On cofinal functors of ∞-bicategories

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Declaration of original work

This thesis contains results which have appeared in the following list of publications:

- [AGDS20]: F. Abellán García, T. Dyckerhoff & W.H. Stern. *A relative 2-nerve.* Algebr. Geom. Topol. 20-6 (2020), 3147-3182
- [AGS22]: F. Abellán García & W.H. Stern. *Theorem A for marked 2-categories.* J. Pure & Applied Algebra, Vol. 226, Issue 9, 2022
- [AG22]: F. Abellán García. *Marked colimits and higher cofinality.* Homotopy Relat. Struct. 17, 1–22 (2022).
- [AGS21]: F. Abellán García & W.H. Stern. *Enhanced twisted arrow categories.* arXiv:2009.11969
- [AGS22I]: F. Abellán García & W.H. Stern. *2-Cartesian fibrations I: A model for ∞-bicategories fibred in ∞-bicategories.* arXiv: 2106.03606
- [AGS22II]: F. Abellán García & W.H. Stern. *2-Cartesian fibrations II: Higher cofinality.* arXiv: 2201.09589

I declare that I have made substantial and original contributions to all parts of the above joint works and that the information derived from the literature which appears in this thesis has been clearly referenced. The main results of this thesis appeared originally in the preprints [AGS22I], [AGS22II] (which correspond to chapters 2-4 in this document) with the exception of the last chapter which contains unpublished original work. The results presented in the aforementioned publications generalize [AGDS20], [AGS22] and [AG22]. Moreover, the proof of the main theorem of this thesis Theorem 4.0.31, is the result of a generalization of the results I developed in the single authored paper [AG22]. Whereas most of the work has been done in joint publications the core ideas and proofs of this thesis were already developed in [AG22]. I declare that my contributions constitute 60% of the results of the aforementioned papers and that none of the works above are part of the doctoral dissertation of Walker H. Stern.

Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel – insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen – benutzt habe. Alle Stellen, die wörtlich oder sinngemäß aus Veröffentlichungen entnommen wurden, sind als solche kenntlich gemacht. Ich versichere weiterhin, dass ich die Arbeit vorher nicht in einem anderen Prüfungsverfahren eingereicht habe und die eingereichte schriftliche Fassung der auf dem elektronischen Speichermedium entspricht.

Hamburg, den 20. April 2022

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Summary

In this thesis we investigate the notion of cofinal functor of ∞-bicategories and establish foundational results in the theory of ∞-bicategories along the way.

We start with an introductory section where we present and motivate the main results achieved to later move into the main body of the thesis which is structured as follows:

- In Chapter 1 we review the relevant $(\infty, 1)$-categorical theory that will be later generalized to the $(\infty, 2)$-categorical realm.
- In Chapter 2 we construct a model structure on the category of marked biscaled simplicial sets over a scaled simplicial set S which models outer 2-Cartesian fibrations: An $(\infty, 2)$-categorical upgrade of the notion of Cartesian fibration.
- In Chapter 3 we prove an ∞-bicategorical Grothendieck construction relating outer 2-Cartesian fibrations and contravariant functors with values in ∞-bicategories.
- In Chapter 4 we characterize cofinal functors of ∞-bicategories via generalizations of the conditions of Quillen's Theorem A.
- In Chapter 5 we provide applications of our cofinality criterion as well pointing out the next steps in the research programme of the author.

Zusammenfassung

In dieser Arbeit untersuchen wir den Begriff eines kofinalen Funktors zwischen ∞-Bikategorien. Weiterhin zeigen wir grundlegende Resultate in der Theorie von ∞-Bikategorien.

Nach der Einleitung, in der die Hauptresultate motiviert und gesammelt sind, folgt der Hauptteil der Arbeit, der wie folgt strukturiert ist:

- In Kapitel 1 führen wir das nötige $(\infty, 1)$-kategorielle Hintergrundmaterial ein, welches in den späteren Kapiteln auf $(\infty, 2)$-Kategorien verallgemeinert wird.
- In Kapitel 2 konstruieren wir eine Modellstruktur auf der Kategorie der markierten biskalierten simplizialen Mengen über einer skalierten simplizialen Menge. Diese Modellstruktur modelliert 2-Cartesische Faserungen und ist daher eine $(\infty, 2)$-kategorielle Verallgemeinerung des Begriffs einer Kartesischen Faserung.
- In Kapitel 3 beschreiben wir eine ∞-bikategorielle Grothendieck-Konstruktion, welche äußere 2-Kartesische Faserungen und kontravariante Funktoren mit Werten in ∞-Bikategorien in Verbindung setzt.
- In Kapitel 4 charakterisieren wir kofinale Funktoren zwischen ∞-Bikategorien über eine Verallgemeinerung von Quillens Theorem A.
- In Kapitel 5 beschreiben wir Anwendungen des Konfinalitätskriteriums und schildern die nächsten Schritte im Forschungsprogramm des Autors.

Contents

Introduction

In this thesis we study colimits in ∞-bicategories[1] and the universal properties that characterize them. This section begins with an informal discussion of the notion of universal property and its relevance throughout mathematics. We discuss the theory of ∞-categorical colimits and cofinality to motivate the main result of this document: A characterization of cofinal functors of ∞-bicategories.

Spaces, categories, universal properties and coherence

Since its early days [EM45], an important contribution of category theory is the notion of *universal property*. Informally speaking, universal properties can be understood as the characterizing features of a mathematical object. This fundamental property of a mathematical object is sufficient to determine it up to adequate notion of equivalence, in other words, if a pair of mathematical objects X and Y satisfy a given universal property then both objects must be equivalent $X \sim Y$, even more, they must be equivalent in a *canonical way*. Let us illustrate this concept with the following elementary example: Suppose that we are given three sets A, B, X and a pair of functions $i_A : A \to X$ and $i_B : B \to X$. We will further suppose that X satisfies the following property

- For every pair of functions $f_A : A \to Y$, $f_B : B \to Y$ there exists a *unique* map $f : X \to Y$ such that $f_A = f \circ i_A$ and such that $f_B = f \circ i_B$.

A direct inspection reveals that any such set X as above must be isomorphic to the disjoint union $A \coprod B$. The main advantage of this a priori seemly complicated definition of the disjoint union of two sets is that it can be implemented in any category. For example, if we assume that A, B, X are all R-algebras with R a commutative ring and that all of the morphisms above are morphisms of R-algebras then X is equivalent to the tensor product $A \otimes_R B$. Consequently, we see that by abstracting the essential information present in a mathematical construction using the language of category theory we can implement a certain definition in a wide variety of contexts. This has allowed a fruitful interplay between different areas of mathematics. For example, in algebraic topology where topological spaces are studied by means of algebraic methods, the language of category theory was quickly adopted as an efficient way of dealing with the theory of algebraic invariants.

A central source of universal properties is the theory of colimits in categories. Given a functor $F : C \to A$, the colimit of F is an object $\operatorname{colim}_A F \in A$ satisfying

[1] We will use the terminology ∞-category (resp. ∞-bicategory) to denote $(\infty, 1)$-categories (resp. $(\infty, 2)$-categories).

a certain universal property which we will explain later in the introduction. This notion is defined in such a way that it is invariant with respect to isomorphisms of functors. For example, if $F, G : C \to \mathrm{Top}$ are functors with values in topological spaces and we have a natural transformation $\alpha : F \Rightarrow G$ which is pointwise an homeomorphism then we have a homeomorphism of spaces $\mathrm{colim}_C F \simeq \mathrm{colim}_C G$. However, from the point of view of homotopy theory where we are only interested in spaces up to weak homotopy equivalence, the notion of equivalence of functors is given by natural transformations which are pointwise weak homotopy equivalences. Therefore, in order for the universal property of the colimit to be meaningful in the homotopical sense both colimits should be *homotopy equivalent* whenever both functors are equivalent in this weaker sense. Unfortunately, the classical theory of colimits is unable to capture this homotopical behaviour (it rather fails spectacularly at doing so) as witnessed by the next example.

Let Λ^2_0 be the category consisting in 3 objects $0, 1, 2$ and morphisms $0 \to 1$ and $0 \to 2$. Suppose we are given a functor $F : \Lambda^2_0 \to A$ which we diagrammatically represent as

$$\begin{array}{ccc} X_0 & \xrightarrow{f_{01}} & X_1 \\ \big\downarrow{\scriptstyle f_{02}} & & \\ X_2 & & \end{array}$$

The colimit of F is given by an object $P \in A$ together with a pair of morphisms $u : X_1 \to P$ and $v : X_2 \to P$ such that $u \circ f_{01} = v \circ f_{02}$ satisfying the following universal property

- For every pair of morphisms $\theta : X_1 \to Y$, $\nu : X_2 \to Y$ such that $\theta \circ f_{01} = \nu \circ f_{02}$ there exists a *unique* morphism $\gamma : X \to Y$ such that $\theta = \gamma \circ u$ and such that $\nu = \gamma \circ v$.

Let us consider a couple of diagrams in $G_i : \Lambda^2_0 \to \mathrm{Top}$ for $i = 1, 2$ given by

$$\begin{array}{ccc} \{*\} \coprod \{*\} & \xrightarrow{\iota} & I \\ \big\downarrow{\scriptstyle \iota} & & \\ I & & \end{array} \qquad\qquad \begin{array}{ccc} \{*\} \coprod \{*\} & \longrightarrow & \{*\} \\ \big\downarrow & & \\ \{*\} & & \end{array}$$

where I denotes the unit interval, $\{*\}$ denotes the point and ι is the inclusion of the end-points. It is immediate to see that both diagrams are weakly equivalent since the interval is a contractible space. A simple inspection now reveals that the colimit of G_1 is homeomorphic to S^1 whereas the the colimit of G_2 is again a point.

There are various ways of solving this issue which involve producing homotopical upgrades of the ordinary theory of categories. In this document we will make use of two of these approaches: Model categories and ∞-categories.

Quillen ([Qui67]) introduced model categories in order to give an axiomatic framework to study homotopy theory in categories. The main idea behind of that of a model category, is that certain categories (for example, Top) can be equipped with distinguished classes of morphisms which allow us to perform homotopy theoretic constructions in that particular category. These classes of morphisms are abstractions of their topological counterparts: Inclusions of CW-complexes, fibrations and weak homotopy equivalences. There is a robust theory of colimits in model categories, homotopy colimits, which addresses completely the question of invariance up to weak equivalence.

Let $F : \Lambda^2_0 \to \mathrm{Top}$ be a diagram with values in CW-complexes. The theory of homotopy colimits tells us that in order to compute the homotopy colimit we must replace one of the morphisms in the diagram with an inclusion of CW-complexes and then compute the ordinary colimit. Going back to the previous example we see that the colimit of G_1 already represents the homotopy colimit. To compute the homotopy colimit of G_2 we replace the original diagram with

$$\begin{array}{ccc} \{*\} \coprod \{*\} & \xrightarrow{\iota} & I \\ \downarrow & & \\ \{*\} & & \end{array}$$

and compute the ordinary colimit, which is as expected given by S^1.

Although model categories are up to this day a useful tool in category theory (and will be used in the coming chapters) this thesis is concerned with a different way of solving the previous problem: ∞-categories. Let $\mathcal{S} \subset \mathrm{Top}$ be the full subcategory of CW-complexes. An essential feature of $\mathcal{S}$ is that given a weak homotopy equivalence $f : X \to Y$ then there exists a morphism $g : Y \to X$ which is a homotopy inverse to f. Observe that the notion of homotopy can be understood as a 2-morphism: A morphism between morphisms. Moreover, since we can reverse the direction of the homotopy it follows that each homotopy is in fact an invertible 2-morphism. We can keep extracting higher categorical information from $\mathcal{S}$ by observing that we can produce homotopies between homotopies and so on. As it happened before, these n-morphisms are always invertible for $n \geqslant 2$. Abstracting this phenomena present in the category of spaces we can give the next informal definition of an ∞-category:

Definition. *An ∞-category $\mathcal{C}$ consists in a collection of objects and for every $n \geqslant 1$ a collection of n-morphisms such that every n-morphism with $n \geqslant 2$ is invertible.*

The theory of ∞-categories offers a vast generalization to the theory of ordinary categories. However, in order to have access to this greater generality we must rebuild the classical theory in a way that takes into account the existence of the higher dimensional morphisms. Whereas in ordinary category theory we mostly deal with morphisms and equations among them in form of commutative diagrams, the ∞-categorical theory considers diagrams commuting up to higher-dimensional coherence data. The success of this homotopy-coherent approach to category theory can be partly attributed to its ability to capture sophisticated universal properties in a homotopy invariant way.

Returning to our example let us try to understand how to compute the colimit of G_2 using the theory of ∞-categories. In this context, the ∞-categorical colimit of a functor $F : \Lambda^2_0 \to \mathcal{S}$ is given by an object $X \in \mathcal{S}$ fitting into a (homotopy-coherent) diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f_{01}} & X_1 \\ {\scriptstyle f_{02}}\downarrow & \searrow^{\theta} & \downarrow{\scriptstyle \alpha_1} \\ X_2 & \xrightarrow{\alpha_2} & X \end{array}$$

where $X_i = F(i)$, $f_{ij} = F(i \to j)$ and such that θ is homotopic to $\alpha_1 \circ f_{01}$ and to $\alpha_2 \circ f_{02}$. In order for the previous diagram to represent the ∞-categorical colimit it must satisfy a certain universal property. We will not give a precise definition of the ∞-categorical universal property of the colimit in the introduction but we will illustrate it by analyzing the case where our diagram is given by $F = G_2$.

Let us suppose we are given a homotopy-coherent diagram

$$\begin{array}{ccc} \{*\} \coprod \{*\} & \xrightarrow{t} & * \\ {\scriptstyle t}\downarrow & \searrow^{\theta} & \downarrow{\scriptstyle \alpha_1} \\ * & \xrightarrow{\alpha_2} & X. \end{array}$$

Note that a homotopy between $\alpha_i \circ t$ and θ is precisely given by a morphism $u : I \to X$ such that:

- The restriction of $u : I \to X$ to the endpoints equals θ.
- The value of u at the middle point of I equals α_i.

Combining both homotopies we obtain a morphism $S^1 \to X$. This shows that we have a canonical choice of colimit diagram

$$\begin{array}{ccc} \{*\} \coprod \{*\} & \xrightarrow{t} & * \\ {\scriptstyle t}\downarrow & \searrow^{\theta} & \downarrow{\scriptstyle \alpha_1} \\ * & \xrightarrow{\alpha_2} & S^1. \end{array}$$

such that any other homotopy-coherent diagram is determined up to homotopy by a morphism $S^1 \to X$. Note that in this approach the use of higher dimensional morphisms (homotopies) is the key ingredient in order to show that S^1 satisfies the ∞-categorical universal property of the colimit.

There exist several equivalent models for defining ∞-categories. We will mostly be interested in the model given by simplicial sets satisfying the inner horn lifting condition (see [Lur09a, Definition 1.1.2.4]). Recall that the simplex category Δ is the full subcategory of Cat, the ordinary 1-category of categories, spanned by the posets

$$[n] = \{0 < 1 < \cdots < n\}, \quad \text{for } n \geqslant 0$$

The category of simplicial sets Set_Δ is defined to be the presheaf category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$. We define the (ordinary category) of ∞-categories denoted by Cat_∞ as the full subcategory on those simplicial sets satisfying the inner horn lifting condition. The next definition allows us to formalize that ∞-categories generalize ordinary 1-categories.

Definition. *There is a fully faithful functor* $\mathrm{N} : \mathrm{Cat} \to \mathrm{Cat}_\infty$ *which sends a category* C *to the simplicial set* $\mathrm{N}(C)$ *whose* n*-simplices are given by functors* $[n] \to C$*.*

Up to this point, we have considered spaces simply as a source of inspiration for giving a sensible definition of what an ∞-category should be. It is natural then to ask ourselves the next question:

Q: What is the role of the ∞-category of spaces in the theory of ∞-categories?

The answer to this question is a particularly satisfactory one: Spaces are to ∞-categories what sets are to ordinary 1-categories. Let us unravel the last claim by carefully translating what sets are into relation to 1-categories into the ∞-categorical realm.

Sets	Spaces
The category Set is the prototypical example of 1-category	The ∞-category $\mathcal{S}$ is the prototypical example of ∞-category
For every pair of objects x, y in a category C we have a set $\mathrm{Hom}_C(x, y)$ of morphisms from x to y	For every of objects x, y in an ∞-category $\mathcal{C}$ we have a space $\mathcal{C}(x, y)$ of morphisms from x to y

We further see that we can view the category Set as the full subcategory of Cat on those categories such that each morphism is an identity morphism. Similarly, we can be view $\mathcal{S}$ as the full subcategory of the ∞-category of ∞-categories $\mathbb{C}\mathrm{at}_\infty$, consisting in those ∞-categories such that each morphism is given by an equivalence. We summarize this discussion in the following diagram

$$\begin{array}{ccc} \mathrm{Set} & \longrightarrow & \mathrm{Cat} \\ \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & \mathbb{C}\mathrm{at}_\infty \end{array} \tag{1}$$

where the vertical maps are all fully-faithful inclusions of ∞-categories. Later in this document we will extend the following commutative diagram to accommodate the 2-dimensional theory. We will view ∞-categories a as an special instance of an even more general entity called an ∞-bicategory which also simultaneously generalizes the classical notion of 2-category.

Colimits in ∞-categories

From ordinary colimits to ∞-categorical colimits

To better understand, how the theory of ∞-categorical colimits relates to the strict 1-categorical theory let us start by reviewing the definition of a colimit in an ordinary category.

Let $F : C \to A$ be a functor of ordinary categories and recall that a cone for F is given by the following data

- An object $a \in A$ known as the "tip of the cone".

- For every $c \in C$ a morphism $\alpha_c : F(c) \to a$ such that for every morphism $e : c \to c'$ the equality $\alpha_c = \alpha_{c'} \circ F(e)$ holds.

We will denote the data of a cone as pair $(a, \{\alpha_c\}_{c \in C})$. The colimit of F is the *universal cone* $(a, \{\alpha_c\}_{c \in C})$ satisfying the following universal property:

$*$ For every cone $(b, \{\beta_c\}_{c \in C})$ there exists a unique morphism $u : a \to b$ in A such that $\beta_c = u \circ \alpha_c$ for every $c \in C$.

Let $\underline{*}_C : C \to \mathcal{S}$ be the constant functor that sends each object to one-point space Δ^0. For simplicity, we will assume that C is an ordinary category. It follows by direct inspection that the colimit of $\underline{*}_C$ is a discrete space (or set) $\pi_0|C|$ which is obtained from the set of objects of C by identifying a pair of objects $x \sim y$ whenever there exists a zig-zag of morphisms connecting both objects

$$x \leftarrow a_0 \to a_1 \to \cdots \leftarrow a_n \to y$$

Without a doubt the value of the colimit is not particularly interesting: It just measures the set of connected components of the category C. One could argue that this is an expected output, after all this is the colimit of a "boring" functor. This is actually not the case, $\pi_0|C|$ is the set of connected components of a space $|C|$ which is the result of taking the correct ∞-categorical colimit of $\underline{*}_C$. In order to understand this new notion of colimit let us explain what a cone in this context should look like.

We define category $C^{\triangleright}$ with objects given by $\mathrm{Ob}(C^{\triangleright}) = \mathrm{Ob}(C) \coprod \{*\}$. The Hom-sets of this category are given by

$$\mathrm{Hom}_{C^{\triangleright}}(x, y) = \begin{cases} \mathrm{Hom}_C(x, y), & \text{if } x, y \in C \\ \{*\}, & \text{if } y = * \\ \varnothing, & \text{otherwise} \end{cases}$$

Observe that an ordinary functor $c_F : C^{\triangleright} \to A$ that restricts to F when evaluated at C is precisely the data of a cone with tip $c_F(*)$. Going back to our original example let us analyze what a cone for the constant value functor looks like with this new definition $c_* : C^{\triangleright} \to \mathcal{S}$. We would like to remark that from this point on we will be viewing $\mathcal{S}$ as an ∞-category otherwise both cone definitions would agree. Let $X = c_*(*)$ be the value of the functor on the cone point. First, let us observe that for every object $c \in C$ we have a morphism $c \to *$ in $C^{\triangleright}$ defining a morphism $\alpha_c : \Delta^0 \to X$ of spaces which amounts to specifying a point in the space X. Similarly given a morphism $u : c \to c'$ in C we have a 2-simplex in $C^{\triangleright}$ which gets mapped under c_* to a homotopy $\Delta^1 \to X$ between α_c and $\alpha_{c'}$ this is nothing more than specifying a path in the space X. Carefully unpacking all the information specified by c_* we see that the data of this functor is equivalent to the data a functor $C \to X$ where we are viewing X as an ∞-category where all morphisms are equivalences. One can show that the ∞-categorical colimit of the constant point value functor is given by formally inverting all of the morphisms of C to obtain a space. This space turns out to be equivalent to the *geometric realization* of C which is usually denoted by $|C|$. The ordinary colimit of the functor is precisely the set of path components of the space $|C|$. Thus we see that the colimit of our "boring functor" is a rather interesting space which can be regarded as a higher generalization of the groupoid completion[2] of the category C. Whereas the groupoid completion $|C|_{\leqslant 1}$ (viewed as a space) has vanishing homotopy groups for $i \geqslant 2$ the space $|C|$ can have highly complicated homotopy groups: A prominent example is given by Quillen's Q-construction [Qui73] where the higher homotopy groups of the geometric realization of an ordinary category compute the higher algebraic K-groups.

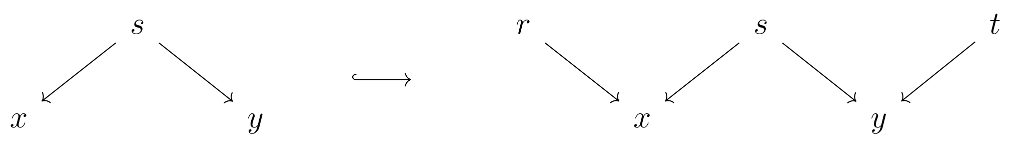
We will postpone the precise definition of the universal property of the colimit in this ∞-categorical context to Chapter 1 where we will review the main ∞-categorical constructions that will be generalized later in this document to the ∞-bicategorical realm. We would like to emphasize that showing that a certain cone is universal in the ∞-categorical sense is not a simple task. That is the reason why it is important to develop the necessary technology that will assist us in the computation of ∞-categorical colimits.

Cofinality

Among the tools devised to simplify the computation colimits perhaps the most important is the notion of a *cofinal* functor of ∞-categories. Suppose that we are given a pair of functors $F : \mathcal{D} \to \mathcal{A}$, $f : \mathcal{C} \to \mathcal{D}$. It follows that via restriction along f we can produce from the universal cone for F a cone for the diagram $F \circ f$. The universal property of the colimit implies the existence of a morphism

$$\operatorname*{colim}_{\mathcal{C}} F \circ f \longrightarrow \operatorname*{colim}_{\mathcal{D}} F$$

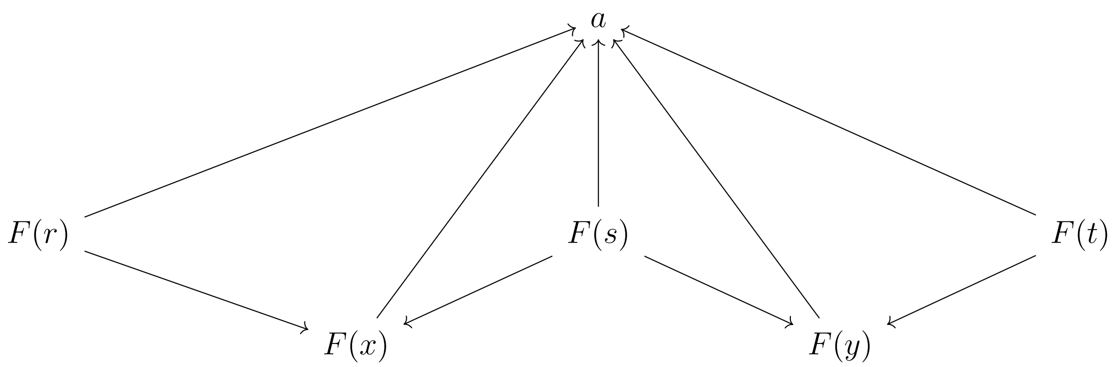
which we refer to as the *canonical comparison map*. We will say that f is cofinal if for every functor F as above the canonical comparison map is an equivalence in $\mathcal{A}$. To better understand this idea, let us look at a simple example. Let $f : \mathcal{C} \to \mathcal{D}$ denote a fully faithful inclusion of categories depicted diagrammatically as



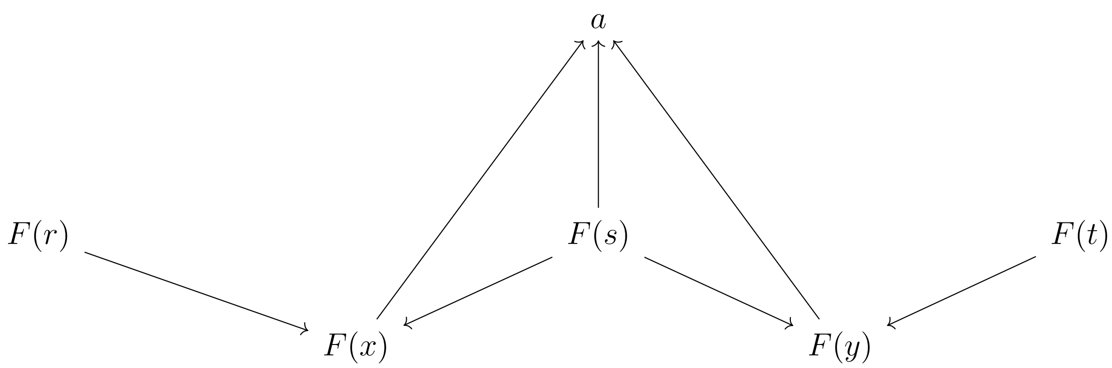
Here the letters r, x, s, y and t denote the objects of the category $\mathcal{D}$ and the arrows denote the (unique) morphisms between these objects. Given a diagram

[2]By groupoid completion we mean the left adjoint to the inclusion of (strict) groupoids into categories.

$F : \mathcal{D} \to \mathcal{A}$ let us consider a cone for F which we can also draw as



We already know that restricting the diagram will yield a cone for $F \circ f$. Let us suppose that we have a cone for the functor for $F \circ f$. Then it is immediate to produce the next diagram in $\mathcal{A}$



Moreover, to obtain a cone for F we only need to pick some composites which are, since we are working with ∞-categories, unique up to contractible choice. This behaviour extends all the way up to the n-simplices of the category of cones $\mathrm{Con}(F)$ and $\mathrm{Con}(Ff)$ which control the colimit of both functors. With some effort one can show that the morphism induced by restriction $\mathrm{Con}(F) \to \mathrm{Con}(Ff)$ is an equivalence of ∞-categories (it is in fact a trivial fibration) which in turn implies that the canonical comparison map between the colimits is an equivalence. The proof of this later fact is essentially independent of the functor F, which shows that f is indeed a cofinal functor.

In practice, to show that a morphism is cofinal by tracking its behaviour on the associated categories of cones is not a feasible task. The power of the theory of cofinal functors is due to the existence of a checkable criterion.

The conditions that characterize cofinal functors date back to Quillen ([Qui73]) and his famous Theorem A. Before commenting on this important result let us introduce some preliminary notation.

Definition. *Let $f : C \to D$ be a functor of ordinary categories. For every $d \in D$ we define a category $C_{d/}$ whose objects are given by an object $c \in C$ together with a morphism $u : d \to f(c)$. A morphism from $u : d \to f(c)$ to $v : d \to f(c')$ is given by a morphism $\alpha : c \to c'$ such that $v = f(\alpha) \circ u$. We will refer to $C_{d/}$ as the comma category.*

The categories $C_{d/}$ for $d \in D$ completely control cofinality for the functor f. Historically though, these categories where utilized by Quillen to give sufficient conditions for a functor $f : C \to D$ of ordinary categories to induce a homotopy equivalence upon passage to geometric realizations.

Theorem. *Let $f : C \to D$ be a functor of ordinary categories and assume that for every $d \in \mathcal{D}$ the geometric realization of the category $|C_{d/}| \simeq *$ is contractible. Then f induces a homotopy equivalence upon passage to geometric realizations*

$$|f| : |C| \xrightarrow{\simeq} |D|$$

The previous theorem has been extensively used in algebraic K-theory where it historically originated. Theorem A is an essential ingredient of the proof of the additivity theorems of Quillen's Q-construction ([Qui73]) and Waldhausen's S-construction ([Wald83]).

The original theorem of Quillen is nowadays interpreted as a cofinality statement. It was proved by Joyal ([Joy]) that a functor of ∞-categories is cofinal if and only if the conditions of Quillen's Theorem A (after taking adequate ∞-categorical comma categories) are satisfied. Going back to our example of the constant point valued functor we see that Theorem A can be recovered after noting that the geometric realization of a category can be obtained as the colimit of the constant point valued functor.

The conditions of Theorem A are in practice relatively easy to check due to two main reasons:

1. Many categories are contractible: Categories with terminal or initial objects are contractible and so are filtered categories, categories with products and many other examples.

2. If our starting functor $f : C \to D$ is a functor of ordinary categories then we need to check contractibility of an ordinary category.

Let us verify that the conditions of Theorem A are satisfied in the example we discussed above. Note that given a fully-faithful functor of ∞-categories $f : \mathcal{C} \to \mathcal{D}$[3] it follows that we have an equivalence $\mathcal{C}_{d/} \simeq \mathcal{D}_{d/}$ for every d in the essential image of f. Note that we are denoting by $\mathcal{D}_{d/}$ the comma category associated to the identity functor. We can also see that for every $d \in \mathcal{D}$ the identity morphism on d defines an initial object in $\mathcal{D}_{d/}$ which in turn implies that $|\mathcal{D}_{d/}| \simeq *$. This reduces the computations in our example to show that $C_{d/}$ for $d = r, t$. It follows immediately after inspection that in both cases $C_{d/} = *$ is precisely the point.

We can also characterize cofinality with respect to ordinary colimits: Given a functor of ordinary categories $f : C \to D$ it follows that f is strict cofinal if and only if $\pi_0|C_{d/}| = *$ for every $d \in D$. We finish this section by studying an example of strict cofinal functor which is not cofinal in the ∞-categorical sense. Let Δ denote the simplex category and consider the (non-full) inclusion

$$\Gamma = \left([1] \underset{d_0}{\overset{d_1}{\rightrightarrows}} [0] \right) \hookrightarrow \Delta^{\mathrm{op}}$$

[3] In general we use roman majescules C, D to denote ordinary categories and caligraphic majescules $\mathcal{C}, \mathcal{D}$ to denote ∞-categories.

It is easy to see that for every $n \geqslant 0$ the category $\Gamma_{[n]/}$ is non-empty and path connected. Note that Δ^{op} has an initial object which implies that $|\Delta^{\mathrm{op}}| \simeq *$. We further see that $|\Gamma| \simeq S^1$. Since cofinal functors must induce homotopy equivalences upon passage to geometric realizations it follows that our functor cannot be cofinal.

When 2-morphisms are no longer invertible: The theory of ∞-bicategories

Much in the same way as sets organize themselves into a category, the collection of (ordinary) categories is the prototypical example of a (strict) 2-category. This 2-category which we denote by $\mathbb{C}\mathrm{at}$ has as objects categories, 1-morphisms are given by functors and 2-morphisms are given by natural transformations. In a similar way, given a pair of ∞-categories we can define the functor ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ whose n-simplices are given by morphisms of simplicial sets $\sigma : \mathcal{C} \times \Delta^n \to \mathcal{D}$. Before delving into how to generalize the ∞-categorical theory into the realm of $(\infty,2)$-categories (also called ∞-bicategories in our preferred model) we must first ask the next question:

Q Why should we care about ∞-bicategories ?

It is worth giving a convincing answer to this question, after all, computations with ∞-categories are daunting in comparison with their ordinary counterparts. There are several ways to answer this question but here we will comment the main points that motivated the work of the author:

1. **Better structural understanding of the theory of ∞-categories:** Many constructions in ordinary category theory are nothing more than certain general 2-categorical construction particularized to $\mathbb{C}\mathrm{at}$. For example, the notion of adjunction of categories is precisely an adjunction internal to $\mathbb{C}\mathrm{at}$. A general 2-dimensional theory will help us to understand the specifics of $\mathbb{C}\mathrm{at}_\infty$ the ∞-bicategory of ∞-categories.

2. **Computations with ∞-categories can be simplified using the 2-dimensional theory:** Some constructions in ∞-category theory are essentially controlled by the theory of ∞-bicategories. This is better understood through the next example which plays a major role in this thesis. Let $\mathcal{C}$ be an ∞-category and suppose we are given a collection of morphisms $E \subset \mathcal{C}_1$ containing every identity. We will say that an edge of $\mathcal{C}$ is marked if it belongs to E and will use a superscript notation to denote the pair $\mathcal{C}^\dagger := (\mathcal{C}, E)$. Pairs as before will be refered to as *marked ∞-categories*. The localization of $\mathcal{C}$ at the collection of edges E is another ∞-category $L_W\left(\mathcal{C}^\dagger\right)$ equipped with a morphism $\iota : \mathcal{C} \to L_W\left(\mathcal{C}^\dagger\right)$ satisfying the following universal property: For every ∞-category $\mathcal{A}$ the functor induced by restriction
$$\iota^*\colon\ \mathrm{Fun}(L_W\left(\mathcal{C}^\dagger\right), \mathcal{A}) \longrightarrow \mathrm{Fun}(C, \mathcal{A})$$
is fully-faithful with essential image given by those functors $F : \mathcal{C} \to \mathcal{A}$ mapping each element of E to an equivalence in $\mathcal{A}$. Many important

categories are obtained via localizations, for example, the ∞-category of spaces is obtained as the ∞-categorical localization of the ordinary category of spaces at the set of weak homotopy equivalences. Another important example, this case in homological algebra is given by localizing the category of chain complexes $\mathrm{Ch}(\mathcal{A})$ on an abelian category with respect to quasi-isomorphisms. The resulting ∞-category is a generalization of the derived category $\mathcal{D}(A)$.

In this thesis we will develop an ∞-bicategorical theory of cofinality which in particular can be used to give sufficient conditions for a marking-preserving functor $f : \mathcal{C}^\dagger \to \mathcal{D}^\dagger$ to induce an equivalence upon passage to ∞-categorical localizations.

3. **Some important 1-categories admit a 2-categorical enhancement:** The main example we would like to discuss here is that the simplex category Δ. We can give a poset structure on the set of maps $\mathrm{Hom}_\Delta([n],[m])$ by declaring $f \leqslant g$ if and only if for every $i \in [n]$ we have $f(i) \leqslant g(i)$. It is easy to see that composition in the simplex category is compatible with the pointwise order on the Hom-sets thus producing a 2-category which we denote by $\mathbb{\Delta}$. The pointwise order of monotone morphisms is used in Waldhausen's original proof of the Additivity Theorem ([Wald83]). This hints at the possibility that the 2-categorical structure of Δ could play a role in K-theory.

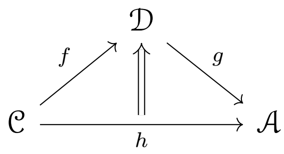
   The study of 2-simplicial objects, that is, functors $\mathbb{\Delta}^{\mathrm{op}} \to \mathbb{A}$ with values in an ∞-bicategory was the main motivation of the author to develop the 2-categorical technology presented in this thesis. In [Dyck21] Dyckerhoff utilizes 2-simplicial objects in stable ∞-categories to produce a categorification of the celebrated Dold-Kan correspondence ([Dold58]). To develop a rich theory of categorified homological algebra it will be necessary to understand the simplex 2-category as having a certain universal property which we will explain at the end of the introduction.

We would like to implement a model for ∞-bicategories that resembles as much as possible the simplicial model for ∞-categories. A practical way of implementing models for (∞,n)-categories is to use simplicial sets equipped with additional decorations known *complicial sets* [Ver08]. We will take a similar approach to that of Verity used by Lurie in [Lur09b]: The framework of *scaled simplicial sets.*

Definition. *A scaled simplicial set is given by a pair (X, C_X) where X is a simplicial set and $C_X \subseteq X_2$ is a subset of the set of 2-simplices containing every degenerate 2-simplex. We refer to the elements of C_X as* thin triangles *or* scaled triangles. *A morphism of scaled simplicial sets $(X,C_X) \to (Y,C_Y)$ is a morphism of the underlying simplicial sets such that $f(C_X) \subseteq C_Y$.*

The idea behind the definition is that the collection of thin triangles should represent those 2-morphisms which are invertible. For example, in the ∞-

bicategory $\mathbb{C}\mathrm{at}$ the data of a 2-simplex σ can be represented as



where we have a natural transformation $H_\sigma : \mathcal{C} \times \Delta^1 \to \mathcal{A}$ between h and $g \circ f$. The collection of thin simplices of $\mathbb{C}\mathrm{at}_\infty$ is given precisely by those triangles such that H_σ is an equivalence of functors. One then defines ∞-bicategories as those scaled simplicial sets satisfying some adequate right-lifting properties. We can organize ∞-bicategories into another ∞-bicategory that we denote by $\mathbb{B}\mathrm{icat}_\infty$.

The next family of 2-categories (where the notation $\mathbb{O}^n$ makes reference to the notion of oriental appearing in [Str87]) will be play a very important role in this document and can be used to extend diagram 1 to accommodate the 2-categorical theory as we will soon explain.

Definition 0.0.1. Let $n \geqslant 0$ and define a 2-category $\mathbb{O}^n$ as follows:

- Objects are given by the elements of the poset $[n]$.
- For every $i, j \in [n]$ the category $\mathbb{O}^n(i,j)$ is either empty if $i > j$ or given by the poset of subsets $S \subseteq [n]$ such that $\min(S) = i$ and $\max(S) = j$ ordered by inclusion. The non-trivial composition functors for $i \leqslant j \leqslant k$ are induced by union of subsets

  $$\mathbb{O}^n(i,j) \times \mathbb{O}^n(j,k) \longrightarrow \mathbb{O}^n(i,k), \;\; (S,T) \longmapsto S \cup T.$$

  The action on morphisms of the composition functors is the obvious one since union preserves our given order.

We think of the 2-categories $\mathbb{O}^n$ as thickened versions of the usual ordinals $[n]$ and use them to produce for every 2-category $\mathbb{D}$ a scaled simplicial set (in fact an ∞-bicategory) via the *scaled nerve* functor. The scaled simplicial set $\mathrm{N}^{\mathrm{sc}}(\mathbb{D})$ has as n-simplices the set of 2-functors $\mathrm{Fun}(\mathbb{O}^n, \mathbb{D})$. Observe that a functor $u : \mathbb{O}^2 \to \mathbb{D}$ specified by the following data:

- Three objects d_0, d_1, d_2.
- Three morphisms $u_{ij} : d_i \to d_j$ for $0 \leqslant i < j \leqslant 2$ and a composite morphism $u_{12} \circ u_{01} : d_0 \to d_2$.
- A 2-morphism $\alpha : u_{02} \Rightarrow u_{12} \circ u_{01}$.

Then we define the collection of thin triangles of $\mathrm{N}^{\mathrm{sc}}(\mathbb{D})$ to consist in those functors $\mathbb{O}^2 \to \mathbb{D}$ such that its associated 2-morphism is invertible in $\mathbb{D}$. The scaled nerve construction allows us produce the following diagram

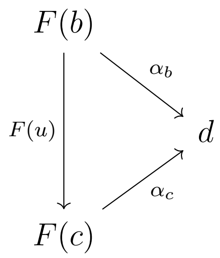
$$\begin{array}{ccccc} \mathrm{Set} & \longrightarrow & \mathrm{Cat} & \longrightarrow & 2\mathbb{C}\mathrm{at} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & \mathbb{C}\mathrm{at}_\infty & \longrightarrow & \mathbb{B}\mathrm{icat}_\infty \end{array}$$

where $2\mathbb{C}\mathrm{at}$ denotes the strict 2-category of 2-categories. We would like to remark that right-most functor is faithful but not full: Given a pair of 2-categories $\mathbb{C}$ and $\mathbb{D}$ a map of scaled simplicial sets $f : \mathrm{N}^{\mathrm{sc}}(\mathbb{C}) \to \mathrm{N}^{\mathrm{sc}}(\mathbb{D})$, then f corresponds to a lax unital functor $\hat{f} : \mathbb{C} \to \mathbb{D}$. In particular, it preserves composition of 1-morphisms up to invertible 2-morphism.

Colimits in ∞-bicategories

As one tries to generalize even strict colimits to 2-categories one runs into an immediate problem: which definition of colimit to use. Loosely speaking, any definition of a colimit should come equipped with a universal cone. However, if we consider a strict 2-functor $F : \mathbb{C} \to \mathbb{D}$, we run into an issue defining cones over F. A cone over F with tip d should consist of:

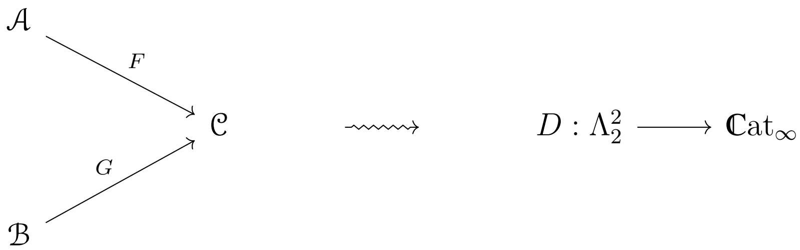
- For every object $c \in \mathbb{C}$, a morphism $\alpha_c : F(c) \to d$.
- For every morphism $u : b \to c$ in $\mathbb{C}$, a diagram



  that commutes appropriately.

It is here that the definition flounders — there are multiple 2-categorical notions which could be described as the diagram "commuting appropriately", and each yields different notions of colimit. If one requires the triangles to commute up to non-invertible 2-morphism, for instance, one obtains the notion of a *lax* colimit. If, on the other had, one requires commutativity up to invertible 2-morphism, the corresponding notion of colimit is the *pseudo-colimit*.

Before continuing with our general discussion we present some examples. Let Λ^2_2 be the poset consisting in three objects $0, 1, 2$ and morphisms $1 \to 2$ and $0 \to 2$. Suppose we are given a diagram



and let us compute the lax limit of D. We informally describe the lax limit which we denote by $\mathcal{A} \times^{\flat}_{\mathcal{C}} \mathcal{B}$ as follows:

- Objects are given the following data: A triple of objects $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $c \in \mathcal{C}$ together with morphisms $\alpha_a : F(a) \to c$ and $\alpha_b : G(b) \to c$.

- A morphism between $(a, b, c, \alpha_a, \alpha_b) \to (a', b', c', \alpha_{a'}, \alpha_{b'})$ is given by morphisms $a \to a'$, $b \to b'$ and $c \to c'$ and a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc} F(a) & \xrightarrow{\alpha_a} & c & \xleftarrow{\alpha_b} & G(b) \\ \downarrow & & \downarrow & & \downarrow \\ F(a') & \xrightarrow{\alpha_{a'}} & c' & \xleftarrow{\alpha_{b'}} & G(b'). \end{array}$$

This construction is known as the lax pullback and has been used in [LT19] to understand the failure of excision for any localising invariant. The pseudolimit $\mathcal{A} \times^{\sharp}_{\mathcal{C}} \mathcal{B}$ (which coincides with the usual ∞-categorical pullback) is the full subcategory of $\mathcal{A} \times^{\flat}_{\mathcal{C}} \mathcal{B}$ on those tuples $(a, b, c, \alpha_a, \alpha_b)$ such that both α_a and α_b are equivalences.

Even in simpler examples the theory of (co)limits in ∞-bicategories is capable of capturing interesting phenomena. For example, given an exact functor of stable ∞-categories $F : \mathcal{A} \to \mathcal{B}$ (which be view as a diagram $E : \Delta^1 \to \mathbb{S}\mathrm{t}$ with values in the ∞-bicategory of stable ∞-categories and exact functors) the lax limit of E is a stable ∞-category which corresponds to a semiorthogonal decomposition (see [BK89]) of $\mathcal{A}$ and $\mathcal{B}$ along the gluing functor F. In particular, this allows us to view (in the setting of stable ∞-categories) the notion of semiorthogonal decomposition along a gluing functor as being characterized by a certain 2-dimensional universal property.

One traditional way of resolving the multiplicity of definitions of 2-dimensional colimits is by defining the more general notion of *weighted colimits*, which specialize to each of the above cases (see for example [Kel06]). However, in the past two years, a different approach has become relevant due to its amenability to applications in simplicial models for higher categories: *marked colimits*. In defining marked colimits, one considers the 2-category $\mathbb{C}$ to be equipped with a collection of *marked* 1-morphisms, and then requires that the chosen 2-morphism making the triangle above commute is invertible whenever u is a marked morphism. This resolution of the above issue loses nothing in comparison to Cat-weighted or $\mathbb{C}\mathrm{at}_\infty$-weighted limits, as the two theories turn out to be equivalent (see [AG22, Theorem 4.7] and [GHL21a, Section 5] for more details). Although this definition of 2-categorical limit is in fact a novel concept in the study of ∞-category theory its use in the strict 2-categorical realm was already established as in seen in [DS18].

The degree to which marked colimits are well-suited to higher-categorical settings is underlined by the fact that, within the span of a year, three groups independently arrived at more or less the same definition: the author in [AGS22] and [AG22]; Berman in [Berm21]; and Gagna, Harpaz, and Lanari in [GHL21a]. The last of these three provides a complete definition of marked limits and colimits in terms of marked-scaled simplicial sets. Their definitions coincide with those given in [AGS22], [AG22], and [Berm21] whenever both versions apply.

Before diving into presenting the main results obtained in this thesis let us illustrate how the theory of marked colimits gives a new perspective to familiar constructions in ∞-category theory. First of all we will introduce some notation that will be used throughout this document: Given a marked ∞-bicategory $\mathbf{C}^{\dagger}$ we will denote the marked colimit of a functor $F : \mathbf{C} \to \mathbb{A}$ as $\mathrm{colim}^{\dagger}_{\mathbf{C}} F$.

Ordinary colimit	∞-categorical colimit	∞-bicategorical colimit
Connected components of $\lvert C \rvert$	∞-groupoid completion $\lvert \mathrm{C} \rvert$	partial ∞-groupoid completion $L_W(C^\dagger)$

Let $\mathcal{C}^\dagger$ be a marked ∞-category and recall the constant point valued functor $\underline{*}_{\mathcal{C}} : \mathcal{C} \to \mathcal{S} \subset \mathbb{C}\mathrm{at}_\infty$ discussed before which we now view as taking values in ∞-categories. Then after some unraveling we see that the data of a marked cone for $\underline{*}_{\mathcal{C}}$ with tip an ∞-category $\mathcal{D}$ is equivalent to specifying a functor $\mathcal{C} \to \mathcal{D}$ which maps every marked edge in $\mathcal{C}^\dagger$ to an equivalence in $\mathcal{D}$. This identifies the universal property of the marked colimit with that of the localization of $\mathcal{C}$ at the collection of marked edges thus yielding an equivalence of ∞-categories

$$L_W(\mathcal{C}^\dagger) \xrightarrow{\simeq} \operatorname*{colim}^\dagger_{\mathcal{C}} \underline{*}_{\mathcal{C}}$$

Note that we can identify the geometric realization $|\mathcal{C}|$ (also known as the ∞-groupoid completion) as the localization of $\mathcal{C}$ at the collection of *all* edges namely $L_W(\mathcal{C}^\sharp)$.

We summarize in the next table the different kinds of colimits studied so far for a constant point valued functor $\underline{*}_C : C \to \mathrm{Set} \subset \mathcal{S} \subset \mathbb{C}\mathrm{at}_\infty$ indexed by an ordinary 1-category:

This computation will be generalized in Theorem 4.1.1. Recall (Chapter 3 in [Lur09a]) that for any functor of ∞-categories $F : \mathcal{C} \to \mathbb{C}\mathrm{at}_\infty$ there is a coCartesian fibration $\mathrm{Un}^{\mathrm{co}}_{\mathcal{C}}(F) \to \mathcal{C}$ called the (coCartesian) *unstraightening* of F. We can now state the result.

Theorem (4.1.1)**.** *Let $\mathcal{C}^\dagger$ be a marked ∞-category. Given $F \colon \mathcal{C} \longrightarrow \mathbb{C}\mathrm{at}_\infty$ there is an equivalence of ∞-categories*

$$L_W\left(\mathrm{Un}^{\mathrm{co}}_{\mathcal{C}}(F)^\dagger\right) \simeq \operatorname*{colim}^\dagger_{\mathcal{C}} F$$

where $L_W\left(\mathrm{Un}^{\mathrm{co}}_{\mathcal{C}}(F)^\dagger\right)$ denotes the ∞-categorical localization at the collection of coCartesian edges lying over marked edges of $\mathcal{C}^\dagger$

Let us return to the example of the (lax) pullback $D : \Lambda^2_2 \to \mathbb{C}\mathrm{at}_\infty$ and suppose that Λ^2_2 comes equipped with a marking consisting on the edge $0 \to 2$ which corresponds to the functor $G : \mathcal{B} \to \mathcal{C}$. In this case we denote the marked limit by $\mathcal{A} \overset{\rightarrow}{\times}_{\mathcal{C}} \mathcal{B}$. One can show that the marked limit is given by the subcategory consisting in tuples $(a, b, c, \alpha_a, \alpha_b)$ such that α_b is an equivalence in $\mathcal{C}$. In this situation we can describe the ∞-category $\mathcal{A} \overset{\rightarrow}{\times}_{\mathcal{C}} \mathcal{B}$ as follows:

- Objects are given by triples (a, b, α) where $a \in \mathcal{A}$, $b \in \mathcal{A}$ and $\alpha : F(a) \to G(b)$.
- A morphism $(a, b, \alpha) \to (a', b', \alpha')$ is given by morphisms $a \to a'$, $b \to b'$ together with a commutative diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{\alpha} & F(b) \\ \downarrow & & \downarrow \\ F(a') & \xrightarrow{\alpha'} & F(b') \end{array}$$

One analogously defines the ∞-category $\mathcal{A}\overset{\leftarrow}{\times}_{\mathcal{C}}\mathcal{B}$ which corresponds to the marked limit of $D:\Lambda^2_2\to\mathbb{C}\mathrm{at}_\infty$ where we are marking the edge $0\to 1$ in Λ^2_2. We summarize our discussion of marked pullbacks in the next commutative diagram

$$\begin{array}{ccc} \mathcal{A}\times^{\sharp}_{\mathcal{C}}\mathcal{B} & \longrightarrow & \mathcal{A}\overset{\rightarrow}{\times}_{\mathcal{C}}\mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{A}\overset{\leftarrow}{\times}_{\mathcal{C}}\mathcal{B} & \longrightarrow & \mathcal{A}\times^{\flat}_{\mathcal{C}}\mathcal{B} \end{array}$$

consisting in fully faithful functors of ∞-categories.

We conclude the section with an example of lax colimit where the indexing diagram is a 2-category. Let $\mathbb{Q}^2$ denote the free living 2-morphism which we view pictorially as

$$0 \; \underset{g}{\overset{f}{\rightrightarrows}} \; 1 \qquad (\Downarrow \text{ from } f \text{ to } g)$$

and consider a diagram $T:\mathbb{Q}\to\mathbb{C}\mathrm{at}_\infty$ which amounts to a pair of functors of ∞-categories $F,G:\mathcal{A}\to\mathcal{B}$ together with a natural transformation $F\Rightarrow G$. For the sake of simplicity let us assume that T takes values in ordinary categories. We consider the 2-category $\mathbb{E}\mathrm{l}(T)$ obtained as the 2-categorical Grothendieck construction of the functor T which is given by:

- Objects are given by pairs (ε,x) where ε is an object of $\mathbb{Q}$ and $x\in T(\varepsilon)$.
- A morphism $(\varepsilon_0,x)\to(\varepsilon_1,y)$ is given by a pair (u,s) consisting of a morphism $u:\varepsilon_0\to\varepsilon_1$ in $\mathbb{Q}$ and a morphism $s:T(u)(x)\to y$.
- A 2-morphism $(u,s)\Rightarrow(v,t)$ is given by a 2-morphism $u\overset{\theta}{\Rightarrow}v$ and a commutative diagram

$$\begin{array}{ccc} T(u)(x) & & \\ {\scriptstyle T(\theta)_x}\downarrow & \overset{s}{\searrow} & \\ & & y \\ & \underset{t}{\nearrow} & \\ T(v)(x) & & \end{array}$$

We claim that the lax colimit of T is the ∞-category obtained by formally inverting the 2-morphisms of $\mathbb{E}\mathrm{l}(T)$. Observe that $\mathbb{E}\mathrm{l}(T)$ comes equipped with a canonical functor $\mathbb{E}\mathrm{l}(T)\to\mathbb{Q}$. We define $\mathbb{E}\mathrm{l}(T)_{\leqslant 1}$ by means of the pullback diagram

$$\begin{array}{ccc} \mathbb{E}\mathrm{l}(T)_{\leqslant 1} & \xrightarrow{i} & \mathbb{E}\mathrm{l}(T) \\ \downarrow & & \downarrow \\ \Delta^1 & \xrightarrow{f} & \mathbb{Q} \end{array}$$

Note that the only non-trivial 2-morphisms in $\mathbb{E}\mathrm{l}(T)$ are those living over the unique non-invertible morphism in $\mathbb{Q}$. In particular, it follows that $\mathbb{E}\mathrm{l}(T)_{\leqslant 1}$ is a strict 1-category. We further note that $\mathbb{E}\mathrm{l}(T)_{\leqslant 1}$ is, by the previous theorem, the lax colimit of $T \circ f$.

We define a functor $r : \mathbb{E}\mathrm{l}(T) \to \mathbb{E}\mathrm{l}(T)_{\leqslant 1}$ which is the identity on those objects and morphisms living over f and that sends an edge $(g, s) : (0, x) \to (1, y)$ to the morphism over f given by the composite $T(f)(x) \to T(g)(x) \to y$. It is immediate to see that this defines a functor which collapses the non-identity 2-morphisms of $\mathbb{E}\mathrm{l}(T)$. One can show that there is a lax natural transformation between $i \circ r$ and the identity on $\mathbb{E}\mathrm{l}(T)$ which in turn implies that i induces an equivalence after inverting the 2-morphisms of $\mathbb{E}\mathrm{l}(T)$. In the next section we will show that the map $f : \Delta^1 \to \mathbb{Q}$ is cofinal in the ∞-bicategorical sense thus proving our claim.

Higher cofinality: The main theorem

Let $f : \mathbf{C}^\dagger \to \mathbf{D}^\dagger$ be a marking-preserving functor of ∞-bicategories. We say that f is *marked cofinal* if for every diagram $F : \mathbf{D} \to \mathbb{A}$ the canonical comparison map

$$\operatorname{colim}^\dagger_{\mathbf{C}} Ff \xrightarrow{\ \simeq\ } \operatorname{colim}^\dagger_{\mathbf{D}} F$$

is an equivalence in $\mathbb{A}$. The central theorem of this thesis gives a positive answer to the following question

Q: Can we efficiently characterize marked cofinal functors of ∞-bicategories?

Moreover, our characterization of cofinal functors is given by a direct generalization of the conditions of Quillen's Theorem A. Let us introduce some preliminary notation to better understand our main result.

Definition 0.0.2. Let $f : \mathbb{C}^\dagger \to \mathbb{D}^\dagger$ be a functor of marked 2-categories. Given an object $d \in \mathbb{D}$ we define the marked comma 2-category $\mathbb{C}^\dagger_{d\nearrow}$ as follows:

- Objects are given by pairs (u, c) where $u : d \to f(c)$ is a morphism in $\mathbb{D}$ with source d and c is an object of $\mathbb{C}$.
- A 1-morphism from $u : d \to f(c)$ to $v : d \to f(c')$ is given by a 1-morphism $\alpha : c \to c'$ in $\mathbb{C}$ and a 2-morphism $f(\alpha) \circ u \Rightarrow v$.
- A 2-morphism in $\mathbb{C}^\dagger_{d\nearrow}$ is given by a 2-morphism $\varepsilon : \alpha \Rightarrow \beta$ such that the diagram below commutes

$$\begin{array}{ccc} f(\alpha)\circ u & & \\ {\scriptstyle f(\varepsilon)*u}\Big\Downarrow & \searrow\!\!\!\!\searrow & v \\ f(\beta)\circ u & \nearrow\!\!\!\!\nearrow & \end{array}$$

- A morphism in $\mathbb{C}^\dagger_{d\nearrow}$ is marked precisely when $\alpha : c \to c'$ is marked in $\mathbb{C}^\dagger$ and the associated 2-morphism is invertible.

If the functor f is the identity on the marked 2-category $\mathbb{D}^\dagger$ we will use the notation $\mathbb{D}^\dagger_{d\nearrow}$.

We would like to point out that an analogous construction can be performed for functors of general ∞-bicategories but for the purpose of clarity we choose to present here the strict version. We refer the reader to Chapter 4 for the general definition. We are ready to present our main theorem.

Theorem (4.0.31)**.** *Let $f : \mathbb{C}^\dagger \to \mathbb{D}^\dagger$ be a marking-preserving functor of ∞-bicategories. Then the following statements are equivalent:*

1. *The functor f is marked cofinal.*
2. *For every $d \in \mathbb{D}$ the functor f induces an equivalence of ∞-categorical localizations $L_W(\mathbb{C}^\dagger_{d\nearrow}) \to L_W(\mathbb{D}^\dagger_{d\nearrow})$.*

First, let us derive some corollaries from the previous theorem. Let us suppose that $\mathbb{C}^\dagger = \mathcal{C}^\sharp$, $\mathbb{D}^\dagger = \mathcal{D}^\sharp$, that is, both ∞-bicategories are actually ∞-categories with all morphisms being marked. Then since for every $d \in \mathcal{D}$ the ∞-category $\mathcal{D}_{d/}$ has an initial object it follows that the ∞-categorical localization at all morphisms is given by the geometric realization functor and thus we see that $L_W(\mathcal{D}^\sharp_{d/}) \simeq |\mathcal{D}_{d/}| \simeq *$. Then the second statement in our theorem collapses to:

- For every $d \in \mathcal{D}$, the geometric realization of the comma category $|\mathcal{C}_{d/}| \simeq *$ is contractible.

In order words, our theorem recovers the characterization of cofinal functors of ∞-categories due to Joyal. In a similar way as the original theorem of Quillen can be recovered from the main cofinality statement, in our situation we can obtain the following generalization of Quillen's Theorem A.

Corollary (4.0.32)**.** *Let $f : \mathbb{C}^\dagger \to \mathbb{D}^\dagger$ be a marking-preserving functor of ∞-bicategories and suppose that the following condition holds*

- *For every $d \in \mathbb{D}$ the functor f induces an equivalence of ∞-categorical localizations $L_W(\mathbb{C}^\dagger_{d\nearrow}) \to L_W(\mathbb{D}^\dagger_{d\nearrow})$.*

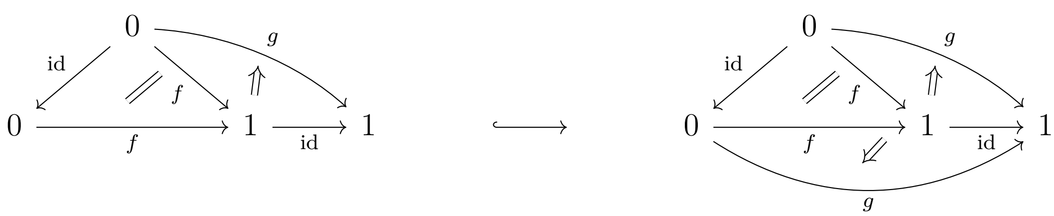
Then f induces an equivalence upon passage to ∞-categorical localizations

$$L_W(f) : L_W\left(\mathbb{C}^\dagger\right) \xrightarrow{\simeq} L_W(\mathbb{D}^\dagger).$$

We conclude this section by describing simple applications of our cofinality criterion. First, let us return to the final example in the previous section, namely, the free living 2-morphism $\mathbb{Q}$. There we claimed that the inclusion $f : \Delta^1 \to \mathbb{Q}$ of the initial object in the mapping category $\mathbb{Q}(0,1)$ is cofinal. We will show that the morphism

$$f_0 \colon \Delta^1_{0\nearrow} \longrightarrow \mathbb{Q}_{0\nearrow}$$

induces an equivalence after localizing the 2-morphisms in $\mathbb{Q}_{0\nearrow}$. We visualize the morphism f_0 schematically as follows

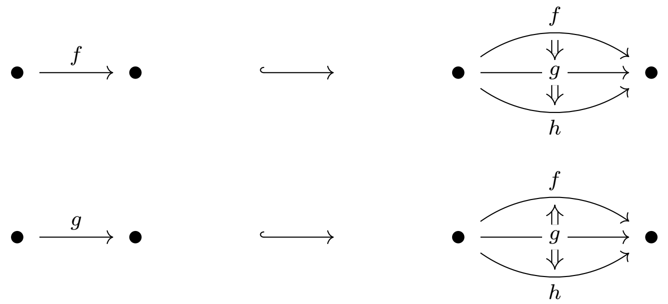


Note that f_0 induces isomorphisms in mapping categories except in the case:

$$\mathrm{Map}_{\Delta^1_{0\nearrow}}(\mathrm{id}, g) \to \mathrm{Map}_{\mathbb{Q}_{0\nearrow}}(\mathrm{id}, g).$$

In this latter instance however, the morphism on mapping categories can be identified with the inclusion of the initial vertex of Δ^1 which clearly becomes an equivalence after localizing 2-morphisms. Since $\Delta^1_{1\nearrow} \simeq \mathbb{Q}_{1\nearrow} \simeq *$ we conclude that $f : \Delta^1 \to \mathbb{Q}$ is cofinal.

We provide further examples of cofinal maps which are similar in spirit to the previous example.



We can, in a similar way as before, verify that the morphisms above are (lax) cofinal. However, it will instructive to see those examples as being part of a general class of cofinal maps.

Let K be an ordinary category. We define a 2-category $\mathbb{2}[K]$ as follows:

- We have a pair of objects $0, 1$.
- The mapping categories are given by $\mathbb{2}[K](\varepsilon, \varepsilon) = \Delta^0$ for $\varepsilon \in \{0, 1\}$, $\mathbb{2}[K](0, 1) = K$ and $\mathbb{2}[K](1, 0) = \varnothing$.

Given a functor of 1-categories $p : K \to S$ we obtain a (strict) 2-functor $\mathbb{p} : \mathbb{2}[K] \to \mathbb{2}[S]$. We will study study the morphisms

$$\mathbb{p}_i \colon \mathbb{2}[K]_{i\nearrow} \longrightarrow \mathbb{2}[S]_{i\nearrow}, \quad \text{for } i = 0, 1.$$

The case $i = 1$ is trivial since the functor $\mathbb{p}_1$ is an isomorphism. We turn our attention to the case $i = 0$. Observe that since $\mathbb{p}$ is an isomorphism on objects it follows that both $\mathbb{2}[K]_{0\nearrow}$ and $\mathbb{2}[S]_{0\nearrow}$ have the same objects which are precisely represented by objects $s \in S$ together with the identity morphism on the object 0. Let $s_1, s_2 \in S$. Then $\mathbb{p}_0$ induces an isomorphism

$$\mathbb{2}[K]_{0\nearrow}(s_1, s_2) \simeq \mathbb{2}[S]_{0\nearrow}(s_1, s_2) \simeq \mathrm{Hom}_S(s_1, s_2)$$

We similarly obtain

$$\mathbb{2}[K]_{0\nearrow}(s, \mathrm{id}) \simeq \mathbb{2}[S]_{0\nearrow}(s, \mathrm{id}) \simeq \varnothing, \quad \mathbb{2}[K]_{0\nearrow}(\mathrm{id}, \mathrm{id}) \simeq \mathbb{2}[S]_{0\nearrow}(\mathrm{id}, \mathrm{id}) \simeq \Delta^0$$

The final case to analyze then yields

$$K_{/s} = \mathbb{2}[K]_{0\nearrow}(\mathrm{id}, s) \longrightarrow \mathbb{2}[S]_{0\nearrow}(\mathrm{id}, s) = S_{/s}$$

where $K_{/s}$ is the dual version[4] of $K_{s/}$ whose objects are given by morphisms $u : p(k) \to s$ in S and whose morphisms are given by maps $k \to k'$ making

[4]These categories control the dual notion of coinitial functor: A functor is f is coinitial if restriction along f preserves limits

the obvious diagram commute. We can now use our characterization of cofinal functors of ∞-bicategories to arrive at the following result.

Proposition. *Let $p : K \to S$ be a functor of ordinary 1-categories. Then $\mathbb{p} : 2[K] \to 2[S]$ is a cofinal functor of 2-categories (with respect to the minimal marking) if and only if $p^{\mathrm{op}} : K^{\mathrm{op}} \to S^{\mathrm{op}}$ is a cofinal functor of 1-categories.*

Fibrations of ∞-bicategories & the Grothendieck construction

The proof of our cofinality theorem required some technology to be develop, much of which is of independent interest. In this section we will introduce the foundational ∞-bicategorical theory developed to assist in the proof of our main theorem.

The main foundational results obtained in this thesis are the construction of a model structure modeling ∞-bicategories fibred in ∞-bicategories (Chapter 2) and its corresponding Grothendieck construction (Chapter 3). Before diving deeper into the discussion of these results let us point out the role that they will play in the theory of ∞-bicategorical colimits and cofinality.

In the ∞-categorical theory, there is an alternative characterization of cofinal functors which admits an easy generalization to the ∞-bicategorical context. Namely, a functor $f : \mathcal{C} \to \mathcal{D}$ of ∞-categories is cofinal if the induced morphism

$$\begin{array}{ccc} \mathcal{C}^{\sharp} & \xrightarrow{\quad f \quad} & \mathcal{D}^{\sharp} \\ & \searrow \quad \swarrow & \\ & \mathcal{D} & \end{array}$$

is a weak equivalence in the Cartesian model structure on $(\mathrm{Set}_{\Delta}^{+})_{/\mathcal{D}}$. In Chapter 2, we will construct a model structure modeling what we call outer 2-Cartesian fibrations. Given a marking preserving functor of ∞-bicategories $f : \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ we will define marked cofinal functors as those inducing equivalences (see Definition 4.0.21) in the 2-Cartesian model structure over $\mathbb{D}$.

The connection between the Grothendieck construction and the ∞-bicategorical theory of colimits was already exposed in the previous section where it was heavily utilised as a universal recipe for computing marked colimits in $\mathbb{C}\mathrm{at}_{\infty}$. Moreover, the Grothendieck construction plays a crucial role in our proof of our cofinality statement since it allows us to derive important formal properties of the functor $f : \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ when viewed as a morphism in the 2-Cartesian model structure (see Proposition 4.0.27).

The ∞-bicategorical Grothendieck construction

In its most basic form, for 1-categories fibred in sets, the Grothendieck construction predates Grothendieck's work on the subject (see, e.g. [MM00, pg. 44] for discussion). In this context, the Grothendieck construction reconstitutes the information of a functor of 1-categories

$$F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

into the associated *category of elements*, a category whose objects consist of an object $c \in \mathcal{C}$, and an element $x \in F(c)$, and whose morphisms are morphisms in $\mathcal{C}$ whose associated maps of sets preserve the chosen element.

The Grothendieck construction in its modern form emerged as a tool to study descent (see, e.g. [Groth71]). In this case, it takes the form of an equivalence

$$\mathrm{Fib}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ps}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}),$$

for any category $\mathcal{C}$, between the category of *fibred categories over* $\mathcal{C}$, and the category of *pseudo-functors* $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$. The underlying idea is that certain conditions on a functor $p : \mathcal{D} \to \mathcal{C}$ mean that the fibres of p vary *(pseudo-)functorially* in $\mathcal{C}$. Indeed, the original definition of a fibred category, in [Groth60], was what we today would call a pseudo-functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$. More precisely, an assignment of a category $F(x) \in \mathrm{Cat}$ for every $x \in \mathcal{C}$, a functor $F(f) : F(y) \to F(x)$ for every morphism $f : x \to y$ in $\mathcal{C}$, and natural isomorphisms $F(g) \circ F(f) \cong F(f \circ g)$ for every composable pair of morphisms, satisfying additional coherence conditions.

The Grothendieck construction, as first exposed in [Groth71], reformulates the data of a pseudo-functor into a *Cartesian fibration*. Given a functor $P : \mathcal{F} \to \mathcal{C}$, a morphism $f : x \to y$ in $\mathcal{F}$ is called *Cartesian* if, for every $g : z \to y$ in $\mathcal{F}$, and every commutative diagram

$$\begin{array}{ccccc} & & P(x) & & \\ & \overset{h}{\nearrow} & & \overset{P(f)}{\searrow} & \\ P(z) & & \xrightarrow[P(g)]{\qquad\qquad} & & P(y) \end{array}$$

in $\mathcal{C}$, there is a unique morphism $\tilde{h} : z \to x$ with $P(\tilde{h}) = h$, such that $f \circ \tilde{h} = g$. The functor P is said to be a *Cartesian fibration* if, for every $f : c \to P(y)$ in $\mathcal{C}$, there is a Cartesian morphism $\tilde{f} : x \to y$ in $\mathcal{F}$ such that $P(\tilde{f}) = f$.

The equivalence between pseudo-functors $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ and Cartesian fibrations over $\mathcal{C}$ is achieved by constructing a Cartesian fibration $P : \mathrm{El}(F) \to \mathcal{C}$ as follows:

- The objects of $\mathrm{El}(F)$ consist of pairs (c, x), where $c \in \mathcal{C}$, and $x \in F(c)$.
- A morphism $(f, \tilde{f}) : (c, x) \to (d, y)$ consists of a morphism $f : c \to d$ in $\mathcal{C}$, together with a morphism $\tilde{f} : x \to F(f)(y)$ in $F(x)$.

The Cartesian morphisms of $\mathrm{El}(F)$ are precisely those $(f, \tilde{f})$ such that $\tilde{f}$ is an isomorphism.

More recent incarnations of the Grothendieck construction have focused on ∞-categorical variants. By their very nature, functors of ∞-categories generalize pseudo-functors of $(2,1)$-categories, so that higher Grothendieck constructions now take the form of equivalences

$$\mathcal{C}\mathrm{art}(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbb{C}\mathrm{at}_\infty)$$

of ∞-categories. This equivalence was proven by Lurie in [Lur09a], using model-categorical techniques.

The basic form of the proof goes as follows: Given an ∞-category $\mathcal{C}$, presented as a quasi-category, Lurie defines *marked simplicial sets* over $\mathcal{C}$ to be pairs (X, M_X) consisting of a simplicial set $X \in \mathrm{Set}_\Delta$ and a subset $M_X \subset X_1$ of *marked edges* containing all degenerate edges, equipped with a morphism $p : X \to \mathcal{C}$ of simplicial sets. Requiring maps to preserve these marked edges yields a category $(\mathrm{Set}_\Delta^+)_{/\mathcal{C}}$. Lurie then constructs a model structure on this category, the fibrant objects of which satisfy lifting properties akin to those defining 1-categorical Cartesian fibrations. In particular, the corresponding model structure on $\mathrm{Set}_\Delta^+ \cong (\mathrm{Set}_\Delta^+)_{/\Delta^0}$ models ∞-categories.

With these model structures in place, one can consider the category $(\mathrm{Set}_\Delta^+)^{\mathfrak{C}[\mathcal{C}]^{\mathrm{op}}}$ of simplicially-enriched functors $\mathfrak{C}[\mathcal{C}]^{\mathrm{op}} \to \mathrm{Set}_\Delta^+$, and equip it with the projective model structure. The ∞-categorical Grothendieck construction then takes the form of a Quillen equivalence

$$\mathrm{St}_{\mathcal{C}} : (\mathrm{Set}_\Delta^+)_{/\mathcal{C}} \rightleftarrows (\mathrm{Set}_\Delta^+)^{\mathfrak{C}[\mathcal{C}]^{\mathrm{op}}} : \mathrm{Un}_{\mathcal{C}}$$

between these two model categories.

In the ∞-categorical context, Grothendieck constructions have become an indispensable tool, as the added computational complexity of ∞-categorical constructions renders many ad-hoc constructions of functors nearly impossible to work with. It is often far easier to work with the fibration associated to a functor of ∞-categories than with the functor itself. Examples of such applications include the study of monoidal ∞-categories in [Lur11] and [Lur17] and the approach to lax colimits presented in [GHN15]. The study of higher forms of cofinality presented in this thesis, is another case in which it is essential to use the Grothendieck construction.

Recall that the ∞-categorical Grothendieck construction comes in *two* variances. One can either consider the aforementioned *Cartesian* fibrations of ∞-categories over $\mathcal{C}$, or consider *coCartesian* fibrations over $\mathcal{C}$. The former correspond to ∞-functors

$$F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbb{C}\mathrm{at}_\infty$$

whereas the latter correspond to ∞-functors

$$F\colon \mathcal{C} \longrightarrow \mathbb{C}\mathrm{at}_\infty$$

Additionally, if one treats the case of functors

$$F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S} \subset \mathbb{C}\mathrm{at}_\infty .$$

valued in ∞-groupoids (spaces), one obtains more restrictive variants of Cartesian/coCartesian fibrations, called *right fibrations* and *left fibrations*, respectively, in [Lur09a, Ch. 2].

Each variance can be obtained from the other by appropriate dualization procedures, and so, in practice it is only necessary to prove one correspondence to obtain the other. In the world of ∞-bicategories, where there are *four* possible variances, a similar principle applies, although the dualization procedures can become more complicated. As a result, we have focused on a single variance in our exploration of the ∞-bicategorical Grothendieck construction.

In Chapter 3 we will provide a complete ∞-bicategorical Grothendieck construction. Loosely speaking, for every scaled simplicial set S, we provide an equivalence of $(\infty, 2)$-categories (or simply ∞-bicategories as in [Lur09b])

$$2\mathbb{C}\mathrm{art}(S) \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathbb{B}\mathrm{icat}_\infty)$$

between 2-Cartesian fibrations[5] over S, and ∞-bifunctors $S^{\mathrm{op}} \to \mathbb{B}\mathrm{icat}_\infty$ with values in ∞-bicategories.

To understand this construction on an intuitive level, it is helpful to first consider the strict 2-categorical variant, developed by Buckley in [Buc14]. In this setting, we consider strict 2-functors $p : \mathbb{C} \to \mathbb{D}$. A 1-morphism $f : c \to \overline{c}$ in $\mathbb{C}$ is called *Cartesian* if, for every $a \in \mathbb{C}$ there is a pullback square of categories

$$\begin{array}{ccc} \mathbb{C}(a,c) & \xrightarrow{f_*} & \mathbb{C}(a,\overline{c}) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P} \\ \mathbb{D}(P(a),P(c)) & \xrightarrow[P(f)_*]{} & \mathbb{D}(P(a),P(\overline{c})) \end{array}$$

A 2-morphism $\alpha : f \Rightarrow g$ in $\mathbb{C}(x, y)$ is called *coCartesian* if it is a coCartesian 1-morphism for the map

$$P\colon \mathbb{C}(x,y) \longrightarrow \mathbb{D}(P(x),P(y)).$$

The functor P is then called a 2-Cartesian fibration if it admits Cartesian lifts of all 1-morphisms, and coCartesian lifts of all 2-morphisms.

As a first step towards our ∞-bicategorical Grothendieck construction we will construct a model structure which models an ∞-bicategorical variant of the above definitions. To keep track of the data of (1) invertible 2-morphisms, (2) Cartesian 1-morphisms, and (3) coCartesian 2-morphisms in the simplicial setting, we consider a 3-part decoration on simplicial sets. Given a simplicial set $X \in \mathrm{Set}_\Delta$, we define a *marking and biscaling* on X to consist of

- As in [Lur09b], invertible 2-morphisms are encoded as a collection $T_X \subset X_2$ of 2-simplices, which is required to contain degenerate simplices. The 2-simplices in T_X are called *thin* 2-simplices.
- As in [Lur09a], Cartesian 1-morphisms are encoded as a collection $M_X \subset X_1$ of 1-simplices, which is required to contain the degenerate 1-simplices. The 1-simplices in M_X are called *marked* 1-simplices.
- The coCartesian 2-morphisms are encoded as a collection $C_X \subset X_2$. Since every invertible 2-morphism should be coCartesian, we require that $T_X \subset C_X$. We refer to the 2-simplices in C_X as *lean* 2-simplices.

A tuple $(X, M_X, T_X \subset C_X)$ is referred to as a *marked-biscaled simplicial set* (or **MB** *simplicial set* for short). We denote the category of **MB** simplicial sets by $\mathrm{Set}_\Delta^{\mathbf{mb}}$. The main model category theoretic result of this thesis can be summarized as,

Theorem. *Let (S, T_S) be a scaled simplicial set.*

[5] What we call 2-Cartesian fibrations are called *outer 2-Cartesian fibrations* in [GHL21b]. Because we focus on a single variance, we trim the terminology for ease of reading.

1. *There is a left proper, combinatorial, simplicial model structure on*[6] $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/(S,\sharp,T_S\subset\sharp)}$*, called the* 2-Cartesian model structure*.*

2. *If $S=\Delta^0$ is the terminal scaled simplicial set, the resulting model structure models ∞-bicategories.*

3. *If (S,T_S) is the scaled nerve of a strict 2-category $\mathbb{D}$, every 2-Cartesian fibration of strict 2-categories $P:\mathbb{C}\to\mathbb{D}$ gives rise to a fibrant object of $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/(S,\sharp,T_S\subset\sharp)}$.*

This result can be found in Theorem 2.2.43, Theorem 2.2.44 and Theorem 2.3.32. In the second of these results, our decoration becomes highly redundant. In a fibrant object, the marked 1-morphisms correspond to equivalences, the thin 2-simplices correspond to invertible 2-morphisms, but the lean 2-simplices are identical to the thin 2-simplices. To simplify our later computations, we rectify this redundancy by also considering *marked-scaled* simplicial sets, i.e., triples (X,M_X,T_X) consisting of a simplicial set X, a collection of marked 1-simplices M_X, and a collection of thin 2-simplices T_X. The category of marked-scaled simplicial sets is denoted by $\mathrm{Set}_\Delta^{\mathbf{ms}}$. We will formalize the fact that marked-scaled simplicial sets should also model ∞-bicategories, as seen in the next theorem.

Theorem. *There is a left proper, combinatorial,* Set_Δ^+*-enriched model structure on* $\mathrm{Set}_\Delta^{\mathbf{ms}}$*. Moreover, it is Quillen equivalent to the 2-Cartesian model structure on* $\mathrm{Set}_\Delta^{\mathbf{mb}}$*, and thus models ∞-bicategories.*

The existence of the model structure is proved Theorem 2.4.7 and we show in Proposition 2.4.20 that is Set_Δ^+-enriched.

The main construction of this thesis yields a functor for each scaled simplicial set S

$$\mathbb{S}\mathrm{t}_S\colon (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}\longrightarrow \mathrm{Fun}(\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}},\mathrm{Set}_\Delta^{\mathbf{ms}})$$

called the *bicategorical straightening over S*. The functor itself is simply a more highly decorated version of previous straightening functors (e.g., that of [Lur09a]), and is discussed in detail in Chapter 2. We then show that $\mathbb{S}\mathrm{t}_S$ admits a right adjoint $\mathbb{U}\mathrm{n}_S$ which we call the (bicategorical) *unstraightening over S*. As already mentioned, the category $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ carries a model structure which models 2-Cartesian fibrations. If we equip the category of Set_Δ^+-enriched functors $\mathfrak{C}[S]^{\mathrm{op}}\to\mathrm{Set}_\Delta^{\mathbf{ms}}$ with the projective model structure, we obtain an enriched model category which models the ∞-category of ∞-bifunctors $S^{\mathrm{op}}\to\mathbb{B}\mathrm{icat}_\infty$. The main technical result of this paper is that this adjunction is in fact a Quillen equivalence.

Theorem (3.2.85)**.** *Let S be a scaled simplicial set. Then the bicategorical straightening-unstraightening adjunction defines a Quillen equivalence*

$$\mathbb{S}\mathrm{t}_S:(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}\rightleftarrows(\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}[S]^{\mathrm{op}}}:\mathbb{U}\mathrm{n}_S$$

between the model structure on (outer) 2-Cartesian fibrations over S and the projective model structure on Set_Δ^+*-enriched functors* $\mathfrak{C}[S]^{\mathrm{op}}\to\mathrm{Set}_\Delta^{\mathbf{ms}}$ *with values in marked-scaled simplicial sets.*

[6] We denote by $(S,E_S,T_S\subset C_S)=(S,\sharp,T_S\subset\sharp)$ the **MB** simplicial set such that every edge belongs to E_S and every triangle belongs to C_S

Observe that both model categories are in fact Set_Δ^+-enriched categories. After performing elementary explicit verifications we prove that the functor $\mathbb{U}\mathrm{n}_S$ is compatible with the (co)tensoring yielding an upgrade of the previous theorem to an intrinsic bicategorical result.

Theorem (3.2.90)**.** *The bicategorical straightening is a left Quillen equivalence for any scaled simplicial set S. Moreover, the functor $\mathbb{U}\mathrm{n}_S$ provides an equivalence of ∞-bicategories*

$$2\mathbb{C}\mathrm{art}(S) \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathbb{B}\mathrm{icat}_\infty).$$

A relative 2-nerve

Although it is desirable to have an ∞-bicategorical Grothendieck construction that works in the most the general context possible, many practical applications make use of ∞-bicategories which arise as scaled nerves of strict 2-categories. We provide a version of the Grothendieck construction better suited to this particular situation in the appendix. In this context, we define an explicit version $\chi_{\mathbb{C}}$ of the unstraightening functor over $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$, which we call the *relative 2-nerve.*

Theorem (3.3.20)**.** *Let $\mathbb{C}$ be a strict 2-category. Then there is a Quillen equivalence*

$$\phi_{\mathbb{C}} : (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \rightleftarrows (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathbb{C}^{(\mathrm{op},-)}} : \chi_{\mathbb{C}}$$

and an equivalence of left-derived functors[7] $\mathrm{L}\mathbb{S}\mathrm{t}_{\mathbb{C}} \overset{\simeq}{\Rightarrow} \mathrm{L}\phi_{\mathbb{C}}$.

As in the ∞-categorical setting (see Section 3.2.5 in [Lur09a]) the benefits of a relative nerve construction are twofold: on the one hand, the relative 2-nerve is particularly computationally tractable and well-suited to explicit examples; on the other, the relative 2-nerve allows us to compare our ∞-bicategorical Grothendieck construction to preexisting strict Grothendieck constructions. We apply our relative nerve construction to obtain a comparison with the Grothendieck construction appearing in [Buc14]. The strict 2-categorical Grothendieck construction of [Buc14] takes the form of an equivalence

$$\mathbb{E}\mathrm{l}\colon\ \mathrm{Fun}^{\mathrm{ps}}(\mathbb{C}^{\mathrm{op}}, \mathrm{Cat}_2) \longrightarrow 2\,\mathrm{Cart}$$

for a 2-category $\mathbb{C}$. The final result of our appendix shows that the relative 2-nerve coincides with $\mathbb{E}\mathrm{l}$ for every strict 2-functor with values in 2-categories.

