Radiation of moving quarks in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory at Finite Temperature

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Abstract

In this thesis I study implications of finite temperature T on the supersymmetric $\mathcal{N} = 4$ Yang-Mills theory in the presence of Wilson loop defects. External particles moving through the gauge background of the field theory lose energy upon acceleration. The thus induced bremsstrahlung function B can be related to the coefficient h of the one-point function of the stress energy tensor, $B = 3h$ at zero temperature. I will first describe thermal corrections to a Ward identity at zero coupling. These can be studied with the help of finite temperature and defect bootstrap techniques, especially the operator product expansion. I will thus be able to show analytically that the relation between bremsstrahlung and stress tensor continues to hold in the finite temperature, non-interacting setting algebraically.
This result is verified in a weak coupling perturbative calculation. At leading order and in an expansion around small and high temperature I discover the existence of a regularization scheme in which the exact formula holds.
I additionally discuss effects of the interaction terms on the thermal theory. Different approaches are presented and I discuss how this might affect the thermal Broken Ward Identity and the relation between bremsstrahlung and stress tensor.
These considerations are supplemented with the second order calculation of the circular and straight line Wilson loop in a weak coupling expansion.

Zusammenfassung

In dieser Dissertation untersuche ich die Implikation von endlicher Temperatur T auf die supersymmetrische $\mathcal{N} = 4$ Yang-Mills-Theorie in der Gegenwart von Wilsonschlaufendefekten. Externe Teilchen, welche sich in einem feldtheoretischen Eichhintergund bewegen, verlieren aufgrund von Beschleunigung Energie. Die hierdurch induzierte Bremsstrahlungsfunktion, B, kann in Verbindung zum Einpunktkoeffizienten, h, des Energie-Impuls-Tensors gebracht werden, sodass $B = 3h$ bei Temperatur null gilt. Ich werde zunächst die interaktionsfreien, thermalen Korrekturen einer Wardidentität darstellen. Diese können dann mithilfe von Techniken des thermalen sowie des konformen Bootstraps, insbesonder der Operatorproduktentwicklung, untersucht werden. Somit werde ich in der Lage sein analytisch zu zeigen, dass die Relation zwischen Bremsstrahlung und Energie-Impuls-Tensor in einem thermalen Hintergund bestehen bleibt, wenn die Kopplungskonstante auf null gesetzt wird.

Dieses Ergebnis wird in einer Näherungsrechnung bestätigt. In führender Ordnung und einer Entwicklung um niedrige sowie hohe Temperaturen entdecke ich die Existenz eines Regularisationsschemas, in dem die exakte Formel weiterhin ihre Gültigkeit behält.

Darüber hinaus betrachte ich Effekte von Interaktionen auf die Theorie bei endlicher Temperatur. Ich stelle verschiedene ansätze vor und diskutiere deren mögliche Implikationen bezüglich der thermalen, gebrochenen Wardidentität sowie der Relation zwischen Bremsstrahlung und Energie-Impuls-Tensor.

Diese Betrachtungen werden um die Berechnung der nächstfolgenden Ordnung der zirkularen sowie geradlinigen Wilsonschlaufen in einer Entwicklung um die schwache Kopplungskonstante ergänzt.

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$\mathbb{R}$	Real numbers
$\mathbb{Z}$	Integer numbers
$\mathbb{N}$	Positive integers
$\mathrm{i}^2 = -1$	Imaginary unit
B	Bremsstrahlung
h	Coefficient of the stress tensor
β	Inverse temperature
$\mathcal{N}$	Number of preserved supersymmetries
Q^I_α	Poincare supercharge
$S_{I\alpha}$	Conformal supercharge
$\mathcal{Q}^I_\alpha$	Preserved Poincare supercharge on WL
$\mathcal{S}_{I\alpha}$	Preserved conformal supercharge on WL
$\mathfrak{Q}^I_\alpha$	Broken Poincare supercharge
$\mathfrak{S}_{I\alpha}$	Broken conformal supercharge
ζ^α_I	Spinor parameter of supersymmetry transformations
$\Delta_\mathcal{O}$	Conformal dimension of an operator $\mathcal{O}$
N	Number of gauge charges
g	Yang-Mills coupling
$\lambda = g^2 N$	't-Hooft coupling
f^{abc}	Structure constant
$\eta_{\mu\nu}$	Metric tensor
A_μ	Gauge field
λ^I_α	Spinor field
Φ_r	Real scalar field
D_μ	Covariant derivative including gauge field
∇_μ	Covariant derivative including gravitational interaction
$\Delta(x)$	Propagator
α	Fermionic phase, proportional to odd integers
$\hat{\alpha},\ \tilde{\alpha}$	Bosonic phase, proportional to even integers
$\omega_n = {}^{(2n+1)\pi}\!/_{\beta}$	Fermionic Matsubara frequency
$\hat{\omega}_n = {}^{2n\pi}\!/_{\beta}$	Bosonic Matsubara frequency

W	Wilson loop operator
$\bar{\mathcal{P}}$	Path-ordering
C	Contour of the Wilson loop
$\mathfrak{t}$	Parameter of the Wilson loop contour
R	Radius of the circula Wilson loop
$\tilde{R}$	Length parameter for the straight line
x_μ	4-dimensional coordinate ($\mu = 0, \dots, 4$)
x_i	3-dimensional coordinate ($i =, 2, 3$)
$\tau = \mathrm{i}x_0$	Coordinate along the compactified time dimension
$y = x_3$	Coordinate along the straight Wilson line
X_m	Coordinate orthogonal to the straight Wilson line ($m = 0, 1, 2$)
P_μ	4-momentum
p_i	3-momentum
$\hat{P}_{\mu\nu}$	Transverse polarization tensor
$\hat{Q}_{\mu\nu}$	Longitudinal polarization tensor
ε	Cut-off variable
$\mu, \nu = 0, \dots, 3$	4-dimensional spacetime index
$i, j = 1, 2, 3$	3-dimensional spatial index
$m, n = 0, 1, 2$	spacetime index for components orthogonal to the defect
$a, b = 1, \dots, N$	Index of the gauge group $\mathfrak{su}(N)_\alpha$
$\alpha, \beta = 1, 2$	Left spinor index of $\mathfrak{su}(2)_\alpha$
$\dot{\alpha}, \dot{\beta} = 1, 2$	Right spinor index of $\mathfrak{su}(2)_{\dot{\alpha}}$
$I, J = 1, \dots, 4$	R-symmetry index of $\mathfrak{su}(4)$ or $\mathfrak{usp}(4)$
$r, s = 1, \dots, 6$	R-symmetry index of $\mathfrak{so}(6)$ or $\mathfrak{so}(5)$
QFT	Quantum field theory
SYM	Supersymmetric Yang-Mills (theory)
CFT	Conformal field theory
SCFT	Superconformal field theory
OPE	Operator product expansion
KMS	Kudo-Martin-Schwinger (condition)
AdS	Anti de-Sitter space
WL	Wilson loop
BPS	Bogomol'nyi-Prasad-Sommerfield
BWI	Broken Ward identity
IR	infrared
UV	ultraviolet
KK	Kaluza-Klein (reduction)
HTL	Hard Thermal Loops

Contents

Chapter 1

Introduction

Quantum field theories are a powerful and widely used tool in theoretical physics for the last decades. They are used to describe a plethora of different phenomenon in modern physics. One of the most notable examples is the Standard Model of Particle Physics which describes high energy interaction between the known elementary particles. Predictions from the Standard Model agree with the experimental measurement to very high accuracy. One example is the measurement of magnetic moments and fine structure constant [2]. Despite its success which should in no way be underestimated the Standard Model still leaves some questions unanswered, including - but not limited to - the connection between high energy quantum field theories and gravity [3, 4]. The missing link between the Standard Model (SM) and gravity can be considered as a hierarchy problem [5]. Comparing the electroweak SM interaction and gravitational couplings it turns out that the latter ones are smaller by several orders of magnitude. Therefore the Standard Model is in fact an effective field theory for energies under a cut-off at around 1 TeV [6]. The Standard Model includes a process of spontaneous symmetry breaking which is induced by the introduction of the Higgs boson [7–10]. While gauge and chiral symmetry protect fermions and vector bosons from getting large mass corrections through self-energies the scalar Higgs is a priori not protected in such a way. The quadratically divergent self-energy of the Higgs boson can technically be renormalized but this requires a very precise fine-tuning to obtain the experimentally derived mass of $m_\phi = 125.10 \pm 0.14\,\mathrm{GeV}$ [11, 12]. This fine tuning seems unnatural and a successful extension of the Standard Model should explain these inconsistencies.

One such approach is the Minimally Supersymmetric Standard Model (MSSM) [13–15] which is enriched with an additional fermionic supersymmetry. The corresponding field transformations relate particles of different statistics with each other. Under the supersymmetry transformations bosonic particles get mapped to fermionic ones and vice versa. In such a model the divergences occurring in the self-energy calculations of the Higgs mass can be canceled by the supersymmetric partners of the SM particles. While supersymmetry thus might be a suitable solution to the hierarchy problem, experimental evidence of supersymmetric particles could not be obtained. In fact, if supersymmetry exists in nature, it has to be spontaneously broken [13]. As the addition of the supersymmetry facilitates calculations to a great extend, many supersymmetric toy models are being studied. They provide a more approachable framework to study and understand quantum field theories in general.

One of the most important ingredients of the Standard Model are gauge theories which were first introduced to describe electromagnetism. Especially the non-abelian Yang-Mills (YM) theories [16] are studied in various different settings and in all numbers of dimensions. While the mathematically rigorous definition of Yang-Mills theories remains mysterious and is one of the Millennium Problems [17, 18] the weak coupling expansion has yielded much insight on the structure of all gauge theories. The amazingly accurate predictions of the Standard Model being but one example of this. Despite the many interesting results the complexity of most gauge theories does not allow for predictions beyond the weak coupling regime which is why our focus will be on supersymmetric YM theories.

While supersymmetric Yang-Mills theories are studied in a great variety of different settings the most successful one is the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM) in four space-time dimensions and planar limit [19–23]. It preserves the maximal amount of supersymmetry in a four dimensional YM theory that does not include gravitational effects. The beta function of this massless toy model vanishes and it is possible to compute large number of quantities exactly.

