}\right| \tag{3.3.10}$$

for all $ij \in J^{(2)}$*. If, on the other hand,* (3.3.10) *fails for all* $ij \in J^{(2)}$*, then we say* Φ *and* Ψ *are* δ-disjoint*.*

For $\mu > 0$ and $\delta \in (0, 1]$ the notion of δ-intersecting defines a reflexive and symmetric relation on the μ-holes on J. However, maybe somewhat surprisingly, the next lemma shows that this relation is also transitive on holes with width bigger than $1/3$ in wicked reduced hypergraphs, if one passes to an appropriate subset of J. This justifies the shorthand notation

$$\Phi \equiv_{\delta,J} \Psi$$

for δ-intersecting holes on J.

Lemma 3.3.10. *For every* $\varepsilon > 0$ *there exists* $\mu > 0$ *such that for every* $t \in \mathbb{N}$ *the following holds. Suppose* $\mathcal{A}$ *is an* ε*-wicked reduced hypergraph with index set* I *and for sufficiently large* $J \subseteq I$ *we are given* $(\mu, |J|, 1/3 + \varepsilon)$*-holes* Φ*,* Ψ*, and* Ω *with*

$$\Phi \equiv_{\varepsilon,J} \Psi \qquad \textit{and} \qquad \Psi \equiv_{\varepsilon,J} \Omega\,.$$

Then there is a subset $J_\star \subseteq J$ *of size* t *such that* $\Phi \equiv_{\varepsilon,J_\star} \Omega$*.*

Proof. Given $\varepsilon > 0$ we fix auxiliary integers t_1, t_2, t_3 and we set μ to satisfy the hierarchy

$$\varepsilon^{-1} \ll t_3 \ll t_2 \ll t_1,\, \mu^{-1}\,. \tag{3.3.11}$$

Let $t \in \mathbb{N}$ and let $\mathcal{A}$ be an ε-wicked reduced hypergraph with index set I and for sufficiently large $J \subseteq I$ let Φ, Ψ, and Ω be $(\mu, |J|, 1/3 + \varepsilon)$-holes such that Φ and Ψ, as well as, Ψ and Ω are ε-intersecting.

Consider an auxiliary 2-colouring of the pairs $ij \in J^{(2)}$ depending on whether

$$|\Phi^{ij} \cap \Omega^{ij}| \leqslant \varepsilon|\mathcal{P}^{ij}| \tag{3.3.12}$$

or not. Since J is sufficiently large, there is a subset $J_1 \subseteq J$ of size $\max\{t_1, t\}$ such that (3.3.12) either holds or fails for every $ij \in J_1^{(2)}$. If (3.3.12) fails, we set $J_\star = J_1$ and are done. Consequently, we may assume that (3.3.12) holds for every $ij \in J_1^{(2)}$ and from which we shall derive a contradiction to the assumption that $\mathcal{A}$ does not support $K_5^{(3)}$.

First we note that for all $i<j<k$ from J_1 and every $P^{ij} \in \mathcal{P}^{ij}$ and $P^{jk} \in \mathcal{P}^{jk}$ the $(1/3+\varepsilon, \boldsymbol{\wedge})$-density of $\mathcal{A}$ and the given width of the holes Φ and Ω together with (3.3.12) imply

$$\begin{aligned}\left|N(P^{ij}, P^{jk}) \cap (\Phi^{ik} \cup \Omega^{ik})\right| &\geqslant \left|N(P^{ij}, P^{jk})\right| + \left|\Phi^{ik}\right| + \left|\Omega^{ik}\right| - \left|\mathcal{P}^{ik}\right| - \left|\Phi^{ik} \cap \Omega^{ik}\right| \\ &\geqslant 2\varepsilon\left|\mathcal{P}^{ik}\right|. \end{aligned} \tag{3.3.13}$$

We define the reduced subhypergraph $\mathcal{A}_1 \subseteq \mathcal{A}$ on J_1 with vertex set $V(\mathcal{A}_1) = V(\mathcal{A})$ and with edges defined for every $i<j<k$ in J_1 by

$$E(\mathcal{A}_1^{ijk}) = E\big(\mathcal{A}^{ijk}[\Phi^{ij} \cap \Psi^{ij}, \Phi^{ik} \cup \Omega^{ik}, \Psi^{jk} \cap \Omega^{jk}]\big)\,.$$

Since Φ and Ψ, as well as, Ψ and Ω are ε-intersecting, we infer from (3.3.13) for every $i<j<k$ in J_1 that

$$\left|E(\mathcal{A}_1^{ijk})\right| = \sum_{\substack{P^{ij} \in \Phi^{ij} \cap \Psi^{ij} \\ P^{jk} \in \Psi^{jk} \cap \Omega^{jk}}} \left|N_{\mathcal{A}}(P^{ij}, P^{jk}) \cap (\Phi^{ik} \cup \Omega^{ik})\right| \geqslant 2\varepsilon^3 |\mathcal{P}^{ij}||\mathcal{P}^{jk}||\mathcal{P}^{jk}|$$

and, hence, $\mathcal{A}_1$ is $(2\varepsilon^3, \therefore)$-dense.

We consider the ε-exceptional left and right cherries (see Definition 3.3.2) of the holes Φ, Ψ, and Ω (restricted to J_1), i.e., for every $i<j<k$ in J_1 we set

$$\mathscr{L}^{ijk} = \mathscr{L}^{ijk}(\Psi, \varepsilon) \cup \mathscr{L}^{ijk}(\Omega, \varepsilon) \quad \text{and} \quad \mathscr{R}^{ijk} = \mathscr{R}^{ijk}(\Phi, \varepsilon) \cup \mathscr{R}^{ijk}(\Psi, \varepsilon)\,.$$

We infer from (3.3.1) that

$$|\mathscr{L}^{ijk}| \leqslant \frac{2\mu}{\varepsilon}|\mathcal{P}^{ij}||\mathcal{P}^{ik}| \quad \text{and} \quad |\mathscr{R}^{ijk}| \leqslant \frac{2\mu}{\varepsilon}|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,.$$

By the choice of μ we can apply Lemma 3.2.5 to $\mathcal{A}_1$ with t_2, $2\varepsilon^3$, and $\frac{2\mu}{\varepsilon}$ in place of t, μ and μ'. This yields an $J_2 \subseteq J_1$ of size t_2 and three transversals $\mathcal{Q}(J_2)$, $\mathcal{R}(J_2)$, and $\mathcal{S}(J_2)$ avoiding the exceptional cherries from $\mathscr{L}^{ijk}$ and $\mathscr{R}^{ijk}$ for every $ijk \in J_2^{(3)}$. Furthermore, for every $i<j<k$ in J_2 we have

$$Q^{ij}R^{ik}S^{jk} \in E(\mathcal{A}_1^{ijk}) = E\big(\mathcal{A}^{ijk}[\Phi^{ij} \cap \Psi^{ij}, \Phi^{ik} \cup \Omega^{ik}, \Psi^{jk} \cap \Omega^{jk}]\big)\,. \tag{3.3.14}$$

We fix disjoint subsets $K, L, M \subseteq J_2$ such that K' and M' have size $\lfloor t_2/3 \rfloor$, L has size t_3, and for every $(k, \ell, m) \in K \times L \times M$ we have $k<\ell<m$. Note that by definition

$R(K', M') \subseteq \Phi \cup \Omega$ and hence there exists a $\Pi \in \{\Phi, \Omega\}$, which contains more then half of $R(K', M')$. Therefore, an application of the Kővari-Sós-Turán Theorem (see [40]) leads to subsets $K \subseteq K'$ and $M \subseteq M'$, each of size t_3, such that

$$R^{km} \in \Pi^{km} \text{ for every } k \in K, \text{ and } m \in M. \tag{3.3.15}$$

Owing to (3.3.14), the restricted transversals $\mathcal{Q}(K, L)$, $\mathcal{R}(K, M)$, and $\mathcal{S}(L, M)$ form an inhabited triple in $\mathcal{A}$. We derive a contradiction by Lemma 3.3.6 and for that we shall show that two of the pairs $\mathcal{Q}(K, L)\mathcal{R}(K, M)$, $\mathcal{Q}(K, L)\mathcal{S}(L, M)$, and $\mathcal{R}(K, M)\mathcal{S}(L, M)$ have ε-intersecting links.

First, we recall that, independent of the chosen Π, the pair $\mathcal{Q}(K, L)\mathcal{S}(L, M)$ consists of transversals inside the hole Ψ and both avoid the exceptional left and right cherries from Ψ. Hence, for all $k \in K$, $\ell\ell' \in L^{(2)}$, and $m \in M$ we have

$$\left|N_{\mathcal{A}}(Q^{k\ell}, Q^{k\ell'}) \cap \Psi^{\ell\ell''}\right| < \varepsilon\left|\mathcal{P}^{\ell\ell'}\right| \quad \text{and} \quad \left|N_{\mathcal{A}}(S^{lm}, S^{\ell' m}) \cap \Psi^{\ell\ell'}\right| < \varepsilon\left|\mathcal{P}^{\ell\ell'}\right|.$$

Consequently, the $(1/3 + \varepsilon, \wedge)$-density of $\mathcal{A}$ and the width of Ψ imply

$$\left|N_{\mathcal{A}}(Q^{k\ell}, Q^{k\ell'}) \cap N_{\mathcal{A}}(S^{\ell m}, S^{\ell' m})\right| > \varepsilon\left|\mathcal{P}^{\ell\ell'}\right|$$

for every $k \in K$, $\ell\ell' \in L^{(2)}$, and $m \in M$, i.e., the pair $\mathcal{Q}(K, L)\mathcal{S}(L, M)$ has ε-intersecting links.

If $\Pi = \Phi$, then $\mathcal{Q}(K, L)$ and $\mathcal{R}(K, M)$ are both transversals in Φ (see (3.3.15)) and both $\mathcal{Q}$ and $\mathcal{R}$ avoid the exceptional right cherries of Φ. As before, this implies that the pair $\mathcal{Q}(K, L)\mathcal{R}(K, M)$ has ε-intersecting links. Consequently, Lemma 3.3.6 gives rise to the contradiction that $\mathcal{A}$ supports a $K_5^{(3)}$.

Analogously, if $\Pi = \Omega$, then $\mathcal{R}(K, M)$ and $\mathcal{S}(L, M)$ are both transversals in Ω and since both $\mathcal{R}$ and $\mathcal{S}$ avoid the exceptional left cherries of Ω, the pair of transversals has ε-intersecting links, which leads to the same contradiciton. □

Another application of Ramsey's theorem leads to the following corollary.

Corollary 3.3.11. *For every $\varepsilon > 0$ there exists $\mu > 0$ such that for all integers t, $r \geqslant 2$ the following holds. Suppose $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I and for sufficiently large $J \subseteq I$ we are given $(\mu, |J|, 1/3 + \varepsilon)$-holes $\Phi_1, \ldots, \Phi_r$.*

Then there is a subset $J_\star \subseteq J$ of size t such that

(*i*) *for all ϱ, $\varrho' \in [r]$ the holes Φ_ϱ and $\Phi_{\varrho'}$ are either ε-intersecting or ε-disjoint on $J_\star$*

(*ii*) *and $\equiv_{\varepsilon,J_\star}$ is an equivalence relation on $\{\Phi_1, \ldots, \Phi_r\}$ with at most two equivalence classes.*

Proof. For $\varepsilon \in (0,1]$ let $\mu > 0$ be given by Lemma 3.3.10. For fixed $t, r \geqslant 2$ let $t' \geqslant t$ be sufficiently large for an application of Lemma 3.3.10 with ε, μ, and with 2 in place of t.

For a given ε-wicked reduced hypergraph $\mathcal{A}$ and $(\mu, |J|, 1/3 + \varepsilon)$-holes $\Phi_1, \ldots, \Phi_r$ we impose that the size of J is larger than the $2^{\binom{r}{2}}$-colour Ramsey number for graph cliques on t' vertices, i.e.,

$$|J| \longrightarrow (t')^2_{|\Xi|} \quad \text{for} \quad \Xi = \left\{\xi = (\xi_{\varrho\varrho'})_{\varrho\varrho' \in [r]^{(2)}} : \xi_{\varrho\varrho'} \in \{0,1\} \text{ for } \varrho\varrho' \in [r]^{(2)}\right\}. \tag{3.3.16}$$

We assign to a pair $ij \in J^{(2)}$ the colour $\xi = (\xi_{\varrho\varrho'})_{[r]^{(2)}}$ with $\xi_{\varrho\varrho'} = 1$ signifying

$$\left|\Phi_\varrho^{ij} \cap \Phi_{\varrho'}^{ij}\right| > \varepsilon\left|\mathcal{P}^{ij}\right|$$

and $\xi_{\varrho\varrho'} = 0$ otherwise. Owing to (3.3.16) there exists a subset $J_\star \subseteq J$ of size at least $t' \geqslant t$ and a colour $\xi^\star = (\xi^\star_{\varrho\varrho'})_{\varrho\varrho' \in [r]^{(2)}}$ such that all pairs of $J_\star$ were assigned $\xi^\star$. Note that assertion (*i*) follows directly from the definition of the colouring, i.e., Φ_ϱ and $\Phi_{\varrho'}$ are ε-intersecting on $J_\star$ if $\xi^\star_{\varrho\varrho'} = 1$ and ε-disjoint otherwise.

Obviously the relation $\equiv_{\varepsilon,J_\star}$ is reflexive and symmetric. Moreover, our choice of t' allows us to invoke Lemma 3.3.10 and the transitivity follows from the definition of the colouring. Since all holes have width at least $1/3 + \varepsilon$ at least two among any choice of three holes must share at least $\varepsilon|\mathcal{P}^{ij}|$ vertices in $\mathcal{P}^{ij}$ for any $ij \in J_\star^{(2)}$ and, hence, $\equiv_{\varepsilon,J_\star}$ has at most two equivalence classes. $\square$

3.3.4 Unions of equivalent holes

The next lemma shows that the union of equivalent holes of width bigger than 1/3 is still a hole on a suitable subset of the index set. This will be crucial in the proof of Proposition 3.1.5. Roughly speaking, we will start with two disjoint holes of width bigger than $1/3$ and then every other hole of width bigger than $1/3$ can be united with one of the two starting holes. We shall ensure that the union will be a larger hole and, hence, after a bounded number of unions we arrive at two holes. These two holes can be used later to define the two colouring φ asserted by Proposition 3.1.5 (see § 3.3.5 and § 3.3.6 for details).

Lemma 3.3.12. *For every μ, $\varepsilon > 0$ there exists $\nu > 0$ such that for every $t \in \mathbb{N}$ the following holds. Suppose $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I and for a sufficiently large subset $J \subseteq I$ we are given two $(\nu, |J|, 1/3 + \varepsilon)$-holes Φ and Ψ on J such that $\Phi \equiv_{\varepsilon,J} \Psi$.*

Then, there exists a subset $J_\star \subseteq J$ of size at least t such that $\Phi \cup \Psi$ is a μ-hole on $J_\star$.

Proof. Let $\mu > 0$ and $\varepsilon > 0$ be given. We may assume that $\varepsilon \leqslant 2/3$ and we let $\mu_\star > 0$ be a sufficiently small auxiliary constant so that Corollary 3.3.11 applies with ε. Moreover, we fix integers $t_4 \leqslant t_3 \leqslant t_2 \leqslant t_1$ and $\nu > 0$ so that

(*1*) t_4 is sufficiently large to apply Corollary 3.3.11 with ε, $\mu_\star$, $r = 4$, and 2 in place of t,

(*2*) t_3 is sufficiently large to apply Corollary 3.3.8 with t_4, ε, $\mu_\star$, and $1/3 + \varepsilon$ in place of t, δ, μ, and d,

(*3*) t_2 is sufficiently large and $\nu \leqslant \min\{\mu, \mu_\star\}$ is sufficiently small so that Lemma 3.2.5 applies with $3t_3$, $\mu/8$, and $2\nu/\varepsilon$ in place of t, μ, and μ',

(*4*) and $t_1 \longrightarrow (t_2)^3_8$.

Finally, for $t \in \mathbb{N}$ let $J \subseteq I$ be sufficiently large so that

$$|J| \longrightarrow (t')^3_2 \quad \text{for } t' = \max\{t, t_1\}\,.$$

Given $(\nu, |J|, 1/3 + \varepsilon)$-holes Φ and Ψ on J let

$$\mathscr{L} = \mathscr{L}(\Phi, \varepsilon) \cup \mathscr{L}(\Psi, \varepsilon) \quad \text{and} \quad \mathscr{R} = \mathscr{R}(\Phi, \varepsilon) \cup \mathscr{R}(\Psi, \varepsilon)$$

be their ε-exceptional left and right cherries. For later reference we recall that (3.3.1) yields

$$|\mathscr{L}^{ijk}| \leqslant \frac{2\nu}{\varepsilon}|\mathcal{P}^{ij}||\mathcal{P}^{ik}| \quad \text{and} \quad |\mathscr{R}^{ijk}| \leqslant \frac{2\nu}{\varepsilon}|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,. \tag{3.3.17}$$

We begin with an application of Ramsey's theorem for hypergraphs and consider a 2-colouring of the triples $ijk \in J^{(3)}$ depending on whether

$$e(\Phi^{ij} \cup \Psi^{ij}, \Phi^{ik} \cup \Psi^{ik}, \Phi^{jk} \cup \Psi^{jk}) > \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}| \tag{3.3.18}$$

or not. Owing to the size of J, there exists a subset $J_1 \subseteq J$ of size $t' \geqslant \max\{t, t_1\}$ such that either (3.3.18) holds or fails for all $ijk \in J_1^{(3)}$. Note that in case it fails we would be done and, hence, we may assume that (3.3.18) holds for every $ijk \in J_1^{(3)}$ and in the remainder we shall derive a contradiction from this assumption.

First we observe that inequality (3.3.18) implies that for at least one the eight possible tuples $(\Pi_1, \Pi_2, \Pi_3) \in \{\Phi, \Psi\}^3$ we have

$$e(\Pi_1^{ij}, \Pi_2^{ik}, \Pi_3^{jk}) > \frac{\mu}{8}|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}| \tag{3.3.19}$$

for every $ijk \in J_1^{(3)}$. Actually, since Φ and Ψ are ν-holes and $\nu \leqslant \mu$ inequality (3.3.19) can neither hold for $e(\Phi^{ij}, \Phi^{ik}, \Phi^{jk})$ nor for $e(\Psi^{ij}, \Psi^{ik}, \Psi^{jk})$. Thus, we may define an auxiliary 6-colouring of the triples ijk in $J_1^{(3)}$ depending on which of the six available tuples in $\{\Phi, \Psi\}^3$ satisfies (3.3.19), where we fix some choice in an arbitrary way in case several choices satisfy (3.3.19). In view of *(4)* there is a subset $J_2 \subseteq J_1$ of size t_2 such that for every $ijk \in J_2^{(3)}$ inequality (3.3.19) holds for $e(\Pi_1^{ij}, \Pi_2^{ik}, \Pi_3^{jk})$ for the same tuple $(\Pi_1, \Pi_2, \Pi_3) \in \{\Phi, \Psi\}^3$ for every $ijk \in J_2^{(3)}$.

Consequently, the reduced subhypergraph $\mathcal{A}' \subseteq \mathcal{A}$ defined for every $i < j < k$ in J_2 by

$$P^{ij}P^{ik}P^{jk} \in E(\mathcal{A}') \qquad \Longleftrightarrow \qquad P^{ij}P^{ik}P^{jk} \in E(\mathcal{A}[\Pi_1^{ij}, \Pi_2^{ik}, \Pi_3^{jk}]) \tag{3.3.20}$$

is $(\mu/8, ∴)$-dense on J_2. Due to (3.3.17) and our choice of t_2 and ν in *(3)*, Lemma 3.2.5 ensures a subset $J_3 \subseteq J_2$ of size $3t_3$ and an inhabited triple of transversals $\mathcal{Q}(J_3)\mathcal{R}(J_3)\mathcal{S}(J_3)$ where each transversal avoids the sets of exceptional left and right cherries $\mathscr{L}$ and $\mathscr{R}$ of Φ and Ψ.

Since $\mathcal{Q}(J_3)\mathcal{R}(J_3)\mathcal{S}(J_3)$ is an inhabited triple, we have $Q^{ij}R^{ik}S^{jk} \in E(\mathcal{A}')$ for every $i < j < k$ in J_3 and, therefore, the definition of $\mathcal{A}'$ in (3.3.20) implies

$$Q^{ij} \in \Pi_1\,, \quad R^{ik} \in \Pi_2\,, \quad \text{and} \quad S^{jk} \in \Pi_3 \tag{3.3.21}$$

for all $i < j < k$ in J_3.

Fix disjoint subsets of indices $K_3, L_3, M_3 \subseteq J_3$ each of size t_3 and such that for every $(k, \ell, m) \in K \times L \times M$ it holds that $k < \ell < m$. Clearly, the restricted transversals $\mathcal{Q}(K_3, L_3)$, $\mathcal{R}(K_3, M_3)$, and $\mathcal{S}(L_3, M_3)$ still form an inhabited triple of transversals. Therefore, the choice of t_3 in *(2)* allows an application of Corollary 3.3.8, which yields subsets $K_4 \subseteq K_3$, $L_4 \subseteq L_3$, and $M_4 \subseteq M_3$ each of size t_4 satisfying properties *(i)* and *(ii)* of Corollary 3.3.8.

Next we shall show that all three pairs of restricted transversals $\mathcal{Q}(K_4, L_4)\mathcal{R}(K_4, M_4)$, $\mathcal{Q}(K_4, L_4)\mathcal{S}(L_4, M_4)$, and $\mathcal{R}(K_4, M_4)\mathcal{S}(L_4, M_4)$ have ε-intersecting links. However, this contradicts property *(i)* of Corollary 3.3.8, which allows only one pair of transversals with ε-intersecting links and this contradiction concludes the proof of Lemma 3.3.12. Below we show that the pair $\mathcal{Q}(K_4, L_4)\mathcal{R}(K_4, M_4)$ has an ε-intersecting link the proof for the other pairs follows verbatim the same lines.

Fix some $\ell \in L_4$ and $m \in M_4$. Property *(ii)* of Corollary 3.3.8 tells us that $\Lambda(\mathcal{Q}, K_4, \ell)$ and $\Lambda(\mathcal{R}, K_4, m)$ are $(\mu_\star, t_4, 1/3 + \varepsilon)$-holes on K_4. Moreover, since $\nu \leqslant \mu_\star$ also Φ and Ψ are $(\mu_\star, t_4, 1/3 + \varepsilon)$-holes on K_4 and, therefore, the choice of t_4 in *(1)* and an application of Corollary 3.3.11 yields a subset $K_\star \subseteq K_4$ of size at least two such that $\equiv_{\varepsilon, K_\star}$ defines an equivalence relation with at most two equivalent classes on the $\mu_\star$-holes

$$\Lambda(\mathcal{Q}, K_\star, \ell)\,, \quad \Lambda(\mathcal{R}, K_\star, m)\,, \quad \Pi_1\,, \quad \text{and} \quad \Pi_2\,.$$

In view of (3.3.21) we have $\mathcal{Q}(K_\star, L_4) \subseteq \Pi_1$ and $\mathcal{R}(K_\star, M_4) \subseteq \Pi_2$ and since $\mathcal{Q}$ and $\mathcal{R}$ avoid the exceptional cherries from $\mathscr{L}$ and $\mathscr{R}$ we infer

$$|N(Q^{k\ell}, Q^{k'\ell}) \cap \Pi_1^{kk'}| < \varepsilon |\mathcal{P}^{kk'}| \qquad \text{and} \qquad |N(R^{km}, R^{k'm}) \cap \Pi_2^{kk'}| < \varepsilon |\mathcal{P}^{kk'}|$$

for k, $k' \in K_\star$. Consequently,

$$\Pi_1 \text{ and } \Lambda(\mathcal{Q}, K_\star, \ell) \text{ are } \varepsilon\text{-disjoint} \quad \text{and} \quad \Pi_2 \text{ and } \Lambda(\mathcal{R}, K_\star, m) \text{ are } \varepsilon\text{-disjoint}$$

Either $\Pi_1 = \Pi_2$ or by assumption of the lemma we have $\Pi_1 \equiv_{\varepsilon,K_\star} \Pi_2$ and since $\equiv_{\varepsilon,K_\star}$ has only two equivalence classes, we arrive at

$$\Lambda(\mathcal{Q}, K_\star, \ell) \equiv_{\varepsilon,K_\star} \Lambda(\mathcal{R}, K_\star, m)\,.$$

Therefore, property *(i)* of Corollary 3.3.8 yields the same conclusion for $K_4 \supseteq K_\star$, i.e., $\Lambda(\mathcal{Q}, K_4, \ell)$ and $\Lambda(\mathcal{R}, K_4, m)$ are ε-intersecting. Finally, since $\ell \in L_4$ and $m \in M_4$ were arbitrary, we infer the promised assertion that the pair of transversals $\mathcal{Q}(K_4, L_4)\mathcal{S}(L_4, M_4)$ has ε-intersecting links. □

We finish this subsection with the following corollary that follows from the application of Corollary 3.3.11 and Lemma 3.3.12. We will use it for the proof of the lemma presented in the following section.

Corollary 3.3.13. *For every μ, $\varepsilon > 0$ there exists $\nu > 0$ such that for every $t \in \mathbb{N}$ the following holds. Suppose $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I and for a sufficiently large subset $J \subseteq I$ we are given three $(\nu, |J|, 1/3 + \varepsilon)$-holes Φ, Ψ, and Ω such that Φ and Ψ are ε-disjoint.*

Then, there exists a subset $J_\star \subseteq J$ of size at least t such that one of the following holds

(A) $\Phi \cup \Omega$ is a $(\mu, t, 1/3 + \varepsilon)$-hole ε-disjoint with Ψ

(B) or $\Psi \cup \Omega$ is a $(\mu, t, 1/3 + \varepsilon)$-hole ε-disjoint with Φ.

Proof. Given μ and $\varepsilon > 0$ we fix an auxiliary positive constant $\mu' \leqslant \mu$ small enough to apply Corollary 3.3.11 with ε. We fix $\nu \leqslant \mu'$ to be small enough to apply Lemma 3.3.12 with ε and μ'. Finally, given $t \in \mathbb{N}$ we fix positive integers $t_2 \leqslant t_1$ such that: t_2 is large enough to apply Corollary 3.3.11 with t and $r = 3$ and t_1 is large enough to apply Lemma 3.3.12 with t_2 in place of t. Let $\mathcal{A}$ as in the lemma and consider a set $J \subseteq I$ large enough for an application of Corollary 3.3.11 with $r = 3$ and t_1 in place of t.

Apply Corollary 3.3.11 with $r = 3$ to find a subset $J_1 \subseteq J$ of size t_1 in which $\equiv_{\varepsilon,J_1}$ is an equivalence relation on $\{\Phi, \Psi, \Omega\}$ with at most two equivalence classes. Since Φ and Ψ are ε-disjoint without loss of generality we may assume

$$\Omega \equiv_{\varepsilon,J_1} \Phi\,. \tag{3.3.22}$$

Moreover, part *(i)* of Corollary 3.3.11 implies that Ω and Ψ are ε-disjoint on J_1.

An application of Lemma 3.3.12 yields the existence of a set $J_2 \subseteq J_1$ of size t_2 on which

$$\Omega \cup \Phi \text{ is a } (\mu', t_2, 1/3 + \varepsilon)\text{-hole.}$$

Since Ψ is ε-disjoint with both Φ and Ω, we have that $\Omega \cup \Phi$ and Ψ are 2ε-disjoint. However, to prove *(A)* we need an other application of Corollary 3.3.11 for the holes Φ, Ψ and $\Phi \cup \Omega$. Through this application we obtain a subset $J_\star \subseteq J_2$ of size t in which $\equiv_{\varepsilon, J_\star}$ is an equivalence relation with at most two equivalent classes. Since Φ and Ψ are ε-disjoint and obviously $\Phi \cup \Omega \equiv_{\varepsilon, J_\star} \Omega$ alternative *(A)* follows.

In the case in which $\Omega \equiv_{\varepsilon, J_1} \Psi$ instead of (3.3.22) alternative *(B)* follows with the same argument. $\square$

3.3.5 Two large disjoint holes

In this section we establish the existence of two essentially disjoint holes such that most cherries in each hole have a large neighbourhood in the other hole. For that we consider the following sets of unwanted cherries. Given μ-holes Φ and Ψ on J, $\varepsilon > 0$, and indices $ijk \in J^{(3)}$ a cherry $(P^{ij}, P^{ik}) \in \mathcal{P}^{ij} \times \mathcal{P}^{ik}$ is ε-*bad* if

$$\begin{aligned} &(P^{ij}, P^{ik}) \in \Phi^{ij} \times \Phi^{ik} \text{ and } |N(P^{ij}, P^{ik}) \smallsetminus \Psi^{jk}| \geqslant \varepsilon |\mathcal{P}^{jk}| \\ \text{or} \quad &(P^{ij}, P^{ik}) \in \Psi^{ij} \times \Psi^{ik} \text{ and } |N(P^{ij}, P^{ik}) \smallsetminus \Phi^{jk}| \geqslant \varepsilon |\mathcal{P}^{jk}|\,. \end{aligned}$$

For $i < j < k$ we denote the sets of all ε-bad *left*, *middle*, and *right* cherries by

$$\mathscr{I}^{ijk}(\Phi, \Psi, \varepsilon) \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{ik}, \mathscr{C}^{ijk}(\Phi, \Psi, \varepsilon) \subseteq \mathcal{P}^{ij} \times \mathcal{P}^{jk}, \text{ and } \mathscr{D}^{ijk}(\Phi, \Psi, \varepsilon) \subseteq \mathcal{P}^{ik} \times \mathcal{P}^{jk},$$

where the letters $\mathscr{I}$, $\mathscr{C}$, and $\mathscr{D}$ come from the initials of the words “left”, “central”, and “right” in Spanish.

The following lemma shows that given two disjoint holes Φ and Ψ of width at least $1/3 + \varepsilon$ it holds that (for a large subset of indices) either there are few bad cherries or there is a third hole Ω of width $1/3 + \varepsilon$ with a positive proportion of vertices outside of Φ and Ψ. In the latter case an application of Corollary 3.3.13 yields two disjoint holes $\Phi_\star$ and $\Psi_\star$ whose sum of widths increased.

Lemma 3.3.14. *For every μ, $\varepsilon \geqslant \gamma > 0$ and $t \in \mathbb{N}$ there is $\nu > 0$ such that the following holds. Suppose $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I and for sufficiently large $J \subseteq I$ we are given ε-disjoint $(\nu, |J|, 1/3 + \varepsilon)$-holes Φ and Ψ.*

Then, there exists a subset $J_\star \subseteq J$ of size t such that one of the following holds

(*A*) *there exist two ε-disjoint $(\mu, t, 1/3 + \varepsilon)$-holes $\Phi_\star$ and $\Psi_\star$ such that*

$$|\Phi_\star^{ij} \cup \Psi_\star^{ij}| \geqslant |\Phi^{ij} \cup \Psi^{ij}| + \frac{\gamma}{2}|\mathcal{P}^{ij}|$$

for every $ij \in J_\star^{(2)}$,

(*B*) *or for all $i < j < k$ in $J_\star$ the sets of γ-bad cherries satisfy*

$$|\mathscr{I}^{ijk}(\Phi, \Psi, \gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|, \quad |\mathscr{D}^{ijk}(\Phi, \Psi, \gamma)| \leqslant \mu|\mathcal{P}^{ik}||\mathcal{P}^{jk}|,$$
$$\text{and} \quad |\mathscr{C}^{ijk}(\Phi, \Psi, \gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{jk}|.$$

Proof. Given μ, $\varepsilon \geqslant \gamma > 0$, and $t \in \mathbb{N}$ we fix auxiliary integers $t_1, \dots, t_6$, and we choose ν to satisfy

$$\varepsilon^{-1}, \mu^{-1}, \gamma^{-1}, t \ll t_6 \ll \dots \ll t_1, \nu^{-1}\,.$$

Let $\mathcal{A}$, $J \subseteq I$, Φ, and Ψ be as in the statement of the lemma. In particular we have $t_1 \ll |J|$. Consequently, if (*B*) fails to be true, an application of Ramsey's theorem with four colours tell us that there exists a subset $J_1 \subseteq J$ of size at least t_1 such that one of the following cases holds for every $i < j < k$ in J_1

$$|\mathscr{I}^{ijk}(\Phi, \Psi, \gamma)| > \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|\,, \quad |\mathscr{D}^{ijk}(\Phi, \Psi, \gamma)| > \mu|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,,$$
$$\text{or} \quad |\mathscr{C}^{ijk}(\Phi, \Psi, \gamma)| > \mu|\mathcal{P}^{ij}||\mathcal{P}^{jk}|\,.$$

We analyse each case separately.

First Case: $|\mathscr{I}^{ijk}(\Phi, \Psi, \gamma)| > \mu\,|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$ for every $i < j < k$ in J_1**.**

Consider the sets of bad cherries restricted to the holes Φ and Ψ defined by

$$\mathscr{I}_\Phi^{ijk} = \mathscr{I}^{ijk}(\Phi, \Psi, \gamma) \cap \Phi^{ij} \times \Phi^{ik} \quad \text{and} \quad \mathscr{I}_\Psi^{ijk} = \mathscr{I}^{ijk}(\Phi, \Psi, \gamma) \cap \Psi^{ij} \times \Psi^{ik}$$

for every $i < j < k$ in J_1. Note that $|\mathscr{I}_\Phi^{ijk}| > \frac{\mu}{2}|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$ or $|\mathscr{I}_\Psi^{ijk}| > \frac{\mu}{2}|\mathcal{P}^{ij}||\mathcal{P}^{ik}|$. Then, with out loss of generality, and through an other application of Ramsey's theorem we may assume that there is a set $J_2 \subseteq J_1$ of size at least t_2 such that

$$|\mathscr{I}_\Phi^{ijk}| > \frac{\mu}{2}|\mathcal{P}^{ij}||\mathcal{P}^{ik}|, \tag{3.3.23}$$

for every $i < j < k$ in J_2.

Consider an auxiliary reduced hypergraph $\mathcal{A}'$ with $V(\mathcal{A}') = V(\mathcal{A})$ and with edges defined for every $i < j < k$ in J_2 by

$$(P^{ij}, P^{ik}, P^{jk}) \in E(\mathcal{A}') \quad \Longleftrightarrow \quad (P^{ij}, P^{ik}) \in \mathscr{I}_\Phi^{ijk} \smallsetminus \mathscr{L}^{ijk}(\Phi, \gamma/2)$$

and notice that $\mathcal{A}'$ is not necessarily a subhypergraph of $\mathcal{A}$.

Observe that for every $i < j < k$ in J_2, due to (3.3.1) we have

$$|\mathscr{L}^{ijk}(\Phi, \gamma/2)| \leqslant \frac{2\nu}{\gamma}|\mathcal{P}^{ij}||\mathcal{P}^{ik}| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|,$$

which together with (3.3.23) yield that $\mathcal{A}'$ is $(\mu, \therefore)$-dense.

Moreover, since we have

$$|\mathscr{L}^{ijk}(\Phi, \gamma/2)| \leqslant \frac{2\nu}{\gamma}|\mathcal{P}^{ij}||\mathcal{P}^{ik}| \qquad \text{and} \qquad |\mathscr{R}^{ijk}(\Phi, \gamma/2)| \leqslant \frac{2\nu}{\gamma}|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,,$$

our choice of constants allows us to apply Lemma 3.2.5. Thus, we obtain a subset $J_3 \subseteq J_2$ of size t_3 and transversals $\mathcal{Q}(J_3)$, $\mathcal{R}(J_3)$, and $\mathcal{S}(J_3)$ that avoid $\mathscr{L}(\Phi, \gamma/2)$ and $\mathscr{R}(\Phi, \gamma/2)$ and form an inhabited triple of transversals in $\mathcal{A}'$.

First, since the triple $\mathcal{QRS}$ is inhabited, we have $Q^{ij}R^{ik}S^{jk} \in E(\mathcal{A}')$ for every three indices $i < j < k$ in J_3. This is to say

$$(Q^{ij}, R^{ik}) \in \mathscr{I}_\Phi^{ijk} \smallsetminus \mathscr{L}^{ijk}(\Phi, \gamma/2). \tag{3.3.24}$$

We remark that the reduced hypergraph $\mathcal{A}'$ and the transversal $\mathcal{S}$ are not relevant for the rest of the proof.

By the definitions of $\mathscr{I}_\Phi^{ijk}$ and $\mathscr{L}^{ijk}(\Phi, \gamma/2)$, (3.3.24) tell us that

$$|N(Q^{ij}, R^{ik}) \smallsetminus \Psi^{jk}| \geqslant \gamma|\mathcal{P}^{jk}| \qquad \text{and} \qquad |N(Q^{ij}, R^{ik}) \cap \Phi^{jk}| < \frac{\gamma}{2}|\mathcal{P}^{jk}|.$$

Moreover, we have that $|N(Q^{ij}, R^{ik})| \geqslant (1/3+\varepsilon)|\mathcal{P}^{jk}|$ and therefore we obtain

$$|N(Q^{ij}, R^{ik}) \smallsetminus (\Phi^{jk} \cup \Psi^{jk})| \geqslant \frac{\gamma}{2}|\mathcal{P}^{jk}|\,, \tag{3.3.25}$$

for every $i < j < k$ in J_3.

Second, since the transversals $\mathcal{Q}$ and $\mathcal{R}$ avoid $\mathscr{R}(\Phi, \gamma/2)$ we have that for every fixed indices $i < j < k < \ell$ in J_3 the neighbourhoods

$$|N(Q^{ik}, Q^{jk}) \cap \Phi^{ij}| \leqslant \frac{\gamma}{2}|\mathcal{P}^{ij}| \leqslant \frac{\varepsilon}{2}|\mathcal{P}^{ij}| \quad \text{and} \quad |N(R^{i\ell}, R^{j\ell}) \cap \Phi^{ij}| \leqslant \frac{\gamma}{2}|\mathcal{P}^{ij}| \leqslant \frac{\varepsilon}{2}|\mathcal{P}^{ij}|\,.$$

Since Φ has width $1/3+\varepsilon$ and by the $\mathord{\wedge}$-density, this implies that

$$|N(Q^{ik}, Q^{jk}) \cap N(R^{i\ell}, R^{j\ell})| \geqslant \varepsilon|\mathcal{P}^{ij}|\,. \tag{3.3.26}$$

In order to prove *(A)* we consider for every $x < i < j$ in J_3 the set

$$\Omega_x^{ij} = N(Q^{xi}, R^{xj}) \subseteq \mathcal{P}^{ij}. \tag{3.3.27}$$

Observe that if there is a subset $J'_\star \subseteq J_3$ of size at least t_4+1 and such that for every $x < i < j < k$ in $J'_\star$ we have

$$e(\Omega_x^{ij}, \Omega_x^{ik}, \Omega_x^{jk}) \leqslant \nu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|, \tag{3.3.28}$$

then the set $\Omega = \bigcup_{i<j\in J'_\star \smallsetminus \{x_0\}} \Omega_{x_0}^{ij}$ with $x_0 = \min J'_\star$ is a $(\nu, t_4, 1/3+\varepsilon)$-hole. An application of Corollary 3.3.13 implies that there is a subset $J_\star \subseteq J'_\star$ of size at least t in which $\Omega \cup \Phi$ and Ψ or $\Omega \cup \Psi$ and Φ are two ε-disjoint μ-holes. By taking $\Phi_\star = \Phi \cup \Omega$ and $\Psi_\star = \Psi$ in the first case or $\Phi_\star = \Phi$ and $\Psi_\star = \Psi \cup \Omega$ in the second, (3.3.25) implies that *(A)* follows.

Therefore, we may assume that (3.3.28) does not hold, and by an application of Ramsey's theorem for 4-uniform hypergraphs there is a set $J_4 \subseteq J_3$ of size at least t_4 such that

$$e(\Omega_x^{ij}, \Omega_x^{ik}, \Omega_x^{jk}) > \nu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}| \tag{3.3.29}$$

for every four indices $x < i < j < k$ in J_4.