Theorem (3.3.21)**.** *Let $F\colon \mathbb{C}^{(\mathrm{op},-)} \longrightarrow 2\mathrm{Cat}$ be a 2-functor, and let $\tilde{F}$ denote the composite*

$$\mathbb{C}^{(\mathrm{op},-)} \longrightarrow 2\mathrm{Cat} \longrightarrow \mathrm{Set}_\Delta^{\mathbf{ms}}$$

Then there is an equivalence

$$\begin{array}{ccc} \chi_{\mathbb{C}}(\tilde{F}) & \xrightarrow{\simeq} & (\mathrm{N}_2(\mathbb{E}\mathrm{l}(F)), M, T \subset C) \\ & \searrow \quad \swarrow & \\ & \mathrm{N}^{\mathrm{sc}}(\mathbb{C}) & \end{array}$$

of 2-Cartesian fibrations over $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$.

[7] see section 3.3.2 for a precise definition of $\mathbb{S}\mathrm{t}_{\mathbb{C}}$

Application: 2-simplicial objects and generic adjunctions

The main motivation to develop this ∞-bicategorical theory of fibrations and cofinality was to establish that the simplex 2-category $\mathbb{\Delta}$ satisfies a 2-dimensional universal property. Let us recall that a morphism $f : [n] \to [m]$ in Δ is given by an order preserving map between the totally ordered sets $[n]$ and $[m]$. The 2-categorical enhancement of Δ can be explained in simple terms: Given a pair of morphisms $f, g : [n] \to [m]$ we say that $f \leqslant g$ if and only if for every $i \in [n]$, we have $f(i) \leqslant g(i)$. This equips the Hom-sets of Δ with the structure of a poset. The resulting 2-category is precisely $\mathbb{\Delta}$. We will call a functor of ∞-bicategories $F : \mathbb{\Delta}^{\mathrm{op}} \to \mathbb{A}$ a 2-simplicial object in $\mathbb{A}$.

In [Dyck21] Dyckerhoff makes extensive use of 2-simplicial objects, to prove a categorified version of the Dold-Kan correspondence.

Theorem. *The categorified normalized chains functor $\mathcal{C}$ furnishes an equivalence*

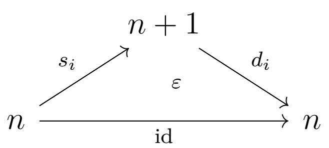
$$\mathcal{C} : \mathcal{S}t_{\mathbb{\Delta}} \rightleftarrows \mathrm{Ch}_{\geqslant 0}(\mathcal{S}t) : \mathcal{N}$$

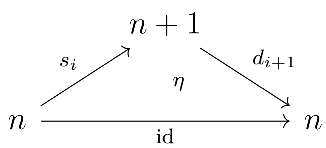
between the ∞-category $\mathcal{S}t_{\mathbb{\Delta}}$ of 2-simplicial stable ∞-categories and the ∞-category $\mathrm{Ch}_{\geqslant 0}(\mathcal{S}t)$ of connective chain complexes of stable ∞-categories with explicit inverse given by the categorified Dold-Kan nerve $\mathcal{N}$.

In the program proposed in [Dyck21] to categorify homological algebra, we see that the study of 2-simplicial objects will play a relevant role. It is therefore desirable to have a systematic way of analyzing the bicategory $\mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathbb{A})$ for an arbitrary $\mathbb{A}$.

Let d_i (resp. s_i) denote the face (resp. degeneracy) operators in $\mathbb{\Delta}^{\mathrm{op}}$. One easily checks that the inequalities $\mathrm{id} \leqslant s_i d_i$ and $s_i d_{i+1} \leqslant \mathrm{id}$ show that the identity 2-morphisms $d_i s_i = \mathrm{id}$ and $\mathrm{id} = d_{i+1} s_i = \mathrm{id}$ are the counit and the unit respectively of a pair of adjunctions exhibiting $d_i \dashv s_i$ and $s_i \dashv d_{i+1}$. The universal property we are after identifies these adjunctions as the generators of $\mathbb{\Delta}^{\mathrm{op}}$ as a 2-category.

In order to state the universal property we fix some notation. Observe that we have equalities $d_i s_i = \mathrm{id}$ and $d_{i+1} s_i$ which we view as thin 2-simplices





Then we can formulate the universal property of $\mathbb{\Delta}^{\mathrm{op}}$ in the following terms.

Conjecture. *Let $\mathbb{A}$ be an ∞-bicategory. Then restriction along ι induces an equivalence of ∞-bicategories*

$$\iota^* \colon \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathbb{A}) \xrightarrow{\simeq} \mathrm{Fun}^{\dashv}(\Delta^{\mathrm{op}}, \mathbb{A})$$

between the ∞-bicategory of 2-simplicial objects in $\mathbb{A}$ and the ∞-bicategory of simplicial objects in $\mathbb{A}$ that send d_i and s_i (resp d_{i+1} and s_i) to an adjunction $F(d_i) \dashv F(s_i)$ with counit given by $F(\varepsilon)$ (resp. $F(s_i) \dashv F(d_{i+1})$ with unit given by $F(\eta)$).

We would like to point out that the previous conjecture has been proved by the author (in an unpublished note) in the case when $\mathbb{A}$ is replaced by a strict 2-category $\mathbb{A}$ using the well-established theory of 2-categorical Kan extensions. The proof of the general ∞-bicategorical statement will appear in [AG22b].

The necessary technology for the proof of the ∞-bicategorical statement is not fully developed: We need a theory of ∞-bicategorical Kan extensions in the language of scaled simplicial sets. Recall that given a functor of ∞-categories $f : \mathcal{C} \to \mathcal{D}$ and a sufficiently cocomplete ∞-category $\mathcal{A}$ we can produce a left adjoint to the restriction functor

$$f_! : \operatorname{Fun}(\mathcal{C}, \mathcal{A}) \rightleftarrows \operatorname{Fun}(\mathcal{D}, \mathcal{A}) : f^*$$

Given a functor $G : \mathcal{C} \to \mathcal{A}$ the value of $f_! G(d)$ is computed by the colimit of the functor

$$\mathcal{C}_{/d} \longrightarrow \mathcal{C} \xrightarrow{G} \mathcal{A}$$

where $\mathcal{C}_{/d}$ is the category having as objects morphisms $u : f(c) \to d$ with $c \in \mathcal{C}$ and with morphisms given by commutative diagrams

$$\begin{array}{ccc} f(c) & \xrightarrow{f(\alpha)} & f(c') \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle v} & \\ & d & \end{array}$$

Moreover the counit $\eta : f_! \circ f^* \Rightarrow \mathrm{id}$ gives for every $T : \mathcal{D} \to \mathcal{A}$ a natural transformation $\eta_T : f_! f^* T \Rightarrow T$. In order to show that f^* is fully-faithful it is enough to show that η_T is an equivalence of functors for every T. Let us explain how to use the theory of cofinality in this context. For every $d \in \mathcal{D}$ we consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{C}_{/d} & \xrightarrow{\rho} & \mathcal{D}_{/d} & & \\ \downarrow{\scriptstyle p_{\mathcal{C}}} & & \downarrow{\scriptstyle p_{\mathcal{D}}} & & \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} & \xrightarrow{T} & \mathcal{A} \end{array}$$

We observe that the colimit of $\hat{T} = T \circ f \circ p_{\mathcal{C}}$ is precisely the value of $f_! f^* T(d)$. We can also see that since $\mathcal{D}_{/d}$ has a terminal object given by the identity morphism we have a canonical comparison map

$$\operatorname{colim}_{\mathcal{C}_{/d}} \hat{T} = \operatorname{colim}_{\mathcal{C}_{/d}}(T \circ p_{\mathcal{D}} \circ \rho) \longrightarrow \operatorname{colim}_{\mathcal{D}_{/d}}(T \circ p_{\mathcal{D}}) \simeq T(d)$$

After unraveling the definitions we can see that the morphism above can be identified with the component of η_T at the object d. Therefore it follows that if the map $\mathcal{C}_{/d} \to \mathcal{D}_{/d}$ is cofinal for every $d \in \mathcal{D}$ then the restriction functor f^* must be fully-faithful.

In Chapter 5 we will explain how to generalize the previous statement to the setting of ∞-bicategories conditional to the existence to a well-behaved theory of Kan extensions of ∞-bicategories. In particular we will show in[8]

[8] We are using the notation ⧫ to denote statements which are proved conditional to the existence of a 2-dimensional theory of Kan extensions

Proposition♦ 5.0.4 that given a functor $f:\mathbb{C}\to\mathbb{D}$ of ∞-bicategories such that for every $d\in\mathbb{D}$ the induced morphism

$$f_d\colon \mathbb{C}^{\natural}_{\nearrow d} \longrightarrow \mathbb{D}^{\natural}_{\nearrow d}$$

is cofinal with respect to the marking given by the Cartesian edges of the 2-Cartesian fibration $\mathbb{C}^{\natural}_{\nearrow d}\to\mathbb{C}$ (resp. $\mathbb{D}^{\natural}_{\nearrow d}\to\mathbb{D}$) then it follows that *for every* ∞-bicategory $\mathbb{A}$ the restriction functor

$$f^*\colon\ \mathrm{Fun}(\mathbb{D},\mathbb{A}) \longrightarrow \mathrm{Fun}(\mathbb{C},\mathbb{A})$$

is fully faithful. This will allow us to prove the following theorem:

Theorem♦ (5.2.1)**.** *Let $\mathbb{A}$ be an ∞-bicategory. Then the restriction functor $\iota:\Delta^{\mathrm{op}}\to\mathbb{\Delta}^{\mathrm{op}}$ induces a fully-faithful functor of ∞-bicategories*

$$\iota^*\colon\ \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}},\mathbb{A}) \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}},\mathbb{A})$$

It is clear by analogy with the ∞-categorical case that in order to identify the essential image of the restriction functor ι^* it will suffice to show that for every functor $F:\Delta^{\mathrm{op}}\to\mathbb{A}$ satisfying the adjunction property we have a natural equivalence $F\Rightarrow\iota^*\iota_!F$. We will show in Theorem♦ 5.0.2 how to reduce the question of essential surjectivity to the case where $\mathbb{A}=\mathbb{C}\mathrm{at}_\infty$ but we will postpone the computation for future work.

As a warm-up for the proof of essential surjectivity for $\mathbb{\Delta}$ we will do a careful study of the 2-category $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$ which incorporates several features of $\mathbb{\Delta}$.

Definition. *We define a 2-category $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$ consisting in:*

- *A pair of objects $-$ and $+$.*
- *The mapping categories $\mathbb{A}\mathrm{dj}^{\mathrm{R}}(x,y)$ are given by the terminal category $*$ except when $x=y=-$, in which case $\mathbb{A}\mathrm{dj}^{\mathrm{R}}(-,-)$ is given by a unique morphism which we denote as $\eta:\mathrm{id}_-\Rightarrow RL$ where L is the unique object in $\mathbb{A}\mathrm{dj}^{\mathrm{R}}(-,+)$ and R is the unique object in $\mathbb{A}\mathrm{dj}^{\mathrm{R}}(+,-)$.*

We depict the 2-category $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$ diagrammatically as follows:

$$- \underset{R}{\overset{L}{\rightleftarrows}} + \ , \qquad \mathrm{id}_- \overset{\eta}{\Longrightarrow} RL$$

This data is required to satisfy the following relations:

- $LR=\mathrm{id}_+$.
- $L*\eta=\mathbb{1}_L$*, where the notation $*$ stands for horizontal composition.*
- $\eta*R=\mathbb{1}_R$.

This definition yields a 2-category that we will call the walking adjunction with fully faithful right adjoint.

We will denote by Adj^R the underlying 1-category of $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$. We will show that for $\varepsilon \in \{-,+\}$ the morphism

$$(\mathrm{Adj}^R)^{\natural}_{\nearrow\varepsilon} \longrightarrow (\mathbb{A}\mathrm{dj}^{\mathrm{R}})^{\natural}_{\nearrow\varepsilon}$$

is marked cofinal. Apart from this fully-faithfulness computation we will show that given $F : \mathrm{Adj}^R \to \mathbb{C}\mathrm{at}_\infty$ sending the unique morphism $R : + \to -$ to a fully faithful right adjoint we have equivalences

$$\underset{\mathrm{Adj}^{\mathrm{R}}}{\overset{\natural}{\mathrm{colim}}}\, F \circ p_\varepsilon \simeq F(\varepsilon)$$

where $p_\varepsilon : (\mathrm{Adj}^R)^{\natural}_{\nearrow\varepsilon} \to \mathrm{Adj}^R$. This will imply that the conditions of Theorem♦ 5.0.2 are satisfied yielding the next result.

Theorem♦ (5.1.1)**.** *Let $\mathbb{A}$ be an ∞-bicategory. Then restriction along ι : $\mathrm{Adj}^{\mathrm{R}} \to \mathbb{A}\mathrm{dj}^{\mathrm{R}}$ induces an equivalence of ∞-bicategories*

$$\mathrm{Fun}(\mathbb{A}\mathrm{dj}^{\mathrm{R}}, \mathbb{A}) \xrightarrow{\simeq} \mathrm{Fun}^{\dashv}(\mathrm{Adj}^{\mathrm{R}}, \mathbb{A})$$

where $\mathrm{Fun}^{\dashv}(\mathrm{Adj}^{\mathrm{R}}, \mathbb{A})$ the full subcategory on those functors $F : \mathrm{Adj}^R \to \mathbb{C}\mathrm{at}_\infty$ sending the unique morphism $R : + \to -$ to a fully faithful right adjoint.

Chapter 1

Fibrations and colimits of ∞-categories

The aim of this chapter is to introduce the concepts that will be later generalized to the setting of ∞-bicategories. This is not intended as an exhaustive review of the theory of fibrations and colimits in ∞-categories but as a warm-up discussion. We will refer the reader to [Lur09a] and [Cis19] for an extensive treatment of the theory of ∞-categories. We would also like to point out that in order to avoid a duplication of preliminary sections we will postpone our recapitulation of basic definitions and constructions to the next chapter.

1.1 Marked simplicial sets and the Cartesian model structure

In this section we will review the basics of the theory of marked simplicial sets and the key steps necessary for the construction of the Cartesian model structure. We will generalize this section to the setting of ∞-bicategories in chapter 2.

Definition 1.1.1. A *marked simplicial set* is a pair (X, E_X) consisting of a simplicial set X together with a subset $E_X \subseteq X_1$ of the set of 1-simplices containing the every degenerate edge. We say that an edge of X is marked if it belongs to E_X.

A morphism of marked simplicial sets $(X, E_X) \to (Y, E_Y)$ is given by a map of the underlying simplicial sets $f : X \to Y$ such that $f(E_X) \subseteq E_Y$. We denote by Set_Δ^+ the category of marked simplicial sets.

Remark 1.1.2. We will frequently use a superscript notation $X^\dagger := (X, E_X)$ to denote a marked simplicial set. The notation $X^\flat := (X, \flat)$ will denote a marked simplicial set where only the degenerate edges are marked. Analogously the notation $X^\sharp := (X, \sharp)$ will denote a marked simplicial set in which *all* edges are marked. We observe that given a marked simplicial set (X, E_X) the set of marked edges is precisely given by the set of maps $(\Delta^1, \sharp) \to (X, E_X)$.

Remark 1.1.3. We will abuse notation and define the marking of a simplicial set X by just specifying the marked *non-degenerate edges*.

Remark 1.1.4. It is not hard to verify explicitly that the category Set_Δ^+ is locally presentable in the sense of [AR94, Definition 1.17]. However, we will present here a somewhat conceptual proof of this fact which will easily generalize to the bicategorical setting.

Definition 1.1.5. We define a category $\Delta_{\mathbf{M}}$ by appending to the simplex category Δ an extra object which we denote by $[1]_+$ and morphisms

$$i_+ : [1] \to [1]_+ \quad , \qquad s_+^0 : [1]_+ \to [0]$$

such that $s_+^0 \circ i_+ = s^0$. We observe that we have a fully-faithful functor $R : \mathrm{Set}_\Delta^+ \to \mathrm{Fun}(\Delta_{\mathbf{M}}^{\mathrm{op}}, \mathrm{Set})$ which sends the pair (X, E_X) to the functor $R(X)$ such that $R(X)([n]) = X_n$ and $R(X)([1]_+) = E_X$. The functor i_+ gets mapped under $R(X)$ to the inclusion $E_X \subseteq X_1$ and similarly the map s_+^0 gets mapped to the inclusion of degenerate edges into E_X. We note that the functor R has essential image those functors $F : \Delta_{\mathbf{M}}^{\mathrm{op}} \to \mathrm{Set}$ sending the map i_+ to a monomorphism.

Remark 1.1.6. Note that $\mathrm{Fun}(\Delta_{\mathbf{M}}^{\mathrm{op}}, \mathrm{Set})$ is locally presentable since it is a presheaf category. In order to show that Set_Δ^+ is locally presentable one easily checks the following:

- The functor R in the previous definition admits a left adjoint.
- The essential image of R is stable under directed colimits.

The conclusion follows from [AR94, Theorem 1.46].

Definition 1.1.7. Let (X, E_X) and (Y, E_Y) be a pair of marked simplicial sets. We define the product $(X, E_X) \times (Y, E_Y)$ as follows:

- The underlying simplicial set is given by the product of the underlying simplicial sets $X \times Y$.
- An edge in the product $\Delta^1 \to X \times Y$ is declared to be marked if *both* of its projections are marked in the target.

Definition 1.1.8. Given a pair of marked simplicial sets $X, Y \in \mathrm{Set}_\Delta^+$ (where we have omitted the notation specifying the marking for clarity) we define the a marked simplicial set $\mathrm{Fun}^+(X, Y)$ characterized by the following universal property

$$\mathrm{Hom}_{\mathrm{Set}_\Delta^+}\Big(Z, \mathrm{Fun}^+(X, Y)\Big) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta^+}(Z \times X, Y)$$

Definition 1.1.9. The set of *generating marked anodyne morphisms* $\mathbf{M}$ is the set of maps of marked simplicial sets consisting of:

A1) The inner horn inclusions

$$(\Lambda_i^n, \flat) \to (\Delta^n, \flat) \quad , \quad n \geqslant 2 \quad , \quad 0 < i < n.$$

A2) The right horn inclusions

$$\Big(\Lambda_n^n, \Delta^{\{n-1,n\}}\Big) \to \Big(\Delta^n, \Delta^{\{n-1,n\}}\Big) \quad , \quad n \geqslant 2.$$

A3) The inclusion of the terminal vertex

$$(\Delta^0, \sharp) \to (\Delta^1, \sharp).$$

S1) The map

$$\left(\Delta^2, \{\Delta^{\{0,1\}}, \Delta^{\{1,2\}}\}\right) \to \left(\Delta^2, \sharp\right).$$

K) For every Kan complex K the map

$$(K, \flat) \to (K, \sharp).$$

A map of marked simplicial sets is said to be *marked anodyne* (or **M**-anodyne) if it belongs to the weakly saturated closure (see [Lur09a, Definition A.1.2.2]) of **M**.

Remark 1.1.10. Observe that the morphisms of type K do not form a set. However, in the previous definition it suffices to allow K to range over a set of representatives for all isomorphism classes of Kan complexes with only countably many simplices as noted in [Lur09a, Remark 3.1.1.3].

Definition 1.1.11. Let $f : (X, E_X) \to (Y, E_Y)$ be a map of marked simplicial sets. We say that f is a **M**-fibration if it has the right lifting property against the class of marked anodyne morphisms.

Remark 1.1.12. Let $X^\natural \to \Delta^0$ be a **M**-fibration. The existence of lifts against of morphisms of type $A1$) in Definition 1.1.9 guarantees that the underlying simplicial set of $X^\natural$ is an ∞-category. Lifts against morphisms of type $A2$) imply that marked edges are equivalences. Finally the existence of lifts against morphisms of type K) guarantees that every equivalence is marked.

Definition 1.1.13. Let $f : (X, E_X) \to (Y, E_Y)$ be a map of marked simplicial sets. We say that f is a *cofibration* if it is a monomorphism in Set_Δ^+. Equivalently, the underlying map of simplicial sets $X \to Y$ is a monomorphism in Set_Δ.

The next technical result is of key importance in the construction of the Cartesian model structure. In the next chapter we will prove a generalization of this result Proposition 2.2.14, which will also be essential in establishing the 2-Cartesian model structure.

Proposition 1.1.14. *Let $f : (A, E_A) \to (B, E_B)$ be a marked anodyne morphism. Given a cofibration $g : (X, E_X) \to (Y, E_Y)$ then the pushout-product*[1]

$$f \wedge g : B \times X \coprod_{A \times X} A \times Y \to B \times Y$$

is marked anodyne.

Proof. A proof of this fact we can found in [Lur09a, Proposition 3.1.2.3.]. □

We can now derive some formal properties from Proposition 1.1.14.

[1] We have omitted the marking in the notation for readability

Lemma 1.1.15. *Let $(Y, E_Y) \to (S, E_S)$ be an $\boldsymbol{M}$-fibration. Then for every marked simplicial set (X, E_X) the induced morphism*

$$\mathrm{Fun}^+(X, Y) \longrightarrow \mathrm{Fun}^+(X, S)$$

is an $\boldsymbol{M}$-fibration.

Proof. Given a marked anodyne morphism $A \to B$ we can construct a the pair of adjoint lifting problems

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Fun}^+(X,Y) \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & \mathrm{Fun}^+(X,S) \end{array} \qquad \begin{array}{ccc} A \times X & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ B \times X & \longrightarrow & S \end{array}$$

The solution of the right-most lifting problem follows from Proposition 1.1.14 in the special case where the cofibration is of the form $\varnothing \to X$. □

Definition 1.1.16. Let $p : X \to S$ and $q : Y \to S$ be a pair of morphisms of marked simplicial sets where q is a **M**-fibration. We define the mapping ∞-category over S as the following pullback

$$\begin{array}{ccc} \mathrm{Map}_S(X,Y) & \longrightarrow & \mathrm{Fun}^+(X,Y) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{\;p\;} & \mathrm{Fun}^+(X,S) \end{array}$$

where the left-most vertical map is also a **M**-fibration by Lemma 1.1.15.

Remark 1.1.17. The objects of $\mathrm{Map}_S(X, Y)$ are given by morphisms of marked simplicial sets $\alpha : X \to Y$ such that $q \circ \alpha = p$. Similarly the morphisms of the mapping ∞-category are given by fibrewise homotopies.

Proposition 1.1.18. *Let $p : X \to S$ be a $\boldsymbol{M}$-fibration. Suppose that we are given maps $A \xrightarrow{u} B \to S$ such that u is a cofibration (resp. $\boldsymbol{M}$-anodyne). Then the induced morphism*

$$u^* \colon \mathrm{Map}_S(B, X) \longrightarrow \mathrm{Map}_S(A, X)$$

is a fibration in the Joyal model structure (resp. trivial fibration).

Proof. Given a morphism $K \to L \to S$ we observe that we have adjoint lifting problems

$$\begin{array}{ccc} K & \longrightarrow & \mathrm{Map}_S(B,X) \\ \downarrow & \nearrow & \downarrow \\ L & \longrightarrow & \mathrm{Map}_S(A,X) \end{array} \qquad \begin{array}{ccc} B \times K \coprod_{A \times K} A \times L & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ B \times L & \longrightarrow & S \end{array}$$

If u is a cofibration and $K \to L$ is **M**-anodyne then the existence of dotted arrow follows from Proposition 1.1.14. It follows that u^* satisfies the conditions of [Lur09a, Corollary 2.4.6.5] that is, u^* is an inner fibration and an isofibration which implies that u^* is a fibration in the Joyal model structure. The other case follows immediately. □

Definition 1.1.19. Let S be a simplicial set. We denote by $(\mathrm{Set}_\Delta^+)_{/S}$ the category of marked simplicial sets over $(S, \sharp)$. We say that an object $p : X \to S$ in $(\mathrm{Set}_\Delta^+)_{/S}$ is a *Cartesian fibration* if p is a $\mathbf{M}$-fibration.

Definition 1.1.20. We have the following distinguished classes of morphisms in $(\mathrm{Set}_\Delta^+)_{/S}$:

- A morphism $A \xrightarrow{u} B \to S$ is said to be a *cofibration* if u is a monomorphism in Set_Δ^+.
- A morphism $A \xrightarrow{u} B \to S$ is said to be a *Cartesian equivalence* if for every Cartesian fibration $X \to S$ the induced map

  $$u^* \colon \operatorname{Map}_S(B, X) \longrightarrow \operatorname{Map}_S(A, X)$$

  is a categorical equivalence, i.e, a weak equivalence in Joyal's model structure.
- We say that a morphism is a trivial cofibration if it is simultaneously a cofibration and a Cartesian equivalence.

Remark 1.1.21. Observe that it follows from Proposition 1.1.18 that if u is a trivial cofibration then the induced morphism $u^* : \operatorname{Map}_S(B, X) \to \operatorname{Map}_S(A, X)$ is a trivial fibration in Joyal's model structure.

Remark 1.1.22. Let $p : X \to S$ be a Cartesian fibration and suppose we are given a lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle u}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & S \end{array}$$

such that the map u is a trivial cofibration. Let us view the composite $A \to X \to S$ as an object in $\rho \in \operatorname{Map}_S(A, X)$. Then it follows that a solution to the lifting problem is given by a preimage of ρ under the map

$$u^* \colon \operatorname{Map}_S(B, X) \longrightarrow \operatorname{Map}_S(A, X).$$

Note that the previous remark implies that u^* is a trivial fibration and therefore the preimage of ρ exists. We conclude that the dotted arrow also exists.

Up to this point we have defined the necessary classes of morphisms for the construction of the Cartesian model structure. Moreover, Remark 1.1.22 will allow us to identify the fibrant objects. Before formally stating the existence of the Cartesian model structure we need to introduce a result which is essential both for the construction of the model structure and for the ∞-categorical Grothendieck construction that we will review later in this chapter.

Proposition 1.1.23. *Suppose that we have a morphism*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of Cartesian fibrations over a simplicial set S. Then the following statements are equivalent:

i) The map f is a Cartesian equivalence.

ii) There exists a morphism $g : X \to Y$ over S which is an inverse up to homotopy for f.

iii) For every $s \in S$ the induced morphism on fibres $f_s : X_s \to Y_s$ is a categorical equivalence.

Proof. See [Lur09a, Proposition 3.1.3.5]. □

Remark 1.1.24. The importance of the previous proposition is twofold: First it is of practical relevance since checking that a map is an equivalence upon passage to fibres is usually way easier than verifying that it induces an equivalence on mapping ∞-categories. The second reason is that if we want Cartesian fibrations to be able to model $\mathbb{C}\mathrm{at}_\infty$-valued functors, it is necessary that equivalences are detected fibrewise. In other words, under the Grothendieck construction a map of Cartesian fibrations corresponds to a natural transformation $\eta : F \Rightarrow G$ of functors $S^{\mathrm{op}} \to \mathbb{C}\mathrm{at}_\infty$. Since η is an equivalence if and only if $\eta_c : F(c) \to G(c)$ is an equivalence of ∞-categories we see that in the realm of fibrations we need "pointwise criterion" for detecting weak equivalences as well.

The proof of Proposition 1.1.23 found in [Lur09a] will be greatly generalized later in this document (see Proposition 2.2.39). We comment the main differences with respect to the bicategorical case. First let us point out that in both situations the implications $i) \implies ii) \implies iii)$ follow quite easily from each other. The key part of the argument is to show that $iii) \implies i)$. One sees that it is sufficent in order to show the claim to restrict our attention to the case where the base is $S = \Delta^n$. In the setting of ∞-categories we are quite lucky: The base Δ^n is itself an ∞-category so we have plenty of structure to play with. This used heavily by Lurie in his proof of the fibrewise criterion for equivalences. However, we would like to point out that when working in the ∞-bicategorical situation the (flat scaled) simplices Δ^n are no longer fibrant which increases the complexity of the proof in this situation.

We are ready to state the main result of this section whose ∞-bicategorical generalization will appear as Theorem 2.2.43 in the next chapter.

Theorem 1.1.25. *Let S be a simplicial set. There exists a simplicial combinatorial model structure on $\left(\mathrm{Set}_\Delta^+\right)_{/S}$ which may be described as follows:*

C) The cofibrations are given by those morphisms $X \to Y$ which are monomorphisms in Set_Δ^+.

W) The weak equivalences are given by Cartesian equivalences (see Definition 1.1.20).

F) The fibrations in $\left(\mathrm{Set}_\Delta^+\right)_{/S}$ are given by those maps having the right lifting property against every morphism which is simultaneously a cofibration and a weak equivalence.

Proof. See [Lur09a, Proposition 3.1.3.7]. □

Remark 1.1.26. We would like to remind that reader that the notion of Cartesian fibration can be dualized to obtain coCartesian fibrations. More precisely, a map of marked simplicial sets $X^{\natural} \to S^{\sharp}$ is a coCartesian fibration if and only if $(X^{\natural})^{\mathrm{op}} \to (S^{\sharp})^{\mathrm{op}}$ is a Cartesian fibration. There is consequently a model structure on $\left(\mathrm{Set}_{\Delta}^{+}\right)_{/S}$ which models coCartesian fibrations. One can construct the model structure starting with the dual class of marked anodyne morphisms and repeating each step in this section.

1.2 The ∞-categorical Grothendieck construction

In this section we will review one of the main theorems of [Lur09a] and probably one of the most important tools in the study of ∞-categories: The Grothendieck construction. Given a simplicial set S, the Grothendieck construction yields an equivalence of ∞-categories

$$\mathrm{St}_S : \mathfrak{Cart}(S) \rightleftarrows \mathrm{Fun}(S^{\mathrm{op}}, \mathbb{C}\mathrm{at}_{\infty}) : \mathrm{Un}_S$$

between the ∞-category of Cartesian fibrations over S and the ∞-category of functors $S^{\mathrm{op}} \to \mathbb{C}\mathrm{at}_{\infty}$. The functors realizing this equivalence are known as straightening and unstraightening respectively.

When dealing with homotopy coherent situations the study of functor categories becomes substantially more complicated than in the ordinary setting. This is the main reason why, the use of Cartesian fibrations is an indispensable tool that allows us to effectively work with functors with values in ∞-categories.

The equivalence between these two ∞-categories is obtained by exhibiting a Quillen equivalence between certain model categories modelling the left and right-hand side respectively. On the left, we will use the Cartesian model structure discussed in the previous section. We will now define the model category which will model the right-hand side of our equivalence.

Remark 1.2.1. We will identify $\mathrm{Set}_{\Delta}^{+}$ with $\left(\mathrm{Set}_{\Delta}^{+}\right)_{/\Delta^0}$ and view it as a model category using Theorem 1.1.25. It follows that the fibrant objects of $\mathrm{Set}_{\Delta}^{+}$ are given by ∞-categories where the marking is given by the equivalences. It is easy to verify that this category is Quillen equivalent to the category of simplicial sets equipped with the Joyal model structure.

Definition 1.2.2. We will endow $\mathrm{Set}_{\Delta}^{+}$ with the structure of a Set_{Δ}-enriched cateogory as follows. Given a pair of marked simplicial sets X, Y we define $\mathrm{Map}^{\simeq}(X,Y)$ to be the simplicial subset of $\mathrm{Fun}^{+}(X,Y)$ (see Definition 1.1.8) given by those simplices $\sigma : \Delta^n \to \mathrm{Fun}^{+}(X,Y)$ mapping each edge of Δ^n to a marked edge in $\mathrm{Fun}^{+}(X,Y)$. Note that if X, Y are fibrant marked simplicial sets then $\mathrm{Map}^{\simeq}(X,Y)$ is a Kan complex.

Theorem 1.2.3 (A.3.3 [Lur09a]). *Let $\mathfrak{C}$ be a Set_{Δ}-enriched category. We endow the category of simplicially enriched functors $\mathrm{Fun}(\mathfrak{C}, \mathrm{Set}_{\Delta}^{+})$ with the* projective model structure *which is uniquely characterized by the following classes of morphisms:*

W) *A natural transformation* $F \Rightarrow G$ *is a* weak equivalence *if for every* $c \in \mathcal{C}$ *the morphism* $F(c) \to G(c)$ *is a weak equivalence in* Set_Δ^+.

F) *A natural transformation* $F \Rightarrow G$ *is a* fibration *if for every* $c \in \mathcal{C}$ *the morphism* $F(c) \to G(c)$ *is a fibration in* Set_Δ^+.

The ∞-categorical Grothendieck construction is realized in the next theorem which appears as Theorem 3.2.0.1 in [Lur09a].

Theorem 1.2.4. *Let* S *be a simplicial set,* $\mathcal{C}$ *a simplicial category and* $\phi : \mathfrak{C}[S] \to \mathcal{C}$ *(see section 1.1.5 in [Lur09a] for a definition of* $\mathfrak{C}$*) a functor between simplicial categories. Then there exists a pair of adjoint functors*

$$\mathrm{St}_\phi : \left(\mathrm{Set}_\Delta^+\right)_{/S} \rightleftarrows \left(\mathrm{Set}_\Delta^+\right)^{\mathcal{C}^{\mathrm{op}}} : \mathrm{Un}_\phi$$

with the following properties:

1. *The functors* $(\mathrm{St}_\phi, \mathrm{Un}_\phi)$ *determine a Quillen adjunction between the category* $\left(\mathrm{Set}_\Delta^+\right)_{/S}$ *(with the Cartesian model structure) and the category* $\left(\mathrm{Set}_\Delta^+\right)^{\mathcal{C}^{\mathrm{op}}}$ *(with the projective model structure).*
2. *If* ϕ *is an equivalence of simplicial categories, then* $(\mathrm{St}_\phi, \mathrm{Un}_\phi)$ *is a Quillen equivalence.*

Remark 1.2.5. There is a subclass of the class of Cartesian fibrations known as right fibrations. A Cartesian fibration $p : X^\natural \to S$ is said to be a *right fibration* if every edge in $X^\natural$ is Cartesian. Under the Grothendieck construction right fibrations correspond to those functors $F : S^{\mathrm{op}} \to \mathbb{C}\mathrm{at}_\infty$ that factor through $\mathcal{S}$, the ∞-category of spaces.

Remark 1.2.6. It is also worth pointing out for the sake of completeness that there exists a coCartesian version (see Remark 1.1.26) of the straightening-unstraightening construction which models an equivalence of ∞-categories

$$\mathrm{St}_S^{\mathrm{co}} : \mathrm{co}\mathfrak{C}\mathrm{art}(S) \rightleftarrows \mathrm{Fun}(S, \mathbb{C}\mathrm{at}_\infty) : \mathrm{Un}_S^{\mathrm{co}}$$

between the ∞-category of coCartesian fibrations over S and the ∞-category of covariant functors $F : S \to \mathbb{C}\mathrm{at}_\infty$. A coCartesian fibration whose edges are all coCartesian is called a left fibration [Lur09a, Section 2.1] and corresponds via the coCartesian version of the Grothendieck construction to a covariant functor with values in spaces.

Before doing a quick overview of the main constructions involved in the previous theorem we would like to point out that in Chapter 3 we will produce a generalization of Theorem 1.2.4 to the ∞-bicategorical setting in Theorem 3.2.85.

Let us start by recalling that we have a Quillen equivalence

$$\mathfrak{C} : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Cat}_{\mathrm{Set}_\Delta} : \mathrm{N}$$

between the Joyal model structure on the category of simplicial sets and the Bergner model structure on the category of simplicially enriched categories. The functor $\mathfrak{C}$ is completely determined by specifying the values $\mathfrak{C}[\Delta^n] = \mathbb{O}^n$. We refer the reader to Definition 2.1.7 for a description of the ordinal 2-categories $\mathbb{O}^n$ for $n \geqslant 0$.

Suppose we are given an object in $\left(\mathrm{Set}_\Delta^+\right)_{/S}$ represented by a morphism $p : X \to S$ and a map of simplicial categories $\mathfrak{C}[S] \to \mathcal{C}$. Let $X^\triangleright := X * \Delta^0$ (see section 1.2.8 in [Lur09a]). We will denote by $*$ the cone point of $X^\triangleright$. We can now consider the pushout diagram of simplicial sets

$$\begin{array}{ccc} X & \longrightarrow & X^\triangleright \\ \downarrow{\scriptstyle p} & & \downarrow \\ S & \longrightarrow & X_S \end{array}$$

To define the functor $\mathrm{St}_\phi(p) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}_\Delta^+$ we consider a pushout diagram of Set_Δ-categories

$$\begin{array}{ccc} \mathfrak{C}[S] & \longrightarrow & \mathfrak{C}[X_S] \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ \mathcal{C} & \longrightarrow & X_\phi \end{array}$$

The functor $\mathrm{St}_\phi(p)$ will be given by a decorated version of the restricted representable functor

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}_\Delta, \;\; c \longmapsto X_\phi(c, *)$$

which we will explain immediately. Let $e : \Delta^1 \to X$ be a marked edge. This edge defines a 2-simplex $e * \Delta^0 : \Delta^2 \to X^\triangleright$. Applying the functor $\mathfrak{C}$ we obtain a morphism on mapping spaces

$$\mathbb{O}^2(0,2) \longrightarrow X_S(a, *)$$

which determines an edge in $\hat{e} : \Delta^1 \to X_S(a, *)$. We will declare the image of that edge to be marked in $X_\phi(p(a), *)$. To finish the definition of the straightening functor we need to extend the marking by functoriality. In other words, given a pair of elements $c, c' \in \mathcal{C}$, an edge $u : \Delta^1 \to \mathcal{C}(c, c')$ and a marked edge $e : \Delta^1 \to X_\phi(c, *)$ then its image under the morphism

$$\mathcal{C}(c, c') \times X_\phi(c', *) \longrightarrow X_\phi(c, *)$$

is also marked in the target. The marking on $X_\phi(c, *)$ is thus the unique marking containing those edges arising from marked edges in X which is closed under functoriality.

In Chapter 3 we will give an upgrade of this construction to yield an ∞-bicategorical enhacement of St_ϕ which we will denote by $\mathbb{S}\mathrm{t}_\phi$. We will show in Proposition 3.2.15 that our ∞-bicategorical straightening functor restricts under adequate conditions to the functor St_ϕ discussed in this section. We refer the reader to section 3.2.1 in [Lur09a] for an in depth discussion of the ∞-categorical straightening functor.

Definition 1.2.7. Let $p : X^\natural \to S$ be a Cartesian fibration. We say that p is the Cartesian fibration classifying the functor $F : S^{\mathrm{op}} \to \mathfrak{C}\mathrm{at}_\infty$ if there exists an equivalence of Cartesian fibrations $X^\natural \xrightarrow{\simeq} \mathrm{Un}_S(F)$ and similarly in the coCartesian case.

1.3 Colimits in ∞-categories

As pointed out in the introduction, the theory of colimits in ∞-categories is a powerful tool that allows us to study a wide range of homotopy-coherent universal properties. The goal of this section is to give an overview of how one sets up the theory of colimits in ∞-categories and to introduce the classical notion of cofinality as well as the conditions that characterize cofinal functors.

Let us suppose that we are given a map of simplicial sets $F : K \to \mathcal{A}$ where $\mathcal{A}$ is an ∞-category. For every object $a \in \mathcal{A}$ we denote by $\underline{a} : K \to \mathcal{A}$ the constant functor on the object a. A cone for the functor F with tip a is given by a natural transformation $F \Rightarrow \underline{a}$. Observe that we have a functor of ∞-categories $c_K : \mathcal{A} \to \mathrm{Fun}(K, \mathcal{A})$ which sends a simplex $\sigma : \Delta^n \to \mathcal{A}$ to the composite

$$K \times \Delta^n \longrightarrow \Delta^0 \times \Delta^n \xrightarrow{\sigma} \mathcal{A}$$

and note that given $a \in \mathcal{A}$ we have $c_K(a) = \underline{a}$. One can then define the colimit of F to be an object $\mathrm{colim}_K F \in \mathcal{A}$ corepresenting the functor

$$\mathcal{A} \xrightarrow{c_K} \mathrm{Fun}(K, \mathcal{A}) \xrightarrow{\mathrm{Nat}_K(F,-)} \mathcal{S}$$

where $\mathrm{Nat}_K(-,-)$ is the mapping space functor in $\mathrm{Fun}(K, \mathcal{A})$. To make this definition more computationally tractable one defines a coCartesian fibration classifying the previous functor and translates the representability condition to the language of fibrations of ∞-categories. Note that since the functor $\mathrm{Nat}_K(F, c_K(-))$ takes values in spaces it is classified by a left fibration.

Definition 1.3.1. Let $p : X \to \mathcal{A}$ be a left fibration and let $a : \Delta^0 \to \mathcal{A}$. We say that p is represented by the object a if there exists a morphism over $\mathcal{A}$

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{\quad u \quad} & X \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{A} & \end{array}$$

such that u is an equivalence in the coCartesian model structure.

Let us unravel the previous definition. Given $a : \Delta^0 \to \mathcal{A}$ we can construct a left fibration $\mathcal{A}_{a/} \to \mathcal{A}$ whose simplices $\hat{\sigma} : \Delta^n \to \mathcal{A}_{a/}$ are given by morphisms $\sigma : \Delta^{n+1} \to \mathcal{A}$ such that $\sigma(0) = a$. It is not hard to see that this left fibration is classified by the corepresentable functor

$$\begin{aligned} \mathcal{A}(a,-) \colon \mathcal{A} &\longrightarrow \mathcal{S} \\ a' &\longmapsto \mathcal{A}(a,a') \end{aligned}$$

Note that we have a morphism $\iota_a : \Delta^0 \to \mathcal{A}_{a/}$ which selects the identity morphism on a. We claim that ι_a is an equivalence in the coCartesian model structure. Since every edge in $\mathcal{A}_{a/}$ is coCartesian it will suffice to show that ι_a is in the weakly saturated hull of morphisms of type

L) $(\Lambda^n_0, \sharp) \to (\Delta^n, \sharp)$ for $n \geqslant 1$.

To achieve this goal we produce a filtration by defining Z_n to be the simplicial subset of $\mathcal{A}_{a/}$ containing the $(n+1)$-simplices $\hat{\sigma} : \Delta^{n+1} \to \mathcal{A}_{a/}$ such that its associated $\sigma : \Delta^{n+2} \to \mathcal{A}$ is of the form $s_0(\theta)$ for some $\theta : \Delta^{n+1} \to \mathcal{A}$ such that $\theta(0) = a$. Note that by construction Z_n contains every n-simplex of $\mathcal{A}_{a/}$. We set the convention $Z_{-1} = \Delta^0$. Note that $\mathcal{A}_{a/}$ can be obtained as the directed colimit of the Z_n's which in turn implies that in order to show the claim the next lemma will suffice.

Lemma 1.3.2. *For every $\ell \geqslant 0$ the morphism $Z_{\ell-1} \to Z_\ell$ is in the weakly saturated hull of morphisms of type L).*

Proof. Let $\hat{\sigma} : \Delta^{\ell+1} \to Z_\ell$ such that $\hat{\sigma}$ does not factor through $Z_{\ell-1}$. We claim that we have a pullback diagram

$$\begin{array}{ccc} \Lambda_0^{\ell+1} & \longrightarrow & \Delta^{\ell+1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \hat{\sigma}} \\ Z_{\ell-1} & \longrightarrow & Z_\ell \end{array}$$

If the claim holds then we can add the simplex $\hat{\sigma}$ via a pushout of a morphism of type L). Repeating this process for every simplex then the result will follow. To verify that the claim holds we note that for $i > 0$ $d_i(\hat{\sigma})$ is represented by the simplex $d_{i+1}(s_0(\theta)) = s_0(d_i(\theta))$ which factors through $Z_{\ell-1}$. □

In particular we now see that if $p : X \to S$ is a representable left fibration then we have an equivalence of left fibrations

$$\begin{array}{ccc} \mathcal{A}_{a/} & \xrightarrow{\quad v \quad} & X \\ & \searrow \quad \swarrow{\scriptstyle p} & \\ & \mathcal{A} & \end{array}$$

which implies that the functor $F : \mathcal{A} \to \mathcal{S}$ classifying p is representable in the expected sense.

Definition 1.3.3. Let $F : K \to \mathcal{A}$ be a functor and let $\mathrm{Con}(F) \to \mathcal{A}$ denote the left fibration classifying the functor $\mathrm{Nat}_K(F, c_K(-))$. The colimit of F is an object $\Delta^0 \to \mathrm{Con}(F)$ which represents the left fibration in the sense of Definition 1.3.1.

Example 1.3.4. We give models for the left fibration $\mathrm{Con}(F) \to \mathcal{A}$. We can construct the desired left fibration as the pullback

$$\begin{array}{ccc} \mathrm{Con}(F) & \longrightarrow & \mathrm{Fun}(K, \mathcal{A})_{F/} \\ \big\downarrow & & \big\downarrow \\ \mathcal{A} & \xrightarrow{\ c_K\ } & \mathrm{Fun}(K, \mathcal{A}) \end{array}$$

This is the approach for the category of cones described in [Cis19]. For the sake of completeness we explain the model used in [Lur09a] which is denoted by $\mathcal{A}_{F/}$. A simplex $\hat{\sigma} : \Delta^n \to \mathcal{A}_{F/}$ is given by a morphism

$$\sigma : K * \Delta^n \to \mathcal{A}$$

such that the restriction $\sigma_{|K} = F$. Restriction along the other factor induces a morphism $\mathcal{A}_{F/} \to \mathcal{A}$. Let us point out that if $K = \Delta^0$ and the functor F selects an object $a \in \mathcal{A}$ then $\mathcal{A}_{F/} = \mathcal{A}_{a/}$; in particular the colimit is (as expected) the object $a \in \mathcal{A}$.

Remark 1.3.5. There is a dual theory of limits which we briefly describe here for completeness. Given a functor $F : K \to \mathcal{A}$ where $\mathcal{A}$ is an ∞-category the limit of F is given by an object $a \in \mathcal{A}$ representing the functor

$$\operatorname{Nat}_K(c_K(-), F) \colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{S}, \;\; a \longmapsto \operatorname{Nat}_K(\underline{a}, F)$$

The theory is constructed in a totally analogous way using instead of left fibrations, the dual notion of right fibrations and its associated notion of representability.

1.3.1 Cofinality

The theory of cofinality is an essential tool for computing colimits. Cofinality can be understood as a way to identify which diagram shapes are equivalent when it comes to computing colimits. The main theorem of this thesis is the characterization of cofinal functors of ∞-bicategories which will appear in Theorem 4.0.31. In this section we aim to review the ∞-categorical theory of cofinality as well as the characterization theorem that will be later generalized in this document.

Suppose that we are given a morphism of simplicial sets $f : L \to K$ and a diagram functor $F : K \to \mathcal{A}$. It is not hard to see that restriction along f induces a morphism over $\mathcal{A}$

$$\begin{array}{ccc} \operatorname{Con}(F) & \xrightarrow{\quad f^* \quad} & \operatorname{Con}(Ff) \\ & \searrow \quad \swarrow & \\ & \mathcal{A} & \end{array} .$$

In particular if both the colimit of F and of Ff exist we obtain a morphism

$$\operatorname*{colim}_L Ff \longrightarrow \operatorname*{colim}_K F$$

which we call the *canonical comparison map*. The theory of cofinality identifies those functors $f : K \to L$ for which the canonical comparison map is always an equivalence.

Remark 1.3.6. In this section we will restrict attention to cofinal functors of ∞-categories. It is possible to work with cofinal maps of arbitrary simplicial sets, however, the characterization of cofinal functors requires some fibrancy assumptions (see [Lur09a, Theorem 4.1.3.1]). Since our characterization in the ∞-bicategorical case is restricted to functors of ∞-bicategories we will take the analogous assumptions in this section.

Definition 1.3.7. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that f

is cofinal if the induced morphism

$$\begin{array}{ccc} \mathcal{C}^\sharp & \xrightarrow{f} & \mathcal{D}^\sharp \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \mathrm{id}} & \\ & \mathcal{D} & \end{array}$$

is a weak equivalence in the Cartesian model structure.

The first task we need to perform is to connect the previous definition with the theory of colimits. To this end we will need to introduce a key construction that will play relevant role in the proof of the bicategorical statement.

Definition 1.3.8. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Let us consider a pullback diagram

$$\begin{array}{ccc} \mathrm{F}(\mathcal{C}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{D}) \\ \downarrow & & \downarrow{\scriptstyle ev_1} \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array}$$

where the right-most vertical morphism is given by evaluation at the terminal vertex in Δ^1. We have a morphism $\mathrm{F}(f) : \mathrm{F}(\mathcal{C}) \to \mathcal{D}$ which is induced by evaluation at the initial vertex. We equip $\mathrm{F}(\mathcal{C})$ with a marking by declaring an edge to be marked if and only if its image in $\mathcal{C}$ under ev_1 is an equivalence in $\mathcal{C}$. We denote the resulting marked simplicial set by $\mathrm{F}(\mathcal{C})^\natural$.

Remark 1.3.9. Let $f : \mathcal{C} \to \mathcal{D}$ be as before and let $\mathcal{C}^\natural$ denote the marked simplicial set where we are marking the equivalences in $\mathcal{C}$. Observe that we have a morphism $\gamma_\mathcal{C} : \mathcal{C}^\natural \to \mathrm{F}(\mathcal{C})^\natural$ over $\mathcal{D}$ which sends a simplex $\Delta^n \xrightarrow{\sigma} \mathcal{C}$ to the simplex

$$\Delta^1 \times \Delta^n \to \Delta^0 \times \Delta^n \xrightarrow{\sigma} \mathcal{C} \xrightarrow{f} \mathcal{D}$$

We will show in Theorem 4.0.17 (in a much more general case) that $\gamma_\mathcal{C}$ is **M**-anodyne. We note that it follows from Corollary 2.4.7.1 in [Lur09a] that $\mathrm{F}(f) : \mathrm{F}(\mathcal{C})^\natural \to \mathcal{D}$ is a Cartesian fibration. Therefore we see that $\mathrm{F}(f)$ is a fibrant replacement for the object $\mathcal{C}^\natural \to \mathcal{D}$ in the Cartesian model structure. Following the terminology of [GHN15] we will call $\mathrm{F}(f)$ the *free Cartesian fibration* on the functor f.

Remark 1.3.10. Let $f : \mathcal{C} \to \mathcal{D}$ and recall that for every $d \in \mathcal{D}$ we have a left fibration $\mathcal{C}_{d/} \to \mathcal{C}$ obtained as the pullback

$$\begin{array}{ccc} \mathcal{C}_{d/} & \longrightarrow & \mathcal{D}_{d/} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array}$$

It is not hard to see that the fibres of $\mathrm{F}(f) : \mathrm{F}(\mathcal{C})^\natural \to \mathcal{D}$ are equivalent to the comma categories $\mathcal{C}_{d/}$. In particular we can see the Cartesian fibration $\mathrm{F}(f)$ as being classified by the functor

$$\mathfrak{C}_{\mathcal{D}/} : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathfrak{C}\mathrm{at}_\infty, \; d \longmapsto \mathcal{C}_{d/}$$

We will use the free Cartesian fibration to characterize cofinal functors. Let us consider a pushout diagram

$$\begin{array}{ccc} \mathcal{C}^\natural & \longrightarrow & \mathcal{C}^\sharp \\ \big\downarrow & & \big\downarrow \\ \mathrm{F}(\mathcal{C})^\natural & \longrightarrow & \mathrm{F}(\mathcal{C})^\diamond \end{array}$$

where the right-most vertical morphism is again **M**-anodyne. It is an easy exercise to verify that $\mathrm{F}(\mathcal{C})^\diamond \to \mathrm{F}(\mathcal{C})^\sharp$ is also marked anodyne. In particular we obtain a commutative diagram over $\mathcal{D}$

$$\begin{array}{ccc} \mathcal{C}^\sharp & \xrightarrow{f} & \mathcal{D}^\sharp \\ \big\downarrow \simeq & & \big\downarrow \simeq \\ \mathrm{F}(\mathcal{C})^\sharp & \longrightarrow & \mathrm{F}(\mathcal{D})^\sharp \end{array}$$

where the marked simplicial set on the bottom-right corner corresponds to the free fibration of the identity functor on $\mathcal{D}$. The previous diagram makes apparent that the functor f is cofinal if and only if the bottom horizontal morphism is a Cartesian equivalence.

Remark 1.3.11. It is not hard to see (and we will verify this explicitly in the ∞-bicategorical case) that the fibrant replacement of $\mathrm{F}(\mathcal{C})^\sharp$ is classified by the functor

$$|\mathfrak{C}_{\mathcal{D}/}|\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{S},\ d \longmapsto |\mathcal{C}_{d/}|$$

which send $d \in \mathcal{D}$ to the geometric realization of the category $\mathcal{C}_{d/}$. Similarly we have a functor classifying $\mathrm{F}(\mathcal{D})^\sharp$

$$|\mathfrak{D}_{\mathcal{D}/}|\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{S},\ d \longmapsto |\mathcal{D}_{d/}|$$

The morphism over $\mathcal{D}$

$$\begin{array}{ccc} \mathrm{F}(\mathcal{C})^\sharp & \longrightarrow & \mathrm{F}(\mathcal{D})^\sharp \\ & \searrow \quad \swarrow & \\ & \mathcal{D} & \end{array}$$

induces a natural transformation $\eta_f : |\mathfrak{C}_{\mathcal{D}/}| \Rightarrow |\mathfrak{D}_{\mathcal{D}/}|$. Consequently, we see that f is cofinal if and only if η_f is a levelwise equivalence.

It is easy to construct a natural transformation $\mathcal{D}_{d/} \times \Delta^1 \to \mathcal{D}_{d/}$ between the constant functor on the object represented by the identity morphism on d and the identity functor on $\mathcal{D}_{d/}$. This shows that $|\mathcal{D}_{d/}| \simeq *$. We have proven the following result.

Theorem 1.3.12. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then f is cofinal if and only if for every $d \in \mathcal{D}$ the geometric realization of the comma category $|\mathcal{C}_{d/}| \simeq *$.*

The previous theorem is precisely the Joyal's characterization of cofinal functors of ∞-categories. However, in order for this characterization to be of practical use we still need to relate the notion of cofinality to the theory of colimits. We finish the section giving a quick overview of how one builds such connection.

Lemma 1.3.13. *Let $F : \mathcal{D} \to \mathcal{A}$ be a diagram. There is an equivalence of functors*

$$\mathrm{Nat}_{\mathcal{D}}(F, \underline{a}) \xRightarrow{\simeq} \mathrm{Nat}_{\mathcal{D}^{\mathrm{op}}}\Big(|\mathfrak{D}_{\mathcal{D}/}|, \mathcal{A}(F(-), a)\Big)$$

which is natural in $\mathcal{A}$.

Proof. Observe that since each of the categories $\mathcal{D}_{d/}$ is contractible it follows that $|\mathfrak{D}_{\mathcal{D}/}|$ is equivalent to the constant functor on the point which we will denote by $\underline{*}$. Using [GHN15, Proposition 5.1] we can produce equivalences

$$\mathrm{Nat}_{\mathcal{D}^{\mathrm{op}}}\left(\underline{*}, \mathcal{A}(F(-), a)\right) \simeq \lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathcal{S}(*, \mathcal{A}(F(d), a)) \qquad \simeq \lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathcal{S}(F(d), a)$$
$$\lim_{\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}} \mathcal{S}(F(d), a) \simeq \mathrm{Nat}_{\mathcal{D}}(F, \underline{a})$$

which are natural in $\mathcal{A}$ thus completing the proof. □

Lemma 1.3.14. *Let $F : \mathcal{D} \to \mathcal{A}$ be a diagram and let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then there is an equivalence of functors*

$$\mathrm{Nat}_{\mathcal{C}}(Ff, \underline{a}) \xRightarrow{\simeq} \mathrm{Nat}_{\mathcal{D}^{\mathrm{op}}}\Big(|\mathfrak{C}_{\mathcal{D}/}|, \mathcal{A}(F(-), a)\Big)$$

which is natural in $\mathcal{A}$.

Proof. The lemma follows immediately after noting that $|\mathfrak{C}_{\mathcal{D}/}|$ can be identified with the left Kan extension $f_!|\mathfrak{C}_{\mathcal{C}/}| \simeq |\mathfrak{C}_{\mathcal{D}/}|$. □

Remark 1.3.15. Recall that we have a natural transformation $\eta_f : |\mathfrak{C}_{\mathcal{D}/}| \Rightarrow |\mathfrak{D}_{\mathcal{D}/}|$. Tracing through the equivalences in the previous lemmas we can see that η_f induces the natural transformation

$$\mathrm{Nat}_{\mathcal{D}}(F, \underline{-}) \Rightarrow \mathrm{Nat}_{\mathcal{C}}(Ff, \underline{-})$$

In particular it follows that if η_f is a levelwise equivalence then the comparison map

$$\operatorname*{colim}_{C} Ff \xrightarrow{\simeq} \operatorname*{colim}_{\mathcal{D}} F$$

is always an equivalence in $\mathcal{A}$.

Proposition 1.3.16. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then the following statements are equivalent:*

1. *The functor f is cofinal.*
2. *For every diagram $F : \mathcal{D} \to \mathcal{A}$ the canonical comparison map (whenever defined)*

$$\operatorname*{colim}_{\mathcal{C}} Ff \xrightarrow{\simeq} \operatorname*{colim}_{\mathcal{D}} F$$

   is an equivalence in $\mathcal{A}$.

Proof. Let us suppose that f is cofinal. Then it follows that $\eta_f : |\mathfrak{C}_{\mathcal{D}/}| \Rightarrow |\mathfrak{D}_{\mathcal{D}/}|$ must be an equivalence of functors. The conclusion follows from Remark 1.3.15. To prove the converse given $d \in \mathcal{D}$ we consider the functor

$$\mathcal{D}(d, -) \colon \mathcal{D} \longrightarrow \mathcal{S}$$

and its restriction along f, namely $\mathcal{D}(d, f(-))$. It follows from Corollary 3.3.4 in [Lur09a] that the comparison map

$$\operatorname*{colim}_{\mathcal{C}} \mathcal{D}(d, f(-)) \xrightarrow{\simeq} \operatorname*{colim}_{\mathcal{D}} \mathcal{D}(d, -)$$

can be identified with the morphism on geometric realizations

$$|\mathcal{C}_{d/}| \xrightarrow{\simeq} |\mathcal{D}_{d/}|$$

This implies that η_f must be a levelwise equivalence and thus f is cofinal. $\square$

Chapter 2

Fibrations of ∞-bicategories

In this chapter we construct a model structure on the category of marked biscaled simplicial sets over a scaled simplicial set S. These highly decorated simplicial sets will be used to model a bicategorical notion of fibration that we will call (outer) 2-Cartesian fibrations. After constructing the model structure we will compare our notion of 2-Cartesian fibration with certain notions of fibrations of Set_Δ^+-enriched categories. This will allow us to show that our definition restricts to the well-known definition of 2-Cartesian fibration already established in the strict 2-categorical literature. The majority of this chapter is based on [AGS22I] where the results we present here originally appeared.

Later in this document we will prove an ∞-bicategorical Grothendieck construction relating our definition of 2-Cartesian fibrations with contravariant functors $S^{\mathrm{op}} \to \mathbb{B}\mathrm{icat}_\infty$ taking values in ∞-bicategories.

2.1 Preliminaries

Recapitulating even the basics of the theory of quasi-categories, ∞-bicategories, and the various types of fibrations between them would take more space than the rest of the paper. Consequently, we here confine ourselves to fixing some notational conventions, and establishing definitions for later reference. Where possible, we will follow the notational conventions established in [Lur09a] and expanded in [Lur09b] and [Lur17]. In referring to the works of Gagna, Harpaz, and Lanari [GHL20, GHL21a, GHL19], we will endeavor to either follow their notation, or explain where our conventions differ.

Definition 2.1.1. The inclusion of $\Delta \subset \mathrm{Cat}$ defines a functor $\mathrm{N} : \mathrm{Cat} \to \mathrm{Set}_\Delta$ which sends a category C to the simplicial set $\mathrm{N}(C)$ whose n-simplices are given by functors $\sigma : [n] \to C$.

Definition 2.1.2. We say that a 2-simplex $\sigma : \Delta^2 \to X$ is *left degenerate* if its restriction $\sigma|_{\Delta^{\{0,1\}}}$ is a degenerate simplex in X.

Definition 2.1.3. Let $n \geqslant 0$. Given $0 \leqslant i \leqslant j \leqslant n$ we denote by $\Delta^{[i,j]}$ the nerve of the subposet of $[n]$ consisting of those objects ℓ such that $i \leqslant \ell \leqslant j$.

Definition 2.1.4. A *scaled simplicial set* is given by a pair (X, C_X) where X is a simplicial set and $C_X \subseteq X_2$ is a subset of the set of 2-simplices containing every degenerate 2-simplex. We refer to the elements of C_X as *thin triangles* or *scaled* *triangles*. A morphism of scaled simplicial sets $(X, C_X) \to (Y, C_Y)$ is a morphism of the underlying simplicial sets $f : X \to Y$ such that $f(C_X) \subseteq f(C_Y)$. We denote by $\mathrm{Set}_\Delta^{\mathbf{sc}}$ the category of scaled simplicial sets.

Notation. Given a simplicial set X we denote by $X_\flat := (X, \flat)$ the scaled simplicial set whose thin triangles are precisely the degenerate 2-simplices. We similarly denote by $X_\sharp := (X, \sharp)$ the scaled simplicial set where *all* triangles are thin.

Remark 2.1.5. By 2-category we mean a category enriched over the symmetric monoidal category of categories. Similarly the notion of 2-functor will refer to an enriched functor. We denote by 2Cat the ordinary 1-category of strict 2-categories.

Definition 2.1.6. We define a functor $\underline{\mathrm{N}} : 2\mathrm{Cat} \to \mathrm{Cat}_{\mathrm{Set}_\Delta^+}$ with values in the category of Set_Δ^+-enriched categories which sends an strict 2-category $\mathbb{D}$ to the Set_Δ^+-enriched category $\underline{\mathrm{N}}(\mathbb{D})$ defined as follows:

- The objects of $\underline{\mathrm{N}}(\mathbb{D})$ are given by the objects of $\mathbb{D}$.
- Given a pair of objects $x, y \in \mathbb{D}$ we define a marked simplicial set $\underline{\mathrm{N}}(\mathbb{D})(x, y)$ with underlying simplicial set given by $\mathrm{N}(\mathbb{D}(x, y))$ (see Definition 2.1.1), where the marking is given by the equivalences in $\mathbb{D}(x, y)$.

We call $\underline{\mathrm{N}}$ the Hom-wise nerve functor. Note that since the functor N is fully faithful it follows that the Hom-wise nerve $\underline{\mathrm{N}}$, is also fully faithful.

Definition 2.1.7. Let $n \geqslant 0$ and define a 2-category $\mathbb{O}^n$ as follows:

- Objects are given by the elements of the poset $[n]$.
- For every $i, j \in [n]$ the category $\mathbb{O}^n(i, j)$ is either empty if $i > j$ or given by the poset of subsets $S \subseteq [n]$ such that $\min(S) = i$ and $\max(S) = j$ ordered by inclusion. The non-trivial composition functors for $i \leqslant j \leqslant k$ are induced by union of subsets

$$\mathbb{O}^n(i, j) \times \mathbb{O}^n(j, k) \longrightarrow \mathbb{O}^n(i, k), \;\; (S, T) \longmapsto S \cup T.$$

  The action on morphisms of the composition functors is the obvious one since union preserves our given order.

This definition extends to a functor $\mathbb{O}^\bullet : \Delta \to 2\mathrm{Cat} \to \mathrm{Cat}_{\mathrm{Set}_\Delta^+}$ where the last functor is given by the Hom-wise nerve.

Remark 2.1.8. We will abuse notation and denote by $\mathbb{O}^n$ the 2-category defined in Definition 2.1.7 together with its image under the Hom-wise nerve.

Definition 2.1.9. Let $\mathcal{C}$ be a Set_Δ^+-enriched category. We define a scaled simplicial set $\mathrm{N}^{\mathrm{sc}}(\mathcal{C})$ whose n-simplices are given by functors of Set_Δ^+-enriched categories $\mathbb{O}^n \to \mathcal{C}$. A 2-simplex $\mathbb{O}^2 \to \mathcal{C}$ is thin if and only if it factors through $\mathbb{O}^2_\sharp \to \mathcal{C}$ where $\mathbb{O}^2_\sharp$ denotes the Set_Δ^+-enriched category obtained from $\mathbb{O}^2$ by maximally marking all mapping spaces.

The definition extends to a functor $\mathrm{N}^{\mathrm{sc}} : \mathrm{Cat}_{\mathrm{Set}_\Delta^+} \to \mathrm{Set}_\Delta^{\mathbf{sc}}$ which has as left adjoint which we denote by $\mathfrak{C}^{\mathbf{sc}} : \mathrm{Set}_\Delta^{\mathbf{sc}} \to \mathrm{Cat}_{\mathrm{Set}_\Delta^+}$. It follows from [Lur09b, Theorem 4.2.7] that the adjunction

$$\mathfrak{C}^{\mathbf{sc}} : \mathrm{Set}_{\Delta}^{\mathbf{sc}} \rightleftarrows \mathrm{Cat}_{\mathrm{Set}_{\Delta}^{+}} : \mathrm{N}^{\mathrm{sc}}$$

is a Quillen equivalence between the model structure of scaled simplicial sets and the model structure on $\mathrm{Set}_{\Delta}^{+}$-enriched categories (see Definition A.3.2.1 in [Lur09a] with $\mathbf{S} = \mathrm{Set}_{\Delta}^{+}$).

Definition 2.1.10. The set of *generating scaled anodyne maps* $\mathbf{S}$ is the set of maps of scaled simplicial sets consisting of:

(i) the inner horn inclusions

$$\left(\Lambda_i^n, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^n, \{\Delta^{\{i-1,i,i+1\}}\}\right) \quad , \quad n \geqslant 2 \quad , \quad 0 < i < n;$$

(ii) the map

$$(\Delta^4, T) \to (\Delta^4, T \cup \{\Delta^{\{0,3,4\}},\ \Delta^{\{0,1,4\}}\}),$$

where we define

$$T \overset{\text{def}}{=} \{\Delta^{\{0,2,4\}},\ \Delta^{\{1,2,3\}},\ \Delta^{\{0,1,3\}},\ \Delta^{\{1,3,4\}},\ \Delta^{\{0,1,2\}}\};$$

(iii) the set of maps

$$\left(\Lambda_0^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right) \quad , \quad n \geqslant 3.$$

A general map of scaled simplicial sets is said to be *scaled anodyne* if it belongs to the weakly saturated closure of $\mathbf{S}$.

Definition 2.1.11. We say that a map of scaled simplicial sets $p : X \to S$ is a weak $\mathbf{S}$-fibration if it has the right lifting property with respect to the class of scaled anodyne maps.

Definition 2.1.12. We say that a scaled simplicial set $X := (X, C_X)$ is an ∞-bicategory if the unique map $X \to \Delta^0$ is a weak $\mathbf{S}$-fibration.

Example 2.1.13. For every 2-category $\mathbb{D}$ the scaled nerve functor yields a ∞-bicategory $\mathrm{N}^{\mathrm{sc}}(\mathbb{D})$.

In general, we will denote fibrant objects in $\mathrm{Set}_{\Delta}^{\mathbf{sc}}$ using blackboard characters, e.g. $\mathbb{D}$. We will use undecorated roman majescules, e.g. X, to denote objects of any category, adding explicit decorations as necessary for clarity.

Definition 2.1.14. Consider the cosimplicial object

$$\begin{aligned} Q\colon \Delta &\longrightarrow \mathrm{Set}_{\Delta}^{\mathrm{sc}} \\ [n] &\longmapsto \Delta^0 \coprod_{\Delta^n} (\Delta^n \star \Delta^0), \end{aligned}$$

equipped with the minimal scaling. Given an ∞-bicategory $X \in \mathrm{Set}_{\Delta}^{\mathrm{sc}}$, for any $a, b \in X$, we define a simplicial set $X(a,b)$ whose n-simplices are maps $Q_n \to X$ which send the first vertex to a and the second to b. It was shown in [GHL19, Proposition 2.24] that $X(a,b)$ is a model for the mapping ∞-category from a to b in X.

2.2 The model structure on marked-biscaled simplicial sets

2.2.1 Marked biscaled simplicial sets and MB-anodyne morphisms.

Definition 2.2.1. A *marked biscaled* simplicial set (**MB** simplicial set) is given by the following data

- A simplicial set X.
- A collection of edges $E_X \subseteq X_1$ containing all degenerate edges. We will refer to the elements of this collection as *marked edges*
- A collection of triangles $T_X \subseteq X_2$ containing all degenerate triangles. We will refer to the elements of this collection as *thin triangles*.
- A collection of triangles $C_X \subseteq X_2$ such that $T_X \subseteq C_X$. We will refer to the elements of this collection as *lean triangles*.

We will denote such objects as triples $(X, E_X, T_X \subseteq C_X)$. A map $(X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ is given by a map of simplicial sets $f : X \to Y$ which maps marked edges in X (resp. thin triangles, resp. lean triangles) to marked edges in Y (resp. thin triangles, resp. lean triangles). We denote by $\mathrm{Set}_\Delta^{\mathbf{mb}}$ the category of **MB** simplicial sets.

Notation. Let $(X, E_X, T_X \subseteq C_X)$ be an **MB** simplicial set. If the collection E_X consists only of degenerate edges then we will use the notation $(X, E_X, T_X \subseteq C_X) = (X, \flat, T_X \subseteq C_X)$ and do similarly for the collection T_X. If C_X consists only of degenerate triangles we fix the notation $(X, E_X, T_X \subseteq C_X) = (X, E_X, \flat)$. In an analogous fashion we use the symbol "$\sharp$" to denote a collection containing all edges (resp. all triangles). Finally, we will employ the notation (X, E_X, T_X) whenever we have $T_X = C_X$.

Remark 2.2.2. We will often abuse notation when defining the collections E_X (resp. T_X, resp. C_X) and just specify its non-degenerate edges (resp. triangles).

Definition 2.2.3. We define a category $\Delta_{\mathbf{MB}}$ by appending to the simplex category Δ three objects $[1]_+$, $[2]_t$ and $[2]_l$ and morphisms

$$[1] \xrightarrow{i_+} [1]_+, \qquad [2] \xrightarrow{i_l} [2]_l \xrightarrow{i_l} [2]_t,$$

$$s_+^0 : [1]_+ \to [0], \qquad s_t^i : [2]_t \to [1], \quad \text{for } i = 0, 1$$

such that $s_+^0 \circ i_+ = s^0$ and such that $s_t^i \circ i_t \circ i_l = si$. We can produce a functor $R : \mathrm{Set}_\Delta^{\mathbf{mb}} \to \mathrm{Fun}(\Delta_{\mathbf{MB}}^{\mathrm{op}}, \mathrm{Set})$ which sends an **MB** simplicial set $(X, E_X, T_X \subseteq C_X)$ to the functor $R(X)$ which maps $[1]_+$ to the collection of marked edges, $[2]_l$ to the collection of lean 2-simplices and $[2]_t$ to the collection of thin triangles. The functor $R(X)$ maps the new morphisms to the obvious inclusions

$$E_X \subseteq X_1, \qquad T_X \subseteq C_X \subseteq X_2$$

between the collections and to the inclusion of degenerate edges (resp. triangles) into the marked edges (resp. thin simplices).

Remark 2.2.4. It follows by direct inspection that the functor $R : \mathrm{Set}_\Delta^{\mathbf{mb}} \to \mathrm{Fun}(\Delta_{\mathbf{MB}}^{\mathrm{op}}, \mathrm{Set})$ is fully faithful with essential image those presheaves mapping the morphisms i_+, i_l and i_t to monomorphisms in Set. It is also straightforward to verify that R has a left adjoint L. This implies that the category of **MB** simplicial sets is a reflective subcategory of a presheaf category. Since the essential image of R is stable under directed colimits we can use Theorem 1.46 in [AR94] to show that $\mathrm{Set}_\Delta^{\mathbf{mb}}$ is *locally presentable* (see [AR94, Definition 1.17]).

Remark 2.2.5. Let $X, Y \in \mathrm{Set}_\Delta^{\mathbf{mb}}$. The product $X \times Y \in \mathrm{Set}_\Delta^{\mathbf{mb}}$ is given by the underlying product of the simplicial sets equipped with the following decorations:

- An edge $\Delta^1 \to X \times Y$ (resp. triangle) is declared marked (resp. thin resp. lean) if and only if its image in X and its image in Y is marked (resp. thin resp. lean).

Remark 2.2.6. Observe that we have a functor, $L : \mathrm{Set}_\Delta^{\mathbf{sc}} \longrightarrow \mathrm{Set}_\Delta^{\mathbf{mb}}$ sending a scaled simplicial set (X, T_X) to $(X, \flat, T_X)$ which is left adjoint to the forgetful functor U sending $(X, E_X, T_X \subseteq C_X)$ to (X, T_X).

Definition 2.2.7. The set of *generating marked-biscaled anodyne maps* **MB** is the set of maps of **MB** simplicial sets consisting of:

(A1) The inner horn inclusions

$$\left(\Lambda^n_i, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\right) \quad , \quad n \geqslant 2 \quad , \quad 0 < i < n;$$

which are a direct generalization of the inner-horns of ∞-categories. For $n = 2$ these morphisms guarantee the existence of composites of 1-morphisms.

(A2) The map

$$(\Delta^4, \flat, T) \to (\Delta^4, \flat, T \cup \{\Delta^{\{0,3,4\}},\ \Delta^{\{0,1,4\}}\}),$$

where we define

$$T \overset{\text{def}}{=} \{\Delta^{\{0,2,4\}},\ \Delta^{\{1,2,3\}},\ \Delta^{\{0,1,3\}},\ \Delta^{\{1,3,4\}},\ \Delta^{\{0,1,2\}}\};$$

These morphisms encode a general 2-out-of-3 property for thin triangles.

(A3) The set of maps

$$\left(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \{\Delta^{\{0,1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \{\Delta^{\{0,1,n\}}\}\right) \quad , \quad n \geqslant 2.$$

These maps force left-degenerate (Definition 2.1.2) lean-scaled triangles to represent coCartesian edges of the mapping category. For $n = 2$ this requires the existence of p-coCartesian lifts of edges in the mapping category of the base.

(A4) The set of maps

$$\left(\Lambda^n_n, \{\Delta^{\{n-1,n\}}\}, \flat \subset \{\Delta^{\{0,n-1,n\}}\}\right) \to \left(\Delta^n, \{\Delta^{\{n-1,n\}}\}, \flat \subset \{\Delta^{\{0,n-1,n\}}\}\right) \quad , \quad n \geqslant 2.$$

This forces the marked morphisms to be p-Cartesian with respect to the given thin and lean triangles.

(A5) The inclusion of the terminal vertex

$$\left(\Delta^0, \sharp, \sharp\right) \to \left(\Delta^1, \sharp, \sharp\right).$$

This requires p-Cartesian lifts of morphisms in the base to exist.

(S1) The map

$$\left(\Delta^2, \{\Delta^{\{0,1\}}, \Delta^{\{1,2\}}\}, \sharp\right) \to \left(\Delta^2, \sharp, \sharp\right),$$

requiring that p-Cartesian morphisms compose across thin triangles.

(S2) The map

$$\left(\Delta^2, \flat, \flat \subset \sharp\right) \to \left(\Delta^2, \flat, \sharp\right),$$

which requires that lean triangles over thin triangles are, themselves, thin.

(S3) The map

$$\left(\Delta^3, \flat, \{\Delta^{\{i-1,i,i+1\}}\} \subset U_i\right) \to \left(\Delta^3, \flat, \{\Delta^{\{i-1,i,i+1\}}\} \subset \sharp\right) \quad , \quad 0 < i < 3$$

where U_i is the collection of all triangles except i-th face. This and the next two generators serve to establish composability and limited 2-out-of-3 properties for lean triangles.

(S4) The map

$$\left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset U_0\right) \to \left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \sharp\right)$$

where U_0 is the collection of all triangles except the 0-th face.

(S5) The map

$$\left(\Delta^3, \{\Delta^{\{2,3\}}\}, \flat \subset U_3\right) \to \left(\Delta^3, \{\Delta^{\{2,3\}}\}, \flat \subset \sharp\right)$$

where U_3 is the collection of all triangles except the 3-rd face.

(E) For every Kan complex K, the map

$$\left(K, \flat, \sharp\right) \to \left(K, \sharp, \sharp\right).$$

which requires that every equivalence is a marked morphism.

A map of **MB** simplicial sets is said to be **MB**-anodyne if it belongs to the weakly saturated closure of **MB**.

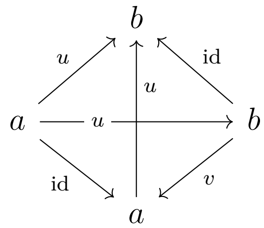
Remark 2.2.8. We would like to point out that a priori the collection (E) is not a set. This issue can be solved by allowing K to range over a set of representatives for all isomorphism classes of Kan complexes with only countably many simplices as explained in [Lur09a, Remark 3.1.1.3].

Definition 2.2.9. Let $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ be a map of **MB** simplicial sets. We say that f is an **MB**-fibration if it has the right lifting property against the class of **MB**-anodyne morphisms.

Lemma 2.2.10. *Let $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ be a* ***MB****-fibration and denote by X_y the fibre of f over $y \in Y$. Then X_y is an ∞-bicategory with precisely the equivalences marked.*

Proof. Observe that it follows from (S2) that the thin triangles and the lean triangles of X_y must coincide. Since X_y has the right lifting property against maps (A1)-(A3) it follows that X_y is an ∞-bicategory. It follows from (E) that all equivalences must be marked.

Now we will show every marked edge is an equivalence. Let $u : a \to b$ be a marked edge in X_y and let $s : \Lambda^2_2 \to X_y$ be the map that sends the edge $1 \to 2$ to u and the edge $0 \to 2$ to the identity morphism on b. It follows that we can provide an extension of s to a thin 2-simplex $\sigma : \Delta^2 \to X_y$ which provides a morphism $v : b \to a$ such that $u \circ v \simeq \mathrm{id}$. To finish the proof we construct a morphism $(\Lambda^3_3, \Delta^{\{2,3\}}, \sharp) \to X_y$ as depicted by the diagram below



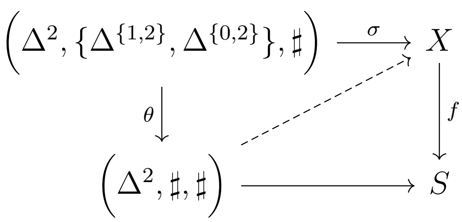
where the only non-degenerate triangle is given by the 0-th face which is precisely σ. An extension of this map to $(\Delta^3, \Delta^{\{2,3\}}, \sharp)$ where we scale the face missing the vertex 3 using a morphism of type (S5) yields a thin 2-simplex exhibiting $v \circ u \simeq \mathrm{id}$. □

Lemma 2.2.11. *The morphism*

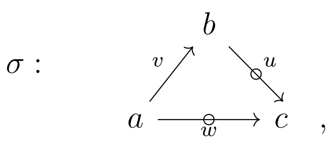
$$\theta\colon \Big(\Delta^2, \{\Delta^{\{1,2\}}, \Delta^{\{0,2\}}\}, \sharp\Big) \longrightarrow \Big(\Delta^2, \sharp, \sharp\Big)$$

is ***MB****-anodyne.*

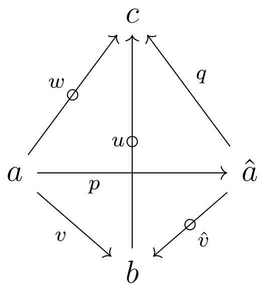
Proof. We first note that, given an **MB**-fibration $f : (X, E_X, T_X \subseteq C_X) \to (S, \sharp, T_S)$, we can find a lift of θ as follows. Suppose we have a lifting problem



where the top arrow corresponds to the thin 2-simplex



Since $f : X \to S$ is an **MB**-fibration, we can choose a marked lift $\hat{v} : \hat{a} \to b$ of $f(v)$. Using a lift of type (A1) to compose u and $\hat{v}$ and a lift of type (A4) to obtain a morphism from a to $\hat{a}$, we can obtain a Λ^3_2-horn, all of whose sides are thin-scaled. We can fill this to a maximally thin-scaled 3-simplex using a pushout of type (A1) and a pushout of type (A2). This three-simplex has the form



Since every triangle is scaled, we can apply lifts of type (S1) to show that q is marked. This implies that p is an equivalence in the fibre over $f(a)$, and so p is marked. Thus, a lift of type (S1) shows that v is marked as desired.

To finish the proof we use the small object argument to produce a factorization

$$\left(\Delta^2, \{\Delta^{\{1,2\}}, \Delta^{\{0,2\}}\}, \sharp\right) \xrightarrow{\alpha} X \xrightarrow{\beta} \left(\Delta^2, \sharp, \sharp\right)$$

where the first morphism is **MB**-anodyne and the second morphism is an **MB**-fibration. The first part of the proof then implies that there exists a section $\gamma : (\Delta^2, \sharp, \sharp) \to X$ such that $\gamma \circ \theta = \alpha$ and such that $\mathrm{id} = \beta \circ \gamma$. This shows that θ is a retract of an **MB**-anodyne morphism and thus the claim holds. □

Definition 2.2.12. We say that a map $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ of **MB** simplicial sets is a cofibration if its underlying map of simplicial sets is a cofibration. Equivalently, a cofibration of **MB** simplicial sets is given by a monomorphism in the category $\mathrm{Set}_\Delta^{\mathbf{mb}}$

Remark 2.2.13. The generators of the class of cofibrations are given by

(C1) $\left(\partial\Delta^n, \flat, \flat\right) \to \left(\Delta^n, \flat, \flat\right)$ for $n \geqslant 0$ where $\partial\Delta^0 = \varnothing$

(C2) $\left(\Delta^1, \flat, \flat\right) \to \left(\Delta^1, \sharp, \flat\right)$.

(C3) $\left(\Delta^2, \flat, \flat\right) \to \left(\Delta^2, \flat, \flat \subset \sharp\right)$.

(C4) $\left(\Delta^2, \flat, \flat \subset \sharp\right) \to \left(\Delta^2, \flat, \sharp\right)$.

Note that (C4) and (S2) are the same morphism.

Proposition 2.2.14. *Let $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ be a cofibration and let $g : (A, E_A, T_A \subseteq C_A) \to (B, E_B, T_B \subseteq C_B)$ be an **MB**-anodyne morphism. Then the pushout-product*

$$f \wedge g : X \times B \coprod_{X \times A} Y \times A \longrightarrow Y \times B$$

*is **MB**-anodyne.*[1]

[1] Note that this proposition is about the pushout-product of marked biscaled simplicial sets. For readability, we have omitted the marking and biscaling from the notation in the conclusion.

Before embarking on our proof of the pushout-product, we will tackle one particularly recalcitrant case by itself. As it so happens, a case nearly precisely dual to this one also occurs in checking the pushout-product. To save paper (and the reader's eyesight), we will only provide the proof of one of these cases, trusting that it will be apparent how to dualize the argument.