One of the most important discoveries in recent decades for high energy theoretical physics was the relation between $\mathcal{N} = 4$ SYM theory in four dimensions at weak coupling and supergravity in the Anti-de-Sitter space $\mathrm{AdS}_5 \times S^5$ [24, 25] discovered by Juan Maldacena. The conjecture usually referred to as 'AdS/CFT-correspondence' or 'holography' allows for computations to be done in a weak as well as in a strong coupling expansion and to compare these results with each other [26, 27].

Another asset of $\mathcal{N} = 4$ SYM is that on top of being maximally supersymmetric the theory is conformally invariant. Conformal invariance is a feature closely linked to scale invariance meaning that the theory behaves identical at arbitrary distances. The combination of Lorentz-symmetry, supersymmetry and conformal symmetry leads to large superconformal algebra (SCA) $\mathfrak{psu}(2, 2|4)$ under which the theory is conserved. This leads to a plethora of tools available to study $\mathcal{N} = 4$ SYM including weak coupling perturbative expansions, gravitational calculations in AdS-space, as well as superconformal bootstrap techniques, localization and integrability [28–34].

In recent years an additional focus has been put onto defects in $\mathcal{N} = 4$ SYM. The introduction of p-dimensional extended objects such as lines and branes, as well as boundary defects partially breaks the superconformal symmetry. While the SCA along the defect is preserved it is broken in the bulk, the directions orthogonal to the defect. Wilson lines are one example for such a defect which preserves one-dimensional superconformal symmetry along the line and a three dimensional rotational symmetry perpendicular to it (see Section 1.1). Considering a variety of operators along the defect and in the bulk a bremsstrahlung function B can be defined which was shown by localization techniques to be related to the coefficient h of the stress-energy tensor of $\mathcal{N}=4$ SYM in the presence of said Wilson line [35, 36],

$$B = 3h\ . \tag{1.1}$$

The above relation could in fact even be extended to theories with $\mathcal{N}=2$ supersymmetry [37, 38]. Also general multipoint functions in the presence of the multipoint defect have been of large interest in modern research. It has been shown that a protected sector of defect excitations can be found in which multipoint correlators can still be expressed by a generalized bremsstrahlung function [39, 40]. Similar results include line defects in ABJM [41] and $\mathcal{N}=2$ theories [42].
Exact results have, however, not been obtained in any theory that does not include supersymmetry.

The intuitive definition of any quantum field theory is that of free particles in an empty space. Therefore, the Standard Model as well as the $\mathcal{N}=4$ SYM theory are defined at a vanishing temperature. Thermal field theories, however have been studies thoroughly in past decades as thermal effects are of great interest for experimental measurements and theory such as dark matter freeze-out and phase transitions in the early universe. These effects include the introduction of a thermal mass for all particles in a theory. By a folk-theorem thermal masses are proportional to the temperature T and positive. Henceforth any broken symmetry introduced by a Higgs-mechanism eventually gets restored at high enough temperatures. At a certain critical temperature a phase transition between spontaneously broken and restored symmetry can be observed [43]. Finite temperature further provides insights on the phases of hot QCD and the Quark-Gluon plasma [44–48].
A thermodynamic system is classically described by an ensemble. As the mechanics of quantum field theories allow for particles to be created and annihilated at any arbitrary point in space and time it is intuitive to define a thermal QFT in a Grand Canonical Ensemble with fixed temperature $\beta = {}^1\!/\!{}_T$, volume and chemical potential μ while correspondingly the energy, particle number N and pressure are variables. The ensemble then defines a partition function $\mathcal{Z}$ given in terms of a density matrix

$$\mathcal{Z} = \mathrm{tr}(\rho) = \mathrm{tr}\left(e^{-\beta(H-\mu N)}\right)\ , \tag{1.2}$$

where H is the time-independent Hamiltonian of the theory. This yields that the theory is in fact defined in a thermodynamic equilibrium. While more general theories are studied in the literature we shall restrict ourselves to this equilibrium case and further assume the chemical potential to vanish.

The thermal QFT is based on a theory defined on some $\mathbb{R}^d$ and conversely the partition function further has an expression using the path integral formalism. To make this accessible the time dimension is compactified on a circle with radius of the inverse temperature β. This straight forwardly defines a thermal manifold $\mathcal{M}_\beta = S^1_\beta \times \mathbb{R}^{d-1}$ on which the thermal theory thus lives. The time-like components of fields can therefore be expressed in discrete Matsubara modes. While it is generally possible to include real time effect (Keldysh formalism) we will mostly focus on the imaginary time formalism here.
From the compactification of the time dimension follows that the respective coordinate τ is periodic and $\tau \sim \tau + \beta$ can be identified. This yields boundary conditions for all fields in the theory. These boundary conditions are completely fixed by the commutation relations of the QFT and the definition of the thermodynamic ensemble. All bosonic fields commute with each other and they are periodic under translations around the thermal circle

$$\phi(\tau, \vec{x}) = \phi(\tau + \beta, \vec{x}) \ . \tag{1.3}$$

Meanwhile all fields which obey fermionic statistics anti-commute and therefore they are anti-periodic

$$\psi(\tau, \vec{x}) = -\psi(\tau + \beta, \vec{x}) \ . \tag{1.4}$$

These are the Kudo-Martin-Schwinger (KMS) conditions for any thermal theory [49, 50]. Following these different periodicity conditions it follows readily that any supersymmetry that was introduced at zero temperature must be broken in the thermal theory [51–53]. Similarly the temperature is an explicit scale introduced to the theory thus breaking conformal invariance. However, locally the thermal manifold $\mathcal{M}_\beta$ looks like flat space and henceforth techniques of conformal field theories remain to be valid to some extend (see Section 1.2). Two operators in a proximity of each other can be expanded in an operator product expansion (OPE) as commonly used in zero temperature CFTs. The distance between the two operator insertions must be smaller than the inverse temperature for the OPE to hold. Furthermore, bootsrap techniques can be applied to the thermal CFT [54–56].

1.1 Wilson Loops

One consistency check for the AdS/CFT-correspondence is the computation of supersymmetric Wilson loops (WL). Wilson loops were first introduced by Kenneth Wilson [57] in 1974 as a gauge invariant observable to measure confinement in lattice theories. Their generalization to supersymmetric theories, especially with respect to the $\text{AdS}_5/\text{CFT}_4$-correspondence was introduced 24 years later by Juan Maldacena [58]. For a generic Yang-Mills theory they describe the holonomy of the gauge connection upon parallel transport along a closed path C as a path-ordered exponential:

$$W(C) = \mathrm{tr}_R \overleftarrow{\mathcal{P}} \exp\left[\mathrm{i} \oint_C \mathrm{d}x^\mu \, A_\mu\right] . \tag{1.5}$$

The trace is evaluated with respect to the gauge group. $A_\mu(x)$ is the gauge field and $\overleftarrow{\mathcal{P}}$ is the path-ordering operator. The Wilson loop corresponds to a heavy quark being moved along a closed loop in a gauge background A_μ. In the quantum gauge theory we assume the quark to have an infinite mass and it can thus be integrated out of the interacting theory.

One important observation made early is that the Wilson loop of a rectangular loop measures the quark-antiquark-potential. Therefore consider a rectangle $T \times R$. The side T is in the time direction and R is in the spatial direction. In the limit $T \to \infty$ while keeping $R = const.$ the two lines along the time direction look like a quark and an anti-quark. The Wilson loop thus yields the potential $V(R)$ between the quark and the anti-quark [59, 60]

$$\langle W(\square)\rangle \simeq e^{-T\,V(R)} . \tag{1.6}$$

Therefore the rectangular Wilson loop is an order parameter for confinement phase transitions. For example confinement can be seen from the '*area law*' $\langle W(\square)\rangle \simeq e^{-TR}$ (linear potential) [59].

Witten related the Wilson loop in 2+1 dimensional Chern-Simons theory to mathematical knot theory and found new implications for 1+1 dimensional Wess-Zumino-Witten models [61]. In supersymmetric gauge theories the Wilson loop is generally interesting as a non-local gauge-invariant operator. Furthermore, for some specific contours the Wilson loop can preserve a certain amount of supersymmetry and one is sometimes able to calculate the Wilson loop exactly. These exact results can be obtained using supersymmetric localization which leads to solvable matrix models. Examples are the straight line which is identically one due to SUSY and the circle computed in [62–64]. Moreover, Wilson loops and their relation to 't Hooft loops (the magnetic dual of Wilson loops) provide tests for S-duality [65–67].

1.1.1 $\mathcal{N} = 4$ Maldacena-Wilson loops

As introduced above the Wilson loop is originally defined for a generic $\mathrm{U}(N)$ gauge theory

$$W(C) = \frac{1}{N} \mathrm{tr}\, \overleftarrow{\mathcal{P}} \exp\left[i \oint_C \mathrm{d}x^\mu \, A_\mu\right] . \tag{1.7}$$

with a closed contour C parameterized by $x(\tau)$ and $A_\mu(x)$ the gauge field of the theory. $\overleftarrow{\mathcal{P}}$ is the τ-path-ordering and the trace is over the fundamental representation. It describes the propagation of a heavy particle ($M \to \infty$) on a closed loop C. We are mainly interested in the expectation value of the Wilson loop $\langle W(C)\rangle$.