Take disjoint subsets X_4, K_4, L_4, and M_4 of J_4 each of them of size $\lfloor |J_4|/4 \rfloor$ and such that for every $(x,k,\ell,m) \in X_4 \times K_4 \times L_4 \times M_4$ we have $x < k < \ell < m$. Observe that, for every fixed $x \in X_4$, even if we only consider the edges in the restricted constituents $\mathcal{A}^{k\ell m}[\Omega_x^{k\ell}, \Omega_x^{km}, \Omega_x^{\ell m}]$ for every $(k,\ell,m) \in K_4 \times L_4 \times M_4$, the resulting reduced hypergraph $\mathcal{A}_x$ is $(\nu, \therefore)$-tridense, because of (3.3.29). Then, an application of Lemma 3.2.7 with $r = 1$ yields the existence of subsets $X_5 \subseteq X_4$, $K_5 \subseteq K_4$, $L_5 \subseteq L_4$, and $M_5 \subseteq M_4$ each of them of size t_5, and transversals $\mathcal{T}(K_5, L_5)$, $\mathcal{U}(K_5, M_5)$, and $\mathcal{V}(L_5, M_5)$ such that

$$T^{k\ell} U^{km} V^{\ell m} \in E(\mathcal{A}_x), \tag{3.3.30}$$

for every $(k,\ell,m) \in K_5 \times L_5 \times M_5$ and $x \in X_5$. Observe that this means $T^{k\ell} \in \Omega_x^{k\ell}$, $U^{km} \in \Omega_x^{km}$, and $V^{\ell m} \in \Omega_x^{\ell m}$. Moreover, recalling (3.3.27) we have

$$Q^{xk} R^{x\ell} T^{k\ell} \in E(\mathcal{A}^{xk\ell}), \quad Q^{xk} R^{xm} U^{km} \in E(\mathcal{A}^{xkm}), \quad \text{and} \quad Q^{x\ell} R^{xm} V^{\ell m} \in E(\mathcal{A}^{x\ell m}).$$

In other words all three triples of transversals

$$\mathcal{Q}(X_5, K_5)\mathcal{R}(X_5, L_5)\mathcal{T}(K_5, L_5), \qquad \mathcal{Q}(X_5, K_5)\mathcal{R}(X_5, M_5)\mathcal{U}(K_5, M_5),$$
$$\text{and} \qquad \mathcal{Q}(X_5, L_5)\mathcal{R}(X_5, M_5)\mathcal{V}(L_5, M_5)$$

are inhabited. Note that here we consider restrictions of the transversals $\mathcal{Q}$ and $\mathcal{R}$ on different subsets of indices. Moreover, from (3.3.30) we infer that the triple of transversals

$$\mathcal{T}(K_5, L_5)\mathcal{U}(K_5, M_5)\mathcal{V}(L_5, M_5)$$

is also inhabited.

We iteratively apply Corollary 3.3.8 four times to these triples of inhabited transversals. After these four applications we obtain index sets X_6, K_6, L_6, and M_6 each of them of size at least 2, which satisfy *(i)* and *(ii)* of Corollary 3.3.8 for all those four inhabited triples of transversals.

We shall show that the two pairs of restricted transversals $\mathcal{T}(K_6, L_6)\mathcal{U}(K_6, M_6)$ and $\mathcal{U}(K_6, M_6)\mathcal{V}(L_6, M_6)$ have ε-intersecting links which contradicts *(i)* of Corollary 3.3.8 and concludes the proof.

First we show that the pair $\mathcal{T}(K_6, L_6)\mathcal{U}(K_6, M_6)$ has ε-intersecting links. Because of (i) we only need to prove that for some $k, k' \in K_6$, $\ell \in L_6$, and $m \in M_6$ we have

$$|N(T^{k\ell}, T^{k'\ell}) \cap N(U^{km}, U^{k'm})| \geqslant \varepsilon |\mathcal{P}^{kk'}|. \tag{3.3.31}$$

Fix $k, k' \in K_6$, $\ell \in L_6$, and $m \in M_6$, and consider $\mathcal{Q}(X_6, K_6)\mathcal{R}(X_6, L_6)\mathcal{T}(K_6, L_6)$. Because of (3.3.26) we have that for every $x, x' \in X_6$

$$|N(Q^{xk}, Q^{x'k}) \cap N(R^{x\ell}, R^{x'\ell})| \geqslant \varepsilon |\mathcal{P}^{xx'}|.$$

By (i) of Corollary 3.3.8 this implies that the whole pair $\mathcal{Q}(X_6, K_6)\mathcal{R}(X_6, L_6)$ has ε-intersecting links. Again by (i) this can hold for at most one pair of transversals, and the other two must have ε-disjoint links. Then, the pair $\mathcal{Q}(X_6, K_6)\mathcal{T}(K_6, L_6)$ has ε-disjoint links, and for every $x \in X_6$ we have

$$|N(Q^{xk}, Q^{xk}) \cap N(T^{k\ell}, T^{k'\ell})| < \varepsilon |\mathcal{P}^{kk'}|,$$

and we may some fix $x \in X_6$.

With a similar argument for $\mathcal{Q}(K_6, X_6)\mathcal{R}(X_6, M_6)\mathcal{U}(K_6, M_6)$ we obtain an analogous inequality

$$|N(Q^{xk}, Q^{xk'}) \cap N(U^{km}, U^{k'm})| < \varepsilon |\mathcal{P}^{kk'}|.$$

Finally since both neighbourhoods $N(T^{k\ell}, T^{k'\ell})$ and $N(U^{km}, U^{k'm})$ have small intersection with $N(Q^{xk}, Q^{xk'})$, and by the $(\mathord{\wedge}, 1/3 + \varepsilon)$-density condition, (3.3.31) follows.

The proof that the pair $\mathcal{U}(K_6, M_6)\mathcal{V}(L_6, M_6)$ has ε-intersecting links follows among the same lines, by considering the triples of transversals $\mathcal{Q}(X_6, K_6)\mathcal{R}(X_6, M_6)\mathcal{U}(K_6, M_6)$ and $\mathcal{Q}(X_6, L_6)\mathcal{R}(X_6, M_6)\mathcal{V}(L_6, M_6)$. This finishes the proof of the first case.

Second Case: $|\mathscr{D}^{ijk}(\Phi, \Psi, \gamma/2)| > \mu\, |\mathcal{P}^{ik}||\mathcal{P}^{jk}|$ for every $i < j < k$ in J_1**.**

The proof is identical to the first case after simply reversing the order of the indices, which exchanges the notions of left and right cherries.

Third Case: $|\mathscr{C}^{ijk}(\Phi, \Psi, \gamma/2)| > \mu\, |\mathcal{P}^{ij}||\mathcal{P}^{jk}|$ for every $i < j < k$ in J_1**.**

As before we consider the set of ε-bad cherries restricted to the different holes $\mathscr{C}_\Phi^{ijk}$ and $\mathscr{C}_\Psi^{ijk}$ and following the same Ramsey argument we find a subset $J_2 \subseteq J$ of size at least t_2 for which we may assume that

$$|\mathscr{C}_\Phi^{ijk}| > \frac{\mu}{2}|\mathcal{P}^{ij}||\mathcal{P}^{jk}|,$$

for every $i < j < k$ in J_2.

Again we define a reduced hypergraph $\mathcal{A}'$ with $V(\mathcal{A}') = V(\mathcal{A})$ and this time the edges are given by

$$(P^{ij}, P^{ik}, P^{jk}) \in E(\mathcal{A}') \quad \Longleftrightarrow \quad (P^{ij}, P^{jk}) \in \mathscr{C}_\Phi^{ijk} \smallsetminus \mathscr{M}^{ijk}(\Phi, \gamma/2),$$

for $i < j < k$ in J_2 (see Definition 3.3.2). Similarly as in the first case, by an application of Lemma 3.2.5 we obtain a set $J_3 \subseteq J_2$ of size t_3 and transversals $\mathcal{Q}$ and $\mathcal{S}$ which satisfy $(Q^{ij}, S^{jk}) \in \mathscr{C}_\Phi^{ijk} \smallsetminus \mathscr{M}^{ijk}(\Phi, \gamma/2)$ for every $i < j < k$ in J_3. This is to say, the following variant of (3.3.25) holds

$$|N(Q^{ij}, S^{jk}) \smallsetminus (\Phi^{ik} \cup \Psi^{ik})| \geqslant \frac{\gamma}{2}|\mathcal{P}^{ik}|. \tag{3.3.32}$$

Moreover, because of Lemma 3.2.5 transversals $\mathcal{Q}$ and $\mathcal{S}$ avoid the exceptional cherries from $\mathscr{L}(\Phi, \varepsilon/4)$ and $\mathscr{R}(\Phi, \varepsilon/4)$. With this we can deduce the following version of (3.3.26). For every $i < j < k < \ell$ in J_2 we have

$$|N(Q^{ij}, Q^{ik}) \cap N(S^{j\ell}, S^{k\ell})| \geqslant \varepsilon|\mathcal{P}^{jk}| \quad \text{and} \quad |N(S^{ik}, S^{i\ell}) \cap N(S^{jk}, S^{j\ell})| \geqslant \varepsilon|\mathcal{P}^{k\ell}|. \tag{3.3.33}$$

For every $i < x < j$ in J_3 define the set of vertices

$$\Omega_x^{ij} = N(Q^{ix}, S^{xj}) \subseteq \mathcal{P}^{ij}. \tag{3.3.34}$$

Observe that if there is a subset $J_\star' \subseteq J_3$ of size at least t_4 and such that for every five indices $i < x < j < y < k$ in $J_\star'$ we have

$$e(\Omega_x^{ij}, \Omega_x^{ik}, \Omega_y^{jk}) \leqslant \nu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}| \tag{3.3.35}$$

then, we can establish *(A)* as follows: consider $J''_\star$ to be every second element in $J'_\star$ and let $i^+ = \min\{j > i : j \in J'_\star\}$. Thus, the set $\Omega = \bigcup_{i<j\in J''_\star} \Omega^{ij}_{i^+}$ is a $(\nu, t_4, 1/3 + \varepsilon)$-hole. As in the first case, an application of Corollary 3.3.13 yields a set $J_\star \subseteq J''_\star$ of size t in which either $\Phi \cup \Omega$ and Ψ or Φ and $\Psi \cup \Omega$ are two ε-disjoint μ-holes. Because of (3.3.32) those two holes signify that *(A)* holds.

Therefore, we may assume that (3.3.35) does not hold, and by an application of Ramsey's theorem now for 5-uniform hypergraphs there exists a subset $J_4 \subseteq J_3$ such that for every $i < x < j < y < k$ in J_4.

$$e(\Omega^{ij}_x, \Omega^{ik}_x, \Omega^{jk}_y) > \nu|\mathcal{P}^{ij}||\mathcal{P}^{ik}||\mathcal{P}^{jk}|. \tag{3.3.36}$$

Take now sets of indices K_4, X_4, L_4, Y_4, and M_4 of size $\lfloor J_4/5 \rfloor$ and such that for every $(k, x, \ell, y, m) \in K_4 \times X_4 \times L_4 \times Y_4 \times M_4$ we have $k < x < \ell < y < m$. By (3.3.36) all reduced hypergraphs $\mathcal{A}_{(x,y)}$ given by the restrictions $\mathcal{A}^{k\ell m}_{(x,y)} = \mathcal{A}^{k\ell m}[\Omega^{k\ell}_x, \Omega^{km}_x, \Omega^{\ell m}_y]$ are $(\nu, \therefore)$-tridense for every $(x, y) \in X_4 \times Y_4$. We can apply Lemma 3.2.7 this time with $r = 2$. This application yields the existence of subsets $X_5 \subseteq X_4$, $Y_5 \subseteq Y_4$, $K_5 \subseteq K_4$, $L_5 \subseteq L_4$, and $M_5 \subseteq M_4$ each of size t_5, and transversals $\mathcal{T}(K_5, L_5)$, $\mathcal{U}(K_5, M_5)$, and $\mathcal{V}(L_5, M_5)$ such that for every $(k, \ell, m) \in K_5 \times L_5 \times M_5$ and $(x, y) \in X \times Y$ we have $T^{k\ell}U^{km}V^{\ell m} \in \mathcal{A}^{k\ell m}_{(x,y)}$. In particular, the triple of transversals

$$\mathcal{T}(K_5, L_5)\mathcal{U}(K_5, M_5)\mathcal{V}(L_5, M_5) \tag{3.3.37}$$

is inhabited. Moreover, this implies that $T^{k\ell} \in \Omega^{k\ell}_x$, $U^{km} \in \Omega^{km}_x$, and $V^{\ell m} \in \Omega^{\ell m}_y$, and due to (3.3.34) we obtain the edges

$$Q^{kx}T^{k\ell}S^{x\ell} \in E(\mathcal{A}^{kx\ell}), \quad Q^{kx}U^{km}S^{xm} \in E(\mathcal{A}^{kxm}), \quad \text{and} \quad Q^{\ell y}V^{\ell m}S^{ym} \in E(\mathcal{A}^{\ell ym}).$$

This is to say, the triples of transversals

$$\mathcal{Q}(K_5, X_5)\mathcal{T}(K_5, L_5)\mathcal{S}(X_5, L_5)\,, \quad \mathcal{Q}(K_5, X_5)\mathcal{U}(K_5, M_5)\mathcal{S}(X_5, M_5)\,,$$
$$\text{and} \quad \mathcal{Q}(L_5, Y_5)\mathcal{V}(L_5, M_5)\mathcal{S}(Y_5, M_5) \tag{3.3.38}$$

are all inhabited.

Again, we apply Corollary 3.3.8 iteratively four times to the triples of transversals from (3.3.37) and (3.3.38). Thus we obtain sets K_6, X_6, L_6, Y_6, and M_6 each of them of size at least t_6, satisfying *(i)* and *(ii)* of Corollary 3.3.8 for those triples of transversals. We show that the two pairs of restricted transversals $\mathcal{T}(K_6, L_6)\mathcal{U}(K_6, M_6)$ and $\mathcal{U}(K_6, M_6)\mathcal{V}(L_6, M_6)$ have ε-intersecting links which as in the first case contradicts *(i)* of Corollary 3.3.8 and concludes the proof.

The proof for the pair $\mathcal{T}(K_6, L_6)\mathcal{U}(K_6, M_6)$ follows from the same arguments presented in the first case. However, for $\mathcal{U}(K_6, M_6)\mathcal{V}(L_6, V_6)$ we proceed slightly different.

Because of *(i)* of Corollary 3.3.8 it is enough to prove that for some $k \in K_6$ and $m \in M_6$ it holds that

$$\Lambda(\mathcal{U}, M_6, k) \equiv_{\varepsilon, M_6} \Lambda(\mathcal{V}, M_6, \ell)\,. \tag{3.3.39}$$

First, consider the triple $\mathcal{Q}(K_6, X_6)\mathcal{U}(K_6, M_6)\mathcal{S}(K_6, M_6)$, and observe that (3.3.33) implies that

$$\Lambda(\mathcal{Q}, X_6, k) \equiv_{\varepsilon, X_6} \Lambda(\mathcal{S}, X_6, m)$$

for every $k \in K_6$ and $m \in M_6$. This means that the pair of transversals $\mathcal{Q}(K_6, X_6)\mathcal{S}(X_6, M_6)$ has ε-intersecting links. Because of *(i)* of Corollary 3.3.8 at most one of the three pairs of transversals can have ε-intersecting links, and the rest must have ε-disjoint links. In particular, for every $k \in K_6$ and $x \in X_6$ we have

$$\Lambda(\mathcal{U}, M_6, k) \;\text{ and } \Lambda(\mathcal{S}, M_6, x) \text{ are } \varepsilon\text{-disjoint } (\mu, t_6, 1/3 + \varepsilon)\text{-holes.} \tag{3.3.40}$$

Reasoning analogously for the triple of transversals $\mathcal{Q}(Y_6, L_6)\mathcal{V}(L_6, M_6)\mathcal{S}(Y_6, M_6)$ we can deduce that for every $\ell \in L_6$ and $y \in Y_6$

$$\Lambda(\mathcal{V}, M_6, \ell) \text{ and } \Lambda(\mathcal{S}, M_6, y) \text{ are } \varepsilon\text{-disjoint } (\mu, t_6, 1/3 + \varepsilon)\text{-holes.} \tag{3.3.41}$$

Moreover, because of (3.3.33), for every $x \in X_6$ and $y \in Y_6$ we obtain

$$\Lambda(\mathcal{S}, M_6, x) \equiv_{\varepsilon, M_6} \Lambda(\mathcal{S}, M_6, y). \tag{3.3.42}$$

Observe that, by *(ii)* of Corollary 3.3.8, all three relations (3.3.40), (3.3.41), and (3.3.42) concern μ-holes in M_6 of width at least $1/3 + \varepsilon$. Then, by an application of Lemma 3.3.11 with $r = 4$ we obtain a subset $M_7 \subseteq M_6$ of size at least two such that $\equiv_{\varepsilon,M_7}$ is an equivalence relation with at most two equivalence classes. Therefore, since (3.3.40), (3.3.41), and (3.3.42) are closed under subsets of indices, we conclude

$$\Lambda(\mathcal{U}, M_7, k) \equiv_{\varepsilon,M_7} \Lambda(\mathcal{V}, M_7, \ell).$$

By *(i)* of Corollary 3.3.8 this implies (3.3.39). □

We can iteratively apply Lemma 3.3.14 to eventually arrive at alternative *(B)*. If after an application of Lemma 3.3.14 to a pair of ε-disjoint ν-holes we conclude that *(A)* holds, then we obtain two ε-disjoint μ-holes with $\mu > \nu$ and on a smaller set of indices, but for which the sum of the widths is larger. With a suitable choice of constants we can apply Lemma 3.3.14 again and repeat this procedure finitely many times. In each time we increase the sum of the widths by $\gamma/2$ and therefore after at most $4\gamma^{-1}$ iterations alternative *(B)* must hold. Thus, we obtain the following corollary.

Corollary 3.3.15. *For every μ, $\varepsilon \geqslant \gamma > 0$ and $t \in \mathbb{N}$ there is $\nu > 0$ such that the following holds. Suppose $\mathcal{A}$ is an ε-wicked reduced hypergraph with index set I and for sufficiently large $J \subseteq I$ we are given ε-disjoint $(\nu, |J|, 1/3 + \varepsilon)$-holes Φ and Ψ.*

Then, there exists a subset $J_\star \subseteq J$ of size t and ε-disjoint $(\mu, t, 1/3 + \varepsilon)$-holes $\Phi_\star$ and $\Psi_\star$ such that for all $i < j < k$ in $J_\star$ the sets of γ-bad cherries satisfy

$$|\mathscr{I}^{ijk}(\Phi_\star, \Psi_\star, \gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|\,, \quad |\mathscr{D}^{ijk}(\Phi_\star, \Psi_\star, \gamma)| \leqslant \mu|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,,$$
$$\textit{and} \qquad |\mathscr{C}^{ijk}(\Phi_\star, \Psi_\star, \gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{jk}|\,. \qquad \square$$

3.3.6 Bicolourisation

In this section we use the previous results on holes to find a suitable bipartition of the vertices. Through this partition and some modifications of the hypergraph we construct the bicoloured reduced hypergraph $\mathcal{A}_\star$ stated in Proposition 3.1.5.

Roughly speaking, $\mathcal{A}_\star$ will be the preimage of a random homomorphism (see Definition 3.2.6) from the given wicked reduced hypergraph $\mathcal{A}$ restricted to the symmetric difference of the two holes $\Phi_\star$ and $\Psi_\star$ provided by Corollary 3.3.15. The bicolouring φ of $V(\mathcal{A}_\star)$ is defined through the holes $\Phi_\star$ and $\Psi_\star$.

For the application of Corollary 3.3.15 we need to establish the existence of two essentially disjoint holes. Those will be provided by Corollary 3.3.8. Moreover, the inhabited triple of transversals required for the application of Corollary 3.3.8 will be given by Theorem 3.2.3. Below we give the details of this proof.

Proof of Proposition 3.1.5. Given ε and t let γ, $\mu > 0$ and $\ell \in \mathbb{N}$ be such that

$$\gamma = \frac{\varepsilon}{12} \qquad \text{and} \qquad \varepsilon, t^{-1} \gg \ell^{-1} \gg \mu \gg t_2^{-1} \gg t_1^{-1}\,.$$

Then, let ν be given by Corollary 3.3.15 and let $t_2 \geqslant t$ to be large enough for such an application. Let t_1 to be sufficiently large to apply Corollary 3.3.8 with $t = t_2$, $\delta = \varepsilon$, $\mu = \nu$, and $d = 1/3 + \varepsilon$.

Recalling that by definition every $(d, \wedge)$-dense reduced hypergraphs is in particular $(d, \therefore)$-dense, we let $\mathcal{A}$ be an ε-wicked reduced hypergraph with a sufficiently large index set I, so that we can apply Theorem 3.2.3 with $t = 3t_1$ and $\mu = 1/3+\varepsilon$. Consequently, we obtain a subset $I_1 \subseteq I$ of size $3t_1$ and an inhabited triple of transversals $\mathcal{Q}(I_1)\mathcal{R}(I_1)\mathcal{S}(I_1)$.

Fix an arbitrary partition $K_1 \cup L_1 \cup M_1$ of I_1 with partition classes of size t_1. Corollary 3.3.8 applied to the inhabited triple of transversals $\mathcal{Q}(K_1, L_1)\mathcal{R}(K_1, M_1)\mathcal{S}(L_1, M_1)$ yields subsets $K_2 \subseteq K_1$, $L_2 \subseteq L_1$, and $M_2 \subseteq M_1$ of size t_2 satisfying properties *(i)* and *(ii)* of the corollary.

Without loss of generality, we may assume that $\mathcal{Q}(K_2, L_2)\mathcal{R}(K_2, M_2)$ has ε-disjoint links. Thus, by arbitrarily fixing $\ell \in L_2$ and $m \in M_2$ we obtain ε-disjoint ν-holes Φ and Ψ defined by

$$\Phi = \Lambda(\mathcal{Q}, K_2, \ell) \quad \text{and} \quad \Psi = \Lambda(\mathcal{R}, K_2, m)\,.$$

Next, we apply Corollary 3.3.15 to obtain a set $J_3 \subseteq K_2$ of size t and ε-disjoint μ-holes $\Phi_\star$ and $\Psi_\star$ such that for every $i < j < k$ in J_3

$$|\mathscr{I}^{ijk}(\Phi_\star, \Psi_\star, \gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{ik}|\,, \quad |\mathscr{D}^{ijk}(\Phi_\star, \Psi_\star, \gamma)| \leqslant \mu|\mathcal{P}^{ik}||\mathcal{P}^{jk}|\,,$$
$$\text{and} \quad |\mathscr{C}^{ijk}(\Phi_\star, \Psi_\star, \gamma)| \leqslant \mu|\mathcal{P}^{ij}||\mathcal{P}^{jk}|\,. \tag{3.3.43}$$

We define a bicoloured reduced hypergraph $\mathcal{A}_1$ which satisfies the minimum codegree conditions required by the proposition, for almost every monochromatic pair. For every pair $ij \in J_3^{(2)}$ consider the colour classes

$$\mathfrak{R}^{ij} = \Phi_\star^{ij} \smallsetminus \Psi_\star^{ij} \quad \text{and} \quad \mathfrak{B}^{ij} = \Psi_\star^{ij} \smallsetminus \Phi_\star^{ij}\,,$$

and let $\mathfrak{R} = \bigcup_{ij \in J_3^{(2)}} \mathfrak{R}^{ij}$ and $\mathfrak{B} = \bigcup_{ij \in J_3^{(2)}} \mathfrak{B}^{ij}$. Define the reduced hypergraph $\mathcal{A}_1$ on the index set J_3 with vertex classes $\mathcal{P}_1^{ij} = \mathfrak{R}^{ij} \cup \mathfrak{B}^{ij} \subseteq \mathcal{P}^{ij}$ for every $ij \in J_3^{(2)}$ and with edges given by

$$E(\mathcal{A}_1) = E(\mathcal{A}[\mathfrak{R} \cup \mathfrak{B}]) \smallsetminus (E(\mathfrak{R}) \cup E(\mathfrak{B}))\ .$$

Now we show that monochromatic cherries which are not γ-bad have large codegree (in $\mathcal{A}_1$). For indices $i < j < k$ in J_3, consider a cherry $(R^{ij}, R^{ik}) \in \mathfrak{R}^{ij} \times \mathfrak{R}^{ik}$ such that $(R^{ij}, R^{ik}) \notin \mathscr{I}^{ijk}(\Phi_\star, \Psi_\star, \gamma)$, then we have

$$|N_{\mathcal{A}}(R^{ij}, R^{ik}) \smallsetminus \Psi_\star^{jk}| \leqslant \gamma |\mathcal{P}^{jk}|.$$

Consequently, since $\Phi_\star$ and $\Psi_\star$ are ε-disjoint and using the $\text{\Lambda}$-density condition we conclude that

$$\begin{aligned}
|N_{\mathcal{A}_1}(R^{ij}, R^{ik})| &= |N_{\mathcal{A}}(R^{ij}, R^{ik}) \cap \mathfrak{B}^{jk}| \\
&\geqslant |N_{\mathcal{A}}(R^{ij}, R^{ik})| - |N_{\mathcal{A}}(R^{ij}, R^{ik}) \smallsetminus \Psi_\star^{jk}| - |\Phi_\star^{jk} \cap \Psi_\star^{jk}| \\
&\geqslant \left(\frac{1}{3} + \varepsilon\right) |\mathcal{P}^{jk}| - \gamma |\mathcal{P}^{jk}| - |\Phi_\star^{jk} \cap \Psi_\star^{jk}| \\
&= \left(\frac{1}{3} + \varepsilon - \gamma\right) \left(|\mathcal{P}^{jk}| - |\Phi_\star^{jk} \cap \Psi_\star^{jk}|\right) - \left(\frac{2}{3} - \varepsilon + \gamma\right) |\Phi_\star^{jk} \cap \Psi_\star^{jk}| \\
&\geqslant \left(\frac{1}{3} + \varepsilon - \gamma - \left(\frac{2}{3} - \varepsilon + \gamma\right) \frac{\varepsilon}{1 - \varepsilon}\right) \left(|\mathcal{P}^{jk}| - |\Phi_\star^{jk} \cap \Psi_\star^{jk}|\right) \\
&\geqslant \left(\frac{1}{3} + \frac{\varepsilon}{4}\right) |\mathcal{P}_1^{jk}|\,, \qquad (3.3.44)
\end{aligned}$$

where the last inequality comes from our choice of γ and from $\mathcal{P}_1^{jk} \subseteq \mathcal{P}^{jk} \smallsetminus (\Phi_\star^{jk} \cap \Psi_\star^{jk})$.

We can deduce analogous inequalities for cherries $(R^{ik}, R^{jk}) \notin \mathscr{D}^{ijk}(\Phi_\star, \Psi_\star, \gamma)$ and for cherries $(R^{ij}, R^{jk}) \notin \mathscr{C}^{ijk}(\Phi_\star, \Psi_\star, \gamma)$. Similarly, we obtain those bounds for non-bad cherries in $\mathfrak{B}$. As a result, the reduced hypergraph $\mathcal{A}_1$ satisfies a minimum codegree condition for all monochromatic pairs that are not γ-bad cherries.

Finally, similar as in [50, Lemma 4.2] we define the reduced hypergraph $\mathcal{A}_\star$ by taking preimages of a random homomorphism $h \in \mathfrak{A}(\mathcal{A}_1, \ell)$ (see Definition 3.2.6). We show that there is a choice of $h \in \mathfrak{A}(\mathcal{A}_1, \ell)$ for which the associated reduced hypergraph $\mathcal{A}_h$ with index set J_3 and vertex set $V(\mathcal{A}_h) = \bigcup_{ij \in J_3^{(2)}} \mathcal{P}^{ij}_\bullet$ satisfies the desired properties.

First, observe that for any choice of the map h, since $\mathcal{A}_1 \subseteq \mathcal{A}$ does not support a $K_5^{(3)}$, neither does $\mathcal{A}_h$. Moreover, the bicolouring $V(\mathcal{A}_1) = \mathfrak{R} \cup \mathfrak{B}$ of $\mathcal{A}_1$ induces a bicolouring $\varphi_h \colon V(\mathcal{A}_h) \longrightarrow \{\text{red}, \text{blue}\}$ of $\mathcal{A}_h$ defined by

$$\varphi_h(P^{ij}_\bullet) = \text{red} \iff h(P^{ij}_\bullet) \in \mathfrak{R} \quad \text{and} \quad \varphi_h(P^{ij}_\bullet) = \text{blue} \iff h(P^{ij}_\bullet) \in \mathfrak{B}\,.$$

Therefore, it is left to prove that with positive probability we have

$$\tau_2(\mathcal{A}_h, \varphi_h) \geqslant \frac{1}{3} + \frac{\varepsilon}{8}\,. \tag{3.3.45}$$

For indices $i < j < k$ in J_3 and a cherry $(P^{ij}_\bullet, P^{ik}_\bullet) \in \mathcal{P}^{ij}_\bullet \times \mathcal{P}^{ik}_\bullet$ let $\mathcal{X} = \mathcal{X}(P^{ij}_\bullet, P^{ik}_\bullet)$ be the event

$$\varphi_h(P^{ij}_\bullet) = \varphi_h(P^{ik}_\bullet) \quad \text{and} \quad |N_{\mathcal{A}_h}(P^{ij}_\bullet, P^{ik}_\bullet)| < \Big(\frac{1}{3} + \frac{\varepsilon}{8}\Big)|\mathcal{P}^{jk}_\bullet|\,.$$

This is to say, $\mathcal{X}$ is the event in which the pair $(P^{ij}_\bullet, P^{jk}_\bullet)$ violates condition (3.3.45). Note that for monochromatic pairs that are not γ-bad, (3.3.44) tells us that the expected size of their neighbourhood is large. More precisely, if

$$\varphi_h(P^{ij}_\bullet) = \varphi_h(P^{ik}_\bullet) \quad \text{and} \quad (h(P^{ij}_\bullet), h(P^{ik}_\bullet)) \notin \mathscr{I}^{ijk}(\Phi_\star, \Psi_\star, \gamma)$$

then,

$$\mathbb{E}\left(|N_{\mathcal{A}_h}(P^{ij}_\bullet, P^{ik}_\bullet)|\right) \geqslant \left(\frac{1}{3} + \frac{\varepsilon}{4}\right)|\mathcal{P}^{jk}_\bullet|\,.$$

Therefore, by Chernoff's inequality, we obtain

$$\mathbb{P}\left(\mathcal{X} \,\middle|\, \varphi_h(P^{ij}_\bullet) = \varphi_h(P^{ik}_\bullet) \text{ and } (h(P^{ij}_\bullet), h(P^{ik}_\bullet)) \notin \mathscr{I}^{ijk}(\Phi_\star, \Psi_\star, \gamma)\right) \leqslant \exp\left(-\tfrac{\varepsilon^2 \ell}{128}\right).$$

Consequently, in view of (3.3.43) we can bound the probability of $\mathcal{X}$ by

$$\begin{aligned}\mathbb{P}(\mathcal{X}) &= |\mathcal{P}^{ij}_1|^{-1}|\mathcal{P}^{ik}_1|^{-1} \sum_{(P^{ij}, P^{ik}) \in \mathcal{P}^{ij}_1 \times \mathcal{P}^{ik}_1} \mathbb{P}\left(\mathcal{X} \,\middle|\, h(P^{ij}_\bullet) = P^{ij} \text{ and } h(P^{ik}_\bullet) = P^{ik}\right) \\ &\leqslant \mu + \exp\left(-\tfrac{\varepsilon^2 \ell}{128}\right).\end{aligned}$$

Analogous inequalities can be deduced for monochromatic cherries which are not in the central bad cherries $\mathscr{C}^{ijk}(\Phi_\star, \Psi_\star, \gamma)$ or the right bad cherries $\mathscr{D}^{ijk}(\Phi_\star, \Psi_\star, \gamma)$.

Finally, since there are at most $3\ell^2\binom{t}{3}$ cherries to consider, we arrive at

$$\mathbb{P}\left(\tau_2(\mathcal{A}_h, \varphi_h) < \frac{1}{3} + \frac{\varepsilon}{8}\right) \leqslant 3\ell^2\binom{t}{3}\left(\mu + \exp\left(-\tfrac{\varepsilon^2\ell}{128}\right)\right).$$

Owing to the hierarchy $\mu \ll \ell^{-1} \ll t^{-1}$ this probability is smaller than 1, and therefore there is a map $h \in \mathfrak{A}(\mathcal{A}_1, \ell)$ for which $\mathcal{A}_h$ has the desired properties. □

3.4 $K_5^{(3)}$ in bicoloured reduced hypergraphs

In this section we establish Proposition 3.1.6 and show that bicoloured reduced hypergraphs with minimum monochromatic codegree density bigger than 1/3 support a $K_5^{(3)}$.

In the proof we shall use the following type of neighbourhoods in reduced hypergraphs $\mathcal{A}$. For two vertices P, $P' \in V(\mathcal{A})$ and a subset $\mathcal{U} \subseteq V(\mathcal{A})$ we denote by $N_{\mathcal{U}}(P, P')$ the neighbourhood restricted to U. Similarly, for two subsets $\mathcal{U}, \mathcal{U}' \subseteq V(\mathcal{A})$ we write $N_{\mathcal{U}\times\mathcal{U}'}(P)$ for the set of pairs in $\mathcal{U} \times \mathcal{U}'$ that together with P form a hyperedge in $\mathcal{A}$, i.e.,

$$N_{\mathcal{U}}(P, P') = \{U \in \mathcal{U} \colon PP'U \in E(\mathcal{A})\}$$
$$\text{and} \quad N_{\mathcal{U}\times\mathcal{U}'}(P) = \{(U, U') \in \mathcal{U} \times \mathcal{U}' \colon PUU' \in E(\mathcal{A})\}.$$

Proof of Proposition 3.1.6. Given $\varepsilon > 0$ we fix a sufficiently small auxiliary constant ξ with $0 < \xi \ll \varepsilon$ such that $\frac{1/6+\varepsilon}{\xi}$ equals to some integer s. Moreover, let I be a sufficiently large index set such that its cardinality satisfies the partition relation $|I| \longrightarrow (5)^2_s$, i.e., it is as least as large as the s-colour Ramsey number for the graph clique K_5. Let $\mathcal{A}$ be a bicolored reduced hypergraph with index set I and vertex classes $\mathcal{P}^{ij}$ for $ij \in I^{(2)}$ and let $\varphi\colon V(\mathcal{A}) \longrightarrow \{\text{red}, \text{blue}\}$ satisfy $\tau_2(\mathcal{A}, \varphi) \geqslant 1/3 + \varepsilon$.

For every $ij \in I^{(2)}$ set

$$\mathfrak{R}^{ij} = \varphi^{-1}(\text{red}) \cap \mathcal{P}^{ij} \qquad \text{and} \qquad \varrho_{ij} = \frac{|\mathfrak{R}^{ij}|}{|\mathcal{P}^{ij}|}$$

and, analogously, we define $\mathfrak{B}^{ij} = \varphi^{-1}(\text{blue}) \cap \mathcal{P}^{ij}$ and $\beta_{ij} = |\mathfrak{B}^{ij}|/|\mathcal{P}^{ij}|$. In view of (3.1.1), the assumption on $\tau_2(\mathcal{A},\varphi)$ implies that all ϱ_{ij}, $\beta_{ij} \in [1/3+\varepsilon, 2/3-\varepsilon]$. Splitting the interval $[1/3+\varepsilon, 2/3-\varepsilon]$ into s intervals of length 2ξ, the size of I yields a subset $J \subseteq I$ of size 5 such that all β_{ij} with $ij \in J^{(2)}$ are in the same interval. Let β be the centre of this interval and set $\varrho = 1-\beta$. We thus arrive at

$$\beta_{ij} = \beta \pm \xi \qquad \text{and} \qquad \varrho_{ij} = \varrho \pm \xi$$

for all $ij \in J^{(2)}$. Without loss of generality we may assume $\beta \leqslant \varrho$, which implies

$$\frac{1}{3} + \varepsilon \leqslant \beta - \xi < \beta \leqslant \frac{1}{2} \leqslant \varrho < \varrho + \xi \leqslant \frac{2}{3} - \varepsilon\,. \tag{3.4.1}$$

For $ijk \in J^{(3)}$ the codegree condition translates for red vertices $R^{ij} \in \mathfrak{R}^{ij}$ and $R^{ik} \in \mathfrak{R}^{ik}$ to

$$\begin{aligned} |N_{\mathfrak{B}^{jk}}(R^{ij}, R^{ik})| = d(R^{ij}, R^{jk}) &\geqslant \left(\frac{1}{3}+\varepsilon\right)|\mathcal{P}^{jk}| \\ &\geqslant \left(\frac{1}{3}+\varepsilon\right)\left(\frac{1}{\beta+\xi}\right)|\mathfrak{B}^{jk}| \geqslant \left(\frac{1}{3\beta}+\frac{\varepsilon}{2}\right)|\mathfrak{B}^{jk}|\,, \end{aligned} \tag{3.4.2}$$

where we used $\xi \ll \varepsilon, \beta$ for the last inequality. Similarly, for blue vertices we have

$$|N_{\mathfrak{R}^{jk}}(B^{ij}, B^{ik})| \geqslant \left(\frac{1}{3\varrho}+\frac{\varepsilon}{2}\right)|\mathfrak{B}^{jk}|\,. \tag{3.4.3}$$

We may rename the indices in J and assume that $J = \mathbb{Z}/5\mathbb{Z}$. We shall show that $\mathcal{A}$ restricted to J supports a $K_5^{(3)}$. For that we have to find ten vertices $P^{ij} \in \mathcal{P}^{ij}$ one for every $ij \in J^{(2)}$ such that for all of the ten triples $ijk \in J^{(3)}$ the vertices P^{ij}, P^{ik}, and P^{jk} span a hyperedge in the constituent $\mathcal{A}^{ijk}$. For every $i \in J = \mathbb{Z}/5\mathbb{Z}$ we will select $P^{i,i+1}$ from $\mathfrak{B}^{i,i+1}$ and $P^{i,i+2}$ from $\mathfrak{R}^{i,i+2}$. Since $\mathcal{A}$ contains no monochromatic triples as hyperedges, it is easy to see that up to symmetry this choice for the colour classes is unavoidable, as it corresponds to the unique 2-colouring of $E(K_5)$ with no monochromatic triangle.

The rest of the proof is based on several averaging arguments relying on the minimum degree condition. For generic vertices from $\mathfrak{R}$ and $\mathfrak{B}$ we shall use capital letters R and B. In the process we will make appropriate choices to fix the ten special vertices that signify the supported $K_5^{(3)}$. For those vertices we will use small letters r and b depending on its colour.

We begin with the selection of $r^{14} \in \mathfrak{R}^{14}$. Applying (3.4.3) to all pairs of vertices $B^{15} \in \mathfrak{B}^{15}$ and $B^{45} \in \mathfrak{B}^{45}$ implies that the total number of hyperedges in $\mathcal{A}^{145}$ crossing the sets $\mathfrak{R}^{14}$, $\mathfrak{B}^{15}$, and $\mathfrak{B}^{45}$ is at least

$$|\mathfrak{B}^{15}||\mathfrak{B}^{45}| \cdot \left(\frac{1}{3\varrho} + \frac{\varepsilon}{2}\right)|\mathfrak{R}^{14}|\,.$$

Consequently, we can fix some vertex $r^{14} \in \mathfrak{R}^{14}$ such that

$$|N_{\mathfrak{B}^{15}\times\mathfrak{B}^{45}}(r^{14})| \geqslant \left(\frac{1}{3\varrho} + \frac{\varepsilon}{2}\right)|\mathfrak{B}^{15}||\mathfrak{B}^{45}|\,. \tag{3.4.4}$$

The following claim fixes the four vertices b^{12}, b^{34} and r^{13}, r^{24}.