We we first prove two quick lemmata, which will somewhat ease the coming proof.

Construction 2.2.15. Let $m \geqslant 2$ and consider a list of vertices $\vec{i} = \{i_1, \dots i_{k+1}\}$ of Δ^m with $k < m$. We denote by $\Lambda^m_{\vec{i}}$ the simplicial subset of Δ^m whose non-degenerate simplices are given by subsets $J \subset [n]$ satisfying the following property

- There exists $\ell \in [m]$ such that $\ell \notin J$ and $\ell \notin \vec{i}$.

Lemma 2.2.16. *Let $\vec{i} = \{i_1, \dots i_{k+1}\}$ be a list of non-consecutive vertices of Δ^m which does not contain $0, m$. We define a biscaling $T_{\vec{i}}$ on Δ^m by declaring that $\Delta^{\{i-1,i,i+1\}}$ is thin for every $i \in \vec{i}$. Then the morphism*

$$(\Lambda^m_{\vec{i}}, \flat, T_{\vec{i}}) \longrightarrow (\Delta^m, \flat, T_{\vec{i}})$$

is in the weakly saturated hull of morphisms of type (A1) for $m \geqslant 2$.

Proof. We proceed by induction on the length of $\vec{i}$. When length$(\vec{i}) = 1$, this is simply a morphism of type (A1).

Now suppose that this holds for length$(\vec{i}) < k + 1$ and let $i_1, \dots, i_{k+1}$ be a $k + 1$-tuple satisfying the hypotheses above. Define $\vec{j} = \vec{i} \setminus \{i_1\}$, and consider the $m - 1$-simplex

$$\sigma : \Delta^{m-1} \to \Delta^m$$

given by the i_1-th face map. Then $\sigma \cap \Lambda^m_{\vec{i}} = \Lambda^{m-1}_{\vec{j}}$, and so, by the inductive hypothesis, we can fill this simplex to obtain a new simplicial subset

$$\Lambda^m_{\vec{i}} \subset X \subset \Delta^m.$$

We then see that X will consist of precisely those subsimplices of Δ^m which either (a) skip i_1 or (b) skip a vertex j not belonging to $\vec{i}$. More simply put, X consists precisely those simplices which skip a vertex not contained in $\{i_2, \dots, i_{k+1}\}$. Consequently,

$$X = (\Lambda^m_{\vec{i} \setminus \{i_1\}}, \flat, T_{\vec{i}})$$

and so, by the inductive hypothesis, this map is in the weakly saturated hull of morphisms of type (A1). □

Lemma 2.2.17. *Let $\vec{i} := \{0, i_1, \dots, i_{k+1}\}$ be a set of distinct vertices of $\Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0$ with $m \geqslant 2$ such that*

- $1 < i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{k+1} < n$
- *The simplex $\{0, 1, m\}$ is lean-scaled.*

Then the map

$$(\Lambda^m_{\vec{i}}) \coprod_{\Delta^{\{0,1\}}} \Delta^0 \to \Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

*is **MB**-anodyne.*

Proof. We once again proceed by induction on the length of $\vec{i}$. If $\vec{i} = \{0\}$, then this is a morphism of type (A3). If $\vec{i} = \{0, i_1\}$, then we can fill the simplex obtained by deleting i_1 using a pushout of type (A3), the resulting inclusion is again an inclusion of type (A3).

We now assume, inductively, that the statement holds for any $\vec{i}$ of length less than $k+2$, and let $\vec{i} = \{0, i_1, \ldots, i_{k+1}\}$. Consider the simplex $\sigma : \Delta^{m-1} \to \Delta^m$ obtained by deleting i_1. Then we see that

$$(\Lambda^m_{\vec{i}} \coprod_{\Delta^{\{0,1\}}} \Delta^0) \cap \sigma = \Lambda^{m-1}_{\vec{i}\setminus i_1} \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

so that, by the inductive hypothesis, we can fill σ using an **MB**-anodyne morphism. The resulting simplicial subset X in

$$(\Lambda^m_{\vec{i}}) \coprod_{\Delta^{\{0,1\}}} \Delta^0 \to X \to \Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

consists of precisely those subsimplices of Δ^m which skip i_1 or which skip an element not in $\vec{i}$. More precisely

$$X = \Lambda^m_{\vec{i}\setminus\{i_1\}} \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

and thus, by the inductive hypothesis,

$$X \to \Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

is **MB**-anodyne, completing the proof. □

Proposition 2.2.18. *Denote by*

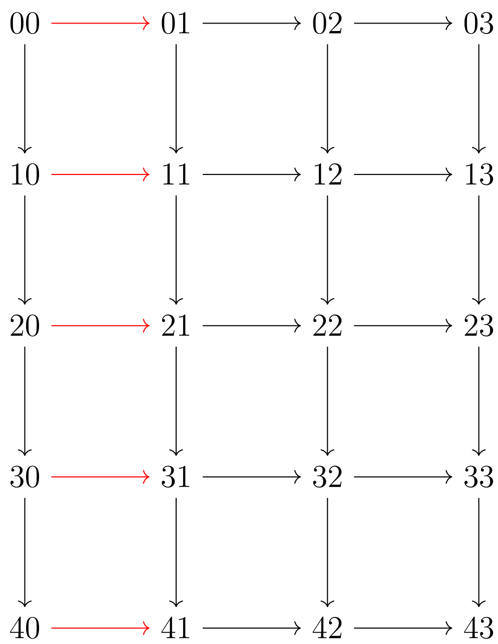
$$f\colon (\partial\Delta^n, \flat, \flat) \longrightarrow (\Delta^n, \flat, \flat)$$

a morphism of type (C1), and by

$$g\colon \left(\Lambda^m_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \{\Delta^{\{0,1,m\}}\}\right) \longrightarrow \left(\Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \{\Delta^{\{0,1,m\}}\}\right)$$

*a morphism of type (A3). Then the pushout-product $f \wedge g$ is **MB**-anodyne.*

Before beginning the proof, we create a diagram for reference. We visualize the product of the targets as a grid, with some simplices which get collapsed.



In the diagram above, we are looking at $\Delta^4 \times \Delta^3$, and the 1-simplices in red are those which get collapsed.

Proof. Since the case $n = 0$ is simply the original type (A3) morphism, we may, without loss of generality, assume $n \geqslant 1$. To prove the claim we will provide a filtration

$$X_0 \to X_1 \to \cdots \to X_{k-1} \to X_k = \Delta^n \times \left(\Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0 \right)$$

$$X_0 = \partial\Delta^n \times \left(\Delta^m \coprod_{\Delta^{\{0,1\}}} \Delta^0 \right) \coprod_{\partial\Delta^n \times \left(\Lambda^m_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0\right)} \Delta^n \times \left(\Lambda^m_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0 \right)$$

and show that each step $X_\alpha \to X_{\alpha+1}$ is **MB**-anodyne. Let us remind the reader that the marking and biscaling on $\Delta^n \times \Delta^m$ is determined by the universal property of the product as discussed in Remark 2.2.5 and each object in the filtration carries the inherited marking and biscaling from $\Delta^n \times \Delta^m$.

We begin by fixing some notation for the $n+m$ simplices in $\Delta^n \times \Delta^m$. We denote the objects of $(a, b) \in \Delta^n \times \Delta^m$ simply as ab according to the diagram above. A non-degenerate simplex $\sigma : \Delta^k \to \Delta^n \times \Delta^m$ is specified by a sequence of vertices $\{a_i b_i\}_{i=0}^k$ such that $a_i < a_{i+1}$ or $b_i < b_{i+1}$. The non-degenerate simplices of maximal dimension are precisely those such that either $a_{i+1} = a_i$ and $b_{i+1} = b_i + 1$ or $a_{i+1} = a_i + 1$ and $b_{i+1} = b_i$.

Let $\sigma : \Delta^k \to \Delta^n \times \Delta^m$ with vertex sequence given by $\{a_i b_i\}_{i=0}^k$. Then σ factors through X_0 if *at least one* of the following conditions is satisfied:

- There exists $j \in [n]$ such that $a_i \neq j$ for $0 \leqslant i \leqslant k$. In other words, the path in our grid determined by the vertex sequence skips the j-th row.
- There exists $j \in [m]$ such that $j \neq 0$ and $b_i \neq j$ for $0 \leqslant i \leqslant k$. As before, this means that the path determined by the vertex sequence skips the j-th column.

The next step in our proof is to define a total order on the set of non-degenerate simplices of maximal dimension. Once this order is provided $\{\sigma_1 < \sigma_1 < \cdots < \sigma_k\}$ we will define X_ℓ as the subsimplicial set of $\Delta^n \times \Delta^m$ containing the non-degenerate simplices θ of maximal dimension such that $\theta \leqslant \sigma_\ell$. Let $\theta, \sigma : \Delta^{m+n} \to \Delta^n \times \Delta^m$ be two distinct simplices of maximal dimension with associated vertex sequences $\{a_i b_i\}_{i=0}^{m+n}$ and $\{c_i d_i\}_{i=0}^{m+n}$. By maximality it follows that $a_0 b_0 = c_0 d_0 = 00$. Let $1 \leqslant \nu < m+n$ be the first index such that $a_\nu b_\nu \neq c_\nu d_\nu$. Then we say that $\theta < \sigma$ if $b_\nu < d_\nu$.

We observe that the decorations of $\Delta^n \times \Delta^m$ are already contained in X_0 unless $n = 1$ and $m = 2$. We will deal with this case separately. Let us suppose that $n = 1$ and $m = 2$ then it follows that every triangle in $\Delta^1 \times \Delta^2$ is lean. The filtration in this case is given by

$$X_0 \to X_1 \to X_2 \to \Delta^1 \times \left(\Delta^2 \coprod_{\Delta^{\{0,1\}}} \Delta^0 \right)$$

Let $\sigma_1 : \Delta^3 \to X_1$ be simplex specified by $00 \to 10 \to 11 \to 12$. We observe that the restriction of σ_1 to X_0 is given by $(\Lambda^3_1)^\dagger := (\Lambda^3_1, \Delta^{\{1,2\}}, \Delta^{\{0,1,2\}} \subset \sharp)$. We observe that the morphism

$$(\Lambda^3_1)^\dagger := (\Lambda^3_1, \Delta^{\{1,2\}}, \Delta^{\{0,1,2\}} \subset \sharp) \to (\Delta^3, \Delta^{\{1,2\}}, \Delta^{\{0,1,2\}} \subset \sharp) = (\Delta^3)^\dagger$$

is **MB**-anodyne since can be obtained via pushouts from a morphism of type (A1) and a morphism of type (S3). It follows that we have a pushout diagram

$$\begin{array}{ccc} (\Lambda^3_1)^\dagger & \longrightarrow & (\Delta^3)^\dagger \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & X_1 \end{array}$$

which shows that the first step is **MB**-anodyne. Now we consider the simplex $\sigma_2 : 00 \to 01 \to 11 \to 12$ in X_2. The restriction of σ_2 is given by $(P, \Delta^{\{0,1\}}, \Delta^{\{0,1,2\}} \subset \sharp)$ where P is the union inside of Δ^3 of the face that skips 1 and the face that skips 3. We can add the 0-th face using a pushout along a morphism of type (A1) thus yielding

$$(P, \Delta^{\{0,1\}}, \Delta^{\{0,1,2\}} \subset \sharp) \to (\Lambda^3_2, \Delta^{\{0,1\}}, V \subset \sharp) \to (\Delta^3, \Delta^{\{0,1\}}, V \subset \sharp)$$

where $V = \{\Delta^{\{0,1,2\}}, \Delta^{\{1,2,3\}}\}$. The first map is in the weakly saturated hull of morphisms of type (A1) and the second is in the weakly saturated hull of morphisms of type (A1) and (S3). It follows by an analogous reasoning that $X_1 \to X_2$ is **MB**-anodyne.

The last 3-simplex to add is given by $\sigma_3 = 00 \to 01 \to 02 \to 12$ which we view as a map

$$\sigma_3 : \Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0 \to \Delta^1 \times \left(\Delta^2 \coprod_{\Delta^{\{0,1\}}} \Delta^0 \right).$$

As before we compute the restriction of σ_3 to X_2 which is precisely given by $A^\diamond = (\Lambda^3_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \sharp)$. We define $B^\diamond = \left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \Delta^{\{1,2,3\}} \subset \sharp\right)$.

It follows by direct inspection that we have a pushout square

$$\begin{array}{ccc} A^{\diamond} & \longrightarrow & (B)^{\diamond} \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & (\Delta^2 \coprod_{\Delta^{\{0,1\}}} \Delta^0) \end{array}$$

so it will suffice to show that the top horizontal morphism is **MB**-anodyne. We construct the following factorization

$$\left(\Lambda^3_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \sharp\right) \to \left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \sharp\right) \to \left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \Delta^{\{1,2,3\}} \subset \sharp\right)$$

where we note that the first map is in the weakly saturated hull of morphisms of type (A3) and (S4). The second morphism is in the weakly saturated hull of morphisms of type (S2) and so the claim holds.

From this point on we will assume that X_0 contains all the decorations. We proceed by cases. First we will assume that $\sigma_\alpha : \Delta^{n+m} \to X_\alpha$ satisfies $\sigma_\alpha(0 \to 1) = 00 \to 10$. Just as we did before we will compute the restriction of σ_α to $X_{\alpha-1}$. Let $\{a_i b_i\}_{i=0}^{n+m}$ be the vertex sequence associated to σ_α. We define $\vec{i} = \{0 < i < n+m \mid a_{i-1} < a_i, \;\; a_i = a_{i+1}\}$ and observe that the restriction of σ_α to $X_{\alpha-1}$ is precisely given by $\Lambda^{n+m}_{\vec{i}}$ as in Construction 2.2.15. It follows by construction that for every $j \in \vec{i}$ the triangle $\{i-1, i, i+1\}$ is thin. Consequently we can apply Lemma 2.2.16 to show that $X_{\alpha-1} \to X_\alpha$ is **MB**-anodyne.

To finish the proof we consider a morphism $\sigma_\alpha : \Delta^{m+n} \coprod_{\Delta^{\{0,1\}}} \Delta^0 \to X_\alpha$ such that $\sigma_\alpha(0 \to 1) = 00 \to 01$. Now we define $\vec{i} = \{0 < i < n+m \mid a_{i-1} < a_i, \;\; a_i = a_{i+1}\} \cup \{0\}$ and observe that $1 \notin \vec{i}$. It follows that the restriction of σ_α to $X_{\alpha-1}$ is given by $\Lambda^{n+m}_{\vec{i}}$ and that the conditions of Lemma 2.2.17 apply. Therefore we see that the morphism $X_{\alpha-1} \to X_\alpha$ is **MB**-anodyne and thus the proof is finished. □

While a significant majority of the cases of the pushout-product remain, all of the remaining cases involve far less difficulty than this one. We can now turn to the main event.

Proof (of Proposition 2.2.14). The proof will consist of the usual rigmarole — checking on pairs of generators. While there are 44 cases in all, the vast majority of these turn out to be trivial, extremely simple or even already known. The two cases dealt with by the preceding propositions are by far the most complicated cases.

We will label our cases first by the generating cofibration, and then by the generating **MB**-anodyne morphism.

(C1) The cofibration is of the form $\left(\partial\Delta^n, \flat, \flat\right) \to \left(\Delta^n, \flat, \flat\right)$.

- (A1) Since the marking is trivial, and the thin and lean scalings agree, we can consider only the thin scalings. Case (1A) from 3.1.8 in [Lur09b] then shows that this can be obtained as a pushout of morphisms of type (A1) and morphisms for the type from remark 3.1.4 in [Lur09b].

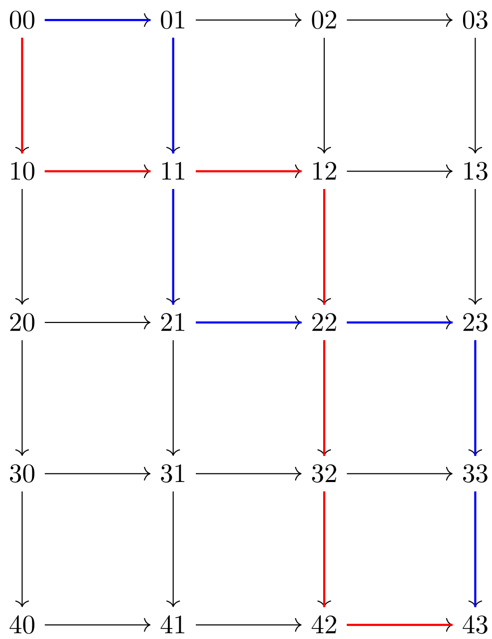


Figure 2.1: Above, we depict in blue and in red two simplices of maximal dimension. Note that in our ordering the simplex depicted by the red path is smaller than the simplex in blue.

(A2) This is precisely case (1B) from 3.1.8 [Lur09b]

(A3) This is Proposition 2.2.18.

(A4) The dual of the argument given for Proposition 2.2.18 suffices once we have replaced "degenerate 1-simplices" with "marked 1-simplices".

(A5) We note that the map of underlying simplicial sets is

$$Y_0 := (\Delta^n \times \{1\}) \coprod_{\partial\Delta^n \times \{1\}} (\partial\Delta^n \times \Delta^1) \to \Delta^n \times \Delta^1$$

We can define a sequence of $n+1$ simplices in $\Delta^n \times \Delta^1$ via the maps

$$\sigma_k \colon [n+1] \longrightarrow [n] \times [1]$$
$$i \longmapsto \begin{cases} (i,0) & i \leqslant k \\ (i-1,1) & i > k \end{cases}$$

We then define Y_i inductively as $Y_{i-1} \cup \sigma_{i-1}$ (following [Lur09a, 2.1.2.6]). We see that the morphism $Y_{i-1} \longrightarrow Y_i$ is a pushout with a Λ^{n+1}_{i+1}-horn. It will thus suffice for us to note two things:

* When $i < n$, the 2-simplex $\sigma_i|_{\Delta^{\{i-1,i,i+1\}}}$ is the simplex

$$(i,0) \to (i,1) \to (i+1,1)$$

in $\Delta^{\{i-1,i\}} \times \Delta^1$, and thus is necessarily thin-scaled. We thus obtain a pushout of type (A1).

* when $i = n$, the 2-simplex $\sigma_n|_{\Delta^{\{0,n-1,n\}}}$ is the simplex

$$(0,0) \to (n,0) \to (n,1)$$

in $\Delta^{\{0,n\}} \times \Delta^1$, and thus is necessarily thin-scaled. Moreover, the morphism $\sigma_{n+1}|_{\Delta^{\{n-1,n\}}}$ is

$$(n,0) \to (n,1)$$

and thus is marked. Hence, we obtain a pushout of type (A4).

(S1) This is an isomorphism when $n \geqslant 1$, and is a morphism of type (S1) when $n = 0$.

(S2) This is an isomorphism on underlying marked lean scaled simplicial sets, and thus in the saturated hull of morphisms of type (S2).

(S3) We will treat the case $i = 2$ — the case $i = 1$ follows virtually identically. When $n > 2$ this is an isomorphism and when $n = 0$, this is a morphism of type (S3). This means that we may consider the following two cases:

* If $n = 2$, we note that this is an isomorphism on the underlying marked simplicial sets, and indeed differs only in the lean-scaling. The only missing lean-scaled simplex is $00 \to 11 \to 23$ in $\Delta^2 \times \Delta^3$. We may expand this to a 3-simplex $00 \to 11 \to 12 \to 23$. It is easily checked that this 3-simplex gives us a pushout of type (S3) (with $i = 1$), showing that the morphism is **MB**-anodyne.
* If $n = 1$, we again have that the source and target differ only in their lean-scaling. It is easy to check that the missing simplices are the simplices $00 \to 11 \to 13$ and $00 \to 01 \to 13$ in $\Delta^1 \times \Delta^3$. In the former case, we can extend to the 3-simplex $00 \to 11 \to 12 \to 13$ and scale the desired 2-simplex with a pushout of type (S3), and in the latter case we can extend to the 3-simplex $00 \to 01 \to 02 \to 13$ and scaled the desired 2-simplex with a pushout of type (S3).

(S4) This case is almost dual to the next one and left as an exercise.

(S5) When $n \geqslant 2$, this is an isomorphism. When $n = 0$, this is a morphism of type (S5). When $n = 1$, we get the identity on underlying marked simplicial sets

$$(\Delta^3)^{\dagger} \times (\Delta^1)^{\flat} \to (\Delta^3)^{\dagger} \times (\Delta^1)^{\flat}, \;\; (\Delta^3)^{\dagger} = (\Delta^3, \Delta^{\{2,3\}})$$

The lean scaling on the target is maximal. The missing scaled simplices in the source are $00 \to 10 \to 21$, $00 \to 11 \to 21$. One can then note that the 3-simplex $00 \to 11 \to 21 \to 31$ is of type (S5), and can thus be filled. Similarly, the 3-simplex $00 \to 10 \to 21 \to 31$ is of type (S5), and can be filled.

(E) If $n \geqslant 1$, this is an isomorphism. If $n = 0$, this is again a morphism of type (E).

(C2) The cofibration is of the form $\left(\Delta^1, \flat, \flat\right) \to \left(\Delta^1, \sharp, \flat\right)$.

(A1) This is isomorphism on underlying marked, lean-scaled simplicial sets, and thus **MB**-anodyne.

(A2) This is an isomorphism.

(A3) This is an isomorphism.

(A4) This is an isomorphism.

(A5) This gives us the inclusion

$$(\Delta^1 \times \Delta^1, E_\dagger, \sharp) \longrightarrow (\Delta^1 \times \Delta^1, \sharp, \sharp)$$

Where $E_\dagger$ is the marking containing $\Delta^1 \times \{0\}$, $\Delta^1 \times \{1\}$, and $\{1\} \times \Delta^1$. A pushout of type (S1) marks the diagonal, and a pushout by the morphism

$$\left(\Delta^2, \{\Delta^{\{1,2\}}, \Delta^{\{0,2\}}\}, \sharp\right) \to \left(\Delta^2, \sharp, \sharp\right)$$

marks the remaining edge. By Lemma 2.2.11, this is **MB**-anodyne.

(S1) This is the identity on $(\Delta^2 \times \Delta^1)$ the underlying simplicial sets. Moreover, every triangle in both simplicial sets is thin scaled. The only 1-simplex which is not marked in the source is $00 \to 21$, and the target is maximally marked. We can add the remaining marked edge using a pushout of type (S1).

(S2) This is an isomorphism.

(S3) This is an isomorphism.

(S4) This is an isomorphism.

(S5) This is an isomorphism.

(E) The source and target of the pushout-product differ only in their marking. However, every edge which is marked in the target by not in the source will be the product of a non-degenerate edge in K and the non-degenerate edge in Δ^1. Consequently, it will be the diagonal in a square $\Delta^1 \times \Delta^1 \subset \Delta^1 \times K$. Since every other 1-simplex of this square will be marked, the diagonal can be marked with a pushout of type (S1).

(C3) The cofibration is of the form $\left(\Delta^2, \flat, \flat\right) \to \left(\Delta^2, \flat, \flat \subset \sharp\right)$.

(A1) When $n > 2$, this is an isomorphism. If $n = 2$, this is an isomorphism on the underlying marked thin-scaled simplicial sets, so we can consider only the lean scaling.

The target is maximally lean scaled. In the source, there are precisely three 2-simplices which are not lean scaled:

$$00 \to 12 \to 22 \tag{2.1}$$

$$00 \to 11 \to 22 \tag{2.2}$$

$$00 \to 10 \to 22 \tag{2.3}$$

For the first, we can extend to the 3-simplex $00 \to 02 \to 12 \to 22$, and obtain obtain a pushout of type (S3) with $i = 1$. For the third, we can extend to the 3-simplex $00 \to 10 \to 20 \to 22$, and obtain a pushout of type (S3) with $i = 2$. For the second, we can then extend to the 3-simplex $00 \to 10 \to 11 \to 22$, and obtain a pushout of type (S3) (with $i = 1$).

(A2) The pushout-product is an isomorphism on underlying marked thin-scaled simplicial sets, so once again we consider the lean triangles. The underlying simplicial sets are both $\Delta^2 \times \Delta^4$. There are two triangles which are lean in the target, but not the source, namely:

$$00 \to 13 \to 24 \tag{2.4}$$

$$00 \to 11 \to 24 \tag{2.5}$$

For (4), if we extend to the 3-simplex $00 \to 03 \to 13 \to 24$, we obtain a pushout of type (S3) with $i = 1$. For (5), if we extend to the 3-simplex $00 \to 11 \to 21 \to 24$, we obtain a pushout of type (S3) with $i = 2$.

(A3) This is an isomorphism when $n > 2$. When $n = 2$, we first note that we can neglect the thin scaling and the marking. Since this is the case, we consider the corresponding inclusion of lean-scaled simplicial sets. The underlying map is

$$\mathrm{id}\colon \Delta^2 \times (\Delta^2 \coprod \Delta^0) \longrightarrow \Delta^2 \times (\Delta^2 \coprod \Delta^0)$$

and the target carries a maximal scaling. The only unscaled simplex in the source is

$$00 \longrightarrow 11 \longrightarrow 22$$

We can then consider the simplex

$$00 \to 01 \to 11 \to 22$$

Since $00 \to 01$ is degenerate, we can scale the remaining simplex via a pushout of type (S4).

(A4) This is an isomorphism when $n > 2$. When $n = 2$, we again note that is sufficient only to consider the marking and the lean scaling since the source of our morphism already contains every thin triangle. In this case, we obtain an isomorphism on the underlying simplicial set $\Delta^2 \times \Delta^2$. The markings are identical on the source and target, so we are again left to consider only the lean scaling. The target is maximally scaled, and the only unscaled simplex in the source is $00 \to 11 \to 22$. Considering the 3-simplex

$$00 \to 11 \to 21 \to 22,$$

we note that $21 \to 22$ is marked. Thus, a pushout of type (S5) suffices.

(A5) The underlying map of simplicial sets is the identity on $\Delta^2 \times \Delta^1$. It is, as above, and isomorphism on the marking and thin-scaling. There are precisely three simplices which we need to lean-scale:

$$00 \to 11 \to 21 \tag{2.6}$$

$$00 \to 10 \to 21 \tag{2.7}$$

$$00 \to 10 \to 20 \tag{2.8}$$

For (6), we can extend to the 3-simplex $00 \to 01 \to 11 \to 21$, and then obtain the desired scaling via a pushout of type (S3) with $i = 1$. For (7), we can extend to the 3-simplex $00 \to 10 \to 11 \to 21$ and obtain the desired scaling via a pushout of type (S3) with $i = 2$. Finally, for (8), we can extend to the 3-simplex $00 \to 10 \to 20 \to 21$, and obtain a pushout of type (S5) (since the morphism $20 \to 21$ is marked).

(S1) This is an isomorphism.

(S2) This is an isomorphism on the underlying marked lean-scaled simplicial sets, and thus a sequence of pushouts of type (S2).

(S3) In both cases, the underlying map of simplicial sets is the identity on $\Delta^2 \times \Delta^3$, and in both cases, there is only one 2-simplex we need to lean scale.

- When $i = 2$, the missing scaling is on $00 \to 11 \to 23$. We can extend to the 3-simplex $00 \to 11 \to 21 \to 23$, and scale the missing 2-simplex using a pushout of type (S3) with $i = 2$.
- When $i = 1$, the missing scaling is on $00 \to 12 \to 23$. We can extend to the 3-simplex $00 \to 02 \to 12 \to 23$, and scale the missing 2-simplex using a pushout of type (S3) with $i = 1$.

(S4) This is effectively dual to the next case.

(S5) On the underlying marked simplicial sets, this is the identity on the marked simplicial set

$$(\Delta^3, \{\Delta^{\{2,3\}}\}) \times (\Delta^2)^\flat.$$

The only simplex which is lean-scaled in the target but not the source is $00 \to 11 \to 22$. However, if we consider the 3-simplex

$$00 \to 11 \to 22 \to 32$$

in $\Delta^3 \times \Delta^2$ whose edge $22 \to 32$ is marked, we obtain a pushout of type (S5) giving the desired scaling.

(E) This is an isomorphism.

(C4) The cofibration is of the form $\left(\Delta^2, \flat, \flat \subset \sharp\right) \to \left(\Delta^2, \flat, \sharp\right)$.

(A1)-(E) All of these are, necessarily, isomorphisms on the underlying marked lean-scaled simplicial sets (since, forgetting about thin simplices, the morphisms of type (C4) are isomorphisms of marked lean-scaled simplicial sets), since every thin triangle in the target is lean scaled in the source we see that the morphisms are **MB**-anodyne. □

Though the preceding arguments may seem an abuse of the reader's patience, now that the pushout-product is established, we can freely use it without directly working with these technicalities. In particular, we gain access to well-behaved mapping spaces, mapping categories, and mapping bicategories for $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ — a key convenience in the work to come.

Definition 2.2.19. Given two **MB** simplicial sets $(K, E_K, T_K \subseteq C_K)$, $(X, E_X, T_X \subseteq C_X)$ we define another **MB** simplicial set denoted by $\mathrm{Fun}^{\mathbf{mb}}(K, X)$ and characterized by the following universal property

$$\mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{mb}}}\Big(A, \mathrm{Fun}^{\mathbf{mb}}(K, X)\Big) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{mb}}}\Big(A \times K, X\Big).$$

As a direct consequence of Proposition 2.2.14 we obtain the following corollary.

Corollary 2.2.20. *Let* $f : (X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ *be an* ***MB****-fibration. Then for every* $K \in \mathrm{Set}_\Delta^{\mathbf{mb}}$ *the induced morphism* $\mathrm{Fun}^{\mathbf{mb}}(K, X) \to \mathrm{Fun}^{\mathbf{mb}}(K, Y)$ *is a* ***MB****-fibration.*

Definition 2.2.21. Let $f : X \to Y$ be a **MB**-fibration and consider another map of **MB** simplicial sets $g : K \to Y$. The previous corollary and Lemma 2.2.10 allow us to define an ∞-bicategory $\mathrm{Map}_Y(K, X)$ by means of the pullback square

$$\begin{array}{ccc} \mathrm{Map}_Y(K,X) & \longrightarrow & \mathrm{Fun}^{\mathbf{mb}}(K,X) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{g} & \mathrm{Fun}^{\mathbf{mb}}(K,Y) \end{array}$$

Proposition 2.2.22. *Let* $f : X \to Y$ *be a* ***MB****-fibration. Suppose that we are given morphisms of* ***MB*** *simplicial sets*

$$L \xrightarrow{h} K \xrightarrow{g} Y$$

such that h *is a cofibration (resp.* ***MB****-anodyne). Then the induced morphism*

$$h^* \colon \mathrm{Map}_Y(K, X) \longrightarrow \mathrm{Map}_Y(L, X)$$

is a fibration of scaled simplicial sets (resp. trivial fibration).

Proof. Suppose that h is a cofibration and let $A \to B$ be a **MB**-anodyne morphism. To show that h^* has the right lifting property against the class of scaled anodyne maps we consider the adjoint lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Map}_Y(K,X) \\ \downarrow & & \downarrow{\scriptstyle h^*} \\ B & \longrightarrow & \mathrm{Map}_Y(L,X) \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} L \times B \coprod_{L\times A} K \times A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ K \times B & \longrightarrow & Y \end{array}$$

and conclude that the dotted arrow exists due to Proposition 2.2.14.

Note that according to Lemma 2.2.10 the marking on both ∞-bicategories is precisely given by equivalences. Therefore using (A5) in Definition 2.2.7 we see that h^* is an isofibration. We can conclude from the construction of the model structure on $\mathrm{Set}_\Delta^{\mathbf{sc}}$ as a Cisinski model structure in [GHL19] that h^* is a fibration of ∞-bicategories. The case where h is a **MB**-anodyne follows immediately from Proposition 2.2.14. □

2.2.2 The model structure

Let $S \in \mathrm{Set}_\Delta^{\mathrm{sc}}$. For the rest of the section we will denote $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ the category of **MB** simplicial set over $(S, \sharp, T_S \subset \sharp)$. In this section we will establish the existence of model structure on $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ using a refinement of Jeff Smith's theorem due to Lurie [Lur09a, Prop. A.2.6.13].

Definition 2.2.23. We say that an object $\pi : X \to S$ in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is an *outer* 2-*Cartesian* fibration if it is an **MB**-fibration.

Remark 2.2.24. We will frequently abuse notation and refer to outer 2-Cartesian as *2-Cartesian fibrations.*

Remark 2.2.25. Given a scaled simplicial set (S, T_S) we will frequently abuse notation and denote the **MB** simplicial set $(S, \sharp, T_S \subset \sharp)$ simply by S.

Definition 2.2.26. Let $\pi : X \to S$ be a morphism of **MB** simplicial sets. Given an object $K \to S$, we define $\mathrm{Map}_S^{\mathrm{th}}(K, X)$ to be the **MB** simplicial subset of $\mathrm{Map}_S(K, X)$ consisting only of the thin triangles. Note that if π is a 2-Cartesian fibration this is precisely the underlying ∞-category of $\mathrm{Map}_S(K, X)$.

We similarly denote by $\mathrm{Map}_S^{\simeq}(K, X)$ the **MB** simplicial subset consisting of thin triangles and marked edges. As before, we note that if π is a 2-Cartesian fibration, the simplicial set $\mathrm{Map}_S^{\simeq}(K, X)$ can be identified with the maximal Kan complex in $\mathrm{Map}_S(K, X)$.

Definition 2.2.27. We define a functor $I\colon \mathrm{Set}_\Delta^+ \longrightarrow \mathrm{Set}_\Delta^{\mathbf{mb}}$ mapping a marked simplicial set (K, E_K) to the **MB** simplicial set $(K, E_K, \sharp)$. If K is maximally marked we adopt the notation $I(K^\sharp) = K_\sharp^\sharp$.

Remark 2.2.28. Note that we can endow the $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ with the structure of a Set_Δ^+-enriched category by means of $\mathrm{Map}_S^{\mathrm{th}}(-,-)$. In addition given $K \in \mathrm{Set}_\Delta^+$ and $\pi : X \to S$ we define $K \otimes X := I(K) \times X$ equipped with a map to S given by first projecting to X and then composing with π. This construction shows that $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is tensored over Set_Δ^+. One can easily show that $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is also cotensored over Set_Δ^+.

In a similar way one can use $\mathrm{Map}_S^{\simeq}(-,-)$ to endow $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ with the structure of a Set_Δ-enriched category. In this case the cotensor is given by $K \otimes X = I(K^\sharp) \times X$.

Definition 2.2.29. Let $L \xrightarrow{h} K \xrightarrow{p} S$ be a morphism in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$. We say that h is a cofibration when it is a monomorphism of **MB** simplicial sets. We will call h a weak equivalence if for every 2-Cartesian fibration $\pi : X \to S$ the induced morphism

$$h^*\colon \mathrm{Map}_S(K, X) \longrightarrow \mathrm{Map}_S(L, X)$$

is a bicategorical equivalence.

Definition 2.2.30. Given two **MB** simplicial sets $p : X \to S$ and $q : Y \to S$ over S, we call a morphism

$$\begin{array}{ccc} (\Delta^1, \sharp, \sharp) \times X & \xrightarrow{\quad h \quad} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

a *marked homotopy over* S from $h|_{\{0\}\times X}$ to $h|_{\{1\}\times X}$. We say that a morphism $f : X \to Y$ is a *marked homotopy equivalence* if there is a morphism $g : Y \to X$ over S and marked homotopies from $f \circ g$ to id_Y and from $g \circ f$ to id_X.

Proposition 2.2.31. *Suppose we are given a pushout diagram in* $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$

$$\begin{array}{ccc} L & \xrightarrow{v} & K \\ \downarrow{\scriptstyle u} & & \downarrow \\ R & \xrightarrow{w} & P \end{array}$$

where u is a cofibration and v is a weak equivalence. Then w is also a weak equivalence.

Proof. Let $\pi : X \to S$ be a 2-Cartesian fibration. Then it follows that we have a pullback diagram of fibrant scaled simplicial sets

$$\begin{array}{ccc} \mathrm{Map}_S(P,X) & \xrightarrow{w^*} & \mathrm{Map}_S(R,X) \\ \downarrow & & \downarrow{\scriptstyle u^*} \\ \mathrm{Map}_S(K,X) & \xrightarrow{v^*} & \mathrm{Map}_S(L,X) \end{array}$$

where u^* is a fibration according to Proposition 2.2.22 and v^* is a bicategorical equivalence. Since this pullback already represents the homotopy pullback it follows that w^* is also a bicategorical equivalence. □

Proposition 2.2.32. *Let $L \xrightarrow{h} K \xrightarrow{p} S$ be a morphism in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$. Then the following are equivalent*

i) The map $h : L \to K$ is a weak equivalence.

ii) For every 2-Cartesian fibration $\pi : X \to S$ the induced morphism

$$\mathrm{Map}_S^{\mathrm{th}}(K,X) \xrightarrow{\simeq} \mathrm{Map}_S^{\mathrm{th}}(L,X)$$

is an equivalence of ∞-categories.

iii) For every 2-Cartesian fibration $\pi : X \to S$ the induced morphism

$$\mathrm{Map}_S^{\simeq}(K,X) \xrightarrow{\simeq} \mathrm{Map}_S^{\simeq}(L,X)$$

is a homotopy equivalence of Kan complexes.

Proof. The implications i) $\implies$ ii) $\implies$ iii) are obvious. To show *iii)* $\implies$ *i)* we apply the small object argument to factor the morphism p (resp. $q = p \circ h$)

$$K \longrightarrow F_K \longrightarrow S$$

where the first morphism is **MB**-anodyne and the second has the right lifting property against the class of **MB**-anodyne morphisms and similarly for q. In particular we obtain 2-Cartesian fibrations $\pi_K : F_K \to S$ and $\pi_L : F_L \to S$. The functoriality of the small object argument implies the existence of a commutative diagram over S

$$\begin{array}{ccc} L & \longrightarrow & F_L \\ \downarrow & & \downarrow{\scriptstyle \varphi} \\ K & \longrightarrow & F_K \end{array}$$

Using Proposition 2.2.22 we obtain for every 2-Cartesian fibration $\pi : X \to S$ a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_S(F_K, X) & \longrightarrow & \mathrm{Map}_S(F_L, X) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Map}_S(K, X) & \longrightarrow & \mathrm{Map}_S(L, X) \end{array}$$

where the horizontal morphisms are trivial fibrations of ∞-bicategories. This shows that the map $F_L \to F_K$ satisfies condition *iii)*. It will therefore suffice to show that $F_L \to F_K$ is a weak equivalence.

We observe that we have an equivalence of Kan complexes

$$\mathrm{Map}_S^{\simeq}(F_K, F_L) \xrightarrow{\simeq} \mathrm{Map}_S^{\simeq}(F_L, F_L)$$

It follows that we have a morphism $\gamma : F_K \to F_L$ over S and a homotopy (again over S) expressing $\gamma \circ \varphi \sim \mathrm{id}_{\pi_L}$. Observe that both $\varphi \circ \gamma$ and id_{π_K} get mapped under

$$\mathrm{Map}_S^{\simeq}(F_K, F_K) \xrightarrow{\simeq} \mathrm{Map}_S^{\simeq}(F_L, F_K)$$

to equivalent objects. Using our hypothesis it follows that $\varphi \circ \gamma \sim \mathrm{id}_{\pi_K}$. To finish the proof we observe that given a 2-Cartesian fibration $X \to S$ we can use the morphism γ to construct an inverse up to marked homotopy for the map

$$\mathrm{Map}_S(F_K, X) \longrightarrow \mathrm{Map}_S(F_L, K)$$

thus concluding the proof. □

Lemma 2.2.33. *Let $L \xrightarrow{h} K \xrightarrow{p} S$ be a morphism in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ such that $p : K \to S$ and $p \circ h : L \to S$ are 2-Cartesian fibrations. Then the conditions i)-iii) in Proposition 2.2.32 are additionally equivalent to*

iv) The morphism f is a marked homotopy equivalence over S.

Proof. The equivalence of iv) and iii) is purely formal, so the result follows from Proposition 2.2.32. □

Definition 2.2.34. We say that a morphism $L \xrightarrow{h} K \longrightarrow S$ is a trivial fibration it it has the right lifting property against the class of cofibrations.

Remark 2.2.35. Observe that every trivial fibration is in particular a weak equivalence. Indeed, if $h : L \to K$ has the right lifting property against all cofibrations we can produce a section $s : K \to L$ (over S) and an a marked homotopy $L \times (\Delta^1)^\sharp \to L$ between the identity on $L \to S$ and $s \circ h$. This provides us with a deformation retract on the mapping ∞-bicategories.

Definition 2.2.36. Suppose we have a morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \Delta^n_\flat & \end{array}$$

of 2-Cartesian fibrations over $\Delta^n_\flat$, for $n \geqslant 1$, and a commutative diagram

$$\begin{array}{ccc} (\partial\Delta^m, \flat, \flat) & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow f \\ (\Delta^m, \flat, \flat) & \xrightarrow{\beta} & Y \end{array}$$

such that $r = q \circ \beta : \Delta^m \to \Delta^n$ is surjective.

We define $j_\beta \in [m]$ to be the largest element such that $r(j_\beta) < r(m)$. We additionally define a simplicial subset $S^{m+1}_{j_\beta} \subset \Delta^{m+1}$ to be the union of:

- all m-simplices of Δ^{m+1} *other than* the faces missing $j_\beta + 2$ or $j_\beta + 1$;
- the $(m-1)$ simplex which misses *both* $j_\beta + 2$ and $j_\beta + 1$.

See Figure 2.2 for a geometric interpretation. We equip Δ^{m+1} with a marking and biscaling as follows:

- The only non-degenerate marked edge is given by $j_\beta + 1 \to j_\beta + 2$.
- A 2-simplex is lean if it contains the edge $j_\beta + 1 \to j_\beta + 2$.
- A 2-simplex is thin if it is lean and its image in $\Delta^n_\flat$ under the morphism $r \circ s_{j_\beta}$ is degenerate where s_{j_β} denotes the j_β-th degeneracy map.

We denote the resulting **MB** simplicial set by $(\Delta^{m+1}, E_\beta, T_\beta \subseteq C_\beta)$ and view it as an object of $(\mathrm{Set}^{\mathbf{mb}}_{\Delta})_{/\Delta^n_\flat}$ by means of the map $r \circ s_{j_\beta}$. We similarly denote $(S^{m+1}_{j_\beta}, E^S_\beta, T^S_\beta \subseteq C^S_\beta)$ the **MB** simplicial set obtained from the inherited decorations.

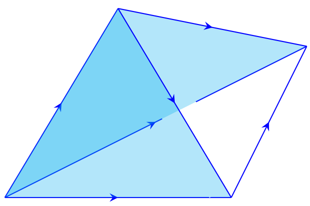


Figure 2.2: The simplicial subset $S^3_1 \subset \Delta^3$.

Lemma 2.2.37. *Let $n \geqslant 1$. Suppose we are given a morphism $f : X \to Y$ of 2-Cartesian fibrations over $\Delta^n_\flat$ and a lifting problem*

$$\begin{array}{ccc} (\partial\Delta^m, \flat, \flat) & \xrightarrow{\alpha} & X \\ \downarrow & \nearrow & \downarrow f \\ (\Delta^m, \flat, \flat) & \xrightarrow{\beta} & Y \end{array}$$

as in Definition 2.2.36. Suppose further that f satisfies condition ii) from 2.2.38.

Then there exists a commutative diagram

$$\begin{array}{ccc} (S^{m+1}_{j_\beta}, E^S_\beta, T^S_\beta \subseteq C^S_\beta) & \xrightarrow{\varepsilon} & X \\ \downarrow & & \downarrow f \\ (\Delta^{m+1}, E_\beta, T_\beta \subseteq C_\beta) & \xrightarrow{\theta} & Y \end{array}$$

such that the following conditions hold:

1. *The restriction of θ to be face missing $j_\beta + 1$ equals β and similarly, the restriction of ε to face missing $j_\beta + 1$ equals α.*
2. *Let ξ denote the restriction of θ to the face missing $j_\beta + 2$. Then either $j_\xi = j_\beta + 1$ if $j_\beta < m - 1$ or ξ factors through Δ^{n-1} and similarly for ε.*

Proof. We start the proof by fixing the notation $\alpha(i) = x_i$ (resp. $\beta(i) = y_i$). Let us pick a marked morphism $e : \hat{x}_{j_\beta} \to x_{j_\beta+1}$. To ease notation, let us just denote j_β simply by j. We define **MB** simplicial sets

$$B^m_j = (\Delta^m, \flat, \flat) \coprod_{\Delta^{\{j+1\}}} (\Delta^1, \sharp, \sharp) \quad , \quad \partial B^m_j = (\partial\Delta^m, \flat, \flat) \coprod_{\Delta^{\{j+1\}}} (\Delta^1, \sharp, \sharp).$$

For the rest of the proof we will omit the marking and biscalings to ease the notation. Note that we have commutative diagrams

$$\begin{array}{ccc} B^m_j & \longrightarrow & Y \\ {\scriptstyle \gamma^m_j}\downarrow & & \downarrow{\scriptstyle q} \\ \Delta^{m+1} & \xrightarrow{r \circ s_j} & \Delta^n \end{array} \qquad \begin{array}{ccc} \partial B^m_j & \longrightarrow & X \\ {\scriptstyle \iota^m_j}\downarrow & & \downarrow{\scriptstyle p} \\ S^{m+1}_j & \longrightarrow & \Delta^n \end{array}$$

where bottom horizontal map in the second diagram is the restriction of $r \circ s_j$ to S^{m+1}_j. We claim that the left vertical maps in both diagrams are **MB**-anodyne. Once this is proven, we let θ be a solution to the left-most commutative square. Note that we can form another diagram

$$\begin{array}{ccc} \partial B^m_j & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ S^{m+1}_j & \longrightarrow & Y \end{array}$$

where bottom horizontal map is the composite $S^{m+1}_j \longrightarrow \Delta^{m+1} \xrightarrow{\theta} Y$. Since f has the right lifting property against **MB**-anodyne morphisms our result follows.

First we will prove the family of cases where $j = m-1$ by using induction on m. The case $m = 1$ is obviously true. Suppose that our claim holds for $m-1$ and let us prove the case m. Let $W^m_{-1} = B^m_{m-1}$ and define for $0 \leqslant i \leqslant m-1$ a **MB** simplicial subset $W^m_i \subset \Delta^{m+1}$ (with the decorations defined in Definition 2.2.36) consisting in those simplices that are either in W^m_{i-1} or are contained in the i-th face for $0 \leqslant i \leqslant m-1$. This yields a filtration

$$W^m_{-1} \longrightarrow W^m_0 \longrightarrow \cdots \longrightarrow W^m_{m-1} = \Lambda^{m+1}_{m+1}$$

We similarly set $\partial W^m_{-1} = \partial B^m_{m-1}$ and produce an analogous filtration by adding step-wise the faces $0 \leqslant i \leqslant m-1$

$$\partial W^m_{-1} \longrightarrow \partial W^m_0 \longrightarrow \cdots \longrightarrow \partial W^m_{m-1} = S^{m+1}_j$$

It will then suffice to show that each step in both filtrations is **MB**-anodyne. Let $0 \leqslant i \leqslant m-1$ then we can produce a pushout squares

$$\begin{array}{ccc} W^{m-1}_{i-1} & \longrightarrow & \Delta^m \\ \downarrow & & \downarrow \\ W^m_{i-1} & \longrightarrow & W^m_i \end{array} \qquad \begin{array}{ccc} W^{m-1}_{i-1} & \longrightarrow & \Delta^m \\ \downarrow & & \downarrow \\ \partial W^m_{i-1} & \longrightarrow & \partial W^m_i \end{array}$$

where the morphism $W^{m-1}_{i-1} \to W^m_{i-1}$ is given by the restriction of the inclusion of the i-th face $\Delta^m \to W^m_i$ to W^m_{i-1} and similarly for the other diagram. The claim now follows from the inductive hypothesis.

The general proof will employ induction on j and each case will be proved using induction on m. Note that given $j \geqslant 0$ the ground case for the induction on m is given by $m = j+1$. In particular we have proved all the ground cases already. Now we will deal with ground case of the induction on j, namely $j = 0$. Assume the claim to hold for $m-1 \geqslant 1$ and let us prove the case m. Let $Z^m_{m+2} = B^m_0$ and define for every $3 \leqslant i \leqslant m+1$ a **MB** subsimplicial set $Z^m_{i-1} \subset \Delta^{m+1}$ consisting in those simplices that are either contained in Z^m_i or are contained in the $(i-1)$-th face of Δ^{m+1}. We similarly denote $\partial Z^m_{m+2} = \partial B^m_0$ and consider a pair of filtrations

$$Z^m_{m+2} \longrightarrow Z^m_{m+1} \longrightarrow \cdots \longrightarrow Z^m_3 \longrightarrow \Lambda^{m+1}_2$$

$$\partial Z^m_{m+2} \longrightarrow \partial Z^m_{m+1} \longrightarrow \cdots \longrightarrow \partial Z^m_3 \longrightarrow S^m_{\bar{j}}$$

where the last step in both filtrations is given by attaching the face missing 0. A similar argument as above shows that the claim follows from the inductive hypothesis for the every step except the last one. To prove that the last map in both filtrations is **MB**-anodyne we consider a pushout diagram

$$\begin{array}{ccc} \Lambda^m_1 & \longrightarrow & \Delta^m \\ \downarrow & & \downarrow \\ Z^m_3 & \longrightarrow & \Lambda^{m+1}_2 \end{array}$$

where the morphism $\Lambda^{m-1}_1 \to Z^m_3$ is the restriction to Z^m_3 of the inclusion of the 0-th face into Λ^{m+1}_2. Note that the triangle $\{0,1,2\}$ must be already be thin if $m > 3$ or it can be chosen to be thin since it lies above a degenerate triangle in Δ^n. The analogous conclusion also holds for ∂Z^m_3. Finally let us assume the claim holds for $j-1 \geqslant 0$. The proof of this final inductive hypothesis is a mix of both previous cases. We will give a sketch here and leave the details for the interested reader. The idea is to add stepwise to B^n_j (resp. ∂B^n_j) the faces missing i for $n \leqslant i \leqslant j+3$. One can check that at each step this result map is **MB**-anodyne using the induction hypothesis. Then we add the faces missing ℓ for $0 \leqslant \ell \leqslant j$ and again we find that each step in this process is **MB**-anodyne.

In the case of ∂B_j^m we have already reached S_j^{m+1}. For B_j^m after this process we reach Λ_{j+2}^{m+1} where the triangle $\{j+1, j+2, j+3\}$ must be thin since it is lean by construction and lies over a thin triangle. The conclusion now follows. □

Proposition 2.2.38. *Given a diagram of the form*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

where both p and q are 2-Cartesian fibrations, then the following statements are equivalent:

i) The map f is a trivial fibration.

*ii) The map f has the right lifting property against **MB**-anodyne maps and for every $s \in S$ the induced map on fibres $f_s \colon X_s \xrightarrow{\simeq} Y_s$ is a bicategorical equivalence.*

Proof. The implication $i) \implies ii)$ is clear. Now suppose that $ii)$ holds. Then we immediately see that for every $s \in S$ the map f_s is a trivial fibration of scaled simplicial sets. First we will show that we can lift the maps

$$\Big(\partial\Delta^m, \flat, \flat\Big) \longrightarrow \Big(\Delta^m, \flat, \flat\Big) \quad , \quad m \geqslant 0.$$

Suppose we are given a lifting problem of the form

$$\begin{array}{ccc} \partial\Delta^m & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \Delta^m & \xrightarrow{\beta} & Y \end{array}$$

and let κ_β be the smallest integer such that $q \circ \beta : \Delta^m \to \Delta^{\kappa_\beta} \to S$. We will use induction on κ_β. Note that when $\kappa_\beta = 0$ the lifting problem occurs in one of the fibres and thus the solution exists. Suppose the claim holds for $0 < \kappa_\beta - 1 \leqslant m - 1$. We will assume without loss of generality that $S = \Delta^{\kappa_\beta}$. Let us remark that by construction the map $r = q \circ \beta : \Delta^m \to \Delta^{\kappa_\beta}$ must be surjective. Let $j_\beta \in [m]$ be the biggest element such that $r(j_\beta) < r(m) = \kappa_\beta$. We can now use Lemma 2.2.37 to produce a commutative diagram

$$\begin{array}{ccc} S_{j_\beta}^{m+1} & \xrightarrow{\varepsilon} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \Delta^{m+1} & \xrightarrow{\theta} & Y \end{array}$$

satisfying the conditions of the lemma. It follows from the proof Lemma 2.2.37 that the triangle $\theta(\{j_\beta, j_\beta+1, j_\beta+2\})$ must be scaled. Restricting this diagram along the face missing $j_\beta + 2$ yields another commutative square

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \Delta^m & \xrightarrow{\xi} & Y \end{array}$$

We claim that our original lifting problem admits a solution if this later lifting problem admits a solution. Indeed, given a solution of this later lifting problem we can produce a commutative diagram

$$\begin{array}{ccccc} S^{m+1}_{j_\beta} & \longrightarrow & \Lambda^{m+1}_{j_\beta+1} & \longrightarrow & X \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ \Delta^{m+1} & = & \Delta^{m+1} & \longrightarrow & Y \end{array}$$

where the dotted arrow exists since the triangle $\theta(\{j_\beta, j_\beta+1, j_\beta+2\})$ is scaled. It follows from Lemma 2.2.37 that the restriction of this solution to the face missing $j_\beta+1$ is a solution for our original lifting problem.

We can further see that if $j_\beta = n-1$ then ξ must factor through $\Delta^{\kappa_\beta - 1}$ and the existence of the solution follows from the inductive hypothesis. If $j_\beta < m-1$ it follows that $q \circ \xi$ must be surjective and that $j_\xi > j_\beta$ so we can keep applying Lemma 2.2.37 until we obtain the solution. The inductive step is proved and the claim holds.

To finish the proof we must show that f detects marked edges and lean (resp. thin) triangles. Let $e : \Delta^1 \to X$ be such that $f(e)$ is marked. Let us denote $e(i) = x_i$ for $i \in \{0,1\}$ and similarly denote $f(x_i) = y_i$. Pick a marked lift $\widetilde{e} : \hat{x}_0 \to x_1$ and observe that we can produce a 2-simplex $\sigma : \Delta^2 \to X$ such that $\sigma|_{\Delta^{\{1,2\}}} = \widetilde{e}$ and $\sigma|_{\Delta^{\{0,2\}}} = e$. It follows from Lemma 2.2.11 that $f(\sigma)$ is fully marked and since its restriction to $\Delta^{\{0,1\}}$ lies in $Y_{q(y_0)}$ that particular edge must be an equivalence. However f detects equivalences in the fibres so it follows that $\sigma|_{\Delta^{\{0,1\}}}$ is marked in X. The claim follows from Definition 2.2.7 (S1).

Suppose we are given $\varphi : \Delta^2 \to X$ such that $f(\varphi)$ is lean-scaled in Y. As usual we will assume without loss of generality that $S = \Delta^2_\flat$ a minimally scaled 2-simplex. We can additionally assume that φ is not contained in some X_i for $i \in [2]$, otherwise the claim follows immediately. Let $s : \Delta^2 \xrightarrow{\varphi} X \xrightarrow{p} S = \Delta^2_\flat$ and define define j_φ as the biggest integer such that $s(j_\varphi) < s(2)$. Then a totally analogous argument to that of Lemma 2.2.37 shows that we can produce a 3-simplex $T : \Delta^3 \to X$ such that:

- The restriction of T to the face missing $j_\varphi + 1$ equals φ.
- The edge $j_\varphi + 1 \to j_\varphi + 2$ is marked.
- Every triangle of T containing the edge $j_\varphi + 1 \to j_\varphi + 2$ is lean.

We claim that by construction $f(T)$ must be fully lean-scaled in Y. There are two cases to study: $j_\varphi = 0$ and $j_\varphi = 1$. If $j_\varphi = 0$ then it follows that every triangle in $f(T)$ is lean except the 2-nd face. However the triangle given by the vertices $\{1,2,3\}$ is lean by construction and lies over an edge. Since lean triangles lying over thin triangles are themselves thin it follows that we can lean-scale the missing face using a morphism of type (S3). If $j_\varphi = 1$ then it follows that every triangle in $f(T)$ is lean except the 3-rd face. We can lean-scale this face using a morphism of type (S5).

We proceed now by cases:

a) The map s is given by $a \to a \to b$. Note that in this case we have $j_\varphi = 1$ and let us consider $T : \Delta^3 \to X$ as before. We see that the face missing 3 is contained in X_a and since its image in Y is lean (it is in fact thin) it follows that it must be lean in X. It follows that we can scale $\varphi : \Delta^2 \to X$ using a morphism of type (S3) since the triangle $\{1, 2, 3\}$ gets mapped under T to a thin 2-simplex.

b) The map s is given by $a \to b \to b$. Now we see that we can scale the face missing 2 in $T : \Delta^3 \to X$ using the previous case. We can scale $\varphi : \Delta^2 \to X$ using a morphism of type (S3) since the triangle $\{0, 1, 2\}$ gets mapped under T to a thin 2-simplex.

c) The map s is given by $a \to b \to c$. It follows that we can scale the face missing 3 in $T : \Delta^3 \to X$ using case b). Since the triangle $\{1, 2, 3\}$ gets mapped under T to a thin 2-simplex we can scale $\varphi : \Delta^2 \to X$ using a morphism of type (S3).

To prove that f detects thin triangles we only need to observe that if the image of a 2-simplex $\varphi : \Delta^2 \to X$ gets mapped under f to a thin triangle then by the discussion for lean-triangles it follows that φ is a lean in X. We can then thin-scale φ using a morphism of type (S2). □

Proposition 2.2.39. *Suppose we are given a morphism of 2-Cartesian fibrations*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

Then the following are equivalent

i) The map f is a weak equivalence.

ii) For every $s \in S$ the induced morphism $f_s : X_s \to Y_s$ is an equivalence of scaled simplicial sets.

Proof. The implication $i) \implies ii)$ is clear since we can construct an inverse up to homotopy for f as we did in the proof of Proposition 2.2.32. To prove the converse we will apply the small object argument and obtain a factorization of f

$$X \xrightarrow{u} L \xrightarrow{v} Y$$

where the map u is **MB**-anodyne and v has the right lifting property against the class of **MB**-anodyne maps. It follows from Proposition 2.2.22 that u must be a weak equivalence. Now we observe that $L \to S$ must be a 2-Cartesian fibration. It follows from 2-out-of-3 that the induced morphism on fibres $L_s \to Y_s$ must be a bicategorical equivalence for every $s \in S$. We can now apply Proposition 2.2.38 to obtain that v must be a trivial fibration. This finishes the proof. □

Definition 2.2.40. Recall from [Lur09a, A.2.6.10] that a class of morphisms W in a presentable category $\mathcal{A}$ is *perfect* if it satisfies the following conditions

1. Every isomorphism belongs to W.

2. Given a pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ if any two of the morphisms f, g and $g \circ f$ belong to W, then so does the third.
3. The class W is stable under filtered colimits.
4. There exists a (small) subset $W_0 \subseteq W$ such that every morphism belonging to W can be obtained as a filtered colimit of morphisms belonging to W_0.

Lemma 2.2.41. *The class of weak equivalences in* $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ *is perfect in the sense of Definition 2.2.40.*

Proof. Using the small object argument we produce a functor (which preserves filtered colimits)

$$T\colon (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} \longrightarrow (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$$

equipped with a natural transformation $\mathrm{id} \Rightarrow T$ such that for every $K \in (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ the map $K \to T(K)$ is **MB**-anodyne and $T(K)$ is a 2-Cartesian fibration. It follows that a morphism $h : K \to L$ is a weak equivalence if and only if $T(h)$ is a weak equivalence. We finally consider the composite

$$W_S\colon (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} \xrightarrow{T} (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} \longrightarrow \prod_{s\in S} \mathrm{Set}_\Delta^{\mathbf{mb}} \longrightarrow \prod_{s\in S} \mathrm{Set}_\Delta^{\mathrm{sc}}$$

where the second functor is given by taking pullback along each fibre and the second functor is a product of forgetful functors. It follows from a simple inspection that W_S preserves filtered colimits. Let $\mathcal{E}_S = \prod_{s\in S} \mathcal{E}$ where $\mathcal{E}$ denotes the collection of weak equivalences in $\mathrm{Set}_\Delta^{\mathrm{sc}}$. Since $\mathcal{E}$ is perfect then so is $\mathcal{E}_S$. We claim that the collection of weak equivalences in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is precisely given by $W_S^{-1}(\mathcal{E}_S)$. Once this is proved the result will follow from [Lur09a, A.2.6.12].

Let $\mathbb{E}$ denote the collection of weak equivalences in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ and let $\alpha : X \to Y$ be a morphism. Let us suppose that $\alpha \in W_S^{-1}(\mathcal{E}_S)$ then it follows from Proposition 2.2.39 that $T(\alpha) \in \mathbb{E}$. Since $\alpha \in \mathbb{E}$ if and only if $T(\alpha) \in \mathbb{E}$ it follows that $W_S^{-1}(\mathcal{E}_S) \subseteq \mathbb{E}$. The converse follows easily. □

Lemma 2.2.42. *Let $p : X \to S$ and $n \geqslant 0$. Then the morphism $r : X \times (\Delta^n)^\sharp_\sharp \to X$ given by projection to X is a weak equivalence.*

Proof. Note that the inclusion of the terminal object $t_n : (\Delta^0)^\sharp_\sharp \to (\Delta^n)^\sharp_\sharp$ induces a section $s : X \to X \times (\Delta^n)^\sharp_\sharp$. Since our class of weak equivalences satisfies 2-out-of-3 it follows that it is enough to show that s is a weak equivalence. We will show that the map t_n is **MB**-anodyne. Then the claim will follow from Proposition 2.2.14.

We prove that t_n is **MB**-anodyne using induction on n. If $n = 1$ then t_1 is the generator (A5). We define for $0 \leqslant i \leqslant n-1$ a **MB** subsimplicial set $A_i \subset (\Delta^n, \sharp, \sharp)$ consisting in those simplices that are contained in the j-th face fo $j \leqslant i$. This produces a filtration

$$\Delta^0 \to A_0 \to \cdots \to A_{n-2} \to A_{n-1} = (\Lambda^n_n, \sharp, \sharp) \to (\Delta^n, \sharp, \sharp).$$

It is easy to verify that each step in this filtration is **MB**-anodyne. □

Theorem 2.2.43. *Let S be a scaled simplicial set. Then there exists a left proper combinatorial simplicial model structure on* $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$, *which is characterized uniquely by the following properties:*

C) A morphism $f : X \to Y$ in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is a cofibration if and only if f induces a monomorphism betwee the underlying simplicial sets.

F) An object $X \in (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$ is fibrant if and only if X is a 2-Cartesian fibration.

Proof. We will use [Lur09a, Prop. A.2.6.13] to deduce the existence of a left proper combinatorial model structure in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$. Lemma 2.2.41 shows that the class of weak equivalences is perfect. We proved in Proposition 2.2.31 that weak equivalences are stable under pushouts along cofibrations. It is also immediate to see that trivial fibrations are in particular weak equivalences so the conditions of [Lur09a, Prop. A.2.6.13] apply. Now we wish to show that this model structure is compatible with the simplicial structure. This follows from [Lur09a, Prop. A.3.1.7] coupled with Lemma 2.2.42.

It is clear that every **MB**-anodyne morphism is a trivial cofibration which implies that every fibrant object is a 2-Cartesian fibration. To show that every 2-Cartesian fibration defines a fibrant object we consider a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\beta} & S \end{array}$$

where i is a general trivial cofibration and $p : X \to S$ is a 2-Cartesian fibration. We consider the induced morphism of mapping ∞-bicategories

$$i^* \colon \mathrm{Map}_S(B, X) \longrightarrow \mathrm{Map}_S(A, X)$$

and observe that due to Proposition 2.2.22 the induced morphism is simultaneosly a bicategorical equivalence and a fibration. Therefore i^* is trivial fibration of ∞-bicategories. The solution to our lifting problem is obtained by taking a preimage of the object $\alpha \in \mathrm{Map}_S(A, X)$. □

Theorem 2.2.44. *The adjunction presented in Remark 2.2.6*

$$L : \mathrm{Set}_\Delta^{\mathrm{sc}} \rightleftarrows \mathrm{Set}_\Delta^{\mathbf{mb}} : U$$

*is a Quillen equivalence where the right-hand side is equipped with the model structure of **MB** simplicial sets over the point constructed in Theorem 2.2.43.*

Proof. First we will show that L preserves cofibrations and trivial cofibrations. The case of cofibrations is immediate. Now let us suppose that $(A, T_A) \to (B, T_B)$ is a trivial cofibration of scaled simplicial sets. Let $\mathbb{D}$ be a fibrant object in $\mathrm{Set}_\Delta^{\mathbf{mb}}$ and note that as stated before $\mathbb{D}$ is an ∞-bicategory with all the equivalences marked. It is immediate that the morphism

$$\mathrm{Fun}^{\mathbf{mb}}(L(B), \mathbb{D}) \longrightarrow \mathrm{Fun}^{\mathbf{mb}}(L(A), \mathbb{D}))$$

can be identified with the analogous morphism

$$\mathrm{Fun}^{\mathbf{sc}}(B, U(\mathbb{D})) \longrightarrow \mathrm{Fun}^{\mathbf{sc}}(A, U(\mathbb{D}))$$

between the underlying scaled simplicial sets. It follows that $L \dashv U$ is a Quillen adjunction. Note that $U \circ L = \text{id}$. To conclude the proof suppose that $\mathbb{B}$ is a fibrant mb simiplicial set. In particular, we need to show that the map

$$(\mathbb{B}, \flat, T_{\mathbb{B}}) \longrightarrow (\mathbb{B}, E_{\mathbb{B}}, T_{\mathbb{B}})$$

is a weak equivalence. However the above morphism is a pushout of a morphism of type (E) in Definition 2.2.7. □

2.3 2-Cartesian fibrations over a fibrant base.

The goal of this section is to give a characterization of 2-Cartesian fibrations in the specific case where $S \in \text{Set}_{\Delta}^{\text{sc}}$ is an ∞-bicategory. For the rest of this section we will fix a functor of ∞-bicategories $p : X \to S$.

Definition 2.3.1. Let $p\colon X \longrightarrow S$ be a weak **S**-fibration (Definition 2.1.11). We call a left-degenerate (Definition 2.1.2) 2-simplex $\sigma : \Delta^2 \to X$, *p-coCartesian* if there exists a solution for any lifting problem of the form

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{f} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & S \end{array}$$

provided $f|_{\Delta^{\{0,1,n\}}} = \sigma$.

Remark 2.3.2. Recall the definition of the mapping ∞-category $X(a, b)$ described in Definition 2.1.14. Let $\sigma : \Delta^2 \to X$ be a p-coCartesian simplex such that $\sigma(0) = a$ and $\sigma(1) = b$. Since σ is left-degenerate it can be viewed as an edge in $X(a, b)$. We can further observe that by definition the dotted arrow in the diagram

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{\rho} & X(a,b) \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

exists provided the restriction of ρ to $\Delta^{\{0,1\}}$ is precisely σ. This shows that p-coCartesian triangles define coCartesian edges in the mapping space. We wish to show that this property precisely characterizes coCartesian triangles. The proof of this later fact will involve a little bit of work.

Lemma 2.3.3. *Let X be an ∞-bicategory and consider an n-simplex $\sigma : \Delta^n \to X$ with $n \geqslant 2$. Suppose that there exists some $0 < k < n$ such that the restriction of σ to $\Delta^{[0,k]}$, see Definition 2.1.3, is degenerate on $\sigma(0) = a$. Then there exists a morphism*

$$\hat{\sigma}\colon \Delta^{n+1} \longrightarrow X$$

with the following properties:

- *The restriction of $\hat{\sigma}$ to its $(k+1)$-face equals σ.*
- *The restriction of $\hat{\sigma}$ to $\Delta^{[0,k+1]}$ is degenerate on a.*

- *For every $k+2 \leqslant j \leqslant n+1$ the 2-simplex $\Delta^{\{k+1,k+2,j\}}$ is thin in X.*

Proof. Our first observation is that if $k=n-1$ then we can define $\hat{\sigma}=s_{n-1}(\sigma)$ and this provides the desired solution. We will assume for the rest of the proof that $n-k>1$. We define a simplicial subset $\iota : R^n_k \to \Delta^{n+1}$ consisting precisely of those simplices $\theta : \Delta^k \to \Delta^{n+1}$ satisfying *at least one* of the following conditions

a) The simplex θ skips the vertex $k+1$.

b) The simplex θ skips the vertex $n+1$.

c) The simplex θ is one of the triangles $\Delta^{\{k+1,k+2,j\}}$ for $k+2<j\leqslant n+1$

We endow Δ^{n+1} with a scaling by scaling those triangles contained in $\Delta^{[0,k+1]}$ in addition to the triangles $\Delta^{\{k+1,k+2,j\}}$ for $k+2<j\leqslant n+1$. The proof will be performed in two steps: First we will show that ι is an scaled anodyne morphism. Finally, we will produce an extension of σ to R^n_k.

We inductively define scaled simplicial subsets $A^n_{(k,i)} \subset \Delta^{n+1}$ (where Δ^{n+1} carries the scaling defined above) consisting in those simplices that either belong to $A^n_{(k,i-1)}$ or are contained in the face missing i for $1\leqslant i\leqslant k$ and where we are using the convention $A^n_{(k,0)}=R^n_k$. Let $B^n_{(k,n)} \subset \Delta^{n+1}$ be the simplicial subset whose simplices either belong to $A^n_{(k,k)}$ or factor through the n-th face. We inductively define $B^n_{(k,j)}$ from $B^n_{(k,j+1)}$ by adding the face missing j for $k+3\leqslant j\leqslant n$ with the convention $A^n_{(k,k)}=B^n_{(k,n+1)}$. This yields a filtration

$$R^n_k \longrightarrow A^n_{(k,1)} \longrightarrow \cdots \longrightarrow A^n_{(k,k)} \longrightarrow B^n_{(k,n)} \longrightarrow \cdots \longrightarrow B^n_{(k,k+3)} \longrightarrow \Delta^{n+1}$$

We wish to show that each step in the filtration is given by a scaled anodyne morphism. Note that $B^n_{(k,k+3)}$ contains all faces except the face missing 0 and the face missing $k+2$. Since the triangle $\Delta^{\{k+1,k+2,k+3\}}$ is thin it is easy to verify that the last step in our filtration is scaled anodyne. We observe that we can produce pushout diagrams

$$\begin{array}{ccc} A^{n-1}_{(k-1,i)} & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow d_{i+1} \\ A^n_{(k,i)} & \longrightarrow & A^n_{(k,i+1)} \end{array} \qquad \begin{array}{ccc} B^{n-1}_{(k,j-1)} & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow d_{j+1} \\ B^n_{(k,j)} & \longrightarrow & B^n_{(k,j-1)} \end{array}$$

where the morphism $A^{n-1}_{(k-1,i)} \to A^n_{(k,i)}$ (resp.$B^{n-1}_{(k,j-1)} \to B^n_{(k,j)}$) is the restriction of the inclusion of the i-th face (resp $(j-1)$-th face) into $A^n_{(k,i-1)}$ (resp. $B^n_{(k,j)}$). Suppose that each step in our filtration is scaled anodyne for $\kappa \leqslant n-1$. Then it follows that each $A^{n-1}_{(k,i)} \to \Delta^n$ is scaled anodyne. Therefore we can use the pushout diagrams above to show that each step in the filtration is scaled anodyne for $\kappa=n$. The ground case we need to show is $n=3$ and $k=1$. In this setting the filtration is of the form

$$R^3_2 \longrightarrow A^3_{(1,1)} \longrightarrow \Delta^4.$$

Note that in this case $k+3=n+1$ so the filtration terminates at $A^3_{(1,1)}$. In particular, the morphism $B^3_{(1,4)}=A^3_{(1,1)} \to \Delta^4$ is scaled anodyne. To verify that the first morphism is scaled anodyne we add to R^3_2 the face that misses the vertices 0 and 1 by taking a pushout along the morphism $(\Lambda^2_1,\sharp) \to (\Delta^2,\sharp)$ obtaining a factorization

$$R^3_2 \to Q \to A^3_{(1,1)}$$

It follows that the restriction of the face missing 1 to Q is given by a horn Λ^3_2 where the triangle $\{1,2,3\}$ is thin. The ground case now follows.

To finish the proof we need to produce the extension from $\sigma : \Delta^n \to X$ to a map $\rho : R^n_k \to X$. We define L^n_k as the subsimplicial of R^n_k consisting in those simplices satisfying conditions a) or c). We define $\rho(k{+}1 \to k{+}2) = \sigma(k \to k{+}1)$ and extend σ to L^n_k by picking the obvious composites of morphisms. Note that if $n-k=2$ then we can produce the desired extension by just setting $d_{n+1}(\rho) = s_k(d_n(\sigma))$. Therefore will assume that L^n_k already contains those simplices that factor through $\Delta^{[0,k+2]}$. To finish the proof we will show that $L^n_k \to R^n_k$ is scaled anodyne. We consider morphisms

$$\alpha_{k+j} : \Delta^{[0,k+j]} \to \Delta^{[0,n]} \subset R^n_k \quad , \qquad \text{for } 3 \leqslant j \leqslant n-k.$$

Let us set $C^n_{(k,2)} = L^n_k$. We define inductively $C^n_{(k,j)}$ by attaching the simplices α_{k+j} to $C^n_{(k,j-1)}$. We obtain our final filtration

$$L^n_k \longrightarrow C^n_{(k,3)} \longrightarrow \cdots \longrightarrow C^n_{(k,n-k)} = R^n_k.$$

Note that we have pushout diagrams

$$\begin{array}{ccc} R^{[0,k+j]}_k & \longrightarrow & \Delta^{[0,k+j]} \\ \downarrow & & \downarrow{\scriptstyle \alpha_k} \\ C^n_{(k,j)} & \longrightarrow & C^n_{(k,j+1)} \end{array}$$

where the top horizontal morphism is scaled anodyne by the first part of this proof. The result follows. □

Proposition 2.3.4. *Let $p : X \to S$ be a weak **S**-fibration. Then a left-degenerate triangle $\sigma : \Delta^2 \to X$ with $\sigma(0) = a$ and $\sigma(2) = b$ is coCartesian if and only if it defines as coCartesian edge in the mapping space $X(a,b)$.*

Proof. It is immediate that if σ is coCartesian then it defines a coCartesian edge in the corresponding mapping space. For the converse let $n \geqslant 3$ and consider a lifting problem

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow{\alpha} & S \end{array}$$

such that $f|_{\Delta^{\{0,1,n\}}} = \sigma$. We define $1 \leqslant k \leqslant n-1$ to be the biggest integer such that the restriction of f to $\Delta^{[0,k]}$ is degenerate on a. Note that if $n-k=1$ then the lifting problem takes place in the mapping space $X(a,b)$ and the solution is guaranteed. We define a subsimplicial set $P^n_k \subset \Delta^{n+1}$ consisting of those simplices $\rho : \Delta^k \to \Delta^{n+1}$ satisfying *at least one* of the following conditions

a) The simplex ρ skips the vertex $n+1$.

b) The simplex ρ skips a pair of vertices $(k+1, i)$ with $i \neq 0$.

c) The simplex ρ factors through $\Delta^{\{k+1,k+2,j\}}$ with $k+2 < j \leqslant n+1$.

Now we can apply Lemma 2.3.3 to the simplex α to obtain a map $\hat{\alpha} : \Delta^{n+1} \to S$ satisfying the conditions stated in the lemma. Our first goal is to produce a commutative diagram

$$\begin{array}{ccccc} & & \overset{f}{\frown\!\!\!\longrightarrow} & & \\ \Lambda^n_0 & \longrightarrow & P^n_k & \xrightarrow{\hat{f}} & X \\ \downarrow & & \downarrow & \overset{\varepsilon}{\nearrow} & \downarrow p \\ \Delta^n & \xrightarrow{d_{k+1}} & \Delta^{n+1} & \xrightarrow{\hat{\alpha}} & S \end{array}$$

since any dotted arrow as above will provide a solution to the original lifting problem. We define $\hat{f}$ as follows:

- On simplices satisfying condition b) the value of $\hat{f}$ is completely determined by f.
- We want to define the map $\hat{f}$ on simplices satisfying condition a). We consider a simplex
$$\sigma_{k+1} : \Delta^{[0,k+1]} \to \Lambda^n_0 \to X$$
We define the image $\Delta^{[0,k+2]} \to P^n_k$ in X to be the value of the k-th degeneracy operator on σ_{k+1}. This is compatible with the morphism $\hat{\alpha}$ as seen in the proof of Lemma 2.3.3. Moreover, if $n-k=2$ this completes the definition of $\hat{f}$ on simplices satisfying a). Let us suppose that $n-k>2$ and let $M^n_k \subset P^n_k$ be the simplicial subset consisting in those simplices satisfying b) in addition to those simplices contained in $\Delta^{[0,k+2]}$. It follows from the previous discussion that we have a commutative diagram
$$\begin{array}{ccccc} & & \overset{f}{\frown\!\!\!\longrightarrow} & & \\ \Lambda^n_0 & \longrightarrow & M^n_k & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow{d_{k+1}} & \Delta^{n+1} & \xrightarrow{\hat{\alpha}} & S \end{array}$$
To finally construct $\hat{f}$ it will be enough to show that $M^n_k \to P^n_k$ is scaled anodyne.

We define $Q^n_k \subset P^n_k$ as the simplicial subset consisting in those simplices satisfying condition a) and b). We will show that each step in the factorization

$$M^n_k \to Q^n_k \to P^n_k$$

is scaled anodyne. It is easy to see that $Q^n_k \to P^n_k$ is scaled anodyne since we can add the missing triangles by taking the adequate composites. To show the claim for $M^n_k \to Q^n_k$ we proceed in an almost identical way as in the proof of Lemma 2.3.3 we produce a filtration by inductively adding to M^n_k the simplex $\Delta^{[0,k+j]}$ for $3 \leqslant j \leqslant n-k$. We leave the standard verification that each step in this filtration is scaled anodyne to the interested reader.It follows that the desired extension $\hat{f} : P_k^n \to X$ exists.

To finish the proof we will construct the dotted arrow ε above. Let S_k^n be the simplicial subset of Δ^{n+1} consisting in those simplices belonging to P_k^n in addition to the faces that skip the vertices i for $1 \leqslant i \leqslant k$. A totally analogous argument as that for Lemma 2.3.3 shows that the map inclusion $P_k^n \to S_k^n$ is scaled anodyne. We can now add the faces that skip the vertices $k+2 \leqslant j \leqslant n$ to obtain a new simplicial set T_k^n. We observe that T_k^n only misses the $(k+1)$-face and the 0-face since the triangle $\Delta^{\{k,k+1,k+2\}}$ must be thin. It is easy to see that $T_k^n \to \Delta^{n+1}$ is scaled anodyne. To finish the proof we need to provide a solution to the lifting problem

$$\begin{array}{ccc} S_k^n & \longrightarrow & X \\ \big\downarrow & \overset{\varphi}{\nearrow} & \big\downarrow p \\ T_k^n & \longrightarrow & S \end{array}$$

We define $D^n_{(k,n)}$ by adding to S_k^n the face missing the vertex n. We define $D^n_{(k,j-1)}$ by adding to $D^n_{(k,j)}$ the face missing j for $k+2 \leqslant j \leqslant n$. This produces a filtration

$$S_k^n \longrightarrow D^n_{(k,n)} \longrightarrow \cdots D^n_{(k,k+2)} = T_k^n$$

We will show how to produce the solution by extending the map stepwise. As usual, we produce a pushout diagram

$$\begin{array}{ccc} D^{n-1}_{(k,j-1)} & \longrightarrow & \Delta^n \\ \big\downarrow & & \big\downarrow \\ D^n_{(k,j)} & \longrightarrow & D^n_{(k,j-1)} \end{array}$$

Now we observe that if $n-k=2$ then original filtration is of the form

$$S^n_{n-2} \longrightarrow D^n_{(n-2,n)} = T^n_{n-2}$$

the previously depicted pushout diagram particularizes now to

$$\begin{array}{ccc} \Lambda^n_0 & \longrightarrow & \Delta^n \\ \big\downarrow & & \big\downarrow \\ S^n_{n-2} & \longrightarrow & T^n_{n-2} \end{array}$$

where the left-most Λ_0^n represents an 0-horn in the mapping space and thus the existence of the extension is guaranteed. An inductive argument shows that we can produce the map φ and the proof is concluded. □

Definition 2.3.5. We say that $p : X \to S$ is *locally fibred* if it satisfies the conditions

i) The map $p : X \to S$ is a weak **S**-fibration.

ii) For every left-degenerate $\widetilde{\sigma} : \Delta^2 \to S$ together with $\tau : \Delta^1 \to X$ such that $\widetilde{\sigma}|_{\Delta^{\{0,2\}}} = p(\tau)$, then there exists a left-degenerate simplex $\sigma : \Delta^2 \to X$ such that σ is coCartesian and $p(\sigma) = \widetilde{\sigma}$.

The following proposition follows immediately from our definitions.

Proposition 2.3.6. *Let $p : X \to S$ be locally fibred. The given $a, b \in X$ a pair of objects it follows that the induced morphism on mapping spaces*

$$p_{a,b} \colon X(a,b) \longrightarrow S(p(a), p(b))$$

is a coCartesian fibration of ∞-categories.

Definition 2.3.7. Let $\sigma, \tau : \Delta^2 \to X$ be a pair of 2-simplices such that τ is left-degenerate. We say τ is the *left-degeneration* of σ if there exists a 3-simplex $\rho : \Delta^3 \to X$ with the following properties:

- The face $d_3(\rho)$ equals $s_0(d_2(\sigma))$.
- The face $d_2(\rho)$ equals τ.
- The face $d_1(\rho)$ equals σ.
- The face $d_0(\rho)$ is thin in X.

Remark 2.3.8. We remark that if X is an ∞-bicategory then the left-degeneration of a 2-simplex always exists. It is trivial to see that every left-degenerate triangle is its own left-degeneration.

Definition 2.3.9. We say that a triangle $\sigma : \Delta^2 \to X$ is *coCartesian* if its left-degeneration is coCartesian. We denote the collection of coCartesian triangles by C_X.

Lemma 2.3.10. *Let $p : X \to S$ be locally fibred. Suppose that we are given a 2-simplex $\sigma : \Delta^2 \to X$ such that σ is p-coCartesian and its image under p is thin in S. Then σ is a thin simplex of X.*

Proof. If σ is left-degenerate the claim follows immediately from Proposition 2.3.6 since σ represents a coCartesian edge in the mapping space $X(a,b)$ whose image in $S(p(a), p(b))$ is an equivalence. To show the general case we let τ be the left-degeneration of ρ witnessed by a 3-simplex $\rho : \Delta^3 \to X$. Note that since S is an ∞-bicategory it follows that $p(\tau)$ is thin in S since every face of $p(\rho)$ is thin except possibly the 2-face. Using the first part of the proof we see that τ must be thin in X. It follows that we can scale σ in X. □

Definition 2.3.11. Let $p : X \to S$ be a weak **S**-fibration. We say that the collection of coCartesian triangles C_X, is a *functorial family* if the following holds:

- Let $0 < i < 3$ and suppose we are given a three simplex $\rho : \Delta^3 \to X$ such that the face $\Delta^{\{i-1,i,i+1\}}$ is thin and all of the faces of ρ are coCartesian except possibly the face missing i. Then the image of ρ only consists in coCartesian triangles.