Consider this Wilson loop in 4d $\mathcal{N} = 4$ SYM theory and in string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$. In AdS space a theory with gauge group $\mathrm{U}(N)$ can be build by stacking N D3-branes at the same point [65]. Consider instead $(N+1)$ D3-branes where one of the branes is separated from the others. This yields that the original gauge group $\mathrm{U}(N+1)$ is broken to a $\mathrm{U}(N)\times\mathrm{U}(1)$ in a Higgs-like mechanism [65–67]. The Wilson loop is interpreted as a string connecting the single brane with the set of N D3-branes. However, the string must also have an extension on the 5-sphere S^5 parametrized by coordinates n^r which correspond to the six scalars Φ^r of $\mathcal{N} = 4$ SYM which is why we introduce the Maldacena-Wilson loop [58]

$$\langle W(C)\rangle = \left\langle \frac{1}{N}\mathrm{tr}\,\bar{\mathcal{P}}\,\exp\left[\oint_C \mathrm{d}t\,(iA_\mu\dot{x}^\mu + n^r\Phi_r|\dot{x}|)\right]\right\rangle . \tag{1.8}$$

For simplicity we consider n^r = constant here. The Wilson loop expectation value can be obtained by considering the amplitude of the massive particle that is the loop. The amplitude can be computed in two ways. It can be considered as the WL expectation value with a prefactor of the exponentiated mass. Furthermore, it is also given by the string partition function with boundary along the contour C. Comparing both formulations one finds

$$\langle W(C)\rangle = e^{-(S_{\text{string}}[C]-M\,L(C))} . \tag{1.9}$$

$L(C)$ is the length of the Wilson loop. The string action is proportional to the area of the worldsheet in $\mathrm{AdS}_5\times\mathrm{S}^5$ which thus diverges. This divergence, however, cancels against the divergence coming from $M\to\infty$ with the latter limit chosen appropriately.
The string we are considering has a tension and following the action principle it wants to have minimal area. In the very simple case of flat space the tension together with the boundary conditions would imply that the worldsheets spans the surface enclosed by C. AdS space, however, introduces a gravitational field dragging the string away from the boundary. Eventually this gravitational pull is stopped by the tension at some finite value. The free heavy boson which we subtract above, on the other hand, stretches along the full AdS_5 in a straight line parallel to C. We want to calculate thus the string action which is nothing else but the tension times the area of the string world sheet fixed to the boundary. The mass term cancels the divergence thus giving the renormalized area from which the Wilson loop is readily computed

$$\mathrm{Area}_{\text{ren.}}(C) = \lim_{\epsilon\to 0}\left(\mathrm{Area}(C)|_{Z\geq\epsilon} - \frac{1}{\epsilon}L(C)\right) , \quad \langle W(C)\rangle = e^{-\frac{\sqrt{\lambda}}{2\pi}\mathrm{Area}_{\text{ren.}}(C)} . \tag{1.10}$$

At weak coupling the calculation of the Maldacena-Wilson loop is done via Feynman diagrams. The general expression (1.8) can be expanded around small values of the 't Hooft coupling. This leads to a power series expansion of the exponential with the fields contracted among themselves or with fields coming from the action.

We will consider this weak coupling expansion more closely for the circular WL in the next subsection.

1.1.2 Circular Maldacena-Wilson loop in $\mathcal{N} = 4$

We consider a circular loop with boundary parameterization

$$C_{\text{loop}} : \; x^{\mu}(\mathsf{t}) = (0\,,\, R\cos(\mathsf{t})\,,\, R\sin(\mathsf{t})\,,\, 0) \quad , \quad \mathsf{t} \in [0, 2\pi]\,. \tag{1.11}$$

The area of the circular loop is calculated from the Nambu-Goto action (for details see [67])

$$\text{Area}_{\text{ren.}}(\bigcirc) = \lim_{\epsilon\to 0}\left[\text{Area}(\bigcirc)|_{Z\geq\epsilon} - \frac{1}{\epsilon}L(\bigcirc)\right] = \lim_{\epsilon\to 0}\left[2\pi\left(\frac{R}{\epsilon} - 1\right) - \frac{2\pi R}{\epsilon}\right] = -2\pi\,. \tag{1.12}$$

And therefore we find for the circular Wilson loop at strong coupling

$$\langle W(\bigcirc)\rangle = e^{\sqrt{\lambda}}\,. \tag{1.13}$$

As is expected for a scale invariant theory the final result does not depend on the radius of the circle. In [62] Erickson, Semenoff and Zarembo managed to calculate the line on the gauge theory side exactly. They find

$$\langle W(\bigcirc)\rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \xrightarrow{\lambda>>1} \sqrt{\frac{2}{\pi}}\,\lambda^{-3/4}\, e^{\sqrt{\lambda}}\,. \tag{1.14}$$

The leading behavior for $\lambda >> 1$ matches in both calculations thus providing a first consistency check of the AdS/CFT correspondence using the Wilson loop. Let us review the main steps of the calculation in [62].
Therefore expand the exponential of the Wilson loop (1.8) in a power series

$$\begin{aligned}\langle W(C)\rangle \approx 1 + \frac{\operatorname{tr}(T^a T^b)}{N}\int_0^{2\pi} \mathrm{d}\mathsf{t}_1 \int_0^{\mathsf{t}_1} \mathrm{d}\mathsf{t}_2 \Big[|\dot{x}_1||\dot{x}_2| n^I n^J \langle \Phi_{I\,1}\Phi_{J\,2}\rangle \\ - \dot{x}_1^{\mu}\dot{x}_2^{\nu}\left\langle A^a_{\mu\,1} A^b_{\nu\,2}\right\rangle\Big] + \mathcal{O}(\lambda^2)\,. \end{aligned}\tag{1.15}$$

The linear term involves only one-point functions which are zero. The quadratic term, which is displayed above, yield Wick contractions between scalar and vector fields. Considering the bare propagators these contributions yield the leading corrections. Subleading terms involve corrections to the propagators which come from the expansion of the action as well as cubic and quartic terms from the Wilson loop exponential.
We can evaluate the integrals by inserting the bare propagators in coordinate space,

$$\left\langle \Phi^a_I(x_1)\Phi^b_J(x_2)\right\rangle = \frac{g^2}{4\pi^2}\frac{\delta^{ab}\delta_{IJ}}{(x_1 - x_2)^2}\,, \tag{1.16a}$$

$$\left\langle A^a_\mu(x_1)A^b_\nu(x_2)\right\rangle = \frac{g^2}{4\pi^2}\frac{\delta^{ab}\eta_{\mu\nu}}{(x_1-x_2)^2} \, . \tag{1.16b}$$

For the trace of the generators we use the convention $\mathrm{Tr}(T^aT^a) = {}^{N^2}\!/_2$ and thus find

$$\langle W(C)\rangle \approx 1 + \frac{\lambda}{8\pi^2}\int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \frac{|\dot{x}_1||\dot{x}_2| - \dot{x}_1^\mu \dot{x}_{2\mu}}{(x_1-x_2)^2} + \mathcal{O}(\lambda^2) \, . \tag{1.17}$$

This first term is depicted in Figure (1.1a). Inserting the parameterization (1.11) we see that the contributions of the nummerator and denominator cancel up to a factor $^1\!/_2$

$$\frac{|\dot{x}_1||\dot{x}_2| - \dot{x}_1^\mu \dot{x}_{2\,\mu}}{(x_1-x_2)^2} = \frac{R^2 - R^2\sin(\mathfrak{t}_1)\sin(\mathfrak{t}_2) - R^2\cos(\mathfrak{t}_1)\cos(\mathfrak{t}_2)}{(R\cos(\mathfrak{t}_1) - R\cos(\mathfrak{t}_2))^2 + (R\sin(\mathfrak{t}_1) - R\sin(\mathfrak{t}_2))^2} = \frac{1}{2} \, . \tag{1.18}$$

The remaining integral is readily evaluated to give a factor of $2\pi^2$ and we can write down the Wilson loop up to order λ

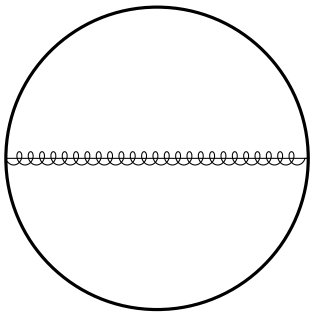
$$\langle W(C)\rangle = 1 + \frac{\lambda}{8} + \mathcal{O}(\lambda^2) \, . \tag{1.19}$$

There are three types diagrams contributing to the next order, shown in the Figure (1.1). The diagram in Figure 1.1a is the insertion of a single bare propagator contributing at order λ. This is precisely the term considered in the above calculations. Further expanding the WL exponential yields the insertion of two bare propagators (Figure 1.1b) coming from the quartic terms. As mentioned previously the expansion of the action allows for the insertion of self-energies to the leading order diagram. This is shown in Figure 1.1c. The three-point vertices from the action can be Wick contracted with the cubic terms of the expansion to form the vertex diagram 1.1d. These three diagrams all contribute at the same order λ^2 in a small coupling expansion. Note that non-planar diagrams like one with two crossing bare propagators are subleading in the large-N limit we consider.
By calculating the relevant self-energies in $\mathcal{N}=4$ SYM and inserting into the circular loop one can eventually show that the self-energy (1.1c) and vertex diagrams (1.1d) give the same contribution up to a sign and thus mutually cancel [62].

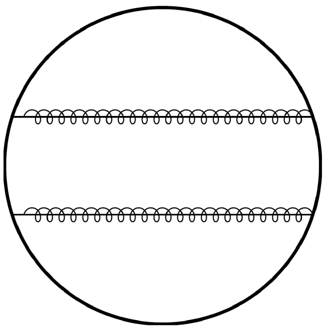
The ladder diagram (1.1b) is thus the only one contributing. Such diagrams will appear at all orders in perturbation theory and they can be resummed to an exact result.
From the expansion in Equation (1.15) it can be seen that the term of order $2n$ will be

$$\begin{aligned}&\frac{1}{N}\int_0^{2\pi}\mathrm{d}\mathfrak{t}_1\int_0^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2 \dots \int_0^{\mathfrak{t}_{2n-1}}\mathrm{d}\mathfrak{t}_{2n} \times \\ &\qquad \times \mathrm{tr}\left\langle\left(iA_{\mu\,1}\dot{x}_1^\mu + \Phi_{I\,1}|\dot{x}_1|n^I\right) \dots \left(iA_{\mu\,2n}\dot{x}_{2n}^\mu + \Phi_{I\,2n}|\dot{x}_{2n}|n^I\right)\right\rangle\end{aligned} \tag{1.20}$$

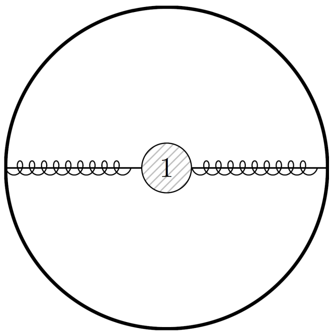
Figure 1.1: The diagrams show insertions into the Wilson loop at low order. The thicker line is the circular loop of the heavy particle. The curly and straight line respectively represent a gauge and a scalar Boson inserted. We must sum over all possible combinations allowed by the present vertices.