Claim 1. *There exist blue vertices $b^{12} \in \mathfrak{B}^{12}$, $b^{34} \in \mathfrak{B}^{34}$ and red vertices $r^{13} \in \mathfrak{R}^{13}$, $r^{24} \in \mathfrak{R}^{24}$ such that*

(*i*) *$b^{12}r^{14}r^{24}$ and $r^{13}r^{14}b^{34}$ are hyperedges in $\mathcal{A}$*

(*ii*) *and $|N_{\mathfrak{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24}, b^{34})| \geqslant \left(1 - \frac{1}{3\beta}\right)|\mathfrak{B}^{23}|$.*

Proof. Owing to (3.4.2) for every $R^{13} \in \mathfrak{R}^{13}$ we have $d(R^{13}, r^{14}) \geqslant \left(\frac{1}{3\beta} + \frac{\varepsilon}{2}\right)|\mathfrak{B}^{34}|$ and, hence, there is a vertex $b^{34} \in \mathfrak{B}^{34}$ such that

$$|N_{\mathfrak{R}^{13}}(r^{14}, b^{34})| \geqslant \left(\frac{1}{3\beta} + \frac{\varepsilon}{2}\right)|\mathfrak{R}^{13}| \geqslant \frac{\varrho}{3\beta}|\mathcal{P}^{13}|\,. \tag{3.4.5}$$

Similarly, we can fix a vertex $r^{24} \in \mathfrak{R}^{24}$ such that

$$|N_{\mathfrak{B}^{23}}(r^{24}, b^{34})| \geqslant \frac{1}{3\varrho}|\mathfrak{B}^{23}|\,. \tag{3.4.6}$$

Recalling that $|\mathfrak{R}^{13}| \leqslant (\varrho + \xi)|P^{13}|$ for every $B^{12} \in \mathfrak{B}^{12}$ and $B^{23} \in \mathfrak{B}^{23}$ we have

$$\begin{aligned}
\left|N_{\mathfrak{R}^{13}}(B^{12}, B^{23}) \cap N_{\mathfrak{R}^{13}}(r^{14}, b^{34})\right| &\geqslant \left(\frac{1}{3} + \varepsilon\right)|\mathcal{P}^{13}| + \left|N_{\mathfrak{R}^{13}}(r^{14}, b^{34})\right| - |\mathfrak{R}^{13}| \\
&\geqslant \left|N_{\mathfrak{R}^{13}}(r^{14}, b^{34})\right| - \left(\varrho + \xi - \frac{1}{3} - \varepsilon\right)|\mathcal{P}^{13}| \\
&\overset{(3.4.5)}{\geqslant} \left(1 - 3\beta + \frac{\beta}{\varrho}\right)\left|N_{\mathfrak{R}^{13}}(r^{14}, b^{34})\right|\,.
\end{aligned}$$

Hence, the number of hyperedges crossing $N_{\mathfrak{B}^{12}}(r^{14}, r^{24})$, $N_{\mathfrak{B}^{23}}(r^{24}, b^{34})$, and $N_{\mathfrak{R}^{13}}(r^{14}, b^{34})$ is at least

$$|N_{\mathfrak{B}^{12}}(r^{14}, r^{24})||N_{\mathfrak{B}^{23}}(r^{24}, b^{34})| \cdot \left(1 - 3\beta + \frac{\beta}{\varrho}\right) |N_{\mathfrak{R}^{13}}(r^{14}, b^{34})| .$$

Consequently, there exist $b^{12} \in N_{\mathfrak{B}^{12}}(r^{14}, r^{24})$ and $r^{13} \in N_{\mathfrak{R}^{13}}(r^{14}, b^{34})$ such that

$$\begin{aligned} |N_{\mathfrak{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24}, b^{34})| &\geqslant \left(1 - 3\beta + \frac{\beta}{\varrho}\right) |N_{\mathfrak{B}^{23}}(r^{24}, b^{34})| \\ &\overset{(3.4.6)}{\geqslant} \left(\frac{1}{3\varrho} - \frac{\beta}{\varrho} + \frac{\beta}{3\varrho^2}\right) |\mathfrak{B}^{23}| \\ &\geqslant \left(1 - \frac{1}{3\beta}\right) |\mathfrak{B}^{23}| , \end{aligned}$$

where the last inequality follows from the identity $\varrho = 1 - \beta$. □

The next claim fixes the four vertices b^{15}, b^{45} and r^{25}, r^{35}. Together with Claim 1 this fixes all vertices except b^{23} and both claims guarantee those seven hyperedges supporting a $K_5^{(3)}$ that do not involve b^{23}.

Claim 2. *There exist blue vertices $b^{15} \in \mathfrak{B}^{15}$, $b^{45} \in \mathfrak{B}^{45}$ and red vertices $r^{25} \in \mathfrak{R}^{25}$, $r^{35} \in \mathfrak{R}^{35}$ such that $b^{12}b^{15}r^{25}$, $r^{13}b^{15}r^{35}$, $r^{14}b^{15}b^{45}$, $r^{24}r^{25}b^{45}$, and $b^{34}r^{35}b^{45}$ are hyperedges in $\mathcal{A}$.*

Proof. Consider the following sets of pairs in $\mathfrak{B}^{15} \times \mathfrak{B}^{45}$.

$$\begin{aligned} G_1 &= \{(B^{15}, B^{45}) \in \mathfrak{B}^{15} \times \mathfrak{B}^{45} \colon N_{\mathfrak{R}^{25}}(b^{12}, B^{15}) \cap N_{\mathfrak{R}^{25}}(r^{24}, B^{45}) \neq \varnothing\} \\ \text{and} \quad G_2 &= \{(B^{15}, B^{45}) \in \mathfrak{B}^{15} \times \mathfrak{B}^{45} \colon N_{\mathfrak{R}^{35}}(b^{13}, B^{15}) \cap N_{\mathfrak{R}^{35}}(b^{34}, B^{45}) \neq \varnothing\} . \end{aligned}$$

Note that for every $B^{15} \in \mathfrak{B}^{15}$ there is some $R^{25} \in N_{\mathfrak{R}^{25}}(b^{12}, B^{15})$ and we have

$$|N_{\mathfrak{B}^{45}}(r^{24}, R^{25})| \overset{(3.4.2)}{\geqslant} \frac{1}{3\beta}|\mathfrak{B}^{45}| .$$

Clearly, $\{B^{15}\} \times N_{\mathfrak{B}^{45}}(r^{24}, R^{25}) \subseteq G_1$ and, hence, we establish

$$|G_1| \geqslant \frac{1}{3\beta}|\mathfrak{B}^{15}||\mathfrak{B}^{45}| . \tag{3.4.7}$$

A symmetric argument yields the same bound for G_2. Combining (3.4.7) and the same bound for G_2 with (3.4.4) leads to

$$|G_1| + |G_2| + |N_{\mathfrak{B}^{15}\times\mathfrak{B}^{45}}(r^{14})| \geqslant \left(\frac{2}{3\beta} + \frac{1}{3\varrho} + \frac{\varepsilon}{2}\right) |\mathfrak{B}^{15}||\mathfrak{B}^{45}| \overset{(3.4.1)}{>} 2\,|\mathfrak{B}^{15}||\mathfrak{B}^{45}|\,,$$

Consequently, we can fix a pair $(b^{15}, b^{45}) \in G_1 \cap G_2 \cap N_{\mathfrak{B}^{15}\times\mathfrak{B}^{45}}(r^{14})$. Moreover, having fixed b^{15} and b^{45} this defines a vertex $r^{25} \in \mathfrak{R}^{25}$ from the non-empty intersection considered in the definition of G_1. Similarly, G_2 leads to our choice of $r^{35} \in \mathfrak{R}^{35}$.

Since $(b^{15}, b^{45}) \in N_{\mathfrak{B}^{15}\times\mathfrak{B}^{45}}(r^{14})$, the hyperedge $r^{14}b^{15}b^{45}$ exists in $\mathcal{A}$ and the other four hyperedges are a result of the definition of G_1 and G_2. $\square$

As mentioned above, Claims 1 and 2 fix all vertices except $b^{23} \in \mathfrak{B}^{23}$ and all hyperedges not involving b^{23}. For the three remaining hyperedges it suffices to show

$$N_{\mathfrak{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24}, b^{34}) \cap N_{\mathfrak{B}^{23}}(r^{25}, r^{35}) \neq \varnothing\,.$$

Claim 1 *(ii)* and (3.4.2) imply

$$\begin{aligned}
&\left|N_{\mathfrak{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24}, b^{34}) \cap N_{\mathfrak{B}^{23}}(r^{25}, r^{35})\right| \\
&\qquad \geqslant \left|N_{\mathfrak{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24}, b^{34})\right| + \left|N_{\mathfrak{B}^{23}}(r^{25}, r^{35})\right| - \left|\mathfrak{B}^{23}\right| \\
&\qquad \overset{(3.4.2)}{\geqslant} \left(1 - \frac{1}{3\beta} + \frac{1}{3\beta} + \frac{\varepsilon}{2} - 1\right) |\mathfrak{B}^{23}| > 0\,.
\end{aligned}$$

Hence a choice for $b^{23} \in N_{\mathfrak{B}^{23}}(b^{12}, r^{13}) \cap N_{\mathfrak{B}^{23}}(r^{24}, b^{34}) \cap N_{\mathfrak{B}^{23}}(r^{25}, r^{35})$ exists and, therefore, $\mathcal{A}$ restricted to J supports a $K_5^{(3)}$. $\square$

3.5 Concluding Remarks

We close with a few related open problems and possible future directions for research.

3.5.1 Turán problems for cliques in ⅄-dense hypergraphs

In view of Theorems 2.1.2 and 2.1.3 for cliques $K_\ell^{(3)}$ with $\ell \leqslant 16$ vertices only the cases $\ell = 9$ and 10 are still unresolved and closing the bounds

$$\frac{1}{2} \leqslant \pi_{\text{⅄}}(K_9^{(3)}) \leqslant \pi_{\text{⅄}}(K_{10}^{(3)}) \leqslant \frac{2}{3}$$

would be interesting.

Determining the value $\pi_{\wedge}(K_\ell^{(3)})$ for large values of ℓ might be a challenging problem and one may first focus on the asymptotic behaviour. For every $\ell \geqslant 3$ Theorem 2.1.2 tells us

$$\pi_{\wedge}(K_\ell^{(3)}) \leqslant 1 - \frac{1}{\log_2(\ell)}\,. \tag{3.5.1}$$

For a lower bound we consider the following well known random construction.

Example 3.5.1. *For $r \geqslant 2$ we consider random hypergraphs $H_\varphi = (V, E_\varphi)$ with the edge set defined by the non-monochormatic triangles of a random r-colouring $\varphi\colon V^{(2)} \longrightarrow [r]$ for a sufficiently large vertex set V. It is easy to check that for any fixed $\eta > 0$ with high probability such hypergraphs H_φ are $(\eta, \frac{r-1}{r}, \wedge)$-dense. On the other hand, if ℓ is at least as large as $R(3; r)$, the r-colour Ramsey number for graph triangles, then every such H_φ is $K_\ell^{(3)}$-free.*

Consequently, Example 3.5.1 yields

$$\pi_{\wedge}(K_\ell^{(3)}) \geqslant 1 - \frac{1}{r}\,, \text{ whenever } \ell \geqslant R(3; r)$$

and using the simple upper bound $R(3; r) \leqslant 3\,r!$ we arrive at

$$\pi_{\wedge}(K_\ell^{(3)}) \geqslant 1 - \frac{\log_2 \log_2(\ell)}{\log_2(\ell)} \tag{3.5.2}$$

for sufficiently large ℓ. Comparing the bounds in (3.5.1) and (3.5.2) suggests the following problem.

Problem 3.5.2. *Determine the asymptotic behaviour of $1 - \pi_{\wedge}(K_\ell^{(3)})$.*

3.5.2 Turán problems for hypergraphs with uniformly dense links

As discussed in the introduction there is a small difference between Theorem 2.1.3 and Corollary 2.1.5. Below we briefly elaborate on these differences.

In this work we study $\wedge$-dense hypergraphs, which are defined by the lower bound condition (1.4.3) in Definition 1.4.3. Requiring in addition a matching upper bound, i.e., replacing (1.4.3) by

$$\big|e_{\wedge}(P,Q) - d\,|\mathcal{K}_{\wedge}(P,Q)|\big| \leqslant \eta |V|^3\,,$$

leads to the notion of $(\eta, d, \wedge)$-*quasirandom* hypergraphs. Clearly, we can transfer the definition of $\pi_{\wedge}(F)$ in Definition 2.1.1 and define the Turán-density $\pi'_{\wedge}(F)$ by restricting to $\wedge$-quasirandom hypergraphs H

$$\begin{aligned}\pi'_{\wedge}(F) = \sup\{d \in [0,1] \colon &\text{ for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists an } F\text{-free},\\ &(\eta, d, \wedge)\text{-quasirandom hypergraph with at least } n \text{ vertices}\}\,.\end{aligned}$$

By definition we have $\pi'_{\wedge}(F) \leqslant \pi_{\wedge}(F)$ for every hypergraph F and one may wonder if this inequality is sometimes strict.

For $K_5^{(3)}$ it is easy to check that the lower bound construction in Example 2.1.4 yields $K_5^{(3)}$-free $(\eta, 1/3, \wedge)$-quasirandom hypergraphs for every fixed $\eta > 0$ and, hence,

$$\pi'_{\wedge}(K_5^{(3)}) = \pi_{\wedge}(K_5^{(3)}) = \frac{1}{3}\,.$$

On the other hand, the lower bound construction for $K_6^{(3)}$ from [55] is given by Example 3.5.1 for $r = 2$. In those hypergraphs H_φ we can take P and Q to be the pairs in colour 1 and 2 respectively and get

$$e_{\wedge}(P,Q) = |\mathcal{K}_{\wedge}(P,Q)|\,,$$

i.e., they have relative density 1. Therefore, the hypergraphs H_φ are only $(\eta, 1/2, \wedge)$-dense, but not $(\eta, 1/2, \wedge)$-quasirandom. In fact, we are not aware of any matching quasirandom lower bound construction for $\pi_{\wedge}(K_6^{(3)})$ and it seems possible that $\pi'_{\wedge}(K_6^{(3)})$ is strictly smaller than $\pi_{\wedge}(K_6^{(3)})$ suggesting the following general problem.[1]

[1] We remark that for the concepts of $\therefore$-dense/quasirandom hypergraphs there is no difference for the corresponding Turán-densities, as every $\therefore$-dense hypergraph contains large $\therefore$-quasirandom hypergraphs of at least the same density.

Problem 3.5.3. *Characterise hypergraphs F with $\pi'_{\therefore}(F) < \pi_{\therefore}(F)$, if there are any.*

Recalling the discussion after Theorem 2.1.3 we note that $\therefore$-dense and $\therefore$-quasirandom hypergraphs can be characterised through properties of their link graphs. As mentioned in Section 2.1 a hypergraph in which all links are quasirandom is $\therefore$-dense. More generally, for every $\eta > 0$ there exists some $\varrho > 0$ such that for every $d \in [0,1]$ and every sufficiently large hypergraph H, in which all but at most $\varrho|V|$ links are (d,ϱ)-quasirandom, is $(\eta, d, \therefore)$-quasirandom. In fact, the opposite implication holds with the quantification of ϱ and η exchanged, and therefore both properties are essentially equivalent. Similarly, $\therefore$-density is equivalent, in the same sense as above, to the property of having bidense (see definition in (1.4.2)) links for almost all vertices.

Finally, one may also consider hypergraphs having just (ϱ, d)-*dense* link graphs, where the lower bound (1.4.2) is only applied to the special cases $X = Y$. On the hypergraph level this would be equivalent to restricting to the cases $P = Q$ in (1.4.3) in Definition 1.4.3. Based on this concept we define

$$\begin{aligned}\pi''_{\therefore}(F) = \sup\big\{d \in [0,1] \colon &\text{ for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists an } F\text{-free hypergraph}\\ &H = (V,E) \text{ with } |V| \geqslant n \text{ and all but at most } \varrho|V| \text{ vertices}\\ &\text{have } (\varrho, d)\text{-dense link graphs}\big\}\end{aligned}$$

for every hypergraph F. Since having (ϱ, d)-dense link graphs is a weaker property we have the trivial inequality

$$\pi''_{\therefore}(F) \geqslant \pi_{\therefore}(F)$$

for every hypergraph F and one may ask for which hypergraphs F this inequality is strict.

3.5.3 Turán problems for $\therefore$-dense hypergraphs with bounded number of colours

In view of Proposition 3.1.6 one may consider a variant of $\pi_{\therefore}(F)$ restricted to large hypergraphs $H = (V, E)$ with "bounded colouring number" defined in the following sense: There is colouring of $V^{(2)}$ with a bounded number of colours such that for every hyperedge $e \in E$ not all pairs in $e^{(2)}$ have same colour. Below we briefly discuss a corresponding problem for reduced hypergraphs.

We say a reduced hypergraph $\mathcal{A}$ with index set I is *r-coloured* by $\varphi\colon V(\mathcal{A}) \longrightarrow [r]$ if there are no monochromatic hyperedges, i.e., $\big|\{\varphi(xy), \varphi(xz), \varphi(yz)\big| \geqslant 2$ for every hyperedge $xyz \in E(\mathcal{A})$. Moreover, such a colouring is *balanced* if $|\mathcal{P}^{ij}| \in r\mathbb{N}$ for every $ij \in I^{(2)}$ and

$$\big|\mathcal{P}^{ij} \cap \varphi^{-1}(\varrho)\big| = \frac{|\mathcal{P}^{ij}|}{r}$$

for every $\varrho \in [r]$ and $ij \in I^{(2)}$. Given an (not necessarily balanced) r-colouring φ we define the *minimum codegree density*

$$\delta_2(\mathcal{A}, \varphi) = \min\Big\{\frac{|N(P^{ij}, P^{ik}) \cap \varphi^{-1}(\varrho)|}{|\mathcal{P}^{jk} \cap \varphi^{-1}(\varrho)|}\colon ijk \in I^{(3)}, P^{ij} \in \mathcal{P}^{ij}, P^{ik} \in \mathcal{P}^{ik}, \varrho \in [r], \\ \text{and } \big|\big\{\varphi(P^{ij}), \varphi(P^{ik}), \varrho\big\}\big| \geqslant 2\Big\}.$$

We remark that for a 2-colouring φ in the definition of $\delta_2(\mathcal{A}, \varphi)$ we also consider pairs of vertices (P^{ij}, P^{ik}) with different colours, which is one of the main differences compared to the definition of $\tau_2(\mathcal{A}, \varphi)$ in (3.1.2). In addition, we measure the codegree neighbourhood with respect to the size of colour class $\varphi^{-1}(\varrho)$ in $\mathcal{P}^{jk}$ instead of all of $\mathcal{P}^{jk}$. For balanced 2-colourings φ we therefore have

$$2 \cdot \tau_2(\mathcal{A}, \varphi) \geqslant \delta_2(\mathcal{A}, \varphi)\,. \tag{3.5.3}$$

For integers $r \geqslant 1$ and $\ell \geqslant 3$ we consider the following Turán-type parameter

$$\pi^{\mathrm{rd}}_{\boldsymbol{\wedge},r}(K_\ell^{(3)}) = \sup\big\{d \in [0,1]\colon \text{for every } t \in \mathbb{N} \text{ there is a balanced } r\text{-coloured,} \\ \text{reduced hypergraph } (\mathcal{A}, \varphi) \text{ with index set of size at least } t \\ \text{and } \delta_2(\mathcal{A}, \varphi) \geqslant d, \text{ which does not support } K_\ell^{(3)}\big\}.$$

Remark 3.5.4. For the corresponding parameter $\pi_{\boldsymbol{\wedge},r}(K_\ell^{(3)})$ one considers $K_\ell^{(3)}$-free hypergraphs $H = (V, E)$ for which there is a quasirandom r-colouring $\varphi\colon V^{(2)} \longrightarrow [r]$ such that the pairs of every hyperedge receive at least two different colours. In this context, the $\pmb{\wedge}$-density with respect φ is at least d, if for all colours ϱ_P, ϱ_Q, $\varrho \in [r]$ with $|\{\varrho_P, \varrho_Q, \varrho\}| \geqslant 2$ and all monochromatic sets of pairs P and Q with colours ϱ_P and ϱ_Q at least, up to an additive error term of $o(|V|^3)$, a d proportion of the triangles xyz with $xy \in P$, $xz \in Q$ and $\varphi(yz) = \varrho$ are hyperedges in H. Defining $\pi_{\pmb{\wedge},r}(K_\ell^{(3)})$ accordingly and following the proof of [50, Theorem 3.3] then yields $\pi_{\pmb{\wedge},r}(K_\ell^{(3)}) = \pi_{\pmb{\wedge},r}^{\mathrm{rd}}(K_\ell^{(3)})$ (see [9] for details).

Given a balanced $(r+1)$-coloured reduced hypergraph $(\mathcal{A}, \varphi)$ we may simply remove the vertices from $\varphi^{-1}(r+1)$ and we obtain a balanced r-coloured reduced hypergraph with the same minimum codegree density. Consequently, for every $r \geqslant 1$ and $\ell \geqslant 3$ we have

$$\pi_{\pmb{\wedge},r}^{\mathrm{rd}}(K_\ell^{(3)}) \geqslant \pi_{\pmb{\wedge},r+1}^{\mathrm{rd}}(K_\ell^{(3)})\,.$$

Note that if $\delta_2(\mathcal{A}, \varphi) \geqslant d$ for some balanced r-colouring φ, then $\mathcal{A}$ is $(d - 1/r, \pmb{\wedge})$-dense and, consequently, with the notation following [50, Theorem 3.3] we have

$$\lim_{r \longrightarrow \infty} \pi_{\pmb{\wedge},r}^{\mathrm{rd}}(K_\ell^{(3)}) \leqslant \pi_{\pmb{\wedge}}^{\mathrm{rd}}(K_\ell^{(3)})\,.$$

In the other direction, considering random balanced r-colourings of reduced $\pmb{\wedge}$-dense hypergraphs $\mathcal{A}$ that do not support $K_\ell^{(3)}$ with monochromatic hyperedges removed, establishes the opposite inequality and we arrive at

$$\lim_{r \longrightarrow \infty} \pi_{\pmb{\wedge},r}^{\mathrm{rd}}(K_\ell^{(3)}) = \pi_{\pmb{\wedge}}^{\mathrm{rd}}(K_\ell^{(3)})\,.$$

In view of Remark 3.5.4 and [50, Theorem 3.3] this shows

$$\lim_{r \longrightarrow \infty} \pi_{\pmb{\wedge},r}(K_\ell^{(3)}) = \pi_{\pmb{\wedge}}(K_\ell^{(3)})\,. \tag{3.5.4}$$

This way one may consider $\pi_{\pmb{\wedge},r}(K_\ell^{(3)})$ as the multipartite version of $\pi_{\pmb{\wedge}}(K_\ell^{(3)})$ in the similar spirit as the multipartite extremal problems for graphs and (3.5.4) can be considered as a variant of [12, Theorem 1] in this context.

For small values of r we note that

$$\pi_{\pmb{\wedge},r}(K_\ell^{(3)}) = 1\,, \text{ whenever } \ell \geqslant R(3;r)\,.$$

In fact, this easily follows by considering the random hypergraphs H_φ from Example 3.5.1. For $r = 1$ and 2 we have $\pi_{\wedge,1}(K_\ell^{(3)}) = 1$ for every $\ell \geqslant 3$ and $\pi_{\wedge,2}(K_\ell^{(3)}) = 1$ for every $\ell \geqslant 6$. Moreover, one can show that

$$\pi_{\wedge,2}(K_4^{(3)}) = 0$$

(see [9] for details) and Proposition 3.1.6 combined with (3.5.3) and Remark 3.5.4 yields

$$\pi_{\wedge,2}(K_5^{(3)}) = \pi_{\wedge,2}^{\mathrm{rd}}(K_5^{(3)}) \leqslant \frac{2}{3}\,. \tag{3.5.5}$$

We note that owing to more restrictive definition of $\delta_2(\mathcal{A},\varphi)$ compared to $\tau_2(\mathcal{A},\varphi)$, the upper bound of (3.5.5) can be proved more easily than the proof of Proposition 3.1.6 by simply exploiting that every pair of vertices in $K_5^{(3)}$ has codegree three. A slightly more refined argument allows us to improve this upper bound from 2/3 to the reciprocal of the golden ratio. In the other direction we have a lower bound construction establishing

$$\frac{1}{2} \leqslant \pi_{\wedge,2}(K_5^{(3)}) < 0.618$$

(see [9]), which leaves the following problem open.

Problem 3.5.5. *Determine* $\pi_{\wedge,2}(K_5^{(3)})$.

Chapter 4

Hamilton cycles in uniformly dense hypergraphs

In this chapter we present the proof of Theorem 2.2.2, which is based in the *Absorption Method*. In the following section we introduce the method and its three main parts: the *Almost Covering Lemma* (see Lemma 4.1.2), the *Connecting Lemma* (see Lemma 4.1.4), and the *Absorbing Path Lemma* (see Lemma 4.1.3). The proofs of those lemmata are given in Sections 4.3, 4.4, and 4.5. In Section 4.2 we collect some preliminary observations needed for the main proof. In Section 4.6 we discuss the necessary changes to the main proof in order to prove Theorem 2.2.3. We close with a few concluding remarks in Section 4.7.

The work corresponding to this chapter was done in collaboration with Araújo and Schacht [5].

4.1 Absorption Method

In [58], Rödl, Ruciński and Szemerédi introduced the Absorption Method, which turned out to be a very useful approach for embedding spanning cycles in hypergraphs. This method reduces the problem to finding an almost spanning cycle with a small special path in it, called the *absorbing path*. The absorbing path A can absorb any small set of vertices into a new bigger path, with the same ends as A, completing the almost spanning cycle into a Hamilton cycle.

The almost spanning cycle will be composed from smaller paths, which will be connected to longer paths. For that it would be useful if any given two pairs of vertices (x, y) and (w, z), being the ends of such smaller paths, can be connected by a short path. However, in view of the assumptions of Theorem 2.2.2, it is easy to see that not *any* pair of pairs can be connected in this way (in particular, there could be pairs with codegree zero). For that we introduce the following notion of *connectable pairs* and we will show that for those pairs there actually exist connecting paths between them (see Lemma 4.1.4 below).

Definition 4.1.1. Let $H = (V, E)$ be a hypergraph. We say that $(x, y) \in V \times V$ is *β-connectable in H* if the set

$$Z_{xy} = \{z \in V : xyz \in E(H) \text{ and } d(yz) \geqslant \beta|V|\},$$

has size at least $\beta|V|$. Moreover, we say that an (a, b)-(c, d)-path is *β-connectable* if the pairs (b, a) and (c, d) are β-connectable.

Observe that the starting pair of the path is asked to be β-connectable in the inverse direction that as it appears in the path.

The proof of Theorem 2.2.2 splits into three lemmata. Let H be a $(\varrho, 1/4+\varepsilon, \therefore)$-dense hypergraph on n vertices, with $1/n \ll \varrho \ll \varepsilon$. First we prove that such hypergraphs can be almost covered by a collection of 'few' paths. We remark that this is even true under the weaker assumption of non-vanishing $\therefore$-density. A straight forward proof is presented in Section 4.3.

Lemma 4.1.2 (Almost Covering Lemma)**.** *For all $d, \gamma \in (0, 1]$ there exist $\varrho, \beta > 0$, and n_0 such that in every $(\varrho, d, \therefore)$-dense hypergraph H on $n \geqslant n_0$ vertices there exists a collection of at most $1/\beta$ disjoint β-connectable paths, that cover all but at most $\gamma^2 n$ vertices of H.*

Next we discuss how to find an absorbing path, which contains a collection of several smaller structures, called *absorbers*. For $v \in V$, we call $A_v \subseteq H$ an absorber for v if both A_v and $A_v \cup \{v\}$ span paths with same ends (we say that A_v absorbs v). The main difficulty is to define the absorbers in such a way that we can prove that every vertex is contained in many of them. In Section 4.5 we see that the absorbers considered here are in fact more complicated and absorb sets of three vertices instead of one. This leads to a divisibility issue which we consider separately in Lemma 4.5.4. Going further, we can find a relatively small collection of paths which can absorb any sufficiently small given set of vertices. After finding this collection we connect them together to form one path with the absorption property described in the following lemma.

Lemma 4.1.3 (Absorbing Path Lemma)**.** *For every $\varepsilon > 0$ there exist $\varrho, \beta, \gamma' > 0$ and n_0 such that the following is true for every positive $\gamma \leqslant \gamma'$ and every $(\varrho, 1/4 + \varepsilon, \therefore)$-dense hypergraph $H = (V, E)$ on $n \geqslant n_0$ vertices with $\delta_1(H) \geqslant \varepsilon n^2$.*

For every $R \subseteq V$, with $|R| \leqslant 2\gamma^2 n$, there exists a β-connectable path A with $V(A) \subseteq V \smallsetminus R$ and $|V(A)| \leqslant \gamma n$, such that for every $U \subseteq V(H) \smallsetminus A$ with $|U| \leqslant 3\gamma^2 n$, the hypergraph $H[V(A) \cup U]$ has a Hamilton path with the same ends as A.

The set of vertices R in Lemma 4.1.3 will act as a reservoir of vertices that will be used later for connecting the paths mentioned in Lemmata 4.1.2 and 4.1.3, without interfering with the vertices already used by those paths.

The next lemma justifies Definition 4.1.1 and shows that between every two β-connectable pairs there exist several short paths connecting them. As it was said before, this is used for connecting the absorbers in the proof of Lemma 4.1.3. Moreover, observe that all paths mentioned in Lemma 4.1.2 and 4.1.3 are β-connectable. This allows us to connect them together into an almost spanning cycle and the absorbing path in this cycle will absorb all the remaining vertices to complete the Hamilton cycle.

Lemma 4.1.4 (Connecting Lemma)**.** *For every $\varepsilon, \beta > 0$ there exist $\varrho, \alpha > 0$ and n_0 such that for every $(\varrho, 1/4 + \varepsilon, \therefore)$-dense hypergraph H on $n \geqslant n_0$ vertices the following holds.*

For every pair of disjoint ordered β-connectable pairs of vertices $(x, y), (w, z) \in V \times V$ there exists an integer $\ell \leqslant 15$ such that the number of (x, y)-(z, w)-paths with ℓ inner vertices is at least αn^ℓ.

In view of the construction given in Example 2.2.1, one can see that the $1/4$ in the $\therefore$-density assumption in Lemma 4.1.4 cannot be dropped. In that example, there are two classes of pairs that cannot be connected by a path (namely the pairs in G and in $\overline{G}$), even though they are β-connectable. Hence, $\therefore$-density of at least $1/4$ is required for Lemma 4.1.4.

Also Lemma 4.1.3 requires $\therefore$-density bigger than $1/4$. In the proof of Lemma 4.1.3 this assumption will be crucial for connecting the so-called absorbers to a path, which makes use of Lemma 4.1.4. Moreover, the type of absorbers used here, leads to a 'divisibility issue'. This is addressed in Lemma 4.5.4 for which we also employ the same density assumption.

We now deduce Theorem 2.2.2 from Lemmata 4.1.2 – 4.1.4.

Proof of Theorem 2.2.2. Given $\varepsilon > 0$ we apply Lemma 4.1.3 and obtain ϱ_1, β_1 and γ'. Lemma 4.1.2 applied with $d = 1/4$ and $\gamma = \min\{\gamma', \varepsilon/2\}$ yields ϱ_2 and β_2. Applying Lemma 4.1.4 with ε and

$$\beta = \frac{1}{8}\min\{\beta_1, \beta_2\},$$

reveals α and ϱ_3. Finally we set

$$\varrho = \min\{\varrho_1, \varrho_2/8, \varrho_3\},$$

and let n be sufficiently large. Having fixed all constants, let H be a $(\varrho, 1/4+\varepsilon, \therefore)$-dense hypergraph on n vertices.

We consider a random set $R \subseteq V$, in which each vertex is present independently with probability γ^2. For every positive integer $\ell \leqslant 15$ consider two pairs $(x, y), (w, z) \in V \times V$ between which there are at least αn^ℓ paths with ℓ inner vertices. Let $Y = Y(\ell, (x, y), (z, w))$ count the number of such paths whose inner vertices are contained in R. We point out that Y is a function determined by n independent random variables, each of which can influence the value of Y by at most $n^{\ell-1}$. Therefore a standard application of Azuma's inequality (see [33, Section 2.4]) implies that

$$\mathbb{P}\left(Y \leqslant \frac{\gamma^{2\ell}}{2} \cdot \alpha n^\ell\right) = \exp(-\Omega(n)) < \frac{1}{2} \cdot \frac{1}{15n^4}, \tag{4.1.1}$$

for any fixed ℓ, (x, y), and (w, z). Moreover, by Markov's inequality we have that

$$\mathbb{P}\big(|R| \geqslant 2\gamma^2 n\big) \leqslant \frac{1}{2}. \tag{4.1.2}$$

Therefore there exists a realisation of R, which from now on will take over the name R, that is not in the event considered in (4.1.2) and in any of the events considered in (4.1.1) (all 4-tuples of vertices and values of ℓ). Since $\gamma' < \gamma$, $\varrho < \varrho_1$, and $|R| < 2\gamma^2 n$, Lemma 4.1.3 ensures that we can find a β_1-connectable absorbing path A of size smaller than γn and which does not intersect R.

Let $V' = V \smallsetminus (V(A) \cup R)$. Since $|V(A) \cup R| \leqslant 3\gamma n \leqslant n/2$, the induced hypergraph $H[V']$ is $(8\varrho, 1/4 + \epsilon, \therefore)$-dense. In particular, $H[V']$ is $(8\delta, 1/4 + \varepsilon, \therefore)$-dense and since $8\varrho \leqslant \varrho_2$, Lemma 4.1.2 implies that there exists a collection of at most $1/\beta_2$ paths with β_2-connectable ends in $H[V']$ that cover all but at most $\gamma^2 n$ vertices.

Set $t = \lfloor 1/\beta_2 + 1 \rfloor$ and let $(P_i)_{i \in [t]}$ be any cyclic ordering of such paths together with the absorbing path. Assume that we were able to find connections in R between the paths $P_1, P_2, \ldots, P_i$, using inner vertices from R only. Moreover, we make sure that each connection is made with at most 15 inner vertices. Let C_i be the path that begins with P_1 and ends in P_i using those connections. Therefore

$$|V(C_i) \cap R| \leqslant t \cdot 15 = o(n).$$

Now, we want to show that we can connect P_i with P_{i+1} to construct C_{i+1}. Observe that all the paths from $(P)_{i \in [t]}$ are β-connectable. This follows from the choice $\beta \leqslant \beta_1$ for the absorbing path A. From the paths given by Lemma 4.1.2 we know that they are β_2-connectable in $H[V']$. Owing to $\beta \leqslant \beta_2/2$ and $|V'| \geqslant n/2$ the β-connectibility follows.

Let (x_i, y_i) be the ending pair of P_i and (z_i, w_i) the starting pair P_{i+1}. Lemma 4.1.4 implies that, for some $\ell_i \leqslant 15$, there exist at least αn^{ℓ_i} (x_i, y_i)-(z_i, w_i)-paths, each with ℓ_i inner vertices. By the choice of R, the number of (x_i, y_i)-(z_i, w_i) paths of length $\ell_i + 2$ whose inner vertices lie in R is at least $\gamma^2 \alpha n^{\ell_i}/2$. Since at most $|V(C_i) \cap R| n^{\ell_i - 1} = o(n^{\ell_i})$ such paths contain a vertex from C_i, for sufficiently large n large enough we can find one path disjoint from C_i.

Finally, consider C_t the final cycle obtained in this process, by connecting P_t to P_1. As C_t includes all the paths in the almost covering the number of vertices not covered by C_t is at most

$$|V \smallsetminus V(C_t)| \leqslant |R| + \gamma^2 n \leqslant 3\gamma^2 n.$$

This finishes the proof, since A can absorb these vertices into a new path with the same endings. □

4.2 Preliminary results and basic definitions

In this section we collect preliminary results and introduce necessary notation. Let $\eta, d \in [0,1]$ and let $G = (V_1 \cup V_2, E)$ be a bipartite graph, we say that G is (η, d)-*regular* if for every two sets of vertices $X \subseteq V_1$ and $Y \subseteq V_2$ we have

$$|e(X,Y) - d|X||Y|| \leqslant \eta |V_1||V_2|\,.$$

It is easy to see that every dense graph contains a linear sized bipartite regular subgraph, with almost the same density. That can be proved by a simple application of Szemerédi's Regularity Lemma or alternatively by a more direct density increment argument (see [43]).

Lemma 4.2.1. *For all η, $d > 0$ there exists some $\mu > 0$ such that for every n-vertex graph G with $e(G) \geqslant dn^2/2$, there are disjoint sets V_1, $V_2 \subseteq V(G)$, with $|V_1| = |V_2| = \lceil \mu n \rceil$ such that the bipartite induced subgraph $G[V_1, V_2]$ is (η, d')-regular for some $d' \geqslant d$.* □

For a hypergraph $H = (V, E)$ recall its *shadow* ∂H is the subset of $V^{(2)}$ of those pairs that are contained in some edge of H. For disjoint sets of vertices V_1, $V_2 \subseteq V$ with a slight abuse of notation we write $\partial H[V_1, V_2]$ for the set of ordered pairs in $V_1 \times V_2$ that correspond to unordered pairs in the shadow, i.e.,

$$\partial H[V_1, V_2] = \left\{(v_1, v_2) \in V_1 \times V_2 \colon \{v_1, v_2\} \in \partial H\right\}.$$

Given $\varrho, d > 0$, a set of ordered pairs of vertices $P \in V^2$, and a subset $X \subseteq V$ we say that H is $(\varrho, d, \therefore)$-*dense over* (X, P) if for every subset of vertices $X' \subseteq X$ and every subset of pairs $P' \subseteq P$ we have

$$e(X', P') \geqslant d\,|X'||P'| - \varrho\,|X||P|\,,$$

which is a version of $\therefore$-density restricted to P and X. For the next lemma we also need the following concept of restricted vertex neighbourhood. Given a vertex $v \in V$ and a set of ordered pairs $P \in V^2$ we define its *neighbourhood restricted to* P by

$$N(v, P) = \{(x, y) \in P \colon vxy \in E\}\,.$$

Lemma 4.2.2. *Let $H = (V, E)$ be a hypergraph, $X \subseteq V$ be a set of vertices, and $P \subseteq V^2$. If H is $(\varrho, d, \therefore)$-dense over (X, P) for some constants ϱ, $d > 0$, then*

$$\left|\left\{x \in X \colon |N(x, P)| < (d - \sqrt{\varrho})|P|\right\}\right| < \sqrt{\varrho}\,|X|\,.$$

Proof. Let $X' \subseteq X$ be the vertices with less than $(d - \sqrt{\varrho})|P|$ neighbour pairs in P. The definition of X' and the $(\varrho, d, \therefore)$-density of H over (X, P) provide the following upper and lower bounds on $e(X', P)$

$$d|X'||P| - \varrho|X||P| \leqslant e(X', P) \leqslant (d - \sqrt{\varrho})|P| \cdot |X'|$$

and the desired bound on $|X'|$ follows. □

The following result asserts that every hypergraph contains a subhypergraph with almost the same density and such that every pair of vertices with positive codegree has at least $\Omega(|V|)$ neighbours. This fact can be proved by removing iteratively the edges which contain a pair with small codegree and we omit the details.

Lemma 4.2.3. *For every $\beta > 0$ and every n-vertex hypergraph H there is a hypergraph $H_\beta \subseteq H$ on the same vertex set with $e(H_\beta) \geqslant e(H) - \beta n^3$ such that for every pair of vertices x, y either $d_{H_\beta}(x, y) = 0$ or $d_{H_\beta}(x, y) \geqslant \beta n$. In particular, if we have $d_{H_\beta}(x, y) > 0$, then (x, y) is β-connectable in H.* □

Let F and F' be two hypergraphs. We say that F *contains a homomorphic copy of* F' if there is a function $\varphi : V(F') \to V(F)$ such that for every edge $xyz \in E(F')$ we have that $\varphi(x)\varphi(y)\varphi(z) \in E(F)$. We denote this fact as $F' \xrightarrow{hom} F$ and we recall the following well known consequence from Erdős [18].