Definition 2.3.12. Let $p : X \to S$ be a locally fibred morphism. We say that p is *functorially fibred* if the collection of coCartesian triangles is functorial.

Lemma 2.3.13. *Let $p : X \to S$ be a functorially fibred map. Given a left-degenerate three simplex $\rho : \Delta^3 \to X$ such that all of its faces except possibly the 0-face belong to C_X then it follows that the 0-face must also belong to C_X.*

Proof. Let us first suppose that restriction of ρ to $\Delta^{[0,2]}$ is a degenerate simplex of $\rho(0)$. Then ρ defines a 2-simplex in the mapping space $X(a,b)$ where all edges are coCartesian except the edge $1 \to 2$. By the limited 2-out-of-3 property of coCartesian edges it follows that $1 \to 2$ is also coCartesian. Then the result follows from Proposition 2.3.4.

We suppose now that $\Delta^{[0,2]}$ is not degenerate on $\rho(0)$. We apply Lemma 2.3.3 to obtain a simplex $\Xi : \Delta^4 \to X$. Note that 4-th face of Ξ can be chosen to be $s_1(d_3(\rho))$. It follows that every triangle in the 4-th face of Ξ is coCartesian. We further note that the triangle $\Delta^{\{1,2,4\}}$ is the left-degeneration of $d_0(\rho)$. We claim that every triangle in $d_1(\Xi) = \sigma$ is coCartesian: First we observe that $d_0(\sigma)$ is thin and that $d_1(\sigma) = d_1(\rho)$. Since every triangle of $d_4(\Xi)$ is coCartesian we see that $d_3(\sigma)$ is coCartesian. It follows that every triangle of σ is coCartesian except possibly the 2-nd face. Since the map is functorially fibred the claim follows.

To finish the proof we consider $d_3(\Xi) = \theta$ and observe that the restriction of θ to $\Delta^{[0,2]}$ is degenerate of $\theta(0) = \rho(0)$. Moreover it follows that $d_1(\theta) = d_2(\sigma)$. We see that every triangle of θ is coCartesian except possible the 0-th face. We can apply now the first part of the proof to conclude. □

Definition 2.3.14. Let $p : X \to S$ be a functorially fibred map. We say that an edge $e : \Delta^1 \to X$ is *strongly* p-cartesian (resp. p-Cartesian) if every lifting problem

$$\begin{array}{ccc} \Lambda^n_n & \xrightarrow{f} & X \\ \downarrow & \overset{\hat{f}}{\nearrow} & \downarrow p \\ \Delta^n & \xrightarrow{\sigma} & S \end{array}$$

admits a solution for $n \geqslant 3$ provided the following conditions are satisfied:

i) $f|_{\Delta^{\{n-1,n\}}} = e$.

ii) $f|_{\Delta^{\{0,n-1,n\}}}$ is coCartesian (resp. thin).

In the case $n = 2$, we distinguish two cases:

- If $f|_{\Delta^{\{1,2\}}} = e$ is *strongly* p-Cartesian the solution of the lifting problem $\hat{f}$ exists and defines a coCartesian triangle in X.
- If $f|_{\Delta^{\{1,2\}}} = e$ is p-Cartesian and $\sigma : \Delta^2 \to S$ is thin then the solution of the lifting problem $\hat{f}$ exists and defines a thin triangle in X.

Definition 2.3.15. We say that a functorially fibred map $p : X \to S$ is an outer 2-fibration or **O2**-fibration, if every degenerate edge is *strongly* p-Cartesian.

Remark 2.3.16. Recall from [GHL21a, Definition 2.1.1] that a map of scaled simplicial sets $p : X \to S$ is a *weak fibration* if it has the right lifting property against the following types of maps

i) The inner horn inclusions

$$\left(\Lambda^n_i, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^n, \{\Delta^{\{i-1,i,i+1\}}\}\right) \quad , \quad n \geqslant 2 \quad , \quad 0 < i < n;$$

ii) The left-horn inclusions

$$\left(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right) \quad , \quad n \geqslant 2.$$

iii) The right-horn inclusions

$$\left(\Lambda^n_n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0, \{\Delta^{\{0,n-1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{n-1,n\}}} \Delta^0, \{\Delta^{\{0,n-1,n\}}\}\right) \quad , \quad n \geqslant 2.$$

Observe that an **O2**-fibration is a *weak fibration* in the terminology of [GHL21a].

Lemma 2.3.17. *Let $p : X \to S$ be a* **O2***-fibration. Given a 3-simplex $\rho : \Delta^3 \to X$ such that*

- *The restriction $\rho|_{\Delta^{\{2,3\}}}$ is a p-Cartesian edge.*
- *Every face of ρ belongs to C_X except possibly the face missing 3.*

Then every face of ρ belongs to C_X.

Proof. Let us fix some notation before diving into the proof. We denote $a = \rho(0)$, $c = \rho(2)$, $d = \rho(3)$ and $\rho|_{\Delta^{\{2,3\}}} = \alpha$. First let us assume that $\rho|_{\Delta^{\{0,1\}}}$ is degenerate on a. Using Proposition 2.3.3 in [GHL21a] we obtain a homotopy pullback diagram

$$\begin{array}{ccc} X(a,c) & \xrightarrow{\alpha\circ -} & X(a,d) \\ \downarrow & & \downarrow \\ S(p(a),p(c)) & \xrightarrow{p(\alpha)\circ -} & S(p(a),p(d)) \end{array}$$

Let $\varepsilon \in X(a,c)$ denote the morphism represented by the 3-face in ρ. We claim that our hypothesis imply that its image under postcomposition with α must be coCartesian. Let $\rho(0 \to 2) = u$ and $\rho(1 \to 2) = v$ and pick composites $\alpha \circ u$ and $\alpha \circ v$ represented by the corresponding thin 2-simplices. We construct a 3-simplex $\tau : \Delta^3 \to X$ such that $d_3(\tau) = d_3(\rho)$, $d_0(\tau) = \alpha \circ v$ and $d_1(\tau) = \alpha \circ u$. This definitions gives a $\Lambda^3_2 \to X$ such that the triangle $\{1,2,3\}$ is thin and thus we can pick an extension to Δ^3 to yield the desired τ. It is clear that $d_2(\tau) = \gamma$ is the image of ε under post-composition with α.

We now apply Lemma 2.3.3 to ρ to obtain a 4-simplex $\nu : \Delta^4 \to X$. Observe that the 0-th face of $d_3(\nu)$ is the left-degeneration of $d_0(\rho)$ which implies that it must be coCartesian. We see that every face of $d_3(\nu)$ must be coCartesian except possibly the face missing 1. Since the triangle $\{0,1,2\}$ is thin it follows every face of $d_3(\nu)$ is coCartesian.

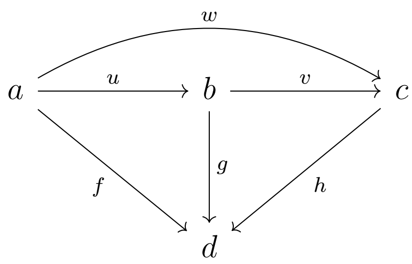
To finish the proof of the claim we construct a map $\overline{\kappa} : \Lambda^4_3 \to X$ as follows:

- The 4-th face is given by $s_0(d_3(\rho))$.
- The 1-st face is given by $d_1(\nu)$.
- The 0-th face is given by τ.
- The 2-nd face is given by picking a lift of the morphism $\Lambda^3_2 \to X$ which sends the 0-th face to $\alpha \circ u$, the 1-st face to $d_1(\rho)$ and the 3-rd face to $s_0(u)$. Let us note that the 2-nd face of any extension must be coCartesian.

Since the triangle $\{2,3,4\}$ is thin we can pick an extension $\kappa : \Delta^4 \to X$. It follows that every face of $d_3(\kappa)$ is coCartesian except possibly the face missing 0 which is precisely γ. The claim now follows from Lemma 2.3.13.

Since $X(a,c)$ can be expressed as a homotopy pullback it follows that ε must be coCartesian as an edge in $X(a,c)$. The claim now follows from Proposition 2.3.4

To prove the general version of the lemma we will reduce it to the previous case. We will fix once and for all the notation regarding ρ by means of the diagram below



Let us consider Λ^2_1 sitting inside the 3-face of ρ and another such horn sitting inside the 2-face of ρ. Let σ_3 (resp. σ_2) denote the corresponding thin 2-simplices obtained by extending the horns. We denote the 1-face of these thin simplices by $v \circ u$ (resp. $g \circ u$). We define a morphism

$$\Lambda^3_2 \longrightarrow X$$

by sending the 0-face to σ_3, the 1-face to $d_3(\rho)$, the 3-face to $s_0(u)$. Since X is an ∞-bicategory we can produce a lift to a 3-simplex that we call θ_4. Observe that if $d_2(\theta_4)$ belongs to C_X then every face of θ_4 is coCartesian except possibly the 1-face. Since $\{0,1,2\}$ is thin (in fact degenerate) it follows that $d_3(\rho) \in C_X$. We define a morphism

$$\Lambda^3_1 \longrightarrow X$$

by sending the 0-face to $d_0(\rho)$, the 2-face to σ_2 and the 3-face to σ_3. We extend this horn to a 3-simplex that we call θ_0. By construction it follows that every face of θ_0 is coCartesian except possibly the face missing 1. Since the triangle $\{0,1,2\}$ is thin by definition we see that every face of θ_0 belongs to C_X. Finally, let us define

$$\Lambda^3_2 \longrightarrow X$$

by sending the 0-face to σ_2, the 1-face to $d_2(\rho)$ and the 3-face to $s_0(u)$. We call θ_3 the extension of this horn to a 3-simplex. We observe that every face of θ_3 belongs to C_X.

Let $\theta_1 = \rho$ and observe that the 3-simplices θ_i for $i \in [4]$, $i \neq 2$ assemble into a Λ^4_2 and that the face $\Delta^{\{1,2,3\}}$ is thin by construction. We take our final extension $\theta : \Delta^4 \to X$ and observe that $d_2(\theta)$ satisfies the conditions of the lemma and its first edge is degenerate. We finish the proof by noting that $d_3 d_2(\theta)$ is coCartesian if and only if $d_3(\rho)$ is. □

Given an **O2**-fibration $p : X \to S$, the condition that an edge e of X be strongly p-Cartesian edge is *prima facie* stronger than the condition that e be p-Cartesian. It turns out, however, that these two notions coincide. To prove this, we must first establish a purely technical result (Corollary 2.3.19). The reader interested only in the characteristics of p-Cartesian edges may safely skip to Proposition 2.3.20.

Lemma 2.3.18. *Let S be an ∞-bicategory. There is a map*

$$E : \coprod_{n\geqslant 2} S_n \longrightarrow \coprod_{n\geqslant 3} S_n$$

which raises the dimension of each simplex by 1, and such that, for $\sigma : \Delta^n \to S$, the map $E(\sigma) : \Delta^{n+1} \to S$ has the following properties.

- *Every triangle in Δ^{n+1} which contains the edge $(n-1) \to n$ is mapped to a thin triangle in S.*
- *The n^{th} face of $E(\sigma)$ is σ.*
- *$E(\sigma)$ sends the triangle $\Delta^{n-1,n,n+1}$ to $s_1(\sigma|_{\Delta^{\{n,n+1\}}})$.*
- *When the dimension of σ is greater than 2, the following identities hold:*

$$d_i E(\sigma) = \begin{cases} E(d_i(\sigma)) & i \leqslant n-2, \quad n > 2 \\ \sigma & i = n \end{cases}$$

Proof. We will prove the lemma by induction on the dimension of a simplex σ. For simplicity, we denote the last edge in the spine of an n-simplex σ by $e_\sigma = \sigma|_{\Delta^{\{n,n+1\}}}$.

We begin by defining E on simplices of dimension 2. Consider the restriction

$$\Xi \colon \Lambda^2_1 \longrightarrow \Delta^2 \xrightarrow{\ \sigma\ } S$$

and pick an extension of Ξ to a thin 2-simplex $\hat{\sigma}$. We fix the notation $\hat{h} = d_1(\hat{\sigma})$ and $h = d_1(\sigma)$. We construct a morphism $\Lambda^3_1 \to S$ as follows:

- The face missing the vertex 0 equals $s_1(e_\sigma)$.
- The face missing the vertex 2 equals σ.
- The face missing the vertex 3 equals $\hat{\sigma}$

Since the triangle $\Delta^{\{0,1,2\}} = \hat{\sigma}$ is thin by construction we can extend this inner horn to a 3-simplex $E(\sigma) : \Delta^3 \to S$.

We then proceed by induction. Suppose we have defined E for $k < n$, and let $\sigma : \Delta^n \to S$. Define a simplicial subset $A_n \subset \Delta^{n+1}$ which consists of all of the n-dimensional faces except two. The face which skips the vertex $(n-1)$, and the face which skips the vertex $n+1$. The final condition on E requires that $E(\sigma)$ restrict to a map

$$\alpha : A_n \longrightarrow S$$

such that

$$d_i E(\sigma) = \begin{cases} E(d_i(\sigma)) & i \leqslant n-2, \quad n > 2 \\ \sigma & i = n. \end{cases}$$

To assure that this definition is valid, we must show that it agrees on shared $(n-1)$-dimensional faces. We must check the case $n = 3$ separately, since we do not have recourse to the above identities for 2-simplices.

- **The case** $n = 3$. We consider the intersection of the i^{th} and j^{th} faces, where $i < j$. We will abusively denote our presumptive definition for the i^{th} face of α by $d_i(\alpha)$

  First suppose that $i = 0$ and $j = 1$. Then

  $$d_0(d_1(\alpha)) = d_0(E(d_1(\sigma)))$$

  By construction, the latter is $s_1(e_{d_1(\sigma)}) = s_1(e(\sigma))$. On the other hand, we have that

  $$d_0(d_0(\alpha)) = d_0(E(d_0(\sigma))) = s_1(e_{d_0(\sigma)}) = s_1(\sigma).$$

  so that the simplices agree on the overlap.

  We then consider $i \leqslant 1$ and $j = 3$. On the one hand,

  $$d_i(d_3(\alpha)) = d_i(\sigma).$$

  On the other hand,

  $$d_2(d_i(\alpha)) = d_2(E(d_i(\sigma))) = d_i(\sigma)$$

  by construction. We thus see that the map $\alpha : A_3 \to S$ is well-defined.

- **The case** $n > 3$. We once again consider $i < j$, and abusively denote the presumptive definition of the i^{th} face of α by $d_i(\alpha)$.

  First suppose $i < j \leqslant n - 2$. We then compute

  $$d_i(d_j(\alpha)) = d_i(E(d_j(\sigma)) = E(d_i(d_j(\sigma))) = E(d_{j-1}(d_i(\sigma))) = d_{j-1}(E(d_i(\sigma))) = d_{j-1}(d_i(\alpha))$$

  so that the definitions of α agree on the intersection of the i^{th} and j^{th} faces. Notice that we have used the final defining identity of $E(\sigma)$ twice, thus necessitating the hypothesis that $n > 3$.

  Finally, suppose $i \leqslant n - 2$ and $j = n$. Then

  $$d_i(d_n(\alpha)) = d_i(\sigma) = d_{n-1}(E(d_i(\sigma))) = d_{n-1}(d_i(\alpha))$$

  as desired. Thus, the map $\alpha : A_n \to S$ is well-defined.

We can then complete the inductive argument. It is easy to see that we have pullback diagram

$$\begin{array}{ccc} \Lambda^n_{n-1} & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow{\scriptstyle d_{n+1}} \\ A_n & \longrightarrow & \Delta^{n+1} \end{array}$$

Note that the triangle $\Delta^{\{n-2,n-1,n\}}$ in A_n gets mapped to a thin triangle in S by the inductive hypothesis. In particular we can extend α to a morphism $\Lambda^{n+1}_{n-1} \to S$. We finish the proof of the lemma by choosing an extension to $E(\sigma) : \Delta^{n+1} \to S$. $\square$

Corollary 2.3.19. *Let $p : X \to S$ be an* **O2***-fibration. Then for each simplex $\theta : \Delta^n \to X$, the simplices $E(\theta)$ and $E(p(\theta))$ from Lemma 2.3.18 can be chosen so that the diagram*

$$\begin{array}{ccc} \Delta^{n+1} & \xrightarrow{E(\theta)} & X \\ \downarrow & & \downarrow p \\ \Delta^{n+1} & \xrightarrow{E(p(\theta))} & S \end{array}$$

commutes and the simplex $E(\theta)$ satisfies the following properties:

i) The map $E(\theta)$ sends every triangle containing the edge $n-1 \to n$ to a thin triangle.

ii) The map $E(\theta)$ sends the triangle $\Delta^{\{n-1,n,n+1\}}$ to $s_1(e)$, where e is the final edge of θ.

iii) The n^{th} face of $E(\theta)$ equals θ.

iv) If the triangle $\Delta^{\{0,n-1,n\}}$ gets mapped under θ to an element of C_X then $E(\theta)$ sends the triangle $\Delta^{\{0,n,n+1\}}$ to an element of C_X.

Proof. This is virtually identical to the proof of Lemma 2.3.18. One simply performs each step of the argument there relative to the fibration p. Property (iv) holds precisely because C_X is a functorial family. □

Proposition 2.3.20. *Let $p : X \to S$ be an* **O2***-fibration. Then an edge $e : \Delta^1 \to X$ is strongly p-Cartesian if and only if it is p-Cartesian.*

Proof. The 'only if' direction is definitional. To show the other direction, let us suppose that e is p-Cartesian and consider a lifting problem

$$\begin{array}{ccc} \Lambda^n_n & \xrightarrow{f} & X \\ \downarrow & \overset{\hat{f}}{\nearrow} & \downarrow p \\ \Delta^n & \xrightarrow{\sigma} & S \end{array}$$

such that $f|_{\Delta^{\{n-1,n\}}} = e$ and $f|_{\Delta^{\{0,n-1,n\}}}$ belongs to C_X. Fix a choice of maps E guaranteed by Corollary 2.3.19.

Define a simplicial subset $B_n \subset \Delta^{n+1}$ to be the subset containing the i^{th} face for $0 \leqslant i \leqslant n-2$, as well as the face which skips *both* the vertices n and $n-1$. We construct a commutative diagram

$$\begin{array}{ccc} B_n & \xrightarrow{\beta} & X \\ \downarrow & & \downarrow p \\ \Delta^{n+1} & \xrightarrow[E(\sigma)]{} & S \end{array}$$

as follows:

- The map β sends the i^{th} face to $E(d_i(f))$ (as constructed above) for $0 \leqslant i \leqslant n-2$.
- The map β sends the face skipping the vertices n and $n-1$ to $d_{n-1}(f)$.

We then consider the pullback diagram

$$\begin{array}{ccc} \Lambda^n_n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow{\scriptstyle d_{n-1}} \\ B_n & \longrightarrow & \Delta^{n+1} \end{array}$$

and observe that by construction the restriction of β along the composite

$$\Lambda^n_n \longrightarrow B_n \xrightarrow{\beta} X$$

maps the final edge of Λ^n_n to an identity morphism and the triangle $\Delta^{\{0,n-1,n\}}$ to a coCartesian triangle.

Let $\widehat{B}_n \subset \Delta^{n+1}$ be the simplicial subset obtained from B_n by adding the face that skips the vertex $n-1$. Since p is an **O2**-fibration we can extend β to a morphism $\gamma : \widehat{B}_n \to X$. We thus obtain a commutative diagram

$$\begin{array}{ccc} \widehat{B}_n & \xrightarrow{\gamma} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^{n+1} & \xrightarrow[E(\sigma)]{} & S \end{array}$$

We can now consider the pullback diagram diagram

$$\begin{array}{ccc} \Lambda^n_n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow{\scriptstyle d_{n+1}} \\ \widehat{B}_n & \longrightarrow & \Delta^{n+1} \end{array}$$

and observe that, since γ maps the final edge of Λ^n_n to e and maps the triangle $\Delta^{\{0,n-1,n\}}$ to a thin triangle, it follows from the fact that e is p-Cartesian that we have a commutative diagram

$$\begin{array}{ccc} \Lambda^{n+1}_n & \xrightarrow{\varepsilon} & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^{n+1} & \xrightarrow[E(\sigma)]{} & S \end{array}$$

Since ε maps $\Delta^{\{n-1,n,n+1\}}$ to a thin triangle, the dotted arrow in the diagram exists. This arrow is the desired morphism $E(f)$, completing the proof. □

Corollary 2.3.21. *Let $p : X \to S$ be a **O2**-fibration and let $\sigma : \Delta^2 \to X$ be a thin 2-simplex as pictured below*

$$\begin{array}{ccccc} & & b & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ a & & \xrightarrow{\quad h \quad} & & c \end{array}$$

Suppose that g is strongly p-Cartesian. Then f is strongly p-Cartesian if and only if h is strongly p-Cartesian.

Proof. By Proposition 2.3.20 it will suffice to prove the claim replacing strongly p-Cartesian with simply p-Cartesian. This is shown in Lemma 2.3.8 and Lemma 2.3.9 in [GHL21a]. □

Corollary 2.3.22. *Let $p : X \to S$ be a* **O2***-fibration. Then an edge $e : b \to c$ in X is strongly p-Cartesian if and only if for every object $a \in X$ post-composition with e induces a homotopy pullback diagram*

$$\begin{array}{ccc} X(a,b) & \xrightarrow{e\circ -} & X(a,c) \\ \downarrow & & \downarrow \\ S(p(a),p(b)) & \xrightarrow{p(e)\circ -} & S(p(a),p(c)) \end{array}$$

Proof. Combine Proposition 2.3.20 with [GHL21a, Prop. 2.3.3]. □

Proposition 2.3.23. *Let $p : X \to S$ be an* **O2***-fibration. Given a pair of objects $a, b \in X$ and an (strongly) p-Cartesian edge $e : a' \to b$ such that $p(a) = p(a')$ we have a pullback diagram in $\mathbb{C}\mathrm{at}_\infty$*

$$\begin{array}{ccc} X_{p(a)}(a,a') & \xrightarrow{e\circ -} & X(a,b) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{p(e)} & S(p(a),p(b)) \end{array}$$

Proof. Let $X_{p(e)} \to \Delta^1$ denote the pullback of $X \to S$ along the map selecting the edge $p(e)$. We claim that we have a pullback diagram of simplicial sets

$$\begin{array}{ccc} X_{p(e)}(a,b) & \longrightarrow & X(a,b) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{p(e)} & S(p(a),p(b)) \end{array}$$

Let $\sigma : \Delta^n \to X_{p(e)}(a,b)$ with associated $(n+1)$-simplex $\overline{\sigma} : \Delta^{n+1} \to X_{p(e)}$. We note that the composite

$$\kappa : \Delta^{n+1} \xrightarrow{\overline{\sigma}} X_{p(e)} \to \Delta^1 \xrightarrow{p(e)} S$$

defines an degenerate $(n+1)$-simplex in S. We can further see that κ represents a simplex $\Delta^n \to S(p(a),p(b))$ which is degenerate on the object $p(e)$. This proves the existence of the commutative diagram above. It is immediate to see that every simplex $\Delta^n \to X(a,b)$ whose image on $S(p(a),p(b))$ is degenerate on $p(e)$ factors through $X_{p(e)}(a,b)$ which implies that the diagram in question is in fact, a pullback diagram.

We observe that it follows from Proposition 2.3.6 that the right-most vertical map is a coCartesian fibration. This in turn implies that this diagram is a pullback diagram in $\mathbb{C}\mathrm{at}_\infty$. Therefore to show our claim we need to verify that the induced morphism

$$X_{p(a)}(a,a') \longrightarrow X_{p(e)}(a,b)$$

is an equivalence of ∞-categories. It is immediate to check that $X_{p(a)}(a,a') = X_{p(e)}(a,a')$. The claim now follows from Corollary 2.3.22. □

Definition 2.3.24. Let $p : X \to S$ be an **O2**-fibration. We say that p is an **O2C**-fibration if for every edge $e : s \to p(x)$ in S there exists a p-Cartesian lift $\hat{e} : \Delta^1 \to X$ such that $p(\hat{e}) = e$.

Remark 2.3.25. The terminology **O2C**-fibration is reminiscent of to the already defined notion of outer 2-Cartesian fibration. We will show that both definitions are equivalent whenever S is an ∞-bicategory in Theorem 2.3.27.

Corollary 2.3.26. *Let $p : X \to S$ be an **O2C**-fibration and let $p^c : X^c \to S$ denote the restriction to p to the simplicial subset X^c consisting only in simplices whose triangles are in C_X. Then p^c is an* outer Cartesian fibration *in the sense of [GHL21a]. In particular, p is an outer Cartesian fibration if and only if all of its triangles belong to C_X.*

Theorem 2.3.27. *Let $p : X \to S$ be a locally fibred map equipped with a coCartesian family of triangles C_X. Let E_X denote the collection of p-cartesian edges. Then p is a **O2C**-fibration if and only if the map $(X, E_X, T_X \subseteq C_X) \to (S, \sharp, T_S \subset \sharp)$ is a 2-Cartesian fibration in the sense of Definition 2.2.23.*

Proof. Let us suppose that p is a **O2C**-fibration. We need to show that p has the right lifting property with respects to the maps of Definition 2.2.7. The only cases that are not hardcoded into the definitions are: (S1), (S2), (S4), (S5), and (E). (S1) follows from Corollary 2.3.21, (S2) follows from Lemma 2.3.10, (S4) follows from Lemma 2.3.13, (S5) follows from Lemma 2.3.17 and finally (E) follows from Corollary 2.3.22. The converse is clear. ☐

Proposition 2.3.28. *Suppose we are given a morphism of 2-Cartesian fibrations*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

Then the following statements are equivalent:

i) For every $s \in S$ the map $f_s : X_s \to Y_s$ is a bicategorical equivalence.

ii) The map f is a bicategorical equivalence.

Proof. The implication $i) \implies ii)$ is a direct consequence of Proposition 2.2.39. To prove the converse let us $u, v \in X$ such that $p(u) = p(v) = s$. Then it follows from Proposition 2.3.23 that we can identify the morphism

$$X_s(u, v) \longrightarrow Y_s(f(u), f(v))$$

with the the fibre over id_s of the map

$$\begin{array}{ccc} X(u, v) & \xrightarrow{f_{uv}} & Y(f(u), f(v)) \\ & \searrow \quad \swarrow & \\ & S(s, s) & \end{array}$$

Since f is a bicategorical equivalence it follows (see [Lur09b, Theorem 4.2.2]) that f_{uv} is a categorical equivalence and we can use [Lur09a, Prop. 3.3.1.5] to show that the map f_s is fully faithful. To finish the proof we will show that f_s is essentially surjective. Let $y \in Y_s$ and pick $x \in X$ together with an equivalence $\alpha : f(x) \to y$. Let us pick an inverse to $p(\alpha)$ namely $\gamma : s \to p(x)$ and a p-Cartesian lift of γ which we call $\beta : \hat{x} \to x$. It is easy to see that β must be an equivalence. To finish the proof we can assemble $f(\beta)$ and α into a Λ^2_1 and construct a extension to $\sigma : \Delta^2 \to X$ such that the edge $\Delta^{\{0,2\}}$ belongs to Y_s. □

2.3.1 Fibrations of simplicially enriched categories

Definition 2.3.29. We say that a Set^+_Δ-enriched category $\mathcal{C}$ is a $\mathbb{C}\mathrm{at}_\infty$-category if it is a fibrant object in the model structure of Set^+_Δ-enriched categories.

Proposition 2.3.30. *Let $f : \mathcal{C} \to \mathcal{D}$ be a fibration of $\mathbb{C}\mathrm{at}_\infty$-categories and recall the functor $\mathrm{N}^{\mathrm{sc}} : \mathrm{Cat}_{\mathrm{Set}^+_\Delta} \to \mathrm{Set}^{\mathbf{sc}}_\Delta$ from Definition 2.1.9. Then the map*

$$\mathrm{N}^{\mathrm{sc}}(f)\colon\ \mathrm{N}^{\mathrm{sc}}(\mathcal{C}) \longrightarrow \mathrm{N}^{\mathrm{sc}}(\mathcal{D})$$

is a functorially fibred morphism if and only if the following hold:

i) For every $x, y \in \mathcal{C}$ the map $\mathcal{C}(x, y) \to \mathcal{D}(f(x), f(y))$ is a coCartesian fibration of ∞-categories.

ii) Let $x, y, z \in \mathcal{C}$ and consider a pair of coCartesian edges $e_1 : \Delta^1 \to \mathcal{C}(x, y)$ and $e_2 : \Delta^1 \to \mathcal{C}(y, z)$. Then the composite

$$\Delta^1 \xrightarrow{e_1 \times e_2} \mathcal{C}(x, y) \times \mathcal{C}(y, z) \longrightarrow \mathcal{C}(x, y)$$

defines a coCartesian edge in the target.

Proof. Observe that since N^{sc} is a right Quillen functor it follows it follows that $\mathrm{N}^{\mathrm{sc}}(f)$ is a fibration in the model structure on scaled simplicial sets. In particular, it is a weak $\mathbf{S}$-fibration. We will show that condition $i)$ is satisfied if and only if $\mathrm{N}^{\mathrm{sc}}(f)$ is locally fibred and that condition $ii)$ is satisfied if and only if the collection of coCartesian triangles is functorial.

Let us suppose that f is functorially fibred and recall the Set^+_Δ-categories $\mathbb{O}^n$ for $n \geqslant 0$ defined Definition 2.1.7. Given a simplex $\Delta^n \to \mathcal{C}(x, y)$, we define a Set^+_Δ-category by means of the pushout

$$\begin{array}{ccc} \mathbb{O}^n & \longrightarrow & \mathbb{O}^0 \\ \downarrow & & \downarrow \\ \mathbb{O}^{n+1} & \longrightarrow & \mathbb{Q}^n \end{array}$$

where the left-most horizontal morphism is induced by the map of posets $d_{n+1} : [n] \to [n + 1]$. We construct a morphism $\hat{l}_\sigma : \mathbb{O}^{n+1} \to \mathcal{C}$ as follows:

- On objects we set $\hat{l}_\sigma(i) = x$ if $0 \leqslant i \leqslant n$ and $\hat{l}_\sigma(n + 1) = y$.
- Given $0 \leqslant i \leqslant j \leqslant n$ the morphism $\mathbb{O}^{n+1}(i, j) \to \mathcal{C}(x, x)$ is constant on the identity on x and similarly for $\mathbb{O}^{n+1}(n + 1, n + 1) \to \mathcal{C}(y, y)$.

- The morphism $\mathbb{O}^{n+1}(i,n+1) \to \mathcal{C}(x,y)$ for $0 \leqslant i \leqslant n$ factors through
$$\mathbb{O}^n(i,n+1) \to \Delta^n \xrightarrow{\sigma} \mathcal{C}(x,y)$$
where the first morphism sends $S \subseteq [n+1]$ to $\max(S \setminus \{n+1\}) \in \Delta^n$.

It is easy to see that our definition of $\hat{l}_\sigma$ factors through the pushout producing a morphism $l_\sigma : \mathbb{Q}^n \to \mathcal{C}$. Since $\mathfrak{C}^{\mathbf{sc}}$ is a left adjoint it follows that $\mathbb{Q}^n \simeq \mathfrak{C}^{\mathbf{sc}}[\Delta^{n+1} \coprod_{\Delta^n} \Delta^0]$.

Suppose we are given a lifting problem

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{u} & \mathcal{C}(x,y) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathcal{D}(f(x),f(y)) \end{array}$$

and let $e : \Delta^1 \to \mathcal{C}(x,y)$ denote the restriction of u to $\Delta^{\{0,1\}} \subset \Lambda^n_0$. Let us further suppose that the morphism $Q^2 \xrightarrow{l_e} \mathcal{C}$ corresponds to a left-degenerate coCartesian triangle in $\mathrm{N}^{\mathrm{sc}}(\mathcal{C})$. We will show that we can construct the dotted arrow in the diagram. We define $\mathbb{Q}^n_0 = \mathfrak{C}^{\mathbf{sc}}[\Lambda^{n+1}_0 \coprod_{\Delta^n} \Delta^0]$ and observe that we can construct another commutative diagram

$$\begin{array}{ccc} \mathbb{Q}^n_0 & \longrightarrow & \mathcal{C} \\ \downarrow & \overset{\omega}{\nearrow} & \downarrow f \\ \mathbb{Q}^n & \longrightarrow & \mathcal{D} \end{array}$$

which admits a solution since its adjoint lifting problem admits one. The definition of the top horizontal morphism is induced by our construction l_σ applied to the left-horn. We provide a solution to the original lifting problem by considering the simplex

$$\Delta^n \xrightarrow{\iota} \mathbb{Q}^n(0,n+1) \xrightarrow{\omega} \mathcal{C}(x,y)$$

where ι sends the vertex i to the subset $[0,i]$. An analogous argument as before shows that we can produce coCartesian lifts of morphisms in the base. We conclude that $i)$ holds.

Let $\sigma : \Delta^2 \to \mathrm{N}^{\mathrm{sc}}(\mathcal{C})$ be a left degenerate simplex whose adjoint morphism $\mathbb{Q}^2 \to \mathcal{C}$ defines a coCartesian edge. Let $n \geqslant 3$ and consider a lifting problem

$$\begin{array}{ccc} \Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0 & \longrightarrow & \mathrm{N}^{\mathrm{sc}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0 & \longrightarrow & \mathrm{N}^{\mathrm{sc}}(\mathcal{D}) \end{array}$$

It follows from unraveling the definitions that we only need to solve the adjoint lifting problem

$$\begin{array}{ccc} \mathbb{P}^n_0 = \mathfrak{C}^{\mathbf{sc}}[\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0](*,n) & \longrightarrow & \mathcal{C}(x,y) \\ \downarrow & & \downarrow \\ \mathbb{P}^n = \mathfrak{C}^{\mathbf{sc}}[\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0](*,n) & \longrightarrow & \mathcal{D}(f(x),f(y)) \end{array}$$

where $*$ denotes the collapsed vertex where the vertices $0, 1$ get mapped onto. We identify $\mathbb{P}^n$ with the nerve of the poset of subsets $S \subseteq [n]$ such that $\min(S) = 0$ and $\max(S) = n$ ordered by inclusion. It follows that $\mathbb{P}^n_0 \subset \mathbb{P}^n$ is the simplicial subset consisting in those simplices $\sigma : \Delta^k \to \mathbb{P}^n$ represented by a chain of inclusions $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k$ satisfying at least one of the the following conditions:

- There exists $1 < j \leqslant n-1$ such that $j \in S_i$ for $0 \leqslant i \leqslant k$.
- There exists $1 \leqslant j \leqslant n-1$ such that $j \notin S_i$ for $0 \leqslant i \leqslant k$.

Note that we can view $\mathbb{P}^n$ geometrically as a $(n-1)$-dimensional cube. Then it follows that $\mathbb{P}^n_0$ is the union of all of the $(n-2)$-dimensional faces of $\mathbb{P}^n$ except the face consisting in subsets S such that $1 \in S$. We will further equip both simplicial sets with a marking given by the edge $0n \to 01n$. Since the image of that particular edge is a coCartesian edge in $\mathcal{C}(x, y)$ it will suffice to show that the inclusion $\mathbb{P}^n_0 \to \mathbb{P}^n$ is an anodyne morphism in the coCartesian model structure.

Let $\sigma_i : S^i_0 \subset S^i_1 \subset \cdots \subset S^i_{n-1}$ for $i = 1, 2$ be a pair of *distinct* non-degenerate simplices of maximal dimension. Observe that by maximality $S^i_0 = 0n$ for $i = 1, 2$. Let $0 \leqslant \nu \leqslant n-2$ be the first index such that $S^1_\nu \neq S^2_\nu$. We say $\sigma_1 < \sigma_2$ if and only if $\max(S^1_\nu \setminus \{n\}) < \max(S^2_\nu \setminus \{n\})$. Let us considered the totally ordered set of non-degenerate simplices of maximal dimension $\{\sigma_1 < \sigma_2 \cdots \sigma_{n!}\}$. We can now produce a filtration

$$\mathbb{P}^n_0 \to X_{n!} \to X_{n!-1} \to \cdots X_2 \to X_1 = \mathbb{P}^n$$

where $X_j \subset \mathbb{P}^n$ consists in those simplices ρ that either factor through $\mathbb{P}^n_0$ or are contained in a non-degenerate simplex of maximal dimension $\rho \subset \sigma_\ell$ for $\ell \geqslant j$. The proof is by now routine and left as an exercise to the reader.

To finish the proof we will show that condition *ii*) holds if and only if the collection of coCartesian triangles is functorial. Let us suppose that *ii*) holds and consider a simplex $\rho_i : \Delta^3 \to \mathrm{N}^{\mathrm{sc}}(\mathcal{C})$ such that the image triangle $\{i-1, i, i+1\}$ is thin in $\mathrm{N}^{\mathrm{sc}}(\mathcal{C})$. Let us assume that every triangle of Δ^3 except the i-th face corresponds via the adjoint map $\alpha_i : \mathbb{O}^3 \to \mathcal{C}$ to a coCartesian edge in the mapping space of $\mathcal{C}$. We consider a pair of commutative diagrams

$$\begin{array}{ccc} 013 & \longrightarrow & 0123 \\ \uparrow \circ & & \simeq \uparrow \\ 03 & \longrightarrow & 023 \end{array} \qquad\qquad \begin{array}{ccc} 013 & \xrightarrow{\simeq} & 0123 \\ \uparrow & & \uparrow \\ 03 & \xrightarrow{\circ} & 023 \end{array}$$

that we interpret as the image of the morphism $\mathbb{O}^3(0, 3) \to \mathcal{C}(\alpha_i(0), \alpha_i(3))$ for $i = 1, 2$. We have circled in both diagrams the coCartesian edges and denoted by "$\simeq$" the equivalences associated to the thin triangle $\{i-1, i, i+1\}$. Note that in the first diagram the edge $013 \to 0123$ can be obtained from the coCartesian edge $13 \to 123$ via precomposition with a degenerate edge. Our assumptions then imply that $013 \to 0123$ is coCartesian and thus the whole diagram must consist of coCartesian edges. Since the edge $03 \to 023$ corresponds to the face missing 1 of ρ_1 the claim holds. The argument for the second diagram is totally analogous.

To finish the proof let us suppose that the collection of coCartesian triangles is functorial. Let $x, y, z \in \mathcal{C}$, we claim that in order to show that the map

$$\gamma_{x,y} \colon \mathcal{C}(x,y) \times \mathcal{C}(y,z) \longrightarrow \mathcal{C}(x,z)$$

preserves coCartesian edges it suffices to prove the particular cases where one of the two morphisms we want to compose is degenerate. Indeed, given $e : \Delta^1 \to \mathcal{C}(x,y) \times \mathcal{C}(y,z)$ determined by a pair of edges $(f \to g, u \to v)$ we can produce a 2-simplex

$$\theta : \Delta^2 \to \mathcal{C}(x,z), \;\; \theta : u \circ f \to v \circ f \to v \circ g$$

such that $d_1(\theta) = \gamma_{x,y}(e)$ and where $d_i(\theta)$ is given by a composition with a degenerate edge for $i = 0, 2$.

Let $f \to g$ be a coCartesian edge in $\mathcal{C}(x,y)$ and let u be an object of $\mathcal{C}(y,z)$. We consider a map $\tau : \mathbb{O}^3 \to \mathcal{C}$ defined as follows:

- We have $\tau(0) = \tau(1) = x$, $\tau(2) = y$ and $\tau(3) = z$.
- The map $\mathbb{O}^3(0,1) \to \mathcal{C}(x,x)$ is degenerate on the identity morphism.
- The map $\mathbb{O}^3(0,2) \to \mathcal{C}(x,y)$ selects the morphism $f \to g$
- The map $\mathbb{O}^3(1,2) \to \mathcal{C}(x,y)$ selects the object g.
- The map $\mathbb{O}^3(2,3) \to \mathcal{C}(y,z)$ selects the object u.
- The map $\mathbb{O}^3(1,3) \to \mathcal{C}(x,z)$ selects the degenerate edge on $u \circ g$.
- The map $\mathbb{O}^3(0,3) \to \mathcal{C}(x,z)$ factors as $\mathbb{O}^3(0,3) \xrightarrow{\pi} \Delta^1 \to \mathcal{C}(x,y)$ where the second morphism selects the edge $u \circ f \to u \circ g$ and the first morphism is determined by $\pi(03) = \pi(023) = 0$ and $\pi(013) = \pi(0123) = 1$.

It follows that the adjoint map $\kappa : \Delta^3 \to \mathrm{N}^{\mathrm{sc}}(\mathcal{C})$ maps the triangle $\{1,2,3\}$ to a thin triangle in $\mathrm{N}^{\mathrm{sc}}(\mathcal{C})$ and that every triangle of κ gets mapped to a coCartesian triangle except possible the face missing 2. Since the collection of coCartesian triangles is functorial it follows that $\{0,1,3\}$ is also a coCartesian triangle. This shows that $u \circ f \to u \circ g$ must be a coCartesian edge in $\mathcal{C}(x,z)$. We leave the completely analogous verification that precomposition with degenerate edges preserves coCartesian edges as an exercise for the reader. □

Definition 2.3.31. Let $f : \mathcal{C} \to \mathcal{D}$ be a map of $\mathbb{C}\mathrm{at}_\infty$-categories. An edge $e : x \to y$ is said to be f-Cartesian if for every $z \in \mathcal{C}$ the following diagram

$$\begin{array}{ccc} \mathcal{C}(z,x) & \longrightarrow & \mathcal{C}(z,y) \\ \downarrow & & \downarrow \\ \mathcal{D}(f(z),f(x)) & \longrightarrow & \mathcal{D}(f(z),f(y)) \end{array}$$

is a homotopy pullback square in Set_Δ^+.

The next theorem follows readily from Proposition 2.3.30.

Theorem 2.3.32. *Let $f : \mathcal{C} \to \mathcal{D}$ be a fibration of $\mathbb{C}\mathrm{at}_\infty$-categories. Then $\mathrm{N}^{\mathrm{sc}}(f)$ is a 2-Cartesian fibration if and only if the following conditions hold:*

i) For every $x, y \in \mathcal{C}$ the map $\mathcal{C}(x, y) \to \mathcal{D}(f(x), f(y))$ is a coCartesian fibration of ∞-categories.

ii) Let $x, y, z \in \mathcal{C}$ and consider a pair of coCartesian edges $e_1 : \Delta^1 \to \mathcal{C}(x, y)$ and $e_2 : \Delta^1 \to \mathcal{C}(y, z)$. Then the composite

$$\Delta^1 \xrightarrow{e_1 \times e_2} \mathcal{C}(x, y) \times \mathcal{C}(y, z) \longrightarrow \mathcal{C}(x, y)$$

defines a coCartesian edge in the target.

iii) For every morphism $e : d \to f(y)$ in $\mathcal{D}$. There exists an f-Cartesian lift $\hat{e} : \hat{d} \to y$ with $f(\hat{e}) = e$.

Remark 2.3.33. We say that a functor of 2-categories $f : \mathbb{C} \to \mathbb{D}$ is a 2-Cartesian fibration if and only if $\underline{\mathrm{N}}(f)$ (see Definition 2.1.6) satisfies the conditions of Theorem 2.3.32. It follows that this definition (after taking the pertinent duals) recovers the notion of 2-fibration presented in [Buc14]. In particular, it follows from Theorem 2.3.32 that our definition generalises the classical notion of a 2-fibration to the realm of ∞-bicategories.

2.4 The model structure on marked scaled simplicial sets

A special case of the model structure of Theorem 2.2.43 of particular interest occurs when $S = \Delta^0$ is the terminal scaled simplicial set. Then, by Theorem 2.2.44, the resulting model structure on $\mathrm{Set}_\Delta^{\mathbf{mb}}$ is Quillen equivalent to the model structure for ∞-bicategories on $\mathrm{Set}_\Delta^{\mathbf{sc}}$. In this case, the data of the two scalings becomes highly redundant — for any fibrant object the two scalings coincide, and heuristically they no longer encode different information.

We can avoid this redundancy by defining a further model structure which includes both markings and scalings, but avoids the redundancies created by a biscaling. The aim of this section is to define this model structure, and relate it to the **MB** model structure.

Definition 2.4.1. A *marked-scaled simplicial set* consists of

- A simplicial set X.
- A collection of edges $E_X \subseteq X_1$ containing all degenerate edges. We call the elements of E_X *marked edges*.
- A collection of triangles $T_X \subseteq X_2$ containing all degenerate triangles. We call the elements of T_X *thin triangles*.

We denote by $\mathrm{Set}_\Delta^{\mathbf{ms}}$ the category of marked-scaled simplicial sets. We view this as a Set_Δ^{+}-enriched category by defining

$$\mathrm{Hom}_{\mathrm{Set}_\Delta^+}(X, \mathrm{Set}_\Delta^{\mathbf{ms}}(Y, Z)) := \mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{ms}}}(X_\sharp \times Y, Z)$$

where $X_\sharp = (X, E_X, \sharp)$.

Before continuing with the construction of the model structure, we briefly digress to explore the relations between $\mathrm{Set}_\Delta^{\mathbf{mb}}$ and $\mathrm{Set}_\Delta^{\mathbf{ms}}$. The primary component of our comparison will be the adjunction:

$$\mathrm{Set}_\Delta^{\mathbf{ms}} \underset{R}{\overset{D}{\rightleftarrows}} \mathrm{Set}_\Delta^{\mathbf{mb}}$$

where D is given on objects by

$$D\colon (X, E_X, T_X) \longmapsto (X, E_X, T_X \subseteq T_X)$$

and R is given on objects by

$$R\colon (Y, E_Y, T_Y \subseteq C_Y) \longmapsto (Y, E_Y, T_Y)$$

We will show that this adjunction becomes a Quillen equivalence once we have equipped $\mathrm{Set}_\Delta^{\mathbf{ms}}$ with the appropriate model structure.

This model structure itself is constructed exactly analogously to the model structure on $\mathrm{Set}_\Delta^{\mathbf{mb}}$. We begin with a set of generating anodyne morphisms:

Definition 2.4.2. The set of *generating* **MS**-*anodyne maps* **MS** is the set of maps of marked-scaled simplicial sets consisting of:

(MS1) The inner horn inclusions

$$\left(\Lambda^n_i, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\right) \quad , \quad n \geqslant 2 \quad , \quad 0 < i < n;$$

(MS2) The map

$$(\Delta^4, \flat, T) \longrightarrow (\Delta^4, \flat, T \cup \{\Delta^{\{0,3,4\}}, \Delta^{\{0,1,4\}}\})$$

where T is defined as in Definition 2.2.7, (A2).

(MS3) The set of maps

$$\left(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \{\Delta^{\{0,1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \{\Delta^{\{0,1,n\}}\}\right) \quad , \quad n \geqslant 2.$$

(MS4) The set of maps

$$\left(\Lambda^n_n, \{\Delta^{\{n-1,n\}}\}, \{\Delta^{\{0,n-1,n\}}\}\right) \to \left(\Delta^n, \{\Delta^{\{n-1,n\}}\}, \{\Delta^{\{0,n-1,n\}}\}\right) \quad , \quad n \geqslant 2.$$

(MS5) The inclusion of the terminal vertex

$$\left(\Delta^0, \sharp, \sharp\right) \longrightarrow \left(\Delta^1, \sharp, \sharp\right)$$

(MS6) The map

$$\left(\Delta^2, \{\Delta^{\{0,1\}}, \Delta^{\{1,2\}}\}, \sharp\right) \to \left(\Delta^2, \sharp, \sharp\right),$$

(MS7) The map

$$\left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, U_0\right) \to \left(\Delta^3 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \sharp\right)$$

where U_0 is the collection of all triangles except the 0-th face.

(MS8) The map

$$\left(\Delta^3, \{\Delta^{\{2,3\}}\}, U_3\right) \to \left(\Delta^3, \{\Delta^{\{2,3\}}\}, \sharp\right)$$

where U_3 is the collections of all triangles except the 3-rd face.

(MSE) For every Kan complex K, the map

$$\left(K, \flat, \sharp\right) \to \left(K, \sharp, \sharp\right).$$

We will call a morphism in $\mathrm{Set}_\Delta^{\mathbf{ms}}$ **MS**-*anodyne* if it lies in the saturated hull of **MS**.

We can immediately obtain two useful lemmata.

Lemma 2.4.3. *The morphism*

$$\left(\Delta^3, \flat, \{\Delta^{\{i-1,i,i+1\}}\} \subset U_i\right) \to \left(\Delta^3, \flat, \{\Delta^{\{i-1,i,i+1\}}\} \subset \sharp\right) \quad , \quad 0 < i < 3,$$

where U_i is the collection of all triangles except i-th face, is **MS**-*anodyne.*

Proof. See [Lur09b, Rmk 3.1.4]. □

Lemma 2.4.4. *The morphism*

$$\theta\colon (\Delta^2, \{\Delta^{\{1,2\}}, \Delta^{\{0,2\}}\}, \sharp) \longrightarrow (\Delta^2, \sharp, \sharp)$$

is **MS**-*anodyne.*

Proof. The proof follows exactly as in Lemma 2.2.11. □

Finally, in total analogy to the marked biscaled case, we can establish a pushout-product axiom, and thereby a model structure.

Proposition 2.4.5. *Let $f : X \to Y$ be an* **MS**-*anodyne morphism in* $\mathrm{Set}_\Delta^{\mathbf{ms}}$*, and let $g : A \to B$ be a cofibration in* $\mathrm{Set}_\Delta^{\mathbf{ms}}$*. The morphism*

$$f \wedge g\colon X \times B \coprod_{X \times A} Y \times A \longrightarrow Y \times B$$

is **MS**-*anodyne.*

Proof. Every case is, mutatis mutandis, the same as the corresponding case in the proof of Proposition 2.2.14. □

As in the marked-biscaled case, we can immediately define several mapping spaces.

Definition 2.4.6. Let $\overline{X} := (X, E_X, T_X)$ be a fibrant marked-scaled simplicial set and $\overline{Y} := (Y, E_Y, T_Y)$ any marked-scaled simplicial set. We can define a marked-scaled simplicial set $\mathrm{Fun}^{\mathbf{ms}}(\overline{Y}, \overline{X})$ via the universal property

$$\mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{ms}}}(\overline{A}, \mathrm{Fun}^{\mathbf{ms}}(\overline{Y}, \overline{X})) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{ms}}}(\overline{A} \times \overline{Y}, \overline{X}).$$

It follows from the pushout-product that this is a fibrant marked-scaled simplicial set, and thus that the underlying scaled simplicial set is an ∞-bicategory. We denote this ∞-bicategory by $\mathrm{Map}_{\mathbf{ms}}(\overline{Y}, \overline{X})$.

We can similarly define

- A marked simplicial set $\mathrm{Map}^{\mathrm{th}}_{\mathbf{ms}}(\overline{Y},\overline{X})$ be the full subsimplicial set of $\mathrm{Fun}^{\mathbf{ms}}(\overline{Y},\overline{X})$ consisting of the thin triangles.
- A simplicial set $\mathrm{Map}^{\simeq}_{\mathbf{ms}}(\overline{Y},\overline{X})$, which consists of precisely the marked edges in $\mathrm{Map}^{\mathrm{th}}_{\mathbf{ms}}(\overline{Y},\overline{X})$.

Finally, we can establish the existence of the model structure:

Theorem 2.4.7. *There is a left-proper combinatorial simplicial model category structure on* $\mathrm{Set}^{\mathbf{ms}}_{\Delta}$ *uniquely characterized by the following properties:*

C) A morphism $f : X \to Y$ in $\mathrm{Set}^{\mathbf{ms}}_{\Delta}$ *is a cofibration if and only if it is a monomorphism on underlying simplicial sets.*

F) An object $X \in$ $\mathrm{Set}^{\mathbf{ms}}_{\Delta}$ *is fibrant if and only if the unique map $X \to \Delta^0$ has the right lifting property with respect to the morphisms in* **MS**.

Remark 2.4.8. It is not hard to see that we can tensor $\mathrm{Set}^{\mathbf{ms}}_{\Delta}$ over $\mathrm{Set}^{+}_{\Delta}$ and Set_{Δ} in a way compatible with the enrichments provided by $\mathrm{Map}^{\mathrm{th}}_{\mathbf{ms}}(-,-)$ and $\mathrm{Map}^{\simeq}_{\mathbf{ms}}(-,-)$, respectively. The latter of these provides the simplicial structure in the preceding proposition.

The weak equivalences in the model structure are precisely those $f : \overline{A} \to \overline{B}$, which satisfy the equivalent conditions for any fibrant marked-scaled simplicial set $\overline{X}$:

- The induced map
$$\mathrm{Map}_{\mathbf{ms}}(\overline{B},\overline{X}) \longrightarrow \mathrm{Map}_{\mathbf{ms}}(\overline{A},\overline{X})$$
is a bicategorical equivalence.
- The induced map
$$\mathrm{Map}^{\mathrm{th}}_{\mathbf{ms}}(\overline{B},\overline{X}) \longrightarrow \mathrm{Map}^{\mathrm{th}}_{\mathbf{ms}}(\overline{A},\overline{X})$$
is a weak equivalence of marked simplicial sets.
- The induced map
$$\mathrm{Map}^{\simeq}_{\mathbf{ms}}(\overline{B},\overline{X}) \longrightarrow \mathrm{Map}^{\simeq}_{\mathbf{ms}}(\overline{A},\overline{X})$$
is a weak equivalence of Kan complexes.

It is not hard to see that the adjunction $D \dashv R$ can be promoted to a simplicial adjunction. By construction, L preserves cofibrations and R preserves fibrant objects, and thus we see that

Lemma 2.4.9. *The adjunction*
$$\mathrm{Set}^{\mathbf{ms}}_{\Delta} \underset{R}{\overset{D}{\rightleftarrows}} \mathrm{Set}^{\mathbf{mb}}_{\Delta}$$
is a simplicial Quillen adjunction.

Further, we can define an adjunction
$$\mathrm{Set}^{\mathbf{sc}}_{\Delta} \underset{G}{\overset{(-)^{\flat}}{\rightleftarrows}} \mathrm{Set}^{\mathbf{ms}}_{\Delta}$$
where $G(X, E_X, T_X) = (X, T_X)$.

Lemma 2.4.10. *The adjunction*

$$\mathrm{Set}_\Delta^{\mathbf{sc}} \underset{G}{\overset{(-)^\flat}{\rightleftarrows}} \mathrm{Set}_\Delta^{\mathbf{ms}}$$

is a Quillen adjunction.

Proof. It is immediate that $(-)^\flat$ preserves cofibrations. Suppose that $f : (X, T_X) \to (Y, T_Y)$ is a weak equivalence. Let (Z, E_Z, T_Z) be a fibrant object in $\mathrm{Set}_\Delta^{\mathbf{ms}}$. It is easy to see that $G(Z, E_Z, T_Z) = (Z, T_Z)$ is a fibrant object in $\mathrm{Set}_\Delta^{\mathbf{sc}}$. We can then note that, by definition, there is an isomorphism of mapping scaled simplicial sets

$$\mathrm{Map}_{\mathbf{sc}}((X, T_X), (Z, T_Z)) \cong \mathrm{Map}_{\mathbf{ms}}((X, \flat, T_X), (Z, E_Z, T_Z)).$$

Thus, since f induces a bicategorical equivalence

$$\mathrm{Map}_{\mathbf{sc}}((Y, T_Y), (Z, T_Z)) \to \mathrm{Map}_{\mathbf{sc}}((X, T_X), (Z, T_Z))$$

we see that the map

$$\mathrm{Map}_{\mathbf{ms}}((Y, \flat, T_Y), (Z, E_Z, T_Z)) \to \mathrm{Map}_{\mathbf{ms}}((X, \flat, T_X), (Z, E_Z, T_Z))$$

induced by $(f)^\flat$ is also an equivalence. We therefore see that $(f)^\flat$ is a weak equivalence in $\mathrm{Set}_\Delta^{\mathbf{ms}}$, as desired. □

Lemma 2.4.11. *The functor G preserves weak equivalences.*

Proof. If, for any ∞-bicategory (Z, T_Z), there exists a set E_Z of marked edges for Z such that (Z, E_Z, T_Z) is a fibrant marked-scaled simplicial set, then this follows from the characterization in terms of mapping ∞-bicategories.

To see that this is the case, let (Z, T_Z) be an ∞-bicategory. Then Z^{th} is an ∞-category, and so we can define a marking E_Z on Z by declaring an edge to be marked if it lies in the maximal Kan complex in Z^{th}. From the definition, it is immediate that (Z, E_Z, T_Z) has the extension property with respect to (MS1), (MS2),(MS3),(MS5),(MS6), and (MSE).

It follows from Corollary 2.3.22 and Corollary 2.3.26 that $Z \to \Delta^0$ is a 2-Cartesian fibration in which the strongly Cartesian edges are precisely the equivalences, and so we see that (Z, E_Z, T_Z) has the extension property with respect to (MS4), (MS7), and (MS8) as well. □

Lemma 2.4.12. *Given a fibrant marked-scaled simplicial set (Y, E_Y, T_Y), the full simplicial subset $Y^{\simeq}$ on the marked edges and scaled triangles is a Kan complex.*

Proof. It is immediate from the definitions that (Y^{th}, E_Y) is a fibrant marked simplicial set, and the lemma follows. □

We now can state and prove the main proposition of this section.

Theorem 2.4.13. *The Quillen adjunctions*

$$\mathrm{Set}_\Delta^{\mathbf{ms}} \underset{R}{\overset{D}{\rightleftarrows}} \mathrm{Set}_\Delta^{\mathbf{mb}}$$

and

$$\mathrm{Set}_\Delta^{\mathbf{sc}} \underset{G}{\overset{(-)^\flat}{\rightleftarrows}} \mathrm{Set}_\Delta^{\mathbf{ms}}$$

are Quillen equivalences.

Proof. By Theorem 2.2.44, the composite adjunction $D \circ (-)^\flat \dashv G \circ R$ is a Quillen equivalence. It thus suffices for us to check that the adjunction $(-)^\flat \dashv G$ is a Quillen equivalence. We will check explicitly that the derived adjunction unit and counit are equivalences.

First, let $(X, T_X) \in \mathrm{Set}_\Delta^{\mathbf{sc}}$. The derived adjunction unit on (X, T_X) is the composite

$$(X, T_X) \longrightarrow G(X, \flat, T_X) \longrightarrow G((X, \flat, T_X)^{\mathrm{fib}})$$

where the superscript fib denotes fibrant replacement. The first of these maps is the identity (since $G(X, \flat, T_X) = (X, T_X)$) and the latter is the image under G of an equivalence of marked-scaled simplicial sets. By Lemma 2.4.11, this is an equivalence.

Now, let $(Y, E_Y, T_Y) \in \mathrm{Set}_\Delta^{\mathbf{ms}}$ be a fibrant object. The derived adjunction counit on (Y, E_Y, T_Y) is the composite

$$(G(Y, E_Y, T_Y)^{\mathrm{cof}})^\flat \longrightarrow G(Y, E_Y, T_Y)^\flat \xrightarrow{\eta_Y} (Y, E_Y, T_Y)$$

Since every scaled simplicial set is cofibrant, the first map is an isomorphism, leaving us to check that the usual adjunction counit η_Y is an equivalence. Note that η_Y is simply the inclusion $(Y, \flat, T_Y) \to (Y, E_Y, T_Y)$.

We have a pushout square

$$\begin{array}{ccc}
(Y^{\simeq}, \flat, \sharp) & \xrightarrow{\psi} & (Y^{\simeq}, \sharp, \sharp) \\
\downarrow & & \downarrow \\
(Y, \flat, T_Y) & \xrightarrow[\eta_Y]{} & (Y, E_Y, T_Y)
\end{array}$$

and, by Lemma 2.4.12 the morphism ψ is a morphism in $\mathbf{MS}$ of type (MSE). Thus, η_Y is $\mathbf{MS}$-anodyne, and is a weak equivalence. □

The Set_Δ^+-enrichment on $\mathrm{Set}_\Delta^{\mathbf{ms}}$

We have already constructed a model structure on the category $\mathrm{Set}_\Delta^{\mathbf{ms}}$ of marked-scaled simplicial sets, and shown that it is a simplicial model category with respect to the mapping spaces $\mathrm{Map}_{\mathbf{ms}}^{\simeq}(-,-)$. However, we will need to consider Set_Δ^+-enriched functors in our analysis of the Grothendieck construction. Our aim in this section is therefore to show that our model structure can, additionally, be viewed as Set_Δ^+-enriched. The following lemma constitutes an easy first check in this direction.

Lemma 2.4.14. *The category* $\mathrm{Set}_\Delta^{\mathbf{ms}}$ *is powered and tensored over* Set_Δ^+ *via the maps*

$$\begin{aligned}
\mathrm{Set}_\Delta^+ \times \mathrm{Set}_\Delta^{\mathbf{ms}} &\longrightarrow \mathrm{Set}_\Delta^{\mathbf{ms}} \\
(K, X) &\longmapsto K_\sharp \times X
\end{aligned}$$

and

$$[-,-]\colon \mathrm{Set}_\Delta^+ \times \mathrm{Set}_\Delta^{\mathbf{ms}} \longrightarrow \mathrm{Set}_\Delta^{\mathbf{ms}}$$
$$(K,X) \longmapsto \mathrm{Fun}^{\mathbf{ms}}(K_\sharp, X)$$

The tensoring and powering is compatible with the mapping spaces $\mathrm{Map}^{\mathrm{th}}_{\mathbf{ms}}(-,-)$.

Our aim throughout the rest of the section will be to show that the tensoring is a left Quillen bifunctor. We will follow the strategy of [GHL20], showing first that the model structure on $\mathrm{Set}_\Delta^{\mathbf{ms}}$ is a Cisinski-Olschok model structure (as with $\mathrm{Set}_\Delta^{\mathbf{sc}}$ in [GHL19]), and then using testing pushout-products with the concomitant interval objects.

We first show that the model structure on $\mathrm{Set}_\Delta^{\mathbf{ms}}$ is Cartesian-closed. This will follow immediately from Proposition 2.4.5 and the following

Lemma 2.4.15. *Let $f : X \to Y$ and $g : A \to B$ be two weak equivalences in* $\mathrm{Set}_\Delta^{\mathbf{mb}}$, *then the product*

$$f \times g\colon X \times A \longrightarrow Y \times B$$

is a weak equivalence.

Proof. Precisely the same argument as in [Lur09b, Lemma 4.2.6] allows us to reduce to the case of the morphism

$$Y \times A \longrightarrow Y \times B$$

where Y, A, and B are all fibrant objects. By the characterization of fibrant objects, this morphism is a weak equivalence if and only if the morphism on underlying scaled simplicial sets is an equivalence, which follows from loc. cit. □

Corollary 2.4.16. *For any cofibrations $f : X \to Y$ and $g : A \to B$, the pushout-product*

$$f \wedge g\colon Y \times A \coprod_{X \times A} X \times B \longrightarrow Y \times B$$

is an equivalence if one of f or g is.

Proof. We can use the small object argument to factor f as

$$X \xrightarrow{\ h\ } Z \xrightarrow{\ k\ } Y$$

where h is **MS**-anodyne. Consequently, k is a weak equivalence. We consider the diagram

$$\begin{array}{ccc} Y \times A \coprod_{X\times A} X \times B & \longleftarrow & Z \times A \coprod_{X\times A} X \times B \\ \downarrow & & \downarrow \\ Y \times B & \longleftarrow & Z \times B \end{array}$$

It follows from the lemma that the bottom horizontal arrow is a weak equivalence, and the top horizontal arrow is the induced map on homotopy colimits by a natural weak equivalence. from Proposition 2.4.5, it follows that the right-hand morphism is an equivalence, and the corollary follows. □

Corollary 2.4.17. *The model structure on* $\mathrm{Set}_\Delta^{\mathbf{ms}}$ *is Cartesian-closed.*

We now wish to show that the Cisinski-Olschok model structure on $\mathrm{Set}_\Delta^{\mathbf{ms}}$ with interval $\Delta^0 \amalg \Delta^0 \to (\Delta^1)^\sharp_\sharp$ and generating anodyne maps the **MS**-anodyne maps is, in fact the model structure constructed in our previous section. We first note that, since one of the morphisms $\Delta^0 \to (\Delta^1)^\sharp_\sharp$ is **MS**-anodyne, it follows that both such morphisms are trivial cofibrations.

Definition 2.4.18. We write $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{CO}}$ for the Cisinski-Olschok model structure on $\mathrm{Set}_\Delta^{\mathbf{ms}}$ with interval $\Delta^0 \amalg \Delta^0 \to (\Delta^1)^\sharp_\sharp$, and generating set of anodyne morphisms the set of **MB**-anodyne morphisms.

For ease, we will write $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{AH}}$ for the model structure previously defined

Proposition 2.4.19. *The two model structures* $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{CO}}$ *and* $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{AH}}$ *coincide.*

Proof. It will suffice to show that the fibrant objects coincide. By construction, every fibrant object of $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{CO}}$ is a fibrant object of $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{AH}}$. However, since $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{AH}}$ is a Cartesian-closed model category, and the interval object for $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{CO}}$ is a cylinder in $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{AH}}$, every anodyne map in $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{CO}}$ is a trivial cofibration in $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{AH}}$. Thus every fibrant object of $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{AH}}$ is a fibrant object of $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{\mathrm{CO}}$. □

As a consequence, we will now drop the unwieldy subscript notation for the model structure on $\mathrm{Set}_\Delta^{\mathbf{ms}}$. We can now prove the following.

Proposition 2.4.20. *The model category* $\mathrm{Set}_\Delta^{\mathbf{ms}}$ *is a* Set_Δ^+*-enriched model category.*

Proof. We need only show that the tensoring satisfies the pushout-product axiom, i.e., that for cofibrations $f : K \to S$ in Set_Δ^+ and $g : X \to Y$ in $\mathrm{Set}_\Delta^{\mathbf{ms}}$, the pushout-product $f \wedge g$ is a trivial cofibration that either f or g is. Since both model structures are Cisinski-Olschok model structures, it suffices to test generating monomorphisms against the two interval inclusions and against the generating anodyne morphisms.

It is immediate from Proposition 2.4.5 that if f (resp. g) is marked (resp. **MS**) anodyne, then $f \wedge g$ is a trivial cofibration. It remains for us to test the cases when f is $\{0\} \to (\Delta^1)^\sharp$ or $\{1\} \to (\Delta^1)^\sharp$, and the cases when g is $\{0\} \to (\Delta^1)^\sharp_\sharp$ or $\{1\} \to (\Delta^1)^\sharp_\sharp$.

However, since the morphisms $\{0\} \to (\Delta^1)^\sharp_\sharp$ or $\{1\} \to (\Delta^1)^\sharp_\sharp$ are trivial cofibrations, and the model structure on $\mathrm{Set}_\Delta^{\mathbf{ms}}$ is Cartesian-closed, this follows immediately. □

Chapter 3

The ∞-bicategorical Grothendieck construction

In this chapter we produce a ∞-bicategorical Grothendieck construction relating 2-Cartesian fibrations over a scaled simplicial set S with contravariant functors $S^{\mathrm{op}} \to \mathbb{B}\mathrm{icat}_\infty$ with values in ∞-bicategories. Our construction can be seen as a direct generalization of Lurie's Cartesian straightening-unstraightening equivalence appearing in [Lur09a]. This chapter is based on [AGS22II].

3.1 Preliminaries: MB-anodyne morphisms and dull subsets

Before proceeding, we here record two variants of the pivot point trick [AGS21, Lem. 1.10] which will be of use later.

Definition 3.1.1. Let $\mathbb{P}(n)$ be the power set of $[n]$. Given $\mathcal{A} \subset \mathbb{P}(n)$ and $X \in \mathbb{P}(n)$ we say that $X \subset \mathcal{A}$ is $\mathcal{A}$-*basal* if it contains precisely one element from each $S \in \mathcal{A}$. We denote the set of $\mathcal{A}$-basal sets by $\mathrm{Bas}(\mathcal{A})$.

Definition 3.1.2. Given subset $\mathcal{A} \subset \mathbb{P}(n)$ such that $\varnothing \notin \mathcal{A}$, and a marked-biscaled simplex $(\Delta^n)^\dagger$, we define a marked-biscaled simplicial subset

$$(\mathcal{S}^{\mathcal{A}})^\dagger = \bigcup_{S \in \mathcal{A}} \Delta^{[n]\setminus S}.$$

Definition 3.1.3. We call a subset $\mathcal{A} \subset \mathbb{P}(n)$ *inner-dull* if the following conditions are satisfied

1. $\mathcal{A}$ does not contain $\varnothing$.
2. There exists $0 < i < n$ such that $i \notin S$ for every $S \in \mathcal{A}$.
3. For any $S, T \in \mathcal{A}$, $S \cap T = \varnothing$.
4. For every $\mathcal{A}$-basal set $X \in \mathbb{P}(n)$ there exists $u, v \in X$ such that $u < i < v$.

We call the element i in the second condition the pivot point.

Definition 3.1.4. Given an inner-dull subset $\mathcal{A} \subset \mathbb{P}(n)$, we define $\mathcal{M}_{\mathcal{A}}$ to be the set of subsets $X \in \mathbb{P}(n)$ satisfying:

A1) X contains the pivot point $i \in X$.

A2) The simplex $\sigma_X : \Delta^X \to (\Delta^n)^\dagger$ does not factor through $(\mathcal{S}^{\mathcal{A}})^\dagger$.

We define $\mathcal{M}^j_{\mathcal{A}} = \{X \in \mathcal{M}_{\mathcal{A}} \mid |X| = j\}$. Note that those elements $X \in \mathcal{M}_{\mathcal{A}}$ of minimal cardinality are of the form $X_0 \cup \{i\}$ for $X_0 \in \operatorname{Bas}(\mathcal{A})$.

Definition 3.1.5. Let $\mathcal{A} \subset \mathbb{P}(n)$ be an inner-dull subset with pivot point i. Given an $\mathcal{A}$-basal subset X we denote by $l^X < u^X$ the pair of consecutive elements such that $l^X < i < u^X$.

Lemma 3.1.6 (The pivot trick). *Let $\mathcal{A} \subset \mathbb{P}(n)$ be an inner-dull subset and let $(\Delta^n)^\dagger$ be a marked biscaled simplex. Suppose that the following conditions hold:*

1. *Every marked edge (resp. thin triangle) which does not contain the pivot point i factors through $(\mathcal{S}^{\mathcal{A}})^\dagger$.*
2. *For every $X \in \operatorname{Bas}(\mathcal{A})$ and every $l^X \leqslant r < i < s \leqslant u^X$ the triangle $\{r, i, s\}$ is thin.*
3. *Let $\sigma = \{a < b < c\}$ be a lean simplex not containing the pivot point i. Then either σ factors through $(\mathcal{S}^{\mathcal{A}})^\dagger$ or we have $a < i < c$ and the simplex $\sigma \cup \{i\}$ is fully lean scaled.*

Then the inclusion

$$(\mathcal{S}^{\mathcal{A}})^\dagger \longrightarrow (\Delta^n)^\dagger$$

is in the weakly saturated hull of morphisms of type (A1) and (S3).

Proof. Observe that since $\mathcal{A}$ is inner-dull it follows that every $\mathcal{A}$-basal set has the same cardinality which we denote ε. For every $\varepsilon \leqslant j \leqslant n$ we define

$$Y_j = Y_{j-1} \cup \bigcup_{X \in \mathcal{M}^j_{\mathcal{A}}} \sigma_X$$

where $Y_{\varepsilon-1} = (\mathcal{S}^{\mathcal{A}})^\dagger$ and we view σ_X as having the inherited decorations. This yields a filtration

$$(\mathcal{S}^{\mathcal{A}})^\dagger \to Y_\varepsilon \to \cdots \to Y_{n-1} \to (\Lambda^n_i)^\dagger \to (\Delta^n)^\dagger$$

We will show that each step of this filtration can be obtained as an iterated pushout along morphisms of type (A1). Let $X \in \mathcal{M}^j_{\mathcal{A}}$ for $\varepsilon \leqslant j \leqslant n-1$ and consider the pullback diagram

$$\begin{array}{ccc} \Lambda^X_i & \longrightarrow & \Delta^X \\ \downarrow & & \downarrow{\scriptstyle \sigma_X} \\ Y_{j-1} & \longrightarrow & Y_j \end{array}$$

We claim that the top horizontal morphism is in the weakly saturated hull of morphisms of type (A1) and (S3). First we notice that the triangle $\{i-1, i, i+1\}$ is thin in Δ^X in virtue of our assumptions. Observe that if the dimension of Δ^X is bigger than 3 then all the possible decorations factor through Λ^X_i. We will therefore assume that the dimension is at most 3 otherwise the claim follows directly. Suppose that $\varepsilon = 2$ then we can have some Δ^X of dimension 2. In this case our assumptions guarantee that the edge that does not have the vertex i cannot be marked. If $\varepsilon = 2$ and the dimension of Δ^X is 3 then it follows that the face that misses the vertex i cannot be thin-scaled. If that face is not lean-scaled then the claim follows immediately. Otherwise our assumptions imply that Δ^X is fully lean scaled the the map $\Lambda^X_i \to \Delta^X$ is a composite of a morphism of type (A1) and a morphism of type (S3). The final case $\varepsilon = 3$ is similar and left as an exercise.

We finish the proof by noting that $X, Y \in \mathcal{M}^j_{\mathcal{A}}$ it follows that $\sigma_X \cap \sigma_Y \in Y_{j-1}$ which implies that the order in which the add the simplices is irrelevant. We conclude that each step in the filtration belongs to the weakly saturated hull of morphisms of type (A1) and (S3). □

We finish the discussion on dull subsets by giving a right-horn variant of the previous construction.

Definition 3.1.7. We call a subset $\mathcal{A} \subset \mathbb{P}(n)$ *right-dull* if the following conditions are satisfied

1. $\mathcal{A}$ does not contain $\varnothing$.
2. For every $S \in \mathcal{A}$, $n \notin S$.
3. For any $S, T \in \mathcal{A}$, $S \cap T = \varnothing$.
4. For every $\mathcal{A}$-basal subset X we have $u, v \in X$ such that $u < v < n$.

In this case we call n the pivot point.

Lemma 3.1.8. *Let $\mathcal{A} \subset \mathbb{P}(n)$ be a right-dull subset. Let $(\Delta^n)^\dagger$ be a marked-biscaled simplex whose thin triangles are degenerate. Suppose that the following conditions holds*

- *For every $\mathcal{A}$-basal subset X and for every $s, r \in [n]$ such that $s \leqslant \min(X) < \max(X) \leqslant r < n$, the triangle $\{s < r < n\}$ is lean, and the edge $r \to n$ is marked.*
- *Let e be a marked edge in $(\Delta^n)^\dagger$ not containing the vertex n. Then e factors through $(\mathcal{S}^{\mathcal{A}})^\dagger$.*
- *Let $\sigma = \{a < b < c\}$ be a lean triangle in $(\Delta^n)^\dagger$ not containing the vertex n. Then either σ factors through $(\mathcal{S}^{\mathcal{A}})^\dagger$ or $\sigma \cup \{n\}$ is fully lean-scaled and $c \to n$ is marked.*

Then $(\mathcal{S}^{\mathcal{A}})^\dagger \to (\Delta^n)^\dagger$ is in the saturated hull of morphisms of type (A4)

Proof. The argument is nearly identical to the proof of Lemma 3.1.6. □

Lemma 3.1.9. *Let $\mathcal{A} \subset \mathbb{P}(n)$ be a right-dull subset. Let $(\Delta^n)^\dagger = (\Delta^n, E_n, T_n \subset C_n)$ be a marked-biscaled simplex such that $(\Delta^n)^\diamond := (\Delta^n, E_n, \flat \subset C_n)$ satisfies the hypothesis of Lemma 3.1.8. Suppose that we are given a morphism*

$$(\Delta^n, E_n, T_n \subset C_r) \longrightarrow (X, \sharp, T_X \subset \sharp)$$

*Then the morphism $(\mathcal{S}^{\mathcal{A}})^\dagger \to (\Delta^n)^\dagger$ is an **MB**-anodyne morphism over (X, T_X).*

Proof. By Lemma 3.1.8 we obtain a pushout diagram

$$\begin{array}{ccc} (\mathcal{S}^{\mathcal{A}})^{\diamond} & \longrightarrow & (\Delta^n)^{\diamond} \\ \downarrow & & \downarrow \\ (\mathcal{S}^{\mathcal{A}})^{\dagger} & \longrightarrow & P \end{array}$$

where the top horizontal morphism is **MB**-anodyne. Note P only differs from $(\Delta^n)^{\dagger}$ in its thin-scaling. Moreover every lean triangle in P whose image in $(\Delta^n)^{\dagger}$ is thin gets mapped to a thin triangle in (X, T_X) so it can be scaled using a morphism of type (S2). □

3.2 The bicategorical Grothendieck construction

Our first step towards an ∞-bicategorical Grothendieck construction is defining the functors which will realize the desired equivalence. These definitions will constitute an upgrade of the straightening and unstraightening constructions of [Lur09a, Section 3.2] to the more highly decorated setting of marked-biscaled simplicial sets and marked-scaled simplicial sets. These functors will define a Quillen equivalence of model categories between $(\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ and a model category we now define.

Definition 3.2.1. Let $\mathcal{C}$ be a $\mathrm{Set}_{\Delta}^{+}$-category. We denote by $(\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathcal{C}}$ the category of $\mathrm{Set}_{\Delta}^{+}$-enriched functors and natural transformations. We endow the category of enriched functors with the projective model structure (See, e.g., [Lur09a, A.3.3.2]).

Definition 3.2.2. Let (Y, E_Y, T_Y) be a marked scaled simplicial set. We define a scaled simplicial set which we denote $(Y^{\triangleright}, T_{Y^{\triangleright}})$ whose underlying simplicial set is given by $Y^{\triangleright} = Y * \Delta^0$ and whose non-degenerate scaled simplices are either those that factor through Y or those of the form $f * \mathrm{id}_{\Delta^0}$ where $f : \Delta^1 \to Y$ belongs to E_Y.

Remark 3.2.3 (Important convention). Let $(X, M_X, T_X \subseteq C_X)$ be an **MB** simplicial set. By the *underlying scaled simplicial set*, we will mean the scaled simplicial set (X, T_X).

Remark 3.2.4 (Notation for ops). Given a simplicial set X with *any* decoration (marking, scaling, etc.), we will denote by X^{op} the opposite simplicial set with the same decoration.

Given an enriched category $\mathcal{C}$ (a $\mathrm{Set}_{\Delta}^{+}$-enriched category, a 2-category, etc.), we will denote $\mathcal{C}^{\mathrm{op}}$ the enriched category with the same objects and $\mathcal{C}^{\mathrm{op}}(x, y) = \mathcal{C}(y, x)$. In the specific case of a 2-category $\mathbb{C}$, we will occasionally write $\mathbb{C}^{(\mathrm{op},-)}$ to denote $\mathbb{C}^{\mathrm{op}}$. We will only rarely make use of the 2-morphism dual $\mathbb{C}^{(-,\mathrm{op})}$.

We now provide the underlying left Quillen functor of our bicategorical Grothendieck construction.

Construction 3.2.5. Fix a scaled simplicial set $S \in \mathrm{Set}_{\Delta}^{\mathbf{sc}}$ and a functor of $\mathrm{Set}_{\Delta}^{+}$-enriched categories $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$. Let $p : X \to S$ be an object of

$(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$. We define a scaled simplicial set X_S via the pushout diagram

$$\begin{array}{ccc} X & \hookrightarrow & X^{\triangleright} \\ {\scriptstyle p}\downarrow & & \downarrow \\ S & \longrightarrow & X_S \end{array}$$

We generically denote both the cone point of $X^{\triangleright}$ and its image in X_S by $*$. We then define a Set_Δ^+-enriched category

$$X_\phi := \mathcal{C} \coprod_{\mathfrak{C}^{\mathrm{sc}}[S]} \mathfrak{C}^{\mathrm{sc}}[X_S].$$

Note that this is equivalently the pushout

$$\begin{array}{ccc} \mathfrak{C}^{\mathrm{sc}}[X] & \longrightarrow & \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}] \\ {\scriptstyle \phi\circ\mathfrak{C}[p]}\downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & X_\phi \end{array}$$

of Set_Δ^+-enriched categories.

Applying the enriched Yoneda embedding on the cone point $*$, this provides a Set_Δ^+-enriched functor

$$\begin{aligned} \mathrm{St}_\phi^+(X)\colon \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathrm{Set}_\Delta^+ \\ s &\longmapsto X_\phi(s,*). \end{aligned}$$

We promote this functor to a Set_Δ^+-enriched functor

$$\mathbb{S}\mathrm{t}_\phi(X)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}_\Delta^{\mathbf{ms}}$$

by equipping its values on objects with a scaling.

After a single, fairly ad-hoc definition, we are able to do this in a highly functorial way. The ad-hoc definition will be a promotion of $\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]$ to a $\mathrm{Set}_\Delta^{\mathbf{ms}}$-enriched category, such that the subcategory $\mathfrak{C}^{\mathrm{sc}}[X] \subset \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]$ has all mapping spaces maximally scaled. We will denote the resulting $\mathrm{Set}_\Delta^{\mathbf{ms}}$-enriched category $\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]_\dagger$. More generally, we will denote scalings on the mapping spaces of a marked-simplicially enriched category $\mathcal{C}$ using subscripts, e.g. $\mathcal{C}_\sharp$ for maximally marked mapping spaces.

We will define the scaling on $\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]$ in three steps:

1. We define the scaling

   $$\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]_\dagger(s,t) := \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](s,t)_\sharp$$

   for $s, t \in X$.

2. We define an auxiliary scaling $P^s_{X^{\triangleright}}$ on each marked simplicial set $\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](s,*)$. iven a map $\sigma\colon \Delta^n \longrightarrow X$, we can pass to the associated $n+1$-simplex $\sigma \star \mathrm{id}_0\colon \Delta^{n+1} \longrightarrow X^{\triangleright}$ and obtain a map of simplicial sets

   $$\mathfrak{C}^{\mathrm{sc}}[\Delta^{n+1}](0, n+1) \longrightarrow \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](\sigma(0),*).$$

Each 2-simplex in $\mathfrak{C}^{\mathrm{sc}}[\Delta^{n+1}](0, n+1)$ is of the form

$$S_0 \cup \{n+1\} \subset S_1 \cup \{n+1\} \subset S_2 \cup \{n+1\}$$

where $S_i \subseteq [n]$ contains 0. We declare the image of such a 2-simplex to be scaled in $\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](s, *)$ precisely when either

- $\max(S_i) = \max(S_j)$ for some $i, j \in \{0, 1, 2\}$; or
- the simplex σ is lean in X (i.e., lies in C_X) and the 2-simplex is $03 \to 013 \to 0123$.

The auxiliary scaling $P^s_{X^{\triangleright}}$ then consists of all such 2-simplices.