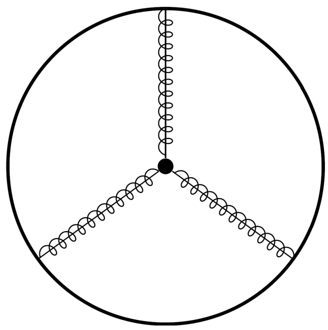
(a) Single propagator insertion. This is the only diagram of order λ computed above.



(b) Ladder diagram: Insertion of two single propagators. This diagram contributes at order λ^2.



(c) Self-energy insertion. The hatched circle in the middle with the 1 stands for any possible one-loop insertion. This diagram contributes at order λ^2.



(d) Vertex insertion. This diagram contributes at order λ^2.

We want to sum only those planar diagrams that have bare internal propagators without any vertex. Therefore any two neighboring fields[1] might be Wick contracted to form a propagator. Each of those n propagators will have a contribution as (1.16a) or (1.16b). Noting that, as before, we may sum gluon and scalar bare propagators we find a contribution of $\lambda/16\pi^2$ due to the cancellation of numerator and denominator because of the chosen parameterization of the circular loop, see also the calculation that led to Equation (1.18). The factor of N arises because the propagators yield a Kronecker delta for the generators and we have $\mathrm{tr}(T^a T^a) = N^2/2$. The $2n$ integrals yield $(2\pi)^{2n}/(2n)!$. Multiplying all these factors yields that the contribution of $2n$ ladder diagrams is

$$\frac{\left(g^2N/4\right)^n}{(2n)!} \times \# \text{ of different diagrams at order } 2n \ . \tag{1.21}$$

We want to find how many different diagrams we have with n internal propagators. Note that we do not need to distinguish between scalar and gluon propagators as this was already included in the above calculation.
For $n = 2$ there are two diagrams due to different naming of the insertion points. The higher order diagrams are then given by having the new propagator inserted between any sub-diagrams of the lower order. Figure (1.2) shows how this is done. Summing over k

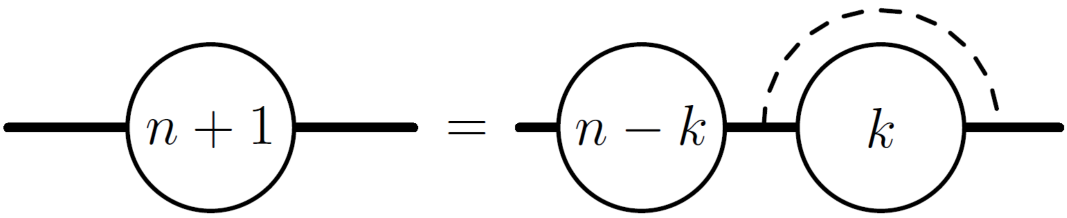


Figure 1.2: Schematic description of how higher order diagrams are obtained from the lower order ones [62].

then gives the number of diagrams with $n+1$ propagators recursively

$$A_{n+1} = \sum_{k=0}^{n} A_{n-k}A_k \ , \quad A_0 = 1 \ . \tag{1.22}$$

All the different A_n might be stored in a generating function

$$f(z) := \sum_{n=0}^{\infty} A_n z^n \ . \tag{1.23}$$

We can then compare the following two quantities:

$$\frac{f(z)-1}{z} = \frac{1}{z}\left(\sum_{n=0}^{\infty} A_n z^n - 1\right) = \frac{1}{z}\sum_{n=1}^{\infty} A_n z^n = \sum_{n=1}^{\infty} A_n z^{n-1} = \sum_{n=0}^{\infty} A_{n+1} z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{n-k}A_k z^n \tag{1.24}$$

[1]Also the first and the last field are neighboring because of the cyclicity of the trace.

$$f^2(z) = \left(\sum_{m=0}^{\infty} A_m z^m\right)\left(\sum_{l=0}^{\infty} A_l z^l\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{n-k}A_k z^n \ . \tag{1.25}$$

For the first quantity we plugged in the recursion formula and for the second one we used the Cauchy product formula

$$\left(\sum_{m=0}^{\infty} a_m z^m\right)\left(\sum_{l=0}^{\infty} b_l z^l\right) = \sum_{n=0}^{\infty} c_n z^n \quad \text{where} \quad c_n = \sum_{k=0}^{n} a_{n-k}b_k \ . \tag{1.26}$$

Therefore we find

$$zf^2(z) = f(z) - 1 \ , \tag{1.27}$$

which is a quadratic equation of $f(z)$ with general solution

$$f_\pm(z) = \frac{1 \pm \sqrt{1-4z}}{2z} \ . \tag{1.28}$$

However the function should also match the power series above, espicially $A_0 = 1$ yields $f(0) = 1$. Computing the limits $f_\pm(z \to 0)$ readily yields that only the solution $f_-(z)$ satisfies this assumption. We can then use the known Taylor expansion for $\sqrt{1+x}$ to find the general expression for all A_n:

$$f(z) = f_-(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!n!} z^n \ . \tag{1.29}$$

Therefore we finally find $A_n = {}^{(2n)!}/_{(n+1)!n!}$. We can now multiply this with the previously motivated prefactor $\frac{\left(g^2 N/4\right)^n}{(2n)!}$ and sum over all n to get the final result

$$\langle W(C)\rangle_{\text{ladder}} = \sum_{n=0}^{\infty} \frac{(\lambda/4)^n}{n!(n+1)!} = -\frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right) \ , \tag{1.30}$$

where I_1 is the modified hyperbolic Bessel function.

Erickson, Semenoff and Zarembo conjectured [62] that the result presented above is exact to all loops. Conversely this means that the cancellation between self-energy and vertex diagrams similar to Figures 1.1c and 1.1d, respectively. This statement was further supported later by Drukker and Gross [63] who also computed the circle by an inversion of the straight line and thus determined the above result as a conformal anomaly. They gave arguments on how to possibly use supersymmetric localization and discovered that the circular loop can be computed using a matrix model. The final proof for the formula above was done by Pestun [64].

1.2 Finite Temperature Bootstrap

The conformal bootstrap is a method used in recent years to study theories with scaling symmetry [68, 69]. One of the most popular example is the bootstrap of the three-dimenional Ising model [70]. Introducing a finite temperature by a compactification of the euclidean time breaks the conformal invariance. However, some bootstrap techniques can still be used to study such theories on $S^1_\beta \times \mathbb{R}^3$. The breaking effectively introduces a scale, namely the inverse temperature $\beta = {}^1\!/\!_T$. One consequence is that one-point functions in the thermal background may aquire a non-zero expectation value. In fact the preserved symmetries demand the operators to be symmetric-traceless tensors of even spin J [54]. Such operators are then fixed by the remaining preserved symmetries to depend only on a one-point coefficient $b_\mathcal{O}$

$$\langle \mathcal{O}^{\mu_1 \cdots \mu_J}(x) \rangle_\beta = \frac{b_\mathcal{O}}{\beta^{\Delta_\mathcal{O}}} \left(e^{\mu_1} \dots e^{\mu_J} - \text{traces} \right) \ . \tag{1.31}$$

In the above equation the e^μ are unit vectors in the direction of the thermal circle. Note that this yields for the stress tensor

$$\langle T^{\mu\nu} \rangle_\beta = \frac{b_T}{\beta^4} \left(e^\mu e^\nu - \frac{1}{d} \eta^{\mu\nu} \right) \ . \tag{1.32}$$

To determine the one-point coefficient of the stress tensor consider the energy density given by [54]

$$E = -\langle T_{00} \rangle = -\left(1 - \frac{1}{d} \right) b_T = -\frac{d-1}{d} b_T \ . \tag{1.33}$$

The energy E must be positive and thus $b_T < 0$. This derivation can be compared to the free energy density

$$F = E - TS = E + T\frac{dF}{dT} = fT^4 \ . \tag{1.34}$$

The last equality follows from dimensional analysis and f is an a priori unknown coefficient. Combining the last two equations yields that the stress stensor coefficient can be obtained from the free energy [54]

$$f = \frac{b_T}{d} < 0 \qquad \Rightarrow \quad b_T = \frac{dF}{T^4} \ . \tag{1.35}$$

For four-dimensional $\mathcal{N} = 4$ SYM the free energy was calculated in [71] and we thus find

$$F^{\mathcal{N}=4} = -\frac{\pi^2 g^2 N^2 T^4}{6} \qquad \Rightarrow \quad b_T^{\mathcal{N}=4} = -\frac{2\pi^2}{3} g^2 N^2 \ . \tag{1.36}$$

1.2.1 Operator Product Expansion

While two-point functions can be fixed using the conformal symmetries at $T = 0$ this is no longer true at finite temperature. Only the one-point functions can be entirely determined. Thermal two-point functions can however be obtained by using an Operator Product Expansion (OPE). These OPEs are known from conformal theories [68, 69, 72]. They describe the product of two operators by a sum of single operators. The convergence of this series was shown to hold if the two operators are in a sphere with flat interior within which no further operators lie. For the thermal theory we additionally need to account for the compactified time dimension. The compactification will spoil the flatness of the OPE interior for large enough distances between the two operators. In the contrary case we can, however, still use the original OPE [54]

$$\phi(x)\phi(0) = \sum_{\mathcal{O}\in\phi\times\phi} f_{\phi\phi\mathcal{O}}|x|^{\Delta_{\mathcal{O}}-2\delta_\phi-J} x_{\mu_1}\dots x_{\mu_J}\mathcal{O}^{\mu_1\dots\mu_J}(0)\ . \tag{1.37}$$

The above equation is the operator product expansion for two scalar fields. In general all operator products statisfy a similar OPE where we would have to account the different indices appropriately. The structure constant $f_{\phi\phi\mathcal{O}}$ is precisely the one from the zero temperature theory. Following the above argumentation this OPE converges for

$$|x| = \sqrt{\tau^2 + |\vec{x}|^2} < \beta\ . \tag{1.38}$$

This means that the largest sphere in which the two operators must lie for the OPE to still be convergent is a sphere that wraps once around the sphere S^1_β, see Figure (1.3). The Operator Product Expansion introduced above can for example be used to find an