Lemma 4.2.4. *For every $\xi > 0$ and k, $\ell \in \mathbb{N}$ there is $\zeta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Let F and F' be hypergraphs such that $|V(F)| = k$ and $|V(F')| = \ell$ and $F' \xrightarrow{hom} F$. If a hypergraph H on $n > n_0$ vertices contains at least ξn^k copies of F, then H contains ζn^ℓ copies of F'.* □

We denote the hypergraph with four vertices and three edges by $K_4^{(3)-}$. We refer to the vertex of degree three as the *apex*. Glebov, Kráľ, and Volec [25] showed that ∴-density bigger than 1/4 yields the existence of a, in fact of many copies of, $K_4^{(3)-}$.

Theorem 4.2.5 (Glebov, Kráľ & Volec, 2016)**.** *For every $\epsilon > 0$ there exist ϱ and $\xi > 0$ such that every sufficiently large $(\varrho, 1/4 + \epsilon, ∴)$-dense n-vertex hypergraph contains ξn^4 copies of $K_4^{(3)-}$.* □

4.3 Almost covering

In this section we present a very straightforward proof of Lemma 4.1.2.

Proof of Lemma 4.1.2. Given $d, \gamma > 0$ take β and ϱ such that

$$\beta = \varrho = \frac{d\gamma^6}{13}.$$

We show that a maximal collection of β-connectable paths, each of which having at least βn vertices, must cover all but at most $\gamma^2 n$ vertices. We do that by showing that every set $X \subseteq V(H)$ with at least $\gamma^2 n$ vertices contains a β-connectable path of size βn. Indeed, the $(\varrho, d, ∴)$-density implies that in such a set X, we have

$$e(X) \geqslant \frac{d|X|^3}{6} - \varrho n^3,$$

where we discounted the ordering of triples. In $H[X]$ we remove, iteratively, every edge that contains an (unordered) pair of vertices with codegree smaller than βn. In this way, we remove at most βn^3 edges and get a hypergraph with at least

$$\begin{aligned} e(X) - \beta n^3 &\geqslant \frac{d|X|^3}{6} - \varrho n^3 - \beta n^3 \\ &\geqslant \left(\frac{d\gamma^6}{6} - \varrho - \beta\right) n^3, \end{aligned}$$

edges. Owing to the choice of β and ϱ this hypergraph is not empty. Now a path with βn vertices can be found in a greedy manner. Moreover, if (x, y) is a pair contained in such path, then we have that the set

$$Z_{xy} = \{z \in V : xyz \in E \text{ and } d(yz) \geqslant \beta n\}$$

has at least βn vertices. □

4.4 Connecting Lemma

We dedicate this section to prove the Connecting Lemma (Lemma 4.1.4). The proof splits into several lemmata. The Connecting Lemma asserts that every ordered connectable pair can be connected to any other ordered connectable pair. In a first step in Lemmata 4.4.1 and 4.4.3 we show that there are many connections between large sets of unordered pairs (without specifying the order of the ending pairs). In fact, these connection can be achieved by paths consisting of only two edges, which we refer to as *lemma:cherries* (see Definition 4.4.2 below). On the price of extending the length by at most two, in Lemma 4.4.4 we establish that one can even fix the order of one of the sets of given pairs. On the other hand, this is complemented by Lemma 4.4.7 showing that there are many pairs of unordered pairs that can be connected in any orientation. We call such pairs of pairs *turnable* (see Definition 4.4.5 below).

For the proof of the Connecting Lemma we can now start with any given connectable pair (x, y) and move to its second neighbourhood, which is a large set of ordered pairs. From that set we shall reach many turnable pairs. Similarly, from any given ending pair (z, w) we also reach many turnable pairs. These paths give the turnable pairs an orientation, but since the turnable pairs can be connected in any orientation, we find the desired (x, y)-(z, w)-paths. The detailed presentation of this argument renders the proof of the Connecting Lemma, which we defer to the end of this section.

Lemma 4.4.1. *For all ξ, $\epsilon \in (0, 1]$ there exist η, $\varrho > 0$ such that the following holds for sufficiently large m.*

Suppose V_1, V_2, V_3 are pairwise disjoint sets of size m and $G = (V_1 \cup V_2, P)$ is an (η, ξ)-regular bipartite graph. If $H = (V_1 \cup V_2 \cup V_3, E)$ is a 3-partite hypergraph that is $(\varrho, 1/4 + \epsilon, \therefore)$-dense over (V_3, P), then

$$\big|\partial H[V_1, V_3]\big| + \big|\partial H[V_2, V_3]\big| \geqslant (1 + \varepsilon)\, m^2 .$$

Proof. Given ξ and ε we set

$$\varrho = \left(\frac{\varepsilon}{21}\right)^2 \qquad \text{and} \qquad \eta \leqslant \frac{\xi\varepsilon}{36} .$$

Let $G = (V_1 \cup V_2, P)$ and $H = (V_1 \cup V_2 \cup V_3, E)$ be given. Since G is bipartite we may view P as a subset of $V_1 \times V_2$ and, hence, as a set of ordered pairs. Lemma 4.2.2 applied to V_3 and P ensures for the hypergraph H that there are at most $\sqrt{\varrho}m$ vertices in V_3 with less than $(1/4 + \epsilon - \sqrt{\varrho})|P|$ neighbour pairs in P. We remove such vertices from V_3 and let V_3' be the resulting subset of V_3.

Consider a fixed vertex $v_3 \in V_3'$. By the definition of V_3', we have

$$|N(v_3, P)| \geqslant \left(\frac{1}{4} + \epsilon - \sqrt{\varrho}\right)|P| \geqslant \left(\frac{1}{4} + \frac{15}{16}\epsilon\right)|P| . \tag{4.4.1}$$

For $i = 1, 2$ we consider the neighbourhood of v_3 in $\partial H[V_i, V_3]$ defined by

$$N_i(v_3) = \big\{v_i \in V_i \colon (v_i, v_3) \in \partial H[V_i, V_3]\big\}$$

and note that

$$|N(v_3, P)| \leqslant e_G\big(N_1(v_3), N_2(v_3)\big) .$$

Consequently, the (η, ξ)-regularity of G yields

$$|N(v_3, P)| \leqslant \xi |N_1(v_3)||N_2(v_3)| + \eta m^2 . \tag{4.4.2}$$

Combining (4.4.1) and (4.4.2) with the lower bound on $|P|$ provided by the regularity of G we obtain

$$4\xi|N_1(v_3)||N_2(v_3)| \geqslant \Big(1 + \frac{15}{4}\epsilon\Big)|P| - 4\eta m^2 \geqslant \Big(1 + \frac{15}{4}\epsilon\Big)(\xi - \eta)m^2 - 4\eta m^2 \geqslant \Big(1 + \frac{7}{2}\epsilon\Big)\xi m^2 ,$$

where the last inequality makes use of the choice of η. Hence, the AM-GM inequality tells us

$$\big(|N_1(v_3)| + |N_2(v_3)|\big)^2 \geqslant 4\,\big|N_1(v_3)\big|\big|N_2(v_3)\big| \geqslant \Big(1 + \frac{7}{2}\epsilon\Big)m^2$$

and, consequently, we arrive at

$$|N_1(v_3)| + |N_2(v_3)| \geqslant \Big(1 + \frac{7}{2}\epsilon\Big)^{1/2} m \geqslant \Big(1 + \frac{11}{10}\epsilon\Big) m\,.$$

Finally, summing for all vertices $v_3 \in V_3'$ we obtain the desired lower bound

$$\begin{aligned}
\big|\partial H[V_1, V_3]\big| + \big|\partial H[V_2, V_3]\big| &\geqslant \sum_{v_3 \in V_3'} \big(|N_1(v_3)| + |N_2(v_3)|\big) \\
&\geqslant \Big(1 + \frac{11}{10}\varepsilon\Big) m \cdot |V_3'| \\
&\geqslant \Big(1 + \frac{11}{10}\varepsilon\Big)\big(1 - \sqrt{\varrho}\big) m^2 \\
&\geqslant (1 + \varepsilon) m^2\,,
\end{aligned}$$

where we used the choice of ϱ for last inequality. □

Paths of length two will play a special rôle in our proof and the following notation will be useful.

Definition 4.4.2. Given a hypergraph $H = (V, E)$ and disjoint sets p, $q \in V^{(2)}$, we say that the edges xyz, $yzw \in E$ form a (p, q)-*cherry*, if $p = \{x, y\}$ and $q = \{z, w\}$.

Moreover, given two sets P, $Q \subseteq V^{(2)}$, we say that edges e, $e' \in E$ form a (P, Q)-*cherry*, if they form a (p, q)-cherry for some disjoint sets $p \in P$ and $q \in Q$.

The next lemma asserts that in $\therefore$-dense hypergraphs with density larger than $1/4$ large sets of pairs induce many cherries.

Lemma 4.4.3. *For every ξ, $\epsilon \in (0, 1]$ there exist ϱ, $\nu > 0$ such that the following holds for every sufficiently large $(\varrho, 1/4 + \varepsilon, \therefore)$-dense hypergraph $H = (V, E)$. For all sets P, $Q \subseteq V^{(2)}$ of size at least $3\xi n^2$ there are at least νn^4 (P, Q)-cherries.*

Proof. Given ξ and ϵ we apply Lemma 4.4.1 and we obtain η and ϱ'. Without loss of generality we may assume that $\eta \leqslant \xi/2$. Moreover, Lemma 4.2.1 applied with η and $d = \xi$ yields some $\mu > 0$ and we fix the desired constants ϱ and ν by

$$\varrho = \frac{\mu^3 \xi}{250}\varrho' \qquad \text{and} \qquad \nu = 9\varrho^2\mu^4\varepsilon\,.$$

Let $H = (V, E)$ and P, $Q \subseteq V^{(2)}$ satisfy the assumptions of the lemma.

We consider a random balanced bipartition of $A \cup B$ of V and let $P_A = \{p \in P : p \subseteq A\}$ and $Q_B = \{q \in Q : q \subseteq B\}$. A standard application of Chebyshev's inequality shows that there exists a balanced partition of V such that $|P_A|$, $|Q_B| \geqslant \xi n^2/2$. We apply Lemma 4.2.1 separately to the graphs (A, P_A) and (B, Q_B) and obtain four pairwise disjoint vertex sets $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ each of size $m \geqslant \mu n/2$ such that the induced bipartite graphs $P[A_1, A_2]$ and $Q[B_1, B_2]$ are both η-regular with density at least ξ.

Next for $i = 1$, 2 we consider the 3-partite subhypergraph $H[B_i, P[A_1, A_2]]$ on $A_1 \cup A_2 \cup B_i$ with the edge set

$$\Big\{\{x, y, z\} \in V^{(3)} : x \in B_i \text{ and } \{y, z\} \in E(P[A_1, A_2])\Big\}\,.$$

Lemma 4.2.3 applied to $H[B_i, P[A_1, A_2]]$ with $\beta = \varrho$ yields a subhypergraph $H^{i,P}_{\varrho}$. We want to prove that $H^{i,P}_{\varrho}$ is $(\varrho', 1/4 + \varepsilon, \therefore)$-dense over $(B_i, P[A_1, A_2])$. Since we removed at most $\varrho(3m)^3$ edges from $H[B_i, P[A_1, A_2]]$ the error term in the $\therefore$-density condition of $H^{i,P}_{\varrho}$ can add up to at most

$$\varrho n^3 + \varrho(3m)^3 \leqslant 28\varrho n^3 \leqslant \varrho' \cdot |B_i| \cdot e(P[A_1, A_2])\,.$$

This implies that $H^{i,P}_{\varrho}$ is $(\varrho', 1/4 + \varepsilon, \therefore)$-dense over $(B_i, P[A_1, A_2])$. Similarly, for $i = 1$, 2 we also define the 3-partite hypergraph $H^{i,Q}_{\varrho}$ with vertex partition $B_1 \cup B_2 \cup A_i$ and note that it is $(\varrho', 1/4 + \varepsilon, \therefore)$-dense over $(A_i, Q[B_1, B_2])$.

Applying Lemma 4.4.1 to the bipartite graph $P[A_1, A_2]$ and the 3-partite hypergraph $H^{1,P}_{\varrho}$ implies

$$\big|\partial H^{1,P}_{\varrho}[A_1, B_1]\big| + \big|\partial H^{1,P}_{\varrho}[A_2, B_1]\big| \geqslant (1 + \varepsilon)m^2.$$

Moreover, three further applications of Lemma 4.4.1 to $P[A_1, A_2]$ with $H^{2,P}_{\varrho}$ and to $Q[B_1, B_2]$ with $H^{1,Q}_{\varrho}$ and with $H^{2,Q}_{\varrho}$ show that

$$\sum_{i=1}^{2}\Big(\big|\partial H^{i,P}_{\varrho}[A_1, B_i]\big| + \big|\partial H^{i,P}_{\varrho}[A_2, B_i]\big|\Big) + \sum_{i=1}^{2}\Big(\big|\partial H^{i,Q}_{\varrho}[B_1, A_i]\big| + \big|\partial H^{i,Q}_{\varrho}[B_2, A_i]\big|\Big) \geqslant 4(1+\varepsilon)m^2.$$

In particular, rearranging the terms shows that

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\Big(\big|\partial H_{\varrho}^{j,P}[A_i,B_j]\big| + \big|\partial H_{\varrho}^{i,Q}[B_j,A_i]\big|\Big) \geqslant 4(1+\varepsilon)m^2$$

and, hence, there are some indices i_0, $j_0 \in \{1,2\}$ such that

$$\big|\partial H_{\varrho}^{j_0,P}[A_{i_0},B_{j_0}]\big| + \big|\partial H_{\varrho}^{i_0,Q}[B_{j_0},A_{i_0}]\big| \geqslant (1+\varepsilon)m^2\,.$$

Consequently, the set of ordered pairs

$$R = \big\{\{y,z\} \in V^{(2)} \colon (y,z) \in \partial H_{\varrho}^{j_0,P}[A_{i_0},B_{j_0}] \text{ and } (z,y) \in \partial H_{\varrho}^{i_0,Q}[B_{j_0},A_{i_0}]\big\}$$

has size at least εm^2.

Finally, we note that every $\{y,z\} \in R$ has positive degree in both hypergraphs $H_{\varrho}^{j_0,P}$ and $H_{\varrho}^{i_0,Q}$ and, hence, these degrees are at least $3\varrho m$. Therefore, there are at least $9\varrho^2 m^2$ distinct vertices $x \in A_{3-i_0}$ and $w \in B_{3-j_0}$ such that xyz and yzw form a (P,Q)-cherry. Summing over all pairs in R yields at least

$$\varepsilon m^2 \cdot 9\varrho^2 m^2 \geqslant \nu n^4$$

(P,Q)-cherries in H. □

The following corollary allows us to find many connections between large sets of unordered pairs and large sets of ordered pairs.

Lemma 4.4.4. *For every ξ, $\epsilon \in (0,1]$ there exist ζ, $\varrho > 0$ such that the following holds for every sufficiently large $(\varrho, 1/4+\varepsilon, \therefore)$-dense n-vertex hypergraph $H = (V,E)$.*

Let $P \subseteq V \times V$ be a set of ordered pairs and let $Q \subseteq V^{(2)}$ be a set of unordered pairs, each of size at least ξn^2. There is an $\ell \in \{2,4\}$ such that there are at least $\zeta n^{\ell+2}$ paths of length ℓ which start with an ordered pair from P and ends in (some ordering of) with a pair from Q.

Proof. Given ξ and ϵ we apply Lemma 4.4.3 with $\xi/6$ and ε and obtain ϱ and ν. Lemma 4.2.4 applied for $\nu/2$, 4, and 6 (in place of ξ, k, and ℓ in Lemma 4.2.4) yields the promised constant $\zeta > 0$. With out loss of generality we may assume that $\zeta < \nu/2$ and let n be sufficiently large.

For a given set of ordered pairs $P \subseteq V \times V$ let $\overline{P}$ be the set of unordered pairs obtained from P by ignoring the order. In particular, $|\overline{P}| \geqslant |P|/2 \geqslant \xi n^2/2$ and Lemma 4.4.3 asserts that there are νn^4 different $(\overline{P}, Q)$-cherries. That is to say there are νn^4 paths on four vertices of the form $xyzw$ where $\{x, y\} \in \overline{P}$ and $\{z, w\} \in Q$.

If for ζn^4 of those cherries we have that $(x, y) \in P$, then the lemma follows with $\ell = 2$. Hence, we may assume that for at least $(\nu - \zeta)n^4 \geqslant \nu n^4/2$ of those paths we (only) have $(y, x) \in P$. Consequently, Lemma 4.2.4 yields ζn^6 blowups of these two edge paths where the vertices y and z are doubled, i.e., H contains at least ζn^6 6-tuples of distinct vertices $(x, y_1, y_2, z_1, z_2, w)$ such that for every $i, j \in \{1, 2\}$ we have

$$(y_i, x) \in P\,, \quad \{z_j, w\} \in Q\,, \quad \text{and} \quad xy_iz_jw \text{ is a path with two edges.}$$

In particular, every such 6-tuple induces a path $y_1xz_1y_2wz_2$ which starts with an ordered pair from P and ends in an unordered pair from Q and this concludes the proof of the lemma. □

For establishing the Connecting Lemma (Lemma 4.1.4) we shall extend Lemma 4.4.4 in such a way that we can connect large sets P and Q, where both of them consist of ordered pairs. For that certain blowups of $K_4^{(3)-}$s will be useful and we introduce the following notation.

Definition 4.4.5. We say a 7-tuple of distinct vertices $(a_1, a_2, a_3, b_1, b_2, c, d) \in V^7$ is a *turn* in a hypergraph $H = (V, E)$ if for every $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$ the set $\{a_i, b_j, c, d\}$ spans a copy of a $K_4^{(3)-}$ in H with a_i being the apex.

Combining Theorem 4.2.5 and Lemma 4.2.4 shows that the hypergraphs with ∴-density bigger than $1/4$ contain many turns. Moreover, we observe that in a turn T the paths

$$a_1b_1ca_2b_2\,, \quad a_1b_1ca_3db_2a_2\,, \quad b_1a_1cda_2b_2\,, \quad \text{and} \quad b_1a_1cb_2a_2 \tag{4.4.3}$$

with at most 3 inner vertices connect the pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$ in all four possible orientations. This motivates the following definition.

Definition 4.4.6. For a hypergraph $H = (V, E)$ we say two disjoint unordered pairs q, $q' \in V^{(2)}$ are (ϑ, L)-*turnable*, if for every ordering (q_1, q_2) of q and every ordering (q'_1, q'_2) of q' there exists some positive integer $\ell \leqslant L$ such that the number of (q_1, q_2)-(q'_1, q'_2)-paths in H with ℓ inner vertices is at least $\vartheta|V|^\ell$.

It follows from (4.4.3) that pairs $\{a_1, b_1\}$ and $\{a_2, b_2\}$ that are contained in $\Omega(|V|^3)$ turns are $(\vartheta, 3)$-turnable for some sufficiently small $\vartheta > 0$. The following variation of this fact, will be useful in the proof of the Connecting Lemma.

Lemma 4.4.7. *For every $\epsilon \in (0, 1]$ there exist ϑ, $\varrho > 0$ such that the following holds for every sufficiently large $(\varrho, 1/4 + \varepsilon, \therefore)$-dense hypergraph $H = (V, E)$.*

There exists a set $Q \subseteq V^{(2)}$ of size at least $\vartheta|V|^2$ such that for every $q \in Q$ there exists a set $Q'(q) \subseteq V^{(2)}$ of size at least $\vartheta|V|^2$ such that q and q' are $(\vartheta, 3)$-turnable for every $q' \in Q'(q)$.

Proof. Let $H = (V, E)$ be a sufficiently large $(\varrho, 1/4 + \epsilon, \therefore)$-dense hypergraph on n vertices. A combined application of Theorem 4.2.5 and Lemma 4.2.4 yields a set $\mathcal{T} \subseteq V^7$ of at least ζn^7 turns $(a_1, a_2, a_3, b_1, b_2, c, d)$ in H for some sufficiently small $\zeta = \zeta(\varepsilon) > 0$ and we shall deduce the conclusion of the lemma for

$$\vartheta = \frac{\zeta}{8}\,.$$

For every pair $(a, b) \in V \times V$ and $i \in \{1, 2\}$ let $\mathcal{T}_i(a, b)$ be the set of such turns where a and b play the rôles of a_i and b_i, respectively. We consider the set

$$\mathcal{T}^\star = \left\{(a, a', a_3, b, b', c, d) \in \mathcal{T} \colon |\mathcal{T}_1(a, b) \cap \mathcal{T}_2(a', b')| \geqslant \zeta n^3/2\right\}$$

and note that $|\mathcal{T}^\star| \geqslant \zeta n^7/2$. By a standard averaging argument there are at least $\zeta n^2/4$ pairs $(a, b) \in V \times V$ for which we have

$$|\mathcal{T}_1(a, b) \cap \mathcal{T}^\star| \geqslant \frac{\zeta}{4} n^5$$

and we denote the set of these ordered pairs by R. Note that for every pair $(a, b) \in R$ there is a set $R'(a, b) \subseteq V \times V$ with

$$|R'(a, b)| \geqslant \frac{\zeta}{4} n^2 \text{ such that } \left|\mathcal{T}_1(a, b) \cap \mathcal{T}_2(a', b')\right| \geqslant \frac{\zeta}{2} n^3 \tag{4.4.4}$$

for every $(a', b') \in R'(a, b)$. Finally, let Q be the set of unordered pairs derived from R, i.e.,

$$Q = \big\{\{q_1, q_2\} \in V^{(2)} \colon (q_1, q_2) \in R\big\}$$

and for every $q = \{q_1, q_2\}$ set

$$Q'(q) = \big\{\{q_1', q_2'\} \in V^{(2)} \colon (q_1', q_2') \in R'(q_1, q_2) \cup R'(q_2, q_1)\big\}\,.$$

Clearly,

$$|Q| \geqslant \frac{|R|}{2} \geqslant \frac{\zeta}{8} n^2 = \vartheta n^2 \quad \text{and} \quad Q'(q) \overset{(4.4.4)}{\geqslant} \frac{\zeta}{8} n^2 = \vartheta n^2$$

and the required number of paths for every orientation of $q \in Q$ and $q' \in Q'(q)$ follows from (4.4.3) and (4.4.4). $\square$

Roughly speaking, the proof of Lemma 4.1.4 follows from Lemmata 4.4.4 and 4.4.7. The definition of connectable pairs allows us to move from the given ordered pairs (x, y) and (w, z), that need to be connected, to large sets of ordered pairs P, P', by considering their second neighbourhoods. Moreover, Lemma 4.4.7 yields sets $Q \subseteq V^{(2)}$ and $Q'(q) \subseteq V^{(2)}$ for every $q \in Q$ of turnable pairs. Applying Lemma 4.4.4 first to P and Q and then to P' and $Q'(q)$ for all $q \in Q$ leads to the desired (x, y)-(z, w)-paths.

Proof of Lemma 4.1.4. For given ϵ, $\beta > 0$ let ϑ and ϱ_1 be the constants provided by Lemma 4.4.7. We set

$$\xi = \min\{\vartheta, \beta^2\}$$

and Lemma 4.4.4 applied with ξ and ε yields ζ and ϱ_2. Finally, we define the promised constants

$$\varrho = \min\{\varrho_1, \varrho_2\} \qquad \text{and} \qquad \alpha = \frac{\zeta^2 \vartheta}{13}\,.$$

Let $H = (V, E)$ be a sufficiently large $(\varrho, 1/4 + \epsilon, \therefore)$-dense hypergraph on n vertices and let (x, y), (w, z) be two disjoint β-connectable pairs. Consider the second neighbourhoods of these pairs defined by

$$P = \{(u, v) \in V \times V \colon xyu,\ yuv \in E\} \quad \text{and} \quad P' = \{(u', v') \in V \times V \colon wzu',\ zu'v' \in E\}\,. \tag{4.4.5}$$

Owing to the β-connectability, both sets P and P' have size at least $\beta^2 n^2 \geqslant \xi n^2$.

Next, let $Q \subseteq V^{(2)}$ and $Q'(q) \subseteq V^{(2)}$ for every $q \in Q$ be the sets of size at least $\vartheta n^2 \geqslant \xi n^2$ provided by Lemma 4.4.7. For every $q \in Q$ we denote by $P_4(q)$ (resp. $P_6(q)$) the number of (u,v)-(q_1,q_2)-paths having 4 (resp. 6) vertices and $(u,v) \in P$ and $\{q_1,q_2\} = q$. Moreover, we normalise these numbers by

$$\eta_P(q) = \max\left\{\frac{P_4(q)}{n^4}, \frac{P_6(q)}{n^6}\right\}$$

and note that Lemma 4.4.4 applied to P and Q ensures

$$\sum_{q \in Q} \eta_P(q) \geqslant \zeta\,. \tag{4.4.6}$$

Analogously, we define $P'_4(q')$, $P'_6(q')$, and $\eta_{P'}(q')$ for every $q' \in \bigcup_{q\in Q} Q'(q)$ and Lemma 4.4.4 applied to P' and $Q'(q)$ implies

$$\sum_{q' \in Q'(q)} \eta_{P'}(q') \geqslant \zeta\,. \tag{4.4.7}$$

for every $q \in Q$. Recall, that the paths accounted for in (4.4.6) and (4.4.7) induce an ordering of the vertices in q and in q'. However, by Lemma 4.4.7 the pairs q and q' are $(\vartheta, 3)$-turnable for every $q \in Q$ and $q' \in Q'(q)$, which means that these pairs can be connected for any possible orientation. Consequently, there is some ℓ with

$$5 \leqslant \ell \leqslant \max\{4,6\} + \max\{1,2,3\} + \max\{4,6\} = 15$$

such that the number of (x,y)-(z,w)-walks in H is at least

$$\frac{n^\ell}{12} \cdot \sum_{q\in Q} \eta_P(q) \cdot \vartheta \cdot \sum_{q' \in Q'(q)} \eta_{P'}(q') \overset{(4.4.7)}{\geqslant} \frac{n^\ell}{12} \cdot \sum_{q \in Q} \eta_P(q) \cdot \vartheta \cdot \zeta \overset{(4.4.6)}{\geqslant} \frac{\zeta^2\vartheta}{12} n^\ell\,.$$

At most $O(n^{\ell-1})$ of these walks might not be a path and, hence, the lemma follows for sufficiently large n. □

4.5 Absorbing path

We dedicate this section to the proof of Lemma 4.1.3. Similarly as in [52] the absorbers we consider here have two parts. Moreover, we use an idea of Polcyn and Reiher [45], which reduces the abundant existence of absorbers to a degenerate Turán problem for the price that we can only absorb exactly three vertices at each time.

Consider the complete 3-partite hypergraph $K^{(3)}_{3,3,3}$ with parts $A_i = \{x_i, y_i, z_i\}$, for every $i = 1, 2, 3$. Note that this hypergraph contains the paths

$$x_1x_2x_3y_1y_2y_3z_1z_2z_3\,, \tag{4.5.1}$$

and

$$x_1x_2x_3z_1z_2z_3\,. \tag{4.5.2}$$

This means that from every copy of $K^{(3)}_{3,3,3}$, ordered as a path like in (4.5.1), we may remove the three inner vertices y_1, y_2, y_3 to obtain a path with the same ends. Since we only consider dense hypergraphs, we can guarantee that many copies $K^{(3)}_{3,3,3}$ exist. In other words, in such a situation the path $x_1x_2x_3z_1z_2z_3$ could absorb the three vertices y_1, y_2, and y_3. However, not every triple might be contained in a $K^{(3)}_{3,3,3}$ and this will be addressed by the second part of the absorbers used here.

Suppose we want to absorb some arbitrary vertices v_1, v_2, and v_3. The idea, similarly as in [52], is to exchange v_i with y_i contained in some $K^{(3)}_{3,3,3}$. Suppose we have found a $K^{(3)}_{3,3,3}$ as described above, but additionally we find a path (as a graph) on four vertices with edges from $N_H(v_i) \cap N_H(y_i)$ disjointly for each $i = 1, 2, 3$. We argue that this whole structure can absorb v_1, v_2, v_3. Indeed, if $a_ib_ic_id_i$ is a path on four vertices with edges from $N_H(v_i) \cap N_H(y_i)$, then both $P(v_i) = a_ib_iv_ic_id_i$ and $P(y_i) = a_ib_iy_ic_id_i$ are paths in the hypergraph and with the same endings. Moreover, the minimum degree and the uniform density imply that for each vertex $v \in V$, most vertices of V have $\Omega(n^2)$ common neighbours with v, which is enough to find such paths.

Therefore, if we choose to absorb v_1, v_2, v_3, we will consider the paths $P(v_1)$, $P(v_2)$, and $P(v_3)$ and the path of $K^{(3)}_{3,3,3}$ as in (4.5.1). On the other hand, if we choose not to absorb them, then we consider the paths $P(y_1)$, $P(y_2)$, and $P(y_3)$ and the path of $K^{(3)}_{3,3,3}$ as in (4.5.2). We will also show that for each triple of vertices, we can find many of these configurations, so that we can choose a small amount of them that still can absorb every triple and also connect them into a single path. Observe that this absorbing path can only absorb sets of vertices with size divisible by three, an issue with which we deal later. First we prove that for every triple there are many absorbers.

Definition 4.5.1. Let $H = (V, E)$ be a hypergraph and $(v_1, v_2, v_3) \in V^3$. We say

$$A = (K, P_1, P_2, P_3) \in V^9 \times V^4 \times V^4 \times V^4\,,$$

with $K = (x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3)$ and $P_i = (a_i, b_i, c_i, d_i)$ is an *absorber for* (v_1, v_2, v_3) if the ordered sets

(i) $x_1x_2x_3y_1y_2y_3z_1z_2z_3$, $x_1x_2x_3z_1z_2z_3$,

(ii) $a_ib_iv_ic_id_i$ and $a_ib_iy_ic_id_i$ for $i = 1, 2, 3$

induce paths in H. All hyperedges of those paths that do not include a vertex from $\{v_1, v_2, v_3\}$ are called *internal edges* of the absorber A.

Formally absorbers are defined to be four tuples. However, sometimes it will be convenient to view them as 21-tuples of vertices.

Lemma 4.5.2. *For all d, $\epsilon \in (0, 1]$ there exist ϱ, $\xi > 0$ such that for sufficiently large n the following holds.*

For every $(\varrho, d, \therefore)$-dense hypergraph H on n vertices with $\delta_1(H) \geqslant \epsilon n^2$ and every triple $T = (v_1, v_2, v_3) \in V(H)^3$ of distinct vertices there are at least ξn^{21} absorbers for T.

Proof. Given d and ϵ we define some auxiliary constant $\zeta = (d/2)^{27}/3$ and set

$$\varrho = \frac{1}{36}\left(\frac{d}{2}\right)^{54} \qquad \text{and} \qquad \xi = \frac{\zeta d^9 \varepsilon^9}{2^{11}}\,.$$

Let $H = (V, E)$ be a $(\varrho, d, \therefore)$-dense hypergraph on n vertices and consider some triple of vertices $T = (v_1, v_2, v_3) \in V^3$.

Three applications of Lemma 4.2.2 each with $X = V$ and for $i \in [3]$ with the set of ordered pairs

$$\big\{(u, w) \colon \{u, w\} \in N_H(v_i)\big\}$$

tells us, that there are at most $3\sqrt{\varrho}n$ *bad vertices* $v \in V$ that may fail to satisfy

$$\big|N_H(v) \cap N_H(v_i)\big| \geqslant (d - \sqrt{\varrho})\big|N_H(v_i)\big| \geqslant (d - \sqrt{\varrho})\delta_1(H) \geqslant \frac{d}{2}\varepsilon n^2 \tag{4.5.3}$$

for some $i \in [3]$. Moreover, the $(\varrho, d, \therefore)$-density of H implies that the edge density of H is at least $d - 2\varrho > d/2$ and since the extremal number of any fixed 3-partite hypergraph is $o(n^3)$ we have $K^{(3)}_{3,3,3} \subseteq H$ for sufficiently large n. In fact, the standard proof of this fact from [18] yields at least $((d/2)^{27} - o(1))n^9$ such copies. Consequently, for sufficiently large n there are at least

$$\left(\left(\frac{d}{2}\right)^{27} - o(1)\right)n^9 - 3\sqrt{\varrho}n \cdot n^8 \geqslant \zeta n^9$$

copies of $K^{(3)}_{3,3,3}$ in H that contain no bad vertex. Let $\mathcal{K} = \mathcal{K}_T \subseteq V^9$ be the set of these $K^{(3)}_{3,3,3}$ in H.

Consider some $K = (x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3) \in \mathcal{K}$. Since none of the vertices of K is bad, for every vertex v from K inequality (4.5.3) holds for every $i \in [3]$. In particular, for every $i \in [3]$ we have $|N_H(y_i) \cap N_H(v_i)| \geqslant d\varepsilon n^2/2$ and it follows from [11] that there exist at least $((d\varepsilon/2)^3 - o(1))n^4$ paths on four vertices with edges from $N_H(y_i) \cap N_H(v_i)$. Consequently, for sufficiently large n, there exist at least

$$|\mathcal{K}| \cdot \left(\Big(\frac{d^3\varepsilon^3}{8} - o(1)\Big)n^4\right)^3 \geqslant \zeta n^9 \cdot \frac{d^9\varepsilon^9}{2^{10}}n^{12} \geqslant 2\xi n^{21}$$

4-tuples $A = (K, P_1, P_2, P_3) \in V^9 \times V^4 \times V^4 \times V^4$ with P_i inducing a path in $N_H(y_i) \cap N_H(v_i)$ for $i = [3]$. Such an A may only fail to be an absorber for T, if it contains some vertex from T itself or if its 21 vertices are not distinct. However, since there are at most $O(n^{20})$ such “degenerate” A’s the lemma follows for sufficiently large n. $\square$

Note that for the proof of Lemma 4.5.2 positive $\therefore$-density was sufficient. However, to address the aforementioned divisibility issue, we will show that the hypergraphs H considered here contain a copy of $C_8(4)$, the 4-blow-up of the cycle on 8 vertices. For the proof of that, we make use of the assumption that the $\therefore$-density of H is bigger than $1/4$.

The $C_8(4)$ is formed by 8 cyclicly ordered independent sets $\{e_i, f_i, g_i, h_i\}_{i\in[8]}$ such that the only edges are the ones with vertices from three consecutive such sets. Note that $C_8(4)$ contains the path

$$e_1e_2\dots e_8f_1f_2\dots f_8g_1g_2\dots g_8h_1h_2\dots h_8. \tag{4.5.4}$$

Moreover, by removing the sets $\{f_i\}_{i\in[8]}$ or $\{f_i, g_i\}_{i\in[8]}$ from the path in (4.5.4) leads to paths with the same ends in $C_8(4)$ with 24 or 16 vertices, respectively. We also remark that 16, 24 and 32 are congruent to 1, 0 and 2 modulo 3, respectively. Therefore, if we connect such a path to the absorbing path, we can decide to remove some of the vertices so that the size of the leftover set is divisible by 3.

Lemma 4.5.3. *For all $\epsilon > 0$ there exist ϱ, $\vartheta > 0$ such that every sufficiently large $(\varrho, 1/4 + \epsilon, \therefore)$-dense hypergraph $H = (V, E)$ contains $\vartheta|V|^{32}$ copies of $C_8(4)$.*

Proof. Given $\epsilon > 0$ we apply Theorem 4.2.5 to obtain ϱ_1 and ξ. Then, the application of Lemma 4.4.3 to $\xi/6$ and ε yields ϱ_2 and ν. Set $\varrho = \min\{\varrho_1, \varrho_2\}$ and let n be sufficiently large.

Let $H = (V, E)$ be a $(\varrho, 1/4 + \epsilon, \therefore)$-dense hypergraph on n vertices. In view of Lemma 4.2.4 it suffices to show that H contains ζn^8 copies of C_8 for some $\zeta > 0$.

Theorem 4.2.5 implies that H contains at least ξn^4 copies of $K_4^{(3)-}$. Let R be the set of ordered pairs (a, x) such that both vertices are contained in at least $\xi n^2/2$ of these $K_4^{(3)-}$ with a being the apex. By double counting we infer $|R| \geqslant \xi n^2/2$.

For every $(a, x) \in R$, let $P_{a,x} \subseteq V^{(2)}$ be those pairs $\{y, z\}$ that span such a copy of $K_4^{(3)-}$ together with a and x. An application of Lemma 4.4.3 to $P = Q = P_{a,x}$ yields at least νn^4 (P, Q)-cherries, i.e., paths with 4 vertices starting and ending at a pair from $P_{a,x}$.

Let F be the hypergraph with vertex set $\{a, x, y, y', z, z'\}$ such that the sets of vertices $\{a, x, y, z\}$ and $\{a, x, y', z'\}$ span copies of $K_4^{(3)-}$ with apex a and it contains a $(\{y, z\}, \{y', z'\})$-cherry. Observe that since y and z (resp. y' and z') play a symmetric role in $K_4^{(3)-}$, regardless of the orientation of the pairs $\{y, z\}$ and $\{y', z'\}$ in the cherry the resulting hypergraph is isomorphic. Without loss of generality we will assume that the cherry is a path of the form $yzy'z'$. By the reasoning above, H contains at least

$$|R| \cdot \nu n^4 \geqslant \frac{\xi}{2}\nu n^6$$

copies of F. We argue that there is a homomorphism of C_8 in F. Indeed, if we consider the vertices of F in the following cyclic ordering

$$xayzy'z'ay'$$

one can check that every consecutive triple forms an edge in F. Since there are at least $\Omega(n^6)$ copies of F in H, then by Lemma 4.2.4 and taking ζ small enough, we have that there are at least ζn^8 copies of C_8. □

We are now ready to prove Lemma 4.1.3.

Proof of Lemma 4.1.3. Given $\varepsilon > 0$ the constants appearing in this proof will satisfy the following hierarchy

$$1 > \varepsilon \gg \xi\,, \vartheta \gg \beta \gg \varrho\,, \alpha \gg \gamma' \geqslant \gamma \gg \frac{1}{n}\,, \tag{4.5.5}$$

where the auxiliary constants ξ, ϑ, and α are provided by Lemmata 4.5.2, 4.5.3, and 4.1.4 and it is easy to check that (4.5.5) complies with the quantification of these lemmata. Let H be a $(\varrho, 1/4+\epsilon, ∴)$-dense hypergraph with $\delta_1(H) \geqslant \epsilon n^2$ and let R be a subset of V with at most $2\gamma^2 n$ vertices. Fix the subhypergraph $H_\beta \subseteq H$ provided by Lemma 4.2.3.