3. We extend the scaling $P^s_{X^{\triangleright}}$ by functoriality. That is, we declare a 2-simplex $\sigma : \Delta^2 \to \mathfrak{C}[X^{\triangleright}](s, *)$ to be scaled if there is a t in X and a 2-simplex

$$\theta = (\theta_1, \theta_2) \colon \Delta^2 \longrightarrow \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](s, t) \times \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](t, *)$$

such that $\theta_2 \circ \theta_1 = \sigma$, where $\theta_2 \in P^s_{X^{\triangleright}}$. We would like to stress to the reader that this also adds scaled 2-simplices in the case where θ_2 is *degenerate*.

We can then define a $\mathrm{Set}^{\mathbf{ms}}_{\Delta}$-enriched variant of X_ϕ to be the pushout of $\mathrm{Set}^{\mathbf{ms}}_{\Delta}$-enriched categories

$$\begin{array}{ccc} \mathfrak{C}^{\mathrm{sc}}[X]_{\sharp} & \longrightarrow & \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]_{\dagger} \\ {\scriptstyle \phi\circ\mathfrak{C}[p]}\downarrow & & \downarrow \\ \mathcal{C}_{\sharp} & \longrightarrow & \overline{X_\phi} \end{array}$$

Unwinding the definitions, we see that a 2-simplex $\sigma : \Delta^2 \to \overline{X_\phi}(s, *)$ is scaled if and only if it satisfies the following condition:

- There is a $t \in X_\phi$ and a 2-simplex

$$\theta = (\theta_1, \theta_2) \colon \Delta^2 \longrightarrow \overline{X_\phi}(s, t) \times \overline{X_\phi}(t, *)$$

  such that (1) $\sigma = \theta_2 \circ \theta_1$, and, (2) θ_2 is either in the image of an element of $P^s_{X^{\triangleright}}$ or is degenerate.

The *bicategorical straightening of* X is then the restriction of the $\mathrm{Set}^{\mathbf{ms}}_{\Delta}$-enriched Yoneda embedding:

$$\begin{aligned} \mathbb{S}\mathrm{t}_\phi(X) \colon \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathrm{Set}^{\mathbf{ms}}_{\Delta} \\ s &\longmapsto \overline{X_\phi}(s, *). \end{aligned}$$

A priori, this is an $\mathrm{Set}^{\mathbf{ms}}_{\Delta}$-enriched functor. However, since we required the mapping spaces in $\mathcal{C}$ to be maximally scaled, this formula in fact defines an $\mathrm{Set}^{+}_{\Delta}$-enriched functor. This construction then yields a functor

$$\mathbb{S}\mathrm{t}_\phi \colon (\mathrm{Set}^{\mathbf{mb}}_{\Delta})_{/S} \longrightarrow (\mathrm{Set}^{\mathbf{ms}}_{\Delta})^{\mathfrak{C}^{\mathrm{op}}}$$

which we call the *(bicategorical) straightening functor*.

Notation. We will denote by $\mathbb{S}\mathrm{t}_S(X)$ the special case in which $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathfrak{C}^{\mathrm{sc}}[S]$ is the identity.

Remark 3.2.6. In line with the philosophy of [Ver08], there should be a model for $(\infty, 3)$-categories on the category of simplicial sets with decorations on 1-, 2-, *and* 3-simplices. The ad-hoc construction of the $\mathrm{Set}_\Delta^{\mathbf{ms}}$-enriched category $\overline{X_\phi}$ above seems likely to fit into some — as-yet-undefined — $(\infty, 3)$-categorical version of the rigidification functor, which turns decorated 3-simplices in scaled 2-simplices in the corresponding mapping space.

Remark 3.2.7. Given a 2-Cartesian fibration $p : X \to S$, we note that if *every* triangle in X is lean, the map $\mathbb{S}\mathrm{t}_S(X)(i) \to \mathbb{S}\mathrm{t}_S(X)(i)_\sharp$ is an equivalence of marked-scaled simplicial sets. More generally, we obtain a diagram

$$\begin{array}{ccc} (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} & \xrightarrow{\mathbb{S}\mathrm{t}_S} & (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}}} \\ {\scriptstyle (-)_{T\subset\sharp}}\uparrow & & \uparrow{\scriptstyle (-)_\sharp} \\ (\mathrm{Set}_\Delta^{\mathbf{ms}})_{/S} & \xrightarrow[\mathrm{St}^+]{} & (\mathrm{Set}_\Delta^{+})^{\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}}} \end{array}$$

which commutes up to natural weak equivalence.

While we will not formalize this statement here, there should be a model structure on $(\mathrm{Set}_\Delta^{\mathbf{ms}})_{/S}$ modeling ∞-bicategories fibred in $(\infty, 1)$-categories, such that St^+ becomes a left Quillen equivalence to the projective model structure. The diagram above would then represent the restriction of our straightening-unstraightening equivalence to this special case.

3.2.1 First properties

Before proceeding to the technical nitty-gritty of the Quillen equivalences, we establish some basic properties of the straightening functor.

Proposition 3.2.8. *Let $S \in \mathrm{Set}_\Delta^{\mathbf{sc}}$ and let $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$ be a Set_Δ^+-enriched functor. Then the following hold*

1. *The straightening functor $\mathbb{S}\mathrm{t}_\phi$ preserves colimits.*

2. *(Base change for scaled functors) Given a morphism of scaled simplicial sets $f : T \to S$ there a diagram*

$$\begin{array}{ccc} (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} & & \\ & \searrow^{\mathbb{S}\mathrm{t}_\phi} & \\ {\scriptstyle f\circ -}\uparrow & & (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathcal{C}^{\mathrm{op}}} \\ & \nearrow_{\mathbb{S}\mathrm{t}_{\phi\circ\mathfrak{C}[f]}} & \\ (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/T} & & \end{array}$$

   which commutes up to natural isomorphism of functors.

3. *(Base change for Set_Δ^+-functors) Given a Set_Δ^+-enriched functor $\psi : \mathcal{C} \to \mathcal{D}$*

there is a diagram

$$\begin{array}{ccc} & & (\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathcal{D}^{\mathrm{op}}} \\ & \nearrow^{\mathbb{S}\mathrm{t}_{\psi\circ\phi}} & \uparrow{\scriptstyle \psi_!} \\ (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/S} & & \\ & \searrow_{\mathbb{S}\mathrm{t}_{\phi}} & \\ & & (\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathcal{C}^{\mathrm{op}}} \end{array}$$

which commutes up to natural isomorphism of functors.

Proof. All three statements hold on the level of St_{ϕ}^{+}, and so the proof amounts to checking scalings. We prove (1), and leave the other two statements to the reader.

It is follows from the definition that

$$\mathrm{St}_{\phi}^{+}\colon (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/S} \longrightarrow (\mathrm{Set}_{\Delta}^{+})^{\mathcal{C}^{\mathrm{op}}}$$

preserves colimits. Since colimits in functor categories are computed pointwise, it will thus suffice to show that, given a diagram

$$D\colon I \longrightarrow (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/S}$$

the scalings on $\mathrm{colim}_I\, \mathbb{S}\mathrm{t}_{\phi}(D(i))$ and $\mathbb{S}\mathrm{t}_{\phi}(\mathrm{colim}_I\, D(i))$ coincide. Indeed, applying the universal property, it will suffice to show that the map

$$\mathbb{S}\mathrm{t}_{\phi}(\operatorname*{colim}_I D(i)) \longrightarrow \operatorname*{colim}_I \mathbb{S}\mathrm{t}_{\phi}(D(i))$$

which is the identity on underlying marked simplicial sets preserves the scalings.

Fix $s \in \mathcal{C}$, we will first show that the map

$$f_s\colon (\mathbb{S}\mathrm{t}_{\phi}(\operatorname*{colim}_I D(i)))(s) \longrightarrow (\operatorname*{colim}_I \mathbb{S}\mathrm{t}_{\phi}(D(i)))(s)$$

preserves the scalings inherited from $P^{s}_{X^{\triangleright}}$. To this end, suppose given a scaled simplex σ in $P^{s}_{(\mathbb{S}\mathrm{t}_{\phi}(\mathrm{colim}_I D(i)))_S}$ which does not come from a lean simplex in the colimit. Tracing through the definition, we note that there must be a simplex $\eta : \Delta^n \to \mathrm{colim}_I(D(i))$ and a simplex $\mu := \{S_0 \cup \{n+1\} \to S_1 \cup \{n+1\} \to S_2 \cup \{n+1\}\}$ in $\mathfrak{C}^{\mathrm{sc}}[\Delta^{n+1}](0, n+1)$ with $\max(S_i) = \max(S_j)$ for some $i, j = 0, 1, 2$ such that σ is the image of μ under the canonical map

$$g_1 : \mathfrak{C}^{\mathrm{sc}}[\Delta^{n+1}](0, n+1) \longrightarrow (\mathbb{S}\mathrm{t}_{\phi}(\operatorname*{colim}_I D(i)))(s)$$

is not scaled.

By the construction of colimits in simplicial sets, this means that there is an $k \in I$ and a simplex $\hat{\eta} : \Delta^n \to D(k)$ such that η factors through the canonical map $D(k) \to \mathrm{colim}_I\, D(i)$ as $\hat{\eta}$. We can then note that $\hat{\eta}$ will yield a map

$$g_2\colon \mathfrak{C}^{\mathrm{sc}}[\Delta^{n+1}](0, n+1) \longrightarrow \mathbb{S}\mathrm{t}_{\phi}(D(k))(s) \longrightarrow (\operatorname*{colim}_I \mathbb{S}\mathrm{t}_{\phi}(D(i)))(s)$$

such that the diagram

$$\begin{array}{ccc} & & (\mathbb{S}\mathrm{t}_{\phi}(\mathrm{colim}_I D(i)))(s) \\ & \overset{g_1}{\nearrow} & \\ \mathfrak{C}^{\mathrm{sc}}[\Delta^{n+1}](0,n+1) & & \Big\downarrow f_s \\ & \underset{g_2}{\searrow} & \\ & & (\mathrm{colim}_I \mathbb{S}\mathrm{t}_{\phi}(D(i)))(s) \end{array}$$

commutes. We thus see that $g_2(\mu) = f_s(g_1(\mu)) = f_s(\sigma)$ is scaled, as desired. The same argument holds, *mutatis mutandis*, for $\sigma \in P^s_{(\mathbb{S}\mathrm{t}_{\phi}(\mathrm{colim}_I D(i)))_S}$ coming from a lean 2-simplex in the colimit.

We can now easily check that the full scalings $T^s_{X_S}$ are preserved by f_s by simply noting that the diagram

$$\begin{array}{ccc} \mathcal{C}(s',s) \times (\mathrm{St}^+_S(\mathrm{colim}_I D(i)))(s) & \xrightarrow{\;\circ\;} & (\mathrm{St}^+_S(\mathrm{colim}_I D(i)))(s') \\ {\scriptstyle f_s}\Big\downarrow & & \Big\downarrow{\scriptstyle f_{s'}} \\ \mathcal{C}(s',s) \times (\mathrm{colim}_I \mathbb{S}\mathrm{t}_S(D(i)))(s) & \xrightarrow[\circ]{} & (\mathrm{colim}_I \mathbb{S}\mathrm{t}_S(D(i)))(s') \end{array}$$

commutes.

To show (2) and (3), we again note that the statements are immediate if we replace $\mathbb{S}\mathrm{t}_{\phi}$ with St^+_{ϕ} (cf. [Lur09b, Rmk 3.5.16] and [Lur09a, Prop 3.2.1.4]). A similar check to the above assures us that the scalings coincide. □

Remark 3.2.9. Note that, in the case where we consider ϕ to be the identity on $\mathfrak{C}^{\mathrm{sc}}[S]$ and are given a morphism $f : T \to S$, combining (2) and (3) in Proposition 3.2.8 yields a diagram

$$\begin{array}{ccc} (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/S} & \xrightarrow{\mathbb{S}\mathrm{t}_S} & (\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}}} \\ {\scriptstyle f\circ -}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathfrak{C}^{\mathrm{sc}}[f]_!} \\ (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/T} & \xrightarrow[\mathbb{S}\mathrm{t}_T]{} & (\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[T]^{\mathrm{op}}} \end{array}$$

which commutes up to natural isomorphism.

Corollary 3.2.10. *Let S be an scaled simplicial set and $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$ a Set^+_{Δ}-enriched functor. Then the straightening functor $\mathbb{S}\mathrm{t}_{\phi}$ has a right adjoint*

$$\mathbb{U}\mathrm{n}_{\phi} \colon \left(\mathrm{Set}_{\Delta}^{\mathbf{ms}}\right)^{\mathcal{C}^{\mathrm{op}}} \longrightarrow (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/S}$$

which we call the (bicategorical) unstraightening functor.

Proof. This follows from the first part in Proposition 3.2.8 using the adjoint functor theorem. □

Let $\Delta^n_{\flat}$ denote the minimally scaled n-simplex and consider $(\Delta^n)^{\flat}_{\flat} = (\Delta^n, \flat, \flat)$ as an object of $(\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/\Delta^n_{\flat}}$ via the identity map. To ease the notation we will denote the straightening of this object as $\mathbb{S}\mathrm{t}_{\Delta^n_{\flat}}(\Delta^n)$.

Definition 3.2.11. Let $n \geqslant 0$ and $0 \leqslant s \leqslant n$. We denote by $L^n(s)$ the poset of subsets $S \subseteq [n]$ such that $\min(S) = s$ ordered by inclusion. Let $\sigma : S_0 \subseteq S_1 \subseteq S_2$ be a 2-simplex in the (nerve of) $L^n(s)$ and denote $s_i = \max(S_i)$ for $i = 0, 1, 2$. We say that σ is *thin* if there exists a pair of indices i, j such that $s_i = s_j$. We endow $L^n(s)$ with a scaling given by thin simplices and with the minimal marking. The resulting marked scaled simplicial set will be denoted by $\mathcal{L}^n_\flat(s)$

Lemma 3.2.12. *Let $n \geqslant 0$ and $0 \leqslant s \leqslant n$. Then there is an isomorphism*

$$\mathbb{S}\mathrm{t}_{\Delta^n_\flat}(\Delta^n)(s) \xrightarrow{\simeq} \mathcal{L}^n_\flat(s)$$

of marked scaled simplicial sets

Proof. Immediate from unraveling the definitions. □

Definition 3.2.13. Let $n \geqslant 0$ and consider a **MB** simplicial set $\Delta^n_T := (\Delta^n, \flat, \flat \subseteq T)$ for some scaling T. Given $0 \leqslant s \leqslant n$ we define a new scaling on $L^n(s)$ (see Definition 3.2.11) by declaring a 2-simplex $S_0 \subseteq S_1 \subseteq S_2$ if and only if the simplex defined by $\max(S_0) \leqslant \max(S_1) \leqslant \max(S_2)$ is lean in Δ^n_T. We denote the resulting scaled simplicial set by $\mathcal{L}^n_T(s)$.

Lemma 3.2.14. *Let $\Delta^n_T := (\Delta^n, \flat, \flat \subseteq T)$ and denote by $\mathbb{S}\mathrm{t}_{\Delta^n_\flat}(\Delta^n_T)$ the straightening of the map $\Delta^n_T \to \Delta^n_\flat$. Then for every $0 \leqslant s \leqslant n$ the canonical map*

$$\mathbb{S}\mathrm{t}_{\Delta^n_\flat}(\Delta^n_T)(s) \longrightarrow \mathcal{L}^n_T(s)$$

is **MS**-*anodyne.*

Proof. The existence of the morphism is clear from the definitions. Suppose that we are given a thin 2-simplex $\sigma : S_0 \subseteq S_1 \subseteq S_2$ in $\mathcal{L}^n_T(i)$. As before, we adopt the convention that $s_i := \max(S_i)$. We will show that σ can be scaled by taking pushouts along **MS**-anodyne morphisms. First let us consider the 3-simplex

$$\theta : S_0 \subseteq S_0 \cup \{s_1\} \subseteq S_0 \cup \{s_1, s_2\} \subseteq S_2$$

We immediately observe that all of its faces are scaled in $\mathbb{S}\mathrm{t}_{\Delta^n_\flat}(\Delta^n)_T(s)$ except the face missing 2. It follows we can scale the remaining face using a pushout along a **MS**-anodyne map of the type described in Lemma 2.4.3. Now we consider another 3-simplex

$$\rho : S_0 \subseteq S_0 \cup \{s_1\} \subseteq S_1 \subseteq S_2$$

Again we observe that all of its faces are scaled except possibly the face missing 1 which is precisely σ. The conclusion easily follows from Lemma 2.4.3 □

Let $S_\sharp$ be a scaled simplicial set and assume every triangle is thin. Denote by S its underlying simplicial set and let $(\mathrm{Set}^+_\Delta)_{/S}$ denote the category of marked simplicial sets over S. We define a functor

$$\iota\colon (\mathrm{Set}^+_\Delta)_{/S} \longrightarrow (\mathrm{Set}^{\mathbf{mb}}_\Delta)_{/S},\ (X, E_X) \longmapsto (X, E_X, \sharp)$$

We view the ∞-categorical straightening functor St_S (see 3.2.1 in [Lur09a]) as a functor with values $(\mathrm{Set}^{\mathbf{ms}}_\Delta)^{\mathfrak{C}^{sc}[S]^{op}}$ by maximally scaling the values of $\mathrm{St}_S X(s)$.

Proposition 3.2.15. *There exists a natural transformation*

$$\varepsilon\colon \mathbb{S}\mathrm{t}_S \circ \iota \Longrightarrow \mathrm{St}_S$$

which is objectwise a weak equivalence of marked scaled simplicial sets.

Proof. The existence of the natural transformation is automatic since both functors only differ on the scaling. It is clear that both functors preserve colimits and that they satisfy base change to respect to morphisms of simplicial sets $S \to T$. In addition, it is routine to verify that both functors respect cofibrations. An standard argument then shows that it suffices to check that the natural transformation is an equivalence (1) when $S = (\Delta^n)^\flat$ with $n \geqslant 0$ and $X \to S$ is the identity morphism, and (2) on $(\Delta^1)^\sharp \to \Delta^1$ when $S = \Delta^1$. This is a direct consequence of Lemma 3.2.14. ☐

We conclude this section with a first step towards showing that the bicategorical straightening is left Quillen.

Proposition 3.2.16. *Let S be a scaled simplicial set and let $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$ be a Set_Δ^+-enriched functor. Then the straightening functor*

$$\mathbb{S}\mathrm{t}_\phi\colon (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} \longrightarrow (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}[\mathcal{C}]^{\mathrm{op}}}$$

preserves cofibrations.

Proof. The generators of the class of cofibrations of marked biscaled simplicial sets are given by

(C1) $\Big(\partial\Delta^n, \flat, \flat\Big) \to \Big(\Delta^n, \flat, \flat\Big)$.

(C2) $\Big(\Delta^1, \flat, \flat\Big) \to \Big(\Delta^1, \sharp, \flat\Big)$.

(C3) $\Big(\Delta^2, \flat, \flat\Big) \to \Big(\Delta^2, \flat, \flat \subset \sharp\Big)$.

(C4) $\Big(\Delta^2, \flat, \flat \subset \sharp\Big) \to \Big(\Delta^2, \flat, \sharp\Big)$.

Note that (C4) and (S2) are the same morphism. Therefore using standard arguments it will suffice to check our claim on those generators.

Let $i : A \to B$ be a cofibration. As stated above it will suffice to check in the case where i is one of the generating cofibrations. Furthermore we can use Proposition 3.2.8 to reduce to the case where S is the underlying scaled simplicial set of B, and ϕ is $\mathrm{id} : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathfrak{C}^{\mathrm{sc}}[S]$. The result follows from a straightforward computation. ☐

3.2.2 Products and tensoring

Before we can proceed to proving that the straightening-unstraightening adjunction is a Quillen equivalence (indeed, before we can prove the straightening is left Quillen), we need to establish the relation of the straightening to the Set_Δ^+-tensoring. We will prove this as a corollary of a more general result — on products of **MB** simplicial sets — which will be of use to us in the sequel.

Let $A, B \in \mathrm{Set}_\Delta^{\mathbf{sc}}$ and consider a pair of objects $X_A \in (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/A}$, $X_B \in (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/B}$ giving rise to $X \in (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/A\times B}$ then we can form a pushout diagram

$$\begin{array}{ccc} \mathfrak{C}^{\mathrm{sc}}[X] & \longrightarrow & \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}] \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ \mathfrak{C}^{\mathrm{sc}}[A] \times \mathfrak{C}^{\mathrm{sc}}[B] & \longrightarrow & X_{A,B} \end{array}$$

where the left-most vertical morphism is the composite $\mathfrak{C}^{\mathrm{sc}}[X] \to \mathfrak{C}^{\mathrm{sc}}[A \times B] \to \mathfrak{C}^{\mathrm{sc}}[A] \times \mathfrak{C}^{\mathrm{sc}}[B]$. Let

$$\mathbb{S}\mathrm{t}_A\, X_A \boxtimes \mathbb{S}\mathrm{t}_B\, X_B : \mathfrak{C}^{\mathrm{sc}}[A]^{\mathrm{op}} \times \mathfrak{C}^{\mathrm{sc}}[B]^{\mathrm{op}} \to \mathrm{Set}_\Delta^{\mathbf{ms}}$$

be the pointwise product of $\mathbb{S}\mathrm{t}_A\, X_A$ and $\mathbb{S}\mathrm{t}_B\, X_B$ and observe there is a canonical natural transformation

$$\varepsilon_X : \mathbb{S}\mathrm{t}_\phi\, X \Rightarrow \mathbb{S}\mathrm{t}_A\, X_A \boxtimes \mathbb{S}\mathrm{t}_B\, X_B$$

We will prove the following theorem:

Theorem 3.2.17. *The map* $\varepsilon_X : \mathbb{S}\mathrm{t}_\phi\, X \Rightarrow \mathbb{S}\mathrm{t}_A(X_A) \boxtimes \mathbb{S}\mathrm{t}_B(X_B)$ *is a pointwise weak equivalence.*

Before proceeding with the proof of the theorem we need to do some preliminary work. First we will do a careful study of the case where $A = (\Delta^n, \flat)$ and $B = (\Delta^k, \flat)$, $X_A = (\Delta^n, \flat, \flat)$ and $X_B = (\Delta^k, \flat, \flat)$. We will assume that the maps $X_A \to A$ and $X_B \to B$ are the identity on the underlying scaled simplicial sets. In this particular situation we will denote $\mathbb{S}\mathrm{t}_\phi\, X(i,j) := \mathbb{P}^{n,k}_{(i,j)}$ and $\mathbb{S}\mathrm{t}_{\Delta^n}\, \Delta^n(i) \times \mathbb{S}\mathrm{t}_{\Delta^k}\, \Delta^k(j) := \mathbb{S}^{n,k}_{(i,j)}$.

Definition 3.2.18. Let $n, k \geqslant 0$ and let $i \in [n], j \in [k]$. We define marked scaled simplicial set $\mathbb{E}^{n,k}_{(i,j)}$ whose underlying simplicial set is given by $\mathfrak{C}[(\Delta^n \times \Delta^k)^{\triangleright}]((i,j),*)$. To define the marking and the scaling we construct a morphism

$$\xi^{n,k}_{(i,j)} : \mathbb{E}^{n,k}_{(i,j)} \longrightarrow \mathbb{S}^{n,k}_{(i,j)}$$

an equip $\mathbb{E}^{n,k}_{(i,j)}$ with the induced marking and scaling. Recall that objects of $\mathbb{E}^{n,k}_{(i,j)}$ are given by a chain or sequence of inequalities $(a_0, b_0) < (a_1, b_1) < \cdots (a_\ell, b_\ell)$ where $a_i \in [n]$ and $b_i \in [n]$ for $i = 0, \ldots, \ell$ and with the property that $(a_0, b_0) = (i, j)$. We will use the notation $C = \{(a_i, b_i)\}_{i=0}^{\ell}$. A morphism between chains $C_1 \to C_2$ is simply given by an inclusion $C_1 \subset C_2$ which we call a refinement of the chain C_1. Then we define $\xi(C) = (S_a, S_b)$ where $S_a = \{a_0, a_1, \ldots, a_\ell\}$ and similarly for S_b.

Remark 3.2.19. It is immediate to see that the map $\xi^{n,k}_{(i,j)}$ constructed before factors as

$$\mathbb{E}^{n,k}_{(i,j)} \longrightarrow \mathbb{P}^{n,k}_{(i,j)} \longrightarrow \mathbb{S}^{n,k}_{(i,j)}$$

where the second morphism is the component of the natural transformation ε_X at the object (i,j) and the first morphism is a canonical collapse map. We will denote the first morphism by $\pi^{n,k}_{(i,j)}$ and the second morphism by $\varepsilon^{n,k}_{(i,j)}$.

Definition 3.2.20. Let $C \in \mathbb{E}^{n,k}_{(i,j)}$ be a chain denoted by $C = \{(a_i, b_i)\}_{i=0}^{\ell}$. We set $|C| = \ell$ and we call it the *length* of the chain.

Definition 3.2.21. Let $C \in \mathbb{E}^{n,k}_{(0,0)}$. We define $\mathcal{E}_C$ to be the full subposet (with the induced marking and scaling) of $\mathbb{E}^{n,k}_{(0,0)}$ consisting of those chains K contained in C.

Definition 3.2.22. Let $C \in \mathbb{E}^{n,k}_{(0,0)}$ be a chain. We say that $K \in \mathcal{E}_C$ is a *rigid chain* if there is no marked morphism in $\mathcal{E}_C$ with source K. We denote the by $\mathcal{E}_C^r$ the full subposet of $\mathcal{E}_C$ on rigid chains.

Lemma 3.2.23. *Let $C \in \mathbb{E}^{n,k}_{(0,0)}$ be a chain and denote by $\mathcal{U}_C$ the image of the morphism $\mathcal{E}_C \to \mathbb{S}^{n,k}_{(0,0)}$. Then $\xi^{n,k}_{(0,0)}$ induces an isomorphism of marked scaled simplicial sets*

$$\xi_C^r : \mathcal{E}_C^r \xrightarrow{\cong} \mathcal{U}_C$$

Proof. The map ξ_C^r is clearly surjective on vertices. Moreover, given a morphism $U \to K$ in $\mathcal{E}_C$, choose a marked morphism $U \to U^r$ to a rigid chain in $\mathcal{E}_C$. Then for every $(a,b) \in U^r \setminus U$, the object $K \cup \{(a,b)\}$ will lie in $\mathcal{E}_C$ over the same element of $\mathcal{U}_C$ as K. We thus obtain a morphism $U^r \to \hat{K}$ lying over the original morphism in $\mathcal{U}_C$, showing that ξ_C^r is surjective on morphisms, and thus on higher simplices.

Moreover ξ_C^r detects and preserves marked edges and thin simplices. It will therefore suffice to show that ξ_C^r is injective. Let $K_i \in \mathcal{E}_C^r$ for $i = 1,2$ such that $\xi_C^r(K_1) = \xi_C^r(K_2)$. Let us denote $K_i = \{(a_j^i, b_j^i)\}_{j=0}^{\ell_i}$ for $i = 1,2$. Without loss of generality let us assume that we have some (a_s^1, b_s^1) such that this pair is not an element in K_2. However, note that since $K_i \subset C$ for $i = 1,2$ then there exists a map $K_2 \to \hat{K}_2$ where $\hat{K}_2$ is obtained from K_2 by appending the element (a_s^1, b_s^1). By construction it follows that $\xi_C^r(K_2) = \xi_C^r(\hat{K}_2)$ since K_2 is rigid it follows that $\hat{K}_2 = K_2$ and therefore $K_1 = K_2$. □

Lemma 3.2.24. *Let $C \in \mathbb{E}^{n,k}_{(0,0)}$. Then the induced morphism*

$$\xi_C \colon \mathcal{E}_C \xrightarrow{\simeq} \mathcal{U}_C$$

is an equivalence of marked scaled simplicial sets.

Proof. Let $\iota : \mathcal{E}_C^r \to \mathcal{E}_C$ denote the obvious inclusion. Using Lemma 3.2.23 we can construct a map $s_C = \iota \circ (\xi_C^r)^{-1}$. It is clear that $\xi_C \circ s_C = \mathrm{id}$. Given K, observe that by construction $s_C \circ \xi_C(K)$ is rigid. Let $K \to K^r$ be a marked edge where K^r is rigid. Since the restriction of ξ_C to rigid objects is injective it follows that $s_C \circ \xi_C(K) = K^r$. This yields a marked homotopy from $s_C \circ \xi_C$ to the identity and the result follows. □

Lemma 3.2.25. *Let $C_i \in \mathbb{E}^{n,k}_{(0,0)}$ for $i = 1,2$. Then there exists a chain K such that the intersection $\mathcal{E}_{C_1} \cap \mathcal{E}_{C_2} = \mathcal{E}_K$.*

Proof. Immediate. □

Proposition 3.2.26. *Let n,k two non-negative integers and consider $i \in [n]$ and $j \in [k]$. Then the morphism*

$$\xi^{n,k}_{(i,j)} \colon \mathbb{E}^{n,k}_{(i,j)} \longrightarrow \mathbb{S}^{n,k}_{(i,j)}$$

is an equivalence of marked scaled simplicial sets.

Proof. First let us observe that the map $\xi^{n,k}_{(i,j)}$ is an isomorphism if either n or k is equal to 0. Using an inductive argument it will suffice to show that the map $\xi^{n,k}_{(0,0)}$ is an equivalence. Note that we can cover $\mathbb{E}^{n,k}_{(i,j)}$ with the subsimplicial sets $\mathcal{E}_C$ where C is a chain of maximal length. Since $\xi^{n,k}_{(0,0)}$ is surjective is covered by the subsimplicial sets $\mathcal{U}_C$. Applying [AGDS20, Lemma 3.2.13], we express $\mathbb{E}^{n,k}_{(0,0)}$ and $\mathbb{S}^{n,k}_{(0,0)}$ as the colimit over the same diagram of two homotopy cofibrant diagrams. We can now identify $\xi^{n,k}_{(0,0)}$ as the map induced by the natural transformation whose components are ξ_C. Therefore using Lemma 3.2.24 it follows that $\xi^{n,k}_{(0,0)}$ is a weak equivalence. □

Definition 3.2.27. Let $\mathbb{O}^{n,k} = \mathfrak{C}[\Delta^n \times \Delta^k]$. We define a marking on $\mathbb{O}^{n,k}((i,j),(a,b))$ by declaring an edge marked if an only if its image in $\mathbb{O}^n(i,a) \times \mathbb{O}^k(j,b)$ is degenerate. If $a,b = n,k$ we set the notation $\mathbb{O}^{n,k}((i,j),(a,b)) = \mathbb{O}^{n,k}_{(i,j)}$.

Lemma 3.2.28. *The canonical morphism $p : \mathbb{O}^{n,k}((i,j),(a,b)) \to \mathbb{O}^n(i,a) \times \mathbb{O}^k(j,b)$ is a weak equivalence of marked simplicial sets.*

Proof. The argument here is virtually identical to that given in Lemma 3.2.23, Lemma 3.2.24, and Proposition 3.2.26. □

Definition 3.2.29. Let $\sigma : K_0 \subset K_1 \cdots \subset K_\ell$ be a simplex in $\mathbb{O}^{n,k}((i,j),(a,b))$ such that $K_0 \neq (i,j)$. Given $(x,y) \in K_0$ then it follows that σ is in the image of the map

$$\gamma_{x,y} \colon \mathbb{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) \longrightarrow \mathbb{O}^{n,k}((i,j),(a,b)).$$

Given a pair of simplices σ_1, σ_2 as above, let (A_i, B_i) denote the preimages of γ_i for $i = 1,2$ under $\gamma_{x,y}$. We define $\mathcal{O}^{n,k}((i,j),(a,b))$ as a quotient of $\mathbb{O}^{n,k}((i,j),(a,b))$ by identifying those simplices σ_1, σ_2 as above such that their corresponding A_i's get identified in $\mathbb{O}^n(i,x) \times \mathbb{O}^k(j,y)$ and $B_1 = B_2$.

Remark 3.2.30. Observe that the previous definition yields a factorization

$$\mathbb{O}^{n,k}((i,j),(a,b)) \xrightarrow{\alpha} \mathcal{O}^{n,k}((i,j),(a,b)) \xrightarrow{\beta} \mathbb{O}^n(i,a) \times \mathbb{O}^k(j,b)$$

Lemma 3.2.31. *The morphisms in Remark 3.2.30 are equivalences of marked simplicial sets.*

Proof. By Lemma 3.2.28, it suffices to show that α is an equivalence. For $(i,j) < (x,y)$, let the *distance* from (x,y) to (i,j) be the maximal length of a chain in $\mathbb{O}^{n,k}((i,j),(x,y))$, using the convention that we count neither (i,j) nor (x,y) towards this length.

It is clear that if the distance from (a,b) to (i,j) is 0, α is an isomorphism. We then proceed by induction. Suppose that that statement is true for all (i,j)

and (a,b) with distance less than r, and let (i,j) and (a,b) be distance r apart. We define a sequence of marked simplicial sets by setting

$$X_0 = \mathbb{O}^{n,k}((i,j),(a,b))$$

and then defining

$$\begin{array}{ccc} \coprod_{d((i,j),(x,y))=\ell} \mathbb{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) & \longrightarrow & X_\ell \\ \downarrow & & \downarrow \\ \coprod_{d((i,j),(x,y))=\ell} \mathbb{O}^{n}(i,x) \times \mathbb{O}^{k}(j,y) \times \mathbb{O}^{n,k}((x,y),(a,b)) & \longrightarrow & X_{\ell+1} \end{array}$$

We then note two facts:

- For $(i,j) < (x,y)$ distance 0 apart, the canonical map
$$\mathbb{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) \to X_0$$
descends through an isomorphism to a map
$$\mathcal{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) \to X_0$$
- For $(i,j) < (x,y)$ distance ℓ apart, the canonical map
$$\mathbb{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) \to X_\ell$$
descends to a map $\mathcal{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) \to X_\ell$, since we have already quotiented out by the relations involving intermediate elements of lesser distance.

We can thus replace the pushout above with the pushout

$$\begin{array}{ccc} \coprod_{d((i,j),(x,y))=\ell} \mathcal{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) & \hookrightarrow & X_\ell \\ \downarrow & & \downarrow \\ \coprod_{d((i,j),(x,y))=\ell} \mathbb{O}^{n}(i,x) \times \mathbb{O}^{k}(j,y) \times \mathbb{O}^{n,k}((x,y),(a,b)) & \longrightarrow & X_{\ell+1} \end{array}$$

where the upper horizontal map is now a cofibration. This means that, by our inductive hypothesis and Lemma 3.2.28, $X_\ell \to X_{\ell+1}$ is an pushout of an equivalence along a cofibration, and thus an equivalence of marked simplicial sets.

Since any intermediate element $(i,j) < (x,y) < (a,b)$ must have distance from (i,j) strictly less than r, we see that $X_r = \mathcal{O}^{n,k}((i,j),(a,b))$. Thus, the composite map

$$\alpha\colon \mathbb{O}^{n,k}((i,j),(a,b)) = X_0 \longrightarrow X_r = \mathcal{O}^{n,k}((i,j),(a,b))$$

is an equivalence, as desired. □

Proposition 3.2.32. *Let $n,k \geqslant 0$ then the morphism $\pi^{n,k}_{(i,j)} : \mathbb{E}^{n,k}_{(i,j)} \to \mathbb{P}^{n,k}_{(i,j)}$ is a weak equivalence of marked scaled simplicial sets.*

Proof. Since $\pi^{n,k}_{(i,j)}$ is an isomorphism whenever either n or k is equal to 0 it follows by an inductive argument that it will suffice to show that $\pi^{n,k}_{(0,0)}$ is an equivalence. We define a sequence of marked scaled simplicial sets beginning with

$$Y_0 = \mathbb{E}^{n,k}_{(0,0)}.$$

Then we define

$$\begin{array}{ccc} \coprod_{d((0,0),(x,y))=\ell} \mathcal{O}^{n,k}((0,0),(x,y)) \times \mathbb{E}^{n,k}_{(x,y)} & \longrightarrow & Y_\ell \\ \downarrow & & \downarrow \\ \coprod_{d((i,j),(x,y))=\ell} \mathbb{O}^n(0,x) \times \mathbb{O}^k(0,y) \times \mathbb{E}^{n,k}_{(x,y)} & \longrightarrow & Y_{\ell+1} \end{array}$$

and observe that the top horizontal morphism is a cofibration. Additionally one sees that the left-most vertical morphism is an equivalence due to Lemma 3.2.31. It follows by construction that $Y_{n+k} = \mathbb{P}^{n,k}_{(0,0)}$ and since each $Y_\ell \to Y_{\ell+1}$ is a weak equivalence the result now follows. □

Corollary 3.2.33. *Let n, k two non-negative integers and consider $i \in [n]$ and $j \in [k]$. Then the morphism*

$$\varepsilon^{n,k}_{(i,j)} : \mathbb{P}^{n,k}_{(i,j)} \longrightarrow \mathbb{S}^{n,k}_{(i,j)}$$

is an equivalence of marked scaled simplicial sets.

Proof of Theorem 3.2.17. As in [Lur09a, 3.2.1.13], it suffices to check this in the special case when $X_A \to A$ and $X_B \to B$ are identity morphisms on underlying simplicial sets, and both A and B are one of the following cases

- The scaled 2-simplex $\Delta^2_\sharp$.
- The unscaled n-simplex $\Delta^n_\flat$.

In the case where $A = \Delta^n_\flat$ and $B = \Delta^k_\flat$, the morphism

$$\varepsilon_X : \mathbb{S}\mathrm{t}_\phi(\Delta^n \times \Delta^k)(i,j) \longrightarrow \left(\mathbb{S}\mathrm{t}_{\Delta^n}(\Delta^n) \boxtimes \mathbb{S}\mathrm{t}_{\Delta^k}(\Delta^k)\right)(i,j)$$

is precisely the morphism

$$\varepsilon^{n,k}_{(i,j)} : \mathbb{P}^{n,k}_{(i,j)} \longrightarrow \mathbb{S}^{n,k}_{(i,j)}$$

and thus is an equivalence of marked-scaled simplicial sets by Corollary 3.2.33. Each other case is a pushout of some $\varepsilon^{n,k}_{(i,j)}$ by a cofibration, and thus is also an equivalence. □

3.2.3 Straightening and anodyne morphisms

This section serves as a stepping-stone to see that the bicategorical straightening is a left Quillen functor. in particular, we will show that $\mathbb{S}\mathrm{t}_S$ preserves **MB**-anodyne morphisms for any $S \in \mathrm{Set}^{\mathbf{sc}}_\Delta$.

Definition 3.2.34. Consider Λ^n_i for $0 \leqslant i \leqslant n$. For every $0 \leqslant s \leqslant n$ we define $\Lambda\mathcal{L}^n_i(s)$ to be the scaled subsimplicial set of $\mathcal{L}^n_\flat(s)$ consisting of those simplices $\sigma : S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n$ satisfying at least one of the following conditions:

- There exists $k \in [n]$ with $k \neq i$ such that, for every $j \in [n]$, $k \notin S_j$.
- There exists some $0 < j \leqslant n$ such that $j \in S_0$ and there exists $0 \leqslant \ell < j$ such that $\ell \neq i$.

Given Δ^n_T as in Definition 3.2.13 we define $(\Lambda\mathcal{L}^n_i)_T(s)$ using the inherited scaling from $\mathcal{L}^n_T(s)$.

Definition 3.2.35. Given a **MB** simplicial set of the form $\Delta^n_T := (\Delta^n, \flat, \flat \subset T)$ for some T, we denote by $(\Lambda^n_i)_T$ the horn with the induced marking and biscaling. We write $\mathbb{S}\mathrm{t}_{\Delta^n_\flat}(\Lambda^n_i)_T$ for the functor associated to the object $(\Lambda^n_i)_T \to \Delta^n_\flat$.

Remark 3.2.36. In some specific instances we will have $\Delta^n_T := (\Delta^n, \flat, \flat \subset T)$ where $T = \Delta^{\{i,j,k\}}$ a chosen 2-simplex in Δ^n. In that situation we will chose a subscript notation $\Delta^n_\dagger = (\Delta^n, \flat, \flat \subset \Delta^{\{i,j,k\}})$. This convention will also applied to previously defined constructions like for example $(\Lambda^n_i)_\dagger$ or $\mathbb{S}\mathrm{t}_{\Delta^n}(\Delta^n)_\dagger$.

Lemma 3.2.37. *Let $\Delta^n_T = (\Delta^n, \flat, \flat \subseteq T)$. Then for every $0 \leqslant s \leqslant n$ the canonical morphism*

$$\mathbb{S}\mathrm{t}_{\Delta^n_\flat}(\Lambda^n_i)_T(s) \xrightarrow{\simeq} (\Lambda\mathcal{L}^n_i)_T(s)$$

is **MS***-anodyne.*

Proof. It is clear that for every $0 < s \leqslant n$ we can pick the morphism to be an isomorphisms on the underlying simplicial sets. We further note that the proof of Lemma 3.2.14 still holds in this setting. Consequently, the claim follows. □

Definition 3.2.38. Let $n \geqslant 0$ and $0 \leqslant s \leqslant n$. We say that a (non-degenerate) simplex σ in $\mathcal{L}^n(s)$ is a *path* if it is of maximal dimension. Let $\mathcal{P}^n_s$ be the set of such paths. We will define an total order on $\mathcal{P}^n_s$ as follows:

Given a path $\sigma : S_0 \subset S_1 \subset S_2 \subset \cdots S_\ell$ one sees that $S_{i+1} \setminus S_i = \{a_{i+1}\}$ consists precisely in one element. Therefore we can identify σ with a list of elements

$$S_\sigma = \{a_i\}_{i=1}^{\ell}.$$

Note that by the maximality of σ, $S_0 = \{s\}$.

Suppose we are given two such lists $S_\sigma = \{a_i\}_{i=1}^{\ell}$ and $S_\theta = \{b_i\}_{i=1}^{\ell}$. We declare $\sigma < \theta$ if for the first index j for which $a_j \neq b_j$ then we have $a_j < b_j$.

Lemma 3.2.39. *Let $\Delta^n_\diamond = (\Delta^n, \flat, \flat \subset \Delta^{\{0,1,n\}})$ and consider the induced morphism $(\Lambda\mathcal{L}^n_0)_\diamond(0) \to \mathcal{L}^n_\diamond(0)$. Collapsing the morphism $0 \to 01$ to a degenerate edge on both sides yields a map of scaled simplicial sets*

$$\Lambda\mathcal{R}^n_0 \longrightarrow \mathcal{R}^n$$

which is scaled anodyne.

Proof. We use the order from Definition 3.2.38 to add simplices to $\Lambda\mathcal{R}^n_0$. We will add simplices in *reverse order*, i.e. for any path σ, we denote by $X^{\geqslant\sigma}$ the scaled simplicial subset of $\mathcal{R}^n$ obtained by adding to $\Lambda\mathcal{R}^n_0$ all paths θ such that $\theta \geqslant \sigma$.

The procedure yields a filtration

$$\Lambda\mathcal{R}^n_0 = X^{\geqslant\sigma_0} \longrightarrow X^{\geqslant\sigma_1} \longrightarrow X^{\geqslant\sigma_2} \longrightarrow \cdots \longrightarrow \mathcal{R}^n$$

where we have labeled our paths σ_i so that $\sigma_i > \sigma_{i+1}$. The proof proceeds by showing that $X^{\geqslant\sigma_{i-1}} \to X^{\geqslant\sigma_i}$ is scaled anodyne for any i.

The proof proceeds by cases. We fix the notation that $S_{\sigma_i} = \{a_k\}_{k=1}^n$.

1. Suppose that $a_1 \neq 1$. We prove this case by showing that the top horizontal map in the pullback diagram

$$\begin{array}{ccc} A_{\sigma_i} & \xrightarrow{\alpha_i} & \Delta^n \\ \downarrow & & \downarrow{\scriptstyle \sigma_i} \\ X^{\geqslant\sigma_{i-1}} & \longrightarrow & X^{\geqslant\sigma_i} \end{array}$$

is itself scaled anodyne.

We see that A_{σ_i} is the union of the following faces of σ_i:

- The face $d_0(\sigma_i)$, since we will have $0 \leqslant 1 < a_1$ in each S_k.
- The face $d_n(\sigma_i)$, since this face will always be missing $a_n \neq 0$.
- The face $d_j(\sigma_i)$ for every j such that $a_{j+1} > a_j$. This is because this will, equivalently, be the j^{th} face of the (greater) simplex with vertex list
$$\{a_1, \ldots, a_{j-1}, a_{j+1}, a_j, a_{j+2}, \ldots, a_n\}.$$

To see that the inclusion of $A_{\sigma_i} \to \Delta^n$ is scaled anodyne, we first note that, by necessity, there is at least one face of Δ^n not contained in A_{σ_i}. We then choose $t \in [n]$ such that $d_t(\Delta^n)$ is not contained in A_{σ_i}.

If we let $j \in [n]$ be the an element such that $j < t$ and $d_j(\Delta^n) \subset A_{\sigma_i}$. Similarly, let $\ell \in [n]$ be the smallest element such that $\ell > t$ and $d_\ell(\Delta^n) \subset A_{\sigma_i}$. A similar argument to [AGS21, Lemma 1.10] (or Lemma 3.1.6) shows that it will suffice to see that the simplex $\Delta^{\{j,t,\ell\}}$ is scaled for every such j, t, and ℓ. It is easy to see that $\max(S_\ell) = \max(S_t)$. Consequently, we see that α_i is scaled anodyne, as desired.

2. Now suppose that $a_1 = 1$, and $S_{\sigma_i} \neq \{1, 2, \ldots, n-1, n\}$. We now must instead consider the pullback diagram

$$\begin{array}{ccc} B_{\sigma_i} & \xrightarrow{\beta_i} & \Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0 \\ \downarrow & & \downarrow \\ X^{\geqslant\sigma_{i-1}} & \longrightarrow & X^{\geqslant\sigma_i} \end{array}$$

as above, we see that B_{σ_i} consists of the faces

- $d_n(\sigma_i)$
- $d_j(\sigma_i)$ for each j such that $a_j < a_{j+1}$.

Since $S_{\sigma_i} \neq \{1, 2, \ldots, n-1, n\}$, there exists some $1 < t < n$ such that B_{σ_i} does not contain the t^{th} face of $\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0$.

We can then consider the pullback diagram

$$\begin{array}{ccc} C_{\sigma_i} & \xrightarrow{\gamma_i} & \Delta^{n-1} \\ \downarrow & & \downarrow d_0 \\ B_{\sigma_i} & \longrightarrow & \Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0 \end{array}$$

an apply precisely the argument from the first case to $t \in [n-1]$ described above to find that γ_i is scaled anodyne. This means that, via a pushout, we may assume that B_{σ_i} contains the 0^{th} face of $\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0$. We can then repeat essentially the same argument, and thereby see that β_i is scaled anodyne.

3. If $S_{\sigma_i} = \{1, 2, \ldots, n-1, n\}$, then the map $X^{\geqslant \sigma_{i-1}} \longrightarrow X^{\geqslant \sigma_i} = \mathcal{R}^n$ is an inclusion

$$\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0 \longrightarrow \Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

where $\Delta^{\{0,1,n\}}$ is scaled. □

Lemma 3.2.40. *Let $\Delta^n_\dagger = (\Delta^n, \flat, \flat \subset \Delta^{\{0,n-1,n\}})$ and consider the induced morphism $(\Lambda\mathcal{L}^n_n)_\dagger(0) \to \mathcal{L}^n_\dagger(0)$. Denote by $\mathcal{T}^n$ (resp. $\Lambda\mathcal{T}^n_n$) the marked scaled simplicial set obtained from $\mathcal{L}^n_n(0)$ (resp. $(\Lambda\mathcal{L}^n_n)_\dagger(0)$) by marking the edges associated to the edge $(n-1) \longrightarrow n$ in Δ^n. Then the associated map*

$$\Lambda\mathcal{T}^n_n \longrightarrow \mathcal{T}^n$$

is **MS**-*anodyne.*

Proof. The argument is nearly identical to the proof of Lemma 3.2.39. Using the same order as in that proof, we produce a filtration

$$\Lambda\mathcal{T}^n_n = X^{\geqslant \sigma_0} \longrightarrow X^{\geqslant \sigma_1} \longrightarrow \cdots \longrightarrow \mathcal{T}^n$$

and show each step is scaled anodyne.

As before, we set $S_{\sigma_i} = \{a_j\}_{j=1^n}$, and consider the pullback diagram

$$\begin{array}{ccc} A_{\sigma_i} & \xrightarrow{\alpha_i} & \Delta^n \\ \downarrow & & \downarrow \sigma_i \\ X^{\geqslant \sigma_{i-1}} & \longrightarrow & X^{\geqslant \sigma_i} \end{array}$$

The case distinction now rests on whether or not $d_n\sigma_i$ factors through A_{σ_i}. The case when it does is formally identical to case (1) from Lemma 3.2.39.

If $d_n(\sigma_i)$ does not factor through A_{σ_i}, then $a_n = n$. There are again two cases, based on whether $S_{\sigma_i} = \{1, 2 \ldots, n-1, n\}$. The case $S_{\sigma_i} \neq \{1, 2 \ldots, n-1, n\}$ is identical to the corresponding case in Lemma 3.2.39. In the case $S_{\sigma_i} = \{1, 2 \ldots, n-1, n\}$ we find that $A_{\sigma_i} = \Lambda^n_n$, the last edge is marked, and $\Delta^{\{0,n-1,n\}}$ is scaled. This is a morphism of type (MS4), and thus is **MS**-anodyne. □

Lemma 3.2.41. *Let* $\Delta^n_{*_i} = (\Delta^n, \flat, \flat \subset \Delta^{\{i-1,i,i+1\}})$ *and consider the induced morphism* $(\Lambda\mathcal{L}^n_i)_{*_i}(0) \to \mathcal{L}_{*_i}(0)$. *Let* $\mathcal{S}^n$ *(resp* $\Lambda\mathcal{S}^n_i$*) denote the marked scaled simplicial set obtained by marking the edges of the form* $S \to S'$ *such that* $i, i+1 \in S$ *but* $i \notin S$ *and such that* $S' = S \cup \{i\}$. *Then the induced morphism*

$$\Lambda\mathcal{S}^n_i \longrightarrow \mathcal{S}^n_i$$

is **MS**-*anodyne.*

Proof. Let $S_\tau = \{1, 2, \ldots, i-1, i+1, \ldots, n-1, n, i\}$ and denote by $\hat{\tau}$ the smallest maximal simplex such that $\hat{\tau} > \tau$. We define a filtration as in Lemma 3.2.39 and Lemma 3.2.40 up until the stage $X^{\geqslant\hat{\tau}}$, yielding

$$\Lambda\mathcal{S}^n_i \longrightarrow X^{\geqslant\sigma_1} \longrightarrow \cdots \longrightarrow X^{\geqslant\hat{\tau}} \longrightarrow \mathcal{S}^n_i.$$

We will first prove that that each step of this factorization is **MS**-anodyne, making a distinction into 2 cases.

We consider the map $X^{\geqslant\sigma_{k-1}} \to X^{\sigma_k}$, and set $S_{\sigma_k} := \{a_j\}_{j=1}^n$. We again form the pullback

$$\begin{array}{ccc} A_{\sigma_k} & \xrightarrow{\alpha_k} & \Delta^n \\ \downarrow & & \downarrow{\scriptstyle \sigma_k} \\ X^{\geqslant\sigma_{k-1}} & \longrightarrow & X^{\geqslant\sigma_k} \end{array}$$

We then have two cases.

1. If S_{σ_k} has as its last entry anything other than i, then A_{σ_k} consists of
   - The face $d_0(\sigma_k)$.
   - The face $d_n(\sigma_k)$.
   - The face $d_j(\sigma_k)$ for each j such that $a_{j+1} > j_j$.

   The argument is then nearly identical to case (1) from Lemma 3.2.39.

2. If the last entry of S_{σ_k} is i, then A_{σ_k} consists of
   - The face $d_0(\sigma_k)$.
   - The face $d_j(\sigma_k)$ for each j such that $a_{j+1} > a_j$.

   The remainder of the argument is nearly identical to case (2) of Lemma 3.2.39.

It now remains only for us to show that $X^{\geqslant\hat{\tau}} \to \mathcal{S}^n_i$ is **MS**-anodyne. For ease of notation, we set $Z := X^{\geqslant\hat{\tau}}$.

We now need to add the remaining simplices. Write $\Sigma^{\leqslant}$ for the set of maximal simplices which are less than or equal to τ. Given $\theta \in \Sigma^{\leqslant}$, we write $S_\theta = \{b_j\}_{j=1}^n$ for the ordered vertex sequence, as usual. We further denote by $\widehat{S_\theta}$ the vertex sequence obtained by removing i. We will call a simplex $\theta \in \Sigma^{\leqslant}$ *disordered* if $\widehat{S_\theta} > \widehat{S_\tau}$. If $\widehat{S_\theta} = \widehat{S_\tau}$, we will call θ *calm.*[1]

Our first order of business is to add the disordered simplices in $\Sigma^{\leqslant}$. For each disordered θ, we define $Z^{\geqslant\theta}$ to be obtained from Z by adding all the disordered

[1] If $\theta \in \Sigma^{\leqslant}$ is calm, then the entries of $\widehat{S_\theta}$ are in the linear order induced by the order on the integers. If θ is disordered, they are not.

simplices σ for which $\sigma \geqslant \theta$. Applying the order induced on disordered simplices, we again obtain a filtration

$$Z \longrightarrow Z^{\geqslant \sigma_1} \longrightarrow Z^{\geqslant \sigma_2} \longrightarrow \cdots \longrightarrow Z^{\geqslant \gamma}$$

where γ is the minimal disordered simplex under the order $<$.

As before, we form a pullback diagram

$$\begin{array}{ccc} B_{\sigma_k} & \xrightarrow{\beta_k} & \Delta^n \\ \downarrow & & \downarrow{\scriptstyle \sigma_k} \\ Z^{\geqslant \sigma_{k-1}} & \longrightarrow & Z^{\geqslant \sigma_k} \end{array}$$

and show that β_k is **MS**-anodyne. Note that B_{σ_k} consists precisely of

- The face $d_0(\sigma_k)$.
- The face $d_n(\sigma_k)$ (since the final entry of S_{σ_k} cannot be i).
- The face $d_j(\sigma_k)$ for each j such that $a_{j+1} > a_j$.

The argument that β_k is anodyne is, by now, routine.

We now turn to adding the calm simplices. Notice that τ is the maximal calm simplex. We now set $Y := Z^{\geqslant \gamma}$, and define a filtration

$$Y \longrightarrow Y^{\leqslant \sigma_1} \longrightarrow Y^{\leqslant \sigma_2} \longrightarrow \cdots \longrightarrow Y^{\leqslant \tau} = \mathcal{S}^n_i$$

By defining $Y^{\leqslant \theta}$ to be the union of Y with all of the calm simplices less than or equal to θ.

For every calm σ_k other than τ itself, we obtain a pullback diagram

$$\begin{array}{ccc} \Lambda^n_\ell & \xrightarrow{\eta_k} & \Delta^n \\ \downarrow & & \downarrow{\scriptstyle \sigma_k} \\ Y^{\leqslant \sigma_{k-1}} & \longrightarrow & Y^{\leqslant \sigma_k} \end{array}$$

where Λ^n_ℓ is an inner horn. If $S_{\sigma_k} = \{1, 2, 3, \ldots, n-1, n\}$, then this is a Λ^n_i, and the scaling on $\Delta^n_{*_i}$ shows us that the simplex $\Delta^{\{i-1,i,i+1\}} \subset \sigma_k$ is scaled. On the other hand, if $S_{\sigma_k} \neq \{1, 2, 3, \ldots, n-1, n\}$, the simplex $\Delta^{\{\ell-1,\ell,\ell+1\}} \subset \Lambda^n_\ell$ is already scaled in $\mathcal{L}^n_\flat(0) \subseteq \mathcal{L}^n_{*_i}(0)$. The morphism η_k is thus a scaled anodyne map, and the pushout is therefore **MS**-anodyne.

We are left only to add τ. However, in this case, we obtain a pullback diagram

$$\begin{array}{ccc} \Lambda^n_n & \xrightarrow{\mu} & \Delta^n \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ Y^{<\tau} & \longrightarrow & \mathcal{S}^n_i \end{array}$$

where the 2-simplex $\Delta^{\{0,n-1,n\}}$ is scaled and the edge $\Delta^{\{n-1,n\}}$ is marked. The result then follows from a pushout of type (MS4). $\square$

Proposition 3.2.42. *Let S be a scaled simplicial set and let $\mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$ be a functor of Set_Δ^+-enriched categories. Consider an **MB**-anodyne morphism $i : A \to B$ in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$. Then for every $s \in S$ then induced map*

$$\mathbb{S}\mathrm{t}_\phi A(s) \longrightarrow \mathbb{S}\mathrm{t}_\phi B(s)$$

is a trivial cofibration of marked scaled simplicial sets.

Proof. As in the proof of Proposition 3.2.16, we can assume that $S = B$, that ϕ is $\mathrm{id} : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathfrak{C}^{\mathrm{sc}}[S]$, and that i is one of the generators in Definition 2.2.7. We proceed to verify each case.

A1) It is immediate that $\mathbb{S}\mathrm{t}_B\, A(s) \to \mathbb{S}\mathrm{t}_B\, B(s)$ is an isomorphism when $s \neq 0$. Lemma 3.2.41 shows that the map is **MS**-anodyne when $s = 0$.

A2) Note that the morphism $\mathbb{S}\mathrm{t}_B A(s) \to \mathbb{S}\mathrm{t}_B B(s)$ is an isomorphism for $s \neq 0$. If $s = 0$ the map is an isomorphism on the underlying marked simplicial sets. Let $\hat{T} = T \cup \Delta^{\{0,1,4\}} \cup \Delta^{\{0,3,4\}}$ (see Definition 2.2.7) and let $\mathcal{L}^4_T(0)$ and $\mathcal{L}^4_{\hat{T}}$ be the simplicial sets defined in Definition 3.2.13 equipped with the marking given by the thin simplices in the base. We obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{S}\mathrm{t}_B A(0) & \longrightarrow & \mathbb{S}\mathrm{t}_B B(0) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{L}^4_T(0) & \longrightarrow & \mathcal{L}^4_{\hat{T}}(0) \end{array}$$

where the vertical morphisms are equivalences due to Lemma 3.2.14. We will show that the bottom morphism is an equivalence. Observe that once we manage to scale the simplices $0 \to 01 \to 014$ and $0 \to 03 \to 034$ then rest of the scaling follows using the argument given in Lemma 3.2.14. We start by considering the 4-simplex given by

$$0 \to 01 \to 012 \to 0123 \to 01234$$

The only faces that are not scaled are precisely $\{0, 01, 01234\}$ and $\{0, 0123, 01234\}$. Consequently we can scale them using a pushout of type (MS2). Now we consider a 3-simplex

$$0 \to 01 \to 014 \to 01234$$

where all of its faces are now scaled except possibly the 3rd face. We further note that we can factor the last morphism as $014 \to 0134 \to 01234$ where both morphisms are marked. Therefore we can assume without loss of generality that the map $014 \to 01234$ is also marked. This allows us to scale the 3rd face using a pushout along a map of type (MS8). Inspecting the 3-simplex

$$0 \to 03 \to 0123 \to 01234$$

we see that we can add to the scaling $\{0, 03, 01234\}$. Finally let us consider

$$0 \to 03 \to 034 \to 01234.$$

As we did before we factor the last map as a composite of marked morphisms $034 \to 0134 \to 01234$. The claim follows by a totally analogous argument as before.

A3) Let $*$ denote the vertex to which 0 and 1 get identified. Then it follows that the induced map $\mathbb{S}\mathrm{t}_B A(s) \to \mathbb{S}\mathrm{t}_B B(s)$ is an isomorphism for $s \neq *$. Lemma 3.2.39 shows that the map is **MS**-anodyne when $s = 0$.

A4) It is immediate that $\mathbb{S}\mathrm{t}_B A(s) \to \mathbb{S}\mathrm{t}_B B(s)$ is an isomorphism for $s \neq 0$. Lemma 3.2.40 shows that the map is **MS**-anodyne when $s = 0$.

S2) The induced map is an isomorphism for every object of Δ^2.

S3) The map is an isomorphism for every $s \in \Delta^3$ such that $s \neq 0$. We will prove the case $i = 1$ leaving the case $i = 2$ as an exercise to the reader. Let $\mathcal{L}^3_{U_1}(0)$ and $\mathcal{L}^3_{\sharp}(0)$ be as in Definition 3.2.13 and equip them with the marking induced by the thin simplex $\Delta^{\{0,1,2\}}$. We obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{S}\mathrm{t}_B A(0) & \longrightarrow & \mathbb{S}\mathrm{t}_B B(0) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{L}^3_{U_1}(0) & \longrightarrow & \mathcal{L}^3_{\sharp}(0) \end{array}$$

where the vertical morphisms are equivalences due to Lemma 3.2.14. Therefore it will enough to show that the bottom morphism is an anodyne map of marked scaled simplicial sets. Consider the simplex $0 \to 01 \to 012 \to 0123$ and observe that all of its faces are scaled except the face missing 1. Therefore we can scale the 1-face using a pushout along an anodyne morphism as described in Lemma 2.4.3. Now we consider $0 \to 02 \to 012 \to 0123$ and we observe that we can scale the face missing 2 by another pushout. Finally we look at $0 \to 02 \to 023 \to 0123$ and we note that the last edge must be marked and that all of the faces are scaled except the face missing the vertex 3. Thefore another pushout along a morphism of type (MS8) yields the result.

S4) & S5) The proof is very similar to the previous case and left to the reader.

A5,S1 & E) Since these maps are always maximally thin scaled we can use Proposition 3.2.15 and apply the pertinent proofs in Proposition 3.2.1.11 in [Lur09a].

□

3.2.4 Straightening over the point

In this section, we will prove two important results. We will show that the the bicategorical straightening functor is left Quillen over any scaled simplicial set, and we will show that the straightening is an equivalence over the point. We fix the notation $\mathbb{S}\mathrm{t}_{\Delta^0} = \mathbb{S}\mathrm{t}_*$.

Definition 3.2.43. We define a an adjunction

$$L : \mathrm{Set}^{\mathbf{mb}}_{\Delta} \rightleftarrows \mathrm{Set}^{\mathbf{ms}}_{\Delta} : R$$

where $L(X, E_X, T_X \subseteq C_X) := (X, E_X, C_X)$ and $R(Y, E_Y, T_Y) = (Y, E_Y, T_Y \subseteq T_Y)$. We note that $L \circ R = \mathrm{id}$ and that the unit natural transformation

$(X, E_X, T_X \subseteq C_X) \to (X, E_X, C_X \subseteq C_X)$ is **MB**-anodyne. It is easy to see that L preserves cofibrations and trivial cofibrations. In particular, we see that $L \dashv R$ is a Quillen equivalence.

Our goal in this section is to construct a natural transformation $\mathbb{S}\mathrm{t}_* \Rightarrow L$ which is a levelwise weak equivalence. By general abstract nonsense, it will suffice to construct morphisms $\alpha_X : \mathbb{S}\mathrm{t}_*(X) \to L(X)$ whenever X is one of the following generators

- $\Delta^n_\flat := (\Delta^n, \flat, \flat)$, for $n \geqslant 0$,
- $\Delta^2_\dagger := (\Delta^2, \flat, \flat \subset \Delta^2)$,
- $\Delta^2_\sharp := (\Delta^2, \flat, \Delta^2)$,
- $(\Delta^1)^\sharp := (\Delta^1, \Delta^1, \flat)$,

and to prove that that the maps α_X are natural with respect to morphisms among generators. The next step is to give a precise description of the straightening functor applied to those generators.

Definition 3.2.44. Let $n \geqslant 0$ and define a simplicial set

$$Q^n := \coprod_{0 \leqslant i \leqslant n} \mathbb{O}^{n+1}(i, n+1) \Big/ \sim$$

where the relation $\sim$ identifies simplices n-simplices $\sigma_1 \in \mathbb{O}^{n+1}(i, n+1)$ and $\sigma_2 \in \mathbb{O}^{n+1}(j, n+1)$ with $i \leqslant j$ whenever σ_1 is in the image of the map

$$\mathbb{O}^{n+1}(i,j) \times \Delta^n \xrightarrow{\mathrm{id} \times \sigma_2} \mathbb{O}^{n+1}(i,j) \times \mathbb{O}^{n+1}(j, n+1) \longrightarrow \mathbb{O}^{n+1}(i, n+1)$$

We further observe that the morphisms

$$\mathbb{O}^{n+1}(i, n+1) \longrightarrow \Delta^n, \;\; S \longmapsto \max\left(S \setminus \{n+1\}\right)$$

assemble into a map $\alpha_n : Q^n \to \Delta^n$. We wish now to upgrade Q^n to a scaled simplcial set. We do so by declaring a triangle $\sigma : \Delta^2 \to Q^n$ thin if and only if its image under p is degenerate in Δ^n. We denote this collection of thin triangles by T_{Q^n}.

Remark 3.2.45. Given an order preserving morphism $f : [n] \to [m]$ then it is straightforward to check that we can produce a commutative diagram

$$\begin{array}{ccc} Q^n & \xrightarrow{Q(f)} & Q^m \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_m} \\ \Delta^n & \xrightarrow{f} & \Delta^m \end{array}$$

Lemma 3.2.46. *We have the following isomorphisms of marked scaled simplicial sets*

- $\mathbb{S}\mathrm{t}_*(\Delta^n_\flat) \simeq (Q^n, \flat, T_{Q^n})$.
- $\mathbb{S}\mathrm{t}_*(\Delta^2_\dagger) = \mathbb{S}\mathrm{t}_*(\Delta^2_\sharp) \simeq (Q^2, \flat, \sharp)$.

- $\mathbb{S}\mathrm{t}_*((\Delta^1)^\sharp) = (Q^1, \sharp, \flat)$.

Lemma 3.2.47. *The morphism*

$$\alpha_n \colon Q^n \longrightarrow \Delta^n_\flat$$

is a weak equivalence of marked scaled simplicial sets.

Proof. We construct a section $s : \Delta^n_\flat \to Q^n$ by sending $i \in [n]$ to the set $[0,i] \cup \{n+1\}$ and note that $\alpha_n \circ s = \mathrm{id}_{\Delta^n}$. To finish the proof we will construct a marked homotopy between id_{Q^n} and $s \circ \alpha_n$.

Let $\sigma : \Delta^k \to Q^n$ and pick a representative $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k$ with $S_j \in \mathbb{O}^{n+1}(i, n+1)$ for $0 \leqslant j \leqslant k$. To ease the notation we will omit the element $n+1$ from the subsets S_j. Let us denote $s_j = \max(S_j)$ and observe that we can produce a diagram $H(-,\sigma) : \Delta^1 \times \Delta^n \to Q^n$

$$\begin{array}{ccccccccc} S_0 & \longrightarrow & S_1 & \longrightarrow & \cdots & \longrightarrow & S_{k-1} & \longrightarrow & S_k \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [i, s_0] & \longrightarrow & [i, s_1] & \longrightarrow & \cdots & \longrightarrow & [i, s_{k-1}] & \longrightarrow & [i, s_k] \end{array}$$

It is straightforward to check that if $\sigma \sim \theta$ then $H(-,\sigma) = H(-,\theta)$. We have constructed now a natural transformation $\Delta^1 \times Q^n \to Q^n$. It is immediate to see that $H(0,-) = \mathrm{id}_{Q^n}$. In addition the fact that the bottom row in the diagram is equivalent to

$$[i, s_0] \longrightarrow [i, s_1] \longrightarrow \cdots \longrightarrow [i, s_{k-1}] \longrightarrow [i, s_k]$$

ensures that $H(1,-) = s \circ \alpha_n$. We conclude the proof by noting that the morphism $S_0 \to [i, s_0]$ gets collapsed to a degenerate edge and thus the homotopy is marked. □

Proposition 3.2.48. *There exists a natural transformation $\alpha : \mathbb{S}\mathrm{t}_* \Rightarrow L$ which is a levelwise weak equivalence.*

Proof. Using Lemma 3.2.46 is immediate to verify that the maps (together with decorated variants) $\alpha_n : Q^n \to \Delta^n$ assemble into a natural transformation $\alpha : \mathbb{S}\mathrm{t}_* \overset{L}{\Rightarrow}$. To check that α is a levelwise equivalence, we note that due to the fact that both $\mathbb{S}\mathrm{t}_*$ and L are left adjoints which preserve cofibrations it will suffice to check on generators. We proceed case by case

- $\alpha_n : (Q^n, \flat, T_{Q^n}) \to (\Delta^n, \flat, \flat)$ is an equivalence due to Lemma 3.2.47.
- $\alpha_2^\sharp : (Q^2, \flat, \sharp) \to (\Delta^2, \flat, \sharp)$ is an equivalence since we can repeat the proof above with maximally scaled simplicial sets.
- $\alpha_1^\sharp : (Q^1, \sharp, \flat) \to (\Delta^1, \sharp, \flat)$ is an isomorphism. □

Theorem 3.2.49. *Let S be an scaled simplicial set, then the bicategorical straightening functor*

$$\mathbb{S}\mathrm{t}_S \colon (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} \longrightarrow (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}}}$$

is a left Quillen functor.

Proof. Given a weak equivalence $f\colon X \longrightarrow Y$ in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$, we can apply fibrant replacement to obtain a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \widetilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \longrightarrow & \widetilde{Y} \end{array}$$

where the horizontal morphisms are **MB**-anodyne, and there vertical morphisms are weak equivalences.

Since $\mathbb{S}\mathrm{t}_S$ preserves **MB**-anodyne morphisms, we may thus assume without loss of generality that X and Y are fibrant objects. By Lemma 2.2.33, f then has a homotopy inverse g. Let

$$H\colon (\Delta^1)^\sharp_\sharp \times X \longrightarrow X$$

be a marked homotopy between $g \circ f$ and id_X over S. Then $\mathbb{S}\mathrm{t}_S(H)$ factors as

$$\mathbb{S}\mathrm{t}_S(X \times (\Delta^1)^\sharp_\sharp) \xrightarrow{\ \varepsilon\ } \mathbb{S}\mathrm{t}_S(X) \boxtimes \mathbb{S}\mathrm{t}_*((\Delta^1)^\sharp_\sharp) \xrightarrow{\ \alpha\ } \mathbb{S}\mathrm{t}_S(X) \boxtimes (\Delta^1)^\sharp_\sharp \longrightarrow \mathbb{S}\mathrm{t}_{\Delta^0_\flat}(K^\sharp)$$

Where ε is an equivalence by Theorem 3.2.17, α is an equivalence by Proposition 3.2.48, and the final map is an equivalence since $(\Delta^1)^\sharp \to \Delta^0$ is an equivalence of marked simplicial sets. Since $\mathbb{S}\mathrm{t}_S(g \circ f)$ and $\mathbb{S}\mathrm{t}_S(\mathrm{id}_X) = \mathrm{id}_{\mathbb{S}\mathrm{t}_S(X)}$ are both sections of $\mathbb{S}\mathrm{t}_S(H)$, they are thus equivalent in the homotopy category. An identical argument shows that $\mathbb{S}\mathrm{t}_S(f \circ g) \simeq \mathrm{id}_{\mathbb{S}\mathrm{t}_S(Y)}$, completing the proof. □

Corollary 3.2.50. *In particular the adjunction*

$$\mathbb{S}\mathrm{t}_* : \left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/\Delta^0} \rightleftarrows \mathrm{Set}_\Delta^{\mathbf{ms}} : \mathbb{U}\mathrm{n}_*$$

is a Quillen equivalence.

Proof. By Proposition 3.2.48, St_* is naturally equivalent to a left Quillen equivalence. The corollary follows immediately. □

3.2.5 Straightening over a simplex

As in [Lur09b, Ch. 2], the proof that our Grothendieck construction is a Quillen equivalence over a general scaled simplicial set will be bootstrapped from a proof over the n-simplices $(\Delta^n)_\flat$. In analogy to the method in op. cit., we will prove this case by constructing a *mapping simplex* for each 2-Cartesian fibration $X \to \Delta^n_\flat$ — a tractible **MB** simplicial set $\mathcal{M}_X \to \Delta^n_\flat$ which is equivalent to X over $\Delta^n_\flat$.[2] The majority of this section is given over to showing that we can decompose a 2-Cartesian fibration $X \to \Delta^n_\flat$ as a homotopy pushout of the restriction of X to Δ^{n-1}, which enables the inductive step of our proof.

Remark 3.2.51. The term "mapping simplex" used above is potentially misleading. in [Lur09a] and [Lur09b], a mapping simplex is a fibration over Δ^n

[2] In contrast to the approach in [Lur09b, Ch. 2], we will not construct this 'mapping simplex' from an enriched functor, but rather as a pushout of **MB** simplicial sets over $\Delta^n_\flat$.

explicitly constructed from a functor $\mathcal{F} : [n] \to \mathrm{Set}_\Delta^+$ or a $\mathcal{F} : \mathfrak{C}[\Delta^n] \to \mathrm{Set}_\Delta^+$. Our construction makes use of no such functor, and thus is not a true mapping simplex in this sense. The abuse of the term mapping simplex in the above exposition should be seen as suggestive of the role this construction fills in our proof — one roughly analogous to the role of the mapping simplex in the proof of the $(\infty, 1)$-categorical Grothendieck construction in [Lur09a, §3.2].

Definition 3.2.52. We define a marked biscaled simplicial set $(\Delta^n)^\diamond := (\Delta^n, E^n_\diamond, \flat \subset \sharp)$ where $E^n_\diamond$ is the collection of all edges containing the vertex n. It is not hard to verify that the inclusion of the terminal vertex $\Delta^{\{n\}} \to (\Delta^n)^\diamond$ is **MB**-anodyne.

For the rest of this section, we fix be a 2-Cartesian fibration $p : X \to \Delta^n$ over the minimally scaled n-simplex. We consider the commutative diagram

$$\begin{array}{ccc} X_n \times \Delta^{\{n\}} & \longrightarrow & X \\ \downarrow & \overset{\alpha}{\nearrow} & \downarrow p \\ X_n \times (\Delta^n)^\diamond & \longrightarrow & \Delta^n \end{array}$$

where X_n denotes the fibre over the vertex n and the dotted arrow exists due to the fact that the left vertical morphism is **MB**-anodyne.

Consider the inclusion morphism $\iota : \Delta^{[0,n-1]} \to \Delta^n$ and equip $\Delta^{[0,n-1]}$ with the structure of an **MB** simplicial set by declaring and edge (resp. triangle) marked (resp. thin, resp. lean) if its image in Δ^n is marked (resp. thin, resp. lean) in $(\Delta^n)^\diamond$. Notice that this amounts to equipping Δ^{n-1} with the minimal marking and thin-scaling, and the maximal lean-scaling.

We denote the restriction of X to Δ^{n-1} by $X|_{\Delta^{n-1}} := X \times_{\Delta^n} \Delta^{n-1}$, and denote the restriction of α to $X_n \times (\Delta^{n-1})^\diamond$ by α'. We construct an **MB** simplicial set $\mathcal{M}_X$ over Δ^n by means of the pushout square

$$\begin{array}{ccc} X_n \times (\Delta^{n-1})^\diamond & \longrightarrow & X_n \times (\Delta^n)^\diamond \\ \downarrow \alpha' & & \downarrow \\ X|_{\Delta^{n-1}} & \longrightarrow & \mathcal{M}_X \end{array}$$

Note that, since the top horizontal map is a cofibration, this is a homotopy pushout square in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\Delta^n_\flat}$. The morphism α and the inclusion $X|_{\Delta^{n-1}} \to X$ yield a cone over this diagram, and thus a canonical morphism $\omega : \mathcal{M}_X \to X$ over Δ^n. The key technical element in this section will be to show that ω is a weak equivalence in the 2-Cartesian model structure.

Definition 3.2.53. Let $\sigma : \Delta^k \to X$. Given $I \subset [k]$, we define

$$F_I(\sigma) = \{\theta : \Delta^I \to \mathcal{M}_X \mid \omega(\theta) = d_I(\sigma)\} \cup \{*\}.$$

Given $J \subset I \subseteq [k]$ and $\theta \in F_I(\sigma)$ such that $\theta \neq *$ we denote by $d_{J,I}(\theta)$ the image of θ in $\mathcal{M}_X$ under the degeneracy operator induced by the inclusion $J \subset I$.

Definition 3.2.54. We define a **MB** simplicial set $\mathcal{L}_X$ whose simplices $\sigma : \Delta^k \to \mathcal{L}_X$ are given by:

- A simplex $\hat{\sigma} : \Delta^k \to X$.
- For every non-empty subset $I \subseteq [k]$ an element $\theta_I \in F_I(\sigma)$. If $\theta_I = *$ we use the empty set notation $\theta_I = \varnothing$.

We impose to this data the following compatibility conditions

H1) Given $J \subset I \subseteq [k]$ and $\theta_I \in F_I(\sigma)$ such that $\theta_I \neq \varnothing$ it follows that $d_{J,I}(\theta_I) = \theta_J$.

H2) Given $I \subseteq [k]$ with $i_m = \max(I)$, then if $p \circ \hat{\sigma}(i_m) \neq n$ it follows that $\theta_I \neq \varnothing$.

H3) Given $I \subseteq [k]$ such that for every $i \in I$ we have $p \circ \hat{\sigma}(i) = n$, then it follows that $\theta_I \neq \varnothing$.

Notice that by construction there is a canonical projection map $v : \mathcal{L}_X \to X$. We equip $\mathcal{L}_X$ with the marking and biscaling induced by v.

Given a simplex $\sigma : \Delta^k \to \mathcal{L}_X$, we refer to the collection $\{\theta_I\}_{I \subset [k]}$ as the *restriction data of* σ.

Lemma 3.2.55. *The projection map $v : \mathcal{L}_X \to X$ is a trivial fibration of **MB** simplicial sets.*

Proof. Since v by definition detects all possible decorations, it will suffice to show that v is a trivial fibration on the underlying simplicial sets. Note that v is a bijection on 0-simplices. Given $k \geqslant 1$ we consider a lifting problem

$$\begin{array}{ccc} \partial\Delta^k & \longrightarrow & \mathcal{L}_X \\ \downarrow & \nearrow & \downarrow \\ \Delta^k & \xrightarrow{\hat{\sigma}} & X \end{array}$$

To produce the dotted arrow we use the bottom horizontal morphism as our choice for simplex in X. If $p \circ \hat{\sigma}(k) \neq n$ or $p \circ \hat{\sigma}$ is constant on n, we set $\theta_{[k]}$ to be the unique preimage of $\hat{\sigma}$ in $\mathcal{M}_X$. If $p \circ \hat{\sigma}(k) = n$ and it is not constant on n, we set $\theta_{[k]} = \varnothing$. The rest of the θ_I are always chosen according to top horizontal morphism. The compatibilities are clearly satisfied. □

We construct a morphism $u : \mathcal{M}_X \to \mathcal{L}_X$ that sends a simplex $\theta : \Delta^k \to \mathcal{M}_X$ to the simplex $\omega(\theta)$ in X. For every $I \subseteq [k]$ we set $\theta_I = d_I(\theta)$. It is clear that u is a cofibration. It is not hard to see that u induces a bijection on the restriction to $\Delta^{[0,n-1]}$ and on the fibre over n.

Remark 3.2.56. Let $\pi = p \circ v : \mathcal{L}_X \to \Delta^n$. Given $\sigma : \Delta^k \to \mathcal{L}_X$ we fix the notation $\overline{\sigma} = \pi \circ \sigma$.

Definition 3.2.57. Let $\sigma : \Delta^k \to \mathcal{L}_X$ be a simplex such that $\overline{\sigma}(k) = n$. Let $\kappa_{\overline{\sigma}}$ be the first element in $[k]$ such that $\overline{\sigma}(\kappa_{\overline{\sigma}}) = n$. We define a full subposet $Z_\sigma \subset [k] \times [n]$ consisting of

- Those vertices of the form $(x, \overline{\sigma}(x))$ with $x < \kappa_{\overline{\sigma}}$.
- Those vertices of the form (x, y) with $x \geqslant \kappa_{\overline{\sigma}}$ and $y \geqslant \overline{\sigma}(0)$.

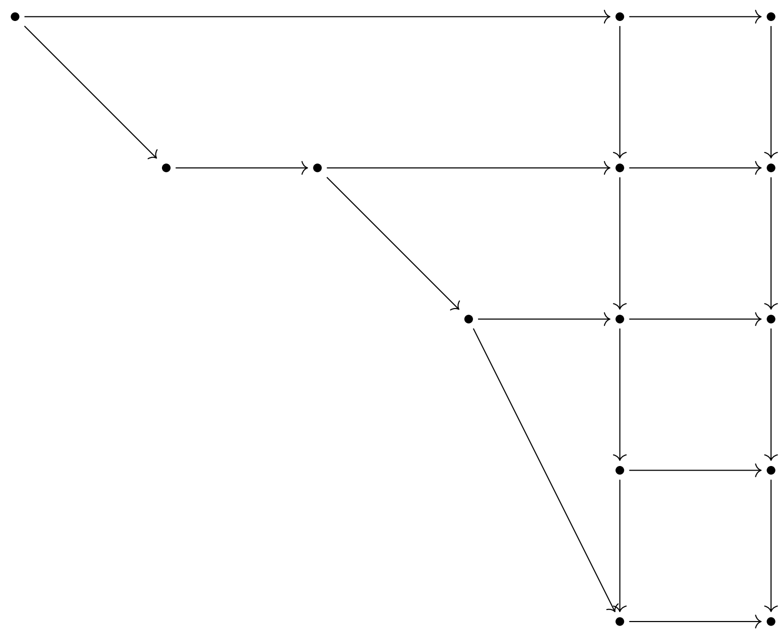


Figure 3.1: The poset Z_σ corresponding to the map $[5] \to [4]$ given by the sequence of values $0, 1, 1, 2, 4, 4$.

We denote by $\mathcal{Z}_\sigma$ the nerve $\mathrm{N}(Z_\sigma)$. Note that the projection $[k] \times [n] \to [n]$ yields a canonical map $\mathcal{Z}_\sigma \to \Delta^n$. We endow $\mathcal{Z}_\sigma$ with the structure of an **MB** simplicial set by declaring an edge $(x_1, y_1) \to (x_2, y_2)$ marked if $x_1 = x_2 \geqslant \kappa_{\overline{\sigma}}$ and $y_2 = n$. A triangle is declared to be lean if the associated 2-simplex in Δ^k is degenerate. Finally we say that a triangle in $\mathcal{Z}_\sigma$ is thin if it is already lean and its image in Δ^n is degenerate.

We call those non-degenerate simplices $\rho : \Delta^\ell \to \mathcal{Z}_\sigma$ which are not contained in any other non-degenerate simplex *essential*.

Remark 3.2.58. The avid reader might complain that our definition of $\mathcal{Z}_\sigma$ only depends on $\overline{\sigma}$ and so should be denoted by $\mathcal{Z}_{\overline{\sigma}}$. The next definition will justify our notation.