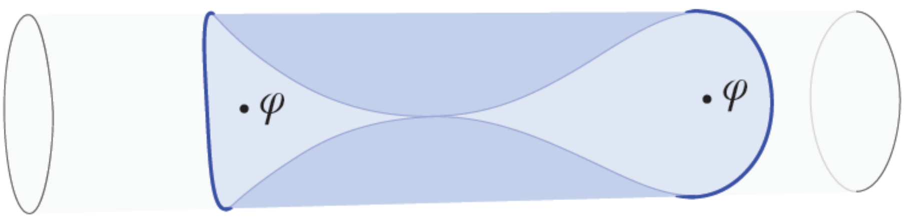


Figure 1.3: The blue sphere shows the maximal region for convergence of the OPE on $S^1_\beta \times \mathbb{R}^3$. The horizontal direction is the $\mathbb{R}^3$ [54].

expression for a scalar two-point function. Therefore consider that the contraction between the x_μ from the OPE and the structure of the one-point function is given by Gegenbauer polynomials

$$|x|^J x_{\mu_1}\dots x_{\mu_J}\left(e^{\mu_1}\dots e^{\mu_J} - \text{traces}\right) = \frac{J!}{2^J(\nu)_J} C_J^{(\nu)}\left(\frac{\tau}{|x|}\right)\ , \tag{1.39}$$

where $\nu = {}^{d-2}\!/_2$ and $(a)_n = {}^{\Gamma(a+n)}\!/_{\Gamma(a)}$ is the Pochhammer symbol. Thus a scalar two-point function at finite temperature can be written as

$$g(x) \coloneqq \langle\phi(x)\phi(0)\rangle_\beta = \sum_{\mathcal{O}\in\phi\times\phi} \frac{a_\mathcal{O}}{\beta^\mathcal{O}} |x|^{\Delta_\mathcal{O}-2\Delta_\phi} C_J^{(\nu)}\left(\frac{\tau}{|x|}\right) , \qquad \text{with} \quad a_\mathcal{O} = \frac{J! f_{\phi\phi\mathcal{O}} b_\mathcal{O}}{2^J (\nu_J)} . \quad (1.40)$$

The resulting expressions $|x|^{\Delta_\mathcal{O}-2\Delta_\phi} C_J^{(\nu)}\left(\frac{\tau}{|x|}\right)$ can be considered as the thermal conformal blocks [54]. They obey crossing equations similar to the zero temperature conformal blocks [73].

1.2.2 KMS condition and clustering

Compactifying the time dimension on an S^1_β yields a periodicity in the τ component. This extends to the operators in the theory. This was first discovered by Kudo, Martin and Schwinger [49, 50] who showed that bosonic fields experience periodic boundary conditions while the boundary conditions for fermions are anti-periodic,

$$\phi(\tau,\vec{x}) = \phi(\tau+\beta,\vec{x}) , \quad \psi(\tau,\vec{x}) = -\psi(\tau+\beta,\vec{x}) . \qquad (1.41)$$

These KMS conditions need to be satisfied by all correlators in the thermal theory which puts further restrictions on the thermal conformal blocks introduced above. Namely

$$g(\tau,\vec{x}) \coloneqq \langle\phi(\tau,\vec{x})\phi(0)\rangle_\beta = \langle\phi(\beta-\tau,\vec{x})\phi(0)\rangle_\beta . \qquad (1.42)$$

This can then be interpreted as a constraint on the thermal coefficients $a_\mathcal{O}$ similar to the crossing equations at zero temperature [54, 73].
Using the SO$(d-1)$ symmetry of $S^1_\beta \times \mathbb{R}^{d-1}$ the spatial $\vec{x}$ can be restricted to a coordinate on a line x_E. Introducing complex coordinates

$$z = \tau + \mathrm{i}x_E , \quad \bar{z} = \tau - \mathrm{i}x_E , \quad z = r\omega , \quad z = r\omega^{-1} , \qquad (1.43)$$

and Wick rotating $x_E \to \mathrm{i}x_L$ we can define a set of two independent variables $(z,\bar{z})$. The KMS condition above can be rewritten as

$$g(z,\bar{z}) = g(1-z, 1-\bar{z}) , \qquad (1.44)$$

in the new coordinates where for convenience we set $\beta = 1$.

The thermal bootstrap including the above thermal blocks (1.40) and the Thermal Crossing Equations (1.44) relies on the convergence of the Operator Product Expansion as shown in Figure 1.3. Outside of the sphere with $|\vec{x}| < \beta$ the above OPE does not hold. However, we can consider the two operators at a large spatial separation $|\vec{x}| \to \infty$. In this limit the correlator between two scalars will be determined by the thermal mass. There is a *folk theorem* at finite temperature saying that all thermal masses are positive $m_{\text{th}} > 0$ yielding that [54]

$$\langle \phi(\tau, \vec{x}) \phi(0) \rangle_\beta \sim \langle \phi \rangle_\beta^2 + \mathcal{O}(e^{-m_{\text{th}}|\vec{x}|}) \qquad \text{for} \quad |\vec{x}| \to \infty \, . \tag{1.45}$$

This in turn means that at large spatial separation we find a clustering. The different operators in the correlator do not interact and we can consider a product of their one-point functions instead.

1.2.3 Lorentzian inversion formula

The methods of the thermal bootstrap as presented in the previous subsections have been used to study different conformal field theories at finite temperature, see for example [56, 68, 74]. In this subsection and the subsequent one we will give an overview of the methods used in these references. They are alternative approaches to CFTs at finite temperature which will not be used in this thesis. They are, however, state of the art results of the thermal bootstrap and hence generally interesting
The operator content of theories in flat space can be studied by considering four-point functions and their conformal blocks. One way of doing this are inversion formulas [74–76]. The inversion formulas are an efficient way of studying crossing equations. As we derived above, in a thermal theory we instead consider two-point functions and their thermal blocks $|x|^{\Delta_{\mathcal{O}}-2\Delta_\phi} C_J^{(\nu)}(\tau/|x|)$. The crossing equations they obey are the KMS conditions (1.44) introduced above. Rewrite the thermal OPE for the scalar two-point function as an integral

$$g(\tau, \vec{x}) = \sum_{J=0}^{\infty} \oint_{-\epsilon - \mathrm{i}\infty}^{-\epsilon + \mathrm{i}\infty} \frac{\mathrm{d}\delta}{2\pi \mathrm{i}} \frac{a(\Delta, J)}{\beta^\Delta} |\vec{x}|^{\Delta - 2\Delta_\phi} C_J^{(\nu)} \left(\frac{\tau}{|\vec{x}|} \right) . \tag{1.46}$$

We have to assume that the function $a(\Delta, J)$ does not grow exponentially fast in the right half-plane. Then we can deform the contour and use the Cauchy theorem to pick up residues. For the above formula to agree with Equation (1.40) these poles have to sit at the physical operator dimensions

$$a(\Delta, J) \sim -\frac{a_{\mathcal{O}}}{\Delta - \Delta_{\mathcal{O}}} \, . \tag{1.47}$$

The above formula can be inverted by integrating against the Gegenbauer polynomials followed by a Laplace transform. Using the coordinates as defined in (1.43) we can invert (1.46) in Lorentzian signature and obtain [54]

$$a(\Delta, J) = (1 + (-1)^J) K_J \int_0^1 \frac{\mathrm{d}\bar{z}}{\bar{z}} \int_0^{1/\bar{z}} \frac{\mathrm{d}z}{z} (z\bar{z})^{\Delta_\phi - \frac{\Delta}{2} - \nu} F_J \left(\sqrt{\frac{\bar{z}}{z}} \right) \mathrm{Disc}[g(z, \bar{z})]$$

$$+\theta(J_0 - J)a_{\text{arcs}}(\Delta, J)\ , \tag{1.48}$$

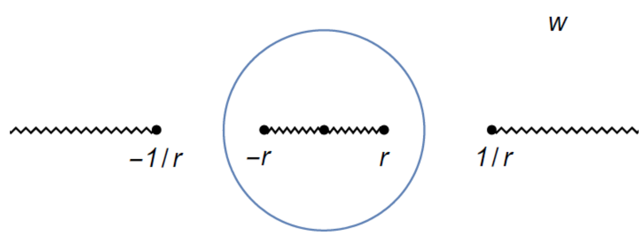
with

$$K_J = \frac{\Gamma(J+1)\Gamma(\nu)}{4\pi\Gamma(J+\nu)}\ , \qquad F_J(\omega) = \omega^{J+d-2}\,{}_2F_1\left(J+d-2, \frac{d}{2}-1, J+\frac{d}{2}\middle|\omega\right)\ . \tag{1.49}$$

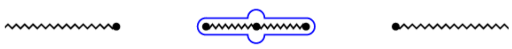
The first inversion included an integral around a unit circle in the $\omega-$plane. The contour then needed to be deformed. To do this in [54] it was claimed for the correlator $g(z = r\omega, \bar{z} = r\omega^{-1})$ to satisfy two important properties. It is analytic in said $\omega-$plane away from the cuts $(-\infty, -1/r)$, $(-r, r)$, $(1/r, \infty)$ for $r < 1$. This analyticity can be conjectured from the convergence of the s-channel and t-channel OPEs [54]. Moreover the function is assumed to be polynomially bounded at large ω. For some constant J_0 the growth is bounded by ω^{J_0}. Due to the symmetry $\omega \to \omega^{-1}$ similarly g is bounded by ω^{-J_0} for small ω.

The integral which was originally around $|\omega| = 1$ can then be deformed separately for terms with ω^J and with ω^{-J} to enclose the cuts mentioned above. These contours are shown in Figure (1.4). For $J \leq J_0$ we pick up non-trivial contributions $a_{\text{arcs}}(\Delta, J)$ from the arc.

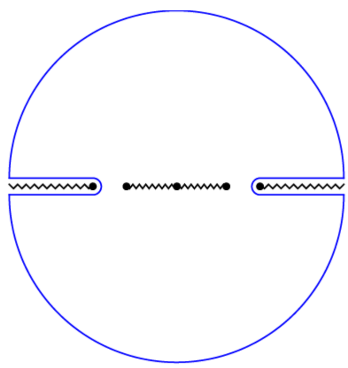
Figure 1.4: The picture shows the different contours. (a) is the original contour while (b) and (c) are the respective deformations mentioned above [54].