For $T \in V^3$, let $\mathcal{A}_T$ be the set of those absorbers for T in H that have no vertex in R and all its 36 internal edges from H_β. It follows from Lemma 4.5.2 applied with $d = 1/4 + \varepsilon$ and ε that

$$|\mathcal{A}_T| \geqslant \xi n^{21} - 21\,|R|n^{20} - 6\cdot 36\big(e(H) - e(H_\beta)\big)n^{18} \geqslant \xi n^{21} - 42\,\gamma^2 n^{21} - 216\,\beta n^{21} \overset{(4.5.5)}{\geqslant} \frac{\xi}{2}n^{21}\,.$$

Let $\mathcal{A} = \bigcup_T \mathcal{A}_T$ be the union over all triples $T \in V^3$ and consider a random collection of absorbers $\mathcal{C} \subseteq \mathcal{A}$ in which each element of $\mathcal{A}$ is present independently with probability

$$p = \frac{\gamma^{4/3} n}{2|\mathcal{A}|}\,.$$

Since $\mathbb{E}|\mathcal{A}| = p|\mathcal{A}|$, Markov's inequality ensures that

$$\mathbb{P}\big(|\mathcal{C}| \geqslant \gamma^{4/3}n\big) \leqslant \frac{1}{2}. \tag{4.5.6}$$

Moreover, for every $T \in V^3$ we have

$$\mathbb{E}|\mathcal{C} \cap \mathcal{A}_T| = p|\mathcal{A}_T| \geqslant \frac{\gamma^{4/3}n}{2|\mathcal{A}|} \cdot \frac{\xi n^{21}}{2} \geqslant \frac{\gamma^{4/3}\xi n}{4} \overset{(4.5.5)}{\geqslant} 4\gamma^2 n\,,$$

Chernoff's inequality combined with the union bound over all triples yields

$$\mathbb{P}\big(\exists T \in V^3\colon |\mathcal{C} \cap \mathcal{A}_T| < 3\gamma^2 n\big) \leqslant o(1)\,. \tag{4.5.7}$$

Letting Y be the number of pairs of distinct absorbers A, $A' \in \mathcal{C}$ that share a vertex we note

$$\mathbb{E}Y = p^2 \cdot n^{21} \cdot 21^2 \cdot n^{20} = \frac{\gamma^{8/3}n^2}{4|\mathcal{A}|^2} \cdot 441 n^{41} \leqslant \frac{441\gamma^{8/3}n}{\xi^2} \overset{(4.5.5)}{\leqslant} \frac{\gamma^2 n}{4}$$

and by Markov's inequality, we have

$$\mathbb{P}(Y \geqslant \gamma^2 n) \leqslant \frac{1}{4}\,. \tag{4.5.8}$$

Consequently, with positive probability none of the bad events from (4.5.6), (4.5.7), and (4.5.8) happen. In particular, there exists a realisation of $\mathcal{C}$ such that

$$|\mathcal{C}| < \gamma^{4/3}n\,, \qquad |\mathcal{C} \cap \mathcal{A}_T| \geqslant 3\gamma^2 n \text{ for every } T \in V^3, \qquad \text{and} \qquad |Y(\mathcal{C})| < \gamma^2 n\,.$$

For every pair of absorbers accounted in $Y(\mathcal{C})$ we remove one of the involved absorbers in an arbitrary way and obtain a subset $\mathcal{B} \subseteq \mathcal{C}$ of pairwise vertex disjoint absorbers satisfying

$$|\mathcal{B}| \leqslant |\mathcal{C}| < \gamma^{4/3}n \qquad \text{and} \qquad |\mathcal{B} \cap \mathcal{A}_T| > |\mathcal{C} \cap \mathcal{A}_T| - \gamma^2 n \geqslant 2\gamma^2 n \text{ for every } T \in V^3.$$

Recall that if the absorbing path would only contain the absorbers from $\mathcal{B}$, then it could only absorb sets U with a cardinality that is divisible by 3. We address this divisibility issue by adding a copy of $C_8(4)$ to the path. Lemma 4.5.3 guarantees at least ϑn^{32} copies of $C_8(4)$ in H. Similarly, as for the estimate of $\mathcal{A}_T$, we infer that there is one such $C_8(4)$ which is vertex disjoint from the set R and from all absorbers from $\mathcal{B}$ and which only contains edges from H_β. In fact, this follows from

$$\begin{aligned}\vartheta n^{32} - 32\,|R|n^{31} - 21\,|\mathcal{B}|n^{31} - 6 \cdot e(C_8(4))\big(e(H) - e(H_\beta)\big)n^{29}& \\ \geqslant \vartheta n^{32} - 64\,\gamma^2 n^{32} - 21\,\gamma^{4/3}n^{32} - 3072\,\beta n^{32} &\overset{(4.5.5)}{>} 0\,.\end{aligned}$$

Fix an ordering of the vertices of such a $C_8(4)$ that induces a path (see, e.g., (4.5.4)) and denote this path by P_C.

In order to obtain the final absorbing path, each absorber $(K, P_1, P_2, P_3) \in \mathcal{B}$ will be viewed as a collection of four paths: $x_1x_2x_3z_1z_2z_3$ and $a_ib_iy_ic_id_i$, for $i = 1, 2, 3$, as in Definition 4.5.1. Therefore, together with joining P_C we have to connect $t = 4|\mathcal{B}| + 1$ paths to build the promised absorbing path A. For each of the connections we will appeal to Lemma 4.1.4 and each application will require to add up at most 15 inner vertices.

Let $(P_i)_{i\in[t]}$ be an arbitrary enumeration of all these paths that need to be connected. We continue in an inductive manner starting with $A_1 = P_1$, let A_j be the already constructed path containing P_i for every $i \leqslant j$. Since every connection requires at most 15 inner vertices and the longest path in $(P_i)_{i\in[t]}$ has 32 vertices we have

$$|V(A_j)| + \sum_{i=j+1}^{t} |V(P_i)| \leqslant 15(j-1) + 32t \leqslant 47t \leqslant 47\big(4|\mathcal{B}| + 1\big) \leqslant 47\big(4\gamma^{4/3}n + 1\big) \leqslant \gamma n\,. \tag{4.5.9}$$

Suppose now that we want to connect P_j, which ends in (x, y), to P_{j+1}, which starts at (z, w). Since all paths P_i with $i \in [t]$ have its edges in H_β, by Lemma 4.2.3 they are β-connectable. Therefore, Lemma 4.1.4 implies that there are at least αn^ℓ paths, with $\ell \leqslant 15$ inner vertices, connecting (x, y) with (z, w) in H. Consequently, in view of (4.5.9) and $|R| \leqslant 2\gamma^2 n$ our choice of γ in (4.5.5) shows that at least one of such connecting paths must be vertex disjoint from

$$V(A_j) \cup \bigcup_{i=j+1}^{t} V(P_i) \cup R\,,$$

which concludes the inductive step and proves the existence of the path A_{j+1}.

Finally, let $A = A_t$ be the final path and let $U \subseteq V \smallsetminus V(A)$ with $|U| \leqslant 3\gamma^2 n$. First we remove 0, 8 or 16 vertices from P_C in A and reallocate them to U to get a set U' with size divisible by three. Moreover $|U'| \leqslant 3\gamma^2 n + 16 \leqslant 3(\gamma^2 n + 6)$ and, hence, U' can be split into at most $\gamma^2 n + 6$ disjoint triples. Since each triple has at least $2\gamma^2 n > \gamma^2 n + 6$ absorbers in A, we can greedily assign one for each and absorb all of them into A. $\square$

4.6 Proof of Theorem 2.2.3

In this section we discuss the few modifications necessary in the proof of Theorem 2.2.2 in order to prove Theorem 2.2.3. Recall that both theorems have the same minimum vertex degree assumption. However, where Theorem 2.2.3 requires the given hypergraph H to be Λ-dense for some positive density, Theorem 2.2.2 requires $\therefore$-density bigger than $1/4$. In other words, the uniform density assumptions of both theorems are incomparable.

The proof of Theorem 2.2.2 consist of three main parts, namely Lemmata 4.1.2–4.1.4. Observe that Lemma 4.1.2 can be applied directly under the conditions of Theorem 2.2.3, but for Lemmata 4.1.3 and 4.1.4 we have the assumption of $\therefore$-density at least $1/4$ which is not provided by Theorem 2.2.3.

We start with the discussion of the Connecting Lemma in the context of Theorem 2.2.3 in the next section and we defer the discussion of the adjustments for the Absorbing Path Lemma (Lemma 4.1.3) to Section 4.6.2.

4.6.1 Connecting Lemma for Theorem 2.2.3

The following lemma will play the rôle of Lemma 4.1.4 in the proof of Theorem 2.2.2.

Lemma 4.6.1 (Connecting Lemma for Λ-density conditions)**.** *For every d, $\beta > 0$ there exist ϱ, $\alpha > 0$ and an n_0 such that for every (ϱ, d, Λ)-dense hypergraph H on $n \geqslant n_0$ vertices the following holds.*

For every $\ell \in \{5, 6, 7\}$ and every pair of disjoint ordered β-connectable pairs (x, y), $(w, z) \in V \times V$, the number of (x, y)-(z, w)-paths with ℓ inner vertices is at least αn^ℓ.

Proof of Lemma 4.6.1 (sketch). We begin with the following observation. For sets of pairs P, $P' \subseteq V \times V$ each of size at least $\Omega(n^2)$ we show that

$$\text{there are at least } \Omega(n^5)\ p\text{-}p'\text{-paths with one inner vertex and } p \in P,\ p' \in P'. \tag{4.6.1}$$

Note that every $(\varrho, d, \wedge)$-dense hypergraph is $(\varrho, d, \therefore)$-dense and in view of Lemma 4.2.2 applied to P and V there is a set $X \subseteq V$ such that $|X| = \Omega(n)$ and for every $x \in X$ we have $|N(x, P)| = \Omega(n^2)$. Similarly, another application of Lemma 4.2.2 to P' and X yields a set $X' \subseteq X$ of size $\Omega(n)$ such that

$$|N(x, P)| = \Omega(n^2) \qquad \text{and} \qquad |N(x, Q)| = \Omega(n^2)$$

for every $x \in X'$. Consequently, a standard averaging argument tells us that each of the sets

$$Q = \big\{(p_2, x) \in V \times X' \colon |\{p_1 \in V \colon (p_1, p_2) \in P \text{ and } p_1p_2x \in E\}| = \Omega(n)\big\}$$

and

$$Q' = \big\{(x, p_1') \in X' \times V \colon |\{p_2' \in V \colon (p_1', p_2') \in P' \text{ and } xp_1'p_2' \in E\}| = \Omega(n)\big\}$$

has size $\Omega(n^2)$. Finally, the $\wedge$-density of H applied to Q and Q' yields $\Omega(n^5)$ p-p'-paths starting in P and ending in P' with an inner vertex from X, i.e., it establishes (4.6.1).

For given connectable pairs (x, y) and (w, z) letting P and P' be their second neighbourhoods as defined in (4.4.5), yields the conclusion of Lemma 4.6.1 for $\ell = 5$.

For $\ell = 6$ we note that $\therefore$-density implies that there are $\Omega(n^2)$ β'-connectable pairs (y, y') with $xyy' \in E$ for sufficiently small $\beta' = \beta'(d) > 0$. Applying the same argument as above for every such pair (y, y') proves the case $\ell = 6$. Finally, for $\ell = 7$ the same reasoning applied to the connectable pairs (y', y'') with xyy', $yy'y'' \in E$ concludes the proof. $\square$

4.6.2 Absorbing Path Lemma for Theorem 2.2.3

Recall that the proof of Lemma 4.1.3 required $\therefore$-density bigger than $1/4$ in only two places:

(i) for the connection of the absorbers to a path and

(ii) in Lemma 4.5.3 for addressing the divisibility issue of the size of the absorbable sets,

while for the abundant existence of the absorbers $\overset{\cdot}{-}$-density d for any $d > 0$ is sufficient (see Lemma 4.5.2). As shown in Section 4.6.1 for the connecting lemma positive $\curlywedge$-density suffices, which addresses (i). Moreover, in Lemma 4.6.1 we are even free to choose the length of the connecting paths, which renders the divisibility issue from (ii) in this context.

4.7 Concluding remarks

We briefly discuss a few open problems for 3-uniform hypergraphs and possible generalisations of Theorems 2.2.2 and 2.2.3 to k-uniform hypergraphs.

4.7.1 Problems for 3-uniform hypergraphs

Theorems 2.2.2 and 2.2.3 concern asymptotically optimal assumptions for uniformly dense hypergraphs that guarantee the existence of Hamilton cycles. The following notation will be useful for the further discussion.

Definition 4.7.1. Given $\star \in \{\therefore, \overset{\cdot}{-}, \curlywedge\}$ and $a \in \{1, 2\}$. We say a pair of reals (d, α) is $(\star, a)$-Hamilton if the following assertion holds:

For every $\varepsilon > 0$ there exist $\varrho > 0$ and n_0 such that every $(\varrho, d+\varepsilon, \star)$-dense hypergraph $H = (V, E)$ with $|V| = n \geqslant n_0$ and $\delta_a(H) \geqslant (\alpha + \varepsilon)\binom{n}{3-a}$ contains a Hamilton cycle.

We remark that we can restrict our attention to Hamilton cycles, since the result of Lenz, Mubayi, and Mycroft [41] asserts that already $(0, 0)$ would be $(\star, a)$-Hamilton for loose cycles for every choice of $\star \in \{\therefore, \overset{\cdot}{-}, \curlywedge\}$ and $a \in \{1, 2\}$. For Hamilton cycles Aigner-Horev and Levy [2] showed that $(0, 0)$ is $(\curlywedge, a)$-Hamilton for $a = 2$ and this was extended by Gan and Han [24] and by Theorem 2.2.3 to $a = 1$. It remains to characterise the minimal pairs (d, α) that are $(\star, a)$-Hamilton for the four combinations $\star \in \{\therefore, \underline{\therefore}\}$ and $a \in \{1, 2\}$.

Example 2.2.1 shows that for (d, α) being $(\underline{\therefore}, 1)$-Hamilton we must have

$$\max\{d, \alpha\} \geqslant \frac{1}{4}\,. \tag{4.7.1}$$

On the other hand, Theorem 2.2.2 asserts that for $d = 1/4$ already $\alpha = 0$ suffices. It would be interesting to determine the smallest value $\alpha_{\underline{\therefore},1}$ such that $d = 0$ suffices. In view of (4.7.1) we have $\alpha_{\underline{\therefore},1} \geqslant 1/4$ and the result from [52] bounds $\alpha_{\underline{\therefore},1}$ by $5/9$. Since all known lower bound constructions for that result are lacking to be $\underline{\therefore}$-dense it seems plausible that $\alpha_{\underline{\therefore},1} < 5/9$.

Similarly, let $\alpha_{\underline{\therefore},2}$ be the infimum over all $\alpha \geqslant 0$ such that $(0, \alpha)$ is $(\underline{\therefore}, 2)$-Hamilton. Here it follows from [58] that $\alpha_{\underline{\therefore},2} \leqslant 1/2$. Moreover, Example 2.2.1 yields a hypergraph with minimum codegree $(1/4 - o(1))n$ that fails to contain a Hamilton cycle. Therefore, we have $\alpha_{\underline{\therefore},2} \geqslant 1/4$ and at this point we are not aware of any reason that excludes the possibility that $\alpha_{\underline{\therefore},2}$ matches this lower bound.

Problem 4.7.2. *Determine $\alpha_{\underline{\therefore},1}$ and $\alpha_{\underline{\therefore},2}$.*

For Hamilton cycles in $\therefore$-dense hypergraphs the problem appears to be more delicate as the following unbalanced version of Example 2.2.1 shows. Instead of a uniformly chosen bipartition of $E(K_{n-2})$ we may colour the edges independently *red* with probability p and *blue* with probability $1 - p$. Let H_p be the resulting hypergraph, where the rest of the construction is carried out in the same way as in Example 2.2.1. By symmetry we may assume $p \geqslant 1/2$ and for the same reasons as in Example 2.2.1 the hypergraph H_p contains no Hamilton cycle. Moreover, for every fixed $\varrho > 0$ we have with high probability that

$$\delta_1(H_p) = \big(\min\{1 - p\,, p^3 + (1 - p)^3\} - \varrho\big)\binom{n}{2} \quad \text{and} \quad \delta_2(H_p) = \big((1 - p)^2 - \varrho\big)n$$

and that H_p is $(\varrho, p^3 + (1 - p)^3, \therefore)$-dense. For p close to 1 this shows that there is no $d < 1$ such that $(d, 0)$ is $(\therefore, a)$-Hamilton for $a \in \{1, 2\}$. In particular, there is no straightforward analogue of Theorem 2.2.2 in this setting.

It would be intriguing if this construction is essentially optimal for every $p \geqslant 1/2$. In such an event it would imply a resolution of the following problems, where the lower bound would be obtained from H_p for $p = 2/3$ and $p = 1/2$.

Problem 4.7.3. *Is it true that:*

(*i*) $(1/3, 1/3)$ *is* $(\therefore, 1)$*-Hamilton?*

(*ii*) $(1/4, 1/4)$ *is* $(\therefore, 2)$*-Hamilton?*

4.7.2 Possible generalisations to k-uniform hypergraphs

The notion of Hamilton cycles straight forwardly extends to k-uniform hypergraphs. Moreover, the definition of uniformly dense hypergraphs is inspired from the theory of quasirandom hypergraphs (see, e.g., [1, 62] and the references therein). Below we briefly recall the generalisation of Definitions 3.2.2 and 1.4.3 for general k-uniform hypergraphs, where we follow the presentation from [51].

Given a nonnegative integer k, a finite set V, and a set $Q \subseteq [k]$ we write V^Q for the set of all functions from Q to V. It will be convenient to identify the Cartesian power V^k with $V^{[k]}$ by regarding any k-tuple $\vec{v} = (v_1, \ldots, v_k)$ as being the function $i \longmapsto v_i$. We denote by $\vec{v} \longmapsto \vec{v} \,|\, Q$ the projection from V^k to V^Q and the preimage of any set $G_Q \subseteq V^Q$ is denoted by

$$\mathcal{K}_k(G_Q) = \left\{ \vec{v} \in V^k \colon (\vec{v} \,|\, Q) \in G_Q \right\}.$$

We may think of $G_Q \subseteq V^Q$ as a directed hypergraph (where vertices in the directed hyperedges are also allowed to repeat). More generally, for a subset $\mathcal{Q} \subseteq \mathscr{P}([k])$ of the power set of $[k]$ and a family $\mathscr{G} = \{G_Q \colon Q \in \mathcal{Q}\}$ with $G_Q \subseteq V^Q$ for all $Q \in \mathcal{Q}$, we define

$$\mathcal{K}_k(\mathscr{G}) = \bigcap_{Q \in \mathcal{Q}} \mathcal{K}_k(G_Q)\,. \tag{4.7.2}$$

Moreover, if $H = (V, E)$ is a k-uniform hypergraph on V, then $e_H(\mathscr{G})$ denotes the cardinality of the set

$$E_H(\mathscr{G}) = \left\{ (v_1, \ldots, v_k) \in \mathcal{K}_k(\mathscr{G}) \colon \{v_1, \ldots, v_k\} \in E \right\}.$$

Now we are ready to state the generalisation of Definitions 3.2.2 and 1.4.3.

Definition 4.7.4. Let ϱ, $d \in (0,1]$, let $H = (V, E)$ be a k-uniform hypergraph on n vertices, and let $\mathcal{Q} \subseteq \mathscr{P}([k])$ be given. We say that H is $(\varrho, d, \mathcal{Q})$-*dense* if for every family $\mathscr{G} = \{G_Q \colon Q \in \mathcal{Q}\}$ associating with each $Q \in \mathcal{Q}$ some $G_Q \subseteq V^Q$ we have

$$e_H(\mathscr{G}) \geqslant d\,|\mathcal{K}_k(\mathscr{G})| - \varrho n^k\,.$$

It is easy to check that for $k = 3$ the following subsets of $\mathscr{P}([3])$

$$\mathcal{Q}_{\therefore} = \big\{\{1\},\{2\},\{3\}\big\}\,, \quad \mathcal{Q}_{\dot{-}} = \big\{\{1\},\{2,3\}\big\}\,, \quad \text{and} \quad \mathcal{Q}_{\wedge} = \big\{\{1,2\},\{1,3\}\big\}$$

correspond to $\therefore$-, $\dot{-}$-, and $\wedge$-dense hypergraphs. More precisely, for every $\star \in \{\therefore, \dot{-}, \wedge\}$ we have that a 3-uniform hypergraph is $(\varrho, d, \star)$-dense if and only if it is $(\varrho, d, \mathcal{Q}_\star)$-dense.

Example 2.2.1 straight forwardly extends to k-uniform hypergraphs. In fact, we may consider a random bipartition $G \cup \overline{G}$ of the $(k-1)$-element subsets of an $(n-2)$-element set and we define a k-uniform hypergraph containing only those hyperedges such that all of its $(k-1)$-element subsets are in the same partition class. Finally, we may add two vertices x and y such that the $(k-1)$-uniform link of x is G and the $(k-1)$-uniform link of y is $\overline{G}$. We remark that for $k = 2$ this construction leads to two disjoint cliques with $\sim n/2$ vertices, which is a lower bound construction for Dirac's Theorem [16] in graphs.

It is easy to check that the resulting k-uniform hypergraph H does not contain a Hamilton cycle and for every fixed $\varrho > 0$ it is $(\varrho, 2^{1-k}, \mathcal{Q})$-dense for

$$\mathcal{Q} = \big\{Q \in [k]^{(k-2)} \colon 1 \in Q\big\} \cup \big\{\{2,\dots,k\}\big\}$$

with high probability for sufficiently large n. Note that for $k = 3$ we have $\mathcal{Q} = \mathcal{Q}_{\dot{-}}$ and H provides a lower bound for Theorem 2.2.2. It seems plausible that the hypergraph H is essentially optimal for $\mathcal{Q}$-dense hypergraphs also for $k > 3$, i.e., that $\mathcal{Q}$-dense k-uniform n-vertex hypergraphs with density bigger than 2^{1-k} and minimum vertex degree $\Omega(n^{k-1})$ contain a Hamilton cycle. This would be an interesting extension of Theorem 2.2.2 to k-uniform hypergraphs.

Moreover, one can check that for

$$\mathcal{Q}' = \big\{\{1,\dots,k-1\},\{1,\dots,k-2,k\}\big\}$$

the hypergraph H constructed above is not $(\varrho, d, \mathcal{Q}')$-dense for any fixed $d > 0$ and sufficiently small $\varrho > 0$. Note that for $k = 3$ we have $\mathcal{Q}' = \mathcal{Q}_{\wedge}$ and, in fact, Theorem 2.2.3 asserts that $(\varrho, d, \mathcal{Q}')$-dense hypergraphs with minimum vertex degree $\Omega(n^2)$ contain a Hamilton cycle for any $d > 0$ and sufficiently small ϱ. We remark that the proof of Theorem 2.2.3 discussed in Section 4.6 extends to k-uniform $\mathcal{Q}'$-dense hypergraphs with an appropriate minimum vertex degree condition.

Chapter 5

Codegree threshold for cycle decompositions

The main goal of this chapter is proving Theorem 2.3.1, but the proofs of Theorem 2.3.4 and Corollaries 2.3.2 and 2.3.3 are included as well.

We start by proving Theorem 2.3.4 in Section 5.1 to then present the short proofs for the Corollaries 2.3.2 and 2.3.3 in Section 5.2.

In Section 5.3 we prove Theorem 2.3.1 by using the technique of *iterative absorption*, which we review there. The technique relies on three main lemmata, the *Vortex Lemma*, *Cover-Down Lemma*, and *Absorbing Lemma*. After some useful tools (Section 5.4), these three lemmata are proved in Sections 5.5, 5.6 and 5.7, respectively. We finish in Section 5.8 with some remarks and open questions.

For this chapter we introduce the following definitions. A *walk* in a hypergraph H is a path that might repeat vertices and a *trail* is a walk that does not repeat edges. We recall that a tour is a cycle that might repeat vertices but not edges. We extend all notations introduced for paths to walks. Given a walk $W = v_1v_2 \dots v_\ell$ we define its *beginning* $b(W)$ and *terminus* $t(W)$ as $\{v_1, v_2\}$ and $\{v_{\ell-1}, v_\ell\}$ respectively (note that this is an unordered version of the starting and ending pairs defined at the beginning of Chapter 1). Moreover, if $\mathcal{C} = \{C_1, \dots, C_r\}$ is a collection of subgraphs of H, sometimes we will let $E(\mathcal{C})$ be the hypergraph whose edges are $\bigcup_{1 \leqslant i \leqslant r} E(C_i)$.

The work corresponding to this chapter was done in collaboration with Sanhueza-Matamala [44].

5.1 Lower bounds

In this section we prove Theorem 2.3.4. The following lemma captures divisibility constraints that tours in hypergraphs must satisfy, and it will be the basis of our constructions. For a hypergraph H, a subgraph $W \subseteq H$ and vertex sets X, Y, Z in $V(H)$, let $W[X, Y, Z]$ be the set of edges xyz in $E(W)$ such that $x \in X$, $y \in Y$, and $z \in Z$.

Lemma 5.1.1. *Let H be a hypergraph with a vertex partition $\{U_0, U_1, U_2\}$, and such that $H[U_0, U_1, U_2] = \varnothing$. If W is a tour in H then*

$$|W[U_1, U_1, U_2]| \equiv |W[U_1, U_2, U_2]| \pmod 3$$

Proof. Let $W = w_1 w_2 \cdots w_r$, in cyclic order, and let $P = \sigma_1 \cdots \sigma_r$ be a cyclic word over the symbols $\{0, 1, 2\}$, where $\sigma_i = j$ if and only if $w_i \in U_j$. Since W is a tour, it does not repeat edges. Thus we have that $|W[U_1, U_1, U_2]|$ is exactly the same as the number of appearances of the patterns $F_1 = \{112, 121, 211\}$ formed by three consecutive symbols in P. Similarly, $|W[U_1, U_2, U_2]|$ is exactly counted by the number of appearances of $F_2 = \{122, 212, 221\}$ consecutively in P. In both cases we count the cyclic appearances of the patterns, i.e. we also consider the patterns formed by $\sigma_{r-1}\sigma_r\sigma_1$ and $\sigma_r\sigma_1\sigma_2$.

Define $\Phi(P)$ as follows. Scan the triples of consecutive symbols of P one by one, and if they belong to $F_1 \cup F_2$, we add the sum of the values of their symbols to $\Phi(P)$. More formally, let $I \subseteq [r]$ be such that $i \in I$ if and only if $\sigma_i\sigma_{i+1}\sigma_{i+2} \in F_1 \cup F_2$ (where the indices are always understood modulo r, i.e. $\sigma_{r+1} = \sigma_1$ and $\sigma_{r+2} = \sigma_2$), and then

$$\Phi(P) = \sum_{i \in I} (\sigma_i + \sigma_{i+1} + \sigma_{i+2}).$$

We aim to show that $\Phi(P) \equiv 0 \bmod 3$. If $I = \varnothing$, this is obvious, and if $I = [r]$ then $\Phi(P)$ sums every symbol of P three times, and thus also $\Phi(P) \equiv 0$. Thus we may assume $I \notin \{\varnothing, [r]\}$. We write I as a disjoint union of intervals of consecutive indices, minimising the number of intervals. Thus, without loss of generality (after shifting W and P cyclically) we can assume $I = I_1 \cup \cdots \cup I_k$, so each I_j is of the form $\{a_j, a_j + 1, \ldots, b_j\}$ for some $a_j \leqslant b_j$ and further we have $a_1 = 1$, $b_j \leqslant a_{j+1} - 2$ for all $1 \leqslant j < k$, and $b_k \leqslant r - 1$. Setting $\Phi_j = \sum_{i \in I_j} (\sigma_i + \sigma_{i+1} + \sigma_{i+2})$ we have $\Phi(P) = \sum_{1 \leqslant j \leqslant k} \Phi_j$, so it is enough to show that $\Phi_j \equiv 0 \bmod 3$ for each j.

Fix an arbitrary $j \in \{1, \ldots, k\}$. For brevity write $a = a_j$ and $b = b_j$ and let $P_j = \sigma_a \sigma_{a+1} \cdots \sigma_{b+1} \sigma_{b+2}$. We claim that P_j begins with two repeated symbols. Since $I_k \subseteq I$, we have $\sigma_a \sigma_{a+1} \sigma_{a+2} \in F_1 \cup F_2$, thus in particular σ_a and σ_{a+1} must be in $\{1, 2\}$. If $\sigma_a \neq \sigma_{a+1}$, then we would have $\sigma_a \sigma_{a+1} = 12$ or $\sigma_a \sigma_{a+1} = 21$. In any case, it cannot happen that $\sigma_{a-1} \in \{1, 2\}$, since then that would imply that $a - 1 \in I$, contradicting the choice of I_k. Thus $\sigma_{a-1} = 0$, and therefore $\sigma_{a-1} \sigma_a \sigma_{a+1} = 012$ or $\sigma_{a-1} \sigma_a \sigma_{a+1} = 021$. But this implies that W contains an edge in $H[U_0, U_1, U_2]$, a contradiction. Thus P_j begins with two repeated symbols, and an analogous argument implies that P_j also ends with two repeated symbols.

If $a = b$, then we would have $\sigma_a \sigma_{a+1} \sigma_{a+2} = 111$ or $\sigma_a \sigma_{a+1} \sigma_{a+2} = 222$, then implying $a \notin I$, a contradiction. Thus $a < b$, and therefore P_j must have the form $P_j = xxQ_j yy$, where $x, y \in \{1, 2\}$ and Q_j is a (possibly empty) word. It is easy to see that every symbol in Q_j is counted three times in Φ. Thus we have

$$\Phi_j = \sum_{a \leqslant i \leqslant b} (\sigma_i + \sigma_{i+1} + \sigma_{i+2}) = x + 2x + 3 \left(\sum_{a+2 \leqslant i \leqslant b} \sigma_i \right) + 2y + y \equiv 0 \bmod 3,$$

and this implies $\Phi(P) \equiv 0 \bmod 3$, as discussed before.

Finally, note that, for $j \in \{1, 2\}$, if $\sigma_i \sigma_{i+1} \sigma_{i+2} \in F_j$, then $\sigma_i + \sigma_{i+1} + \sigma_{i+2} \equiv j \bmod 3$. Thus $\Phi(P) \equiv |W[U_1, U_1, U_2]| + 2|W[U_1, U_2, U_2]| \bmod 3$. But since $\Phi(P) \equiv 0 \bmod 3$ and $2 \equiv -1 \bmod 3$, we deduce $|W[U_1, U_1, U_2]| \equiv |W[U_1, U_2, U_2]| \bmod 3$, as desired. □

To prove Theorem 2.3.4, we will consider alterations of the following hypergraph.

Definition 5.1.2. Let n be divisible by 18 and write $n = 18k$. Consider the hypergraph H_n on n vertices, whose vertex set is partitioned into three clusters V_0, V_1, V_2 whose sizes are n_0, n_1, n_2 respectively, and are defined by

$$n_0 = 6k, \qquad n_1 = 6k - 2, \qquad \text{and} \qquad n_2 = 6k + 2. \tag{5.1.1}$$

Given a vertex $x \in V(H_n)$, the *label* $l(x)$ of x is i if and only if $x \in V_i$. The edge set of H_n is

$$E(H_n) = \{xyz : l(x) + l(y) + l(z) \not\equiv 0 \bmod 3\}.$$

In words, every 3-set is present as an edge in H_n, except for those which are entirely contained in one of the clusters V_i or have non-empty intersection with all three clusters. Usually n will always be clear from context, and for a cleaner notation we will just write $H = H_n$ in the remainder of this section.

We begin our analysis by noting the hypergraph H has large minimum codegree.

Lemma 5.1.3. *Let $n \in 18\mathbb{N}$. Then $\delta_2(H) \geqslant (2n - 12)/3$.*

Proof. Let $x, y \in V(H)$, and set $p = l(x) + l(y)$. By the definition of H, a vertex z will form an edge together with xy whenever $p + l(z) \not\equiv 0 \bmod 3$. This is equivalent to $l(z) \equiv 1 - p \bmod 3$ or $l(z) \equiv 2 - p \bmod 3$. Thus, if $i, j \in \{0, 1, 2\}$ are such that $i \equiv 1 - p \bmod 3$ and $j \equiv 2 - p \bmod 3$, then $N(xy) = (V_i \cup V_j) \smallsetminus \{x, y\}$. A quick case analysis reveals that $|N(xy)|$ is minimised whenever $x \in V_0$, $y \in V_1$ and in such a case $d_H(xy) = n_0 + n_1 - 2 = 12k - 4$. Thus $\delta_2(H) = 12k - 4 = (2n - 12)/3$, as required. $\square$

We note that identities (5.1.1) imply that n_0, n_1, and n_2 are even and that for all $i \in \{0, 1, 2\}$ we have

$$n_i \equiv i \pmod 3, \tag{5.1.2}$$

Given $(i, j, k) \in \{0, 1, 2\}^3$, write $H_{ijk} = H[V_i, V_j, V_k]$.

Lemma 5.1.4. *Let $n \in 18\mathbb{N}$. Then*

(i) for every $x \in V(H)$, $d_H(x) \equiv 1 \bmod 3$ and

(ii) $|H_{112}| \not\equiv |H_{122}| \bmod 3$.

Proof. We begin by noting that $\binom{m}{2} \equiv 2m(m-1)$ mod 3 holds for all integers m. Thus $\binom{m}{2} \equiv 1$ mod 3 if $m \equiv 2$ mod 3, and $\binom{m}{2} \equiv 0$ mod 3 otherwise.

Now let $x \in V_0$. Then the pairs yz such that $xyz \in H$ are those such that

1. $y \in V_0 \setminus \{x\}$ and $z \in V_1 \cup V_2$, of which there are $(n_0 - 1)(n_1 + n_2)$ many,
2. $yz \subseteq V_1$, of which there are $\binom{n_1}{2}$ many, and
3. $yz \subseteq V_2$, of which there are $\binom{n_2}{2}$ many.

Thus we have $d_H(x) = (n_0 - 1)(n_1 + n_2) + \binom{n_1}{2} + \binom{n_2}{2}$. Together with (5.1.2), we have that $d_H(x) \equiv 0 + 0 + 1 \equiv 1$ mod 3. Analogous calculations show that

$$d_H(y) \equiv 0 + 0 + 1 \equiv 1 \text{ mod } 3 \text{ for } y \in V_1 \text{ and}$$
$$d_H(z) \equiv 1 + 0 + 0 \equiv 1 \text{ mod } 3 \text{ for } z \in V_2,$$

thus *(i)* holds.

Finally, the sizes of $|H_{112}|$ and $|H_{122}|$ are $\binom{n_1}{2}n_2$ and $\binom{n_2}{2}n_1$ respectively, which then are easily seen to be equivalent to 0 and 1 modulo 3, respectively, which implies *(ii)*. □

Since H is not quite 3-vertex-divisible, our counterexample will consist actually of a slight alteration of H obtained by removing some sparse subgraph.

Lemma 5.1.5. *Let $n \in 18\mathbb{N}$. Then there exists a perfect matching $F \subseteq H \setminus (H_{112} \cup H_{122})$.*

Proof. Let k be such that $n = 18k$. Let a, b be two distinct vertices in V_2, and let $V_1' = V_1 \cup \{a, b\}$ and $V_2' = V_2 \setminus \{a, b\}$. Note that $|V_0| = |V_1'| = |V_2'| = 6k$. Let $V_0 = \{x_1, \ldots, x_{6k}\}$, $V_1' = \{y_1, \ldots, y_{6k}\}$ and $V_2' = \{z_1, \ldots, z_{6k}\}$, with $y_1 = a$ and $y_2 = b$. Then

$$F = \{y_{2i-1}y_{2i}x_{2i-1} : 1 \leqslant i \leqslant 3k\} \cup \{z_{2i-1}z_{2i}x_{2i} : 1 \leqslant i \leqslant 3k\}$$

is a perfect matching in which every edge intersects V_0 in exactly one vertex. Thus F has no edge in $H_{112} \cup H_{122}$, as required. □

We are now ready to show Theorem 2.3.4.

Proof of Theorem 2.3.4. Consider the hypergraph $H = H_n$ given in Definition 5.1.2, and consider the perfect matching $F \subseteq H \smallsetminus (H_{112} \cup H_{122})$ given by Lemma 5.1.5. Let $\ell' \in \{4, \ldots, \ell + 3\}$ be such that $|E(H - F)| + \ell' \equiv 0 \bmod \ell$. Since $n = 18k \geqslant 3(\ell + 3)$, we have $|V_0| = 6k \geqslant \ell + 3 \geqslant \ell'$. To $H - F$, we add a cycle C of length ℓ', edge-disjoint from $H - F$, which is entirely contained in V_0. We claim the hypergraph

$$H' = (H \smallsetminus F) \cup C$$

has all of the desired properties.

We first check H' is $C_\ell^{(3)}$-divisible. We start by checking H' is 3-vertex-divisible. Indeed, let $x \in V(H')$ be arbitrary. We have $d_H(x) \equiv 1 \bmod 3$ by Lemma 5.1.4*(i)*, we have $d_F(x) = 1$ since F is a perfect matching, and $d_C(x) \equiv 0 \bmod 3$ since C is a cycle on $\ell' \geqslant 4$ vertices. Thus $d_{H'}(x) \equiv 1 - 1 + 0 \equiv 0 \bmod 3$ for all $x \in V(H')$, as required. Moreover, the number of edges of H' is $|E(H')| = |E(H - F)| + \ell'$, which was chosen to be divisible by ℓ, so indeed H' is $C_\ell^{(3)}$-divisible.

Now we check H' has large codegree. It suffices to show $H - F$ has large codegree. Removing a perfect matching from H decreases the codegree of every pair at most by 1, thus by Lemma 5.1.3, we have $\delta_2(H - F) \geqslant \delta_2(H) - 1 \geqslant (2n - 12)/3 - 1 = (2n - 15)/3$.

Now we prove H' does not have a tour decomposition. Since $F \subseteq H \smallsetminus (H_{112} \cup H_{122})$ and $C \subseteq V_0$ we have

$$H'[V_1, V_1, V_2] = H_{112} \quad \text{and} \quad H'[V_1, V_2, V_2] = H_{122}\,.$$

For a contradiction, suppose that $W^1, \ldots, W^r$ are tours forming a tour decomposition in H'. For a walk W, let $W_{112} = H_{112} \cap E(W)$, and let $W_{122} = H_{122} \cap E(W)$. Since the tours are edge-disjoint and cover all edges of H', we have $\sum_{1 \leqslant i \leqslant r} |W_{112}^i| = |H_{112}|$ and $\sum_{1 \leqslant i \leqslant r} |W_{122}^i| = |H_{122}|$. Moreover, Lemma 5.1.1 implies that $|W_{112}^i| \equiv |W_{122}^i| \bmod 3$ for each $1 \leqslant i \leqslant r$. Therefore, $|H_{112}| \equiv |H_{122}| \bmod 3$, but this contradicts Lemma 5.1.4*(ii)*. $\square$

Remark 5.1.6. For sufficiently large values of n, we can make our example vertex-regular instead of $C_\ell^{(3)}$-divisible. This is needed, for instance, when we are looking at decompositions into spanning vertex-disjoint collections of cycles, such as Hamilton cycles.

Start from $H = H_n$, and remove F as before to get to $H' = H - F$ which is 3-vertex-divisible. Every vertex in V_i has the same degree d_i, for all $i \in \{0, 1, 2\}$, and a calculation reveals that $d_1 = d_0 - 9$ and $d_2 = d_0 - 3$. Then, adding three edge-disjoint Hamilton cycles to $H[V_1]$ and one Hamilton cycle to $H[V_2]$ leaves a hypergraph H^* in which every vertex has degree d_0. It can be similarly proved that H^* does not admit any tour decomposition.

5.2 Proof of Corollaries 2.3.2 and 2.3.3

In this short section we deduce Corollaries 2.3.2 and 2.3.3 from Theorem 2.3.1.

Proof of Corollary 2.3.2. Let m be the number of edges of H, and write it as $m = 9q + r$ for some $q \geqslant 1$ and $0 \leqslant r < 9$. Find a cycle C of length $9 + r$ in H: this can be done greedily (see Section 5.4.1 for details). Then, $H' = H - C$ is a 3-divisible graph, its minimum codegree is $\delta_2(H') \geqslant \delta_2(H) - 2 \geqslant (2/3 + \varepsilon/2)n$, and its number of edges is $m - (9 + r) = 9(q - 2)$, which is divisible by 9. By Theorem 2.3.1, H' has a C_9-decomposition, together with C this is a cycle decomposition of H. □

For the proof of Corollary 2.3.3 we use the strategy of Glock, Joos, Kühn, and Osthus [26]. Crucial part of their argument is (using our terminology) to first find a trail W which is *spanning* i.e. every 2-tuple of distinct vertices of H is contained as a sequence of consecutive vertices of W, but at the same time it is sparse (it satisfies $\Delta_2(W) = o(n)$, where $\Delta_2(H)$ denotes the maximum codegree of H among all pairs of vertices).