Definition 3.2.59. Let $\sigma : \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$. We define a subsimplicial set $\mathcal{X}_\sigma \subset \mathcal{Z}_\sigma$ (with the inherited marking and scalings) consisting of those simplices $\{(x_i, y_i)\}_{i=0}^{\ell}$ satisfying at least one of the conditions below

i) We have $y_i = \overline{\sigma}(x_i)$ for $i = 0, \dots, \ell$.

ii) There exists $I \subseteq [k]$ such that $\theta_I \neq \varnothing$ with $\overline{\sigma}(\max(I)) = n$ and $x_i \in I$ for $i = 0, \dots, \ell$.

Definition 3.2.60. Let $\sigma : \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$ and suppose we are given a subset $I \subset [k]$ such that $\theta_I \neq \varnothing$ and such that $\overline{\sigma}(\max(I)) = n$. We construct a morphism

$$\Delta^I \times \Delta^{[\overline{\sigma}(0),n]} \longrightarrow (\Delta^n)^\diamond \times X_n$$

whose component at $(\Delta^n)^\diamond$ is given by $\Delta^I \times \Delta^{[\overline{\sigma}(0),n]} \to \Delta^{[\overline{\sigma}(0),n]} \to (\Delta^n)^\diamond$ and whose component at X_n is given by $\Delta^I \times \Delta^{[\overline{\sigma}(0),n]} \to \Delta^I \to X_n$ where the last morphism is induced from θ_I.

We define a subposet $K_I \subset \Delta^I \times \Delta^{[\overline{\sigma}(0),n]}$ to be the intersection of $\Delta^I \times \Delta^{[\overline{\sigma}(0),n]}$ with $\mathcal{Z}_\sigma$. We denote $\mathcal{K}_I$ the **MB** simplicial set obtained by equipping K_I with the decorations induced from $(\Delta^n)^\diamond \times X_n$.

Remark 3.2.61. Observe that we can construct $\mathcal{X}_\sigma$ as the union of Δ^k and every $\mathcal{K}_I$ inside of $\mathcal{Z}_\sigma$.

Remark 3.2.62. Let $\sigma : \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$. We define a morphism $\widetilde{f}_\sigma : \mathcal{X}_\sigma \to \mathcal{L}_X$ as follows:

- For simplices satisfying condition $i)$ in Definition 3.2.59, $\widetilde{f}_\sigma$ is simply σ.
- For simplices satisfying condition $ii)$ in Definition 3.2.59, $\widetilde{f}_\sigma$ is given by the composite

$$\mathcal{K}_I \longrightarrow \Delta^I \times \Delta^{[\overline{\sigma}(0),n]} \longrightarrow (\Delta^n)^\diamond \times X_n \xrightarrow{\ u\ } \mathcal{L}_X.$$

One observes that this definition is compatible in the various intersections $\mathcal{K}_I \cap \mathcal{K}_J$ thus defining the desired morphism.

Definition 3.2.63. Let $\sigma : \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$. We define a subsimplicial subset $\mathcal{X}_\sigma^\uparrow \to \mathcal{Z}_\sigma$ (with the induced decorations) consisting of those simplices $\{(x_i, y_i)\}_{i=0}^{\ell}$ that are either in $\mathcal{X}_\sigma$ or satisfty the property:

- There exists $j \in [k]$ such that $x_i \neq j$ for every $i = 0, \dots, \ell$ and such that $\overline{\sigma}(d^j(k-1)) = n$.

Remark 3.2.64. Note that we can equivalently define $\mathcal{X}_\sigma^\uparrow$ to consist of those simplices that are either in $\mathcal{X}_\sigma$ or that are contained in the image of

$$\mathcal{Z}_{d_j(\sigma)} \longrightarrow \mathcal{Z}_\sigma$$

where $d_j(\sigma)(k-1) = n$.

Definition 3.2.65. We define an order on the set of essential simplices of $\mathcal{Z}_\sigma$ which we denote by "$\prec$". Let ρ_i for $i = 1, 2$ be two essential simplices determined by the sequence of vertices $\{(x_j^i, y_j^i)\}_{j=0}^{r_i}$ for $i = 1, 2$. Let ε be the first index such that $\rho_1(\varepsilon) \neq \rho_2(\varepsilon)$. We say that $\rho_1 \prec \rho_2$ if precisely one the following conditions is satisfied:

- We have that $x_\varepsilon^1 = \kappa_\sigma$.
- We have $x_\varepsilon^i > \kappa_\sigma$ for $i = 1, 2$ and $y_\varepsilon^1 > y_\varepsilon^2$.

Lemma 3.2.66. *Let $\sigma : \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$. Then the following morphisms are **MB**-anodyne:*

$$\mathcal{X}_\sigma \longrightarrow \mathcal{X}_\sigma^\uparrow \longrightarrow \mathcal{Z}_\sigma.$$

Proof. If σ factors through $\mathcal{M}_X$, then $\mathcal{X}_\sigma = \mathcal{Z}_\sigma$, so we may assume without loss of generality that σ does not factor through $\mathcal{M}_X$.

We proceed by induction on k. Consider $k = 1$, and note that $\mathcal{X}_\sigma = \mathcal{X}_\sigma^\uparrow$. Since σ does not factor through $\mathcal{M}_X$, we may assume $\sigma(0) \neq n$. There is thus some $\ell \geqslant 1$ such that the morphism $\mathcal{X}_\sigma \to \mathcal{Z}_\sigma$ can be identified with the inclusion

$$\psi_\ell \colon \Delta^1 \coprod_{\Delta^0} (\Delta^\ell)^\diamond \longrightarrow (\Delta^{\ell+1})^\dagger$$

Where † indicates marking where $i \to n$ is marked when $i \neq 0$, where every triangle is lean, and only those over degenerate triangles in Δ^n are thin. It follows immediately from Lemma 3.1.9 that this morphism is **MB**-anodyne.

We now suppose that the lemma holds in dimension $k-1$. By the inductive hypothesis,

$$\mathcal{X}_\sigma \longrightarrow \mathcal{X}_\sigma^\uparrow$$

is **MB**-anodyne. Thus, it is sufficient for us to show that the morphism $\mathcal{X}_\sigma^\uparrow \longrightarrow \mathcal{Z}_\sigma$ is **MB**-anodyne.

We will add essential simplices of $\mathcal{Z}_\sigma$ according to the order $\prec$. Given an essential simplex ρ, we denote by N_ρ the **MB** simplicial subset of $\mathcal{Z}_\sigma$ obtained by adding every essential simplex ρ' such that $\rho' \preceq \rho$. We consider a pullback diagram

$$\begin{array}{ccc} Q_\rho & \longrightarrow & \Delta^r \\ \downarrow & & \downarrow \rho \\ N_{\rho'} & \longrightarrow & N_\rho \end{array}$$

and we turn our attention to proving that the top horizontal morphism is **MB**-anodyne. Let us fix the notation $\rho = \{(x_j, y_j)\}_{j=0}^r$ and denote by θ the first index such that $x_\theta = \kappa_\sigma$.

We define three types of vertices $\varepsilon \in [r]$ of ρ.

Anterior vertices are those ε which have $x_\varepsilon < \kappa_\sigma$.

Recumbent vertices are those $\varepsilon \in [r]$ which have $x_\varepsilon > \kappa_\sigma$ and $x_{\varepsilon-1} < x_\varepsilon$. Note that this necessarily implies $y_{\varepsilon-1} = y_\varepsilon$

Plumb vertices are those $\varepsilon \in [r]$ which have $x_\varepsilon \geqslant \kappa_\sigma$ and $x_{\varepsilon-1} = x_\varepsilon$. Note that this necessarily implies that $y_{\varepsilon-1} < y_\varepsilon$. Note also that every ρ has at least one plumb vertex.

Notice that the only vertex which is not anterior, recumbent or plumb is θ. We call a vertex $\varepsilon \in [r]$ a *downturn* if ε is either recumbent or $\varepsilon = \theta$ **and** $x_{\varepsilon+1} = x_\varepsilon$. Note that ρ is uniquely determined by its set of downturns and the fact that it is essential.

We will prove three claims about these types of vertices, which then will enable us to complete the proof.

CLAIM 1: If ε is an anterior vertex, then $d_\varepsilon(\Delta^r) \subset Q_\rho$.

Subproof. Since ε is anterior, it is the only vertex of ρ whose first coordinate is x_ε. Consequently, $d_\varepsilon(\rho)$ factors through $\mathcal{Z}_{d_\varepsilon(\sigma)}$. ■

CLAIM 2: Let ε be a recumbent vertex. Then $d_\varepsilon(\Delta^r) \subset Q_\rho$.

Subproof. There are two cases. If $x_{\varepsilon+1} > x_{\varepsilon}$, then as before $d_{\varepsilon}(\rho)$ factors through $Z_{d_j(\sigma)}$ for some face operator d_j. If, on the other hand, $x_{\varepsilon+1} = x_{\varepsilon}$, then $d_{\varepsilon}(\rho)$ factors through a previous essential simplex. ■

Claim 3: Let X be a set of plumb vertices. Then $d_X(\Delta^r) \nsubseteq Q_{\rho}$.

Subproof. Since $d_X(\rho)$ contains a point with first coordinate j for every $j \in [k]$, we see that $d_X(\rho)$ cannot factor through $\mathcal{Z}_{d_j(\sigma)}$. Similarly, if $d_X(\rho)$ factors through $\mathcal{K}_I$ for $I \subset [k]$, then $I = [k]$, which would imply that σ factors through $\mathcal{M}_X$. Moreover, $d_X(\rho)$ cannot factor through σ, since it contains the vertex (x_θ, y_θ).

Finally, if $\gamma \prec \rho$ is a previous simplex in our factorization, then $d_X(\rho)$ cannot factor through γ, since γ and ρ are uniquely determined by their sequences of downturns, and the only sequence of downturns containing $d_X(\rho)$ determine simplices greater than ρ under the order $\prec$. ■

To finish the proof we will consider two different cases. Each of this cases will be solved used inner-dull subsets (resp. right-dull) subsets. It is important to remark that since we can assume that $\dim(\sigma) > 1$ it follows that all the decorations in $\mathcal{Z}_\sigma$ factor through $\mathcal{X}_\sigma^{\uparrow}$.

The first case is given precisely when the vertex r is recumbent. In this situation it follows that we can use Lemma 3.1.6 where the pivot point is given by the biggest plumb vertex. Since $r \neq \theta$ it follows that if r is not recumbent it must be plumb. In this cases the claim follow from Lemma 3.1.9. □

Definition 3.2.67. Let $\sigma : \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$ and let $\ell = n - \overline{\sigma}(\kappa_{\overline{\sigma}} - 1)$. For every morphism $f_\sigma : \mathcal{Z}_\sigma \to \mathcal{L}_X$ such that its restriction to $\mathcal{X}_\sigma$ equals $\widetilde{f}_\sigma$ as in Remark 3.2.62, we define a $(k+\ell)$-simplex $\mathsf{B}(\sigma) \in \mathcal{L}_X$ to be the composite

$$\Delta^{k+\ell} \xrightarrow{\rho_\sigma} \mathcal{Z}_\sigma \xrightarrow{f_\sigma} \mathcal{L}_X$$

Where

$$\rho_\sigma \colon [k+\ell] \longrightarrow Z_\sigma$$
$$j \longmapsto \begin{cases} (j, \overline{\sigma}(j)) & 0 \leqslant j \leqslant \kappa_{\overline{\sigma}} - 1 \\ (\kappa_{\overline{\sigma}}, \overline{\sigma}(\kappa_{\overline{\sigma}} - 1)) & j = \kappa_{\overline{\sigma}} \\ (\kappa_{\overline{\sigma}}, j), & \kappa_{\overline{\sigma}} < j \leqslant \kappa_{\overline{\sigma}} + \ell \\ (j - \ell, n), & \kappa_{\overline{\sigma}} + \ell + 1 \leqslant j \leqslant k + \ell \end{cases}$$

Remark 3.2.68. We can equivalently characterise the simplex $\mathsf{B}(\sigma)$ in Definition 3.2.67 as the smallest essential simplex of $\mathcal{Z}_\sigma$ under the order $\prec$ of Definition 3.2.65 that does not factor through $\mathcal{X}_\sigma^{\uparrow}$.

Remark 3.2.69. There are two key parameters which we will use to analyze the simplices $\mathsf{B}(\sigma)$, for $\sigma : \Delta^k \to \mathcal{L}_X$. One is the fundamental vertex κ_σ — the first vertex such that $\overline{\sigma}(\kappa_\sigma) = n$. The other is the *terminal size* of σ: the number of vertices $j \in [k]$ such that $\overline{\sigma}(j) = n$. We will denote the terminal size of σ by

$$\nu_\sigma = |\{j \in [k] \mid \overline{\sigma}(j) = n\}|.$$

Notice that the terminal size of $\mathsf{B}(\sigma)$ is *always* equal to the terminal size of σ.

To make use of the simplices $\mathsf{B}(\sigma)$ in an inductive pushout argument, we will need to ensure we can make sufficiently compatible choices of maps

$$f_\sigma\colon \mathcal{Z}_\sigma \longrightarrow \mathcal{L}_X$$

to define our choices of $\mathsf{B}(\sigma)$.

Proposition 3.2.70. *There exists a collection indexed by the simplices of* $\mathcal{L}_X$

$$\mathfrak{I} := \{f_\sigma : \mathcal{Z}_\sigma \to \mathcal{L}_X \mid \sigma : \Delta^k \to \mathcal{L}_X\}.$$

With the following properties:

i) The restriction of f_σ to $\mathcal{X}_\sigma$ equals $\widetilde{f}_\sigma$ as in Remark 3.2.62.

ii) Given a face operator $d_j : [k-1] \to [k]$ such that $\overline{\sigma}(d^j(k-1)) = n$ we have that the composite

$$\mathcal{Z}_{d_j(\sigma)} \longrightarrow \mathcal{Z}_\sigma \xrightarrow{f_\sigma} \mathcal{L}_X$$

equals $f_{d_j(\sigma)}$.

iii) If $\sigma \subseteq \mathsf{B}(\tau)$, where $\tau \subsetneq \sigma$, then $\mathsf{B}(\sigma)$ is degenerate on a simplex $\rho \subseteq \mathsf{B}(\tau)$.

Remark 3.2.71. Of key importance to our argument is the fact that, if $\sigma \subseteq \mathsf{B}(\tau)$ and $\tau \subsetneq \sigma$, then $\overline{\sigma}^{-1}(n)$, $\overline{\tau}^{-1}(n)$, and $\overline{\mathsf{B}(\tau)}^{-1}(n)$ have the same cardinality, and σ, τ, and $\mathsf{B}(\sigma)$ agree on corresponding simplex.

Definition 3.2.72. Set $\Xi_k = \{\sigma : \Delta^r \to \mathcal{L}_X \mid r \leqslant k\}$. We will call a collection

$$\mathfrak{I} := \{f_\sigma : \mathcal{Z}_\sigma \to \mathcal{L}_X\}_{\sigma\in\Xi_k}$$

a *compatible k-collection* if it satisfies conditions i), ii), and iii) above.

Lemma 3.2.73. *Let $\mathfrak{I}_{k-1}$ be a compatible $(k-1)$-collection, and let $\sigma : \Delta^k \to \mathcal{L}_X$ such that σ does not factor through $\mathcal{M}_X$. Suppose that there is a simplex $\tau : \Delta^s \to \mathcal{L}_X$ with $s < k$ such that*

- *There is an inclusion $\sigma \subset \mathsf{B}(\tau)$.*
- *The terminal sizes agree, i.e. $\nu_\sigma = \nu_{\mathsf{B}(\tau)}$.*

Then there is a subsimplex $\gamma \subset \tau$ such that

1. *There is an inclusion $\gamma \subsetneq \sigma$*
2. *There is an inclusion $\sigma \subset \mathsf{B}(\gamma)$*
3. *The terminal sizes ν_σ and $\nu_{\mathsf{B}(\gamma)}$ agree.*

Proof. One need only restrict to the maximal sub-simplex $\tau \cap \sigma$ that factors through τ and σ both. Conditions (1) and (3) are immediate, and it is an easy check to see that $\sigma \subset \mathsf{B}(\tau \cap \sigma)$. □

Lemma 3.2.74. *Let $\mathfrak{I}_{k-1}$ be a compatible $(k-1)$-collection. Let $\sigma : \Delta^k \to \mathcal{L}_X$ and assume that σ does not factor through $\mathcal{M}_X$. Let us suppose there exists a pair $\tau_i : \Delta^{s_i} \to \mathcal{L}_X$ with $s_i < k$ for $i = 1, 2$; such that $\sigma \subseteq \mathsf{B}(\tau_i)$ and $\tau_i \subsetneq \sigma$. Then $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$.*

Proof. We can partition Δ^{s_i} into two parts: $\tau_i^{-1}([0, n-1])$ and $\tau_i^{-1}(n)$. We identify each of these with subsets of Δ^k. By Remark 3.2.71, $\tau_1^{-1}(n) = \tau_2^{-1}(n) = \sigma^{-1}(n)$. Since σ does not factor through $\mathcal{M}_X$, the initial vertex of σ must factor through each τ_i. Since the only vertices j of $\mathsf{B}(\tau_i)$ which are not vertices of τ_i satisfy with $j \geqslant \kappa_{\tau_i}$, we see that for $j \in [k]$ such that $j < \kappa_{\tau_i}$, $\sigma(j)$ must factor through τ_i.

Thus, we see that τ_i obtained from σ by deleting the vertices $j \in [k]$ such that $\kappa_{\tau_i} < j < \kappa_\sigma$. Thus, if $\kappa_{\tau_1} \leqslant \kappa_{\tau_2}$, then $\tau_1 \subseteq \tau_2$. □

Corollary 3.2.75. *Let $\mathfrak{I}_{k-1}$ be a compatible $(k-1)$-collection, and let $\sigma : \Delta^k \to \mathcal{L}_X$ be a simplex that does not factor through $\mathcal{L}_X$. If there is any $\tau : \Delta^s \to \mathcal{L}_X$ with $\sigma \subseteq \mathsf{B}(\tau)$ and $\tau \subsetneq \sigma$, then there is a unique minimal such simplex. Moreover, if there is such a simplex τ with $\sigma = \mathsf{B}(\tau)$, then one such simplex is minimal.*

Proof. The first statement is an immediate consequence of the previous lemma. To prove the second claim suppose that we have $\rho \subseteq \tau$ such that $\mathsf{B}(\tau) \subseteq \mathsf{B}(\rho)$. First we observe that $\nu_\tau = \nu_\rho$. Let μ_ρ be the biggest element of $\mathsf{B}(\rho)$ that does not lie over n and such that μ_ρ and $\mu_\rho - 1$ lie over the same vertex of Δ^n. We similarly define μ_τ as the biggest element of $\mathsf{B}(\rho)$ contained in $\mathsf{B}(\tau)$ satisfying the analogous property as before. Note that such elements always exist by construction. An easy argument then shows that $\mu_\tau = \mu_\rho$ and our claim follows from dimension counting. □

Definition 3.2.76. Suppose given a compatible $(k-1)$-collection $\mathfrak{I}_{k-1}$ and $\sigma : \Delta^k \to \mathcal{L}_X$. If it exists, we call the minimal simplex of Corollary 3.2.75 the *capsule* of σ. We say that σ is *encapsulated* if it admits a capsule.

There is one final fact to establish: that there is a way of choosing a compatible degeneracy to ensure condition iii). Given $\sigma : \Delta^k \to \mathcal{L}_X$ which does not factor through $\mathcal{M}_X$, we denote by $\mathcal{R}_\sigma$ the pullback

$$\begin{array}{ccc} \mathcal{R}_\sigma & \longrightarrow & \Delta^{k+\ell} \\ \downarrow & & \downarrow{\scriptstyle \rho_\sigma} \\ \mathcal{X}_\sigma^{\uparrow} & \longrightarrow & \mathcal{Z}_\sigma \end{array}$$

Given a compatible $(k-1)$-collection $\mathfrak{I}_{(k-1)}$, the compatibilities (1) and (2) allow us to define a map

$$\tilde{f}_\sigma^{\uparrow} : \mathcal{X}_\sigma^{\uparrow} \longrightarrow \mathcal{L}_X$$

for each $\sigma : \Delta^k \to \mathcal{L}_X$ which extends $\tilde{f}_\sigma$, and which agrees with $f_{d_j(\sigma)}$ for each face operator d_j such that $d_j(\sigma)(k-1) = n$.

Lemma 3.2.77. *Let $\mathfrak{I}_{k-1}$ be a compatible $(k-1)$-collection, and suppose that $\sigma : \Delta^k \to \mathcal{L}_X$ is encapsulated with capsule τ. Then for each $\zeta : \Delta^r \to \mathcal{R}_\sigma$ such that ζ hits both $\rho_\sigma(\kappa_\sigma - 1)$ and $\rho_\sigma(\kappa_\sigma)$, then $\tilde{f}_\sigma^{\uparrow}(\zeta)$ is degenerate on those vertices.*

Proof. Note that our assumption means that ζ does *not* factor through σ.

First suppose that ζ factors through $\mathcal{Z}_{d_j(\sigma)}$ for $j \leqslant \kappa_\tau - 1$. Then we note that $d_j(\sigma) \subset \mathsf{B}(d_j(\tau))$, and $\nu_{d_j(\sigma)} = \nu_{\mathsf{B}(d_j(\tau))}$, so by Lemma 3.2.73 and the fact that $\mathcal{I}_{k-1}$ satisfies iii′) we see that $\tilde{f}^{\uparrow}_{\sigma}(\zeta)$ is degenerate. An identical argument holds when $j > \kappa_\sigma$.

If ζ factors through $\mathcal{Z}_{d_j(\sigma)}$ for $\kappa_\tau \leqslant j \leqslant \kappa_\sigma$, then $\sigma(j)$ is not in τ. Thus $\tau \subset d_j(\sigma)$, $d_j(\sigma) \subset \mathsf{B}(\tau)$, and so since $\mathcal{I}_{k-1}$ satisfies condition iii′), ζ is degenerate.

Finally, suppose that ζ factors through $\mathcal{K}^{\sigma}_{I}$ for some $\theta^{\sigma}_{I} \neq \varnothing$. Then $I \cap [s] = J$ has $\theta^{\tau}_{J} \neq \varnothing$, and we can factor $\tilde{f}^{\uparrow}_{\sigma}(\zeta)$ through u as

$$\Delta^r \longrightarrow \Delta^I \times \Delta^{[\sigma(0),n]} \longrightarrow \Delta^n \times X_n$$

By construction, the first factor of this simplex is degenerate at the desired vertex. The second factor can be equivalently factored through $\Delta^J \times \Delta^{[\tau(0),n]}$ and thus is degenerate at the desired vertex as well. Thus $\tilde{f}^{\uparrow}_{\sigma}(\zeta)$ is degenerate. □

Corollary 3.2.78. *Let $\mathcal{I}_{k-1}$ be a compatible $(k-1)$-collection. Suppose that $\sigma : \Delta^k \to \mathcal{L}_X$ is encapsulated, and let τ be the capsule for σ. Then there is a subsimplex $\gamma \subset \mathsf{B}(\tau)$ and a degeneracy operator s_α such that the diagram*

$$\begin{array}{ccc} \mathcal{R}_\sigma & \longrightarrow & \Delta^{k+\ell} \\ \downarrow & & \downarrow{\scriptstyle s_\alpha(\gamma)} \\ \mathcal{X}^{\uparrow}_{\sigma} & \xrightarrow[\tilde{f}^{\uparrow}_{\sigma}]{} & \mathcal{L}_X \end{array}$$

commutes.

With this corollary in hand we can now return to Proposition 3.2.70.

Proof (of Proposition 3.2.70). First let us observe that the choices of f_σ for every $\sigma : \Delta^k \to \mathcal{M}_X \subset \mathcal{L}_X$ are already made since $\mathcal{X}_\sigma = \mathcal{Z}_\sigma$. It is also easy to check that the rest of the conditions hold for those choices. Therefore we can restrict our attention to producing the choices for simplices $\sigma : \Delta^k \to \mathcal{L}_X$ that do not factor through $\mathcal{M}_X$.

We will inductively compatible k-collections $\mathcal{I}_k$ for every $k \geqslant 1$. Before commencing our argument we will make a preliminary definition. Given $\sigma : \Delta^k \to \mathcal{L}_X$ we define $\mathcal{Y}^{\uparrow}_{\sigma}$ to be the subsimplicial subset (with the inherited decorations) of $\mathcal{Z}_\sigma$ whose simplices are those of $\mathcal{X}^{\uparrow}_{\sigma}$ in addition to the simplex $\mathsf{B}(\sigma)$. It follows from the argument given in Lemma 3.2.66 that the inclusion $\mathcal{Y}^{\uparrow}_{\sigma} \to \mathcal{Z}_\sigma$ is **MB**-anodyne.

For every $e : \Delta^1 \to \mathcal{L}_X$ we fix the choice of f_e which is guaranteed by Lemma 3.2.66. In this ground case, there are no conditions to check. Let us consider a triangle $\sigma : \Delta^2 \to \mathcal{L}_X$. Using the previous choices the can extend the map $\tilde{f}_\sigma$ to a morphism

$$f^{\uparrow}_{\sigma} \colon \mathcal{X}^{\uparrow}_{\sigma} \longrightarrow \mathcal{L}_X$$

We distinguish now two cases. Suppose that σ is not contained in some $\mathsf{B}(e)$ for $e : \Delta^1 \to \mathcal{L}_X$. Then we define f_σ to be an extension of $f^{\uparrow}_{\sigma}$ to $\mathcal{Z}_\sigma$. If $\sigma \subseteq \mathsf{B}(e)$ we can assume that $e \subset \sigma$ since otherwise we have $\sigma \in \mathcal{M}_X$. We extend $f^{\uparrow}_{\sigma}$ to a map $\mathcal{Y}^{\uparrow}_{\sigma} \to \mathcal{L}_X$ by sending $\mathsf{B}(\sigma)$ to the following simplex: Let $\sigma_e : \Delta^r \to \mathcal{L}_X$ be the simplex obtained by forgetting every vertex j in $\mathsf{B}(e)$such that $j \leqslant \kappa_{\overline{\sigma}} - 1$ and such that is not in σ. We can now map $\mathsf{B}(\sigma)$ to $s_\alpha(\sigma_e)$ where $\alpha = \kappa_{\overline{\sigma}} - 1$

and consequently condition *iii*) is satisfied. This means that we can construct a compatible 1-collection $\mathfrak{I}_1$.

Now suppose we have a compatible $(k-1)$-collection $\mathfrak{I}_{k-1}$. Let $\sigma : \Delta^k \to \mathcal{L}_X$ be a simplex. If σ is not encapsulated, then we may define f_σ by solving the lifting problem

$$\begin{array}{ccc} \mathcal{X}_\sigma^{\uparrow} & \xrightarrow{\tilde{f}_\sigma^{\uparrow}} & \mathcal{L}_X \\ \downarrow & & \\ \mathcal{Z}_\sigma & & \end{array}$$

using 3.2.66. If σ is encapsulated with capsule τ, we can use Corollary 3.2.78 to define a map

$$\mathcal{Y}_\sigma^{\uparrow} \longrightarrow \mathcal{L}_X$$

which sends $\mathsf{B}(\sigma)$ to the degenerate simplex described in Corollary 3.2.78. Solving the corresponding lifting problem yields an f_σ satisfying i), ii), and iii). Thus, we can extend $\mathfrak{I}_{k-1}$ to a compatible k-collection, as desired. □

Proposition 3.2.79. *The cofibration $u : \mathcal{M}_X \to \mathcal{L}_X$ is **MB**-anodyne.*

Proof. We say that a simplex $\sigma : \Delta^k \to \mathcal{L}_X$ is *wide* if it is not contained in the image of u. Let $\sigma : \Delta^k \to \mathcal{L}_X$ and recall the definition $\nu_\sigma = |\{j \in [k] \mid \overline{\sigma}(j) = n\}|$. We produce a filtration

$$\mathcal{M}_X \to S^1 \to S^2 \to \cdots \to \mathcal{L}_X$$

where S^ℓ consists of those simplices σ in $\mathcal{L}_X$ that either factor through $\mathcal{M}_X$ or they satisfy $\nu_\sigma \leqslant \ell$. We will fix the convention $S^0 = \mathcal{M}_X$. We will show that each step in the filtration is **MB**-anodyne. Let us fix once and for all a choice of $f_\sigma : \mathcal{Z}_\sigma \to \mathcal{L}_X$ for every $\sigma : \Delta^k \to \mathcal{L}_X$ with the properties listed in Proposition 3.2.70. First, let us observe that given $\sigma : \Delta^k \to S^\ell$ it follows that the morphisms f_σ also factor through S^ℓ. We can now define $S^{(\ell,s)}$ to consist in those simplices contained in S^ℓ in addition to the simplices $B(\sigma)$ for $\sigma : \Delta^k \to \mathcal{L}_X$ wide and non-degenerate, such that $k \leqslant s$ and $\nu_\sigma = \ell + 1$. This produces a filtration

$$S^{\ell-1} \to S^{(\ell-1,\ell)} \to S^{(\ell-1,\ell+1)} \to \cdots \to S^\ell$$

We fix the convention $S^\ell = S^{(\ell,\ell)}$. Let us consider a pullback diagram

$$\begin{array}{ccc} A_\sigma & \longrightarrow & \mathcal{Z}_\sigma \\ \downarrow & & \downarrow{\scriptstyle f_\sigma} \\ S^{(\ell,s-1)} & \longrightarrow & S^{(\ell,s)} \end{array}$$

where $\sigma : \Delta^s \to \mathcal{L}_X$ does not factor through $S^{(\ell,s-1)}$. Then it follows by construction that A_σ contains every simplex of $\mathcal{Z}_\sigma$ except the simplex $\mathsf{B}(\sigma)$. To check that the top horizontal morphism is **MB**-anodyne, it suffices to apply Lemma 3.1.6 with pivot point $\kappa_{\overline{\sigma}}$ after observing that the restriction of $\mathsf{B}(\sigma)$ to A_σ consists precisely in the union of the following $(s+\ell-1)$-dimensional faces:

- The face that misses the vertex j for $0 \leqslant j \leqslant \kappa_{\overline{\sigma}} - 1$. This is because this simplex either factors through $\mathcal{Z}_{d_j(\sigma)}$ or it is contained in $\mathcal{M}_X$.

- The face that misses the vertex j for $\kappa_{\overline{\sigma}} + \ell \leqslant j \leqslant s + \ell$. This is because those faces have strictly smaller parameter $\nu_{d_j(\sigma)}$ if $\nu_\sigma > 1$ or they are already in $\mathcal{M}_X$ if $\nu_\sigma = 1$.

To finish the proof we observe that given $\sigma : \Delta^s \to \mathcal{L}_X$ such that σ factors through $S^{(\ell,s-1)}$ but $\nu_\sigma = \ell + 1$ then it follows by condition *iii)* in Proposition 3.2.70 that $\mathsf{B}(\sigma)$ is already contained in $S^{(\ell,s-1)}$. This together with previous discussion implies that $S^{(\ell,s-1)} \to S^{(\ell,s)}$ is **MB**-anodyne. □

We can distill the key upshot of the preceding technical arguments into a single, simple corollary.

Corollary 3.2.80. *For any 2-Cartesian fibration $X \to \Delta^n_\flat$, the square*

$$\begin{array}{ccc} X_n \times (\Delta^{n-1})^\diamond & \longrightarrow & X_n \times (\Delta^n)^\diamond \\ \downarrow & & \downarrow \\ X|_{\Delta^{n-1}} & \longrightarrow & X \end{array}$$

is homotopy pushout.

The equivalence over a simplex

Having now established the necessary preliminaries, we turn to the proof that the straightening is an equivalence over the minimally-scaled simplex. With few exceptions, the arguments from here on out are standard, and follow the general shape of the analogous arguments given in [Lur09a] and [Lur09b]. We begin with a lemma, which allows us to more easily apply the straightening to our homotopy pushout.

Lemma 3.2.81. *Consider the inclusion $(\Delta^{n-1})^\diamond \to (\Delta^n)^\diamond$ as a morphism in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{\Delta^n_\flat}$. Then for every $0 \leqslant i < n$, the induced morphism*

$$\psi\colon \mathbb{St}_{\Delta^n_\flat}((\Delta^{n-1})^\diamond)(i) \longrightarrow \mathbb{St}_{\Delta^n_\flat}((\Delta^n)^\diamond)(i)$$

is an equivalence of marked-scaled simplicial sets.

Proof. To begin, we examine the morphism on underlying marked simplicial sets. Consider the pushouts

$$\begin{array}{ccc} (\Delta^{n-1})^\diamond & \longrightarrow & ((\Delta^{n-1})^\diamond)^\triangleright \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & X \end{array}$$

and

$$\begin{array}{ccc} (\Delta^n)^\diamond & \longrightarrow & ((\Delta^n)^\diamond)^\triangleright \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

and the induced map

$$\phi\colon \mathfrak{C}^{\mathrm{sc}}[X](i,*) \longrightarrow \mathfrak{C}^{\mathrm{sc}}[Y](i,*)$$

We first note that, since $i < n$, we have that $\mathfrak{C}^{\mathrm{sc}}[X](i,*) = \mathfrak{C}[((\Delta^{n-1})^{\diamond})^{\triangleright}](i,*)$.
From the definition, we then have that

$$\mathfrak{C}^{\mathrm{sc}}[X](i,*) \cong \mathrm{N}(\mathbb{P}(\{i+1,\dots,n-1\}))^{\flat}$$

and

$$\mathfrak{C}^{\mathrm{sc}}[Y](i,*) \cong \mathrm{N}(\mathbb{P}(\{i+1,\dots,n\}))^{\dagger}$$

where † indicates the marking in which precisely the non-degenerate morphisms $S \to S \cup \{n\}$ are marked.

We note that, on underlying marked simplicial sets, this means that ϕ can be identified with the morphism

$$\mathfrak{C}^{\mathrm{sc}}[X](i,*) \times \{0\} \longrightarrow \mathfrak{C}^{\mathrm{sc}}[X](i,*) \times (\Delta^1)^{\sharp}.$$

We will show that this yields an equivalence of marked-scaled simplicial sets by showing that both scalings are equivalent to the maximal scaling.

We claim that the morphisms

$$f_n^i\colon \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^n)^{\diamond})(i) \longrightarrow (\mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^n)^{\diamond})(i))_{\sharp}$$

and

$$g_n^i\colon \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^{n-1})^{\diamond})(i) \longrightarrow (\mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^{n-1})^{\diamond})(i))_{\sharp}$$

are **MS**-anodyne. To show that f_n^i is **MS**-anodyne it suffices to apply Lemma 3.2.14. The argument for g_n^i is similar and left as an exercise. We thus obtain, for any $i < n$ a commutative diagram

$$\begin{array}{ccc} \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^{n-1})^{\diamond})(i) & \xrightarrow{\phi} & \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^{n})^{\diamond})(i) \\ {\scriptstyle g}\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\downarrow{\scriptstyle f} \\ \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^{n-1})^{\diamond})(i)_{\sharp} & \xrightarrow[\sim]{\phi_\sharp} & \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^{n})^{\diamond})(i)_{\sharp} \end{array}$$

Showing that ϕ is an equivalence of marked-scaled simplicial sets by 2-out-of-3. □

Lemma 3.2.82. *Let $X \to \Delta^n_\flat$ be a 2-Cartesian fibration, and denote by X_i the fibre over i. Let $\mathbb{S}\mathrm{t}_*$ denote the straightening over Δ^0. Then the map*

$$\psi_i^X\colon \mathbb{S}\mathrm{t}_*(X_i) \longrightarrow \mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X)(i)$$

is an equivalence of marked-scaled simplicial sets.

Proof. Following [Lur09a, 3.2.3.3], we proceed by induction on n. We have already shown the case $n = 0$ in Corollary 3.2.50

By construction, ψ_n is an isomorphism. For $i < n$, we get a canonical commutative diagram

$$\begin{array}{ccc} & \mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X|_{\Delta^{n-1}})(i) & \\ \nearrow & & \searrow{\scriptstyle\gamma_i} \\ \mathbb{S}\mathrm{t}_*(X_i) & \xrightarrow[\psi_i^X]{} & \mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X)(i) \end{array}$$

We can identify the upper-left map with $\psi_i^{X|_{\Delta^{n-1}}}$, and so by the inductive hypothesis, it is an equivalence. It thus suffices for us to show that γ_i is an equivalence.

By Corollary 3.2.80, we get a homotopy pushout diagram

$$\begin{array}{ccc} X_n \times (\Delta^{n-1})^\diamond & \longrightarrow & X_n \times (\Delta^n)^\diamond \\ \downarrow & & \downarrow \\ X|_{\Delta^{n-1}} & \longrightarrow & X \end{array}$$

in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\Delta^n_\flat}$. Applying the left Quillen functor $\mathbb{S}\mathrm{t}_{\Delta^n_\flat}$ yields a homotopy pushout diagram

$$\begin{array}{ccc} \mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X_n \times (\Delta^{n-1})^\diamond) & \longrightarrow & \mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X_n \times (\Delta^n)^\diamond) \\ \downarrow & & \downarrow \\ \mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X|_{\Delta^{n-1}}) & \xrightarrow[\gamma]{} & \mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X) \end{array}$$

We have a commutative diagram

$$\begin{array}{ccc} \mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X_n \times (\Delta^{n-1})^\diamond) & \longrightarrow & \mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X_n \times (\Delta^n)^\diamond) \\ \downarrow & & \downarrow \\ \mathbb{S}\mathrm{t}_*(X_n) \boxtimes \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^{n-1})^\diamond) & \longrightarrow & \mathbb{S}\mathrm{t}_*(X_n) \boxtimes \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^n)^\diamond) \end{array}$$

where the vertical maps are equivalences of marked-scaled simplicial sets by Theorem 3.2.17. It thus suffices to note that, by Lemma 3.2.81, the induced morphism

$$\psi_i\colon \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^{n-1})^\diamond)(i) \longrightarrow \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^n)^\diamond)(i)$$

is an equivalence for any $i < n$. □

Before continuing, we fix some notation to ease the coming discussion. We will in the following theorem denote the straightening-unstraightening equivalence over the point by

$$S : (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\Delta^0_\flat} \rightleftarrows (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{sc}[\Delta^0_\flat]^{op}} : U$$

Proposition 3.2.83. *The Quillen adjunction*

$$\mathbb{S}\mathrm{t}_{\Delta^n_\flat} : (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\Delta^n_\flat} \rightleftarrows (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{sc}[\Delta^n_\flat]^{op}} : \mathbb{U}\mathrm{n}_{\Delta^n_\flat}$$

is a Quillen equivalence.

Proof. As in [Lur09a, Lem. 3.2.3.2], we see that $\mathbb{U}\mathrm{n}_{\Delta^n_\flat}$ reflects weak equivalences between the images of fibrant objects. It is thus sufficient to show that the derived adjunction unit

$$\mathrm{Id} \Rightarrow \mathsf{R}(\mathbb{U}\mathrm{n}_{\Delta^n_\flat}) \circ \mathbb{S}\mathrm{t}_{\Delta^n_\flat}$$

is an equivalence. Since $\mathsf{R}(\mathbb{U}\mathrm{n}_{\Delta^n_\flat})$ preserves weak equivalences and $\mathbb{S}\mathrm{t}_{\Delta^n_\flat}$ preserves trivial cofibrations, it is sufficient to check this for fibrant objects.

Let $X \to \Delta^n_\flat$ be a 2-Cartesian fibration, and let

$$\mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X) \xrightarrow{\ \sim\ } \mathcal{F}$$

be a fibrant replacement in $(\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[\Delta^n_\flat]^{\mathrm{op}}}$. We are thus left to show that the induced map

$$X \longrightarrow \mathbb{U}\mathrm{n}_{\Delta^n_\flat}(\mathcal{F})$$

is an equivalence in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\Delta^n_\flat}$. Since both objects are fibrant, it suffices to show that this map is a fibrewise equivalence.

We can identify $\mathbb{U}\mathrm{n}_{\Delta^n_\flat}(\mathcal{F})$ with $U(\mathcal{F}(i))$. Using the equivalence of Corollary 3.2.50, we see that the map

$$X_i \longrightarrow U(\mathcal{F}(i))$$

is an equivalence if and only if the adjoint map

$$S(X_i) \longrightarrow \mathcal{F}(i)$$

is an equivalence. However, we can factor this map as

$$\begin{array}{ccccc} & & \mathbb{S}\mathrm{t}_{\Delta^n_\flat}(X)(i) & & \\ & \overset{\psi_i^X}{\nearrow} & & \searrow & \\ S(X_i) & & \longrightarrow & & \mathcal{F}(i) \end{array}$$

The upper-right map is an equivalence since $\mathcal{F}$ was a fibrant replacement, and ψ_i^X is an equivalence by Lemma 3.2.82. The proposition is thus proven. □

Corollary 3.2.84. *Consider the scaled simplicial set* $(\Delta^2)_\sharp := \mathrm{N}^{\mathrm{sc}}([2])$. *Then Quillen adjunction*

$$\mathbb{S}\mathrm{t}_{\Delta^n_\sharp} : (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\Delta^n_\sharp} \rightleftarrows (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[\Delta^n_\sharp]^{\mathrm{op}}} : \mathbb{U}\mathrm{n}_{\Delta^n_\sharp}$$

is a Quillen equivalence.

Proof. The key point to note is that base change along the cofibration

$$(\Delta^2)^\sharp_{\flat\subset\sharp} \to (\Delta^2)^\sharp_{\sharp\subset\sharp}$$

induces a fully faithful inclusion

$$(\mathrm{Set}_\Delta^{\mathbf{mb}})^\circ_{/\Delta^2_\flat} \to (\mathrm{Set}_\Delta^{\mathbf{mb}})^\circ_{/\Delta^2_\sharp}$$

and similarly, composition with the induced map $\mathfrak{C}^{\mathrm{sc}}[\Delta^2_\flat] \to \mathfrak{C}^{\mathrm{sc}}[\Delta^2_\sharp]$ induces a fully faithful inclusion

$$\left((\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[\Delta^2_\sharp]^{\mathrm{op}}}\right)^\circ \to \left((\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[\Delta^2_\flat]^{\mathrm{op}}}\right)^\circ$$

and so we obtain a commutative diagram

$$\begin{array}{ccc} \left((\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[\Delta^2_\sharp]^{\mathrm{op}}}\right)^\circ & \xrightarrow{\mathbb{U}\mathrm{n}} & (\mathrm{Set}_\Delta^{\mathbf{mb}})^\circ_{/\Delta^2_\sharp} \\ \big\downarrow & & \big\downarrow \\ \left((\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[\Delta^2_\flat]^{\mathrm{op}}}\right)^\circ & \xrightarrow[\mathbb{U}\mathrm{n}]{} & (\mathrm{Set}_\Delta^{\mathbf{mb}})^\circ_{/\Delta^2_\flat} \end{array}$$

of simplicial categories.

The remainder of the proof is, *mutatis mutandis*, that of [Lur09b, Prop. 3.8.7]. □

3.2.6 Straightening in general

We now prove the main theorem of this paper.

Theorem 3.2.85. *Let $S \in \mathrm{Set}_\Delta^{\mathbf{sc}}$ be a scaled simplicial set, and let $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$ be an equivalence of Set_Δ^+-enriched categories. The Quillen adjunction*

$$\mathbb{S}\mathrm{t}_\phi : (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S} \rightleftarrows (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathcal{C}^{\mathrm{op}}} : \mathbb{U}\mathrm{n}_\phi$$

is a Quillen equivalence.

Coupled with the fact, discussed immediately hereafter, that $\mathbb{U}\mathrm{n}_\phi$ is a Set_Δ^+-enriched functor, this will immediately imply a stronger result — the functor of Set_Δ^+-enriched categories of fibrant-cofibrant objects induces an equivalence of ∞-bicategories.

The argument from here on out is standard, and follows the same path as [Lur09b, Section 3.8]. Our first aim will be to show that, for any scaled simplicial set S, the functor

$$\mathbb{U}\mathrm{n}_\phi\colon (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}}} \longrightarrow (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}$$

is, in fact, an Set_Δ^+-enriched functor.

The Set_Δ^+-enrichment on $\mathbb{U}\mathrm{n}_\phi$ is given as follows. Let $\mathcal{F}, \mathcal{G} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}_\Delta^{\mathbf{ms}}$ be Set_Δ^+-enriched functors, and $K \in \mathrm{Set}_\Delta^+$. A map

$$K \longrightarrow \mathrm{Map}^+(\mathcal{F}, \mathcal{G})$$

is equivalently a map $\mathcal{F} \otimes K \to \mathcal{G}$, where $(\mathcal{F} \otimes K)(s) := \mathcal{F}(s) \times K_\sharp$. We then have a natural map

$$\mathbb{U}\mathrm{n}_\phi(\mathcal{F}) \times K_{\sharp\subset\sharp} \to \mathbb{U}\mathrm{n}_\phi(\mathcal{F}) \times \mathbb{U}\mathrm{n}_*(K_\sharp)$$

Where the second component is induced by the natural transformation $\alpha : \mathbb{S}\mathrm{t}_* \Rightarrow L$. We can then write down a natural composite map

$$\mathbb{U}\mathrm{n}_\phi(\mathcal{F}) \times K_{\sharp\subset\sharp} \to \mathbb{U}\mathrm{n}_S(\mathcal{F}) \times \mathbb{U}\mathrm{n}_*(K_\sharp) \to \mathbb{U}\mathrm{n}_\phi(\mathcal{F} \otimes K) \to \mathbb{U}\mathrm{n}_\phi(\mathcal{G})$$

Which is equivalently a map $K \to \mathrm{Map}^{\mathrm{th}}(\mathbb{U}\mathrm{n}_S(\mathcal{F}), \mathbb{U}\mathrm{n}_S(\mathcal{G}))$. The naturality guarantees that this defines a map of simplicial sets

$$\mathrm{Map}^+(\mathcal{F}, \mathcal{G}) \to \mathrm{Map}^{\mathrm{th}}(\mathbb{U}\mathrm{n}_\phi(\mathcal{F}), \mathbb{U}\mathrm{n}_\phi(\mathcal{G})).$$

Similarly, since the composition maps in both cases are defined via the diagonal $\Delta^n \to \Delta^n \times \Delta^n$, naturality ensures that this defines an enriched functor. A wholly analogous argument shows that $\mathbb{U}\mathrm{n}_S$ can also be viewed as a simplicially-enriched functor.

Proof (of Theorem 3.2.85). The proof is now nearly identical to that of [Lur09b, Prop. 3.8.4]. The argument hangs on the claim that the functor

$$\begin{aligned} F\colon (\mathrm{Set}_\Delta^{\mathbf{sc}})^{\mathrm{op}} &\longrightarrow \mathrm{Cat}_\Delta \\ S &\longmapsto ((\mathrm{Set}_\Delta^{\mathbf{ms}})_f^{\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}}})[\mathcal{W}_S^{-1}] \end{aligned}$$

sends pushouts along cofibrations to homotopy pullbacks, and sends transfinite composites of cofibrations to homotopy limits, which follows from the argument given in loc. cit. □

Corollary 3.2.86. *Let $S \in \mathrm{Set}_\Delta^{\mathbf{sc}}$ be an ∞-bicategory. The Set_Δ^+-enriched functor $\mathbb{U}\mathrm{n}_S$ induces an equivalence of ∞-bicategories*

$$\mathrm{N}^{\mathrm{sc}}\left(\left((\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}}}\right)^{\circ}\right) \longrightarrow \mathrm{N}^{\mathrm{sc}}\left(\left((\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}\right)^{\circ}\right).$$

Proof. This follows immediately from Theorem 3.2.17, Theorem 3.2.85, and [Lur09a, A.3.1.10]. □

One final step is left: to interpret this result internally to marked-scaled simplicial sets.

Definition 3.2.87. The Set_Δ^+-enrichment on $\mathrm{Set}_\Delta^{\mathbf{ms}}$ equips the full subcategory $(\mathrm{Set}_\Delta^{\mathbf{ms}})^\circ$ of fibrant-cofibrant objects with the structure of a fibrant Set_Δ^+-enriched category. We denote by $\mathbb{B}\mathrm{icat}_\infty := \mathrm{N}^{\mathrm{sc}}((\mathrm{Set}_\Delta^{\mathbf{ms}})^\circ)$ the homotopy-coherent scaled nerve of this Set_Δ^+-category (considered as a scaled simplicial set). We refer to $\mathbb{B}\mathrm{icat}_\infty$ as the *∞-bicategory of ∞-bicategories.*

Similarly, for $S \in \mathrm{Set}_\Delta^{\mathbf{sc}}$, we denote by $2\mathbb{C}\mathrm{art}(S) := \mathrm{N}^{\mathrm{sc}}\left(\left((\mathrm{Set}_\Delta^{\mathbf{mb}})_{/S}\right)^{\circ}\right)$ the *∞-bicategory of 2-Cartesian fibrations over S.*

Remark 3.2.88. Formally, considering $\mathrm{Set}_\Delta^{\mathbf{ms}}$ as the category of all $\mathcal{U}$-small marked-scaled simplicial sets for some Grothendieck universe $\mathcal{U}$, the marked-scaled simplicial set $\mathbb{B}\mathrm{icat}_\infty$ is no longer small. We thus resort to fixing a new Grothendieck universe $\mathcal{V}$ in which $\mathcal{U}$, and thus $\mathbb{B}\mathrm{icat}_\infty$, becomes $\mathcal{V}$-small.

Proposition 3.2.89. *Let $\mathcal{C}$ be a small Set_Δ^+-enriched category, S a small scaled simplicial set, $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$ an equivalence of Set_Δ^+-enriched categories, and $\mathbf{A}$ a combinatorial, Set_Δ^+-enriched model category. Endow $\mathbf{A}^{\mathcal{C}}$ with the projective model structure. Then the functor*

$$\mathrm{N}^{\mathrm{sc}}((\mathbf{A}^{\mathcal{C}})^\circ) \to \mathrm{Fun}(S, \mathrm{N}^{\mathrm{sc}}(\mathbf{A}^\circ))$$

is a bicategorical equivalence of scaled simplicial sets.

Proof. The proof is that of [Lur09a, Prop. 4.2.4.4]. The only thing that changes is the exchange of Set_Δ for $\mathrm{Set}_\Delta^{\mathbf{ms}}$, and as both of these are excellent model categories, no further emendation is necessary. □

Corollary 3.2.90. *Let $S \in \mathrm{Set}_\Delta^{\mathbf{sc}}$. There is an equivalence of ∞-bicategories*

$$2\mathbb{C}\mathrm{art}(S) \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathbb{B}\mathrm{icat}_\infty).$$

3.3 The Relative 2-Nerve

There is a special case of most ∞-categorical Grothendieck constructions in which the computation of the right adjoints can be greatly simplified. When the base is suitably strict, it is possible to define a *relative nerve*, which computes the Grothendieck construction of a functor. The aim of this appendix is to provide a relative nerve construction which takes as input a Set_Δ^+-enriched functor

$$F\colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}_\Delta^{\mathbf{ms}}$$

and yields as output a 2-Cartesian fibration $\chi_{\mathbb{C}}(F) \to \mathrm{N}^{\mathrm{sc}}(\mathbb{C})$. In form, this relative nerve will actually seem slightly *more* complicated than the associated straightening functor. However, it will enable us to more easily make the comparison with the strict 2-categorical relative nerve construction of [Buc14]. The particular virtue of our relative 2-nerve construction in this regard is that, given a strict 2-functor

$$F\colon \mathbb{C}^{\mathrm{op}} \longrightarrow 2\mathrm{Cat},$$

we can compute the relative 2-nerve in terms of strict 2-functors into $\mathbb{C}$ and $F(x)$, without first passing to simplicial sets.

In our previous papers [AGDS20] and [AGS22], we defined two variants of the relative 2-nerve, which provided ∞-bicategories fibred in $(\infty,1)$-categories. In this section, we will upgrade the later of these constructions to provide the desired $\chi_{\mathbb{C}}$.

Remark 3.3.1. Our choice of notation $\chi_{\mathbb{C}}$ for the relative 2-nerve of a functor $F : \mathbb{C}^{\mathrm{op}} \to \mathrm{Set}_\Delta^{\mathbf{ms}}$ does in fact collide with the choice of notation in [AGDS20] and [AGS22]. An ideal choice of notation would involved a superscript $\chi_{\mathbb{C}}^{\varepsilon}$ where ε denotes one of the four variances for bicategorical fibrations. We will use this rather abusive notation to improve readibility since we will only consider the *outer Cartesian* variance.

Definition 3.3.2. Given a totally ordered set I, the 2-category $\mathbb{O}^I_{i\nearrow}$ has

- Objects given by subsets $S \subseteq I$ such that $\min(S) = i$.
- Each mapping category $\mathbb{O}^I_{i\nearrow}(S,T)$ is a poset whose objects $\mathcal{U}\colon S \longrightarrow T$ are given by subsets $\mathcal{U} \subseteq I$ such that

  $$\min(\mathcal{U}) = \max(S), \;\; \max(\mathcal{U}) = \max(T), \;\; S \cup \mathcal{U} \subseteq T,$$

  ordered by inclusion.
- Composition is given by union.

These lax slice categories piece together into a 2-functor

$$\begin{aligned}(\mathbb{O}^I)^{\mathrm{op}} &\longrightarrow 2\mathrm{Cat}\\ i &\longmapsto \mathbb{O}^I_{i\nearrow}\end{aligned}$$

so that, in particular for any $J \subset I$ with $i = \min(I)$ and $j = \min(J)$, we have 2-functors

$$\omega_{I,J}\colon \mathbb{O}^I(i,j) \times \mathbb{O}^J_{j\nearrow} \longrightarrow \mathbb{O}^I_{i\nearrow}$$

given on objects by the union of sets. It is an easy check that these functors are injective on objects, 1-morphisms, and 2-morphisms.

The 2-categories $\mathbb{O}^I_{i\nearrow}$ play a central role in our relative nerve construction.

Construction 3.3.3. Let

$$F\colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}_{\Delta}^{\mathbf{ms}}$$

be a $\mathrm{Set}_{\Delta}^{+}$-enriched functor. We define a marked-biscaled simplicial set $\chi_{\mathbb{C}}(F)$ as follows. An n-simplex $\Delta^n \to \chi_{\mathbb{C}}(F)$ consists of

- A simplex $\sigma : \Delta^n_\flat \to \mathrm{N}^{\mathrm{sc}}(\mathbb{C})$.
- For every $\varnothing \neq I \subset [n]$ with $\min(I) = i$, a map of marked-scaled simplicial sets
$$\theta_I\colon\ \mathrm{N}^{\mathrm{ms}}(\mathbb{O}^I_{i\nearrow})^\flat \longrightarrow F(\sigma(i))$$
such that, for every $\varnothing \neq J \subset I \subset [n]$ with $\min(J) = j$ and $\min(i) = i$, the diagram
$$\begin{array}{ccc} \mathrm{N}^{\mathrm{ms}}(\mathbb{O}^I(i,j)) \times \mathrm{N}^{\mathrm{ms}}(\mathbb{O}^J_{j\nearrow}) & \longrightarrow & \mathrm{N}^{\mathrm{ms}}(\mathbb{O}^I_{i\nearrow}) \\ \Big\downarrow{\scriptstyle \mathrm{N}(\sigma)\times\theta_J} & & \Big\downarrow{\scriptstyle \theta_I} \\ \mathrm{N}^{\mathrm{ms}}(\mathbb{C}(\sigma(i),\sigma(j))) \times F(\sigma(j)) & \xrightarrow[F(-)]{} & F(\sigma(i)) \end{array}$$
commutes.

We then define markings and scalings on $\rho_{\mathbb{C}}(F)$.

- A 1-simplex $\Delta^1 \to \chi_{\mathbb{C}}(F)$ is marked if the corresponding map $\theta_{[1]}$: $\mathrm{N}^{\mathrm{ms}}(\mathbb{O}^1_{0\nearrow}) \to F(\sigma(i))$ descends to a map
$$\mathrm{N}^{\mathrm{ms}}(\mathbb{O}^1_{0\nearrow})^\sharp \longrightarrow F(\sigma(0)).$$
- A 2-simplex $\Delta^2 \to \chi_{\mathbb{C}}(F)$ is lean if the corresponding map
$$\mathrm{N}^{\mathrm{ms}}(\mathbb{O}^2_{0\nearrow}) \longrightarrow F(\sigma(0))$$
descends to a map
$$\mathrm{N}^{\mathrm{ms}}(\mathbb{O}^2_{0\nearrow})_\sharp \longrightarrow F(\sigma(0)).$$
- A 2-simplex $\Delta^2 \to \chi_{\mathbb{C}}(F)$ is thin if and only if it is lean *and* the corresponding 2-simplex $\sigma : \Delta^2 \to \mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ is thin.

Note that there is a canonical forgetful functor

$$\begin{aligned} \chi_{\mathbb{C}}(F) &\longrightarrow \mathrm{N}^{\mathrm{sc}}(\mathbb{C}) \\ (\sigma, \{\theta_I\}) &\longmapsto \sigma \end{aligned}$$

which sends thin triangles to thin triangles.

Definition 3.3.4. The *relative bicategorical nerve* over a 2-category $\mathbb{C}$ is the functor

$$\begin{aligned} \chi_{\mathbb{C}}\colon\ (\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{C}^{\mathrm{op}}} &\longrightarrow (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/\,\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \\ F &\longmapsto \chi_{\mathbb{C}}(F) \end{aligned}$$

By the adjoint functor theorem, $\chi_{\mathbb{C}}$ admits a left adjoint, which we will denote by

$$\phi_{\mathbb{C}}\colon (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/\,\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \longrightarrow (\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{C}^{\mathrm{op}}}.$$

Lemma 3.3.5. *The functor* $\chi_{\mathbb{C}}$ *preserves trivial fibrations.*

Proof. We need only check that the lifting problems

$$\begin{array}{ccc} A & \longrightarrow & \chi_{\mathbb{C}}(F) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \chi_{\mathbb{C}}(\mu)} \\ B & \longrightarrow & \chi_{\mathbb{C}}(G) \end{array}$$

have solutions when $\mu : F \Rightarrow G$ is a projective (pointwise) trivial fibration and $f : A \to B$ is a generating cofibration of marked-biscaled simplicial sets. The proof is virtually identical to the proof of [AGS22, Prop. 3.0.11]. □

Corollary 3.3.6. *The functor* $\phi_{\mathbb{C}}$ *preserves cofibrations.*

3.3.1 Identifying $\phi_{\mathbb{O}^n}$

Let $\mathbb{C}$ be a 2-category. Then we can define a 2-functor

$$\mathbb{C}^{\mathrm{op}} \longrightarrow 2\mathrm{Cat}, \ c \longmapsto \mathbb{C}_{c\nearrow}$$

that maps a 1-morphism $f : c \to d$ to the functor $f^* : \mathbb{C}_{d\nearrow} \to \mathbb{C}_{c\nearrow}$ given by precomposition with f. It easy to verify that given a 2-morphism $\alpha : f \Rightarrow g$ we can construct a natural transformation $f^* \Rightarrow g^*$ whose component at an object $u : d \to x$ is given by $\alpha * u$. Passing to Set_Δ^+-enriched categories we thus obtain, for any strict 2-category $\mathbb{C}$, a Set_Δ^+-enriched functor

$$\mathbb{C}_{-\nearrow} : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}_\Delta^{\mathbf{ms}}, \ c \longmapsto \mathrm{N}^{\mathrm{ms}}(\mathbb{C}_{c\nearrow})$$

Definition 3.3.7. In the particular case where $\mathbb{C} = \mathbb{O}^n$, we will denote the functor constructed above by

$$\mathfrak{O}^n : (\mathbb{O}^n)^{\mathrm{op}} \longrightarrow \mathrm{Set}_\Delta^{\mathbf{ms}}$$

Notation. The canonical normal lax functor $\xi_n : [n] \to \mathbb{O}^n$ gives rise to an inclusion of scaled simplicial sets which we denote by

$$p_n : \Delta^n_\flat \longrightarrow \mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n).$$

We will equip Δ^n with the minimal marking and lean scaling, and conventionally view p_n as an object in $\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/\,\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n)}$.

Lemma 3.3.8. *Let* $F : (\mathbb{O}^n)^{\mathrm{op}} \to \mathrm{Set}_\Delta^{\mathbf{ms}}$ *be a* Set_Δ^+*-enriched functor. There is a natural bijection*

$$\mathrm{Nat}_{\mathbb{C}^{\mathrm{op}}}(\mathfrak{O}^n, F) \cong \mathrm{Hom}_{\left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/\,\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n)}}(p_n, \chi_{\mathbb{O}}(F)).$$

Consequently, we have an equivalence of Set_Δ^+*-enriched functors* $\phi_{\mathbb{O}^n}(p_n) \cong \mathfrak{O}^n$.

Proof. Follows immediately from unwinding the definitions. □

Corollary 3.3.9. *Denote by*

$$p_1^\sharp : (\Delta^1)^\sharp \longrightarrow \mathrm{N}^{\mathrm{sc}}(\mathbb{O}^1),$$

$$(p_2)_{\flat\subset\sharp} \colon (\Delta^2)^{\flat}_{\flat\subset\sharp} \longrightarrow \mathrm{N}^{\mathrm{sc}}(\mathbb{O}^2), \ \textit{and}$$

$$(p_2)_{\sharp} \colon (\Delta^2)^{\flat}_{\sharp} \longrightarrow \mathrm{N}^{\mathrm{sc}}(\mathbb{O}^2)_{\sharp}$$

the obvious decorated versions of the p_n. Then

$$\oint_{\mathbb{O}^1}(p_1^{\sharp}) \colon (\mathbb{O}^1)^{\mathrm{op}} \longrightarrow (\mathrm{Set}_{\Delta}^{\mathbf{ms}}), \ i \longmapsto \mathrm{N}^{\mathrm{ms}}(\mathbb{O}^1_{i\nearrow})^{\sharp}$$

$$\oint_{\mathbb{O}^2}((p_2)_{\flat\subset\sharp}) \colon (\mathbb{O}^2)^{\mathrm{op}} \longrightarrow (\mathrm{Set}_{\Delta}^{\mathbf{ms}}), \ i \longmapsto \mathrm{N}^{\mathrm{ms}}(\mathbb{O}^2_{i\nearrow})_{\sharp}, \ \textit{and}$$

$$\oint_{\mathbb{O}^2}((p_2)_{\flat\subset\sharp}) \colon (\mathbb{O}^2)^{\mathrm{op}}_{\sharp} \longrightarrow (\mathrm{Set}_{\Delta}^{\mathbf{ms}}), \ i \longmapsto \mathrm{N}^{\mathrm{ms}}(\mathbb{O}^2_{i\nearrow})^{\dagger}_{\sharp}$$

where † denotes the marking in which the unique morphism $02 \to 012$ *is marked.*

Proof. All identifications except the last are immediate from the definitions. The additional marking in the final case follows from the necessity that the functor have source $\mathbb{O}^2_{\sharp}$. □

Notation. We will denote the three functors above by $(\mathfrak{O}^1)^{\sharp}$, $(\mathfrak{O}^2)_{\flat\subset\sharp}$, and $(\mathfrak{O}_2)_{\sharp}$, respectively.

3.3.2 Identifying $\mathbb{S}\mathrm{t}_{\mathbb{O}^n}$

Our comparison will be with a very specific version of the straightening functor:

Notation. For a 2-category $\mathbb{C}$, we view $\mathbb{C}$ as a $\mathrm{Set}_{\Delta}^{+}$-enriched category. The counit $\varepsilon_{\mathbb{C}} : \mathfrak{C}^{\mathrm{sc}}(\mathrm{N}^{\mathrm{sc}}(\mathbb{C})) \to \mathbb{C}$ is an equivalence of $\mathrm{Set}_{\Delta}^{+}$-enriched categories. We will denote by

$$\mathbb{S}\mathrm{t}_{\mathbb{C}} \colon (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \longrightarrow (\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{C}^{\mathrm{op}}}$$

the relative straightening functor $\mathbb{S}\mathrm{t}_{\varepsilon_{\mathbb{C}}}$.

We now unravel the definitions to characterize $\mathbb{S}\mathrm{t}_{\mathbb{O}^n}(p_n)$. By construction, the underlying functor to $\mathrm{Set}_{\Delta}^{+}$ is given by the Yoneda embedding on the $\mathrm{Set}_{\Delta}^{+}$-enriched category

$$\mathbb{O}^n \coprod_{\mathfrak{C}^{\mathrm{sc}}(\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n))} \mathfrak{C}^{\mathrm{sc}}(\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n)) \coprod_{\mathfrak{C}^{\mathrm{sc}}(\Delta^n_{\flat})} \mathfrak{C}^{\mathrm{sc}}((\Delta^n_{\flat})^{\triangleright})$$

We note that $\mathbb{O}^n = \mathfrak{C}^{\mathrm{sc}}(\Delta^n_{\flat})$, and by the triangle identities for the adjunction $\mathfrak{C}^{\mathrm{sc}} \dashv \mathrm{N}^{\mathrm{sc}}$, we see that the induced map

$$\mathfrak{C}^{\mathrm{sc}}(\Delta^n_{\flat}) \to \mathbb{O}^n$$

is simply the identity. The pushout above thus collapses to simply $\mathfrak{C}^{\mathrm{sc}}((\Delta^n_{\flat})^{\triangleright})$. We can then describe the marked-scaled simplicial set $\mathbb{S}\mathrm{t}_{\mathbb{O}^n}(\Delta^n_{\flat})(i)$ as the poset $\mathcal{L}^n_{\flat}(i)$ described in Definition 3.2.11. To ease the notation let us denote $\mathcal{L}^n_{\flat}(i)$ simply by $\mathcal{L}^n_i$.

Construction 3.3.10. We construct a morphism of marked-scaled simpliial sets $\eta^n_i : \mathcal{L}^n_i \to \mathrm{N}^{\mathrm{ms}}(\mathbb{O}^n_{i\nearrow})$ whose underlying map of simplicial sets is given by (the nerve of) a normal lax functor defined as follows:

- On objects, $S \mapsto S$.
- On morphisms $S \subset T$ is sent to the mophism $\{\max(S), \max(T)\} : S \to T$.

The fact that, for $S \subset T \subset V$, we have

$$\{\max(S), \max(V)\} \subset \{\max(S), \max(T), \max(V)\}$$

gives us our compositors. The fact that if $S = T$, we have $\{\max(S), \max(T)\} = \{\max(S)\}$ gives strict unitality. Since both marked-scaled simplicial sets carry the minimal marking we only need to check that η_i^n preserves the scaling. Let $S_0 \subset S_1 \subset S_2$ be a 2-simplex in the source. If there are i, j such that $\max(S_i) = \max(S_j)$, then it follows immediately that $\{\max(S_0), \max(S_1), \max(S_2)\} = \{\max(S_0), \max(S_2)\}$.

The following lemma follows immediately from our definitions.

Lemma 3.3.11. *The maps η_i^n define natural transformations of Set_Δ^+-enriched functors $\eta^n : \mathbb{S}\mathrm{t}_{\mathbb{O}^n}(\Delta_\flat^n) \to \mathfrak{O}^n$.*

Proposition 3.3.12. *The morphisms $\eta_i^n : \mathbb{S}\mathrm{t}_{\mathbb{O}^n}(\Delta_\flat^n)(i) \to \mathfrak{O}^n(i)$ are equivalences of marked-scaled simplicial sets.*

We will prove this proposition in a series of lemmata. Since both simplicial sets are equipped with the minimal marking, it suffices to show that the map is an equivalence on underlying scaled simplicial sets by Theorem 2.4.13. Since $\mathfrak{O}^n(i) = \mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n_{i\nearrow})$, it suffices to show that the induced map

$$\xi_i^n : \mathfrak{C}^{\mathrm{sc}}[(\mathbb{S}\mathrm{t}_{\mathbb{O}^n}(\Delta_\flat^n))(i)] \longrightarrow \mathbb{O}^n_{i\nearrow}$$

is an equivalence of Set_Δ^+-enriched categories. Since this map is clearly bijective on objects, it suffices to check that the induced morphisms on mapping spaces are equivalences.

In both cases, the mapping spaces are nerves of posets.

- For $S, T \in \mathbb{O}^n_{i\nearrow}$, the mapping space $\mathfrak{O}_i^n(S,T)$ is the poset of chains $U \subset [n]$ such that $\min(U) = \max(S)$, $\max(U) = \max(T)$, and $S \cup U \subset T$. Equivalently, this is the poset $\mathbb{O}^T(\max(S), \max(T))$, equipped with the minimal marking.
- For $S, T \in Q_i^n$, the mapping space $\mathfrak{C}^{\mathrm{sc}}[(\mathbb{S}\mathrm{t}_{\mathbb{O}^n}(\Delta_\flat^n))(i)](S,T)$ is the poset of chains
$$S \subset S_1 \subset \cdots \subset S_k \subset T$$
in $\mathcal{L}_i^n$. An inclusion $\vec{S} \subset \vec{U}$ is marked if and only if, for every $S_i \subset S_{i+1}$ in $\vec{S}$, every $U_\ell \in \vec{U}$ between S_i and S_{i+1}, either $\max(U_\ell) = \max(S_i)$ or $\max(U_\ell) = \max(S_{i+1})$. Notice that $T \setminus S$ could have elements lower than $\max(S)$.

The map

$$\xi_i^n : \mathfrak{C}^{\mathrm{sc}}[(\mathbb{S}\mathrm{t}_{\mathbb{O}^n}(\Delta_\flat^n))(i)](S,T) \to \mathfrak{O}_i^n(S,T)$$

sends a chain $S \subset S_1 \subset \cdots \subset S_k \subset T$ to the chain

$$\xi_i^n(\vec{S}) = \bigcup_{V \in \vec{S}} \{\max(V)\}.$$

Definition 3.3.13. For ease of notation, we define

$$\mathbb{L}_i^n := \mathfrak{C}^{\mathrm{sc}}[(\mathbb{St}_{\mathbb{O}^n}(\Delta_\flat^n))(i)]$$

For any $S, T \in \mathbb{L}_i^n$, we define a full subposet $\mathbb{B}_i^n(S,T) \subset \mathbb{L}_i^n(S,T)$ consisting of chains

$$S \subset [T, s_1] \subset \cdots \subset [T, s_k] \subset T$$

where we define for every $s \in T$ the subset $[T,s] = \{r \in T \mid r \leqslant s\}$.

Lemma 3.3.14. *An inclusion $\vec{S} \subset \vec{U}$ represents a marked morphism in $\mathbb{L}_i^n(S,T)$ if and only if its image under ξ_i^n is degenerate.*

Proof. Immediate from the definition. □

Lemma 3.3.15. *The restriction of the map ξ_i^n*

$$\xi_i^n \colon \mathbb{B}_i^n(S,T) \longrightarrow \mathbb{O}^T(\max(S), \max(T))$$

is an equivalence of marked simplicial sets.

Proof. We define a map

$$\gamma \colon \mathbb{O}^T(\max(S), \max(T)) \longrightarrow \mathbb{B}_i^n(S,T)$$

which sends $\max(S) < s_1 < \cdots < s_k < \max(T)$ to the chain

$$S \subset [T, s_1] \subset \cdots \subset [T, s_k] \subset T.$$

We then note that $\xi_i^n \circ \gamma = \mathrm{id}$. We claim that $\gamma \circ \xi_i^n \leqslant \mathrm{id}$, which yields a marked homotopy $\gamma \circ \xi_i^n$ to id. To prove the claim we note that $\gamma \circ \xi_i^n(\vec{S})$ is given by $\vec{S}$ if $S_1 \neq [T, s_0]$ with $s_0 = \max(S)$ or by $\vec{S} \setminus [T, s_0]$ in which case the existence of the marked morphism $\gamma \circ \xi_i^n(\vec{S}) \to \vec{S}$ follows immediately. □

Lemma 3.3.16. *The inclusion $\iota : \mathbb{B}_i^n(S,T) \to \mathbb{L}_i^n(S,T)$ is an equivalence of marked simplicial sets.*

Proof. Let $s_j \in T$. We define $\mathbb{L}^s \subset \mathbb{L}$ as the full subposet consisting of those chains

$$\vec{S} = S \subset S_1 \subset \cdots \subset S_k \subset T,$$

such that $S_i = [T, s_i]$ whenever $s_i \geqslant s_j$. Note that if $s_j \leqslant s_0 = \max(S)$ then it follows that $\mathbb{L}^s = \mathbb{B}$. Let $T_S = \{s_j \in T \mid s_j \geqslant s_0\}$ and consider a filtration

$$\mathbb{B} = \mathbb{L}^{s_0} \subset \mathbb{L}^{s_1} \subset \cdots \subset \mathbb{L}^{s_m} \subset \mathbb{L}^{s_m+1} = \mathbb{L}, \quad \text{with } s_m = \max(T)$$

Our goal is to show that each step in the filtration is a weak equivalence of marked simplicial sets. We denote by $\iota_j : \mathbb{L}^j \to \mathbb{L}^{j+1}$ for $j = 0, \ldots, s_m$. Let $\vec{S} = S \subset S_1 \subset \cdots \subset S_k \subset T$ be an object of $\mathbb{L}^{j+1}$ we construct a new chain $\pi_j(\vec{S})$ by replacing each S_ℓ with $s_\ell \geqslant s_j$ with its corresponding $[T, s_\ell]$. This definition yields a functor

$$\pi_j \colon \mathbb{L}^{j+1} \longrightarrow \mathbb{L}^j, \; \vec{S} \longmapsto \pi_j(\vec{S})$$

such that $\pi_j \circ \iota_j = \mathrm{id}$. Let $\zeta_j = \iota_j \circ \pi_j$. We construct a functor

$$\theta_j \colon \mathbb{L}^{j+1} \longrightarrow \mathbb{L}^{j+1}$$

that appends to each chain $\vec{S} \in \mathbb{L}^{j+1}$ the object $[T, s_j]$ if there exists some $S_\ell \in \vec{S}$ such that $\max(S_\ell) = s_j$ or leaves the chain untouched otherwise. Note that if $s_j = s_m$ then this functor is the identity. We also observe that we have a natural transformation $\mathrm{id} \leqslant \theta_j$ and $\zeta_j \leqslant \theta_j$ whose components are marked. It follows that each ι_j is a weak equivalence and consequently so is ι.

□

Proof (of Proposition 3.3.12). We simply apply Lemma 3.3.16, Lemma 3.3.15, and 2-out-of-3. □

Turning now to the cases $(p_1)^\sharp$, $(p_2)_{\flat\subset\sharp}$, and $(p_2)_\sharp$, we see that the corresponding straightenings are obtained from $\mathbb{S}\mathrm{t}_{\mathbb{O}^1}(p_1)$ and $\mathbb{S}\mathrm{t}_{\mathbb{O}^2}(p_2)$ by maximally marking or maximally scaling the values of the functors, respectively. We then have the following

Corollary 3.3.17. *The transformations ξ^n, $n = 1, 2$ induce equivalences of enriched functors*

$$(\xi^1)^\sharp \colon \mathbb{S}\mathrm{t}_{\mathbb{O}^1}(p_1^\sharp) \longrightarrow (\mathfrak{O}^1)^\sharp,$$
$$(\xi^2)_{\flat\subset\sharp} \colon \mathbb{S}\mathrm{t}_{\mathbb{O}^2}((p_2)_{\flat\subset\sharp}) \longrightarrow (\mathfrak{O}^2)_{\flat\subset\sharp}, \quad \textit{and}$$
$$(\xi^2)_\sharp \colon \mathbb{S}\mathrm{t}_{\mathbb{O}^2_\sharp}((p_2)_\sharp) \longrightarrow (\mathfrak{O}^2)_\sharp$$

Proof. The morphism $(\xi^1)^\sharp$ is an isomorphism, and it is a quick check to extend the previous arguments to cover the case $(\xi^2)_{\flat\subset\sharp}$. One then notes that, for each $i \in \mathbb{O}^2$, the i-component of $(\xi^2)_\sharp$ is a pushout of the i-component of $(\xi^2)_{\flat\subset\sharp}$ along the inclusion $(\Delta^1)^\flat \to (\Delta^1)^\sharp$, and thus is an equivalence. □

Remark 3.3.18. As in [AGDS20, Prop. 4.1.1] any 2-functor $f : \mathbb{C} \to \mathbb{D}$ yields diagrams

$$\begin{array}{ccc} (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathbb{D}^{\mathrm{op}}} & \xrightarrow{f^*} & (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathbb{C}^{\mathrm{op}}} \\ {\scriptstyle \mathbb{\chi}_{\mathbb{D}}}\downarrow & & \downarrow{\scriptstyle \mathbb{\chi}_{\mathbb{C}}} \\ (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\,\mathrm{N}^{\mathrm{sc}}(\mathbb{D})} & \xrightarrow[\mathrm{N}^{\mathrm{sc}}(f)^*]{} & (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\,\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \end{array}$$

and

$$\begin{array}{ccc} (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathbb{D}^{\mathrm{op}}} & \xleftarrow{f_!} & (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathbb{C}^{\mathrm{op}}} \\ {\scriptstyle \mathbb{\phi}_{\mathbb{D}}}\uparrow & & \uparrow{\scriptstyle \mathbb{\phi}_{\mathbb{C}}} \\ (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\,\mathrm{N}^{\mathrm{sc}}(\mathbb{D})} & \xleftarrow[\mathrm{N}^{\mathrm{sc}}(f)_!]{} & (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\,\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \end{array}$$

which commute up to natural isomorphism.

Theorem 3.3.19. *There exists a unique family of natural weak equivalences*

$$\xi^{\mathbb{C}}(X) \colon \mathbb{S}\mathrm{t}_{\mathbb{C}} \Longrightarrow \mathbb{\phi}_{\mathbb{C}}$$

indexed by pairs $(\mathbb{C}, X)$ consisting of a 2-category $\mathbb{C}$ and $X \in (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\,\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}$ with the following properties.

1. *On the maps p_n for $n \geqslant 0$, $p_1^\sharp$, $(p_2)_{\flat\subset\sharp}$, and $(p_2)_\sharp$, the transformations $\xi^{\mathbb{C}}(X)$ coincide with the transformations ξ^n from Proposition 3.3.12 and Corollary 3.3.17.*

2. *For every map $g : X \to Y$ in $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}$, the diagram*

$$\begin{array}{ccc} \mathbb{S}\mathrm{t}_{\mathbb{C}}(X) & \xrightarrow{\xi^{\mathbb{C}}(X)} & \mathbb{\phi}_{\mathbb{C}}(X) \\ {\scriptstyle \mathbb{S}\mathrm{t}_{\mathbb{C}}(g)}\downarrow & & \downarrow{\scriptstyle \mathbb{\phi}_{\mathbb{C}}(g)} \\ \mathbb{S}\mathrm{t}_{\mathbb{C}}(Y) & \xrightarrow[\xi^{\mathbb{C}}(Y)]{} & \mathbb{\phi}_{\mathbb{C}}(Y) \end{array}$$

commutes

3. *For every 2-functor $f : \mathbb{C} \to \mathbb{D}$, the diagram*

$$\begin{array}{ccc} f_!\mathbb{S}\mathrm{t}_{\mathbb{C}}(X) & \xrightarrow{f_!\xi^{\mathbb{C}}(X)} & f_!\mathbb{\phi}_{\mathbb{C}}(X) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \mathbb{S}\mathrm{t}_{\mathbb{D}}(f_!X) & \xrightarrow[\eta_{\mathbb{C}}(f_!X)]{} & \mathbb{\phi}_{\mathbb{C}}(f_!X) \end{array}$$

commutes.

Proof. This is identical to the proofs of [AGDS20, Prop 4.3.1 and 4.3.1]. □

Corollary 3.3.20. *The adjunction*

$$\mathbb{\phi}_{\mathbb{C}} : (\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\,\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \rightleftarrows (\mathrm{Set}_\Delta^{\mathbf{ms}})^{\mathbb{C}^{\mathrm{op}}} : \mathbb{\chi}_{\mathbb{C}}$$

is a Quillen equivalence.

Proof. Since $\mathbb{\phi}_{\mathbb{C}}$ preserves cofibrations, and is naturally weakly equivalent to $\mathbb{S}\mathrm{t}_{\mathbb{C}}$, it preserves trivial cofibrations, and thus is left Quillen. Moreover, the left-derived functors of $\mathbb{S}\mathrm{t}_{\mathbb{C}}$ and $\mathbb{\phi}_{\mathbb{C}}$ agree, and the former is an equivalence. □

3.3.3 Comparison to the strict case

We now establish a comparison result with the strict 2-categorical case, as worked out by Buckley in [Buc14]. We will heavily leverage two facts to ease the proof of this comparison results

- For a strict 2-functor $F : \mathbb{C}^{\mathrm{op}} \to 2\mathrm{Cat}$, we can describe $\mathbb{\chi}_{\mathbb{C}}(F)$ *entirely* in terms of 2-functors into $\mathbb{C}$ and $F(x)$, for $x \in \mathbb{C}$.
- The Duskin 2-nerve $\mathrm{N}_2(\mathbb{C})$ of any strict 2-category $\mathbb{C}$ is 3-coskeletal.

Making use of these two facts allows us to construct a comparison map by checking a finite number of cases by hand. Once the comparison is established, we can work with strict 2-categories to prove that it is a fibre-wise equivalence.

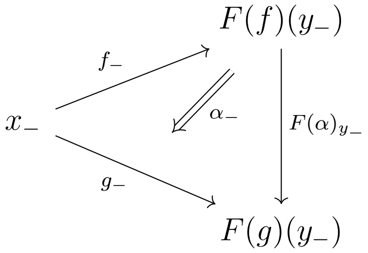
Let us now introduce the 2-categorical Grothendieck construction we wish to compare with. Appropriately dualizing Buckley's construction[3], the *strict 2-categorical Grothendieck construction* of a 2-functor

$$F\colon \mathbb{C}^{\mathrm{op}} \longrightarrow 2\mathrm{Cat}$$

is the 2-category $\mathrm{El}(F)$ which has

[3] Buckley defines a construction that takes as input a functor $F : \mathbb{C}^{(\mathrm{op},\mathrm{op})} \to 2\mathrm{Cat}$ where both 1- and 2-morphisms have been reversed.

- Objects: pairs (x, x_-) with $x \in \mathbb{C}$ and $x_- \in F(x)$.
- Morphisms:
$$(f, f_-)\colon (x, x_-) \longrightarrow (y, y_-)$$
where $f : x \to y$ in $\mathbb{C}$, and $f_- : x_- \to F(f)(y_-)$.
- 2-Morphisms: $(\alpha, \alpha_-) : (f, f_-) \Rightarrow (g, g_-)$, where $\alpha : f \Rightarrow g$ is a 2-morphism in $\mathbb{C}$, and α_- fits in the diagram

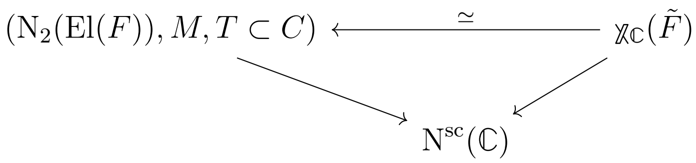


The resulting functor $\mathrm{El}(F) \to \mathbb{C}$ is a 2-Cartesian fibration, where

- (f, f_-) is Cartesian if f_- is an equivalence, and
- (α, α_-) is coCartesian if α_- is an isomorphism.