(a) Initial integration contour $|w| = 1$.



(b) The deformed contour for terms that behave as w^J.



(c) The deformed contour for terms that behave as w^{-J}.

The contour along the cuts yields the discontinuities of the two-point function

$$\text{Disc}[g(z,\bar{z})] := \frac{1}{\mathrm{i}}\left(g(z+\mathrm{i}\epsilon,\bar{z}) - g(z-\mathrm{i}\epsilon,\bar{z})\right) . \tag{1.50}$$

As the discontinuities of thermal two-point functions are often known while their full structure is not, by using the inversion formula (1.48) we can determine the thermal coefficients $a_{\mathcal{O}}$ with relative ease. In [54] this was shown for mean field theory. While the coefficents in this case can also be computed analytically this provides a cross-check for the inversion formula.
In [55] the thermal coefficients where calculated for the strongly coupled three dimensional Ising model.

1.2.4 Holographic thermal correlators

Corrections to the mean field theory correlator [54]

$$G_\Delta(x) := \langle\phi(x)\phi(0)\rangle_\beta^{\text{MFT}} = \sum_{m=-\infty}^{\infty} \frac{1}{((\tau+m)^2 + |\vec{x}|^2)^\Delta} . \tag{1.51}$$

coming from $\lambda\phi^4$ interactions can be studied in AdS_{d+1} space. Using holography the free theory in AdS is dual to the mean field theory (MFT) on the thermal manifold $S^1_\beta \times \mathbb{R}^{d-1}$. Considering quartic interactions in the bulk the leading correction to the MFT correlator can be obtained by an analysis in Mellin space similar to the zero temperature case [77]

$$\langle\phi(x)\phi(0)\rangle_\beta|_{\lambda^1} = \oint_{s_0-\mathrm{i}\infty}^{s_0+\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \Gamma^2(s)\Gamma^2(\Delta-s) M_\beta(s) G_s(x) , \tag{1.52}$$

where $0 < \text{Re}s_0 < \Delta - {}^{(d-1)}\!/_2$ [56]. This expression depends only on the free MFT result $G_s(x)$ and on the Mellin amplitude $M_\beta(s)$. The latter one can be entirely fixed by the zero temperature results for the anomalous dimensions $\gamma_{n,l=0}$ of spin 0 operators obtained in [77] which are analytic in n. The analyticity is an important property for the above formula to work [56]. Additionally the formula for the thermal Mellin amplitudes (1.52) are fixed by some consistency requirements.
First and foremost the KMS condition introduced above (see Equation (1.44))

$$\langle\phi(z,\bar{z})\phi(0)\rangle_\beta = \langle\phi(1-z,1-\bar{z})\phi(0)\rangle_\beta , \tag{1.53}$$

must hold. When using the thermal operator product expansion (1.37) no new operators are exchanged compared to the zero temperature case. Therefore the anomalous dimensions at zero and nonzero temperature must be equal. Furthermore the function $\langle\phi(z = r\omega, \bar{z} = r\omega^{-1})\phi(0)\rangle_\beta$ must be analytic in the ω−plane as shown in Figure 1.4. This was shown in [54]. Similarly for large spatial distances the clustering effect introduced above must be obeyed by the two-point function. Lastly the correlator must satisfy Regge boundedness [56].

Considering the quartic interactions and Witten diagrams in the AdS bulk theory the above formula (1.52) can be proven while an extension including more general interactions was proposed [56]. These interactions are covariant derivatives acting on the quartic term $\lambda_k \nabla^{2k} \phi^4$ where k is summed from 2 to some finite $L > 0$ corresponding to the zero temperature solutions of [77] with maximum spin L. The scalar two-point function can in this case be written at leading order using the new Mellin amplitudes $\mathfrak{M}^L(s, \partial_z, \partial_{\bar{z}})$ which now also depend on derivatives,

$$\langle \phi(x)\phi(0) \rangle_\beta |^{(1)} = \oint_{s_0 - \mathrm{i}\infty}^{s_0 + \mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi \mathrm{i}} \Gamma^2(s)\Gamma^2(\Delta - s)\mathfrak{M}^L(s, \partial_z, \partial_{\bar{z}}) G_s(x) \,, \tag{1.54}$$

$$\mathfrak{M}^L(s, \partial_z, \partial_{\bar{z}}) = \mathfrak{M}^L(s, \partial_{\bar{z}}, \partial_z) \,, \quad \mathfrak{M}^L(s, \partial_z, \partial_{\bar{z}}) = \mathfrak{M}^L(s, -\partial_z, -\partial_{\bar{z}}) \,. \tag{1.55}$$

The constraints on $\mathfrak{M}^L$ follow from the $z \leftrightarrow \bar{z}$ symmetry and the KMS condition (1.44). Again, only the anomalous dimensions of double-trace operators obtained from crossing equations at zero temperature are needed to determine the thermal Mellin amplitudes [56].

1.3 Outline of the thesis

The main focus of this thesis is on the Relation (1.1)

$$B = 3h \,,$$

between the bremsstrahlung of a heavy particle (the Wilson loop) and the coefficient of the stress energy tensor. We will show that it continues to hold in a thermal setting of $\mathcal{N} = 4$ SYM at zero coupling both analytically and perturbatively. For the analytic proof we study a supersymmetric Broken Ward Identity (BWI) and show that in a regime where the thermal operator product expansion can be applied, no relevant thermal corrections arise. The algebraic argument is presented at zero gauge coupling $g = 0$. We will reinforce this exact statement by a perturbative calculation of the bremsstrahlung B and the stress tensor coefficient h at leading order in an expansion around weak 't Hooft coupling λ. Subsequently the coupling is turned on and we study the effects of the interactions to the previous algebraic argument. Additionally, the next-to-leading order of the circular Wilson loop and the straight Wilson line are computed in the thermal setting.

Chapter 2 is a review of known results at zero temperature.
We start by deriving the Relation (1.1) analytically by considering a Ward identity related to the stress tensor multiplet of $\mathcal{N} = 4$ and the displacement multiplet induced by the WL defect (Section 2.1). The resulting two-point functions which are related to the coefficients B and h are studied in a way that simplifies their extension to the thermal calculation in later chapters.

We furthermore derive the bremsstrahlung as a function of the circular Wilson loop which is in a $1/2$ Bogomol'nyi-Prasad-Sommerfield (BPS) state and calculate the one-point function of the stress energy tensor $T_{\mu\nu}$ perturbatively from Feynman diagrams (Section 2.2). Although our calculation will be carried out in a weak-coupling perturbative expansion the resulting identity can - in the zero temperature scenario - be proven exactly as well with a similar calculation.

In Section 3 the above considerations are extended to finite temperature. We restrict ourselves to the non-interacting case by setting the Yang-Mills coupling g to zero.
Once again, we start by considering an algebraic argument related to a Ward identity in Chapter 3.1. First we consider the compactification of the time dimension and show that supersymmetry can be preserved when assuming periodic boundary conditions for all fields in the theory. In a continuative argument we introduce the anti-periodic boundary conditions of the thermal fermions through phase transformations. These yields new terms breaking the supersymmetry of the action and hence the Ward identity. The new Broken Ward Identity then yields corrections compared to the previous chapter. Subsequently these corrections are studied in an operator product expansion. By assuming that inserted operators are sufficiently close to each other, the known OPEs can still be applied such that we are able to simplify the thermal corrections of the BWI. In a final step, the dependence on spacetime and R-symmetry indices of these new relations can be restricted allowing us to show that they ultimately do not affect the relation $B = 3h$ in the given setting.
After obtaining (1.1) exactly we provide a perturbative check in Section 3.2. Zero coupling means in this case that only bare propagator insertions are allowed in the perturbative expansion of the Wilson loop. The leading order, being the only consistent one in this context, will thus be studied in detail. We start with a calculation of the circular Wilson loop at leading order. It will prove convenient to consider this calculation in a small temperature expansion as well as in a high temperature limit. Equipped with the result of the circular Wilson loop we can then derive an expression for the bremsstrahlung similar to the one at zero temperature. We will find that for the straight line WL a scale coefficient has to introduced for dimensional reasons. The resulting bremsstrahlung can then be compared with the stress tensor one-point function in the presence of the circular Wilson loop. We find a scheme for the new coefficient of the straight line which indeed reproduces the Relation (1.1), $B = 3h$. The scheme is dependent on temperature and the radius of the circular loop. This suggests that at every temperature a scheme matching (1.1) can be found

The extension of the algebraic consideration to the full interacting theory is considered in Chapter4. We will follow two different approaches. The goal is to find a supersymmetric piece of the action and a piece breaking the supersymmetry such that the a thermal Broken Ward Identity can be applied similarly to the $g = 0$ case. We consider field redefinitions

depending on a priori arbitrary integers and a dimensional KK-like reduction by using Fourier modes.
We first consider the field redefinitions in Section 4.1. The Yukawa interactions will restrict the fermionic phases mentioned above and furthermore yield the necessity to introduce similar phases for the scalar fields. What is more, these phases are related in a way which breaks the R-symmetry. We study how this breaking arises and possible ways to preserve the largest possible subgroup. Clearly this result alters what we found in the zero coupling discussion of the BWI. While we did not find a relation for the BWI with the broken R-symmetry we give arguments regarding the possibility to again obtain the relation $B = 3h$. Meanwhile, there is a free choice regarding the phases which influences the preserved symmetry group. While we considered only classical arguments before, these choices might be restricted when considering a dynamical calculation. We discuss the steps necessary to determine these restrictions.
The Fourier series representation of fields is considered in Section 4.2. Contrarily to the previous ansatz we can show that R-symmetry is preserved. However, supersymmetry is not preserved in this ansatz and we derive the breaking term. The reason being that Yukawa terms yield interactions between different modes because a shift is introduced.
In Section 4.3 we review the result of the two approaches and discuss the steps necessary to complete these considerations. We expect that eventually we will be able to find a consistent manner in which they can be combined into one convincing ansatz.