Here we state their lemma only for 3-uniform hypergraphs. A hypergraph H on n vertices is *α-connected* if for all distinct $v_1, v_2, v_4, v_5 \in V(H)$, there exist at least αn vertices $v_3 \in V(H)$ such that $v_1v_2v_3v_4v_5$ is a walk in H.

Lemma 5.2.1 ([26, Lemma 5]). *Suppose $n \in \mathbb{N}$ is sufficiently large in terms of α. Suppose H is an α-connected hypergraph on n vertices. Then H contains a spanning trail W satisfying $\Delta_2(W) \leqslant \log^3 n$.*

Proof of Corollary 2.3.3. Take n_0 such that $1/n_0 \ll \varepsilon$. Since $\delta_2(H) \geqslant (2/3+\varepsilon)n$, then H is ε-connected. By Lemma 5.2.1 there exists a spanning trail $W = w_1 \cdots w_r$ satisfying $\Delta_2(W) \leqslant \log^3 n$. Use the ε-connected property of H to close W to a tour, using three extra vertices, while avoiding edges previously used by W (using that $\Delta_2(W) \leqslant \log^3 n$). The resulting $W' = w_1 \cdots w_{r+3}$ is a spanning tour which satisfies $\Delta_2(W') \leqslant 2\log^3 n$. Let $H' = H - W'$. Since W' is a tour and H is 3-vertex-divisible, H' is 3-vertex-divisible as well. Since $\Delta_2(W') \leqslant 2\log^3 n \leqslant \varepsilon n/2$ and $\delta_2(H) \geqslant (2/3+\varepsilon)n$, we deduce $\delta_2(H') \geqslant (2/3+\varepsilon/2)n$. Since n is sufficiently large, Corollary 2.3.2 implies that H' has a cycle decomposition. Fix one of those cycles $C = v_1 v_2 \cdots v_m$ and note that the ordered pair (v_1, v_2) must appear consecutively in some part of W' (since W' is spanning). We may write $W' = W'_1 v_1 v_2 W'_2$ and extend W' by taking $W'_1 v_1 v_2 \cdots v_m v_1 v_2 W'_2$, which is still an spanning tour, but now uses the edges of C in addition to those of W'. Attaching the cycles of the decomposition one by one to W', we obtain the desired Euler tour. □

5.3 Iterative absorption: proof of Theorem 2.3.1

Our proof of Theorem 2.3.1 follows the strategy of *iterative absorption* introduced by Barber, Kühn, Lo, and Osthus [8] and further developed by Glock, Kühn, Lo, Montgomery, and Osthus [27] to study decomposition thresholds in graphs. We base our outline in the exposition of Barber, Glock, Kühn, Lo, Montgomery, and Osthus [7].

The method of iterative absorption is based on three main lemmata, originally called the the Vortex Lemma, Absorbing Lemma, and the Cover-Down Lemma. We will introduce these lemmata first while explaining the global strategy, then we will use them to prove Theorem 2.3.1. The proof of these lemmata will take up the rest of the chapter.

A sequence of nested subsets of vertices $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ is called a (δ, ξ, m)-*vortex in H* if satisfies the following properties.

(*V1*) $U_0 = V(H)$,

(*V2*) for each $1 \leqslant i \leqslant \ell$, $|U_i| = \lfloor \xi |U_{i-1}| \rfloor$,

(*V3*) $|U_\ell| = m$,

(*V4*) $d(x, U_i) = |N(x, U)| \geqslant \delta \binom{|U_i|}{2}$ for each $1 \leqslant i \leqslant \ell$ and $x \in U_{i-1}$, and

(*V5*) $d(xy, U_i) = |N(xy, U)| \geqslant \delta |U_i|$ for each $1 \leqslant i \leqslant \ell$ and $xy \in U_{i-1}^{(2)}$,

where $N(x, U)$ and $N(xy, U)$ correspond to the restricted neighbourhoods

$$\begin{aligned} N(x, U) &= \{yz \in U^{(2)} \colon xyz \in E(H)\}, \\ N(xy, U) &= \{z \in U \colon xyz \in E(H)\}. \end{aligned}$$

The existence of vortices for suitable parameters δ, ξ, and m is stated in the Vortex Lemma.

Lemma 5.3.1 (Vortex Lemma)**.** *Let $\xi, \delta > 0$ and $m' \in \mathbb{N}$ be such that $1/m' \ll \xi$. Let H be a hypergraph on $n \geqslant m'$ vertices with $\delta_2(H) \geqslant \delta$. Then H has a $(\delta - \xi, \xi, m)$-vortex, for some $\lfloor \xi m' \rfloor \leqslant m \leqslant m'$.*

The main idea is to use the properties of the vortex to find a suitable $C_\ell^{(3)}$*-packing*, i.e. a collection of edge-disjoint $C_\ell^{(3)} \subseteq H$. We will find a packing covering most edges of H, and moreover the non-covered edges will lie entirely in U_ℓ. The Absorbing Lemma will provide us with a small structure that we put aside at the beginning, and that will be used to deal with the small remainder left by our $C_\ell^{(3)}$-packing. If $R \subseteq H$ is a subgraph of H, a $C_\ell^{(3)}$*-absorber for R* is a subgraph $A \subseteq H$, edge-disjoint from R, such that both A and $A \cup R$ are $C_\ell^{(3)}$-decomposable.

Lemma 5.3.2 (Absorbing Lemma)**.** *Let $\ell \geqslant 7$, $\varepsilon > 0$, and $n, m \in \mathbb{N}$ such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a hypergraph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Let $R \subseteq H$ be $C_\ell^{(3)}$-divisible on at most m vertices. Then there exists a $C_\ell^{(3)}$-absorber for R in H with at most $(2m\ell)^9$ edges.*

Finally, we construct the desired $C_\ell^{(3)}$-packing step by step through the nested sets of the vortex. More precisely, suppose $U_i \supseteq U_{i+1}$ are two consecutive sets in a vortex of H. The Cover-Down Lemma allows us to find a $C_\ell^{(3)}$-packing which covers every edge of $H[U_i]$, except maybe for some in $H[U_{i+1}]$. Thus the desired packing will be found via iterative applications.

Lemma 5.3.3 (Cover-Down Lemma)**.** *Let $\ell \geqslant 9$ be divisible by 3 or at least 10^7, let $\varepsilon, \mu > 0$, and let $n \in \mathbb{N}$ with $1/n \ll \mu, \varepsilon \ll 1/\ell$. Suppose H is a hypergraph on n vertices, and $U \subseteq V(H)$ with $|U| = \lfloor \varepsilon n \rfloor$, which satisfy*

(*CD1*) $\delta_2(H) \geqslant (2/3 + 2\varepsilon)n$,

(*CD2*) $d_H(x, U) \geqslant (2/3 + \varepsilon)\binom{|U|}{2}$ *for each* $x \in V(H)$,

(*CD3*) $d_H(xy, U) \geqslant (2/3 + \varepsilon)|U|$ *for each* $xy \in V(H)^{(2)}$, *and*

(*CD4*) $d_H(x)$ *is divisible by* 3 *for each* $x \in V(H) \setminus U$.

Then H has a $C_\ell^{(3)}$-decomposable subhypergraph F such that $H - H[U] \subseteq F$, and $\Delta_2(F[U]) \leqslant \mu n$.

Assuming lemmata 5.3.2–5.3.3, we prove Theorem 2.3.1.

Proof of Theorem 2.3.1. It is enough to show that for every $\varepsilon > 0$, there exists n_0 such that every $C_\ell^{(3)}$-divisible hypergraph H on $n \geqslant n_0$ vertices with $\delta_2(H) \geqslant (2/3 + 8\varepsilon)n$ admits a $C_\ell^{(3)}$-decomposition. Given ε and ℓ, we fix m', n_0 such that

$$1/n_0 \ll 1/m' \ll \varepsilon, 1/\ell. \tag{5.3.1}$$

Let H on $n \geqslant n_0$ vertices as before, we are done if we show H has a $C_\ell^{(3)}$-decomposition.

Step 1: Setting the vortex and the absorbers. By Lemma 5.3.1, H has a $(2/3 + 7\varepsilon, \varepsilon, m)$-vortex $U_0 \supseteq \cdots \supseteq U_\ell$, for some m such that $\lfloor \varepsilon m' \rfloor \leqslant m \leqslant m'$.

Let $\mathscr{L}$ be the family of all $C_\ell^{(3)}$-divisible hypergraphs which are subgraphs of $H[U_\ell]$. Since $|U_\ell| = m$, clearly $|\mathscr{L}| \leqslant 2^{\binom{m}{3}}$. Pick an arbitrary hypergraph $L \in \mathscr{L}$. Since $m \leqslant m'$ and (5.3.1), a suitable application of Lemma 5.3.2 yields a $C_\ell^{(3)}$-absorber $A_L \subseteq H \setminus H[U_1]$

of L with at most $(4m\ell)^9$ edges. Since $1/n \ll 1/m, \varepsilon, 1/\ell$, removing the edges of A_L only barely affects the codegree of H, thus we can repeat the argument to obtain an absorber $A_{L'} \subseteq H \setminus H[U_1]$ for some $L' \neq L$, edge-disjoint from A_L. Since the total number of $L \in \mathscr{L}$ is tiny with respect to n, we can iterate this argument to obtain edge-disjoint $C_\ell^{(3)}$-absorbers $A_L \subseteq H \setminus H[U_1]$ for each $L \in \mathscr{L}$. Moreover, each A_L contains at most $(4m\ell)^9$ edges, and hence, the union $A = \bigcup_{L\in\mathscr{L}} A_L \subseteq H \setminus H[U_1]$ contains at most $|\mathscr{L}|(4m\ell)^9 \leqslant 2^{\binom{m}{3}}(4m\ell)^9 \leqslant \varepsilon n$ edges. By construction, we have A is $C_\ell^{(3)}$-decomposable and for every $L \in \mathscr{L}$, $L \cup A$ is $C_\ell^{(3)}$-decomposable.

Let $H' = H \setminus A$ and observe that $\delta_2(H') \geqslant (2/3 + 7\varepsilon)n$ and $U_0 \supseteq \cdots \supseteq U_\ell$ is a $(2/3 + 6\varepsilon, \varepsilon, m)$-vortex for H' (for this, it is crucial that $A \subseteq H \setminus H[U_1]$). Notice that since A and H are $C_\ell^{(3)}$-divisible, we get that H' is $C_\ell^{(3)}$-divisible.

Step 2: The cover-down. Now we aim to find a $C_\ell^{(3)}$-packing in H' using every edge of $H' \setminus H'[U_\ell]$. Let $U_{\ell+1} = \varnothing$. For each $0 \leqslant i \leqslant \ell$ we shall find $H_i \subseteq H'[U_i]$ such that

(a_i) $H' - H_i$ has a $C_\ell^{(3)}$-decomposition,

(b_i) $\delta_2(H_i) \geqslant (2/3 + 4\varepsilon)|U_i|$,

(c_i) $d_{H_i}(x, U_{i+1}) \geqslant (2/3 + 5\varepsilon)\binom{|U_{i+1}|}{2}$ for all $x \in U_i$,

(d_i) $d_{H_i}(xy, U_{i+1}) \geqslant (2/3 + 5\varepsilon)|U_{i+1}|$ for all $x, y \in U_i$, and

(e_i) $H_i[U_{i+1}] = H'[U_{i+1}]$.

For $i = 0$ this can be done by setting $H_0 = H'$. Now suppose H_i satisfying (a_i)–(e_i) is given for some $0 \leqslant i < \ell$, we wish to construct H_{i+1} satisfying (a_{i+1})–(e_{i+1}). Let $H_i' = H_i \setminus H_i[U_{i+2}]$. By ($b_i$)–($d_i$) and $|U_{i+2}| \leqslant \varepsilon|U_{i+1}| \leqslant \varepsilon^2|U_i|$, we have

(*CD1*) $\delta_2(H_i') \geqslant \delta_2(H_i) - |U_{i+2}| \geqslant (2/3 + 3\varepsilon)|U_i|$,

(*CD2*) $d_{H_i'}(x, U_{i+1}) \geqslant d_{H_i}(x, U_{i+1}) - |U_{i+2}|(|U_{i+1}| - 1) \geqslant (2/3 + 3\varepsilon)\binom{|U_{i+1}|}{2}$, for each $x \in U_i$,

(*CD3*) $d_{H_i'}(xy, U_{i+1}) \geqslant d_{H_i'}(xy, U_{i+1}) - |U_{i+2}| \geqslant (2/3 + 4\varepsilon)|U_{i+1}|$ for each $x, y \in U_i$, and

(*CD4*) $d_{H_i'}(x)$ is divisible by 3 for each $x \in U_i \setminus U_{i+1}$.

This allows us to apply Lemma 5.3.3 with $\varepsilon, \varepsilon^4, |U_i|, H_i', U_{i+1}$ playing the rôles of $\varepsilon, \mu, n, H, U$. We get a $C_\ell^{(3)}$-decomposable subgraph $F_i \subseteq H_i'$ such that $H_i' \smallsetminus H_i'[U_{i+1}] \subseteq F_i$ and that $\Delta_2(F_i[U_{i+1}]) \leqslant \varepsilon^4|U_i|$. Let $H_{i+1} = H_i[U_{i+1}] \smallsetminus F_i$, we prove it satisfies the required properties.

Clearly F_i is $C_\ell^{(3)}$-divisible and $F_i \subseteq H_i' \subseteq H_i$. Therefore ($a_i$) implies that the hypergraph $H' - H_{i+1} = (H' - H_i) \cup F_i$ has a $C_\ell^{(3)}$-decomposition, and thus (a$_{i+1}$) holds. Moreover, from (d_i) and since $\Delta_2(F_i[U_{i+1}]) \leqslant \varepsilon^4|U_i| \leqslant \varepsilon^2|U_{i+1}|$, we prove (b$_{i+1}$) by noticing that $\delta_2(H_{i+1}) \geqslant (2/3 + 5\varepsilon)|U_{i+1}| - \varepsilon^2|U_{i+1}| \geqslant (2/3 + 4\varepsilon)|U_{i+1}|$,

By the properties of $(2/3 + 6\varepsilon, \varepsilon, m)$-vortices, we have $d_{H'}(x, U_{i+2}) \geqslant (2/3 + 6\varepsilon)\binom{|U_i|}{2}$ for each $x \in U_{i+1}$, together with $\Delta_2(F_i[U_{i+1}]) \leqslant \varepsilon^2|U_{i+1}|$ and (e_i) we deduce (c$_{i+1}$) holds, and (d$_{i+1}$) can be verified similarly. Finally, since $F_i \subseteq H_i' = H_i \smallsetminus H_i[U_{i+1}]$, we have $F_i[U_{i+2}]$ is empty and therefore $H_{i+1}[U_{i+2}] = H_i[U_{i+2}] = H'[U_{i+2}]$, which verifies (e$_{i+1}$).

Now $H_\ell \subseteq H'[U_\ell]$ is such that $H' \smallsetminus H_\ell$ has a $C_\ell^{(3)}$-decomposition.

Step 3: Finish. Since both H' and $H' \smallsetminus H_\ell$ are $C_\ell^{(3)}$-divisible, we deduce $H_\ell \subseteq H'[U_\ell]$ is $C_\ell^{(3)}$-divisible. Therefore, $H_\ell \in \mathscr{L}$ and by construction of A we know that $H_\ell \cup A$ is $C_\ell^{(3)}$-decomposable. Since H is the edge-disjoint union of $H' \smallsetminus H_\ell$ and $H_\ell \cup A$, and both of them have $C_\ell^{(3)}$-decompositions, we deduce H has a $C_\ell^{(3)}$-decomposition, as desired. □

5.4 Useful tools

We collect various results to be used during the proof of Lemmatas 5.3.2–5.3.3.

5.4.1 Counting path extensions

The following lemma find short trails between prescribed pairs of vertices. For a hypergraph H, a set of vertices $U \subseteq V(H)$, and a set of pairs $G \subseteq V(H)^{(2)}$ let $\delta_2^{(3)}(H; U, G)$ be the minimum of $|N(e_1) \cap N(e_2) \cap N(e_3) \cap U|$ over all possible choices of $e_1, e_2, e_3 \in G$. This is the size of the minimum joint neighbourhood in U of three distinct pairs in G. Also, let $\delta_2^{(3)}(H; U) = \delta_2^{(3)}(H, U, V(H)^{(2)})$ and $\delta_2^{(3)}(H) = \delta_2^{(3)}(H; V(H))$.

Lemma 5.4.1. *Let $\varepsilon > 0$ and $n, \ell \in \mathbb{N}$ be such that $\ell \geqslant 5$ and $1/n \ll \varepsilon, 1/\ell$. Let H be a hypergraph on n vertices, $U \subseteq V(H)$, and $G \subseteq V(H)^{(2)}$ such that the set of pairs $\{uv \in V(H)^{(2)} : u \in U\} \subseteq G$. Suppose $\delta_2^{(3)}(H; U, G) \geqslant 2\varepsilon n$. Then, for every two disjoint pairs v_1v_2 and $v_{\ell-1}v_\ell$ in G there exist at least $(\varepsilon n)^{\ell-4}$ many (v_1, v_2)-$(v_{\ell-1}, v_\ell)$-paths on ℓ vertices, whose internal vertices are in U.*

Proof. Every pair of vertices in G has at least $2\varepsilon n$ neighbours in U. For each $1 \leqslant i \leqslant \ell-3$, since $\{uv \in V(H)^{(2)} : u \in U\} \subseteq G$ we can build a path $v_1v_2 \cdots v_i$ such that $\{v_{i-1}, v_i\} \in G$ by choosing vertices in U greedily. Due to $\delta_2^{(3)}(H; U, G) \geqslant 2\varepsilon n$ we are able to finish the path by choosing $v_{\ell-2}$ as a common neighbour in U of the pairs $v_{\ell-4}v_{\ell-3}$, $v_{\ell-3}v_{\ell-1}$ and $v_{\ell-1}v_\ell$, all of which belong to G. At any step we only need to avoid choosing one of the vertices already chosen so far, which are at most $\ell \leqslant \varepsilon n$. Thus in each step there are at least εn possible choices, which gives the desired bound. $\square$

In the particular for a hypergraph H with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$ a simple application of Lemma 5.4.1 with $U = V(H)$ and $G = V(H)^{(2)}$ implies the existence of many trails of length $\ell \geqslant 5$ between arbitrary pairs of vertices.

Sometimes we want find many paths which also avoid a small prescribed set of vertices or edges, for instance to extend paths into cycles. This is accomplished as follows.

Lemma 5.4.2. *Let $\varepsilon, \mu > 0$ and $n, \ell \in \mathbb{N}$ be such that $\ell \geqslant 5$ and $1/n \ll \mu \ll \varepsilon, 1/\ell$. Suppose that $v_1, v_2, v_{\ell-1}, v_\ell \in V(H)$ and there are at least $2\varepsilon n^{\ell-4}$ many (v_1, v_2)-$(v_{\ell-1}, v_\ell)$-paths on ℓ vertices in H. Let $F \subseteq H$ with $\Delta_2(F) \leqslant \mu n$. Then there are at least $\varepsilon n^{\ell-4}$ many (v_1, v_2)-$(v_{\ell-1}, v_\ell)$-paths on ℓ vertices in $H \smallsetminus F$.*

Proof. The number of (v_1, v_2)-$(v_{\ell-1}, v_\ell)$-paths on ℓ vertices such that $v_1v_2v_3 \in F$ is at most $d_F(v_1v_2)n^{\ell-5} \leqslant \Delta_2(F)n^{\ell-5} \leqslant \mu n^{\ell-4}$. Similar bound are obtained for the paths of the same form such that $v_{\ell-2}v_{\ell-1}v_\ell \in F$, $v_3v_4v_5 \in F$, or $v_{\ell-3}v_{\ell-2}v_{\ell-1} \in F$. Finally, the paths such that $v_jv_{j+1}v_{j+2} \in F$ for some $3 \leqslant j \leqslant \ell - 4$ is at most $|E(F)|n^{\ell-7} \leqslant \mu n^{\ell-4}$. All together, the number of paths destroyed by passing from H to $H \smallsetminus F$ is at most $(\ell - 2)\mu n^{\ell-4} \leqslant \varepsilon n^{\ell-4}$, where the last inequality uses $\mu \ll \varepsilon$. $\square$

The following is an immediate corollary of Lemma 5.4.1 and Lemma 5.4.2.

Corollary 5.4.3. *Let $\varepsilon > 0$ and $n, \ell, \ell' \in \mathbb{N}$ be such that $1/n \ll \mu \ll \varepsilon \ll \varepsilon', 1/\ell, 1/\ell'$ and $\ell \geqslant \ell' + 1$. Let H be a hypergraph on n vertices, $U \subseteq V(H)$ and $G \subseteq V(H)^{(2)}$ such that $\{uv \in V(H)^{(2)} : u \in U\} \subseteq G$. Suppose $\delta_2^{(3)}(H; U, G) \geqslant 2\varepsilon' n$. Let P be a path on ℓ' vertices in H, whose two endpoints are in G. Then there are at least $\varepsilon n^{\ell-\ell'}$ many cycles C on ℓ vertices which contain P and such that $V(C) \smallsetminus V(P) \subseteq U$.*

Note that for a hypergraph H with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$ and a set $W \subseteq V(H)$ with $|W| < \varepsilon n/2$, a simple application of Corollary 5.4.3 with $U = V(H) \smallsetminus W$ and with $G = V(H)^{(2)}$ yields the existence of many cycles containing one fix path P and avoiding the set of vertices W.

5.4.2 Probabilistic tools

We shall use the following concentration inequalities [33, Corollary 2.3, Corollary 2.4, Remark 2.5, Theorem 2.10].

Theorem 5.4.4. *Let X be a random variable which is a sum of n independent $\{0, 1\}$-random variables, or hypergeometric with parameters n, N, M.*

(*i*) *If $x \geqslant 7\mathbb{E}[X]$, then $\mathbb{P}[X \geqslant x] \leqslant \exp(-x)$,*

(*ii*) $\mathbb{P}[|X - \mathbb{E}[X]| \geqslant t] \leqslant 2\exp(-2t^2/n)$, *and*

(*iii*) $\mathbb{P}[|X - \mathbb{E}[X]| \geqslant t] \leqslant 2\exp(-t^2/(3\mathbb{E}[X]))$.

The following lemma allows us to bound the tail probabilities of sums of sequentially-dependent $\{0, 1\}$-random variables by comparing them with binomial random variables. We use the probability-theoretic notion of conditioning in a sequence of random variables, which in our application will take the following form. If $X_1, \ldots, X_i$ are random variables, we denote by $\mathbb{P}[X_i = 1 | X_1, \ldots, X_{i-1}] \leqslant p_i$ the fact that the probability of $X_i = 1$ is always at most p_i, even after conditioning on any possible output of $X_1, \ldots, X_{i-1}$.

Theorem 5.4.5. *Let $X_1, \dots, X_t$ be Bernoulli random variables (not necessarily independent) such that for each $1 \leqslant i \leqslant t$ we have $\mathbb{P}[X_i = 1 | X_1, \dots, X_{i-1}] \leqslant p_i$. Let $Y_1, \dots, Y_t$ be independent Bernoulli random variables such that $\mathbb{P}[Y_i = 1] = p_i$ for all $1 \leqslant i \leqslant t$. If $X = \sum_{i=1}^{t} X_i$ and $Y = \sum_{i=1}^{t} Y_i$, then $\mathbb{P}[X \geqslant k] \leqslant \mathbb{P}[Y \geqslant k]$ for all $k \in \{0, 1, \dots, t\}$.*

The proof of Theorem 5.4.5 was given by Jain [47, Lemma 7] in the particular case where $p_i = p$ for all $1 \leqslant i \leqslant t$. The slightly more general statement of Theorem 5.4.5 follows by mimicking that proof (which goes by induction on t), so we omit it.

5.5 Vortex Lemma

We prove Lemma 5.3.1 by selecting random subsets (cf. [7, Lemma 3.7]).

Proof of Lemma 5.3.1. Let $n_0 = n$ and $n_i = \lfloor \xi n_{i-1} \rfloor$ for all $i \geqslant 1$. In particular, note $n_i \leqslant \xi^i n$. Let ℓ be the largest i such that $n_i \geqslant m'$ and let $m = n_{\ell+1}$. Note that $\lfloor \xi m' \rfloor \leqslant m \leqslant m'$.

Let $\xi_0 = 0$ and, for all $i \geqslant 1$, define $\xi_i = \xi_{i-1} + 2(\xi^i n)^{-1/3}$. Thus we have

$$\xi_{\ell+1} = 2n^{-1/3} \sum_{i=1}^{\ell} (\xi^{-1/3})^i \leqslant 2n^{-1/3} \sum_{i=1}^{\infty} (\xi^{-1/3})^i \leqslant \frac{2(n\xi)^{-1/3}}{1 - \xi^{-1/3}} \leqslant \xi,$$

where in the last inequality we used $1/m' \ll \xi$ and $n \geqslant m'$.

Note that taking $U_0 = V(H)$ yields a $(\delta - \xi_0, \xi, n_0)$-vortex in H. Suppose we have already found a $(\delta - \xi_{i-1}, \xi, n_{i-1})$-vortex $U_0 \supseteq \cdots \supseteq U_{i-1}$ in H for some $i \leqslant \ell + 1$. In particular, $\delta_2(H[U_{i-1}]) \geqslant (\delta - \xi_{i-1})|U_{i-1}|$. Let $U_i \subseteq U_{i-1}$ be a random subset of size n_i. By Theorem 5.4.4, with positive probability we have

$$d(xy, U_i) \geqslant (\delta - \xi_{i-1} - n_i^{-1/3})|U_i| \quad \text{and} \quad d(x, U_i) \geqslant (\delta - \xi_{i-1} - n_i^{-1/3})\binom{|U_i|}{2},$$

for every $x, y \in U_{i-1}$. Since $\xi_{i-1} + n_i^{-1/3} \leqslant \xi_i$, we have found a $(\delta - \xi_i, \xi, n_i)$-vortex for H. In the end, we will have found a $(\delta - \xi_{\ell+1}, \xi, n_{\ell+1})$-vortex for H. Since we have $m = n_{\ell+1}$ and we have established $\xi_{\ell+1} \leqslant \xi$, we are done. $\square$

5.6 Cover-Down Lemma

5.6.1 Extending paths into cycles

More than once during our proof, we will be faced with the following situation: we have a family of (not too many) edge-disjoint paths and we want to extend each of them into a cycle of a given length in such a way that all obtained cycles are edge-disjoint. In this subsection we will prove a lemma which will find such extensions for us.

Given a path P we say that a path or a cycle C is an *extension of* P if $P \subseteq C$. Let H be a hypergraph, for a path $P \subseteq H$ and a pair of vertices $e \in V(H)^{(2)}$ we say that P is of *type* r for e, where $r = \max\{e \cap b(P), e \cap t(P)\}$ (see definition of $b(P)$ and $t(P)$ at the beginning of this chapter). The only possible types are 0, 1, or 2.

We say that a collection of edge-disjoint paths $\mathcal{P}$ in H is γ*-sparse* if, for each $e \in V(H)^{(2)}$ and each $r \in \{0,1,2\}$, $\mathcal{P}$ has at most γn^{3-r} paths P of type r for e.

Lemma 5.6.1 (Extending Lemma). *Let $\varepsilon, \mu, \gamma > 0$ and $n, \ell, \ell' \in \mathbb{N}$ such that $\ell' \geqslant 4$, $\ell \geqslant \ell' + 2$ and $1/n \ll \gamma \ll \mu \ll \varepsilon, 1/\ell$. Let H_1, H_2 be two edge-disjoint hypergraphs on the same vertex set V of size n. Let $\mathcal{P} = \{P_1, \ldots, P_t\}$ be an edge-disjoint collection of paths on ℓ' vertices in H_1 such that*

(a) *$\mathcal{P}$ is γ-sparse and*

(b) *for each $P_i \in \mathcal{P}$, there exists at least $2\varepsilon n^{\ell-\ell'}$ copies of $C_\ell^{(3)}$ in $H_1 \cup H_2$ which extend P_i using extra edges of H_2 only.*

Then, there exists a $C_\ell^{(3)}$-decomposable subgraph $F \subseteq H_1 \cup H_2$, such that

$(i)_F$ *$E(\mathcal{P}) \subseteq F$ and*

$(ii)_F$ *$\Delta_2(F \smallsetminus E(\mathcal{P})) \leqslant \mu n$.*

Proof. The idea is to pick, sequentially, an extension C_i of P_i into an ℓ-cycle, chosen uniformly at random among all the extensions which do not use edges already used by $C_1, \ldots, C_{i-1}$. Since $\mathcal{P}$ is γ-sparse and there are plenty of choices for C_i in each step, we expect that in each step the random choices do not affect the codegree of the graph formed by the unused edges in H_2 by much. This will ensure that, even after removing the edges used by $C_1, \dots, C_{i-1}$, there are still many extensions available for P_i. If all goes well, then we can continue the process until the end, thus achieving $(i)_F$ and $(ii)_F$ by setting $F = \bigcup_{1 \leqslant i \leqslant t} E(C_i)$.

To formalise the above plan, we begin by noting that the removal of a sufficiently sparse hypergraph from H_2, there are still many extensions available for each P_i. Given $G \subseteq H_2$ and $1 \leqslant i \leqslant t$, let $\mathscr{C}_i(G)$ be the set of *G-avoiding cycle-extensions of* P_i, that is, the copies of $C_\ell^{(3)}$ in $H_1 \cup H_2$ which extend P_i and use extra edges from $H_2 \smallsetminus G$ only. By assumption, $|\mathscr{C}_i(\varnothing)| \geqslant 2\varepsilon n^{\ell-\ell'}$, thus Lemma 5.4.2 implies that

$$\text{if } G \subseteq H_2 \text{ is such that } \Delta_2(G) \leqslant \mu n, \text{ then } |\mathscr{C}_i(G)| \geqslant \varepsilon n^{\ell-\ell'}. \tag{5.6.1}$$

We now describe the random process which outputs edge-disjoint extensions C_i of P_i for each $1 \leqslant i \leqslant t$. In the case of success each C_i will be an ℓ-cycle extending P_i. To account for the case of failure, in our description we will allow the degenerate case in which $C_i \smallsetminus P_i$ is empty.

For each $1 \leqslant i \leqslant t$, assume we have already chosen $C_1, C_2, \dots, C_{i-1} \subseteq H_1 \cup H_2$ edge-disjoint graphs, and we describe the choice of C_i. Let $G_{i-1} = \bigcup_{1 \leqslant j < i} E(C_j) \smallsetminus E(P_j)$ correspond to the edges of H_2 used by the previous choices of C_j, which we need to avoid when choosing C_i (note that G_0 is empty). If $\Delta_2(G_{i-1}) \leqslant \mu n$, then by (5.6.1) we have $|\mathscr{C}_i(G_{i-1})| \geqslant \varepsilon n^{\ell-\ell'}$ and we take $C_i \in \mathscr{C}_i(G_{i-1})$ uniformly at random. Otherwise, if $\Delta_2(G_{i-1}) > \mu n$, let $C_i = P_i$.

In any case, the process outputs a collection $C_1, \dots, C_t$ of edge-disjoint cycles or paths which extend P_i. Our task now is to show that with positive probability, there is a choice of $C_1, \dots, C_t$ such that $\Delta_2(G_t) \leqslant \mu n$. This would imply also that each C_i was an ℓ-cycle. Formally, for each $1 \leqslant i \leqslant t$, let $\mathcal{S}_i$ be the event that $\Delta_2(G_i) \leqslant \mu n$. Thus it is enough to show $\mathbb{P}[\mathcal{S}_t] > 0$.

Fix $e \in V^{(2)}$. For each $1 \leqslant i \leqslant t$, let $X_i(e)$ be the random variable which takes the value 1 precisely if e belongs to an edge of $C_i \smallsetminus P_i$, and 0 otherwise. Equivalently, $X_i(e) = 1$ if and only if e belong to the shadow $\partial(C_i \smallsetminus P_i)$. Since $\Delta_2(C_i) \leqslant 2$ for each

$1 \leqslant i \leqslant t$, we have

$$d_{G_i}(e) \leqslant 2 \sum_{j=1}^{i} X_j(e). \tag{5.6.2}$$

For each $1 \leqslant i \leqslant t$, define

$$p_i^*(e) := \min\left\{1, \frac{c}{n^{2-r}}\right\},$$

where $r \in \{0, 1, 2\}$ is such that P_i is of type r for e, and $c := 4\ell\varepsilon^{-1}$.

Claim 3. *For each $e \in V^{(2)}$ and $1 \leqslant i \leqslant t$,*

$$\mathbb{P}[X_i(e) = 1 | X_1(e), X_2(e), \ldots, X_{i-1}(e)] \leqslant p_i^*(e).$$

Proof of the claim. Using conditional probabilities, we separate our analysis depending on whether $\mathcal{S}_{i-1}$ holds or not. Assume first that $\mathcal{S}_{i-1}$ fails. Then the process declares $C_i = P_i$, thus $C_i \smallsetminus P_i$ is empty. Therefore $X_i(e) = 0$ regardless of the values of $X_1(e), \ldots, X_{i-1}(e)$, and we have

$$\mathbb{P}[X_i(e) = 1 | X_1(e), X_2(e), \ldots, X_{i-1}(e), \mathcal{S}_{i-1}^c] = 0 \leqslant p_i^*(e).$$

Now assume that $\mathcal{S}_{i-1}$ holds. Then the set G_{i-1} of edges to be avoided while constructing C_i satisfies $\Delta_2(G_{i-1}) \leqslant \mu n$. By (5.6.1), C_i will be an ℓ-cycle extending P_i selected uniformly at random from the set $\mathscr{C}_i(G_{i-1})$, which has size at least $\varepsilon n^{\ell-\ell'}$; and this will happen no matter the values of $X_1(e), \ldots, X_{i-1}(e)$.

If P_i is of type 2 for e, then we are required to bound a probability by $p_i^*(e) = 1$, which holds trivially. Suppose now that P_i is of type 0 for e, and suppose $P_i = v_1 v_2 \cdots v_{\ell'}$. For $C_i \in \mathscr{C}_i(G_{i-1})$, $C_i \smallsetminus P_i$ is a path of the form $v_{\ell'-1} v_{\ell'} u_1 u_2 \cdots u_{\ell-\ell'} v_1 v_2$. We wish to estimate the number of such paths where $e \in \partial(C_i \smallsetminus P_i)$. Since P_i is of type 0 for e, then $e \in \partial(C_i \smallsetminus P_i)$ can only happen if $e = u_j u_k$ for $|j - k| \leqslant 2$. There are $(\ell - \ell' - 1) - (\ell - \ell' - 2) \leqslant 2\ell$ choices for j, k. Having fixed those, there are two 2 possibilities for assigning e to $\{u_j, u_k\}$, and having fixed those, there are at most n possibilities for each other u_p with $p \notin \{j, k\}$. All together, the number of C_i which extend P_i and such that $e \in \partial(C_i \smallsetminus P_i)$ is certainly at most $4\ell n^{\ell-\ell'-2}$. Thus we have

$$\mathbb{P}[X_i(e) = 1 | X_1(e), X_2(e), \ldots, X_{i-1}(e), \mathcal{S}_{i-1}] \leqslant \frac{4\ell n^{\ell'-\ell-2}}{|\mathscr{C}_i(G_{i-1})|} \leqslant \frac{4\ell}{\varepsilon n^2} = \frac{c}{n^2} = p_i^*(e),$$

as required. Finally, if P_i is of type 1 for e, then similar (but simpler) calculations show that $\mathbb{P}[X_i(e) = 1 | X_1(e), X_2(e), \ldots, X_{i-1}(e), \mathcal{S}_{i-1}] \leqslant \frac{6n^{\ell'-\ell-1}}{|\mathscr{C}_i(G_{i-1})|} \leqslant \frac{c}{n} = p_i^*(e)$, and we are done. ■

Now, we use that $\mathcal{P}$ is γ-sparse to argue $\sum_{i=1}^t p_i^*(e)$ is suitably small. Indeed, for each $r \in \{0,1,2\}$, let t_r be the number of $i \in \{1,\ldots,t\}$ such that P_i is of type r for e. Since $\mathcal{P}$ is γ-sparse, we have $t_r \leqslant \gamma n^{3-r}$ for each $r \in \{0,1,2\}$. Therefore, we have

$$\sum_{i=1}^{t} p_i^*(e) = t_0 \frac{c}{n^2} + t_1 \frac{c}{n} + t_2 \leqslant \gamma c n + \gamma c n + \gamma n \leqslant \frac{\mu}{30} n\,. \tag{5.6.3}$$

where the last inequality follows from the choice of c and $\gamma \ll \mu, \varepsilon$.

We now claim that

$$\mathbb{P}\left[\sum_{i=1}^{t} X_i(e) \geqslant \frac{\mu}{3} n\right] \leqslant \exp\left(-\frac{\mu}{3} n\right). \tag{5.6.4}$$

Indeed, inequality (5.6.3) implies that $7\sum_{i=1}^t p_i^*(e) \leqslant \mu n/3$, so the bound follows from Theorem 5.4.5 combined with Theorem 5.4.4.

For each $e \in V(H)^{(2)}$, let $X_e := \sum_{i=1}^t X_i(e)$. Let $\mathcal{E}$ be the event that $\max_e X_e \leqslant \mu n/3$. By using an union bound over all the (at most n^2) possible choices of e and using (5.6.4), we deduce that $\mathcal{E}$ holds with probability at least $1 - o(1)$.

Now we can show that $\mathcal{S}_t$ holds with positive probability. We shall prove that $\mathbb{P}[\mathcal{S}_t | \mathcal{E}] = 1$, which then will imply $\mathbb{P}[\mathcal{S}_t] \geqslant \mathbb{P}[\mathcal{S}_t|\mathcal{E}]\mathbb{P}[\mathcal{E}] \geqslant 1 - o(1)$. So assume $\mathcal{E}$ holds, that is, $\max_e X_e \leqslant \mu n/3$. Note that $\mathcal{S}_0$ holds deterministically, and suppose $1 \leqslant i \leqslant t$ is the minimum such that $\mathcal{S}_i$ fails to hold. Since $\mathcal{S}_{i-1}$ holds, using (5.6.2) we deduce

$$\begin{aligned}
\Delta_2(G_i) &\leqslant 2 + \Delta_2(G_{i-1}) = 2 + \max_e d_{G_{i-1}}(e) \leqslant 2\left(1 + \max_e \sum_{j=1}^{i-1} X_i(e)\right) \\
&\leqslant 2\left(1 + \max_e X_e\right) \leqslant 2\left(1 + \frac{\mu}{3} n\right) \leqslant \mu n,
\end{aligned}$$

where in the second last inequality we used that $\mathcal{E}$ holds, and in the last inequality we used $1/n \ll \mu$. Thus $\mathcal{S}_i$ holds, a contradiction. □

5.6.2 Well-behaved approximate cycle decompositions

In this section we show the existence of approximate cycle decomposition which are ‘well-behaved’, meaning that the subgraph left by the uncovered edges has small codegree. The argument is different depending on the two settings considered by Theorem 2.3.1, and we start with the former.

When ℓ is divisible by 3, the cycle $C_\ell^{(3)}$ is 3-partite. By a well-known theorem from Erdős [18, Theorem 1], we know that the Turán number of $C_\ell^{(3)}$ is degenerate, i.e. edge-maximal $C_\ell^{(3)}$-free hypergraphs on n vertices have at most $o(n^3)$ edges. This allows us to find an approximate decomposition of any hypergraph H with copies of $C_\ell^{(3)}$ if ℓ is divisible by 3, simply by removing copies of $C_\ell^{(3)}$ greedily until $o(n^3)$ edges remain. This argument alone does not provide us with the ‘well-behavedness’ condition we alluded to earlier, but it is, however, possible to modify such a packing locally to guarantee such a property holds.