Our aim is to prove the following

Theorem 3.3.21. *Let*
$$F\colon \mathbb{C}^{(\mathrm{op},-)} \longrightarrow 2\mathrm{Cat}$$
be a 2-functor, and let $\tilde{F}$ *denote the composite*
$$\mathbb{C}^{(\mathrm{op},-)} \longrightarrow 2\mathrm{Cat} \longrightarrow \mathrm{Set}_{\Delta}^{\mathbf{ms}}$$
Then there is an equivalence



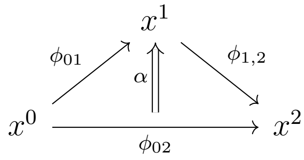
of 2-Cartesian fibrations over $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$.

We begin by showing there is a map in one direction. For ease of notation, given a morphism $\phi : x \to y$ in $\mathbb{C}$, we will write $\phi^* := F(\phi)$. We will employ the same convention for 2-morphisms.

Since the 2-nerve of a 2-category is 3-coskeletal, it suffices to define a map
$$\mathrm{sk}_3(\mathbb{X}_{\mathbb{C}}(F)) \longrightarrow N_2(\mathrm{El}(F))$$
which is compatible with markings and scalings.

On 0- and 1-simplices, the data specified by the simplices in both constructions is identical. A 2-simplex in $\mathbb{X}_{\mathbb{C}}(F)$ consists of the following data:

- A 2-simplex



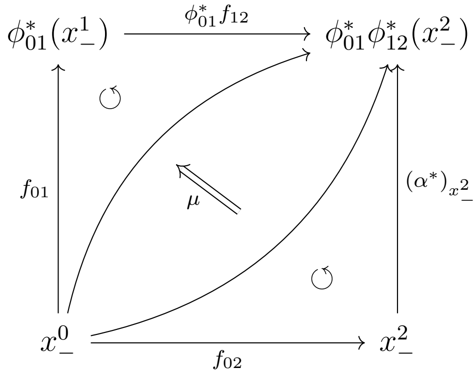
- Three 1-simplices

$$f_{12} \colon x^1_- \longrightarrow \phi^*_{12}(x^2_-)$$
$$f_{01} \colon x^0_- \longrightarrow \phi^*_{01}(x^1_-)$$

and

$$f_{02} \colon x^0_- \longrightarrow \phi^*_{02}(x^2_-)$$

- A diagram



in $\mathbb{C}$.

These data are identical to the data of a 2-simplex in $\mathrm{El}(F)$. It is immediate that markings and scalings coincide under these correspondences.

Finally, we note that 3-simplices in $\mathrm{El}(F)$ are simply compatibility conditions on 2-morphisms. It is a long but easy check to see that, given a 3-simplex in $\chi_{\mathbb{C}}(F)$, the corresponding 2-simplices in $\mathrm{El}(F)$ are compatible. We have thus shown

Proposition 3.3.22. *There is a morphism of 2-Cartesian fibrations*

$$\tau \colon \chi_{\mathbb{C}}(F) \longrightarrow \mathrm{El}(F).$$

The final ingredient in our proof will involve a comparsion of 2-functors.

Definition 3.3.23. We denote by $\pi^n : \mathbb{O}^n_{0\nearrow} \to \mathbb{O}^n$ the canonical projection. This sends $S \mapsto \max(S)$, and sends a morphism $\mathcal{U} : S \to T$ to the set $\mathcal{U}$.

Given a 2-category $\mathcal{C}$, we call a 2-functor

$$f : \mathbb{O}^n_{0\nearrow} \to \mathbb{C}$$

peripatetically constant if it sends every morphism $\mathcal{U} : S \to T$ where $\max(S) = \max(T)$ to an identity, and every 2-morphism between such morphisms to an identity as well. We denote the *set* of peripatetically constant functors $\mathbb{O}^n_{0\nearrow} \to \mathbb{C}$ by

$$\mathrm{Hom}^{\mathrm{PC}}(\mathbb{O}^n_{0\nearrow}, \mathbb{C}).$$

Lemma 3.3.24. *For any $n \geqslant 0$, the 2-functor $\pi^n : \mathbb{O}^n_{0\nearrow} \to \mathbb{O}^n$ induces a bijection*

$$(\pi^n)^* : \mathrm{Hom}(\mathbb{O}^n, \mathbb{C}) \xrightarrow{\cong} \mathrm{Hom}^{\mathrm{PC}}(\mathbb{O}^n_{0\nearrow}, \mathbb{C})$$

Proof. We can define a strict 2-functor

$$\begin{aligned} s^n \colon \mathbb{O}^n &\longrightarrow \mathbb{O}^n_{0\nearrow} \\ j &\longmapsto [0, j] \end{aligned}$$

which acts as the identity on 1- and 2-morphisms. Since $\pi^n \circ s^n = \mathrm{id}$, we have $(s^n)^* \circ (\pi^n)^* = \mathrm{id}$, and thus, π^n_* is injective. It is immediate from unraveling the definitions that the image of $(\pi^n)^*$ is precisely the peripatetically constant functors. □

Corollary 3.3.25. *Let $\mathbb{D}$ be a 2-category, and denote by $*$ the terminal 2-category. There is an isomorphism*

$$\chi_*(\mathbb{D}) \cong \mathrm{N}^{\mathrm{sc}}(\mathbb{D}).$$

Proof. An n-simplex in $\chi_*(\mathbb{D})$ consists of 2-functors

$$\theta_I \colon \mathbb{O}^I_{i\nearrow} \longrightarrow \mathbb{D}$$

for every non-empty $I \subset [n]$, such that for every $J \subset I \subset [n]$, the diagram

$$\begin{array}{ccc} \mathbb{O}^I(i,j) \times \mathbb{O}^J_{j\nearrow} & \longrightarrow & \mathbb{O}^I_{i\nearrow} \\ \downarrow & & \downarrow \\ * \times \mathbb{D} & \longrightarrow & \mathbb{D} \end{array}$$

commute. Such functors are uniquely determined by the map $\theta_{[n]} : \mathbb{O}^n_{0\nearrow} \to \mathbb{D}$, and the commutativity of the diagrams above is equivalent to requiring that $\theta_{[n]}$ be peripatetically constant.

Consequently, we obtain a bijection on sets of n-simplices $\mathrm{N}^{\mathrm{sc}}(\mathbb{D})_n \cong (\chi_*(\mathbb{D}))_n$ by pulling back along π^n. The corollary follows from checking directly that these bijections respect face and degeneracy maps. □

Proof (of Theorem 3.3.21). By Proposition 2.2.39, it will suffice for us to show that this morphism is an equivalence on fibres. By construction, the fibre of $\mathrm{El}(F)$ over $x \in \mathbb{C}$ is precisely the 2-category $F(x)$, and the fibre of $\chi_{\mathbb{C}}(F)$ over x is *also* precisely $\chi_x(F(x)) \cong \mathrm{N}^{\mathrm{sc}}(F(x))$.

It is a quick explicit check that, on 0-,1-,2-, and 3-simplices, the map $\tau : \chi_x(F(c)) \to \mathrm{El}_x(F(x)) \cong \mathrm{N}^{\mathrm{sc}}(F(x))$ agrees with the isomorphism of Corollary 3.3.25. The theorem then follows from 3-coskeletalness. □

Chapter 4

Marked colimits and higher cofinality

In previous work [AG22], [AGS22] we have studied 2-categorical notions of (co)limits in ∞-bicategories. In this section we will prove the cofinality conjecture as stated in [AGS22]. Let us remark that in previous work we only considered specific instances of colimits:

- In [AGS22] we considered diagrams $F : \mathbb{D}^\dagger \to \mathbb{A}$ where both $\mathbb{D}$ and $\mathbb{A}$ are strict 2-categories and $\mathbb{D}^\dagger$ comes equipped with a collection of marked edges.
- In [AG22] we considered diagrams $F : \mathcal{D}^\dagger \to \mathbf{A}$ where $\mathcal{D}^\dagger$ is an ∞-category equipped with a collection of marked edges and $\mathbf{A}$ is an ∞-bicategory.

The general notion of marked colimit has been extensively studied by Gagna, Harpaz and Lanari in [GHL21a]. It was noted by the authors that their notion coincides with ours in the cases we studied as shown in [GHL21a, Remark 5.2.12.]. Moreover, the notion of cofinality studied in [AG22] also agrees with the definition of cofinal functor given in [GHL21a]. However, in the previous document no computational criterion is given to determine whether a functor of marked ∞-bicategories $f : \mathbf{C}^\dagger \to \mathbf{D}^\dagger$ is cofinal. We will provide such criterion as the main result in this section extending the well-known conditions of Quillen's Theorem A.

Recall that in [GHL21a] the authors define in Section 4.1 a marked-scaled version of the Gray product. Given a marked scaled simplicial set K and a ∞-bicategory $\mathbf{D}$ it follows that we have a ∞-bicategory $\operatorname{Fun}^{\mathrm{gr}}(K, \mathbf{D})$ of functors and partially lax natural transformations where the level of laxness is specified by the marking of K (see Remark 4.1.13 in [GHL21a]). Following their notation we will denote the mapping ∞-category of $\operatorname{Fun}^{\mathrm{gr}}(K, \mathbf{D})$ by $\operatorname{Nat}^{\mathrm{gr}}_K(-,-)$.

Remark 4.0.1. The theory of marked colimits comes in two variances: inner and outer colimits. In this document we will only study outer (marked) colimits which we will simply call marked colimits.

Definition 4.0.2. Let $K^\dagger$ be a marked scaled simplicial set and let $F : K \to \mathbb{A}$ be a functor of scaled simplicial sets where K denotes the underlying scaled simplicial set of $K^\dagger$. The outer marked colimit of F denoted by $\operatorname{colim}^\dagger{}_K F$ is an object of $\mathbb{A}$ corepresenting the functor[1]

$$\mathrm{Nat}^{\mathrm{gr}}_K(F, c_K(-)) \colon \mathbb{A} \longrightarrow \mathbb{C}\mathrm{at}_\infty,\ a \longmapsto \mathrm{Nat}^{\mathrm{gr}}_K(F, \underline{a})$$

where $\underline{a} : K \to \mathbb{A}$ is the constant functor with value $a \in \mathbb{A}$.

4.0.1 The free 2-Cartesian fibration

Throughout this section, we fix an ∞-bicategory $\mathbb{D}$, and aim to construct, for each functor of ∞-bicategories $p : \mathbb{X} \to \mathbb{D}$, a 2-Cartesian fibration $\mathbb{F}(p) : \mathbb{F}(\mathbb{X}) \to \mathbb{D}$. We will characterize this fibration as the *free 2-Cartesian fibration on p.*

To construct this 2-Cartesian fibration, we will make use of the Gray tensor products constructed in [GHL20, §2.1] and [GHL21a, §4.1]. To this end, we briefly recall the definitions we will need from [GHL20].

Definition 4.0.3. Let $X, Y \in \mathrm{Set}^{\mathbf{sc}}_\Delta$. We define the *Gray product $X \otimes Y$* to be the scaled simplicial set with underlying simplicial set $X \times Y$, where we declare a 2-simplex (σ_X, σ_Y) to be scaled if and only if the following two conditions are both satisfied

- $(\sigma_X, \sigma_Y) \in T_X \times T_Y$.
- Either the image of σ_X degenerates along $\Delta^{\{1,2\}}$ or σ_Y degenerates along $\Delta^{\{0,1\}}$.

Given scaled simplicial sets X and Y, we define the *Gray functor category* $\mathrm{Fun}^{\mathrm{gr}}(X, Y)$ by the adjunction

$$\mathrm{Hom}_{\mathrm{Set}^{\mathbf{sc}}_\Delta}(S, \mathrm{Fun}^{\mathrm{gr}}(X, Y)) \cong \mathrm{Hom}_{\mathrm{Set}^{\mathbf{sc}}_\Delta}(X \otimes S, Y).$$

There is a dual version, defined by replacing $X \otimes S$ by $S \otimes X$, which we denote by $\mathrm{Fun}^{\mathrm{opgr}}(X, Y)$.

Proposition 4.0.4. *Let $\mathbb{D}$ an ∞-bicategory. Then the map $\mathrm{ev}_0 \colon \mathrm{Fun}^{\mathrm{gr}}(\Delta^1, \mathbb{D}) \longrightarrow \mathbb{D}$ is a 2-Cartesian fibration. The collection of Cartesian edges, thin triangles, and lean (coCartesian) triangles can be described as follows:*

- *An edge represented by a map $e : \Delta^1 \otimes \Delta^1 \to \mathbb{D}$ is Cartesian if and only if it factors through $\Delta^1 \times \Delta^1$ and the restriction to $\Delta^{\{1\}} \times \Delta^1$ is an equivalence in $\mathbb{D}$.*
- *A triangle represented by a map $\sigma : \Delta^1 \otimes \Delta^2 \to \mathbb{D}$ is lean if and only if its restriction to $\Delta^{\{1\}} \times \Delta^2$ is thin in $\mathbb{D}$.*
- *A triangle represented by a map $\sigma : \Delta^1 \otimes \Delta^2 \to \mathbb{D}$ is thin if and only if it is lean and its restriction to $\Delta^{\{0\}} \times \Delta^2$ is thin.*

Proof. Since, over an ∞-bicategory, the definition of 2-Cartesian we provided in [AGS22I] coincides with the notion of *outer 2-Cartesian fibration* from [GHL21b], this follows immediately from [GHL21b, Thm 2.2.6]. □

We can leverage this definition to give an extension of the Gray product, which more fully captures the decoration in this case.

[1] see section 1.3 for a definition of the functor c_K in the context of ∞-categories

Definition 4.0.5. Let $X \in \mathrm{Set}_\Delta^{\mathbf{mb}}$ and denote by $\widetilde{X}$ its underlying scaled simplicial set. We define $\Delta^1 \widehat{\otimes} X \in \mathrm{Set}_\Delta^{\mathbf{ms}}$ extending $\Delta^1 \otimes \widetilde{X}$ by declaring

- A 1-simplex (σ_1, σ_X) is marked if it is degenerate, or if σ_1 is degenerate on $\{1\}$, and σ_X is marked in X.
- A 2-simplex (σ_1, σ_X) is thin if *any* of the following conditions hold.
  - The simplex (σ_1, σ_X) is thin in $\Delta^1 \otimes \widetilde{X}$.
  - The simplex σ_X is lean in X and $\sigma_1(1 \to 2)$ is degenerate on 1.
  - The simplex σ_X is lean in X, $\sigma_X(0 \to 1)$ is marked in X and the simplex σ_1 is of the form $0 \to 0 \to 1$.

For $X, Y \in \mathrm{Set}_\Delta^{\mathbf{ms}}$, we can then define $\mathrm{Fun}^{\widehat{\mathrm{gr}}}(\Delta^1, Y) \in \mathrm{Set}_\Delta^{\mathbf{mb}}$ via the adjunction

$$\mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{mb}}}(S, \mathrm{Fun}^{\widehat{\mathrm{gr}}}(\Delta^1, Y)) \cong \mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathbf{ms}}}(\Delta^1 \otimes S, Y).$$

Remark 4.0.6. For $X = (X, M_X, T_X \subset C_X)$, we will denote by $\{0\}\widehat{\otimes} X$ the full **MS** simplicial subset of $\Delta^1 \widehat{\otimes} X$ corresponding to $\{0\} \times X$. This is isomorphic to the **MS** simplicial set $(X, \flat, T_X)$. Similarly, we denote by $\{1\}\widehat{\otimes} X$ the **MS** simplicial set (X, M_X, C_X).

We can then define the free 2-Cartesian fibration.

Definition 4.0.7. Let $p : \mathbb{X} \to \mathbb{D}$ be a functor of ∞-bicategories. Denote by $\mathbb{X}^\natural = (\mathbb{X}, M_X, T_X \subset T_X)$ the associated **MB** simplicial set in which the equivalences are marked. We define an **MB** simplicial set $\mathbb{F}(\mathbb{X})^\natural$ as the pullback

$$\begin{array}{ccc} \mathbb{F}(\mathbb{X})^\natural & \longrightarrow & \mathrm{Fun}^{\widehat{\mathrm{gr}}}(\Delta^1, \mathbb{D}) \\ \big\downarrow & & \big\downarrow \mathrm{ev}_1 \\ \mathbb{X}^\natural & \xrightarrow[p]{} & \mathbb{D} \end{array}$$

We denote the natural map induced by evaluation at 0 by $\mathbb{F}(p) \colon \mathbb{F}(\mathbb{X})^\natural \longrightarrow \mathbb{D}$.

Proposition 4.0.8. *Let $p : \mathbb{X} \to \mathbb{D}$ be a functor of ∞-bicategories. Then*

$$\mathbb{F}(p) \colon \mathbb{F}(\mathbb{X})^\natural \longrightarrow \mathbb{D}$$

is a 2-Cartesian fibration

This proposition will follow from a somewhat technical lemma.

Lemma 4.0.9. *Let $f \colon A \longrightarrow B$ be an **MB**-anodyne morphism. Then*

$$\Delta^1 \widehat{\otimes} A \coprod_{\{0\}\widehat{\otimes} A} \{0\}\widehat{\otimes} B \longrightarrow \Delta^1 \widehat{\otimes} B$$

is **MS**-*anodyne.*

Proof. As usual, we can check on generating **MB**-anodyne morphisms. Before commencing the proof, we will make a preliminary definition. We say that a morphism of **MS** simplicial sets is of type (♡) if it is in the weakly saturated hull of morphisms of the type described in Lemma 2.4.3. We can now proceed to perfom our case-by-case analysis.

(A2) We can scale $\{1\}\widehat{\otimes}\Delta^4$ using a pushout of type (MS2). The remaining 2-simplices can be scaled using morphisms of type (♡).

(A5) Is a pushout of type (MS1), a pushout of type (MS5), and a pushout of type (MS4).

(S1) Is an iterated pushout of type (MS6) and morphisms of type (♡).

(S2) Is an isomorphism.

(S3) Is a pushout along a morphism of type (♡).

(S4) Is a pushout along morphisms of type (MS7) and (♡).

(S5) Is a pushout along morphisms of type (MS8) and (♡).

The remaining three cases are the horn inclusions.

(A1) Since no morphisms are marked on either side and the lean and thin scalings are identical, this is a consequence of [GHL21a, Prop 4.1.9].

(A4) Let us set the notation $(\Delta^n)^\dagger = (\Delta^n, \Delta^{\{n-1,n\}}, \flat \subset \Delta^{\{0,n-1,n\}})$ and let us similarly define $(\Lambda^n_n)^\dagger$. First we will define an order for the simplices of maximal dimension in $\Delta^1\widetilde{\otimes}(\Delta^n)^\dagger$. Let $\theta_\varepsilon : \Delta^{n+1} \to \Delta^1\widetilde{\otimes}(\Delta^n)^\dagger$ for $\varepsilon \in \{0,1\}$ and let ν_{θ_ε} be the first element in Δ^{n+1} such that the value of θ_ε at ν_{θ_ε} has the first coordinate equal to 1. We say that $\theta_1 < \theta_2$ if and only if $\nu_{\theta_1} < \nu_{\theta_2}$. We further observe every simplex of maximal dimension is uniquely determined by the value ν_θ. Consequently we will denote by θ_i for $i \in \{1, 2, \dots, n+1\}$ the unique simplex of maximal dimension that has $\nu_{\theta_i} = i$. We can produce now a filtration

$$\Delta^1\widehat{\otimes}(\Lambda^n_n)^\dagger \to Y_{n+1} \to Y_n \to \cdots \to Y_2 \to Y_1 = \Delta^1\widetilde{\otimes}(\Delta^n)^\dagger$$

where Y_j is the full **MS** subsimplicial set of $\Delta^1\widetilde{\otimes}(\Delta^n)^\dagger$ containing the simplices of Y_{j+1} in addition to the simplex θ_j. It is an straightforward to see that the first map is a pushout along the inner-horn inclusion $\Lambda^{n+1}_n \to \Delta^{n+1}$. Since the edge $n-1 \to n$ is marked in $(\Delta^n)^\dagger$ it follows that the triangle $\{n-1, n, n+1\}$ is thin in θ_{n+1}. The rest of the morphisms are in the weakly satured hull of morphisms of type (MS4). Each map $Y_{j+1} \to Y_j$ is obtained precisely after taking two pushouts of type (MS4). First we had the face missing $j-1$ of θ_j and then we add the whole simplex missing.

(A3) The argument here is precisely dual to the previous case, replacing 'marked edge" with "degenerate edge" and "(MS4)" with "(MS3)". Note that we also need to reverse the order in which the add the simplices of maximal dimension that are missing.

□

Proof (of 4.0.8). Given a lifting problem

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{F}(\mathbb{X})^{\natural} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ B & \longrightarrow & \mathbb{D} \end{array}$$

where f is **MB**-anodyne, we need to find a diagram

$$\begin{array}{ccc} \{1\}\widehat{\otimes}B & \longrightarrow & \mathbb{X} \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \Delta^1\widehat{\otimes}B & \longrightarrow & \mathbb{D} \end{array}$$

extending the diagram

$$\begin{array}{ccc} \{1\}\widehat{\otimes}A & \longrightarrow & \mathbb{X} \\ \big\downarrow & & {\scriptstyle p}\big\downarrow \\ \Delta^1\widehat{\otimes}A \coprod\limits_{\{0\}\widehat{\otimes}A} \{0\}\widehat{\otimes}B & \longrightarrow & \mathbb{D} \end{array}$$

defined by the lifting problem. We first note that, since $\{1\}\widehat{\otimes}A \to \{1\}\widehat{\otimes}B$ is, in particular, **MS**-anodyne, we can solve the lifting problem on the bottom. It thus remains for us to solve the extension problem

$$\begin{array}{ccc} \{1\}\widehat{\otimes}B \coprod\limits_{\{1\}\widehat{\otimes}B} \Delta^1\widehat{\otimes}A \coprod\limits_{\{0\}\widehat{\otimes}A} \{0\}\widehat{\otimes}B & \longrightarrow & \mathbb{D}^{\natural} \\ \big\downarrow & & \\ \Delta^1\widehat{\otimes}B & & \end{array}$$

However, the morphism on the left fits into a composite.

$$\Delta^1\widehat{\otimes}A \coprod_{\{0\}\widehat{\otimes}A} \{0\}\widehat{\otimes}B \longrightarrow \{1\}\widehat{\otimes}B \coprod_{\{1\}\widehat{\otimes}B} \Delta^1\widehat{\otimes}A \coprod_{\{0\}\widehat{\otimes}A} \{0\}\widehat{\otimes}B \longrightarrow \Delta^1\widehat{\otimes}B$$

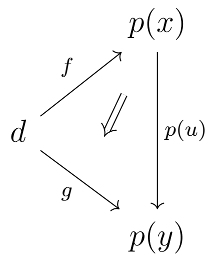
where the first morphism is a pushout by an *MS*-anodyne morphisms, and the composite is one of the morphisms from Lemma 4.0.9. Consequently, the morphism on the left is an **MS** trivial cofibration by 2-out-of-3, and thus the lifting problem has a solution. □

Definition 4.0.10. Let $p : \mathbb{X} \to \mathbb{D}$ be a functor of ∞-bicategories. We denote by $\mathbb{X}_{d\nearrow}$ the fibre of $\mathbb{F}(p)\colon \mathbb{F}(\mathbb{X})^{\natural} \longrightarrow \mathbb{D}$ over $d \in \mathbb{D}$. Note that Proposition 4.0.8 implies that $\mathbb{X}_{d\nearrow}$ is an ∞-bicategory.

Remark 4.0.11. Unwinding the definition of $\mathbb{X}_{d\nearrow}$, we see that a morphism from $f : d \to p(x)$ to $g : d \to p(y)$ is given by a diagram

$$\begin{array}{ccc} d & \xrightarrow{\ f\ } & p(x) \\ \Big\| & \searrow \ \Leftarrow & \big\downarrow{\scriptstyle p(u)} \\ d & \xrightarrow[\ g\ ]{} & p(y) \end{array}$$

which, in a strict 2-category, we could view as a diagram



which justifies our notation. We will later see that, for a functor $F : \mathbb{C} \to \mathbb{D}$ of strict 2-categories, there is an equivalence of ∞-bicategories

$$\mathrm{N}^{\mathrm{sc}}(\mathbb{C}_{d\nearrow}) \simeq \mathrm{N}^{\mathrm{sc}}(\mathbb{C})_{d\nearrow}.$$

connecting our free fibration to more familiar notions.

Remark 4.0.12. Let $p : \mathbb{X} \to \mathbb{D}$ be a functor of ∞-bicategories. There is a cofibration

$$\gamma_{\mathbb{X}} \colon \mathbb{X} \longrightarrow \mathbb{F}(\mathbb{X})^{\natural}$$

over $\mathbb{D}$, which sends a simplex $\Delta^n \to \mathbb{X}$ to the map $\Delta^1 \widehat{\otimes} \Delta^n \to \Delta^0 \otimes \Delta^n \to \mathbb{X} \to \mathbb{D}$.

To simplify our examination of this map, we provide a way of constructing 'almost degenerate' $(n+1)$-simplices in $\mathbb{F}(\mathbb{X})^{\natural}$ from n-simplices in $\mathbb{F}(\mathbb{X})^{\natural}$.

Construction 4.0.13. For every $0 \leqslant j \leqslant n$, we define a map

$$E_j \colon \Delta^1 \times \Delta^{n+1} \longrightarrow \Delta^1 \times \Delta^n$$
$$(m, r) \longmapsto \begin{cases} (m, r) & r \leqslant j \\ (1, r-1) & r > j \end{cases}$$

It is easy to check that this map respects scalings, and thus yields $E_j : \Delta^1 \otimes \Delta^{n+1} \to \Delta^1 \otimes \Delta^n$. Moreover, the induced map $\{1\} \times \Delta^{n+1} \to \{1\} \times \Delta^n$ is precisely the degeneracy map s_j.

Given an n-simplex $\sigma : \Delta^n \to \mathbb{F}(\mathbb{X})$ (possibly having some non-trivial decorations) defined by $\phi^\sigma : \Delta^1 \widehat{\otimes} \Delta^n \to \mathbb{D}$ and $\rho^\sigma : \{1\} \times \Delta^n \to \mathbb{X}$, we define an $(n+1)$-simplex $E_j^*(\sigma)$ by

$$\begin{array}{ccccc} \Delta^1 \otimes \Delta^{n+1} & \xrightarrow{E_j} & \Delta^1 \widehat{\otimes} \Delta^n & \xrightarrow{\phi^\sigma} & \mathbb{D} \\ \uparrow & & \uparrow & & \uparrow p \\ \Delta^{n+1} \times \{1\} & \xrightarrow[s_j]{} & \{1\} \times \Delta^n & \xrightarrow[\rho^\sigma]{} & \mathbb{X} \end{array}$$

We will call $E_j^*(\sigma)$ the j^{th} *extension of* σ.

Given a simplex $\sigma : \Delta^n \to \mathbb{F}(\mathbb{X})$ as above, we will denote by ℓ_0^σ the corresponding map $\{1\} \times \Delta^n \to \mathbb{X}$.

The following lemmata follow immediately from the definition.

Lemma 4.0.14. *Let $\sigma : \Delta^n \to \mathbb{F}(\mathbb{X})$. Then the faces of $E_j^*(\sigma)$ can be written as follows.*

- *If $j = n$, and $s = n+1$, then $d_s(E^*_j(\sigma)) = \sigma$.*
- *If $j+1 < s \leqslant n+1$, then $d_s(E^*_j(\sigma) = E^*_j(d_{s-1}(\sigma))$.*
- *If $0 \leqslant s < j$, then $d_s(E^*_j(\sigma)) = E_{j-1}(d_s(\sigma))$.*
- *If $s = j+1$, then $d_{j+1}(E^*_j(\sigma)) = d_{j+1}(E^*_{j+1}(\sigma))$.*
- *If $s = j$ and $s \neq 0$, then $d_j(E^*_j(\sigma)) = d_j(E^*_{j-1}(\sigma))$*
- *If $s = j = 0$, then $d_0(E^*_0(\sigma)) = \gamma_{\mathbf{X}}(d_0(\ell^\sigma_0))$.*

Lemma 4.0.15. *If $\sigma : \Delta^n \to \mathbb{F}(\mathbf{X})$ is degenerate, then for every $0 \leqslant j \leqslant n$, the j^{th} extension $E^*_j(\sigma)$ is degenerate.*

Lemma 4.0.16. *Let $\sigma : \Delta^{n-1} \to \mathbb{F}(\mathbf{X})$. Then for every $0 \leqslant j \leqslant n-1$ and every $0 \leqslant i \leqslant n$, the simplex $E^*_i(E^*_j(\sigma))$ is degenerate.*

Theorem 4.0.17. *Let $p\colon \mathbf{X} \longrightarrow \mathbb{D}$ be a functor of ∞-bicategories. Then the morphism*

$$\gamma_{\mathbf{X}}\colon \mathbf{X}^\natural \longrightarrow \mathbb{F}(\mathbf{X})^\natural$$

*is **MB**-anodyne over $\mathbb{D}$.*

Proof. Let us start by defining Z_0 to be the subsimplicial set of $\mathbb{F}(\mathbf{X})$ consisting of all of the simplices belonging to $\mathbf{X}$, all of the 0-simplices of $\mathbb{F}(\mathbf{X})$ and all of the possible j-extensions of the 0-simplices. We extend this definition inductively by defining Z_n to consist of all of the simplices of Z_{n-1}, all of the n-simplices of $\mathbb{F}(\mathbf{X})$ and the $(n+1)$-simplices appearing as extensions of n-simplices. We set the convention $Z_{-1} = X$ and we fix the notation

$$\gamma_{n-1}\colon Z_{n-1} \longrightarrow Z_n.$$

Observe that since **MB**-anodyne maps are stable under transfinite composition it will suffice to show that each γ_{n-1} is **MB**-anodyne.

We start by analyzing γ_{-1}. Observe that given an object $\sigma : \Delta^0 \to \mathbb{F}(\mathbf{X})$ we can consider the Cartesian edge $E^*_0(\sigma) : \Delta^1 \to \mathbb{F}(\mathbf{X})$. Since the target of $E^*_0(\sigma)$ is already contained in $\mathbf{X}$ it follows that we can add σ by means of a pushout along a **MB**-anodyne map. Repeating this process for each object in $\mathbb{F}(\mathbf{X})$ conclude that γ_{-1} is **MB**-anodyne.

Now we will tackle the general case for γ_{n-1} with $n \geqslant 1$. Let us pick an order on the set of non-degenerate n-simplices of $\mathbb{F}(\mathbf{X})$ that are not already contained in Z_{n-1}. For every $\sigma \in \mathbb{F}(\mathbf{X})_n$ we define $Z_{n-1}(\sigma)$ as the subsimplicial subset of Z_n containing all of the simplices of Z_{n-1} in addition to the n-simplices $\theta \leqslant \sigma$ and its corresponding extensions. Let $\operatorname{suc}(\sigma)$ be the successor of σ in our chosen order. We will adopt the convention $Z_{n-1}(\varnothing) = Z_{n-1}$ and $\operatorname{suc}(\varnothing) = \sigma_0$ is the first element in our ordering. To show that γ_{n-1} is **MB**-anodyne it will suffice to prove that

$$Z_{n-1}(\sigma) \longrightarrow Z_{n-1}(\rho)$$

is **MB**-anodyne where $\rho = \operatorname{suc}(\sigma)$.

The proof will be divided into three cases. First let us assume that for every $1 \leqslant j \leqslant n$ all of the faces of $E^*_j(\rho)$ are contained in $Z_{n-1}(\sigma)$ except the faces missing $j+1, j$. Applying Lemma 4.0.14 for $j=0$ yields

$$d_s E_0^*(\rho) = \begin{cases} E_0^*\left(d_s(\rho)\right), & \text{if } 1 < s \leqslant n+1 \\ d_1\left(E_1^*(\rho)\right), & \text{if } s = 1 \\ \gamma_X(d_0(\ell_0)), & \text{if } s = 0. \end{cases}$$

which shows that all of the faces of $E_0^*(\rho)$ are already in $Z_{n-1}(\sigma)$ except the 1-face. By construction the triangle $\Delta^{\{0,1,2\}}$ is thin in $E_0^*(\rho)$, which shows that we can add the simplex $E_0^*(\rho)$ via a pushout along a **MB**-anodyne map. Let us denote by V_0 the resulting simplicial set $Z_{n-1}(\sigma) \to V_0 \to Z_{n-1}(\rho)$. Using Lemma 4.0.14 again, we see that all of the faces of $E_1^*(\rho)$ are in V_0 except the 2-face. A similar argument as above shows that we can add $E_1^*(\rho)$ in a **MB**-anodyne way and thus obtaining a new subsimplicial set that we denote V_1. We can repeat this process until we reach V_{n-1}. In our final step we observe that we have a pullback diagram

$$\begin{array}{ccc} \Lambda^{n+1}_{n+1} & \longrightarrow & \Delta^{n+1} \\ \downarrow & & \downarrow{\scriptstyle E_n^*(\rho)} \\ V_{n-1} & \longrightarrow & Z_{n-1}(\rho) \end{array}$$

where the last edge of Λ^{n+1}_{n+1} is 2-Cartesian in $\mathbb{F}(\mathbf{X})$ and the triangle $\Delta^{\{0,n,n+1\}}$ is coCartesian. Therefore we can add $E_n^*(\rho)$ using a pushout along a **MB**-anodyne map and conclude that $Z_{n-1}(\sigma) \to Z_{n-1}(\rho)$ is in this case **MB**-anodyne.

For the second case let us suppose that there exists some $1 \leqslant \alpha \leqslant n$ such that $d_\alpha\left(E_\alpha^*(\rho)\right)$ is already in $Z_{n-1}(\sigma)$ and that for every $j > \alpha$ we have that the faces missing $j+1$, j in $E_j^*(\rho)$ are not contained in $Z_{n-1}(\sigma)$. We claim that for every $0 \leqslant k < \alpha$ the simplex $E_k^*(\rho) \in Z_{n-1}(\rho)$. One can easily check that

$$E_k^*(\rho) = E_k^*(d_\alpha\left(E_\alpha^*(\rho)\right)), \quad 0 \leqslant k < \alpha.$$

In particular, this shows that it will suffice to show that all of the extensions of $d_\alpha\left(E_\alpha^*(\rho)\right)$ are contained in $Z_{n-1}(\sigma)$. To provide a proof of this latter claim we observe that $d_\alpha\left(E_\alpha^*(\rho)\right) \in Z_{n-1}(\sigma)$ if and only at least one of the following conditions hold:

*) The face $d_\alpha\left(E_\alpha^*(\rho)\right)$ is contained in $\mathbf{X}$.

i) The face $d_\alpha\left(E_\alpha^*(\rho)\right)$ is degenerate.

ii) The face $d_\alpha\left(E_\alpha^*(\rho)\right)$ is the extension of an $(n-1)$-simplex.

iii) There exists $\theta \leqslant \sigma$, such that $d_\alpha\left(E_\alpha^*(\rho)\right)$ is a face of an extension of θ

If condition *) holds then it is easy to see that all of the possible extensions are already in $Z_{n-1}(\sigma)$. Using Lemma 4.0.15 and Lemma 4.0.16 we see that the claim holds if the conditions i) or ii) are satisfied. Suppose now that condition iii) holds. We can assume without loss of generality that $d_\alpha(E_\alpha^*(\rho)) = d_\beta(E_\beta^*(\theta))$. A straightforward computation shows that

$$E_j^*(d_\beta(E_\beta^*(\theta))) = \begin{cases} s_j(d_\beta(E_\beta^*(\theta))), & \text{if } j \geqslant \beta \\ E_j^*(\theta), & \text{if } j < \beta \end{cases}$$

so again, the claim holds. We have shown $Z_{n-1}(\sigma) = V_{\alpha-1}$ and thus the previous argument runs exactly the same way.

The last case to analyze is the degenerate situation where ρ is already in $Z_{n-1}(\sigma)$. In this case we need to show that $Z_{n-1}(\sigma) = Z_{n-1}(\rho)$, i.e. we need to show that we already have all of the extensions of ρ. Since $\rho \notin Z_{n-1}$ and it is not degenerate it follows that $\rho = d_\beta(E_\beta(\theta))$ for some $\theta \leqslant \sigma$. Using the same reasoning as before we can see that $E_k(\rho) \in Z_{n-1}(\sigma)$ for all $0 \leqslant k \leqslant n$ and the claim follows.

In order to finish the proof there is one last thing we have to take care of in the filtration, namely, the decorations. We need to show that whenever we add a marked edge (resp. lean, resp. thin triangle) in our filtration we can add the decoration to our filtration in a **MB**-anodyne way. For the marked edges this essentially an specific case of the proof given in Corollary 4.0.20. We leave the rest of the decorations as an exercise for the reader. □

Remark 4.0.18. The morphism $\gamma_{\mathbb{X}} : \mathbb{X}^\natural \to \mathbb{F}(\mathbb{X})^\natural$ we can viewed as the unit of a bicategorical free-forgetful adjunction between the ∞-bicategory of of ∞-bicategories over $\mathbb{D}$ and the ∞-bicategory of 2-Cartesian fibrations over $\mathbb{D}$. We will not pursue this direction further in this document. A detailed study of bicategorical adjunctions is part of the research program of the authors and will appear in future work.

Definition 4.0.19. Let $p\colon \mathbb{X} \longrightarrow \mathbb{D}$ be a functor of ∞-bicategories. Assume that $\mathbb{X}$ comes equipped with a marking containing all of the equivalences and denote the resulting marked ∞-bicategory by $\mathbb{X}^\dagger$. We define new marking on $\mathbb{F}(\mathbb{X})$ as follows. We declare and edge represented by $\Delta^1 \otimes \Delta^1 \to \mathbb{D}$ marked if and only if it factors through $\Delta^1 \times \Delta^1$ and its restriction to $\Delta^{\{1\}} \times \Delta^1$ factors through a marked edge in $\mathbb{X}$. We define marked-scaled simplicial $\mathbb{F}(\mathbb{X})^\dagger$ having the same lean and thin triangles as $\mathbb{F}(\mathbb{X})^\natural$ but equipped with this new collection of marked edges.

Corollary 4.0.20. *Let $p\colon \mathbb{X} \longrightarrow \mathbb{D}$ be a functor of ∞-bicategories. Assume that X comes equipped with a marking (containing the equivalences) and denote the corresponding marked ∞-bicategory by $\mathbb{X}^\dagger$. Then the morphism*

$$\mathbb{X}^\dagger \longrightarrow \mathbb{F}(\mathbb{X})^\dagger$$

*is **MB**-anodyne.*

Proof. Let us consider the pushout diagram

$$\begin{array}{ccc} \mathbb{X} & \longrightarrow & \mathbb{X}^\dagger \\ \downarrow & & \downarrow \\ \mathbb{F}(\mathbb{X})^\natural & \longrightarrow & \mathbb{F}(\mathbb{X})^\diamond \end{array}$$

where it follows from Theorem 4.0.17 that the left-most vertical map is **MB**-anodyne. To finish the proof we just need to show that the morphism $\mathbb{F}(\mathbb{X})^\diamond \to \mathbb{F}(\mathbb{X})^\dagger$ is again anodyne. Let $e : \Delta^1 \otimes \Delta^1 \to \mathbb{D}$ be a marked edge of $\mathbb{F}(\mathbb{X})^\dagger$. First we observe that $E_0^*(e)$ is a thin 2-simplex such that $d_0(E_0^*(e))$ and $d_2(E_0^*(e))$ are marked in $\mathbb{F}(\mathbb{X})^\diamond$. It particular it follows that we can marked the edge

$d_1(E_0^*(e))$ using a pushout along a morphism of type (S1). Using a pushout along a morphism of the type described in [AGS22I, Lem. 3.7] we can mark all edges in $E_1^*(e)$. We conclude the proof after noting that $d_2(E_1^*(e)) = e$. □

4.0.2 Marked colimits and cofinality

We now turn to our main result of this section, a criterion for higher cofinality. We will not here recapitulate the theory of higher (co)limits expounded in [GHL21a], but see Remark 4.0.22 for details on the connection with $(\infty, 2)$-categorical colimits.

Definition 4.0.21. Let $X^\dagger, Y^\dagger$ be a pair of marked-scaled simplicial sets and consider a marking preserving functor, $f : X^\dagger \to Y^\dagger$. We say that f is a marked cofinal functor if the associated functor of marked-biscaled simplicial sets

$$f\colon (X, E_X, T_X \subset \sharp) \longrightarrow (Y, E_Y, T_Y \subset \sharp)$$

is a weak equivalence in model structure of **MB** simplicial sets over Y.

Remark 4.0.22. The theory of marked colimits in ∞-bicategories was independently developed by Berman in [Berm21], the present author in [AGDS20] and [AG22], and Gagna, Harpaz, and Lanari in [GHL21a]. The latter provides a full characterization of marked (co)limits in ∞-bicategories, including the four variances which arise from changing the directions of 2-morphisms in the corresponding notion of cone. The theory of marked colimits described in [AG22] corresponds to the case of *outer colimits* in the language of [GHL21a].

By [GHL21a, Thm 5.4.4], a functor of marked ∞-bicategories $f : \mathbb{C}^\dagger \to \mathbb{D}^\dagger$ is *outer cofinal* — i.e., pullback along f preserves outer colimits — if and only if f is marked cofinal in the sense of Definition 4.0.21 above (compare [GHL21a, Defn 4.3.3] to [AGS22I, Defn 3.25] and [AGS22I, Prop. 3.28] to see that these conditions do indeed coincide).

Remark 4.0.23. Let $f : (\mathbb{C}, E_{\mathbb{C}}, T_{\mathbb{C}}) \to (\mathbb{D}, E_{\mathbb{D}}, T_{\mathbb{D}})$ be a functor of marked ∞-bicategories. Observe that in order to see if f is cofinal we can assume that the markings of both ∞-bicategories contain all equivalences. Indeed, this follows easily after taking pushouts along morphisms of type (E). Consequently for the rest of the section we will assume that the markings satisfy this property.

Lemma 4.0.24. *Let $f : \mathbb{C}^\dagger \to \mathbb{D}^\dagger$ be a functor of marked ∞-bicategories and recall from Definition 4.0.19 the associated marking on $\mathbb{F}(\mathbb{C})^\dagger = (\mathbb{F}(\mathbb{C}), E_{\mathbb{F}(\mathbb{C})^\dagger}, T_{\mathbb{F}(\mathbb{C})^\dagger})$. Then the induced morphism*

$$(\mathbb{C}, E_{\mathbb{C}^\dagger}, T_{\mathbb{C}^\dagger} \subset \sharp) \longrightarrow (\mathbb{F}(\mathbb{C}), E_{\mathbb{F}(\mathbb{C})^\dagger}, T_{\mathbb{F}(\mathbb{C})^\dagger} \subset \sharp)$$

*is **MB**-anodyne.*

Proof. Let us denote $\mathbb{C}^\dagger_\sharp = (\mathbb{C}, E_{\mathbb{C}^\dagger}, T_{\mathbb{C}^\dagger} \subseteq \sharp)$ and similarly $\mathbb{F}(\mathbb{C})^\dagger_\sharp$. We consider a pushout diagram over $\mathbb{D}$

$$\begin{array}{ccc} \mathbb{C}^\dagger & \longrightarrow & \mathbb{C}^\dagger_\sharp \\ \big\downarrow \simeq & & \big\downarrow \simeq \\ \mathbb{F}(\mathbb{C})^\dagger & \longrightarrow & \mathbb{F}(\mathbb{C})^\dagger_\diamond \end{array}$$

whose vertical morphisms are all weak equivalences. We will show that the induced map $s : \mathbb{F}(\mathbb{C})^{\dagger}_{\diamond} \to \mathbb{F}(\mathbb{C})^{\dagger}_{\sharp}$ is anodyne. Let $\sigma : \Delta^2 \to \mathbb{F}(\mathbb{C})^{\dagger}_{\diamond}$ be a triangle. Note that by construction $E^*_0(\sigma)$ is fully lean scaled. This implies that $E^*_0(\sigma)$ has all faces lean except possibly the face missing the vertex 1. Since the triangle $\{0, 1, 2\}$ is thin we can take a pushout along an **MB**-anodyne morphism to lean scaled the face missing 1.

Now we consider $E^*_1(\sigma)$ and observe that the face missing 0 and 3 are lean. Additionally we see that $d_1(E^*_1(\sigma)) = d_1(E^*_0(\sigma))$ so it follows that all faces are already lean except the face missing the vertex 2. We scale the aforementioned face after noting that $\{1, 2, 3\}$ is a thin triangle. A similar argument then shows that all of the faces of $E^*_2(\sigma)$ are scaled except possible $d_3(E^*_2(\sigma)) = \sigma$. However since the last vertex is marked and the triangle $\{0, 2, 3\}$ is scaled the result follows. □

Definition 4.0.25. Let $f : \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a functor of marked ∞-bicategories. Given $d \in \mathbb{D}$ we denote by $\mathbb{C}^{\dagger}_{d\nearrow}$ the fibre over the object d of the morphism $\mathbb{F}(\mathbb{C})^{\dagger} \to \mathbb{D}$.

Definition 4.0.26. Let $f : \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a functor of marked ∞-bicategories. We define the 2-Cartesian fibration $\mathbb{C}^{\dagger}_{\mathbb{D}\nearrow} \to \mathbb{D}$ to be a fibrant replacement of the object $(\mathbb{F}(\mathbb{C}), E_{\mathbb{F}(\mathbb{C})^{\dagger}}, T_{\mathbb{F}(\mathbb{C})^{\dagger}} \subset \sharp)$ in $(\mathrm{Set}^{\mathbf{mb}}_{\Delta})_{/\mathbb{D}}$.

Proposition 4.0.27. *Let $f : \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a functor of marked ∞-bicategories. Then the following statements are equivalent:*

i) The map f is marked cofinal.

ii) The morphism $(\mathbb{F}(\mathbb{C}), E_{\mathbb{F}(\mathbb{C})^{\dagger}}, T_{\mathbb{F}(\mathbb{C})^{\dagger}} \subset \sharp) \to (\mathbb{F}(\mathbb{D}), E_{\mathbb{F}(\mathbb{D})^{\dagger}}, T_{\mathbb{F}(\mathbb{D})^{\dagger}} \subset \sharp)$ is a weak equivalence.

iii) For every $d \in \mathbb{D}$ we have an equivalence of ∞-categorical localizations $L_W(\mathbb{C}^{\dagger}_{d\nearrow}) \to L_W(\mathbb{D}^{\dagger}_{d\nearrow})$.

Proof. The equivalence $i) \iff ii)$ follows from Lemma 4.0.24 and the functoriality of the free fibration. To finish the proof we will show that $ii) \iff iii)$.

Consider projective-fibrant functors $\mathcal{F}_{\mathbb{C}}, \mathcal{F}_{\mathbb{D}} : \mathfrak{C}^{\mathrm{sc}}[\mathbb{D}]^{\mathrm{op}} \to \mathrm{Set}^{\mathbf{ms}}_{\Delta}$ equipped with equivalences $\mathrm{Un}_{\mathbb{D}}(\mathcal{F}_{\mathbb{C}}) \simeq \mathbb{F}(\mathbb{C})$ and $\mathrm{Un}_{\mathbb{D}}(\mathcal{F}_{\mathbb{D}}) \simeq \mathbb{F}(\mathbb{D})$. We can define new functors $\mathcal{F}^{\dagger}_{\mathbb{C}}$ and $\mathcal{F}^{\dagger}_{\mathbb{D}}$ and a morphism $\mathcal{F}^{\dagger}_{\mathbb{C}} \to \mathcal{F}^{\dagger}_{\mathbb{D}}$ via pushout, e.g.,

$$\begin{array}{ccc} \mathbb{S}\mathrm{t}_{\mathbb{D}}(\mathbb{F}(\mathbb{C})) & \xhookrightarrow{\sim} & \mathcal{F}_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathbb{S}\mathrm{t}_{\mathbb{D}}(\mathbb{F}(\mathbb{C})^{\dagger}) & \xhookrightarrow{\sim} & \mathcal{F}^{\dagger}_{\mathbb{C}} \end{array}$$

We thus see that the induced map on fibrant repacements $\mathsf{R}(\mathbb{U}\mathrm{n}_{\mathbb{D}}(\mathcal{F}^{\dagger}_{\mathbb{C}})) \to \mathsf{R}(\mathbb{U}\mathrm{n}_{\mathbb{D}}(\mathcal{F}^{\dagger}_{\mathbb{C}}))$ is a model for $\mathbb{C}^{\dagger}_{\mathbb{D}\nearrow} \to \mathbb{D}^{\dagger}_{\mathbb{D}\nearrow}$. Moreover, the pushout is computed pointwise. Unraveling the definition, we note that for each $d \in \mathbb{D}$ the square

$$\begin{array}{ccc} \mathbb{S}\mathrm{t}_*(\mathbb{C}_{d\nearrow}) & \xrightarrow{\sim} & \mathbb{S}\mathrm{t}_{\mathbb{D}}(\mathbb{F}(\mathbb{C}))(d) \\ \downarrow & & \downarrow \\ \mathbb{S}\mathrm{t}_*(\mathbb{C}^{\dagger}_{d\nearrow}) & \xrightarrow{\sim} & \mathbb{S}\mathrm{t}_{\mathbb{D}}(\mathbb{F}(\mathbb{C})^{\dagger})(d) \end{array}$$

is a homotopy pushout. We thus have natural equivalences

$$\mathbb{C}^{\dagger}_{d\nearrow} \simeq \mathbb{S}\mathrm{t}_*(\mathbb{C}^{\dagger}_{d\nearrow}) \simeq \mathbb{S}\mathrm{t}_{\mathbb{D}}(\mathbb{F}(\mathbb{C})^{\dagger})(d) \simeq \mathcal{F}^{\dagger}_{\mathbb{C}}(d).$$

Finally, we note that there are canonical natural identifications

$$\mathbb{C}^{\dagger}_{\mathbb{D}\nearrow} \times_{\mathbb{D}} \{d\} \simeq \mathsf{R}(\mathcal{F}^{\dagger}_{\mathbb{C}})(d) \simeq L_W(\mathcal{F}^{\dagger}_{\mathbb{C}}(d))$$

so that we get a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^{\dagger}_{\mathbb{D}\nearrow} \times_{\mathbb{D}} \{d\} & \xrightarrow{\simeq} & L_W(\mathbb{C}^{\dagger}_{d\nearrow}) \\ \downarrow & & \downarrow \\ \mathbb{D}^{\dagger}_{\mathbb{D}\nearrow} \times_{\mathbb{D}} \{d\} & \xrightarrow{\simeq} & L_W(\mathbb{D}^{\dagger}_{d\nearrow}) \end{array}$$

The proposition then follows from [AGS22I, Prop. 4.25]. □

Proposition 4.0.28. *Let $p : \mathbb{X} \to \mathbb{D}$ be a 2-Cartesian fibration such that every triangle in $\mathbb{X}$ is lean. Suppose that for every $d \in \mathbb{D}$ there exists an initial object $i_d \in \mathbb{X}_d$ in the fibre over d. Then the restriction of p to the the marked biscaled simplicial set spanned by initial objects $\hat{p} : \hat{\mathbb{X}} \to \mathbb{D}$ is a trivial fibration of scaled simplicial sets.*

Proof. We first show that $\hat{p}$ is a fibration in the model structure on $\mathrm{Set}^{\mathbf{sc}}_{\Delta}$. Since p is a 2-Cartesian fibration, it is easy to see that $\hat{p}$ has the right lifting property against all scaled anodyne morphisms. By virtue of [GHL19, Cor 6.4] it will suffice to check that $\hat{p}$ is an isofibration. Let $d_0 \to d_1 = p(x_1)$ be an equivalence in $\mathbb{D}$ and pick a lift $x_0 \to x_1$ such that x_1 is initial in the fibre over d_1. Let us pick an initial object $\hat{x}_0$ and consider the composite morphism $u : \hat{x}_0 \to x_1$. We claim that u is an equivalence. Let $\mathcal{D} \subset \mathbb{D}$ denote the underlying ∞-category of $\mathbb{D}$ and let us consider a pullback diagram

$$\begin{array}{ccc} \mathbb{X}_{\mathcal{D}} & \longrightarrow & \mathbb{X} \\ \downarrow & & \downarrow p \\ \mathcal{D} & \longrightarrow & \mathbb{D} \end{array}$$

where the left-most vertical morphism is a Cartesian fibration of ∞-categories. Let $\hat{\mathbb{X}}_{\mathcal{D}}$ denote the restriction to the full subcategory on fibrewise initial objects. Then it follows from [Lur09a, Prop. 2.4.4.9] that the restriction $\hat{\mathbb{X}}_{\mathcal{D}} \to \mathcal{D}$ is a trivial Kan fibration. In particular it detects equivalences and the claim follows.

We have thus reduced our problem to showing that $\hat{p}$ is a bicategorical equivalence. By our hypothesis it follows that $\hat{p}$ is surjective on objects. To finish the proof we will check that for every pair of objects $x, y \in \mathbf{X}$ the induced morphism of mapping ∞-categories

$$\hat{p}_{x,y} \colon \operatorname{Map}_{\hat{\mathbf{X}}}(x,y) \longrightarrow \operatorname{Map}_{\mathbf{D}}(\hat{p}(x), \hat{p}(y))$$

is an equivalence. Note that since every 2-simplex in $\mathbf{X}$ is lean it follows that not only is $p_{x,y}$ a coCartesian fibration, it is also a left fibration. Therefore we reduce our problem to showing that the fibres of $\hat{p}_{x,y}$ are all contractible. This follows from our hypothesis using [AGS22I, Proposition 4.21] □

Lemma 4.0.29. *Let $\mathbf{D}^{\dagger}$ be a marked ∞-bicategory. Then the 2-Cartesian fibration $\mathbf{D}^{\dagger}_{\mathbf{D}^{\nearrow}} \to \mathbf{D}$ satisfies the hypothesis of Proposition 4.0.28.*

Proof. Recall the model for $\mathbf{D}^{\dagger}_{\mathbb{D}^{\nearrow}}$ given in Proposition 4.0.27. As a direct consequence we observe that every triangle in $\mathbf{D}^{\dagger}_{\mathbb{D}^{\nearrow}}$ is lean. We claim that for every $d \in \mathbb{D}$ the identity morphism id_d on d is initial in its corresponding fibre. Note that we can identify the fibre over d with $L_W(\mathbb{D}^{\dagger}_{d^{\nearrow}})$. Since for every object $f : d \to d'$, the mapping ∞-category $\operatorname{Map}_{\mathbb{D}_{d^{\nearrow}}}(\mathrm{id}_d, f)$ is contractible due to Lemma 4.0.30 it follows that id_d is initial in the localisation. □

Lemma 4.0.30. *Let $\mathbf{D}$ be an ∞-bicategory. Let $\mathrm{id}_d : d \to d$ and $e : d \to d'$ be a pair of edges in $\mathbf{D}$ such that id_d is degenerate. Let $r : \Delta^1 \times \Delta^1 \to \Delta^1$ be the morphism that sends every vertex to 0 except $(1,1)$ which gets sent to 1. Then the composite*

$$\eta_e \colon \Delta^1 \times \Delta^1 \xrightarrow{\ r\ } \Delta^1 \xrightarrow{\ e\ } \mathbf{D}$$

defines a terminal object in the mapping ∞-category $\operatorname{Map}_{\mathbf{D}_{d^{\nearrow}}}(\mathrm{id}_d, f)$.

Proof. We will show that every boundary $\partial\alpha : \partial\Delta^n \to \operatorname{Map}_{\mathbf{D}_{d^{\nearrow}}}(\mathrm{id}_d, f)$ such that $\partial\alpha(n) = \eta_e$ can be extended to an n-simplex $\alpha : \Delta^n \to \operatorname{Map}_{\mathbf{D}_{d^{\nearrow}}}(\mathrm{id}_d, f)$.

We define a subsimplicial subset (with the inherited scaling) $\mathcal{S}^{n+1} \subset \Delta^1 \otimes \Delta^{n+1}$ consisting of precisely those simplices σ satisfying at least one of the conditions below:

- The simplex σ is contained in $\Delta^{\{0\}} \times \Delta^{n+1}$.
- Given $j \in [n+1]$ the simplex σ skips vertices of the form (ε, j) with $\varepsilon \in \{0,1\}$.

Unraveling the definitions we see that we need to solve the associated lifting problem

$$\begin{array}{ccc} \mathcal{S}^{n+1} & \xrightarrow{\partial\alpha} & \mathbf{D} \\ \Big\downarrow{\scriptstyle\iota} & \nearrow & \\ \Delta^1 \otimes \Delta^{n+1} & & \end{array}$$

We will abuse notation and denote by $\Delta^1 \otimes \Delta^{n+1}$ the Gray product where we are additionally scaling the triangles $(0,j) \to (0,j+1) \to (1,j+1)$ whenever $j < n$ and the triangle $(n,0) \to (n+1,0) \to (1,n+1,1)$. We will carry this additional scaling to $\mathcal{S}^{n+1}$. Note that by construction $\partial\alpha$ sends those triangles to thin simplices in $\mathbb{D}$. We produce a factorization

$$\mathcal{S}^{n+1} \xrightarrow{u} \mathcal{R}^{n+1} \xrightarrow{v} \Delta^1 \otimes \Delta^{n+1}$$

where $\mathcal{R}^{n+1}$ consists of those simplices of $\Delta^1 \otimes \Delta^{n+1}$ that skip the vertex $(1,1)$. It is easy to see that u is scaled anodyne and that v fits into a pushout square

$$\begin{array}{ccc} \Lambda^{n+1}_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0 & \longrightarrow & \Delta^{n+1} \coprod_{\Delta^{\{0,1\}}} \Delta^0 \\ \downarrow & & \downarrow \\ \mathcal{R}^{n+1} & \xrightarrow{v} & \Delta^1 \otimes \Delta^{n+1} \end{array}$$

since the triangle $\{0,1,n\}$ is thin by construction it follows that v is also scaled anodyne. The result now follows. □

We now arrive at the main theorem of this section, which provides a computational criterion for cofinality.

Theorem 4.0.31. *Let $f : \mathbb{C}^\dagger \to \mathbb{D}^\dagger$ be a functor of marked ∞-bicategories. Then the following statements are equivalent*

1. *The functor f is marked cofinal.*
2. *For every $d \in \mathbb{D}$ the functor f induces an equivalence of ∞-categorical localizations $L_W(\mathbb{C}^\dagger_{d\nearrow}) \to L_W(\mathbb{D}^\dagger_{d\nearrow})$.*
3. *The following conditions hold:*
   - *i) For every $d \in \mathbb{D}$ there exists a morphism $g_d : d \to f(c)$ which is initial in $L_W(\mathbb{C}^\dagger_{d\nearrow})$ and $L_W(\mathbb{D}^\dagger_{d\nearrow})$.*
   - *ii) Every marked morphism $d \to f(c)$ defines an initial object in $L_W(\mathbb{C}^\dagger_{d\nearrow})$.*
   - *iii) For any marked morphism $d \to b$ in $\mathbb{D}$ the induced functor $L_W(\mathbb{C}^\dagger_{b\nearrow}) \to L_W(\mathbb{C}^\dagger_{d\nearrow})$ preserves initial objects.*

Proof. By Proposition 4.0.27 it will suffice to show that 2 holds if and only if 3 holds. Let us suppose that 2 holds. Since by hypothesis the morphism $L_W(\mathbb{C}^\dagger_{d\nearrow}) \to L_W(\mathbb{D}^\dagger_{d\nearrow})$ is an equivalence of ∞-categories we can pick an object $d \to f(c)$ whose image in $L_W(\mathbb{D}^\dagger_{d\nearrow})$ is equivalent to id_d. Since equivalences preserve and detect initial objects we see that condition $i)$ is satisfied. To see that condition $ii)$ holds we just note that every marked morphism in $L_W(\mathbb{D}^\dagger_{d\nearrow})$ is equivalent to id_d. Using again that equivalences detect initial objects the claim follows. For the final condition we consider a commutative diagram

$$\begin{array}{ccc} L_W(\mathbb{C}^\dagger_{b\nearrow}) & \xrightarrow{\simeq} & L_W(\mathbb{D}^\dagger_{b\nearrow}) \\ \downarrow & & \downarrow \\ L_W(\mathbb{C}^\dagger_{d\nearrow}) & \xrightarrow{\simeq} & L_W(\mathbb{D}^\dagger_{d\nearrow}) \end{array}$$

It is now clear that condition $iii)$ holds if the right-most vertical morphism preserves initial objects. We observe that this map sends the identity on b to an object represented by a marked morphism and thus preserves initial objects.

Now let us suppose that the conditions in 3 are satisfied. Using Proposition 4.0.27 we see that it will suffice to show that the induced morphism of fibrant replacements $\mathcal{A}_f : \mathbb{C}^{\dagger}_{\mathbb{D}\nearrow} \to \mathbb{D}^{\dagger}_{\mathbb{D}\nearrow}$ is an equivalence of 2-Cartesian fibrations. Notice that by assumption it follows that $\mathbb{C}^{\dagger}_{\mathbb{D}\nearrow}$ satisfies the hypothesis of Proposition 4.0.28. Let us denote by $\hat{\mathbb{C}}^{\dagger}_{\mathbb{D}\nearrow}$ the full marked-biscaled simplicial set spanned by fibrewise initial objects and similarly for $\hat{\mathbb{D}}^{\dagger}_{\mathbb{D}\nearrow}$. Observe that due to Proposition 4.0.28 we have a section

$$s_f \colon \mathbb{D}_{\sharp} \longrightarrow \hat{\mathbb{C}}^{\dagger}_{\mathbb{D}\nearrow} \longrightarrow \mathbb{C}^{\dagger}_{\mathbb{D}\nearrow} \quad \text{where } \mathbb{D}_{\sharp} = (\mathbb{D}, E_{\mathbb{D}}, T_{\mathbb{D}} \subset \sharp).$$

We can pick the section so that each d gets sent to $g_d : d \to f(c)$ as in condition $i)$. We claim that s_f sends marked edges in $\mathbb{D}^{\dagger}_{\sharp} = (\mathbb{D}, E_{\mathbb{D}^{\dagger}}, T_{\mathbb{D}^{\dagger}} \subset \sharp)$ to Cartesian edges in $\mathbb{C}^{\dagger}_{\mathbb{D}\nearrow}$. Let $e : d \to b$ be a marked edge in $\mathbb{D}^{\dagger}_{\sharp}$ and pick a Cartesian lift of e, $\hat{e} : \Delta^1 \to \mathbb{C}^{\dagger}_{\mathbb{D}\nearrow}$ such that $\hat{e}(1) = g_b$. By condition $iii)$, we have that $\hat{e}(0)$ is initial in the fibre over d. We consider the commutative diagram

$$\begin{array}{ccc} \Lambda^2_2 & \xrightarrow{\sigma} & \mathbb{C}^{\dagger}_{\mathbb{D}\nearrow} \\ \big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow \\ \Delta^2 & \xrightarrow{s_0(e)} & \mathbb{D} \end{array}$$

with $\sigma(1 \to 2) = \hat{e}$ and $\sigma(0 \to 2) = s_f(e)$. The triangle θ is thin by construction and the edge $0 \to 1$ is an equivalence since it is a morphism between initial objects. It follows that $s_f(e)$ is Cartesian. We can now use Lemma 4.0.24 to produce a solution to the lifting problem

$$\begin{array}{ccc} \mathbb{D}^{\dagger}_{\sharp} & \xrightarrow{s_f} & \mathbb{C}^{\dagger}_{\mathbb{D}\nearrow} \\ \big\downarrow{\scriptstyle i_{\mathbb{D}}} & \overset{\mathcal{I}_f}{\nearrow} & \\ \mathbb{D}^{\dagger}_{\mathbb{D}\nearrow} & & \end{array}$$

We claim that $\mathcal{A}_f$ and $\mathcal{I}_f$ are mutually inverse. First we observe that $\mathcal{A}_f \circ s_f$ is a section of $\mathbb{D}^{\dagger}_{\mathbb{D}\nearrow}$ that maps each object $d \in \mathbb{D}$ to an initial object in the fibre. Using the fact that $\hat{\mathbb{D}}^{\dagger}_{\mathbb{D}\nearrow} \to \mathbb{D}$ is a trivial fibration we can construct a homotopy over $\mathbb{D}$,

$$H_{\mathbb{D}} \colon \Delta^1 \times \mathbb{D}_{\sharp} \longrightarrow \mathbb{D}^{\dagger}_{\mathbb{D}\nearrow}$$

between $i_{\mathbb{D}}$ and $\mathcal{A}_f \circ s_f$. Observe that $i_{\mathbb{D}}(d) = \mathrm{id}_d$ so it maps every object to an initial object in the fibre. By construction the components of $H_{\mathbb{D}}$ are morphisms between initial objects and thus equivalences. Let $e : d \to b$ be a marked morphism in $\mathbb{D}^{\dagger}_{\sharp}$ then it follows that $H_{\mathbb{D}}(0 \to 1, e)$ is marked in $\mathbb{D}^{\dagger}_{\mathbb{D}\nearrow}$. We can therefore upgrade the homotopy $H_{\mathbb{D}}$ to a marked homotopy

$H_{\mathbb{D}} : (\Delta^1)^\sharp \times \mathbb{D}^\dagger_\sharp \to \mathbb{D}^\dagger_{\mathbb{D}\nearrow}$. To see that $\mathcal{A}_f \circ \mathcal{I}_f \simeq \mathrm{id}$ it suffices to check that $\mathcal{A}_f \circ \mathcal{I}_f \circ i_{\mathbb{D}} \simeq i_{\mathbb{D}}$, however we have

$$\mathcal{A}_f \circ \mathcal{I}_f \circ i_{\mathbb{D}} = \mathcal{A}_f \circ s_f \simeq i_{\mathbb{D}}.$$

Let us fix some notation $i_f : \mathbb{C}^\dagger_\sharp \to \mathbb{C}^\dagger_{\mathbb{D}\nearrow}$ and $i_{\mathbb{C}} : \mathbb{C}^\dagger_\sharp \to \mathbb{C}^\dagger_{\mathbb{C}\nearrow}$. In order to show that $\mathcal{I}_f \circ \mathcal{A}_f \simeq \mathrm{id}$ we will show that $\mathcal{I}_f \circ \mathcal{A}_f \circ i_f \simeq i_f$. Consider the following pullback square

$$\begin{array}{ccc} f^*\left(\mathbb{C}^\dagger_{\mathbb{D}\nearrow}\right) & \xrightarrow{\phi} & \mathbb{C}^\dagger_{\mathbb{D}\nearrow} \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{f} & \mathbb{D} \end{array}$$

and note that $i_f = \phi \circ \mathcal{B}_f \circ i_{\mathbb{C}}$ where $\mathcal{B}_f : \mathbb{C}^\dagger_{\mathbb{C}\nearrow} \to f^*\left(\mathbb{C}^\dagger_{\mathbb{D}\nearrow}\right)$ is the obvious morphism. We now observe that $\mathcal{I}_f \circ \mathcal{A}_f \circ \phi = \phi \circ f^*(\mathcal{I}_f) \circ f^*(\mathcal{A}_f)$. A similar argument as before shows that

$$f^*(\mathcal{I}_f) \circ f^*(\mathcal{A}_f) \circ \mathcal{B}_f \circ i_{\mathbb{C}} \simeq \mathcal{B}_f \circ i_{\mathbb{C}}.$$

This is due to the fact that both sides of the equation describe sections of $f^*\left(\mathbb{C}^\dagger_{\mathbb{D}\nearrow}\right)$ with values in initial objects. Note that $\mathcal{B}_f \circ i_{\mathbb{C}}(c)$ is initial as a consequence of condition ii). We conclude the proof by finally noting

$$\mathcal{I}_f \circ \mathcal{A}_f \circ i_f = \mathcal{I}_f \circ \mathcal{A}_f \circ \phi \circ \mathcal{B}_f \circ i_{\mathbb{C}} = \phi \circ f^*(\mathcal{I}_f) \circ f^*(\mathcal{A}_f) \circ \mathcal{B}_f \circ i_{\mathbb{C}} \simeq \phi \circ \mathcal{B}_f \circ i_{\mathbb{C}} = i_f$$

We have shown that $\mathcal{I}_f \circ \mathcal{A}_f \simeq \mathrm{id}$ and the theorem now follows. □

We derive as an immediate corollary a ∞-bicategorical upgrade of Quillen's Theorem A.

Corollary 4.0.32. *Let $f : \mathbb{C}^\dagger \to \mathbb{D}^\dagger$ be a functor of marked ∞-bicategories and suppose that for every $d \in \mathbb{D}$ the induced functor*

$$L_W(\mathbb{C}^\dagger_{d\nearrow}) \xrightarrow{\simeq} L_W(\mathbb{D}^\dagger_{d\nearrow})$$

is an equivalence of ∞-categories. Then the functor f induces an equivalence upon passage to ∞-categorical localizations

$$L_W(\mathbb{C}^\dagger) \xrightarrow{\simeq} L_W(\mathbb{D}^\dagger).$$

Proof. By Theorem 4.0.31 it follows that f is marked cofinal. Note that its follows directly from the definitions that the pushforward functor

$$t_* \colon \left(\mathrm{Set}_\Delta^{\mathbf{mb}}\right)_{/\mathbb{D}} \longrightarrow \mathrm{Set}_\Delta^{\mathbf{mb}}$$

preserves weak equivalences. The image of the map $f : (\mathbb{C}, E_{\mathbb{C}}, T_{\mathbb{C}} \subset \sharp) \to (\mathbb{D}, E_{\mathbb{C}}, T_{\mathbb{D}} \subset \sharp)$ under t_* is equivalent to the morphism $t_*(f) : (\mathbb{C}, E_{\mathbb{C}}, \sharp) \to (\mathbb{C}, E_{\mathbb{C}}, \sharp)$. The claim follows after taking a fibrant replacement of the map $t_*(f)$. □

We finish this section by studying the case where $f : \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ is a functor between strict 2-categories equipped with a marking. For the rest of the section we will denote $\mathbf{C}^{\dagger} = \mathrm{N}^{\mathrm{sc}}(\mathbb{C})^{\dagger}$ (resp. $\mathbf{D}^{\dagger} := \mathrm{N}^{\mathrm{sc}}(\mathbb{D})^{\dagger}$) where the marking comes from the marking in $\mathbb{C}^{\dagger}$ (resp. $\mathbb{D}^{\dagger}$). Our goal is to relate (nerves of) the comma 2-categories of Definition 0.0.2 with the fibres of the free 2-Cartesian fibration thus simplifying the conditions of Theorem 4.0.31.

Definition 4.0.33. Let $f : \mathbb{C} \to \mathbb{D}$ be a functor of strict 2-categories. We define a new 2-category $\mathrm{Fr}(\mathbb{C})$ as follows:

- Objects are given by morphisms $u : d_0 \to f(c_0)$ where $d_0 \in \mathbb{D}$ and $c_0 \in \mathbb{C}$.
- A morphism $\varphi_0 : u \to v$ from $u : d_0 \to f(c_0)$ to $v : d_1 \to f(c_1)$ is given by a pair of morphisms $a_0 : d_0 \to d_1$ and $\alpha_0 : c_0 \to c_1$ and a 2-morphism $\theta_{\varphi_0} : f(\alpha) \circ u \Rightarrow v \circ a$.
- A 2-morphism $\varepsilon : \varphi_0 \to \varphi_1$ is given by a pair of 2-morphisms $\psi : a_0 \Rightarrow a_1$ and $\zeta : \alpha_0 \Rightarrow \alpha_1$ such that the followign diagram commutes

$$\begin{array}{ccc} f(\alpha_0) \circ u & \xRightarrow{f(\zeta)*u} & f(\alpha_1) \circ u \\ \Big\Downarrow{\scriptstyle \theta_{\varphi_0}} & & \Big\Downarrow{\scriptstyle \theta_{\varphi_1}} \\ v \circ a_0 & \xRightarrow{v*\psi} & v \circ a_1 \end{array}$$

There is an obvious 2-functor $\mathrm{Fr}(\mathbb{C}) \to \mathbb{D}$ which is easily verified to be a 2-Cartesian fibration. In particular one observes the following:

- A morphism in $\mathrm{Fr}(\mathbb{C})$ is Cartesian if the associated morphism $\alpha : c_0 \to c_1$ is an equivalence in $\mathbb{C}$ and the 2-morphism φ_0 is invertible.
- A 2-morphism in $\mathrm{Fr}(\mathbb{C})$ is coCartesian if the associated 2-morphism $\zeta : \alpha_0 \Rightarrow \alpha_1$ is invertible.

One immediately sees that the fibres of $\mathrm{Fr}(\mathbb{C})$ are precisely the categories $\mathbb{C}_{d\nearrow}$ of Definition 0.0.2.

Remark 4.0.34. As a direct consequence of [AGS22I, Theorem 4.29] we see that the induced morphism $\mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C})) \to \mathbf{D}$ is a 2-Cartesian fibration. We further observe that there is an strict 2-functor $\mathbb{C} \to \mathrm{Fr}(\mathbb{C})$. We will see at the end of the section that $\mathrm{Fr}(\mathbb{C})$ is another model for the free 2-Cartesian fibration on the functor f.

Remark 4.0.35. Suppose we are given a morphism of marked strict 2-categories $f : \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$. Then we can construct a marked 2-category $\mathrm{Fr}(\mathbb{C})^{\dagger}$ by declaring an edge in $\mathrm{Fr}(\mathbb{C})$ to be marked if and only if it is Cartesian or the associated morphism $\alpha : c_0 \to c_1$ is marked in $\mathbb{C}^{\dagger}$ and the 2-morphism φ_0 is invertible. We denote by $\mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C}))^{\dagger}$ the associated **MB** simplicial set.

Before we continue, we must provide a good characterization of the simplices of $\mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C}))$. As it turns out, we can view $\mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C}))$ as a simplicial subset of $\mathbb{F}(\mathbf{C})$, and we will use this to provide an alternate characterization of the simplices of the former. To this end we fix some terminology. Let us call a 2-simplex of $\Delta^1 \widehat{\otimes} \Delta^n$ *contrary* if it is scaled.

An n-simplex σ of $\mathbb{F}(\mathbb{C})$ consists of a commutative diagram

$$\begin{array}{ccc} \Delta^1 \widehat{\otimes} \Delta^n & \xrightarrow{\phi^\sigma} & \mathbb{D} \\ \uparrow & & \uparrow f \\ \{1\} \times \Delta^n & \xrightarrow{\rho^\sigma} & \mathbb{C} \end{array}$$

We will call such a simplex *tame* if it sends contrary 2-simplices to *identities.*[2] By construction, the tame simplices form a simplicial subset of $\mathbb{F}(\mathbb{C})$, which we will denote by $\mathrm{Tame}(\mathbb{C}, \mathbb{D})$. When this is equipped with the marking and biscaling induced by $\mathbb{F}(\mathbb{C})$, we denote it by $\mathrm{Tame}(\mathbb{C}, \mathbb{D})^\dagger$.

Lemma 4.0.36. *There is an isomorphism*

$$\mathrm{Tame}(\mathbb{C}, \mathbb{D})^\dagger \cong \mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C}))^\dagger$$

of marked-biscaled simplicial sets.

Proof. We will prove that the underlying simplicial set $\mathrm{Tame}(\mathbb{C}, \mathbb{D})$ is 3-coskeletal, reducing the proof to a straightforward check on 3-truncations.

Consider a morphism $\partial\Delta^n \to \mathrm{Tame}(\mathbb{C}, \mathbb{D})$ where $n > 3$. This corresponds to a diagram

$$\begin{array}{ccc} \Delta^1 \times \partial\Delta^n & \xrightarrow{\phi} & \mathbb{D} \\ \uparrow & & \uparrow f \\ \{1\} \times \partial\Delta^n & \xrightarrow[\rho]{} & \mathbb{C} \end{array}$$

We now note that $\mathbb{C}$ and $\mathbb{D}$ are, themselves 3-coskeletal, and thus, in particular, they admit unique horn fillers for all horns of dimension 5 or higher. Using, e.g., the filtration of [Lur09a, Prop. 2.1.2.6], we see that ϕ has a unique extension to a map $\Delta^1 \times \Delta^n \to \mathbb{D}$. Since ρ clearly has a unique extension to a map $\Delta^n \to \mathbb{C}$, we can obtain an extension

$$\begin{array}{ccc} \Delta^1 \times \Delta^n & \xrightarrow{\widetilde{\phi}} & \mathbb{D} \\ \uparrow & & \uparrow f \\ \{1\} \times \Delta^n & \xrightarrow[\widetilde{\rho}]{} & \mathbb{C} \end{array} \qquad (*)$$

of the diagram above.

Moreover, since $n > 3$, every 2-simplex of $\Delta^1 \times \Delta^n$ is contained in $\Delta^1 \times \partial\Delta^n$. Consequently, the fact that ϕ arises from a map $\partial\Delta^n \to \mathrm{Tame}(\mathbb{C}, \mathbb{D})$ implies that the diagram $(*)$ defines an n-simplex in $\mathrm{Tame}(\mathbb{C}, \mathbb{D})$.

The remaining low-dimensional checks are left to the reader. □

Remark 4.0.37. Note that the argument above can in fact be repurposed to show that $\mathbb{F}(\mathbb{C})$ is itself 3-coskeletal in our present setting. However, in spite of their equivalence, $\mathbb{F}(\mathbb{C})$ will not be isomorphic to $\mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C}))$, as the former has significantly more 1- and 2-simplices.

[2] Notice that this definition is only sensible because $\mathbb{D} := \mathrm{N}^{\mathrm{sc}}(\mathbb{D})$. Otherwise, there is no good notion of identity 2-simplices which are neither left nor right degenerate.

Remark 4.0.38. By construction, the canonical morphism $\mathbf{C} \to \mathbb{F}(\mathbf{C})$ factors through $\mathrm{Tame}(\mathbf{C}, \mathbf{D})$.

Lemma 4.0.39. *Let $\sigma : \Delta^n \to \mathrm{Tame}(\mathbf{C}, \mathbf{D})$ be an n-simplex. Then each extension $E_j^*(\sigma) : \Delta^{n+1} \to \mathbb{F}(\mathbf{C})$ factors through $\mathrm{Tame}(\mathbf{C}, \mathbf{D})$.*

Proof. This follows immediately from unraveling the definitions. □

Definition 4.0.40. We denote by $\mathrm{Tame}(\mathbf{C}, \mathbf{D})^\natural$ the marking and biscaling induced by $\mathbb{F}(\mathbf{C})^\natural$. Similarly, we denote by $\mathrm{Tame}(\mathbf{C}, \mathbf{D})^\dagger$ the marking and biscaling induced by $\mathbb{F}(\mathbf{C})^\dagger$.

Proposition 4.0.41. *The morphism $\mathbf{C}^\natural \to \mathrm{Tame}(\mathbf{C}, \mathbf{D})^\natural$ is **MB**-anodyne over $\mathbf{D}$.*

Proof. This is identical to the proof of Theorem 4.0.17 once we redefine Z_n to consist of n-simplices of $\mathrm{Tame}(\mathbf{C}, \mathbf{D})$, together with j-extensions of these simplices. □

Theorem 4.0.42. *Let $f : \mathbb{C}^\dagger \to \mathbb{D}^\dagger$ be a functor between marked strict 2-categories, and let $f : \mathbf{C}^\dagger \to \mathbf{D}^\dagger$ denote the induced morphism of **MS** simplicial sets. Then the following hold:*

- *There exists a commutative diagram over $\mathbf{D}$*

$$\begin{array}{ccc} \mathbf{C}^\dagger & \longrightarrow & \mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C})) \\ \downarrow & \nearrow_{\Xi} & \\ \mathbb{F}(\mathbf{C})^\dagger & & \end{array}$$

  such that each morphism in the diagram is a 2-Cartesian equivalence.
- *For every $d \in \mathbb{D}$ the map Ξ induces an equivalence of **MS** simplicial sets $\mathbf{C}^\dagger_{d\nearrow} \xrightarrow{\simeq} \mathrm{N}^{\mathrm{sc}}(\mathbb{C}^\dagger_{d\nearrow})$.*

Proof. First let us assume that the marking on both $\mathbb{C}^\dagger = \mathbb{C}^\natural$ and $\mathbb{D}^\dagger = \mathbb{D}^\dagger$ only consists of equivalences so that the marking on both $\mathbb{F}(\mathbf{C})$ and $\mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C})^\natural)$ is precisely given by Cartesian edges. Recall the filtration defined in Theorem 4.0.17. First we will define a morphism

$$\begin{array}{ccc} \mathbf{C}^\natural & \longrightarrow & \mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C})^\natural) \\ \downarrow & \nearrow_{\Xi_1} & \\ Z_1 & & \end{array}$$

Since $Z_1 \to \mathbb{F}(\mathbf{C})$ is **MB**-anodyne and $\mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C}))$ is a 2-Cartesian fibration, we can pick an extension of Ξ_1 to the desired Ξ. Observe that we can map the objects of Z_1 isomorphically to those of $\mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C}))$. Given $e : \Delta^1 \to Z_1$ we see that this data precisely amounts to morphisms $u_i : d_i \to f(c_i)$ for $i = 0, 1$, $a : d_0 \to d_1$, $\alpha : c_0 \to c_1$ and $g : d_0 \to f(c_1)$ together with a pair of 2-morphisms $\varepsilon : g \overset{\simeq}{\Rightarrow} f(\alpha) \circ u_0$ and $\theta : g \Rightarrow u_1 \circ a$ such that ε is invertible. We can then map e to an edge $\Xi_1(e)$ defined by the same 1-morphisms but with associated 2-morphism $\theta \circ \varepsilon^{-1}$. One perfoms a similar construction for mapping the non-degenerate 2-simplices contained in Z_1 thus giving a definition for Ξ_1.

We can now observe that in the case that $\mathbb{C}^\dagger$ comes equipped with a general marking (containing the equivalences) we have a homotopy pushout $(\mathrm{Set}_\Delta^{\mathbf{mb}})_{/\mathbb{D}}$

$$\begin{array}{ccc} \mathbb{F}(\mathbf{C})^\natural & \longrightarrow & \mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C})^\natural) \\ \downarrow & & \downarrow \\ \mathbb{F}(\mathbf{C})^\dagger & \longrightarrow & \mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C})^\dagger) \end{array}$$

This shows it will suffice to prove the case where only the equivalences are marked. This follows from 2-out-of-3 after noting that $\mathbf{C}^\natural \to \mathrm{N}^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C})^\natural)$ is an equivalence. This follows immediately from Proposition 4.0.41. □

Remark 4.0.43. The significance of this result is twofold. Most importantly, it shows that, when considering diagrams indexed over strict 2-categories, the criteria for marked cofinality can be expressed in terms of the strict slice 2-categories. Consequently, the criteria for cofinality become much easier to explicitly check in this case.

Of lesser significance, but still of interest, is the second consequence. Since we can identify $\mathrm{N}^{\mathrm{sc}}(\mathbb{C}_{d\nearrow})$ and $\mathbf{C}_{d\nearrow}$, the criteria of Theorem 4.0.31 precisely agree with those of [AGS22, Thm 4.0.1]. Theorem 4.0.31 thus generalizes [AGS22, Thm 4.0.1], as expected.