Considering the interactive thermal theory an alternative ansatz is to consider the relation $B = 3h$ perturbatively at the next to leading order. We adress this in Chapter 5. As a starting point we calculate the circular Wilson loop and the straight Wilson line in the thermal setting at order λ^2. This order includes the significantly more complex diagrams of vertex and self-energy insertions to the Wilson loop. It is therefore necessary to preliminary consider the effects of finite temperature and the WL on the gauge propagator and calculated propagators at one loop order. Equipped with these results the circular Wilson loop at order λ^2 is then computed. Due to arising difficulties for the vertex diagram and the stress tensor calculation, some of the results obtained are only numerical. Similarly the straight line Wilson loop is obtained from the one-loop propagators. Interestingly, for the straight line only the self-energy insertion is relevant as the contour prefactor cancels all other diagrams readily. The calculations are again obtained in a small temperature expansion and in a high temperature limit.

We conclude the thesis in Chapter 6 where we recall the main results and give a short recap on the necessary calculation.
Some appendices clarifying certain steps of our considerations are provided. In Appendix A we define Pauli matrices and Clebsch-Gordan coefficients. Furthermore we recall the superconformal algebra relations of $\mathcal{N} = 4$. Following this we discuss the breaking of

the superalgebra by the defect in Appendix B. An overview over the $\mathcal{N} = 4$ stress tensor defect and how it can be obtained from $\mathcal{N} = 2$ multiplets is provided in Appendix C. In Appendix D we derive the supersymmetry transformations of the displacement primary $\mathcal{O}$ and the first descendant ψ of the stress tensor multiplet. Appendix E is a consistency check showing that the displacement-stress-tensor two-point functions we use are consistent with the similar ones in $\mathcal{N} = 2$ presented in [38]. For the finite temperature theory it is necessary to introduce redefinition of some of the fields with phase prefactors. In Appendix F we show that these redefinitions are non-anomalous at zero coupling. Similarly, we show that also the anomalies in the interacting theory are zero in Appendix G. In Appendix H we present the status of the dynamical self-energy calculation necessary to determine the correct choice for the phases introduced in Section 4.1 and further considerations in similar directions. We uncover an inconsistency in the respective approach. Therefore, in Appendix I we suppose a different ansatz. After showing that the R-symmetry is preserved at finite temperature we write down a general Lagrangian which allows us to use Broken Ward Identities. We show that its self-energy can be made consistent when introducing proper masses for scalars and spinors. Lastly the Feynman rules of $\mathcal{N} = 4$ SYM are given in J.

1.4 Conventions

Let us clarify some conventions that are used throughout the thesis.
Define the action of $\mathcal{N} = 4$ as

$$\begin{aligned} S = \frac{1}{2g^2}\mathrm{tr}\int \mathrm{d}^4x \Big[&\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu A^\mu)^2 + D_\mu\Phi^{IJ}D^\mu\bar{\Phi}_{IJ} + \mathrm{i}\bar{\lambda}^I\sigma^\mu D_\mu\lambda_I + \mathrm{i}\lambda_I\bar{\sigma}^\mu\bar{D}_\mu\bar{\lambda}^I + \\ &+\mathrm{i}\bar{\Phi}_{IJ}\left\{\lambda^I,\lambda^J\right\} - \mathrm{i}\Phi^{IJ}\left\{\bar{\lambda}_I,\bar{\lambda}_J\right\} + \frac{1}{2}\left[\Phi^{IJ},\Phi^{KL}\right]\left[\bar{\Phi}_{IJ}\bar{\Phi}_{KL}\right] + \partial_\mu\bar{c}D^\mu c\Big] \\ = \frac{1}{2g^2}\int \mathrm{d}^4x\Big[&\frac{1}{2}F^a_{\mu\nu}F^{a\,\mu\nu} + \left(\partial^\mu A^a_\mu\right)^2 + D_\mu\Phi^a_r D^\mu\Phi^a_r + i\bar{\lambda}^a_I \not{D}\lambda^{Ia} - i\lambda^a_I\bar{\not{D}}\bar{\lambda}^{Ia} \\ &+ if^{abc}\lambda^{Ia}\bar{\Sigma}_{rIJ}\Phi^{rb}\lambda^{Jc} - if^{abc}\bar{\lambda}^a_I\Sigma^{IJ}_r\Phi^{rb}\bar{\lambda}^c_J \\ &+\frac{1}{2}f^{abc}f^{ade}\Phi^b_r\Phi^c_s\Phi^{rd}\Phi^{se} + \partial_\mu\bar{c}^a D_\mu c^a\Big]\,. \end{aligned} \tag{1.56}$$

The trace is taken over the color group SU(N) and we consider only Feynman gauge. We use Minkowski indices $\mu,\nu = 0,\dots,3$ and $\mathfrak{su}(4)_R$ fundamental indices $I,J = 1,\dots,4$. A_μ is the gauge field with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, $\lambda^I, \bar{\lambda}_I$ are the four Weyl spinors, $\Phi^{IJ}, \bar{\Phi}_{IJ}$ are the matrices for the six real scalars and $\bar{c}$ and c are the ghosts. Covariant derivatives of any field are given by $D_\mu\bullet = \partial_\mu\bullet + [A_\mu,\bullet]$ and $(\sigma_\mu,\bar{\sigma}_\mu)$ are the Pauli matrices. The structure constants of the gauge group SU(N) are f^{abc} with fundamental color indices $a, b = 1,\dots,N$. The generators T^a employ the known commutation relation $[T^a, T^b] = \mathrm{i}f^{abc}T^c$. Furthermore

$f^{abc}f^{abc} = N(N-1) \simeq N^2$ in the large N limit, $\operatorname{tr} T^a = 0$ and $T^aT^a = \frac{N}{2}$. We follow the notation of [62, 78] and adapt the Feynman gauge $\xi = 1$ as it is convenient for Wilson loop calculations. The first line of(1.56) is the Lagrangian with a color-trace and $\mathfrak{su}(4)_R$ indices only while in the second line color indices are written explicitly and we use $\mathfrak{so}(6)_R$ fundamental indices $r, s = 1, \ldots, 6$ for the six real scalars. The scalar field matrices are related by

$$\Phi^{IJ} = \frac{1}{2}\epsilon^{IJKL}\bar{\Phi}_{KL} \,, \quad \Phi_{IJ} = \begin{pmatrix} 0 & \phi_1 & \phi_2 & \phi_3 \\ -\phi_1 & 0 & \bar{\phi}_3 & -\bar{\phi}_2 \\ -\phi_2 & -\bar{\phi}_3 & 0 & \bar{\phi}_1 \\ -\phi_3 & \bar{\phi}_2 & -\bar{\phi}_1 & 0 \end{pmatrix}_{IJ} \tag{1.57}$$

and their relation to the six real scalars Φ_r $(r = 1, \ldots, 6)$ is (considering the matrices $\bar{\Sigma}_{rIJ}$ defined below) given by

$$\Phi_r = \bar{\Sigma}_{rIJ}\Phi^{IJ} \,. \tag{1.58}$$

For the R-symmetry we use the Clebsch-Gordan coefficients between $\mathfrak{so}(6)_R$ and $\mathfrak{su}(4)_R$ given in [79, 80] which are given through the six matrices

$$\Gamma_r = \gamma_5 \otimes \begin{pmatrix} 0 & \bar{\Sigma}_r \\ \Sigma_r & 0 \end{pmatrix} , \quad r = 1, \ldots, 6 \,, \tag{1.59}$$

where $\gamma_5 = \sigma^3 \otimes \sigma^0$ and the Σ and $\bar{\Sigma}$ satisfy

$$\Sigma_r^{IJ} = -\Sigma_r^{JI} \,, \quad \bar{\Sigma}_{rIJ} = -\bar{\Sigma}_{rJI} \,, \quad \Sigma_r^{IJ} = \left(\bar{\Sigma}_{rIJ}\right)^\dagger \,, \quad \operatorname{tr}\left(\Sigma_r\bar{\Sigma}_s\right) = -4\delta_{r,s} \,. \tag{1.60}$$

We use the usual form of Pauli matrices σ^μ. The anti-symmetric sigma matrices are given by

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right) \,. \tag{1.61}$$

The $\mathcal{N} = 4$ stress energy tensor is given by [36, 81, 82]

$$T_{\mu\nu} = \frac{1}{g^2}\left(T^V_{\mu\nu}(A) + T^F_{\mu\nu}(\lambda) + T^S_{\mu\nu}(\Phi) + T^{\text{interactions}}_{\mu\nu}(A, \lambda, \Phi)\right) \,, \tag{1.62}$$

$$T^V_{\mu\nu}(A) = \operatorname{tr}\left(-F_{\mu\rho}F_{\nu\sigma}\eta^{\rho\sigma} + \frac{1}{4}\eta_{\mu\nu}(F_{\rho\sigma})^2\right) \,,$$

$$T^F_{\mu\nu}(\lambda) = -\frac{1}{4}\operatorname{tr}\left(\bar{\lambda}_I\gamma_\mu\partial_\nu\lambda^I - \bar{\lambda}_I\gamma_\nu\partial_\mu\lambda^I\right) \,,$$

$$T^S_{\mu\nu}(\Phi) = \mathrm{tr}\left(-\partial_\mu \Phi^{IJ}\partial_\nu \bar{\Phi}_{IJ} + \frac{1}{2}\eta_{\mu\nu}\partial_\rho \Phi^{IJ}\partial^\rho \bar{\Phi}_{IJ} + \frac{1}{6}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right)\Phi^{IJ}\bar{\Phi}_{IJ}\right) .$$

The stress tensor is traced with respect to the colors. The conventions of the stress tensor are the same as in [36][2]. The prefactor $1/g^2$ was included because we defined it also for the action, see the conventions of [62, 78] and our previous definition (1.56).