Lemma 5.6.2 (Well-behaved approximate cycle decompositions, version 1)**.** *Let $\varepsilon, \gamma > 0$ and $n, \ell \in \mathbb{N}$ be such that $\ell \geqslant 9$ is divisible by 3 and $1/n \ll \varepsilon, \gamma, 1/\ell$. Let H be a hypergraph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Then H has a $C_\ell^{(3)}$-packing $\mathcal{C}$ such that $\Delta_2(H \smallsetminus E(\mathcal{C})) \leqslant \gamma n$.*

The proof of Lemma 5.6.2 is not difficult and follows the same lines as similar results included in [8]. However it is somewhat long, thus we defer it to the end of this subsection. Before we consider the second range of ℓ, where $\ell \geqslant 10^7$.

Lemma 5.6.3 (Well-behaved approximate cycle decomposition, version 2)**.** *Let $\varepsilon, \gamma > 0$ and $n, \ell \in \mathbb{N}$ be such that $\ell \geqslant 10^7$ and $1/n \ll \varepsilon, \gamma, 1/\ell$. Let H be a hypergraph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Then H has a $C_\ell^{(3)}$-packing $\mathcal{C}$ such that $\Delta_2(H \smallsetminus E(\mathcal{C})) \leqslant \gamma n$.*

For Lemma 5.6.3 we exploit the connection of *fractional graph decompositions* with their integral counterparts. Given a hypergraph H, let $\mathcal{C}_\ell(H)$ be the family of all ℓ-cycles in H, and given $X \in E(H)$ let $\mathcal{C}_\ell(H, X) \subseteq \mathcal{C}_\ell(H)$ be those cycles which use the edge X. A *fractional $C_\ell^{(3)}$-decomposition* of a hypergraph H is a function $\omega : \mathcal{C}_\ell(H) \to [0, 1]$ such that for every edge $X \in H$ we have $\sum_{C \in \mathcal{C}_\ell(H,X)} \omega(C) = 1$. Joos and Kühn [34] proved the existence of fractional $C_\ell^{(k)}$-decompositions under general conditions. We state their results only in the particular case $k = 3$. A hypergraph H on n vertices is (α, ℓ)-*connected* if for every two ordered edges (s_1, s_2, s_3), $(t_1, t_2, t_3) \in V(H)^3$, there are at least $\alpha n^{\ell-1}/(3!|E(H)|)$ walks with ℓ edges starting at (s_1, s_2, s_3), ending at (t_1, t_2, t_3).

Theorem 5.6.4 (Joos and Kühn [34]). *For all $\alpha \in (0, 1)$, $\mu \in (0, 1/3)$ and $\ell \geqslant 2$, there is n_0 such that the following holds for all $n \geqslant n_0$. Suppose H is an (α, ℓ_0)-connected hypergraph on n vertices with $540\frac{\ell_0}{\alpha}\log\frac{\ell_0}{\alpha}\log\frac{1}{\mu} \leqslant \ell$. Then there is a fractional $C_\ell^{(3)}$-decomposition ω of H with*

$$(1-\mu)\frac{2|E(H)|}{\Delta(H)^\ell} \leqslant \omega(C) \leqslant (1+\mu)\frac{2|E(H)|}{\delta(H)^\ell}$$

for all ℓ-cycles C in H.

To use Theorem 5.6.4 we show that hypergraphs with $\delta_2(H) \geqslant 2n/3$ are (α, ℓ_0)-connected for some suitable α, ℓ_0. The following argument is due to Reiher [34, Lemma 2.3]. We include its proof for completeness and because for 3-uniform hypergraphs we get a better value of α, which turns out to increase the range of ℓ in which one can apply Theorem 5.6.4.

Lemma 5.6.5. *For each $d \geqslant 1/2$, every hypergraph H on n vertices and such that $\delta_2(H) \geqslant (d + o(1))n$ is $(d^2(2d-1)^4, 8)$-connected.*

Proof. Let $V = V(H)$ and $(s_1, s_2, s_3), (t_1, t_2, t_3) \in V^3$ be two arbitrary ordered edges of H. For $z \in V(H)$, let the function $I_z : V^2 \to \{0, 1\}$ be such that $I_z(x_1, x_2) = 1$ if and only if $s_2s_3x_1x_2t_1t_2$ is a path in the link graph of z in H. Let $N = N_H(s_2s_3) \cap N_H(t_1t_2)$ and note that $|N| > (2d-1)n$. Note that if $z_1, z_2 \in N$ (possibly equal) and $(x_1, x_2) \in V^2$ are such that $I_{z_1}(x_1, x_2) = I_{z_2}(x_1, x_2) = 1$, then $s_1s_2s_3z_1x_1x_2z_2t_1t_2t_3$ is a walk from (s_1, s_2, s_3) to (t_1, t_2, t_3) using 8 edges, call such walks *standard*.

First, note that having fixed $z \in N$, the number of pairs $(x_1, x_2) \in V^2$ such that $I_z(x_1, x_2) = 1$ can be bounded as follows: choose $x_1 \in N_H(s_3z)$ arbitrarily (there are at least dn choices) and then $x_2 \in N_H(zx_1) \cap N_H(zt_1)$ (of which there are at least $(2d-1)n$ choices). Thus we have $\sum_{(x_1,x_2)\in V^2} I_z(x_1, x_2) \geqslant d(2d-1)n^2$ for all $z \in N$.

On the other hand, note that for a fixed (x_1, x_2) with $x_1 \neq x_2$, the number of standard walks which use (x_1, x_2) is exactly $(\sum_{z\in N} I_z(x_1, x_2))^2$. Thus the number of standard walks is at least (using Jensen's inequality in the first inequality, and $|N| \geqslant (2d-1)n$ in the third inequality)

$$\sum_{(x_1,x_2)\in V^2} \left(\sum_{z\in N} I_z(x_1,x_2)\right)^2 \geqslant n^2 \left(\frac{1}{n^2}\sum_{z\in N}\sum_{(x_1,x_2)\in V^2} I_z(x)\right)^2$$
$$\geqslant n^2 \left(\frac{1}{n^2}\sum_{z\in N} d(2d-1)n^2\right)^2 \geqslant d^2(2d-1)^4 n^4,$$

as required. □

To prove Lemma 5.6.3 we combine the fractional matching of Theorem 5.6.4 with a nibble-type matching argument. We use a result by Alon and Yuster [4] (but see also Kahn [35] and Ehard, Glock and Joos [17] for variations and extensions). This result states that every k-uniform hypergraph which is almost regular (for its 1-degree) and with bounded maximum 2-degree contains a 'well behaved' matching with respect to a given collection of subsets of vertices. The statement of the theorem is technical, but in our context the conditions are easy to check. We define the parameter $g(H) = \Delta_1(H)/\Delta_2(H)$ for every k-uniform hypergraph H.

Theorem 5.6.6 (Alon and Yuster [4])**.** *For every $\varepsilon > 0$ there is a $\mu > 0$ such that for every sufficiently large n the following holds. Let H be an n-vertex k-uniform hypergraph and let $U_1, \dots, U_t \subseteq V(H)$ be subsets of vertices with $\log t \leqslant g(H)^{1/(3k-3)}$ and such that $|U_i| \geqslant 5g(H)^{1/(3k-3)} \log(g(H)t)$ for every $1 \leqslant i \leqslant t$. Suppose that*

(*a*) $\delta_1(H) \geqslant (1-\mu)\Delta_1(H)$ *and*

(*b*) $\Delta_1(H) \geqslant (\log n)^7 \Delta_2(H)$,

then there is a matching $M \subseteq E(H)$ covering at least $(1-\varepsilon)|U_i|$ vertices from U_i for every $1 \leqslant i \leqslant t$.

Proof of Lemma 5.6.3. Let $\alpha = 4 \times 3^{-6}$ (as in Lemma 5.6.5 for $d = 2/3$) and $\ell_0 = 8$. By Lemma 5.6.5, H is (α, ℓ_0)-connected. A numerical calculation shows that we can fix

$\mu \in (0, 1/3)$ such that $540\frac{\ell_0}{\alpha}\log\frac{\ell_0}{\alpha}\log\frac{1}{\mu} \leqslant 10^7 \leqslant \ell$. Thus Theorem 5.6.4 informs us that there exists a fractional $C_\ell^{(3)}$-decomposition ω of H with

$$\omega(C) \leqslant (1+\mu)\frac{2|E(H)|}{\delta_2(H)^\ell} \leqslant 4\frac{|E(H)|}{\delta_2(H)^\ell} \leqslant \frac{4n^3}{\delta_2(H)^\ell} \leqslant \frac{4 \times 3^\ell}{n^{\ell-3}}$$

for all $C \in \mathcal{C}_\ell(H)$.

Consider the auxiliary ℓ-uniform hypergraph F with vertex set $E(H)$, and an edge for each cycle in $\mathcal{C}_\ell(H)$ corresponding to its set of ℓ edges. Define a random subgraph $F' \subseteq F$ by keeping each edge C with probability $p_C := n^{1/2}\omega(C)$. By the bounds on $\omega(C)$ and $1/n \ll 1/\ell$ we have $p_C \leqslant 1$ for all $C \in \mathcal{C}_\ell(H)$.

For each edge $e \in E(H)$ we have $\mathbb{E}[d_{F'}(e)] = n^{1/2}\sum_{C \in \mathcal{C}_\ell(H,e)}\omega(C) = n^{1/2}$. Moreover, since two distinct edges $e, f \in E(H)$ can participate together in at most $O(n^{\ell-4})$ ℓ-cycles in H, we have $\mathbb{E}[d_{F'}(e,f)] = O(n^{-1/2})$. Standard concentration inequalities (see Theorem 5.4.4 *(i)* and *(iii)*), imply that with high probability $d_{F'}(e) = (1+o(1))n^{1/2}$ for each $e \in V(F')$ and that $\Delta_2(F') = O(\log n)$. This means that $\delta_1(F') \geqslant (1-o(1))\Delta_1(F')$, $g(H) = \Omega(n^{1/2}/\log n)$, and $g(H) = O(n^{1/2})$.

For each 2-set uv of vertices of H, let $U_{uv} \subseteq V(F)$ correspond to the edges in H containing uv. There are at most n^2 such sets and each has size at least $2n/3$. Thus, Theorem 5.6.6 yields a matching M in F' such that at most γn vertices in $V(F')$ are uncovered in each U_{uv}. The matching M in $F' \subseteq F$ translates to a $C_\ell^{(3)}$-packing $\mathcal{C}$ in H and the latter condition implies $\Delta_2(H \setminus E(\mathcal{C})) \leqslant \gamma n$, as desired. □

As mentioned before, we end this subsection with the proof of Lemma 5.6.2.

Proof of Lemma 5.6.2. The proof proceeds in three steps. First, we find $H_p \subseteq H$ by including each edge with probability p, and in the remainder $H_0 = H \setminus H_p$ we find an almost perfect $C_\ell^{(3)}$-packing $\mathcal{C}_0$, let $L_0 = H_0 \setminus E(\mathcal{C}_0)$ be the leftover edges. Second, we correct the leftover L_0 in the vertices incident with $\Omega(n^2)$ many edges of L_0 by constructing cycles with the help of the edges in H_p. This provides us with a new cycle packing $\mathcal{C}_1 \subseteq L_0 \cup H_p$ whose new leftover $L_1 = H_0 \setminus E(\mathcal{C}_0 \cup \mathcal{C}_1)$ satisfies $\Delta_1(L_1) = o(n^2)$. Finally, we correct the new leftover L_1 in a similar way, fixing the pairs incident to $\Omega(n)$

edges in L_1. We get a cycle packing $\mathcal{C}_2 \subseteq L_1 \cup H_p$, and $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ will be the desired cycle packing.

Step 1: Random slice and approximate decomposition. Note that $\delta_2^{(3)}(H) \geqslant 3\varepsilon n$. Now let $p = \gamma/4$, and let $H_p \subseteq H$ be obtained from H by including each edge independently with probability p. Using concentration inequalities (e.g. Theorem 5.4.4) we see that with non-zero probability

$$\Delta_2(H_p) \leqslant 2pn \text{ and } \delta_2^{(3)}(H_p) \geqslant 2\varepsilon pn. \tag{5.6.5}$$

hold simultaneously for H_p. From now on we suppose H_p is fixed and satisfies (5.6.5).

Let $H_0 = H \setminus H_p$. In H_0, construct a $C_\ell^{(3)}$-packing by removing edge-disjoint cycles, one by one, until no longer possible. We get a $C_\ell^{(3)}$-packing $\mathcal{C}_0$ in H_0, let $F_0 = E(\mathcal{C}_0)$. By Erdős' Theorem [18, Theorem 1] there exists $c > 0$ such that $L_0 = H_0 \setminus F_0$ has at most n^{3-3c} edges.

Step 2: Eliminating bad vertices. Let $B_0 = \{v \in V : d_{L_0}(v) \geqslant n^{2-2c}\}$. Since $|L_0| \leqslant n^{3-3c}$, by double-counting we have $|B_0| \leqslant 3n^{1-c}$.

For each $b \in B_0$, let G_b be the subgraph of $L_0(b)$ obtained after removing the vertices of B_0. Note that $L_0(b) - G_0$ has at most $|B_0|n \leqslant 3n^{2-c}$ edges. Now, let $\mathcal{P}_b$ be a maximal edge-disjoint collection of paths of length 3 in G_b. Since every graph on n vertices with at least $n+1$ edges contains a path of length 3, then $G_b - E(\mathcal{P}_b)$ has at most n edges. All together, we deduce that the number of edges in $L_0(b) - E(\mathcal{P}_b)$ satisfies

$$|L_0(b)| - |E(\mathcal{P}_b)| \leqslant 3n^{2-c} + n \leqslant 4n^{2-c}. \tag{5.6.6}$$

Since G_b contains at most n^2 edges, we certainly have $|\mathcal{P}_b| \leqslant n^2$. Let $\mathcal{P}_b$ be a collection of paths on five vertices obtained by replacing each $v_0v_1v_2v_3$ in $\mathcal{P}_b$ with the path $v_0v_1bv_2v_3$ in L_0. Note that any two distinct $P_1, P_2 \in \mathcal{P}_b$ are edge-disjoint, and for two distinct $b, b' \in B_0$, and $P \in \mathcal{P}_b$, $P' \in \mathcal{P}_{b'}$, since $b' \notin V(G_b)$ we have P, P' are edge-disjoint. Thus the union $\mathcal{P} = \bigcup_{b \in B_0} \mathcal{P}_b$ is an edge-disjoint collection of paths on 5 vertices.

Select $\gamma', \mu', \varepsilon'$ such that $1/n \ll \gamma' \ll \mu' \ll \varepsilon' \ll \gamma, \varepsilon, 1/\ell$. We wish to apply Lemma 5.6.1 to extend $\mathcal{P}$ into cycles. We claim $\mathcal{P}$ is γ'-sparse. Let $S \in V(H)^{(2)}$.

Since $|\mathcal{P}| \leqslant |B_0|n^2 \leqslant 3n^{3-c} \leqslant \gamma' n^3$, certainly $\mathcal{P}$ contains at most $|\mathcal{P}| \leqslant \gamma' n^3$ paths of type 0 for S. Now, note that for each $b \in B_0$, $P \in \mathcal{P}_b$ can have at most $2n$ paths of type 1 for S, thus $\mathcal{P}$ has at most $|B_0|2n \leqslant 6n^{2-c} \leqslant \gamma' n^2$ paths of type 1 for S. Analogously, for each $b \in B_0$, $P \in \mathcal{P}_b$ can have at most 1 path of type 2 for S, thus $\mathcal{P}$ has at most $|B_0| \leqslant 3n^{1-c} \leqslant \gamma' n$ paths of type 2 for S. Thus $\mathcal{P}$ is γ'-sparse.

Recall that L_0 is edge-disjoint with H_p. Inequalities (5.6.5) together with $p = \gamma/4$ and $\varepsilon' \ll \gamma, 1/\ell$, show that we can use Corollary 5.4.3 (with $U = V(H)$) and deduce that for each $P \in \mathcal{P}$, there exists at least $\varepsilon' n^{\ell-5}$ copies of $C_\ell^{(3)}$ in $L_0 \cup H_p$ which extend P_i using extra edges of H_p only.

We apply Lemma 5.6.1 with ε', μ', γ', ℓ, 5, L_0, H_p, $\mathcal{P}$ playing the rôle of ε, μ, γ, ℓ, ℓ', H_1, H_2, $\mathcal{P}$ respectively, to obtain a $C_\ell^{(3)}$-decomposable graph $F_1 \subseteq L_0 \cup H_p$ such that $E(\mathcal{P}) \subseteq F_1$ and

$$\Delta_2(F_1 \smallsetminus E(\mathcal{P})) \leqslant \mu' n. \tag{5.6.7}$$

Since F_0, F_1 are edge-disjoint, $F_0 \cup F_1$ is $C_\ell^{(3)}$-decomposable. Let $L_1 = H_0 \smallsetminus (F_0 \cup F_1)$. Note that if $v \notin B_0$, then $d_{L_1}(v) \leqslant d_{L_0}(v) < n^{2-2c}$ by definition. Moreover, if $v \in B_0$, then each edge in $E(\mathcal{P}_v)$ is in F_1, and hence (5.6.6) implies $d_{L_1}(v) \leqslant |L_0(v)| - |E(\mathcal{P}_v)| \leqslant 4n^{2-c}$. Therefore,

$$\Delta_1(L_1) \leqslant 4n^{2-c}. \tag{5.6.8}$$

Step 3: Eliminating bad pairs. Let $f = c/2$ and $B_1 = \{xy \in V^{(2)} : d_{L_1}(xy) \geqslant n^{1-f}\}$. From $|L_1| \leqslant |L_0| \leqslant n^{3-3c} \leqslant n^{3-6f}$ we deduce $|B_1| \leqslant n^{2-4f}$. Now consider B_1 as the set of edges of a graph in V. Each edge of B_1 incident to a vertex x implies that x belongs to at least n^{1-f} edges in L_1, and each of those edges participates in at most two of the edges in B_1 incident to x. So we have $d_{L_1}(x) \geqslant \frac{1}{2} n^{1-f} d_{B_1}(x)$. Together with inequality (5.6.8) we deduce $\Delta(B_1) \leqslant 8n^{1-f}$.

A path P on L_1 is B_1-*based* if $P = zxyw$ and $xy \in B_1$. Let $\mathcal{P}_2$ be a maximal packing of B_1-based paths. For all $xy \in B_1$, it holds that $d_{L_1}(xy) - d_{E(\mathcal{P}_2)}(xy) \leqslant 1$. Otherwise it would exist distinct $z, w \in N_{L_1 \smallsetminus E(\mathcal{P}_2)}(xy)$, and then $zxyw$ would a B_1-based path not in $\mathcal{P}_2$ which contradicts its maximality.

We claim $\mathcal{P}_2$ is γ'-sparse. For each $xy \in B_1$, let $\mathcal{P}_{xy} \subseteq \mathcal{P}_2$ be the paths whose two interior vertices are precisely xy. Clearly $|\mathcal{P}_{xy}| \leqslant n$ and $\mathcal{P}_2 = \bigcup_{xy \in B_1} \mathcal{P}_{xy}$. Let $e \in V^{(2)}$. Since $|\mathcal{P}_2| \leqslant \sum_{xy \in B_1} |\mathcal{P}_{xy}| \leqslant n|B_1| \leqslant n^{3-4f} \leqslant \gamma' n^3$, there are at most $\gamma' n^3$ paths of type 0 for e in $\mathcal{P}_2$. Recall that if $P = zxyw$ is a path of type 1 for e, then we have $|e \cap \{z, x, y, w\}| = 1$. If $xy \in B_1$ satisfies $e \cap \{x, y\} = \varnothing$, then at most two paths in $\mathcal{P}_{xy}$ can be of type 1 for e and therefore there are at most $2|B_1| \leqslant 2n^{2-4f}$ paths of type 1 for e in $\mathcal{P}_2$. We estimate the contribution of the pairs $xy \in B_1$ such that $|e \cap \{x, y\}| = 1$. Each such xy contributes with at most n paths of type 1 for e in $\mathcal{P}_{xy}$. By (5.6.8), the number of such xy is at most $2\Delta(B_1) \leqslant 16n^{1-f}$, thus the total contribution of those pairs is at most $16n^{2-f}$. All together, the total number of paths of type 1 for e in $\mathcal{P}_2$ is at most $2n^{2-4f} + 16n^{2-f} \leqslant \gamma' n^2$. If $e = \{a, b\}$ then $\mathcal{P}_{a,b}$ does not contain any path of type 2 for e, by definition of the path types. Thus the only possible contributions come from the pairs in $\mathcal{P}_{a,x}$ and $\mathcal{P}_{b,y}$ for some $x, y \in V(H)$; and each one of those sets contains at most 1 path of type 2 for e. Thus the total number of pairs of type 2 for e in $\mathcal{P}_2$ is at most $2\Delta(B_1) \leqslant 16n^{1-f} \leqslant \gamma' n$. Thus $\mathcal{P}_2$ is γ'-sparse.

Let $H'_p = H_p \smallsetminus (F_0 \cup F_1)$. Inequalities (5.6.5) and (5.6.7), together with the hierarchies $\mu' \ll \varepsilon' \ll \gamma, 1/\ell$, allow us to use Corollary 5.4.3 with $U = V(H)$, thus for each $P \in \mathcal{P}_2$, there exists at least $\varepsilon' n^{\ell-4}$ copies of $C_\ell^{(3)}$ in $L_1 \cup H'_p$ which extend P using extra edges of H'_p only. Apply Lemma 5.6.1 with the parameters $\varepsilon', \mu', \gamma', \ell, 4, L_1, H'_p, \mathcal{P}_2$ playing the rôles of $\varepsilon, \mu, \gamma, \ell, \ell', H_1, H_2, \mathcal{P}$ respectively, to obtain a $C_\ell^{(3)}$-decomposable $F_2 \subseteq L_1 \cup H'_p$ such that $E(\mathcal{P}_2) \subseteq F_2$ and $\Delta_2(F_2 \smallsetminus E(\mathcal{P}_2)) \leqslant \mu' n$.

We claim that $\Delta_2(L_1 \smallsetminus F_2) \leqslant n^{1-f}$. Indeed, if $xy \in B_1$, $d_{L_1 \smallsetminus F_2}(xy) \leqslant d_{L_1}(xy) \leqslant n^{1-f}$ follows by definition, otherwise, $E(\mathcal{P}_2) \subseteq F_2$ implies $d_{L_1 \smallsetminus F_2}(xy) \leqslant d_{L_1}(xy) - d_{F_2}(xy) \leqslant 1$. Since F_2 and $F_0 \cup F_1$ are edge-disjoint, $F = F_0 \cup F_1 \cup F_2$ is a $C_\ell^{(3)}$-decomposable subgraph of H. We claim $L = H \smallsetminus F$ satisfies $\Delta_2(L) \leqslant \gamma n$. Indeed, an edge not covered by F is either in H_p or in $L_1 \smallsetminus F_2$. Thus we have

$$\Delta_2(L) \leqslant \Delta_2(H_p) + \Delta_2(L_1 \smallsetminus F_2) \leqslant 2pn + n^{1-f} \leqslant \gamma n\,,$$

as required. □

5.6.3 Proof of the Cover-Down Lemma

As a final tool, we borrow the following theorem of Thomassen [61] about path-decompositions of graphs.

Theorem 5.6.7 ([61])**.** *Any* 171*-edge-connected graph* G *such that* $|E(G)|$ *is divisible by* 3 *has a* P_3*-decomposition.*

Proof of Lemma 5.3.3. Let $\gamma_1, p_1, p_2 > 0$ with $\gamma_1 \ll p_1 \ll p_2 \ll \mu, \varepsilon$. For $i \in \{0,1,2,3\}$, say an edge e of H is of *type* i if $|e \cap U| = i$, and let $H_i \subseteq H$ be the edges of H which are of type i. For $i \in \{1,2\}$, let $R_i \subseteq H_i$ be defined by choosing edges independently at random from H_i with probability $3p_i/2$. By assumption, $\delta_2^{(3)}(H;U) \geqslant 3\varepsilon|U|$ (see definition at the beginning of Section 5.4.1).

By Theorem 5.4.4 we get that, for $i \in \{1,2\}$, with non-zero probability, that

$$\Delta_2(R_i) \leqslant 2p_i n, \tag{5.6.9}$$

$$\delta_2^{(3)}(R_1 \cup R_2 \cup H[U];U) \geqslant 2\varepsilon p_1|U|, \text{ and} \tag{5.6.10}$$

$$\delta_2^{(3)}(R_2 \cup H[U];U,G) \geqslant 2\varepsilon p_2|U|, \tag{5.6.11}$$

where $G \subseteq V(H)^{(2)}$ corresponds to the pairs e such that $e \cap U \neq \varnothing$. From now on we assume R_1, R_2 are fixed with those properties.

Let $H' = H - H[U] - R_1 - R_2$. Recall that, by assumption, $\delta_2(H) \geqslant (2/3 + 2\varepsilon)n$ and $|U| = \lfloor \varepsilon n \rfloor$. By our choice of p_1, p_2 and (5.6.9), we deduce that $\delta_2(H') \geqslant (2/3 + \varepsilon/2)n$.

We consider two possible cases depending on the value of ℓ. If $\ell \geqslant 9$ is divisible by 3, then we apply Lemma 5.6.2, otherwise by assumption $\ell \geqslant 10^7$, and we can apply Lemma 5.6.3. In any case, the output is a $C_\ell^{(3)}$-packing $\mathcal{C}$ in H' such that $\Delta_2(H' \setminus E(\mathcal{C})) \leqslant \gamma_1 n$. Let $J = H' \setminus E(\mathcal{C})$ be the edges in H' not covered by $\mathcal{C}$, and for each $i \in \{0,1,2\}$ let J_i be the edges of type i in J. We shall cover the edges in J with cycles of length ℓ and for that we will proceed in three steps, covering the edges of J_0, J_1, and J_2 in order.

Consider each edge in J_0 as a path on three vertices $v_1v_2v_3$, assigning to each edge an arbitrary order and let $\mathcal{P}_0$ be the collection of those paths. Observe that, due to the inequalities $\Delta_2(J_0) \leqslant \Delta_2(J) \leqslant \gamma_1 n$ the collection $\mathcal{P}_0$ is γ_1-sparse. Let $\mu_1, \varepsilon_1 > 0$ be such that $\gamma_1 \ll \mu_1 \ll \varepsilon_1 \ll p_1, \varepsilon$. Equation (5.6.10) and Corollary 5.4.3 imply that each path $P \in \mathcal{P}_0$ can be extended to at least $2\varepsilon_1 n^{\ell-3}$ cycles C, such that $C \setminus P \subseteq R_1 \cup R_2 \cup H[U]$ and $V(C) \setminus V(P) \subseteq U$. Then an application of Lemma 5.6.1 with $\varepsilon_1, \mu_1, 3, J_0, R_1 \cup R_2 \cup H[U], \mathcal{P}_0$ in place of $\varepsilon, \mu, \ell', H_1, H_2, \mathcal{P}$ respectively, implies that there is a $C_\ell^{(3)}$-decomposable subgraph F_0 such that $F_0 \supseteq J_0$, and

$$\Delta_2(F_0 \setminus J_0) \leqslant \mu_1 n. \tag{5.6.12}$$

By construction, F_0 is edge-disjoint with the cycles in $\mathcal{C}$, and then $F_0' = E(\mathcal{C}) \cup F_0$ is $C_\ell^{(3)}$-descomposable. Note that all edges not covered by F_0' lie in $(J_1 \cup J_2) \cup (R_1 \cup R_2) \cup H[U]$.

Let $J_1' = (J_1 \cup R_1) \setminus F_0'$ and $R_2' = (R_2 \cup H[U]) \setminus F_0'$. Let $\gamma_2, \mu_2, \varepsilon_2 > 0$ be such that $p_1 \ll \gamma_2 \ll \mu_2 \ll \varepsilon_2 \ll p_2, \varepsilon$. Since $J_1' \subseteq J_1 \cup R_1 \subseteq J \cup R_1$, we have

$$\Delta_2(J_1') \leqslant \Delta_2(J) + \Delta_2(R_1) \leqslant \gamma n + 2p_1 n \leqslant \gamma_2 n.$$

Since each edge in J_1' is of type 1 in H, we can consider each edge in J_1' as a path $P = v_1v_2v_3$ where $v_2 \in U$ and $v_1, v_3 \notin U$; and let $\mathcal{P}_1$ be the collection of those paths. Then $\Delta_2(J_1') \leqslant \gamma_2 n$ implies $\mathcal{P}_1$ is γ_2-sparse. By (5.6.11) and (5.6.12), together with Corollary 5.4.3, we deduce that each $P \in \mathcal{P}_1$ can be extended to at least $2\varepsilon_2 n^{\ell-3}$ cycles C, such that $C \setminus P \subseteq R_2'$ and $V(C) \setminus V(P) \subseteq U$. Apply Lemma 5.6.1 with $\varepsilon_2, \mu_2, \gamma_2, 3, J_1', R_2', \mathcal{P}_1$ in place of $\varepsilon, \mu, \gamma, \ell', H_1, H_2, \mathcal{P}$ to obtain a $C_\ell^{(3)}$-decomposable subgraph F_1 such that $F_1 \supseteq J_1'$, and

$$\Delta_2(F_1 \setminus J_1) \leqslant \mu_2 n. \tag{5.6.13}$$

By construction, F_1 and F_0' are edge-disjoint, and then $F_1' = F_1 \cup F_0'$ is $C_\ell^{(3)}$-decomposable. Note that the edges not covered by F_1' lie in $J_2 \cup R_2 \cup H[U]$.

Let $J_2' = (J_2 \cup R_2) \setminus F_1'$. Note that each edge in J_2' is of type 2. For each $v \in V(H) \setminus U$, let $G_v = J_2'(v)[U]$, that is, G_v is the link graph of v in J_2' restricted to U. Fix $v \in V(H) \setminus U$. Given $x, y \in U$, the equations (5.6.11) and (5.6.13) imply that x and y have at least $2\varepsilon p_2|U| - 2\mu_2 n \geqslant 171$ common neighbours in G_v, so G_v is 171-edge-connected. Since $v \notin U$, our assumption on H implies that the number of edges of $H(v)$ is divisible by 3. Note that G_v is exactly the link graph of $H \setminus F_1'$ when restricted to U. Therefore, and since F_1' is $C_\ell^{(3)}$-decomposable, the number of edges in G_v is divisible by 3 as well.

By Theorem 5.6.7, G_v has a decomposition into paths $\mathcal{P}_v' = \{P_1, \ldots, P_t\}$, each of length 3. Observe that these paths yields to a collection of (3-uniform) paths in J_2' by substituting each path $P_i = w_1w_2w_3w_4$ in $\mathcal{P}_v'$ by the path $w_1w_2vw_3w_4$. Let $\mathcal{P}_v$ be the collection of paths obtained in this way. Observe that for $u \neq v$ in $V(H) \setminus U$, $\mathcal{P}_v$ and $\mathcal{P}_u$ are edge-disjoint. Let $\mathcal{P}_2 = \bigcup_{v \in V(H) \setminus U} \mathcal{P}_v$. Note that $\mathcal{P}_2$ decomposes J_2' into paths on five vertices.

Let $\gamma_3, \varepsilon_3 > 0$ be such that $p_2 \ll \gamma_3 \ll \varepsilon_3 \ll \mu_3 \ll \mu, \varepsilon$. Recall that $|U| = \lfloor \varepsilon n \rfloor$. Since $J_2' \subseteq J_2 \cup R_2 \subseteq F \cup R_2$, we have $\Delta_2(J_2') \leqslant \Delta_2(R_2) + \Delta_2(J) \leqslant 2p_2n + \gamma_1 n \leqslant \gamma_3 n$, so $\mathcal{P}_2$ is γ_3-sparse. Let $H_2' = H[U] \setminus F_1'$. We have $F_1'[U] = F_1[U] \cup F_0[U]$. By (5.6.12)–(5.6.13), we have $\delta_2(H_2') \geqslant \delta_2(H[U]) - 2\mu_2 n \geqslant (2/3 + \varepsilon/2)|U|$. By Corollary 5.4.3, we deduce each $P \in \mathcal{P}_2$ can be extended to at least $2\varepsilon_2 n^{\ell-5}$ cycles C such that $C \setminus P \subseteq H_2'$. Thus we can apply Lemma 5.6.1 with $\varepsilon_3, \mu_3, \gamma_3, 5, J_2', H_2', \mathcal{P}_2$ playing the rôles of $\varepsilon, \mu, \gamma_3, \ell', H_1, H_2, \mathcal{P}$ respectively, to obtain a $C_\ell^{(3)}$-decomposable subgraph F_2 such that $F_2 \supseteq J_2'$, and

$$\Delta_2(F_2 \cap H_2') \leqslant \mu_3 n. \tag{5.6.14}$$

By construction, F_2 and F_1' are edge-disjoint, and then $F = F_1' \cup F_2$ is $C_\ell^{(3)}$-decomposable. Moreover, all edges not contained in U are covered by F. In fact, we have that

$$H - H[U] = E(\mathcal{C}) \cup J_0 \cup (J_1 \cup R_1) \cup (J_2 \cup R_2) \subseteq E(\mathcal{C}) \cup F_0 \cup F_1 \cup F_2 = F.$$

Finally, inequalities (5.6.12)–(5.6.14) yield that $\Delta_2(F[U]) \leqslant \mu n$, as required. $\square$

5.7 Absorbing Lemma

In this section we prove Lemma 5.3.2. We need to show that, given a sufficiently large H with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$ and a subgraph $R \subseteq H$ on at most m vertices, there is an $C_\ell^{(3)}$-absorber A for R on at most $O(m^9\ell^9)$ edges. We divide the proof in two main parts.

First, in Section 5.7.1 we shall find a bounded-size hypergraph $A_1 \subseteq H$, edge-disjoint from R, which admits a $C_\ell^{(3)}$-decomposition. This subgraph will be chosen such that $R \cup A_1$ contains a *tour decomposition*, that is, a decomposition in which all subgraphs are tours (see Lemma 5.7.1). The second step is to transform the found tour decomposition to a $C_\ell^{(3)}$-decomposition (see details in Section 5.7.2). Finally, in Section 5.7.3 we combine both steps to prove Lemma 5.3.2.

5.7.1 Tour decomposition

The main goal of this subsection is to prove the following lemma.

Lemma 5.7.1. *Let $\ell \geqslant 7$, $\varepsilon > 0$, and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a hypergraph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Let $R \subseteq H$ be $C_\ell^{(3)}$-divisible on at most m vertices. There exists a subgraph $A_1 \subseteq H$, edge-disjoint with R, such that*

(i) A_1 has at most $5m^3\ell^2$ edges,

(ii) $A_1 \cup R$ spans at most $m + 5m^3\ell^2$ vertices,

(iii) A_1 has a $C_\ell^{(3)}$-decomposition, and

(iv) $A_1 \cup R$ has a tour decomposition.

Tour-trail decompositions

We consider decompositions $\mathcal{T} = \{C_1, \ldots, C_t, P_1, \ldots, P_k\}$ in which C_i is a tour for every $i \in [t]$ and P_j is a trail for every $j \in [k]$. In this case we say $\mathcal{T}$ is a *tour-trail decomposition*. Note that every hypergraph has a tour-trail decomposition, since we can consider every edge as a trail on three vertices (by giving it an arbitrary ordering).

For a trail $P = u_1u_2 \cdots u_{k-1}u_k$ we say that the ordered pairs (u_2, u_1) and (u_{k-1}, u_k) are the *tails* of P. Observe that the set of tails of a P depends on the edge-set of P only, i.e. is independent of order in which we transverse the trail. We remark that the tails differ from the starting and ending pairs of P (as defined in Chapter 1) since they have different orderings.

Given H and a tour-trail decomposition $\mathcal{T} = \{C_1, C_2, \dots, C_t, P_1, P_2, \dots, P_k\}$ of some $R \subseteq H$, we define the *residual digraph of* $\mathcal{T}$, denoted as $D(\mathcal{T})$, as the multidigraph on the same vertex set as H, where the arcs correspond to the union of the tails of each trail of $\mathcal{T}$, considered with repetitions. Thus $D(\mathcal{T})$ has exactly $2k$ arcs, counted with multiplicities, if and only if $\mathcal{T}$ has k trails. Outdegrees and indegrees of a vertex x in $D(\mathcal{T})$ are denoted by $d^+{}_{D(\mathcal{T})}(x)$ and $d^-{}_{D(\mathcal{T})}(x)$ respectively, omitting subscripts from the notation if the underlying digraph is clear from context.

Remark 5.7.2. Observe that if $(x, y), (y, x) \in E(D_{\mathcal{T}})$ then, there are trails P_i and P_j in $\mathcal{T}$ that can be merged into a trail (if $i \neq j$) or tour (if $i = j$) which contains all the edges contained in P_i and P_j. Thus there is another tour-trail decomposition $\mathcal{T}'$ of R with less trails than $\mathcal{T}$, obtained from $\mathcal{T}$ by removing P_i, P_j and adding the tour or trail born from joining P_1 and P_2.

We construct A_1 in Lemma 5.7.1 as follows. We begin with an arbitrary tour-trail decomposition $\mathcal{T}_0$ of R, and we will find an increasing sequence of subgraphs $\varnothing = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k \subseteq H$. Each $T_i \smallsetminus T_{i-1}$ will be sufficiently small, $C_\ell^{(3)}$-decomposable and edge-disjoint from T_{i-1}. Moreover, each $T_i \smallsetminus T_{i-1}$ will be an edge-disjoint union of 'gadget' of a prescribed family. More precisely, for each $i > 0$, each $T_i \cup R$ will contain a tour-trail decomposition $\mathcal{T}_i$, obtained from a previous tour-trail decomposition $\mathcal{T}_{i-1}$ of $T_{i-1} \cup R$. As an intermediate step (see Lemma 5.7.6), for some $k' < k$ we will find $T_{k'}$ and a tour-trail decomposition $\mathcal{T}_{k'}$ of $T_{k'} \cup R$ whose residual digraph is Eulerian (with the appropriate definition for directed graphs). At the end, we will have found a hypergraph T_k and a tour-trail decomposition $\mathcal{T}_k$ of $R \cup T_k$ which has an empty residual digraph. Thus $\mathcal{T}_k$ is actually a tour decomposition, and we finish by setting $A_1 = T_k$.

Gadgets

In the following two lemmata we describe the aforementioned gadgets, and their main properties.

First, for a given tour-trail decomposition $\mathcal{T}$ of $R \subseteq H$ and three distinct vertices v_1, v_2, v_3, the following lemma states that there is a subgraph $S_3 = S_3(v_1, v_2, v_3) \subseteq H$ edge-disjoint with R and which contains a $C_\ell^{(3)}$-decomposition. Moreover, there is a tour-trail decomposition of $R \cup S_3$ such that its residual digraph is exactly $D(\mathcal{T})$ with the additional arcs (v_1, v_2), (v_2, v_3), and twice the arc (v_1, v_3). We define the multidigraph $\vec{S}_3(v_1, v_2, v_3) = \{(v_1, v_3), (v_1, v_3), (v_1, v_2), (v_2, v_3)\}$.

For two multidigraphs D_1, D_2, we set the notation $D_1 \sqcup D_2$ to mean the multigraph on $V(D_1) \cup V(D_2)$ obtained by adding all the arcs of D_2 to D_1, considering the multiplicities.

Lemma 5.7.3. *Let $\ell \geqslant 7$, $\varepsilon > 0$ and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a hypergraph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Given three distinct vertices $v_1, v_2, v_3 \in V(H)$, $R \subseteq H$ on at most m vertices, and a tour-trail decomposition $\mathcal{T}$ of R the following holds. There is a subgraph $S_3 = S_3(v_1, v_2, v_3) \subseteq H$, edge-disjoint from R, and a tour-trail decomposition $\mathcal{T}_{S_3} = \mathcal{T}_{S_3}(\mathcal{T}, v_1, v_2, v_3)$ of $R \cup S_3$ such that*

$(i)_{S_3}$ S_3 contains at most 2ℓ edges and $S_3 \cup R$ spans at most $m + 2\ell - 3$ vertices,

$(ii)_{S_3}$ S_3 has a $C_\ell^{(3)}$-decomposition, and

$(iii)_{S_3}$ $D(\mathcal{T}_{S_3}) = D(\mathcal{T}) \sqcup \vec{S}_3(v_1, v_2, v_3)$.