4.1 Computing marked colimits with the Grothendieck construction

It is well known that the Grothendieck construction can be used as a universal recipe to compute the value of the ∞-categorical colimit of a functor $F : \mathcal{C} \to \mathbb{C}\mathrm{at}_\infty$ as stated in [Lur09a, Corollary 3.3.4]. An analogous result was proved Gepner-Haugseng-Nikolaus in [GHN15, Theorem 7.4] showing that the lax colimit (i.e. the marked colimit with respect to the minimal marking) can be also computed by means of the Grothendieck construction.

In this section we will give a result that interpolates between both theorems which was originally proved in [AG22].

Theorem 4.1.1. *Let $F : \mathcal{C} \to \mathbb{C}\mathrm{at}_\infty$ where $\mathcal{C}$ is an ∞-category. Let us suppose that $\mathcal{C}$ comes equipped with a collection a marked edges and denote the resulting marked ∞-category by $\mathcal{C}^\dagger$. Then there is an equivalence of ∞-categories*

$$\operatorname*{colim}_{\mathcal{C}}^{\dagger} F \xrightarrow{\simeq} L_W\left(\mathrm{Un}^{\mathrm{co}}_{\mathcal{C}}(F)^\dagger\right)$$

where $\mathrm{Un}^{\mathrm{co}}_{\mathcal{C}}(F)^\dagger$ denotes the coCartesian straightening of F equipped with a marking consisting in those coCartesian edges lying over a marked edge in $\mathcal{C}^\dagger$.

Remark 4.1.2. As seen in section 5.2 of [GHL21a], given a marked ∞-category $\mathcal{C}^\dagger$ the marked colimit of a functor $F : \mathcal{C} \to \mathbb{C}\mathrm{at}_\infty$ can be computed as the weighted colimit of F where the weight functor is given by the associated functor of the Cartesian fibration $\mathcal{C}^\dagger_{\mathcal{C}/}$ (see Definition 4.0.26). Using Corollary 2 in [AG22] we can compute this weighted colimit as the ∞-categorical colimit of the functor:

$$\mathrm{Tw}(\mathcal{C}) \longrightarrow \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{L_W\left(\mathcal{C}^{\dagger}_{-/}\right) \times F} \mathbb{C}\mathrm{at}_{\infty}$$

Before embarking upon the proof of the theorem we will need some preliminary definitions.

Definition 4.1.3. Let $F\colon \mathcal{C} \longrightarrow \mathbb{C}\mathrm{at}_{\infty}$ be a functor and denote by $\mathcal{F} \longrightarrow \mathcal{C}$ its associated coCartesian fibration. Given $\mathcal{X} \in \mathbb{C}\mathrm{at}_{\infty}$ we define a simplicial set $\Phi^{\mathcal{F}}_{\mathcal{X}}$ over $\mathcal{C}$ via the universal property $\mathrm{Map}_{\mathcal{C}}(K, \Phi^{\mathcal{F}}_{\mathcal{X}}) \simeq \mathrm{Hom}(K \times_{\mathcal{C}} \mathcal{F}, \mathcal{X})$.

Remark 4.1.4. As a special case of (the dual of) Corollary 3.2.2.12 in [Lur09a] we see that $\Phi^{\mathcal{F}}_{\mathcal{X}} \longrightarrow \mathcal{C}$ is a Cartesian fibration. An edge $\Delta^1 \longrightarrow \Phi^{\mathcal{F}}_{\mathcal{X}}$ is Cartesian if and only if the associated functor $\Delta^1 \times_{\mathcal{C}} \mathcal{F} \longrightarrow \mathcal{X}$ maps coCartesian edges in $\Delta^1 \times_{\mathcal{C}} \mathcal{F}$ to equivalences in $\mathcal{X}$.

Proposition 4.1.5. *The Cartesian fibration $\Phi^{\mathcal{F}}_{\mathcal{X}} \longrightarrow \mathcal{C}$ classifies the functor*

$$\mathrm{Fun}(F(-), \mathcal{X})\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbb{C}\mathrm{at}_{\infty}\,.$$

Proof. This is Proposition 7.3 of [GHN15]. □

Definition 4.1.6. Let $\mathcal{C}^{\dagger}$ be a marked ∞-category and consider a Cartesian (resp. coCartesian) fibration $\mathcal{X} \longrightarrow \mathcal{C}$. We equip $\mathcal{X}$ with a marking by declaring an edge marked if and only if it is Cartesian (resp. coCartesian) and its image in $\mathcal{C}$ is marked. We will denote this marked ∞-category over $\mathcal{C}$ by $\mathcal{X}^{\dagger}$.

Remark 4.1.7. Let $\mathcal{C}^{\dagger}$ be a marked ∞-category and consider a functor $F\colon \mathcal{C} \longrightarrow \mathbb{C}\mathrm{at}_{\infty}$. Denote its associated coCartesian fibration by $\mathcal{F}$. Then given $\mathcal{X} \in \mathbb{C}\mathrm{at}_{\infty}$ we have a natural equivalence of ∞-categories

$$\mathrm{Fun}\left(\mathrm{L}_W\left(\mathcal{F}^{\dagger}\right), \mathcal{X}\right) \simeq \mathrm{Fun}(\mathcal{F}^{\dagger}, \mathcal{X}) \simeq \mathrm{Map}_{\mathcal{C}}(\mathcal{C}^{\dagger}, \Phi^{\mathcal{F}}_{\mathcal{X}}).$$

where the first equivalence is the universal property of the localization and the second is given by the universal property of $\Phi^{\mathcal{F}}_{\mathcal{X}}$.

Proposition 4.1.8. *Let $\mathcal{C}$ be an ∞-category. Given $F, G\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_{\infty}$ classified by the Cartesian fibrations $\mathcal{F}$ and $\mathcal{G}$ respectively, there is a natural equivalence of ∞-categories*

$$\mathrm{Map}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \simeq \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}(F(-), G(-))$$

Proof. See Proposition 6.9 in [GHN15]. □

Proof of Theorem 4.1.1. We fix the notation $\mathrm{Un}^{\mathrm{co}}_{\mathcal{C}}(F)^{\dagger} = \mathcal{F}^{\dagger}$. Note that by Remark 4.1.7 and Theorem 4.0.17 we have natural equivalences

$$\mathrm{Fun}\left(L_W(\mathcal{F}^{\dagger}), \mathcal{X}\right) \simeq \mathrm{Map}_{\mathcal{C}}(\mathcal{C}^{\dagger}, \Phi^{\mathcal{F}}_{\mathcal{X}}) \simeq \mathrm{Map}_{\mathcal{C}}(\mathrm{F}(\mathcal{C})^{\dagger}, \Phi^{\mathcal{F}}_{\mathcal{X}}) \simeq \mathrm{Map}_{\mathcal{C}}(\mathcal{C}^{\dagger}_{\mathcal{C}/}, \Phi^{\mathcal{F}}_{\mathcal{X}})$$

where $\mathcal{C}^{\dagger}_{\mathcal{C}/}$ denotes the fibrant replacement[3] of $\mathrm{F}(\mathcal{C})^{\dagger}$ as in Definition 4.0.26. Using Proposition 4.1.8 we produce natural equivalences

$$\mathrm{Map}_{\mathcal{C}}(\mathcal{C}^{\dagger}_{\mathcal{D}/}, \Phi^{\mathcal{F}}_{\mathcal{X}}) \simeq \lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}\left(L_W\left(\mathcal{C}^{\dagger}_{-/}\right), \mathrm{Fun}(F(-), \mathcal{X})\right)$$

[3] We are using the notation $\mathrm{F}(\mathcal{C})$ (instead of $\mathbb{F}(\mathcal{C})$) to denote the free Cartesian fibration as in Definition 1.3.8.

$$\lim_{\mathrm{Tw}(\mathcal{C})^{\mathrm{op}}} \mathrm{Fun}\left(L_W\left(\mathcal{C}^{\dagger}_{-/}\right), \mathrm{Fun}(F(-), \mathcal{X})\right) \simeq \mathrm{Fun}\left(\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} L_W\left(\mathcal{C}^{\dagger}_{-/}\right) \times F, \mathcal{X}\right)$$

The result now follows from the Yoneda lemma and Remark 4.1.2. □

Chapter 5

Applications and future directions

In this section we will apply the theory of marked colimits and higher cofinality to understand 2-dimensional universal properties of certain 2-categories. Unfortunately, some of the theory necessary for the proofs we will present here is still undeveloped. We will prove these statements conditional to the existence of an ∞-bicategorical theory of Kan extensions which is part of the author's current research program.

Definition 5.0.1. Let $\mathbb{D}$ be an ∞-bicategory. We say that $\mathbb{D}$ is *marked cocomplete* if given a marked-scaled simplicial set $K^\dagger$ and a functor $F : K \to \mathbb{D}$ the marked colimit of F exists.

Recall that in the setting of ordinary strict 2-categories given a 2-functor $f : \mathbb{C} \to \mathbb{D}$ and a 2-category $\mathbb{A}$ having all weighted colimits we can produce a 2-categorical adjunction

$$f_! : \mathrm{Fun}(\mathbb{C}, \mathbb{A}) \rightleftarrows \mathrm{Fun}(\mathbb{D}, \mathbb{A}) : f^*$$

such that $f_!$ is left adjoint to the restriction functor f^*. Given a functor $F : \mathbb{C} \to \mathbb{A}$ the value of $f_!F(d)$ is computed as the weighted colimit of F with weight functor given by

$$\begin{aligned} \mathbb{C}^{\mathrm{op}} &\longrightarrow \mathbb{C}\mathrm{at} \\ c &\longmapsto \mathbb{D}(f(c), d) \end{aligned}$$

Using the equivalence between marked colimits and weighted colimits (see section 5.2 in [GHL21a]) we see that the value of $f_!F(d)$ can be alternatively described as the marked colimit of the functor

$$\mathbb{C}^{\natural}_{/d} \longrightarrow \mathbb{C} \xrightarrow{F} \mathbb{A}$$

where the marking $\mathbb{C}^{\natural}_{/d}$ corresponds to those edges whose associated 2-morphism is invertible. The next conjectural theorem (denoted with the symbol ⧫) states that this construction will extend to the setting of ∞-bicategories

Theorem⧫ 5.0.1. *Let $f : \mathbb{C} \to \mathbb{D}$ be a functor of ∞-bicategories. Given a marked cocomplete ∞-bicategory $\mathbb{A}$ we have an adjunction of ∞-bicategories*

$$f_! : \mathrm{Fun}(\mathbb{C}, \mathbb{A}) \rightleftarrows \mathrm{Fun}(\mathbb{D}, \mathbb{A}) : f^*$$

such that $f_!$ is left adjoint to the restriction functor f^*, satisfying the following properties:

K1) Let $F : \mathbb{C} \to \mathbb{A}$ be a functor and let $d \in \mathbb{D}$. We denote by $p_d : \mathbb{C}_{/\!\!/ d} \to \mathbb{C}$ the canonical projection. Observe that p_d is a 2-Cartesian fibration and let us denote by $\mathbb{C}^{\natural}_{/\!\!/ d}$ the marking given by Cartesian edges. Then we have

$$\operatorname*{colim}^{\natural}_{\mathbb{C}_{/\!\!/ d}} F \circ p_d \simeq f_! F(d).$$

K3) The restriction functor f^* is fully faithful if and only if the counit map $\varepsilon_G : (f_! \circ f_*)G \Rightarrow G$ is an equivalence for every functor $G : \mathbb{D} \to \mathbb{A}$.

K4) Given $d \in \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{A}$ let us consider the diagram

$$\begin{array}{ccc} \mathbb{C}^{\natural}_{/\!\!/ d} & \xrightarrow{f_d} & \mathbb{D}^{\natural}_{/\!\!/ d} \\ \downarrow{\scriptstyle p_d} & & \downarrow{\scriptstyle \pi_d} \\ \mathbb{C} & \xrightarrow{f} & \mathbb{D} \end{array}$$

Then it follows that the morphism $f_! f^*(G)(d) \to G(d)$ can be identified as the canonical comparison map

$$\operatorname*{colim}^{\natural}_{\mathbb{C}^{\natural}_{/\!\!/ d}}(G \circ f \circ p_d) = \operatorname*{colim}^{\natural}_{\mathbb{C}^{\natural}_{/\!\!/ d}}(G \circ \pi_d \circ f_d) \longrightarrow \operatorname*{colim}^{\natural}_{\mathbb{D}^{\natural}_{/\!\!/ d}} G \circ \pi_d \simeq G(d)$$

K5) If f^* is fully faithful then its essential is given by those functors $F : \mathbb{C} \to \mathbb{A}$ such that the unit natural transformation $\eta_F : F \Rightarrow f^* f_!(F)$ is an equivalence of functors.

K6) Given $c \in \mathbb{C}$ and $F : \mathbb{C} \to \mathbb{A}$ then the morphism $F(c) \to f^* f_! F(c)$ is induced by the inclusion of the identity on $f(c)$, $\mathrm{id}_{f(c)} : \Delta^0 \to \mathbb{C}^{\natural}_{/\!\!/ f(c)}$.

Remark 5.0.2. Let $f : \mathbb{C} \to \mathbb{D}$ be a functor of ∞-bicategories and let $\mathbb{A}$ be a marked cocomplete ∞-bicategory. It follows from conditions $K3$) and $K4$) in the previous theorem that if for every $d \in \mathbb{D}$ the map

$$f_d \colon \mathbb{C}^{\natural}_{/\!\!/ d} \longrightarrow \mathbb{D}^{\natural}_{/\!\!/ d}$$

is marked cofinal then the restriction functor $f^* : \mathrm{Fun}(\mathbb{D}, \mathbb{A}) \to \mathrm{Fun}(\mathbb{C}, \mathbb{A})$ is fully faithful. Our next goal is to show how that if f_d is cofinal for every $d \in \mathbb{D}$ then the restriction functor is fully faithful *regardless* of the cocompletness assumptions on $\mathbb{A}$. This will also necessitate an extension of the results obtained in [AGS21] which we will not prove here.

Let $\mathcal{C}$ be an ∞-category and let $\mathbb{A}$ be an ∞-bicategory. In [AGS21, Theorem 4.3] we proved that given a pair of functors $F, G : \mathcal{C} \to \mathbb{A}$ the ∞-category of natural transformations can be computed as the limit of functor

$$N_{(F,G)} : \mathrm{Tw}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \xrightarrow{F \times G^{\mathrm{op}}} \mathbb{A} \times \mathbb{A}^{\mathrm{op}} \xrightarrow{\mathbb{A}(-,-)} \mathbb{C}\mathrm{at}_\infty$$

where $\mathrm{Tw}(\mathcal{C})$ denotes the twisted arrow ∞-category [Lur17, Prop. 4.2.3]. We would like to remind the reader that in [AGS21] we constructed a Cartesian fibration classifying the restricted functor

$$\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \xrightarrow{\mathbb{A}(-,-)} \mathbb{C}\mathrm{at}_{\infty}$$

where $\mathcal{A}$ denotes the underlying ∞-category of $\mathbb{A}$. The proof of the previous limit formula depends heavily on this construction which we call the *enhanced twisted arrow category.*

Let us suppose that we are given a fully faithful functor of ∞-bicategories $\omega : \mathbb{A} \to \mathbb{B}$. Then this yields a morphism between the diagrams $N_{(F,G)} \to N_{(\omega \circ F, \omega \circ G)}$ which is levelwise a weak equivalence. Since both colimits model respective ∞-categories of natural transformations it follows that the functor

$$\omega_{*} \colon \mathrm{Fun}(\mathcal{C}, \mathbb{A}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathbb{D})$$

is a fully faithful.

Let us suppose that for every ∞-bicategory $\mathbb{A}$, we can produce a 2-Cartesian fibration $\mathbb{T}\mathrm{w}(\mathbb{A}) \to \mathbb{A} \times \mathbb{A}^{\mathrm{op}}$ classifying the mapping ∞-category functor $\mathbb{A}(-,-) : \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to \mathbb{C}\mathrm{at}_{\infty}$. It is expected that the limit formula for the ∞-category of natural transformations we discussed will generalize to this context thus yielding the next result.

Proposition♦ 5.0.3. *Let $\omega : \mathbb{A} \to \mathbb{B}$ be a fully faithful functor of ∞-bicategories with essential image $\mathbb{B}_0 \subseteq \mathbb{B}$. Then for every ∞-bicategory $\mathbb{C}$ the functor induced by post-composition with ω*

$$\omega_{*} \colon \mathrm{Fun}(\mathbb{C}, \mathbb{A}) \longrightarrow \mathrm{Fun}(\mathbb{C}, \mathbb{B})$$

is fully faithful with essential image given by those functors $F : \mathbb{C} \to \mathbb{B}$ which factor through $\mathbb{B}_0$.

Proposition♦ 5.0.4. *Let $f : \mathbb{C} \to \mathbb{D}$ be a functor of ∞-bicategories and suppose that for every $d \in \mathbb{D}$ the morphism*

$$f_d \colon \mathbb{C}^{\natural}_{/d} \longrightarrow \mathbb{D}^{\natural}_{/d}$$

is marked cofinal. Then for every ∞-bicategory $\mathbb{A}$ the restriction functor

$$f^{*} \colon \mathrm{Fun}(\mathbb{D}, \mathbb{A}) \longrightarrow \mathrm{Fun}(\mathbb{C}, \mathbb{A})$$

is fully faithful.

Proof. By Remark 5.0.2 the claim holds if $\mathbb{A}$ is marked cocomplete. For a general $\mathbb{A}$, we consider the Yoneda embedding $\mathcal{Y} : \mathbb{A} \to \mathbb{C}\mathrm{at}_{\infty}^{\mathbb{A}^{\mathrm{op}}}$ and observe that the target is marked cocomplete. We can construct a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}(\mathbb{D}, \mathbb{A}) & \xrightarrow{f^{*}} & \mathrm{Fun}(\mathbb{C}, \mathbb{A}) \\ \downarrow{\scriptstyle \mathcal{Y}_{*}} & & \downarrow{\scriptstyle \mathcal{Y}_{*}} \\ \mathrm{Fun}(\mathbb{D}, \mathbb{C}\mathrm{at}_{\infty}^{\mathbb{A}^{\mathrm{op}}}) & \xrightarrow{f^{*}} & \mathrm{Fun}(\mathbb{C}, \mathbb{C}\mathrm{at}_{\infty}^{\mathbb{A}^{\mathrm{op}}}) \end{array}$$

where the vertical morphisms are fully faithful by Proposition♦ 5.0.3. Since the bottom horizontal morphism is fully faithful by cocompleteness of the presheaf category it follows that the top horizontal morphism is also fully faithful. □

Theorem♦ 5.0.2. *Let $f : \mathbb{C} \to \mathbb{D}$ be a functor of ∞-bicategories and assume that f is surjective on vertices. Suppose that for every ∞-bicategory $\mathbb{A}$ we have a full subcategory $\mathrm{Fun}^{\diamond}(\mathbb{C}, \mathbb{A}) \subset \mathrm{Fun}(\mathbb{C}, \mathbb{A})$ such that:*

i) For every $G : \mathbb{D} \to \mathbb{A}$ the restricted functor $G \circ f$ belongs to $\mathrm{Fun}^{\diamond}(\mathbb{C}, \mathbb{A})$.

ii) For every functor $\omega : \mathbb{A} \to \mathbb{B}$ and every $F \in \mathrm{Fun}^{\diamond}(\mathbb{C}, \mathbb{A})$ then $\omega \circ F$ belongs to $\mathrm{Fun}^{\diamond}(\mathbb{C}, \mathbb{B})$.

Let us further suppose that the following conditions are satisfied:

1) For every $d \in \mathbb{D}$ the functor $f_d : \mathbb{C}^{\natural}_{/d} \to \mathbb{D}^{\natural}_{/d}$ is marked cofinal.

2) For every functor $F \in \mathrm{Fun}^{\diamond}(\mathbb{C}, \mathbb{C}\mathrm{at}_{\infty})$ the unit morphism $\eta_F : F \Rightarrow f^* f_! F$ is an equivalence.

Then for every ∞-bicategory $\mathbb{A}$, the restriction functor induces an equivalence of ∞-bicategories

$$\mathrm{Fun}(\mathbb{D}, \mathbb{A}) \xrightarrow{\simeq} \mathrm{Fun}^{\diamond}(\mathbb{C}, \mathbb{A})$$

Proof. It follows from Proposition♦ 5.0.4 that condition 1) implies that the restriction functor f^* is always fully faithful. We also see that condition 2) implies that the conclusion of the theorem holds for $\mathbb{A} = \mathbb{C}\mathrm{at}_{\infty}$.

We now prove that the claim holds for $\mathbb{C}\mathrm{at}_{\infty}^{\mathbb{A}^{\mathrm{op}}}$. Let us consider $F \in \mathrm{Fun}^{\diamond}(\mathbb{C}, \mathbb{C}\mathrm{at}_{\infty}^{\mathbb{A}^{\mathrm{op}}})$ and let us show that $\eta_F : F \Rightarrow f^* f_! F$ is an equivalence. First, let us observe that in order to show that η_F is an equivalence it will suffice to show that the image of η_F under the map

$$(\mathrm{ev}_a)_* \colon \mathrm{Fun}^{\diamond}(\mathbb{C}, \mathbb{C}\mathrm{at}_{\infty}^{\mathbb{A}^{\mathrm{op}}}) \longrightarrow \mathrm{Fun}^{\diamond}(\mathbb{C}, \mathbb{C}\mathrm{at}_{\infty})$$

is an equivalence for every $a \in \mathbb{A}$, where $ev_a : \mathbb{C}\mathrm{at}_{\infty}^{\mathbb{A}^{\mathrm{op}}} \to \mathbb{C}\mathrm{at}_{\infty}$ denotes evaluation at a. We further observe that since ev_a preserves all marked colimits we have that $(ev_a)_*(\eta_F) \simeq \eta_{F_a}$ where $F_a = ev_a \circ F$. Condition 2) then implies that η_F is also an equivalence.

For the final case let us consider the Yoneda embedding $\mathcal{Y} : \mathbb{A} \to \mathbb{C}\mathrm{at}_{\infty}^{\mathbb{A}^{\mathrm{op}}}$. Given $F \in \mathrm{Fun}^{\diamond}(\mathbb{C}, \mathbb{A})$ we set the notation $\mathcal{Y} \circ F = \hat{F}$. We claim that $f_! \hat{F}$ is in the essential image of

$$\mathcal{Y}_* \colon \mathrm{Fun}(\mathbb{D}, \mathbb{C}\mathrm{at}_{\infty}) \longrightarrow \mathrm{Fun}(\mathbb{D}, \mathbb{C}\mathrm{at}_{\infty}^{\mathbb{A}^{\mathrm{op}}})$$

To prove the claim we need to show that for every $d \in \mathbb{D}$ the functor $\hat{F}(d)$ is representable. Since the claim is already established for presheaf categories we have an equivalence $\hat{F} \Rightarrow f^* f_! \hat{F}$. Note that by construction $\hat{F}$ is pointwise a representable functor. We conclude that $f_! \hat{F}$ is also pointwise representable after noting that $f : \mathbb{C} \to \mathbb{D}$ is surjective on objects by our assumptions. We can now pick some $G : \mathbb{D} \to \mathbb{A}$ such that $\mathcal{Y} \circ G \simeq f_! \hat{F}$. By fully faithfulness of postcomposition with the Yoneda embedding we obtain an equivalence $F \Rightarrow f^* G$ which shows that F is in the essential image of the restriction functor. □

5.1 Adjunctions with fully faithful right adjoint

In this section we will see a prototypical situation where Theorem♦ 5.0.2 can be applied. Let $\mathbb{A}$ be an ∞-bicategory and consider a pair of morphisms

$$a \underset{R}{\overset{L}{\rightleftarrows}} a'$$

Our goal is to show that 2-categorical data which encodes coherent adjunctions with fully faithful right adjoint in an ∞-bicategory $\mathbb{A}$ can be recovered from the underlying 1-categorical data as *a property* of the morphisms L and R and the counit. To this end we will consider the *walking adjunction* with fully faithful right adjoint $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$ (see Definition 5.1.5 for more details) which we schematically represent as

$$- \underset{R}{\overset{L}{\rightleftarrows}} + \;, \qquad \mathrm{id}_- \overset{\eta}{\Longrightarrow} RL$$

and its underlying 1-category Adj^R which only contains the data of the morphisms L and R and the (invertible) counit $\varepsilon : LR \Rightarrow \mathrm{id}_+$.

We will show that for every ∞-bicategory the restriction functor to the underlying 1-category of $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$

$$\mathrm{Fun}(\mathbb{A}\mathrm{dj}^{\mathrm{R}}, \mathbb{A}) \longrightarrow \mathrm{Fun}(\mathrm{Adj}^R, \mathbb{A})$$

is fully faithful with essential image given by those functors $F : \mathrm{Adj}^R \to \mathbb{A}$ sending the data L, R, η in Adj^R to an adjunction with fully faithful right adjoint where the counit is given by $F(\eta)$.

5.1.1 Preliminaries

Definition 5.1.1. Let $f : x \to y$ be a morphism in $\mathcal{C}$. We define a simplicial set $\mathcal{C}_f$ whose simplices are given by maps $\widetilde{\sigma} : \Delta^{n+2} \to \mathcal{C}$ satisfying the following properties

- The $(n+1)$-face of $\widetilde{\sigma}$ factors through the map $\Delta^{n+1} \xrightarrow{\theta_n} \Delta^1 \xrightarrow{f} \mathcal{C}$ where

$$\theta_n \colon \Delta^{n+1} \longrightarrow \Delta^1, i \longmapsto \begin{cases} 0, & \text{if } i \leqslant n \\ 1, & \text{if } i = n+1 \end{cases}$$

- The image of the edge $n+1 \to n+2$ under $\widetilde{\sigma}$ is degenerate on y.

Given a simplex $\tau : \Delta^m \to \mathcal{C}_f$ and a monotone map $\ell : [n] \to [m]$ we define $\ell^*(\tau)$ to be the composite

$$\ell^* \colon \Delta^{n+2} \longrightarrow \Delta^{m+2} \longrightarrow \mathcal{C}$$

where $\ell^*(n+1) = m+1$, $\ell^*(n+2) = m+2$ and $\ell^*(i) = \ell(i)$ if $i \leqslant n$. This definition comes equipped with a canonical map $p_f \colon \mathcal{C}_f \longrightarrow \mathcal{C}(x, y)$ that only remembers the $(n+2)$-face.

Lemma 5.1.2. *The morphism $p_f\colon \mathcal{C}_f \longrightarrow \mathcal{C}(x,y)$ is a Kan fibration. In particular $\mathcal{C}_f$ is an ∞-groupoid.*

Proof. Since the base of this map is an ∞-groupoid it suffices to show that p_f is a right fibration by [Lur09a, Lem. 2.1.3.3]. Let $0 < i \leqslant n$, after some unraveling we see that we need to provide a solution to the lifting problem

$$\begin{array}{ccc} A_i^n & \xrightarrow{\alpha} & \mathcal{C} \\ \downarrow & \nearrow & \\ \Delta^{n+2} & & \end{array}$$

for $n \geqslant 1$ where A_i^n is the subsimplicial set consisting containing all faces except the face missing i and the face missing $n+1$. We observe that we can extend the map α to Λ_i^{n+2} due to the fact that the $(n+1)$-face must be degenerate on the edge f. Since Λ_i^{n+2} is an inner horn and $\mathcal{C}$ is an ∞-category the desired lift exists. One easily verifies that any lift must satisfy by construction the two conditions in Definition 5.1.1. □

Lemma 5.1.3. *$\mathcal{C}_f$ is a contractible ∞-groupoid.*

Proof. We will show that that we can lift boundary inclusions $\partial\Delta^n \to \Delta^n$ for $n \geqslant 0$ against $\mathcal{C}_f$. For $n = 0$ it is enough to note that $\mathcal{C}_f$ is non empty since we always have an object given by a degenerate 2-simplex on f. For $n \geqslant 1$ we proceed as in the previous lemma and arrive to the following lifting problem

$$\begin{array}{ccc} B^n & \xrightarrow{\beta} & \mathcal{C} \\ \downarrow & \nearrow & \\ \Delta^{n+2} & & \end{array}$$

where B^n contains all the faces except those missing the vertex $n+1$ and $n+2$ respectively. In a similar way as we did before we can extend β to an outer horn Λ_{n+2}^{n+2}. We note that β maps the last edge to an equivalence so the desired lift exists. □

Proposition 5.1.4. *Let $L\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor of ∞-categories. Given $x, y \in \mathcal{C}$ denote by*

$$L_{x,y}\colon \mathcal{C}(x,y) \longrightarrow \mathcal{D}\left(L(x), L(y)\right)$$

the induced morphism on mapping spaces. Let $g \in \mathcal{D}(L(x), L(y))$ and consider the pullback diagram

$$\begin{array}{ccc} L^*(\mathcal{D}_g) & \longrightarrow & \mathcal{D}_g \\ \downarrow & & \downarrow{\scriptstyle p_g} \\ \mathcal{C}(x,y) & \longrightarrow & \mathcal{D}\left(L(x), L(y)\right) \end{array}$$

Then $L_{x,y}$ is an equivalence of ∞-groupoids if and only if $L^*(\mathcal{D}_g) \simeq \Delta^0$ for every $g \in \mathcal{D}\left(L(x), L(y)\right)$.

Proof. By Lemma 5.1.2 it follows that the pullback diagram above is a homotopy pullback. In addition, Lemma 5.1.3 shows that p_g is equivalent to the map selecting the object g in the mapping space. We conclude that $L^*(\mathcal{D}_g)$ represents the homotopy fiber and the result follows. □

5.1.2 The main result

Definition 5.1.5. We define a 2-category $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$ consisting in:

- A pair of objects $-$ and $+$.
- The mapping categories $\mathbb{A}\mathrm{dj}^{\mathrm{R}}(x,y)$ are given by the terminal category $*$ except when $x=y=-$, in which case $\mathbb{A}\mathrm{dj}^{\mathrm{R}}(-,-)$ is given by a unique morphism which we denote as $\eta : \mathrm{id}_- \Rightarrow RL$.

We depict the 2-category $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$ diagrammatically as follows:

$$- \underset{R}{\overset{L}{\rightleftarrows}} + \;, \qquad \mathrm{id}_- \overset{\eta}{\Longrightarrow} RL$$

This data is required to satisfy the following relations:

- $LR = \mathrm{id}_+$.
- $L * \eta = \mathbb{1}_L$.
- $\eta * R = \mathbb{1}_R$.

This definition yields a 2-category that we will call *the walking adjunction with fully faithful right adjoint.*

Definition 5.1.6. Let $\mathrm{Adj}^{\mathrm{R}}$ be the underlying 1-category of $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$. We denote by $\iota\colon \mathrm{Adj}^{\mathrm{R}} \longrightarrow \mathbb{A}\mathrm{dj}^{\mathrm{R}}$ the canonical inclusion functor.

Remark 5.1.7. To ease the notation we will denote the 2-category $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$ simply by $\mathbb{A}$ and similarly we will use the notation A for $\mathrm{Adj}^{\mathrm{R}}$.

Proposition 5.1.8. *Let $x \in \{-,+\}$ and consider the following pullback diagram*

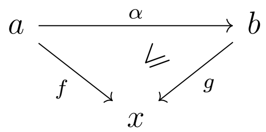
$$\begin{array}{ccc} A^{\natural}_{\nearrow x} & \xrightarrow{\iota_x} & \mathbb{A}^{\natural}_{\nearrow x} \\ \downarrow & & \downarrow \\ A & \xrightarrow{\iota} & \mathbb{A} \end{array}$$

where we are marking those morphisms in $\mathbb{A}_{\nearrow x}$ (resp. $A_{\nearrow x}$) such their associated 2-simplex is thin. Then i_x is marked cofinal.

Before embarking upon the proof of this fact we will need to prepare the necessary notation for the proof. Given $f \in \mathbb{A}^{\natural}_{/x}$ we set

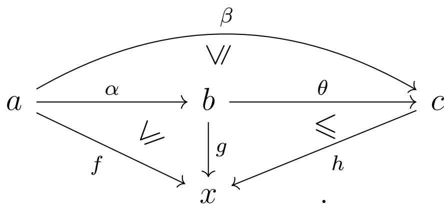
$$\left(\mathbb{A}_{\nearrow x}\right)_{f\nearrow} := \mathbb{A}_{f\nearrow}, \text{ and similarly } \left(A_{\nearrow x}\right)_{f\nearrow} := A_{f\nearrow}.$$

Since all the 2-categories we we will be considering for the proof of Proposition 5.1.8 are enriched in posets we will denote 2-morphisms with “$\leqslant$“. Let $f : a \to x$ and recall that the objects $X \in A_{f\nearrow}$ are given by diagrams



where $f \leqslant g \circ \alpha$. We will use the notation $X = (\alpha, g)$. We say that X is a *marked object* if and only if $f = g \circ \alpha$.

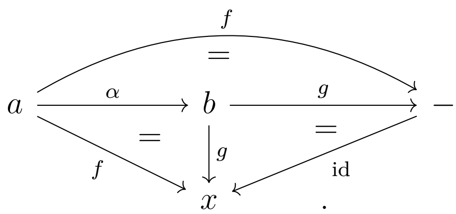
A morphism $(\alpha, g) \xrightarrow{\theta} (\beta, h)$ is given by a 2-commutative diagram



It follows that θ is marked if and only if $g = h \circ \theta$ and $\theta \circ \alpha = \beta$. This choice of marking defines a marked category that we will denote $A^{\diamond}_{f\nearrow}$.

Proof of Proposition 5.1.8. We start by proving the case $x = +$. Observe that for every 2-category $\mathbb{B}$ and every $b \in \mathbb{B}$ the morphism $\Delta^0 \longrightarrow \mathbb{B}^{\natural}_{\nearrow b}$ selecting the identity on b is marked cofinal. Therefore it will suffice to show that id_+ is terminal (in the marked sense) in $A^{\natural}_{\nearrow +}$. A quick inspection shows that all the morphisms in this category are marked and that id_+ is terminal in the usual sense.

The case $x = -$ is slightly more delicate and will require the use of Theorem 4.0.31. We will prove the conditions of the theorem are satisfied for $f \in \{\mathrm{id}_-, R, RL\}$. Let $f \in \{R, RL\}$. Then it follows that all objects of $A^{\diamond}_{f\nearrow}$ are marked. This in turn implies that the conditions of Theorem 4.0.31 reduce to showing that $L_W\left(A^{\diamond}_{f\nearrow}\right) \simeq \Delta^0$. Observe that for every marked object $X \in A^{\diamond}_{f\nearrow}$ we have a canonical choice of marked morphism,



If $f = R$ it follows that $X = (\mathrm{id}_+, R)$ is already initial in $A^{\diamond}_{R\nearrow}$ and consequently we have that the localization of $A^{\diamond}_{R\nearrow}$ is given by Δ^0. If $f = RL$ the object $Y = (L, R)$ is terminal in $A^{\diamond}_{RL\nearrow}$ and the conclusion also holds.

In the case when $f = \mathrm{id}$ (where we are abusing notation by omitting the subscript “-”) we observe that $A^{\diamond}_{\mathrm{id}\nearrow}$ only has one marked object, namely $\nabla = (\mathrm{id}, \mathrm{id})$. Furthermore a similar argument as before shows that all the non-marked objects are equivalent to each other in the localization. Let $\eta = (L, R)$

be one of those non-marked objects and denote by $\mathcal{A}_{f\nearrow} = L_W\left(A^{\diamond}_{f\nearrow}\right)$. We will show that

$$\mathcal{A}_{\mathrm{id}\nearrow}(\nabla,\eta) \simeq \mathcal{A}_{\mathrm{id}\nearrow}(\nabla,\nabla) = *.$$

Suppose we are given a zig-zag of morphisms with source and target ∇. It is easy to see that all the intermediate objects in this zig-zag must be equal to ∇ and that all the morphisms must be the identity. Using the hammock localization as a model for $\mathcal{A}_{id\nearrow}$ ([DK80]) we deduce that $\mathcal{A}_{\mathrm{id}\nearrow}(\nabla,\nabla) = *$. To finish the proof we construct a pair of marking preserving adjoint functors

$$R^* : A^{\diamond}_{\mathrm{id}\nearrow} \rightleftarrows A^{\diamond}_{R\nearrow} : L^*$$

with $R^* \dashv L^*$. Let $Z = (\alpha,\beta) \in A^{\diamond}_{R\nearrow}$ then $L^*(\alpha,\beta) = (\alpha \circ L, \beta)$. The action on morphisms is the obvious one. It is immediate to see that L^* preserves marked morphisms. The definition of R^* given in an analogous way by precomposition with R. It is easy to see that our definitions yield a pair of adjoint functors. Since both functors are marking preserving it follows that the adjunction descends to the localization. We can finish the proof after observing that

$$\mathcal{A}_{\mathrm{id}\nearrow}(\nabla,\eta) = \mathcal{A}_{\mathrm{id}\nearrow}(\nabla, L^*(\mathrm{id}_+, R)) \simeq \mathcal{A}_{R\nearrow}((R,\mathrm{id}_-),(\mathrm{id}_+,R)) \simeq * \qquad \square.$$

Definition 5.1.9. Let $F \in \mathrm{Fun}(A, \mathbb{C}\mathrm{at}_\infty)$ and denote $F(-) = \mathcal{C}_-$, $F(+) = \mathcal{C}_+$. We define an ∞-bicategory, $\mathrm{Fun}^{\dashv}(A, \mathbb{C}\mathrm{at}_\infty)$ consisting in those functors F satisfying the following condition:

- Given $x \in \mathcal{C}_-$ and $y \in \mathcal{C}_+$ the canonical map

  $$\mathcal{C}_-\left(x, F(R)(y)\right) \longrightarrow \mathcal{C}_+\left(F(L)(x), y\right)$$

  is an equivalence of Kan complexes.

Remark 5.1.10. Unraveling the definitions we see that $\mathrm{Fun}^{\dashv}(A, \mathbb{C}\mathrm{at}_\infty)$ consists in those functors mapping L, R to an adjunction $F(L) \dashv F(R)$ with the counit being an equivalence. Let us remark that the previous definition is stable under natural equivalence. In addition, we observe that we can produce a factorization of the restriction functor

$$\iota^*\colon\ \mathrm{Fun}(\mathbb{A}, \mathbb{C}\mathrm{at}_\infty) \longrightarrow \mathrm{Fun}^{\dashv}(A, \mathbb{C}\mathrm{at}_\infty).$$

Remark 5.1.11. When no confusion should arise we will abuse notation by denoting $F(L)$ (resp. $F(R)$) simply by L (resp. R). We further set the notation

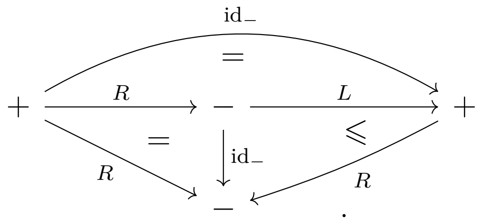
$$F_{/x}\colon A_{\nearrow x} \longrightarrow A \longrightarrow \mathbb{C}\mathrm{at}_\infty$$

for $x \in \{-,+\}$.

Lemma 5.1.12. *Let* $[2]$ *be the usual ordinal category. We define a marking on* $[2]$ *by declaring all morphisms marked* except $1 \to 2$. *The corresponding marked category will be denoted by* $[2]^{\bigstar}$. *We define a marked functor*

$$T\colon [2]^{\bigstar} \longrightarrow A^{\natural}_{\nearrow -}$$

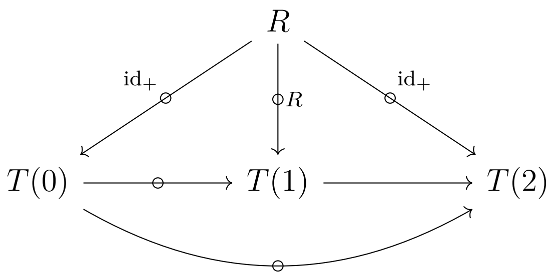
by sending $T(0) = R$*,* $T(1) = \mathrm{id}_-$ *and* $T(2) = R$*. The action on morphisms is specified by the diagram*



Then T *is marked cofinal.*

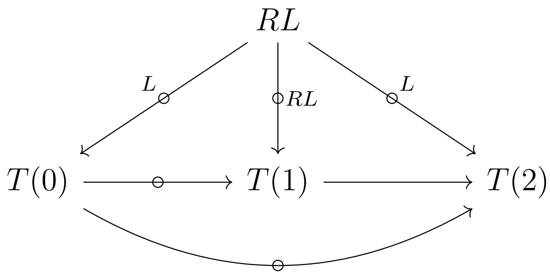
Proof. Although the proof is quite elementary we include it in this document for the sake of completeness. As usual, we will show that the conditions of Theorem 4.0.31 are satisfied. In order to do so, we compute the comma categories $[2]^{\bigstar}_{f\nearrow}$ for $f \in A_{\nearrow -}$.

$f = R$: After inspection we arrive at the diagram



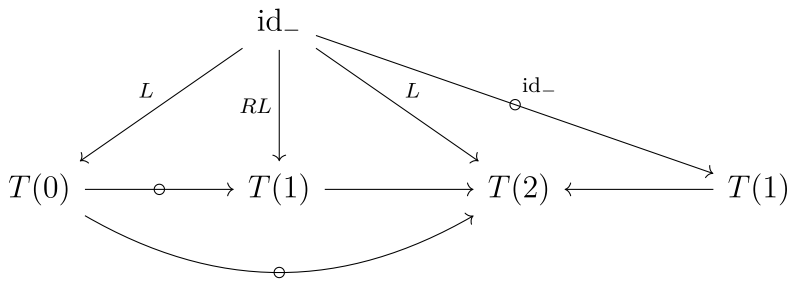
This shows that $L_W\left([2]^{\bigstar}_{R\nearrow}\right) \simeq \Delta^0$.

$f = RL$: In this case we have



Again, this shows that $L_W\left([2]^{\bigstar}_{RL\nearrow}\right) \simeq \Delta^0$.

$f = \mathrm{id}_-$: For the final case we compute



We immediately conclude that $L_W\left([2]^{\bigstar}_{\mathrm{id}\nearrow}\right) \simeq \Delta^1$ so again the conditions of Theorem 4.0.31 are clearly satisfied. This finishes the proof. □

Proposition 5.1.13. *Let $F \in \mathrm{Fun}^{\dashv}(A, \mathbb{C}\mathrm{at}_\infty)$ and let $1_x \colon (\Delta^0)^\sharp \longrightarrow (A_{/x})^\natural$ be the functor selecting the object id_x. Then the morphism induced via restriction along 1_x*

$$\mathcal{C}_x \xrightarrow{\simeq} \operatorname*{colim}^{\natural}_{A_{/x}} F_{/x}$$

is an equivalence of ∞-categories.

Proof. Since the property of belonging to $\mathrm{Fun}^{\dashv}(A, \mathbb{C}\mathrm{at}_\infty)$ is stable under equivalence, we can assume that F arises as the homotopy coherent nerve of a projectively fibrant functor

$$F \colon A \longrightarrow \mathrm{Set}^+_\Delta .$$

In particular, it follows that the composite LR is the identity functor on $\mathcal{C}_+$.

We will analyze each case separately. The case $x = +$ is obvious since 1_+ is marked cofinal as seen in the proof of Proposition 5.1.8. One could be tempted to think that 1_- is also marked cofinal. If that were the case, then id_- would be a terminal object (in the usual sense) in $A_{\nearrow -}$ which is not true. Observe that we can factor 1_- as

$$(\Delta^0)^\sharp \longrightarrow [2]^{\bigstar} \longrightarrow A^{\natural}_{\nearrow -}$$

This fact coupled with Lemma 5.1.12 shows that it suffices to show that the induced morphism

$$\mathcal{C}_- \xrightarrow{\simeq} \operatorname*{colim}^{\bigstar}_{[2]} T \circ F_{/-}$$

is an equivalence of ∞-categories. Let $G = T \circ F_{/-}$ and consider the marked functor $t \colon (\Delta^1)^\sharp \longrightarrow [2]^{\bigstar}$ selecting $0 \to 1$. We further denote $G_{\leqslant 1} = G \circ t$. To finish the proof it will suffice to show that restriction along t induces an equivalence of ∞-categories.

$$\varphi \colon \operatorname*{colim}^{\sharp}_{[1]} G_{\leqslant 1} \longrightarrow \operatorname*{colim}^{\bigstar}_{[2]} G.$$

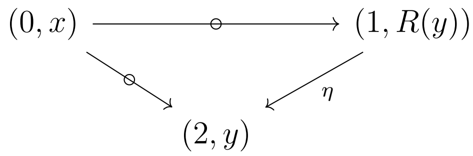
Let $\chi(G)^{\bigstar} \to \Delta^2$ be relative nerve of the functor G (and similarly for $G_{\leqslant 1}$) where the marking is given by those coCartesian edges lying over marked edges in $(\Delta^2)^{\bigstar}$. Restriction along t produces a marked functor

$$\Phi \colon \chi(G_{\leqslant 1})^\natural \longrightarrow \chi(G)^{\bigstar}.$$

where the marking of $\chi(G_{\leqslant 1})^\natural$ is given by all coCartesian edges. According to Theorem 4.1.1 it is only left to show that Φ descends to an equivalence after localization. To achieve our goal we will show that Φ satisfies the dual conditions of Theorem 4.0.31. As customary we will denote the objects in the Grothendieck construction $\chi(G)$ as pairs (i, x) where $i \in [2]$ and $k \in G(i)$. Given such pair (i, x) we consider a pullback square

$$\begin{array}{ccc} \chi(G_{\leqslant 1})^\natural_{/(i,x)} & \longrightarrow & \chi(G)^{\bigstar}_{/(i,x)} \\ \downarrow & & \downarrow \\ \chi(G_{\leqslant 1})^\natural & \longrightarrow & \chi(G)^{\bigstar} \end{array}$$

and observe that in the case where $i \in \{0,1\}$ then top horizontal arrow is an isomorphism. This shows that the conditions of Theorem 4.0.31 are trivially satisfied in this case. Let $(2,y) \in \chi(G)$ and note that every marked morphism with target $(2,y)$ is of the form $(0,x) \to (2,y)$. We further observe that we can produce a marked morphism $\chi(G_{\leqslant 1})^{\natural}_{/(2,y)}$ as follows



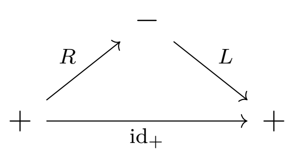
where horizontal morphism is obtained by simply applying R to $x \to y$ and the last object is induced by the identity morphism $LR(y) = y$. We conclude that it suffices to show that $\eta : (1, R(y)) \to (2,y)$ becomes a terminal object in the localization of $\chi(G_{\leqslant 1})^{\natural}_{/(2,y)}$. We will actually prove something stronger, namely that η is already terminal in $\chi(G_{\leqslant 1})_{/(2,y)}$. This is clearly enough since localization maps are cofinal as shown in [Cis19, Prop. 7.1.10].

Let $\gamma : (1,x) \to (2,y)$ be an object of $\chi(G_{\leqslant 1})_{/(2,y)}$ with associated map $u : L(x) \to y$. After some unraveling one sees that $\mathrm{Map}(\gamma, \eta)$[1] can be identified with the model of the homotopy fiber of the map

$$\mathcal{C}_{-}(x, R(y)) \longrightarrow \mathcal{C}_{+}(L(x), y)$$

provided in Proposition 5.1.4 at the object u. By hypothesis this fiber is always contractible thus showing that η is a terminal object. The case $\mathrm{Map}(\kappa, \eta)$ with $\kappa : (0,x) \to (2,y)$ is analogous and left as an exercise to the reader. This finishes the proof. □

Definition 5.1.14. Let $\mathbb{A}$ be an ∞-bicategory. We define $\mathrm{Fun}^{\dashv}(\mathrm{Adj}^{\mathrm{R}}, \mathbb{A})$ as the full subcategory of $\mathrm{Fun}(\mathrm{Adj}^{\mathrm{R}}, \mathbb{A})$ consisting in those functors $F : \mathrm{Adj}^{\mathrm{R}} \to \mathbb{A}$ such that $F(L) \dashv F(R)$ with counit given by the image under F of the 2-simplex



Theorem♦ 5.1.1. *Let $\mathbb{A}$ be an ∞-bicategory then restriction along $\iota : \mathrm{Adj}^{\mathrm{R}} \to \mathbb{A}\mathrm{dj}^{\mathrm{R}}$ induces an equivalence of ∞-bicategories*

$$\mathrm{Fun}(\mathbb{A}\mathrm{dj}^{\mathrm{R}}, \mathbb{A}) \xrightarrow{\simeq} \mathrm{Fun}^{\dashv}(\mathrm{Adj}^{\mathrm{R}}, \mathbb{A})$$

Proof. This follows from the previous discussion together with Theorem♦ 5.0.2. □

5.2 2-simplicial objects in ∞-bicategories

In this section we will study the simplex 2-category $\mathbb{\Delta}$ which is a 2-categorical enhancement of the simplex category Δ. The 2-category $\mathbb{\Delta}$ plays an important

[1] We are using an alternative notation for the mapping space to improve readability

role in the program of Dyckerhoff begun in [Dyck21] to produce a categorified theory of homological algebra. In the aforementioned document 2-simplicial objects (see Remark 5.2.3 below) are used to provide a categorified Dold-Kan correspondence. It is expected that a deep understanding of the simplex 2-category will be useful in developing a solid and usable categorified theory of homological algebra.

The motivation for most of the foundational results in ∞-bicategories that are part of the thesis was to prove that $\mathbb{\Delta}$ satisfies a certain 2-dimensional universal property that we shall explain now. To this end we will introduce some definitions.

Definition 5.2.1. Let us denote by $\mathbb{C}\mathrm{at}$ the 2-category of small categories and by $\mathbb{\Delta} \subset \mathbb{C}\mathrm{at}$ the full 2-subcategory spanned by the standard ordinals $\{[n]\}$, considered as categories. We call $\mathbb{\Delta}$ the *simplex* 2-category.

Remark 5.2.2. First, let us observe that the underlying 1-category of $\mathbb{\Delta}$ is the usual simplex category Δ. In addition, it is straightforward to see from the definition that $\mathbb{\Delta}(n, m)$ is the poset of monotone maps equipped with the pointwise order. We will use the notation $f \leqslant g$ whenever there exists a 2-morphism (which is necessarily unique) between $f, g \in \mathbb{\Delta}(n, m)$.

Remark 5.2.3. Since we will be mainly interested in 2-simplicial objects, i.e. 2-functors $X\colon \mathbb{\Delta}^{(\mathrm{op},-)} \longrightarrow \mathbb{A}$ where $\mathbb{A}$ is an ∞-bicategory, we will work with the 2-category $\mathbb{\Delta}^{(\mathrm{op},-)}$. We set the notation $\mathbb{\Delta}^{\mathrm{op}} = \mathbb{\Delta}^{(\mathrm{op},-)}$.

It is apparent from the definitions that given two face maps $d_i, d_j \in \mathbb{\Delta}^{\mathrm{op}}(n, n-1)$ we have $d_i \leqslant d_j$ if and only if $i \geqslant j$. Dually $s_i \leqslant s_j$ if and only if $i \leqslant j$ for degeneracy maps $s_i, s_j \in \mathbb{\Delta}^{\mathrm{op}}(n-1, n)$. We will show in this section is to show that previous inequalities are the generators of the 2-morphisms in $\mathbb{\Delta}^{\mathrm{op}}$.

Remark 5.2.4. Using the inequality $s_i \leqslant s_{i+1}$ we can postcompose with d_i and obtain.

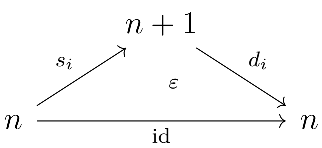
$$\mathrm{id} \leqslant d_i s_{i+1} = s_i d_i. \tag{5.1}$$

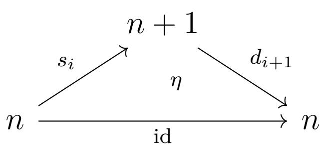
This shows that $d_i \dashv s_i$. Similarly, we can precompose in the inequality $d_{i+2} \leqslant d_{i+1}$ with s_i and obtain

$$d_{i+2} s_i = s_i d_{i+1} \leqslant \mathrm{id} \tag{5.2}$$

thus showing that $s_i \dashv d_{i+1}$.

In order to state our main result we will fix some notation for two important families of thin 2-simplices in $\mathbb{\Delta}^{\mathrm{op}}$ using the diagrams below.





Definition 5.2.5. Given an ∞-bicategory $\mathbb{A}$ we denote by $\mathrm{Fun}^{\dashv}(\Delta^{\mathrm{op}}, \mathbb{A})$ the full subcategory on those simplicial objects in $\mathbb{A}$ that send d_i and s_i (resp d_{i+1} and s_i) to an adjunction $F(d_i) \dashv F(s_i)$ with counit given by $F(\varepsilon)$ (resp. $F(s_i) \dashv F(d_{i+1})$ with unit given by $F(\eta)$).

Definition 5.2.6. We denote by $\iota : \Delta^{\mathrm{op}} \to \mathbb{\Delta}^{\mathrm{op}}$ the canonical inclusion.

The universal property of $\mathbb{\Delta}$ can be then expressed as follows.

Conjecture 5.2.1. *Let $\mathbb{A}$ be ∞-bicategory. Then restriction along ι induces an equivalence of ∞-bicategories*

$$\iota^*\colon \operatorname{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathbb{A}) \xrightarrow{\simeq} \operatorname{Fun}^{\dashv}(\Delta^{\mathrm{op}}, \mathbb{A})$$

In this document we will however, only prove one part of the previous conjecture. We will show that ι^* is fully faithful in Theorem♦ 5.2.1. The proof of this theorem will be very similar (although computationally considerably more challenging) to the proof of Proposition 5.1.8. Before diving into our main computation we establish some auxiliary lemmata.

Lemma 5.2.7. *Let $f, g\colon [n-l] \longrightarrow [n]$ be two injective monotone functions and write them in canonical form,*

$$f = d^{i_1} \cdots d^{i_l}, \;\; g = d^{j_1} \cdots d^{j_l},$$

where $i_1 > i_2 > \cdots > i_l$, $j_1 > j_2 > \cdots > j_l$. Then $f \leqslant g$ if and only if $i_k \geqslant j_k$ for $1 \leqslant k \leqslant l$.

Proof. The only if direction is clear from horizontal composition. Let us assume that $i_l < j_l$. Then we have that $f(i_l) \geqslant i_l + 1 > i_l = g(i_l)$, a contradiction. Assume that $i_k \geqslant j_k$ for $k = l, l-1, \ldots, t+1$ with $t \geqslant 1$ and suppose that $i_t < j_t$. Then we observe that,

$$f(i_t - t - 1) \geqslant i_t + 1 > i_t = g(i_t - t - 1).$$

This is again a contradiction, so $i_t \geqslant j_t$. Repeating inductively this argument shows the result. □

In a very similar way one can prove the dual statement.

Lemma 5.2.8. *Let $f, g\colon [n] \longrightarrow [n-l]$ be two monotone functions only consisting of degeneracy maps and write them in canonical form,*

$$f = s^{i_l} \cdots s^{i_1}, \;\; g = s^{j_l} \cdots s^{j_1},$$

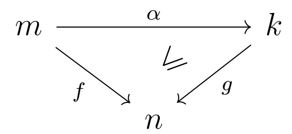
where $i_l < i_{l-1} < \cdots < i_1$, $j_l < j_{l-1} < \cdots < j_1$. Then $f \leqslant g$ if and only if $i_k \leqslant j_k$ for $1 \leqslant k \leqslant l$.

5.2.1 The main computation

In this section we will prove the following statement.

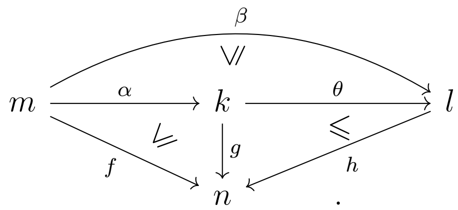
Proposition 5.2.9. *Let $n \geqslant 0$, then the canonical map $\iota_n\colon \left(\Delta^{\mathrm{op}}_{/n}\right)^{\natural} \longrightarrow \left(\mathbb{\Delta}^{\mathrm{op}}_{/n}\right)^{\natural}$ is marked cofinal.*

As we did in the previous section we will show that ι_n satisfies the conditions of Theorem 4.0.31. We will freely borrow the notation that we introduced while dealing with the category $\mathbb{A}\mathrm{dj}^{\mathrm{R}}$. Let us recall that the objects $X \in \Delta^{\mathrm{op}}_{f\nearrow}$ are given by diagrams



where $f \leqslant g \circ \alpha$. We will use the notation $X = (\alpha, g)$. We say that X is *marked object* if and only if $f = g \circ \alpha$.

A morphism $(\alpha, g) \xrightarrow{\theta} (\beta, h)$ is given by a 2-commutative diagram



It follows that θ is marked if and only if $g = h \circ \theta$ and $\theta \circ \alpha = \beta$.

Definition 5.2.10. We define a marked 1-category $\left(\Delta^{\mathrm{op}}_{f\nearrow}\right)^+$ as the full subcategory of $\Delta^{\mathrm{op}}_{f\nearrow}$ consisting on marked objects equipped with the induced marking.

Proposition 5.2.11. *The induced map in ∞-localizations $L_W\left(\left(\Delta^{\mathrm{op}}_{f\nearrow}\right)^+\right) \longrightarrow L_W\left(\Delta^{\mathrm{op}}_{f\nearrow}\right)$ is fully faithful.*

Proof. Let $X, Y \in \Delta^{\mathrm{op}}_{f\nearrow}$ such that $X = (\alpha, g)$, $Y = (\beta, h)$ and suppose that we are given a morphism $X \xrightarrow{\theta} Y$. Then it is immediate from the definitions that $g \circ \alpha \leqslant h \circ \beta$. We also see that if θ is marked then $g \circ \alpha = h \circ \beta$. We can compute $L_W\left(\Delta^{\mathrm{op}}_{f\nearrow}\right)$ using the hammock localization ([DK80]). Suppose that $X, Y \in \left(\Delta^{\mathrm{op}}_{f\nearrow}\right)^+$, and consider an arbitrary zigzag in $\Delta^{\mathrm{op}}_{f\nearrow}$

$$X \longleftarrow\!\!\circ\; A_1 \longrightarrow A_1 \cdots \longrightarrow A_s \longrightarrow Y$$

then it follows from the previous discussion that $A_i \in \left(\Delta^{\mathrm{op}}_{f\nearrow}\right)^+$ for $i = 1, 2, \ldots, s$. This finishes the proof. ☐

The key technical part of this section is to show that $L_W\left(\left(\Delta^{\mathrm{op}}_{\mathrm{id}_n\nearrow}\right)^+\right) \simeq *$. For this purpose, it will be necessary to introduce some definitions.

Remark 5.2.12. Let $X \in \left(\Delta^{\mathrm{op}}_{\mathrm{id}_n\nearrow}\right)^+$ with $X = (\alpha, \overline{\alpha})$ and $\alpha \neq \mathrm{id}_n$. Since $\overline{\alpha} \circ \alpha = \mathrm{id}_n$. It follows that

$$\alpha = s_{\alpha_1} \cdots s_{\alpha_m}, \quad \alpha_1 > \alpha_2 > \cdots > \alpha_m$$

$$\overline{\alpha} = d_{\overline{\alpha}_m} \cdots d_{\overline{\alpha}_1} \quad \overline{\alpha}_m < \overline{\alpha}_{m-1} < \cdots < \overline{\alpha}_1$$

where $d_{\overline{\alpha}_i} s_{\alpha_i} = \mathrm{id}$ for $i = 1, 2, \ldots, m$.

Definition 5.2.13. Let $X \in \left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+$ we define $d(X) \in \mathbb{N}$ by the formula

$$d(X) = \begin{cases} 1, & \text{if } X = (\mathrm{id}_n, \mathrm{id}_n) \\ \frac{1}{m}\sum\limits_{i=1}^{m} \overline{\alpha}_i - \alpha_i, & \text{otherwise.} \end{cases}$$

and call it the *discrepancy* of X.

Definition 5.2.14. Let $\mathfrak{S}^1_n$ be the full subcategory of $\left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+$ on those objects such that $d(X) = 1$. We regard $\mathfrak{S}^1_n$ as marked category by means of the induced marking.

Proposition 5.2.15. *There is an equivalence of ∞-categories $L_W\left(\mathfrak{S}^1_n\right) \simeq *$.*

Proof. Let $\nabla_n \in \mathfrak{S}^1_n$ with $\nabla_n = (\mathrm{id}_n, \mathrm{id}_n)$. It will suffice to show that ∇_n is initial in $\mathfrak{S}^1_n$ and that for every $X \in \mathfrak{S}^1_n$ the unique morphism $\nabla_n \xrightarrow{\circ} X$ is marked. Let $X = (\alpha, \overline{\alpha})$ then we see that any morphism $\nabla_n \xrightarrow{\theta} X$ satisfies $\overline{\alpha} \circ \theta = \mathrm{id}_n$ with $\theta \leqslant \alpha$. Let $\theta = s_{\theta_1} \cdots s_{\theta_m}$ then it follows from Lemma 5.2.8 that $\theta_i \leqslant \alpha_i$ for $i = 1, 2, \ldots, m$. Since $d(X) = 1$ it follows that $\overline{\alpha}_i = \alpha_i + 1$ for $i = 1, 2, \ldots, m$. This finally implies that $\theta_i = \alpha_i$ for all indices $i = 1, 2, \ldots, m$ and thus the proof is finished. □

Our strategy will be to show that the canonical map $i \colon \mathfrak{S}^1_n \longrightarrow \left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+$ induces an equivalence on ∞-localizations.

Remark 5.2.16. Let $X \in \left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+$ with $X = (\alpha, \overline{\alpha})$ such that $d(X) \neq 1$. We express α in canonical form $\alpha = s_{\alpha_1} \cdots s_{\alpha_m}$ and we let i_1 be the first index such that $\overline{\alpha}_{i_1} - \alpha_{i_1} = 0$. We define $\widetilde{X} \in \left(\Delta^{\mathrm{op}}_{\mathrm{id}_n /}\right)^+$ with $\widetilde{X} = (\beta, \overline{\beta})$, $\beta = s_{\beta_1} \ldots s_{\beta_{m-1}}$, $\overline{\beta} = d_{\overline{\beta}_{m-1}} \cdots d_{\overline{\beta}_1}$ as follows

$$\beta_j = \begin{cases} \alpha_j - 1, & \text{if } j < i_1 \\ \alpha_{j+1}, & \text{if } j \geqslant i_1 \end{cases} \qquad \overline{\beta}_j = \begin{cases} \overline{\alpha}_j - 1, & \text{if } j < i_1 \\ \overline{\alpha}_{j+1}, & \text{if } j \geqslant i_1 \end{cases}$$

It follows from the simplicial identities that we have a marked morphism $X \xrightarrow[\circ]{d_{\overline{\alpha}_{i_1}}} \widetilde{X}$. In addition, we see that by construction we get $d(\widetilde{X}) > d(X)$. Repeating this process inductively we can produce a marked morphism

$$\xi_X : X \xrightarrow{\circ} r(X)$$

with $\xi_X = d_{\overline{\alpha}_{i_l}} \cdots d_{\overline{\alpha}_{i_1}}$, where $\{i_1 < \cdots < i_l\} \subseteq \{1, \ldots, m\}$ is the subset of those indices such that $\overline{\alpha}_{i_k} - \alpha_{i_k} = 0$ for $k = 1, \ldots, l$. It is clear that $r(X) \in \mathfrak{S}^1_n$.

Lemma 5.2.17. *Let $\theta \colon X \longrightarrow Y$ be a morphism in $\left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+$ such that $Y \in \mathfrak{S}^1_n$. If $\theta = s_{u_1} \cdots s_{u_s}$ then $X \in \mathfrak{S}^1_n$.*

Proof. Let $X = (\alpha, \overline{\alpha})$ and $Y = (\beta, \overline{\beta})$ with $\beta = s_{\beta_1} \cdots s_{\beta_m}$ we will assume that $\alpha \neq \mathrm{id}_n$ since otherwise $X = \nabla_n$. We will use induction on s. For the case

$s = 1$ we note as usual that $\overline{\beta} \circ s_u \circ \alpha = \mathrm{id}_n$. Let i_1 be the first index such that $d_{\overline{\beta}_{i_1}} s_u = \mathrm{id}$. We construct the object $\widetilde{Y} = (\varepsilon, \overline{\varepsilon})$ with $\varepsilon = s_{\varepsilon_1} \cdots s_{\varepsilon_{m-1}}$ as follows

$$\varepsilon_j = \begin{cases} \beta_j - 1, & \text{if } j < i_1 \\ \beta_{j+1}, & \text{if } j \geqslant i_1 \end{cases} \qquad \overline{\varepsilon}_j = \begin{cases} \overline{\beta}_j - 1, & \text{if } j < i_1 \\ \overline{\beta}_{j+1}, & \text{if } j \geqslant i_1 \end{cases}$$

It is clear from construction that $\widetilde{Y} \in \mathfrak{S}_n^1$ and that we have a marked morphism $Y \xrightarrow{d_{\overline{\beta}_{i_1}}} \widetilde{Y}$. It follows immediately that $\alpha \leqslant \varepsilon$, $\overline{\alpha} \leqslant \overline{\varepsilon}$. Then we can use effectively the same argument as in Proposition 5.2.15 to show that $\alpha = \varepsilon$ and that $\overline{\alpha} = \overline{\varepsilon}$. The induction step is clear and left as an exercise to the reader. □

Remark 5.2.18. Let $\theta \colon X \longrightarrow Y$ be a morphism in $\left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+$ such that $\theta = d_{u_s} \cdots d_{u_1}$. Let $X = (\alpha, \overline{\alpha})$ with $\alpha = s_{\alpha_1} \cdots s_{\alpha_m}$. Since $\overline{\beta} \circ \theta \circ \alpha = \mathrm{id}_n$ (where $Y = (\beta, \overline{\beta})$) it follows that $\theta \circ \alpha$ consists only of degeneracy maps. Given $j \in \{1, \dots, s\}$ let $i_j \in \{1, \dots, m\}$ be the smallest index such that $d_{u_j} s_{\alpha_{i_j}} = \mathrm{id}$. It is an easy exercise to check that $i_1 < \cdots < i_s$.

Lemma 5.2.19. *Let $\theta \colon X \longrightarrow Y$ be a morphism in $\left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+$ as in Remark 5.2.18. Then $\overline{\alpha}_{i_l} \geqslant u_l$ for $l = 1, \dots, s$.*

Proof. We will proceed by induction on s. Let $\theta = d_u$ and suppose for contradiction that $\alpha_{i_1} = \overline{\alpha}_{i_1} = u - 1$. Since $d_u \circ \alpha \leqslant \beta$ we see that Lemma 5.2.8 implies that $\alpha_j - 1 \leqslant \beta_j$, if $j < i_1$. Recall that $\overline{\alpha} \leqslant \overline{\beta} \circ d_u$, so after rearranging $\overline{\beta} \circ d_u$ we obtain the following canonical form

$$\overline{\beta} \circ d_u = d_{\overline{\beta}_{(m-1)}} \cdots d_{\overline{\beta}_k} d_u d_{\overline{\beta}_{(k-1)}-1} \cdots d_{\overline{\beta}_1 - 1}$$

where $\beta_k < u$. We find as consequence of Lemma 5.2.7 that $k < i_1$ and that

$$\overline{\alpha}_{k+1} \geqslant \overline{\beta}_k, \quad \overline{\alpha}_k \geqslant u.$$

Moreover, since $\overline{\alpha}_{k+1} < \overline{\alpha}_k$ we see that $\overline{\alpha}_k > \overline{\beta}_k$. Therefore, we obtain the following system of inequalities

$$\alpha_k - 1 \leqslant \beta_k \tag{5.3}$$

$$\overline{\alpha}_k > \overline{\beta}_k \tag{5.4}$$

$$\overline{\alpha}_k \geqslant u \tag{5.5}$$

$$\overline{\beta}_k < u \tag{5.6}$$

There are two cases to check $\alpha_k = \overline{\alpha}_k$ and $\alpha_k + 1 = \overline{\alpha}_k$. For the first case we can use equations (3), (4) to obtain $\alpha_k = \beta_k + 1$. This together with equations (5), (6) shows that $\alpha_k = u$. Then it follows that $d_u s_{\alpha_k} = \mathrm{id}$ which contradicts minimality of i_1. If $\alpha_k + 1 = \overline{\alpha}_k$ equations (3), (4) imply that $\alpha_k = \beta_k$ and similarly equations (5), (6) imply that $u - 1 = \alpha_k$ which is again a contradiction. We have shown that $\overline{\alpha}_{i_1} \geqslant u$.

Suppose that the claim holds for $s - 1$ and let $\theta = d_{u_s} \cdots d_{u_1}$. Given $k \in \{1, \dots, s\}$ we define $X_k \in \left(\Delta^{\mathrm{op}}_{\mathrm{id}_n /}\right)^+$ with $X_k = (\varepsilon^k, \overline{\varepsilon}^k)$, $\varepsilon^k = s_{\varepsilon_1^k} \cdots s_{\varepsilon_{m-1}^k}$ as follows

$$\varepsilon_j^k = \begin{cases} \alpha_j - 1, & \text{if } j < i_k \\ \alpha_{j+1}, & \text{if } j \geqslant i_k \end{cases} \qquad \overline{\varepsilon}_j^k = \begin{cases} \overline{\alpha}_j - 1, & \text{if } j < i_k \\ \overline{\alpha}_{j+1}, & \text{if } j \geqslant i_k \end{cases}$$

We observe that there is a marked morphism $s_{\alpha_{i_k}} \colon X_k \longrightarrow X$. . It is immediate that

$$\theta \circ s_{\alpha_{i_k}} = d_{u_s} \cdots d_{u_{k+1}} d_{u_{(k-1)}-1} \cdots d_{u_1 - 1}.$$

Then induction hypothesis now shows that $\overline{\alpha}_{i_l} \geqslant u_l$ for $l \in \{1, \ldots, s\} \setminus \{k\}$. Repeating the same argument with $k' \neq k$ gives the result. □

Remark 5.2.20. Let $\theta \colon X \longrightarrow Y$ be a morphism in $\left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+$ such that $\theta = s_{v_1} \ldots s_{v_t} d_{u_s} \cdots d_{u_1}$ and let us define $\{i_1 < \cdots < i_s\} \subseteq \{1, 2, \ldots, m\}$ in an analogous way as before. Then it is also true that $\overline{\alpha_{i_j}} \geqslant u_j$ for every $j = 1, \ldots, s$. Indeed it is easy to construct a morphism $\gamma \colon Y \longrightarrow Z$ such that $\gamma \circ \beta = d_{u_s} \cdots d_{u_1}$ and the conclusion follows from Lemma 5.2.19.

Proposition 5.2.21. *Let $\theta \colon X \longrightarrow Y$ be a morphism in $\left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+$ such that $d(X) \neq 1$ and $Y \in \mathfrak{S}^1_n$. Then there exists a unique morphism $r(\theta) \colon r(X) \longrightarrow Y$ such that $r(\theta) \circ \xi_X = \theta$.*

Proof. First, let us remark that uniqueness of the map $r(\theta)$ follows immediately from the fact that ξ_X is an epimorphism in Δ^{op}. Using Lemma 5.2.17 we see that $\theta = s_{v_1} \cdots s_{v_r} d_{u_s} \cdots d_{u_1}$ since otherwise $X \in \mathfrak{S}^1_n$. Let $X = (\alpha, \overline{\alpha})$ with $\alpha = s_{\alpha_1} \cdots s_{\alpha_m}$ and let $\{i_1 < i_2 < \cdots < i_s\} \subseteq \{1, 2, \ldots, m\}$ as in the proof of Lemma 5.2.19. Since $\overline{\alpha}_{i_1} \geqslant u_1$ (see Remark 5.2.20) we can factor θ as

$$X \xrightarrow{d_{u_1}} X_1 \xrightarrow{\hat{\theta}} Y.$$

with X_1 also as in Lemma 5.2.19. Repeating this procedure we can factor $\theta = \theta_s \circ \theta_d$ where $\theta_s = s_{v_1} \ldots s_{v_r}$ and $\theta_d = d_{u_s} \cdots d_{u_1}$. Using Lemma 5.2.17 we conclude that we can assume that $\theta = d_{u_s} \ldots d_{u_1}$.

It is easy to check that $\alpha_j + 1 = \overline{\alpha_j}$ for every $j \in \{1, \ldots, m\} \setminus \{i_1, i_2, \ldots, i_s\}$. Let $l \in \{1, \ldots, s\}$ such that $\alpha_{i_l} = \overline{\alpha}_{i_l}$ then by definition $\alpha_{i_l} \leqslant u_l$ and by Lemma 5.2.19 $\alpha_{i_l} = \overline{\alpha}_{i_l} \geqslant u_l$ so it follows that $\alpha_{i_l} = u_l$. This shows that after rearranging the face maps we obtain $\theta = r(\theta) \circ \xi_X$ thus finishing the proof. □

Proposition 5.2.22. *There is an equivalence of ∞-categories $L_W\left(\left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+\right) \simeq *$.*

Proof. By Proposition 5.2.15 it will suffice to show that the functor $i \colon \mathfrak{S}^1_n \longrightarrow \left(\Delta^{\mathrm{op}}_{\mathrm{id}_n /}\right)^+$ induces an equivalence on ∞-localizations. We define a functor $\mathcal{R} \colon \left(\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}\right)^+ \longrightarrow \mathfrak{S}^1_n$ whose action on objects is given by

$$\mathcal{R}(X) = \begin{cases} X, & \text{if } d(X) = 1 \\ r(X), & \text{if } d(X) \neq 1. \end{cases}$$

Given $X \xrightarrow{\theta} Y$ we can use Proposition 5.2.21 to define

$$\mathcal{R}(\theta) = \begin{cases} r(\theta), & \text{if } d(Y) = 1 \\ r(\xi_Y \circ \theta), & \text{if } d(Y) \neq 1. \end{cases}$$

It is straightforward to check that $\mathcal{R}$ is a well defined functor and that it maps marked edges to marked edges. Observe that by construction $\mathcal{R} \circ i = \mathbb{1}$. Finally we define a natural transformation $\Xi \colon \mathbb{1} \Longrightarrow i \circ \mathcal{R}$ given by

$$\Xi_X = \begin{cases} \mathrm{id}_X, & \text{if } d(X) = 1 \\ \xi_X, & \text{if } d(X) \neq 1. \end{cases}$$

Finally we note that Ξ_X is always a marked morphism so $\mathcal{R}$ and i descend to mutually inverse functors in the ∞-localizations. □

Now that we have established the preliminary lemmata regarding the categories $\Delta^{\mathrm{op}}_{\mathrm{id}_n \nearrow}$ we turn our attention to the categories $\left(\Delta^{\mathrm{op}}_{\nearrow n}\right)_{f\nearrow}$ for a general morphism $f : [m] \to [n]$.

Lemma 5.2.23. *Let $f \colon [m] \longrightarrow [n]$ be a morphism in Δ^{op}. Let $d_i \colon [m+1] \longrightarrow [m]$ be a face operator with $i > 0$, then the adjunction $s_{i-1} \dashv d_i$ induces the following pair of adjoint functors,*

$$(d_i)^* : \left(\Delta^{\mathrm{op}}_{\nearrow n}\right)_{f\nearrow} \rightleftarrows \left(\Delta^{\mathrm{op}}_{\nearrow n}\right)_{fd_i\nearrow} : (s_{i-1})^*$$

In addition, both functors preserve the markings thus descending to adjoint functors in the localization.

Proof. We define

$$\begin{aligned} (d_i)^* \colon \left(\Delta^{\mathrm{op}}_{\nearrow n}\right)_{f\nearrow} &\longrightarrow \left(\Delta^{\mathrm{op}}_{\nearrow n}\right)_{fd_i\nearrow} \\ (\alpha, g) &\longmapsto (\alpha d_i, g) \end{aligned}$$

and note that is clearly functorial an preserves the marking. Similarly we define,

$$\begin{aligned} (s_{i-1})^* \colon \left(\Delta^{\mathrm{op}}_{\nearrow n}\right)_{fd_i\nearrow} &\longrightarrow \left(\Delta^{\mathrm{op}}_{\nearrow n}\right)_{f\nearrow} \\ (\gamma, h) &\longmapsto (\gamma s_{i-1}, h) \end{aligned}$$

where we define the 2-morphism necessary via $f = fd_i s_{i-1} \leqslant h\gamma s_{i-1}$. We observe that

$$\mathbb{1} = (s_{i-1})^* \circ (d_i)^*,$$

and that we have a natural transformation $(d_i)^* \circ (s_{i-1})^* \Longrightarrow \mathbb{1}$. One easily checks that the snake relations are satisfied. □

Lemma 5.2.24. *Let $f \colon [m] \longrightarrow [n]$ be a morphism in Δ^{op} and let $d_i \colon [n] \longrightarrow [n-1]$ be face operator with $i < n$. The adjunctions $d_i \dashv s_i$ induce the following pair of adjoint functors,*

$$(d_i)_* : \left(\Delta^{\mathrm{op}}_{\nearrow n}\right)_{f\nearrow} \rightleftarrows \left(\Delta^{\mathrm{op}}_{\nearrow n-1}\right)_{d_i f\nearrow} : (s_i)_*$$

preserving the respective markings and thus inducing an adjunction of the localization.

Proof. We define,

$$(s_i)_* \colon \left(\Delta^{\mathrm{op}}_{\nearrow n-1}\right)_{d_i f} \longrightarrow \left(\Delta^{\mathrm{op}}_{\nearrow n}\right)_{f\nearrow}$$
$$(\gamma, g) \longmapsto (\gamma, s_i g)$$

where the 2-morphism is obtained as the composite $f \leqslant s_i d_i f \leqslant s_i g\gamma$. The definition of $(d_i)_*$ and the construction of the unit and the counit is straightforward and left to the reader. □

Remark 5.2.25. In the next proofs we will have to use the mapping space in the categories $\Delta^{\mathrm{op}}_{\nearrow f}$. To ease the notation given a pair of objects $X, Y \in \Delta^{\mathrm{op}}_{\nearrow f}$ we will denote $\Delta^{\mathrm{op}}_{\nearrow f}(X, Y)$ simply by $\mathrm{Map}(X, Y)$ hoping it will be clear from the context in which ∞-category we are considering the mapping space.

Proposition 5.2.26. *Let $n \geqslant 0$ and let $X \in L_W\left(\Delta^{\mathrm{op}}_{\mathrm{id}_n\nearrow}\right)$ be a marked object. Then X is initial.*

Proof. By Proposition 5.2.11 and Proposition 5.2.22 we only need to check that $\mathrm{Map}(X, Y) \simeq *$ when Y is not marked. We will assume $X = \nabla_n$ since all marked objects are equivalent in the localization. The proof will use induction on n. The case $n = 0$ follows from the fact that all the objects of $\Delta^{\mathrm{op}}_{\mathrm{id}_0\nearrow}$ are marked and Proposition 5.2.22. Assume that the claim holds for $n - 1$. It is straightforward to check that every non-marked object is equivalent to an object of the form $Y = (u, v)$ with $u = d_{u_m} \cdots d_{u_1}$ and $v = s_{v_1} \cdots s_{v_m}$ where $v_j \geqslant u_j$ for $j = 1, \ldots, m$. There are two cases to check $v_1 = 0$ and $v_1 > 0$.

If $v_1 = 0$ it follows that $m = 1$ and that $u_1 = 0$. Then we can express $Y = (d_0)^* Z$ with $Z = (\mathrm{id}_{n-1}, s_0)$. An analogous argument as in Lemma 5.2.23 shows that $(s_0)^* \dashv (d_0)^*$ which in turn implies that $\mathrm{Map}(\nabla_n, Y) \simeq \mathrm{Map}((s_0)^*\nabla_n, Z)$. Note since $(s_0)_*$ is a left adjoint it preserves initial objects. Then $(s_0)_*\nabla_{n-1}$ is initial and it follows that every marked object in $\Delta^{\mathrm{op}}_{s_0\nearrow}$ is also initial. This shows that $\mathrm{Map}((s_0)^*\nabla_n, Z) \simeq *$.

For the case $v_1 > 0$ we use that $Y = (s_{v_1})_* Z$ and that $(d_{v_1})_* \dashv (s_{v_1})_*$ to get $\mathrm{Map}(\nabla_n, (s_{v_1})_* Z) \simeq \mathrm{Map}((d_{v_1})_*\nabla_n, Z)$. Since $v_1 > 0$ we have an adjunction $(d_{v_1})^* \dashv (s_{v_1-1})^*$ which shows that $(d_{v_1})^*\nabla_{n-1}$ is initial. This implies $\mathrm{Map}((d_{v_1})_*\nabla_n, Z) \simeq *$, so ∇_n is initial and the proof is finished. □

Theorem♦ 5.2.1. *Let $f \colon m \longrightarrow n$ be a morphism in Δ^{op} with and let $X \in L_W\left(\Delta^{\mathrm{op}}_{f\nearrow}\right)$ be a marked object. Then X is initial. In particular, the map ι_n : $(\Delta^{\mathrm{op}})^{\natural}_{\nearrow n} \to (\mathbb{\Delta}^{\mathrm{op}})^{\natural}_{\nearrow n}$ is marked cofinal and thus for every ∞-bicategory $\mathbb{A}$, the restriction functor*

$$\iota^* \colon \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathbb{A}) \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbb{A})$$

is fully faithful.

Proof. First we will show the case where f is injective. Let s be the number of face maps appearing in the canonical form of f. We will show the claim by applying induction on s. If there are no face maps then $f = \mathrm{id}_n$ for and the claim follows from Proposition 5.2.26. Suppose that the claim holds for $s - 1$ and let $f = d_{u_s} \cdots d_{u_1}$. If $u_1 = 0$ it follows that $f = d_0 \colon n + 1 \longrightarrow n$.

Since $(d_0)_*$ is a left adjoint Proposition 5.2.26 implies that $(d_0)_*\nabla_{n+1}$ is initial and the claim holds. If $u_1 > 0$, let $\hat{f} = d_{u_s} \cdots d_{u_2}$ and consider the marked object $X = (d_{u_s} \cdots d_{u_2}, \mathrm{id}_{m-1})$ in $L_W\left(\Delta^{\mathrm{op}}_{\hat{f}\nearrow}\right)$ which is initial by the induction hypothesis. The result now follows after observing that $(d_{v_1})^*$ is a left adjoint.

For the general case we let $f = s_{v_1} \cdots s_{v_t} d_{u_s} \cdots d_{u_1}$ and proceed by induction on t. If $t = 0$ the f must be injective and the claim holds. Assume the claim to hold for $t-1$ and write $f = s_{v_1} \circ \hat{f}$. We observe that we have an adjunction,

$$(s_{v_1})_* : \left(\Delta^{\mathrm{op}}_{\nearrow n-1}\right)_{\hat{f}\nearrow} \rightleftarrows \left(\Delta^{\mathrm{op}}_{\nearrow n}\right)_{f\nearrow} : (d_{v_1+1})_*$$

which shows that the claim holds for t. Therefore, the map ι_n satisfies the condition of Theorem 4.0.31. The final part of the statement follows from Proposition♦ 5.0.4. □

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