Proof. The minimum codegree condition on H implies that there is a vertex $x \in V(H)$ that lies in $N(v_1v_2) \cap N(v_1v_3) \cap N(v_2v_3)$. Considering the paths v_1v_3x and $v_3xv_2v_1$, two applications of Lemma 5.4.1 yield the existence of two edge-disjoint cycles C_1 and C_2 of length ℓ, edge-disjoint with R, and such that $v_1v_3x \in E(C_1)$ and $v_3xv_2, xv_2v_1 \subseteq E(C_2)$ (transversing the vertices in that order). Then $S_3 = C_1 \cup C_2$, clearly satisfies $(i)_{S_3}$ and $(ii)_{S_3}$. Hence, we only need to prove the existence of a tour-trail decomposition $\mathcal{T}_{S_3}$ of $R \cup S_3$ for which $(iii)_{S_3}$ holds.

For this, consider the trail $P_1 = v_3v_2xv_1v_3$. Observe that $E(S_3) \smallsetminus E(P_1)$ consists exactly in the edges of a trail P_2 whose tails are (v_1, v_2) and (v_1, v_3). Indeed, the edges contained in the set $E(C_2) \smallsetminus \{v_3v_2x, v_2xv_1\}$ form a trail between (v_2, v_1) and (v_3, x), that we may merge with the trail with edges in $E(C_1) \smallsetminus \{xv_1v_3\}$ from (v_3, x) to (v_1, v_3).

Therefore, $\mathcal{T}_{S_3} = \mathcal{T} \cup \{P_1, P_2\}$ is a tour-trail decomposition of $R \cup S_3$. We deduce $(iii)_{S_3}$ by noticing that the tails of P_1 and P_2 are (v_2, v_3) and (v_1, v_3), and (v_1, v_2) and (v_1, v_3) respectively. $\square$

The following is our second gadget. It is designed in such a way that we can add a small subgraph $C_4 \subseteq H$ to some R, such that $R \cup C_4$ contains a tour-trail decomposition in which the residual digraph has an extra directed four-cycle. We introduce the notation $\vec{C}_4(v_1, v_2, v_3, v_4) = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$.

Lemma 5.7.4. *Let $\ell \geqslant 7$, $\varepsilon > 0$ and $n, m \in \mathbb{N}$ such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a hypergraph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Given four distinct vertices $v_1, v_2, v_3, v_4 \in V(H)$, a subgraph $R \subseteq H$ on at most m vertices, and a tour-trail decomposition $\mathcal{T}$ of R the following holds. There is a subgraph $C_4 = C_4(v_1, v_2, v_3, v_4) \subseteq H$, edge-disjoint from R and a tour-trail decomposition $\mathcal{T}_{C_4} = \mathcal{T}_{C_4}(\mathcal{T}, v_1, v_2, v_3, v_4)$ of $R \cup C_4$ such that*

$(i)_{C_4}$ *C_4 has at most 4ℓ edges and $C_4 \cup R$ spans at most $m + 4\ell - 6$ vertices,*

$(ii)_{C_4}$ *C_4 has a $C_\ell^{(3)}$-decomposition, and*

$(iii)_{C_4}$ *$D(\mathcal{T}_{C_4}) = D(\mathcal{T}) \sqcup \vec{C}_4(v_1, v_2, v_3, v_4)$.*

Proof. Two consecutive applications of Lemma 5.7.3 yield the existence of edge-disjoint subgraphs $S_3(v_1, v_2, v_3)$ and $S_3(v_3, v_4, v_1)$. More precisely, first we apply Lemma 5.7.3 to obtain $S_3(v_1, v_2, v_3)$ edge-disjoint from R. Then, we apply it again with $R \cup S_3(v_1, v_2, v_3)$ in place of R to obtain $S_3(v_3, v_4, v_1)$ edge disjoint from $R \cup S_3(v_1, v_2, v_3)$ (here we use $1/n \ll 1/m$, to apply Lemma 5.7.3 to a larger subgraph with at most $m + 2\ell - 6$ vertices). It is not difficult to check that the subgraph $C_4 = S_3(v_1, v_2, v_3) \cup S_3(v_3, v_4, v_1)$ satisfies $(i)_{C_4}$ and $(ii)_{C_4}$

Moreover, in the second application of Lemma 5.7.3 we obtain a tour-trail decomposition $\mathcal{T}'$ of $R \cup C_4$ equal to $\mathcal{T}' = \mathcal{T}_{S_3}\big(\mathcal{T}_{S_3}(\mathcal{T}, v_1, v_2, v_3), v_3, v_4, v_1\big)$, whose residual digraph is given by

$$D(\mathcal{T}') = D(\mathcal{T}) \sqcup \vec{S}_3(v_1, v_2, v_3) \sqcup \vec{S}_3(v_3, v_4, v_1).$$

Observe that $D(\mathcal{T}')$ contains both the arcs (v_1, v_3) and (v_3, v_1) twice. By Remark 5.7.2, we can obtain a tour-trail decomposition $\mathcal{T}_{C_4}$ which satisfies $(iii)_{C_4}$. $\square$

Directed Eulerian tour

Given a directed multigraph D, we can extend the definition of *closed walk* as sequence of non-necessarily distinct vertices $v_1, \ldots, v_\ell$ such that, for every $1 \leqslant i \leqslant \ell$, the arc (v_i, v_{i+1}) is in D (understanding the indices modulo ℓ). A closed walk in which all arcs are distinct is called *tour*, and if every arc in D is covered exactly once, we say that it is an *Eulerian tour*. Directed multigraphs which contain Eulerian tours are called *Eulerian*.

In order to prove Lemma 5.7.1 we first prove that there is a bounded $C_\ell^{(3)}$-decomposable subgraph $T \subset H$, edge-disjoint with R, and such that $R \cup T$ contains a tour-trail decomposition $\mathcal{T}$ for which $D(\mathcal{T})$ is Eulerian.

We say that a directed multigraph D is *strongly connected* if for every two distinct vertices $x, y \in V(D)$ there is a closed walk which includes both. Similarly to the graph case, it is well-known that a directed multigraph D is Eulerian if an only if D is strongly connected and for every vertex $x \in V(D)$ we have $d^-(x) = d^+(x)$.

Now, we establish a crucial property of residual digraphs in 3-vertex-divisible hypergraphs.

Lemma 5.7.5. *Let $H = (V, E)$ be a 3-vertex-divisible hypergraph and let $\mathcal{T}$ be a tour-trail decomposition of H with residual digraph $D(\mathcal{T})$. For every $x \in V$ we have that*

$$d^+(x) \equiv d^-(x) \pmod 3.$$

Proof. For every vertex $x \in V(H)$, we need to show that $d^+(x) - d^-(x) \equiv 0 \bmod 3$ in the digraph $D(\mathcal{T})$. Consider the auxiliary digraph $F(\mathcal{T})$ obtained as follows: for every trail or tour $P = w_1 w_2 \cdots w_\ell$ in $\mathcal{T}$, to $F(\mathcal{T})$ add the arcs (w_i, w_{i+1}) and (w_{i+2}, w_{i+1}) for every $1 \leqslant i \leqslant \ell - 2$ (and for tours, add $(w_{\ell-1}, w_\ell), (w_1, w_\ell), (w_\ell, w_1), (w_2, w_1)$ as well), including all repetitions. In such a way (and since $\mathcal{T}$ is a decomposition) every edge of H contributes with exactly two arcs to $F(\mathcal{T})$. It is straightforward to check $D(\mathcal{T}) \subseteq F(\mathcal{T})$

and, crucially, that

$$d^+{}_{D(\mathcal{T})}(x) - d^-{}_{D(\mathcal{T})}(x) = d^+{}_{F(\mathcal{T})}(x) - d^-{}_{F(\mathcal{T})}(x),$$

so from now on we work with $F(\mathcal{T})$ only.

Let $x \in V(H)$. Each edge xyz in H contributes with two arcs to $F(\mathcal{T})$, which can be of type $\{(x,y),(x,z)\}, \{(y,x),(y,z)\}$, or $\{(z,x),(z,y)\}$. The edges of the first type contribute with 2 to $d^+(x) - d^-(x)$ in $F(\mathcal{T})$. The edges of second and third type contribute with -1 to $d^+(x) - d^-(x)$ in $F(\mathcal{T})$, which is congruent to 2 mod 3. Thus we deduce $d^+(x) - d^-(x) \equiv 2|d_H(x)| \bmod 3$. Since H is 3-vertex-divisible, this is congruent to 0 mod 3, and we are done. □

As mentioned, we find a tour-trail decomposition in which the residual digraph is Eulerian.

Lemma 5.7.6. *Let $\ell \geqslant 7$, $\varepsilon > 0$, and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be 3-divisible hypergraph on n vertices with $\delta(H) \geqslant (2/3 + \varepsilon)n$. Let $R \subseteq H$ be $C_\ell^{(3)}$-divisible in at most m vertices. Then, there exists a subgraph $T \subseteq H$, edge-disjoint from R such that*

$(i)_\circlearrowright$ *T has at most $m^3\ell$ edges and $T \cup R$ spans at most $m + m^3\ell$ vertices,*

$(ii)_\circlearrowright$ *T has a $C_\ell^{(3)}$-decomposition, and*

$(iii)_\circlearrowright$ *there is a tour-trail decomposition $\mathcal{T}_\circlearrowright$ of $T \cup R$ such that $D(\mathcal{T}_\circlearrowright)$ is Eulerian.*

Proof. We will prove that there is a subgraph $T \subseteq H$, edge-disjoint with R, satisfying $(i)_\circlearrowright$ and $(ii)_\circlearrowright$, and such that $T \cup R$ has a tour-trail decomposition $\mathcal{T}$ whose residual digraph satisfies

$$D(\mathcal{T}) \text{ is strongly connected and for every } x \in V \text{ we have } d^-{}_{D(\mathcal{T})}(x) = d^+{}_{D(\mathcal{T})}(x). \tag{5.7.1}$$

It is well-known this implies $D(\mathcal{T})$ is Eulerian, and therefore $(iii)_\circlearrowright$ will also follow.

Consider an arbitrary tour-trail decomposition $\mathcal{T}_0$ of R. Since R spans at most m vertices, it has at most $\binom{m}{3}$ edges. Since each trail in $\mathcal{T}_0$ contributes with two arcs and uses at least one edge of R, we deduce that the number of arcs in $D(\mathcal{T}_0)$, counting repetitions, is at most $2|E(R)| \leqslant 2\binom{m}{3}$. Let $U \subseteq V$ a subset of vertices disjoint from $V(R)$, since $1/n \ll 1/m$ we can assume that $|U| \geqslant n/2$.

Let $V_1, V_2, \ldots, V_k$ be the strongly connected components of $D(\mathcal{T}_0)$, ignoring isolated vertices. Observe that $k \leqslant m$. For each $1 \leqslant i \leqslant k$, take an arbitrary vertex $v_i \in V_i$, and also take vertices $x_i, y_i \in U$, all distinct. Now, apply Lemma 5.7.4 to obtain the gadget $C_4(v_1, v_2, x_1, y_1)$ and the tour-trail decomposition $\mathcal{T}'$ of $R \cup C_4$ whose residual digraph is given by

$$D(\mathcal{T}') = D(\mathcal{T}) \sqcup \vec{C}_4(v_1, v_2, x_1, y_1).$$

Hence, in $D(\mathcal{T}')$ the vertices v_1 and v_2 are strongly connected (and also the new vertices x_1, y_1).

Since $1/n \ll 1/m$ and the four-cycle gadget spans at most $4\ell - 6$ new vertices we may assume that n is large enough for $k-2$ extra iterative applications of Lemma 5.7.4. Therefore we get edge-disjoint subgraphs $C_4(v_i, v_{i+1}, x_i, y_i)$ for every $1 \leqslant i < k$. Consider $T_1 = \bigcup_{i \in [k-1]} C_4(v_i, v_{i+1}, x_i, y_i)$ and $\mathcal{T}_1$ be the tour-trail decomposition of $R \cup T_1$ given by the the last application of Lemma 5.7.4. By construction, it is easy to see that $D(\mathcal{T}_1)$ is strongly connected. Moreover by $(i)_{C_4}$ and $(ii)_{C_4}$ it follows that T_1 is $C_\ell^{(3)}$-decomposable, has at most $4\ell(k-1) \leqslant 4(m-1)\ell$ edges and $R \cup T_1$ spans at most $m + k(4\ell - 6) \leqslant m + 4(m-1)\ell$ vertices.

For the second part of statement (5.7.1) we proceed as follows. For an arbitrary tour-trail decomposition $\mathcal{T}$ of a hypergraph G, define $\Phi(\mathcal{T}) = \sum_{v \in V(H)} |d^-{}_{D(\mathcal{T})}(v) - d^+{}_{D(\mathcal{T})}(x)|$.

Assume $\Phi(\mathcal{T}_1)$ is positive (otherwise we are done). Since $\mathcal{T}_1$ is obtained from $\mathcal{T}_0$ adding only C_4 gadgets, and since $d^-{}_{\vec{C}_4}(x) = d^+{}_{\vec{C}_4}(x)$ we have that

$$\Phi(\mathcal{T}_1) = \Phi(\mathcal{T}_0) \leqslant 2|E(D(\mathcal{T}_0))| \leqslant 4\binom{m}{3}.$$

Let $x \in V$ such that $d^-{}_{D(\mathcal{T}_1)}(x) \neq d^+{}_{D(\mathcal{T}_1)}(x)$, which exists by assumption. Without loss of generality we can assume $d^-{}_{D(\mathcal{T}_1)}(x) - d^+{}_{D(\mathcal{T}_1)}(x) > 0$, and hence we can find $y \in V$ such that $d^+{}_{D(\mathcal{T}_1)}(y) - d^-{}_{D(\mathcal{T}_1)}(y) > 0$. Observe that by Lemma 5.7.5 we have $d^-{}_{D(\mathcal{T}_1)}(x) - d^+{}_{D(\mathcal{T}_1)}(x) = 3r_1$ and $d^+{}_{D(\mathcal{T}_1)}(y) - d^-{}_{D(\mathcal{T}_1)}(y) = 3r_2$ for some $r_1, r_2 \in \mathbb{Z}^+$.

Selecting any unused vertex $u \in U$, an application of Lemma 5.7.3 yields the existence of a subgraph $S_3(x,u,y) \subseteq H$ such that there is tour-trail decomposition $\mathcal{T}''$ of the hypergraph $R \cup T_1 \cup S_3(x,u,y)$ with residual digraph given by

$$D(\mathcal{T}'') = D(\mathcal{T}_1) \sqcup \vec{S}_3(x,u,y).$$

Thus, we have that

$$d^-{}_{D(\mathcal{T}'')}(x) - d^+{}_{D(\mathcal{T}'')}(x) = 3(r_1 - 1) \quad \text{and} \quad d^+{}_{D(\mathcal{T}'')}(y) - d^-{}_{D(\mathcal{T}'')}(y) = 3(r_2 - 1),$$

This is to say, the absolute difference between the indegree and outdegree of x is reduced by 3, similarly with y. Moreover, for every $z \in V \smallsetminus \{x,y\}$ this difference is not altered, that is,

$$d^-{}_{D(\mathcal{T}'')}(z) - d^+{}_{D(\mathcal{T}'')}(z) = d^-{}_{D(\mathcal{T}_1)}(z) - d^+{}_{D(\mathcal{T}_1)}(z).$$

Therefore, we have $\Phi(\mathcal{T}'') = \Phi(\mathcal{T}_1) - 6$. We further note that $D(\mathcal{T}'')$ is still strongly connected.

As before, since $1/n \ll 1/m$ and S_3 spans at most $2\ell - 3$ new vertices, we may assume that n is large enough to apply Lemma 5.7.3 iteratively $\binom{m}{3}$ times. In each step Φ decreases by 6, so after at most $\frac{4}{6}\binom{m}{3} \leqslant \binom{m}{3}$ applications of Lemma 5.7.3 we can obtain a subgraph $T_2 \subseteq H$, edge disjoint with $R \cup T_1$, and such that $R \cup T_1 \cup T_2$ has a tour-trail decomposition $\mathcal{T}$ with $\Phi(\mathcal{T}) = 0$. In particular, $\mathcal{T}$ satisfies (5.7.1). It is easily checked that $T = T_1 \cup T_2$ and $\mathcal{T}_{\circlearrowleft} = \mathcal{T}$ satisfy $(i)_{\circlearrowleft}$ and $(ii)_{\circlearrowleft}$ as well. $\square$

Now we are ready to prove the main lemma of this subsection.

Proof of Lemma 5.7.1. Let $T \subseteq H$ be given by applying Lemma 5.7.6 and let $\mathcal{T}_{\circlearrowleft}$ be a tour-trail decomposition of $R \cup T$ whose residual digraph is Eulerian. Observe that since each trail in $\mathcal{T}_{\circlearrowleft}$ contributes with two arcs in $D(\mathcal{T}_{\circlearrowleft})$, the number of arcs is even. Let $v_1 v_2 \cdots v_{2k}$ be the sequence of the directed Eulerian tour in $D(\mathcal{T}_{\circlearrowleft})$.

Let $U \subseteq V$ be disjoint from $V(R \cup T)$, and let $C = u_1 u_2 \cdots u_{2k}$ be an arbitrary sequence of vertices in U, where for all $1 \leqslant i \leqslant 2k$, $u_i \neq u_{i+1}$ (here, and during the rest of the proof, indices are understood modulo $2k$). We will apply gadgets to $T \cup R$ to find a new tour-trail decomposition $\mathcal{T}_C$ such that $D(\mathcal{T}_C)$ consists precisely of a closed walk in the sequence C. First, we describe the construction for arbitrary C, then we will give a particular choice of C which will allow us to finish the proof.

Since $1/n \ll 1/m, 1/\ell$ and the gadget C_4 contains at most $4\ell - 6$ new vertices we may assume that n is large enough to apply Lemma 5.7.4 iteratively $2k \leqslant m + m^3\ell$ times. More precisely, assume that after $i-1$ applications of the Lemma 5.7.4 we have obtained a sequence of subgraphs $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_{i-1}$ such that T_{i-1} is edge-disjoint with $R \cup T$. Then, we apply Lemma 5.7.4 with $R \cup T \cup T_{i-1}$ in the place of R to obtain a suitable C_4-gadget, edge-disjoint from $R \cup T \cup T_{i-1}$. We take the next subgraph T_i simply as the union of T_{i-1} and the found gadget.

Let $T_0 = \varnothing$, and for $1 \leqslant i \leqslant 2k$, in the ith application of Lemma 5.7.4 we take

$$T_i = T_{i-1} \cup C_4(v_{i+1}, v_i, u_i, u_{i+1}).$$

We obtain a trail-tour decomposition $\mathcal{T}_{2k}$ whose residual digraph is given by

$$D(\mathcal{T}_{2k}) = D(\mathcal{T}_\circlearrowright) \sqcup \bigsqcup_{i \in [2k]} \vec{C}_4(v_{i+1}, v_i, u_i, u_{i+1}).$$

Observe that, for each $1 \leqslant i \leqslant 2k$, $D(\mathcal{T}_{2k})$ contains both (v_i, v_{i+1}) and (v_{i+1}, v_i), the first contributed by $D(\mathcal{T}_\circlearrowright)$ and the second by $\vec{C}_4(v_{i+1}, v_i, u_i, u_{i+1})$. Similarly, for each $1 \leqslant i \leqslant 2k$, two consecutive cycles will contribute with the edges (v_i, u_i) and (u_i, v_i). Following Remark 5.7.2 we can find a tour-trail decomposition $\mathcal{T}'$ of $R \cup T_k$ whose residual digraph removes all of those edges. What remains are precisely the edges (u_i, u_{i+1}) for all $1 \leqslant i \leqslant 2k$, so $D(\mathcal{T}')$ is the closed walk C as desired.

Now we fix a particular choice of C to finish the proof. We select two distinct vertices $x, y \in U$ and take C such that, for each $1 \leqslant i \leqslant 2k$, $u_i = x$ for odd i, and $u_i = y$ if i is even. Thus the closed walk C consists of k arcs from x to y, and k arcs in the opposite direction. By Remark 5.7.2 again, we can find a tour-trail decomposition $\mathcal{T}'$ of $R \cup T_k$ with an empty residual digraph. It is easy to check that we are done by setting $A_1 = T_k$. $\square$

5.7.2 From a tour decomposition to a cycle decomposition

In this section we prove the following lemma, which constructs an absorber given a $C_\ell^{(3)}$-divisible remainder which has a tour decomposition.

Lemma 5.7.7. *Let $\ell \geqslant 7$, $\varepsilon > 0$, and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a hypergraph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Let $R \subseteq H$ be a $C_\ell^{(3)}$-divisible edge-disjoint collection of tours spanning at most m vertices in total. Then, there is a $C_\ell^{(3)}$-absorber A_2 for R, such that $A_2 \cup R$ spans at most $10\binom{m}{3}\ell^2$ edges.*

Given two subgraphs R_1 and R_2, we say that a subgraph $T \subseteq H$ edge-disjoint from R_1 and R_2 is a (R_1, R_2)-*transformer* if $T[V(R_1)], T[V(R_2)]$ are empty and both $T \cup R_1$ and $T \cup R_2$ contain a $C_\ell^{(3)}$-decomposition. Observe that if R_2 has a $C_\ell^{(3)}$-decomposition, then $T \cup R_2$ is an absorber for R_1.

Lemma 5.7.8. *Let $\ell \geqslant 7$, $\varepsilon > 0$, and $n, m \in \mathbb{N}$ be such that $1/n \ll \varepsilon, 1/m, 1/\ell$. Let H be a hypergraph on n vertices with $\delta_2(H) \geqslant (2/3 + \varepsilon)n$. Let $R \subseteq H$ be a tour and $C \subseteq H$ be a cycle. Suppose that R and C are edge-disjoint and have the same number of edges, which is at most m. Then H contains an (R, C)-transformer L with at most $m\ell$ edges and spanning at most $m(\ell - 4)$ vertices.*

Proof. Let $r_1, r_2, \dots, r_m$ and $c_1, c_2, \dots, c_m$ the sequence of vertices of R and C respectively (recall that while C does not contain repetitions, R may contain).

In the following, all operations on the indices are modulo m. We define iteratively the following paths P_i, Q_i for every $i \in [m]$. Apply Lemma 5.4.1 to obtain a path P_i on 5 vertices, edge-disjoint from $R \cup C$, from the pair (r_i, r_{i+1}) to the pair (c_{i-1}, c_i). Similarly, we can obtain a path Q_i on $\ell - 5$ vertices, from the pair (r_i, r_{i-1}) to the pair (c_i, c_{i-1}), edge disjoint from $R \cup C$, and with no interior vertex in common with the paths P_i, P_{i-1}.

We claim that $L = \bigcup_{i \in [m]} (P_i \cup Q_i)$ is the desired transformer. Indeed, observe that the edges of P_i and Q_i together with the edge $r_{i-1}r_ir_{i+1} \in E(R)$ form a cycle of length ℓ, thus $R \cup L$ can be decomposed into those ℓ-cycles. In the same way, the edges of P_{i-1}

and Q_i together with the edge $c_{i-2}c_{i-1}c_i \in E(C)$ form a cycle of length ℓ, and therefore all those cycles form a $C_\ell^{(3)}$-decomposition of $C \cup L$. $\square$

For any $k, \ell \in \mathbb{N}$ we define $B(k, \ell)$ to be the hypergraph resulting from a cycle of length $k\ell$ with vertices in $\{v_1, v_2, \ldots, v_{k\ell}\}$ and identifying all vertices v_i such that $i \equiv 1 \bmod \ell$ and all vertices v_j such that $j \equiv 2 \bmod \ell$. This is to say that $B(k, \ell)$ consists of k copies of cycles of length ℓ glued through exactly two vertices, and those two vertices are consecutive in every cycle. Observe that $B(k, \ell)$ is a tour and admits a $C_\ell^{(3)}$-decomposition.

Now we are ready to prove Lemma 5.7.7.

Proof of Lemma 5.7.7. Consider the tours $T_1, T_2, \ldots, T_k$ in R and note that $k \leqslant \binom{m}{3}/4$ (each tour has at least 4 edges). First, we want to reduce the proof to the case in which there is a single long tour. Suppose $k \geqslant 2$ and take a_i, b_i two consecutive vertices in T_i for $i = \{1, 2\}$. We can apply Lemma 5.4.1 to find a path P_1 on 5 vertices with tails (b_1, a_1) and (a_2, b_2) which is edge-disjoint to R. Similarly, we can find P_2 on $\ell - 5$ vertices with tails (a_1, b_1) and (b_2, a_2), edge-disjoint with R, and sharing no interior vertex with P_1. Starting in (a_1, b_2) and then traversing sequentially T_1, P_1, T_2, and P_2, one can check that $T_1 \cup T_2 \cup P_1 \cup P_2$ forms a tour spanning at most $|V(T_1 \cup T_2)| + \ell - 4$ vertices. Moreover, it is easy to see that $P_1 \cup P_2$ is a cycle of length ℓ. By repeating this argument we can obtain $A' \subseteq H$ edge-disjoint from R, $C_\ell^{(3)}$-decomposable, and such that $R' = R \cup A'$ consists of a single tour spanning at most $m + k(\ell - 4)$ vertices. Observe that since R is $C_\ell^{(3)}$-divisible, then so is R'. Let m' be the number of edges in R' and notice that

$$m' \leqslant \binom{m}{3} + k\ell \leqslant 2\binom{m}{3}\ell\,.$$

Second, observe that by several applications of Lemma 5.4.1 we can find two edge-disjoint subgraphs $B, C \subseteq H$, vertex-disjoint to each other, both of them edge-disjoint with R', and such that B is a copy of $B(m'/\ell, \ell)$ and C is a cycle of length m' (observe that ℓ divides m' since R' is $C_\ell^{(3)}$-divisible).

Now two suitable applications of Lemma 5.7.8 yield the result. More precisely, first apply Lemma 5.7.8 with R' in the rôle of R to obtain a (R', C)-transformer

$L_1 \subseteq H$ with at most $m'\ell$ edges. For the second application of Lemma 5.7.8 observe that, since $R' \cup L_1$ contain at most $m'(\ell+1)$ we may assume n is large enough so that $\delta_2(H \smallsetminus (R' \cup L_1)) \geqslant (2/3 + \varepsilon/2)n$. Hence, another application of Lemma 5.7.8 now with B in the rôle of R and $H \smallsetminus (R' \cup L_1)$ in the rôle of H yields the existence of a (B, C)-transformer $L_2 \subseteq H$ edge disjoint with $R' \cup L_1$.

Putting all this together, and recalling that both A' and B contain a $C_\ell^{(3)}$-decomposition, we have that the hypergraphs

$$R \cup A' \cup L_1 \cup C \cup L_2 \cup B \quad \text{and} \quad A' \cup L_1 \cup C \cup L_2 \cup B$$

contain $C_\ell^{(3)}$-decompositions. To finish the proof take $A_2 = A' \cup L_1 \cup C \cup L_2 \cup B$ and observe that each of the hypergraphs A', L_1, L_2, C, and B contain at most $m'\ell \leqslant 2\binom{m}{3}\ell^2$ edges. □

5.7.3 Proof of Lemma 5.3.2

We can finally give the short proof of Lemma 5.3.2.

Proof of Lemma 5.3.2. Given $R \subseteq H$, an application of Lemma 5.7.1 yields the existence of $A_1 \subseteq H$ edge disjoint from R such that

(*i*) A_1 has at most $5m^3\ell^2$ edges,

(*ii*) $A_1 \cup R$ spans at most $m + 5m^3\ell^2$ vertices,

(*iii*) A_1 has a $C_\ell^{(3)}$-decomposition, and

(*iv*) $A_1 \cup R$ has a tour decomposition.

Then, we apply Lemma 5.7.7 to obtain $A_2 \subseteq H$, which is an absorber of $R \cup A_1$. It is straightforward to check that $A = A_1 \cup A_2$ has the desired properties. □

5.8 Final remarks

A natural question is what happens for the values of ℓ not covered by our Theorem 2.3.1. Our results do not cover $C_\ell^{(3)}$-decompositions for small values of ℓ, i.e. $\ell \leqslant 8$. As in the graph case, for short cycles it is likely that the behaviour of the decomposition threshold is different.

For $\ell = 4$ the cycle $C_4^{(3)}$ is isomorphic to a tetrahedron $K_4^{(3)}$. Since every pair of vertices in $K_4^{(3)}$ has degree 2, the obvious necessary divisibility conditions in a host hypergraph which admits a $C_4^{(3)}$-decomposition are

(*i*) total number of edges divisible by 4,

(*ii*) every vertex degree divisible by 3, and

(*iii*) every codegree divisible by 2.

Say that a hypergraph satisfying all three conditions is $K_4^{(3)}$-*divisible*. We define $\delta_{K_4}^{(3)}$ as the asymptotic minimum codegree threshold ensuring a $K_4^{(3)}$-decomposition over $K_4^{(3)}$-divisible hypergraphs (in analogy to $\delta_{C_\ell}^{(3)}$ taken over $C_\ell^{(3)}$-divisible hypergraphs). The following construction shows that $\delta_{K_4^{(3)}} \geqslant 3/4$.

Example 5.8.1. *Let $k \geqslant 1$ be arbitrary, $d = 6k+2$ and $n = 12k+9$. Let G_1 be an arbitrary d-regular graph on n vertices. Let G be the graph on $2n$ vertices obtained by taking two vertex-disjoint copies of G_1 and adding every edge between vertices belonging to different copies, say those edges are* crossing. *Now, form a hypergraph H as follows. Take a set Z on $2n$ vertices and edges forming a complete 3-uniform graph on Z. Then add two new vertices x_1, x_2. For each $z \in Z$, add the edge x_1x_2z. Identify a copy of the graph G in Z and, for each edge z_1z_2 of G add the edges $z_1z_2x_1$ and $z_1z_2x_2$.*

The hypergraph H has $2n+2 = 24k+20$ vertices and $\delta_2(H) = d+n+1 = 18k+12$ (attained by any pair x_1z with $z \in Z$). It is tedious but straightforward to check H is $K_4^{(3)}$-divisible. To see H is not $K_4^{(3)}$-decomposable, we prove that the link graph $H(x_1)$ is not $C_3^{(2)}$-decomposable. Note $H(x_1)$ is isomorphic to the graph G' obtained from G

by adding an extra universal vertex x. Suppose G' has a triangle decomposition. There are n^2 crossing edges in G, at most n of those can be covered with triangles using x. Thus at least $n(n-1)$ crossing edges are covered with triangles which use one edge in a copy of G_1 and two crossing edges. Thus we need at least $n(n-1)/2$ edges in the two copies of G_1, but this is a contradiction since those copies have $dn < n(n-1)/2$ edges.

What is the smallest ℓ_0 such that $\delta^{(3)}_{C_\ell} = 2/3$ holds for all $\ell \geqslant \ell_0$? The previous example and Theorem 2.3.1 show that $5 \leqslant \ell_0 \leqslant 10^7$. Observe that our Absorbing Lemma works for all $\ell \geqslant 7$. The bottleneck is our use of Theorem 5.6.4 in the Cover-Down Lemma. New ideas are needed to close the gap.

Another natural question asks for optimal codegree conditions for cycle decomposition in k-uniform hypergraphs when $k \geqslant 4$. It is not clear for us if Theorem 2.3.4 indicates the emergence of a pattern where the necessary codegree to ensure cycle decompositions and Euler tours on n-vertex k-uniform hypergraph is substantially larger than $(1/2 + o(1))n$.

Question 5.8.1. Let $k \geqslant 4$. Is there a constant $\eta > 0$ and a k-uniform hypergraph H with $\delta_{k-1}(H) \geqslant \left(\frac{1}{2} + \eta + o(1)\right) n$ not containing a cycle decomposition or an Euler tour?

Problem 5.8.2. *Let $k \geqslant 4$, determine $\delta^{(k)}_{C_\ell}$ for sufficiently large ℓ.*

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Chapter 6

Appendix

6.1 English Summary

In this thesis we proved three results for 3-uniform dense hypergraphs. In each case, we determined conditions for the existence of different kinds of substructures.

In the first one, we show that 3-uniform hypergraphs with the property that all vertices have a quasirandom link graph with density bigger than 1/3 contain a clique on five vertices. This result is asymptotically best possible. With this, we solved an open problem left by Reiher, Rödl, and Schacht [55] about Turán densities in uniformly dense hypergraphs.

For the second problem, we study sufficient conditions for the existence of Hamilton cycles in uniformly dense 3-uniform hypergraphs. Problems of this type were first considered by Lenz, Mubayi, and Mycroft [41] for loose Hamilton cycles and Aigner-Horev and Levy [2] considered it for tight Hamilton cycles for a fairly strong notion of uniformly dense hypergraphs. We focus on tight cycles and obtain optimal results for a weaker notion of uniformly dense hypergraphs. We show that if an n-vertex 3-uniform hypergraph $H = (V, E)$ has the property that for any set of vertices X and for any collection P of pairs of vertices, the number of hyperedges composed by a pair belonging to P and one vertex from X is at least $(1/4 + o(1))|X||P| - o(|V|^3)$ and H has minimum vertex degree at least $\Omega(|V|^2)$, then H contains a tight Hamilton cycle. A probabilistic construction shows that the constant $1/4$ is optimal in this context.

Finally, we show that 3-uniform hypergraphs on n vertices whose codegree is at least $(2/3 + o(1))n$ can be decomposed into tight cycles subject to the trivial necessary divisibility conditions. This result can be used to prove the existence of a tight Euler tour under the same minimum codegree condition. We provide a construction showing that our bounds are best possible up to the $o(1)$ term. All together, our results address recent open problems by Glock, Kühn, and Osthus [29].

6.2 German Summary

In dieser Arbeit werden asymptotisch bestmögliche hinreichende Bedingungen untersucht, welche die Existenz gegebener Unterstrukturen eszwingen.

Im ersten teil zeigen wir, dass 3-uniforme Hypergraphen mit der Eigenschaft, dass alle Ecken einen quasi-zufälligen Linkgraphen mit einer Dichte größer als $1/3$ haben, eine Clique auf fünf Ecken enthalten. Dieses Ergebnis ist asymptotisch bestmöglich. Dies beantwortet eine Frage von Reiher, Rödl und Schacht [55] über Turán-Dichten in gleichmäßig dichten Hypergraphen.

Für das zweite Problem untersuchen wir hinreichende Bedingungen für die Existenz von Hamiltonkreisen in gleichmäßig dichten 3-uniformen Hypergraphen. Probleme dieser Art wurden zuerst von Lenz, Mubayi und Mycroft [41] für lose Hamiltonkreisen und von Aigner-Horev und Levy [2] für enge Hamiltonkreisen für eine ziemlich starke Definition von gleichmäßig dichten Hypergraphen untersucht. Wir konzentrieren uns auf enge Kreise und erhalten optimale Ergebnisse für eine schwächere Definition von gleichmäßig dichten Hypergraphen. Wir zeigen, dass wenn ein 3-uniformer Hypergraph $H = (V, E)$ mit n Ecken die Eigenschaft hat, dass für eine beliebige Menge von Ecken X und für eine beliebige Menge P von Eckenpaaren die Anzahl der Hyperkanten, die aus einem zu P gehörenden Paar und einer Ecke von X zusammengesetzt sind, mindestens $(1/4 + o(1))|X||P| - o(|V|^3)$ beträgt und H einen minimalen Eckengrad von mindestens $\Omega(|V|^2)$ hat, dann enthält H einen engen Hamiltonkreis. Eine probabilistische

Konstruktion zeigt, dass die Konstante 1/4 in diesem Zusammenhang optimal ist.

Schließlich zeigen wir, dass 3-uniforme Hypergraphen auf n Ecken, deren Eckenpaargrad mindestens $(2/3 + o(1))n$ ist, unter den trivialen notwendigen Teilbarkeitsbedingungen in enge Kreise zerlegt werden können. Dieses Ergebnis kann verwendet werden, um die Existenz einer engen Eulertour zu beweisen. Wir liefern auch eine Konstruktion, die zeigt, dass unsere Schranken bis zum Term $o(1)$ bestmöglich sind. Dies adressiert ein offenes Problem von Glock, Kühn und Osthus [29].

6.3 Publications related to this dissertation

Articles and preprints

P. Araújo, S. Piga, and M. Schacht, *Localised codegree conditions for tight Hamilton cycles in* 3*-uniform hypergraphs* (2020), to appear in SIAM Journal on Discrete Mathematics. Available at arXiv:2005.11942. (See [5]).

S. Berger, S. Piga, Chr. Reiher, V. Rödl, and M. Schacht, *Turán density of cliques of order five in* 3*-uniform hypergraphs with quasirandom links*, manuscript (39 pages) (see [10]).

S. Piga and N. Sanhueza-Matamala, *Cycle decompositions in* 3*-uniform hypergraphs* (2021), available at arXiv:2101.12205. (28 pages) (see [44]).

Extended abstracts

P. Araújo, S. Piga, and M. Schacht. *Localised codegree conditions for tight Hamilton cycles in* 3*-uniform hypergraphs*, Acta Mathematica Universitatis Comenianae, 88(3) (2019), 389-394.

S. Berger, S. Piga, Chr. Reiher, V. Rödl, and M. Schacht, *Turán density of cliques of order five in* 3*-uniform hypergraphs with quasirandom links*, to appear in Procedia Computer Science, Proceedings of LAGOS 2021.

S. Piga and N. Sanhueza-Matamala, *Cycle decompositions in* 3*-uniform hypergraphs* (2021), to appear in Procedia Computer Science, Proceedings of LAGOS 2021.

6.4 Declaration of Contributions

This thesis in mainly based on [5], [10], and [44], and most of the work was shared equally among the coauthors.

The results in Chapter 3 follow up a work by Reiher, Rödl, and Schacht [10]. Together with Berger and Schacht, we focused on a two colour case which seemed to capture an interesting 'toy version' of the full problem. We finally arrived to a proof of Proposition 3.1.6. The proof of the reduction to the two colour case (see Proposition 3.1.3) was only found after Reiher joined the group. The extended abstract and the initial version of the manuscript was essentially written by me, with the proofreading of my advisor Mathias Schacht and fellow PhD. Sören Berger. The other authors are currently commenting the last version of the manuscript.

While Araújo was visiting Hamburg University we read [2] by Aigner-Horev and Levy and found the problem that we finally solved in Chapter 4. After discussing the problem with Araújo and Schacht, I discovered a construction which proves that the problem was not 'degenerate' as initially expected. In fact, this construction turned out to be optimal, which was established in Theorem 2.2.2. The first draft of the manuscript was drafted by Araújo and me. Later Schacht proofread that version and the two of us polish the last details of the paper together.

The problem solved in Chapter 5 was motivated by a conjecture by Glock, Kühn, and Osthus [29] and was presented to me by my coauthor Nicolás Sanhueza-Matamala.

Most of the work was concentrated on Lemma 5.3.2 and we were able to prove it after several meetings. In particular, in the process of finding that proof, we arrived to a construction that refutes the initial conjectures in [29] and [26] (see Theorem 2.3.4). We announced the result in an extended abstract, which we wrote together (this included the proof of Theorem 2.3.4). We divided the responsibilities for writing first drafts of the other proofs. I wrote the initial versions of the proofs of Theorem 2.3.1 based on 'iterative absorption' (assuming all lemmata in which it relies) and Lemma 5.3.2. Later we both proofread the whole article and included some minor changes.

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6.6 Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, die vorliegende Dissertation selbst verfasst und keine anderen als die angegebenen Hilfsmittel benutzt zu haben. Darüber hinaus versichere ich, dass diese Dissertation nicht in einem früheren Promotionsverfahren eingereicht wurde.