The Maldacena-Wilson loop is defined as in [62]

$$W(C) = \frac{1}{N}\mathrm{tr}\bar{\mathcal{P}}\exp\left[\oint_C \mathrm{d}\mathfrak{t}\left(\mathrm{i}A_\mu \dot{x}^\mu + \Phi_r|\dot{x}|n^r\right)\right] , \tag{1.63}$$

where $A_\mu = A^a_\mu T^a$ is the gauge field and $\Phi_r = \Phi^a_r T^a$ are the six scalars, they are related to Φ^{IJ} and $\bar{\Phi}_{IJ}$ by the Relation (1.58). We denote the generators of the SU(N) gauge group by T^a as mentioned above. The trace is taken over the adjoint representation of the gauge group. n^r is a six-dimensional unit vector ($n^2 = 1$) on the S^5. $\bar{\mathcal{P}}$ is the $\mathfrak{t}$-path-ordering. The contour of the loop we consider is C.
Two contours are of special interest to us, the circular loop and the straight line. For a circular loop with radius $R \in \mathbb{R}^+$ we can parameterize the contour by [62]

$$C_{\text{loop}} : \; x^\mu(\mathfrak{t}) = (0\,, R\cos(\mathfrak{t})\,, R\sin(\mathfrak{t})\,, 0) \quad , \quad \mathfrak{t} \in [0, 2\pi] . \tag{1.64}$$

Similarly we introduce a length parameter $\tilde{R}$ in the parameterization of the straight line. Similar to [35, 63] we use a hyperbolic tangent function. As we want to eventually consider finite temperature we cannot put the line in the x_0 component[3] and instead place it in the x_3 component.

$$C_{\text{line}} : \; x^\mu(\mathfrak{t}) = \left(0,\, 0,\, 0,\, \tilde{R}\tanh\left(\frac{\mathfrak{t}}{2}\right)\right) \quad , \quad \mathfrak{t} \in (-\infty, \infty) . \tag{1.65}$$

The above $n_r \in S^5$ from the Wilson loop breaks the SO(6)$_R$ symmetry group to SO(5)$_R \simeq$ Sp(4)$_R$. For the latter one it is convenient to define a symplectic metric

$$\Omega^{IJ} = n_r \Sigma^{IJ}_r , \quad \Omega_{IJ} = n_r \bar{\Sigma}_{rIJ} . \tag{1.66}$$

This symplectic metric is an Sp(4)$_R$ singlet [83, 84]. It is used to raise and lower indices and satisfies the following properties

$$\Omega_{IJ} = -\Omega_{JI} , \quad \Omega_{IJ}\Omega^{JK} = -\delta^K_I , \quad \epsilon_{IJKL} = -\Omega_{IJ}\Omega_{KL} + \Omega_{IK}\Omega_{JL} - \Omega_{IL}\Omega_{JK} . \tag{1.67}$$

[2]The convention in [81, 82] differs in a sign for the scalars.
[3]This would yield a Polyakov loop at finite temperature.

Furthermore, note the relation

$$n_r n_s \bar{\Sigma}_{rIJ} \bar{\Sigma}_{sKL} \delta^J_L = n^2 \delta^K_I = \delta^K_I \ . \tag{1.68}$$

Chapter 2

Zero temperature

In this Chapter some known results at zero temperature are reviewed. The relation $B = 3h$ is derived in two different manners which eventually will prove helpful for the thermal computation in the subsequent chapter.
While the Relation (1.1) has first been derived using localization in [35, 36] we will start with a different approach. We want to consider a supersymmetric Ward identity allowing us to relate several two-point functions of operators in the displacement and stress tensor multiplet. Each of these correlators has either B or h as a coefficient and the Ward identity will eventually yield $B = 3h$. This approach has first been used to prove an identical relation for $\mathcal{N} = 2$ supersymmetric theories by [38]. As the relations in $\mathcal{N} = 4$ are similar we can closely to the presented derivation.
In a second section the Relation (1.1) is shown to hold in a weak coupling perturbative expansion. Using the results of [35] the bremsstrahlung of a moving quark can be calculated from the circular Maldacena-Wilson loop. The expectation value of said WL is known from [62]. The one-point function of the stress tensor in the presence of the Wilson loop can be computed from leading order Feynman diagrams reproducing the result found by [81]. Comparing these two calculation indeed reproduces $B = 3h$.
We present all calculations in a way which will make the application of finite temperature effects as easy as possible.

2.1 Supersymmetric Ward Identity

The bremsstrahlung B of a moving heavy quark in $\mathcal{N} = 4$ SYM theory can be related to the coefficient of the stress tensor h, sometimes also called energy at infinity. The relation $B = 3h$ was derived originally by [35, 36]. We will recall a proof of this relation which is similar to the one for an identical identity in $\mathcal{N} = 2$ SYM theories as proposed by [38]. This proof mainly relies on algebraic analysis of the stress tensor and displacement multiplets as well as a Ward identity.

Consider a straight line Wilson loop[1] in 4 dimensional $\mathcal{N} = 4$ SYM defined by

$$W = \frac{1}{N}\mathrm{tr}\overleftarrow{\mathcal{P}}\exp\left[\mathrm{i}\int \mathrm{d}\mathfrak{t}\,(A_3 - \mathrm{i}n_r\Phi^r)\right] . \tag{2.1}$$

Here $n_r \in S^5$ $(r = 1,\dots,6)$ is a vector in the $\mathrm{SO}(6)_R \simeq \mathrm{SU}(4)_R$ symmetry group. Note that we use $x_3 \to y(\mathfrak{t})$ to identify the component along the Wilson loop. This Wilson line is a defect which breaks the original superalgebra $\mathfrak{psu}(2,2|4)$ to the subalgebra $\mathfrak{osp}(4^*|4)$ [85]. The preserved fermionic supercharges are

$$\mathcal{Q}^I_\alpha = \frac{1}{\sqrt{2}}Q^I_\alpha + \frac{1}{\sqrt{2}}\Omega^{IJ}\sigma^3_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}_J\,, \quad \text{and} \quad \mathcal{S}^I_\alpha = -\frac{1}{\sqrt{2}}\Omega^{IJ}S_{J\alpha} + \frac{1}{\sqrt{2}}\sigma^3_{\alpha\dot{\alpha}}\bar{S}^{J\dot{\alpha}}\,, \tag{2.2}$$

analogous to [38]. This choice ensures that the Wilson loop operator is preserved under supersymmetry transformations $\delta W = 0$. For details see Appendix B.

Let us consider the conformal symmetry. Naively scale invariance is broken by the introduction of the defect as the distance between any point in the bulk and the Wilson line becomes a relevant distance scale. However, scale invariance and also conformal invariance remain unbroken on the one-dimensional Wilson loop insertion. This leads to the preservation of the translations P_3, special conformal transformations K_3 and dilatations Δ along the defect. Furthermore, the aforementioned distance between a bulk coordinate and the defect is purely radial one and therefore rotations in the bulk directions orthogonal to the defect are also preserved. The respective Lorentz rotations and boosts are M_{mn} $(m,n = 0,1,2)$.

The introduction of the 'preferred direction' $n_r \in S^5$ breaks the R-symmetry group to $\mathrm{SO}(5)_R \simeq \mathrm{USp}(4)_R$. It is convenient to introduce the $\mathrm{USp}(4)_R$ singlet [83, 86]

$$\Omega_{IJ} = n_r\bar{\Sigma}_{rIJ} \quad , \qquad \Omega^{IJ} = n_r\Sigma^{IJ}_r\,, \tag{2.3}$$

using the Klebsch-Gordan coefficients between SO(6) and SU(4) [2]. The symplectic metric Ω_{IJ} for $\mathrm{USp}(4)_R$ thus follows directly from the Wilson loop and will be an integral part of our calculations. It can be used to raise and lower R-symmetry indices.

The original 15 R-symmetry generators $R^I{}_J$ split into preserved and broken generators. The 10 symmetric R-symmetry generators

$$\mathcal{R}^{IK} = \frac{1}{2}R^I{}_J\Omega^{JK} + \frac{1}{2}R^K{}_J\Omega^{JI}\,, \tag{2.4}$$

are preserved by the $\mathrm{USp}(4)_R$ symmetry group while the five anti-symmetric generators

$$\mathfrak{R}^{IK} = \frac{1}{2}R^I{}_J\Omega^{JK} - \frac{1}{2}R^K{}_J\Omega^{JI}\,, \tag{2.5}$$

[1] Note that compared to [38] the Wilson line is placed in a different spacetime coordinate.

[2] See also Appendix A

are broken [83]. Note that by construction $\Omega_{IK}\mathfrak{R}^{IK} = 0$. This breaking into symmetric and anti-symmetric components is straight forward and matches with the dimension of the preserved R-symmetry group.

Let us gather all operators of the $\mathfrak{psu}(2,2|4)$ algebra. We generally use calligraphic letters for most of the preserved generators while broken ones are written in a fraktur font. The preserved generators thus are

$$\mathcal{Q}^I_\alpha\,, \quad \mathcal{S}^I_\alpha\,, \quad \Delta\,, \quad P_3\,, \quad K_3\,, \quad M_{mn}\,, \quad \mathcal{R}^{IJ}\,. \tag{2.6}$$

They define the preserved $\mathfrak{osp}(4^*|4)$ algebra. The broken generators are

$$\mathfrak{Q}^I_\alpha\,, \quad \mathfrak{S}^I_\alpha\,, \quad \mathfrak{P}_m\,, \quad \mathfrak{K}_m\,, \quad \mathfrak{M}_{3m}\,, \quad \mathfrak{R}^{IJ}\,. \tag{2.7}$$

We want to use a Ward identity similar to Equation (33) in [38]. Therefore we need to take a closer look at the displacement multiplet and the stress tensor multiplet in $\mathcal{N} = 4$. This is done in chapter 2.1.1. We will then discuss the zero temperature Ward identity in chapter 2.1.2.

2.1.1 Stress tensor and displacement multiplet

The idea of our calclation is to use a Ward identity similar to the one used in [37, 38] to find a relation for the Wilson loop. Therefore we will need the bulk-to-defect two-point functions between operators from the stress-tensor and the displacement muliplet, respectively.

The displacement multiplet is closely linked to the broken supercharges. As mentioned above, when placing the straight line a relevant distance scale is introduced. Therefore translational symmetry is broken in the components orthogonal to the Wilson line. With translations no longer a symmetry of the theory it is convenient to introduce a new operator called displacement operator $\mathbb{D}$ that can be derived from the components of the stress tensor which are no longer conserved. The displacement operator is only well defined inside a correlator. For any operator χ of the theory the displacement operator is

$$\partial_\mu \langle T^{\mu m}(x_0,x_1,x_2,x_3)\chi(z)\rangle = \delta(x_0)\delta(x_1)\delta(x_2)\,\langle \mathbb{D}^m(x_3)\chi(z)\rangle\ . \tag{2.8a}$$

As we saw above, due to the introduction of the Wilson line defect also half of the supercharges are broken. This leads to the supercurrent $\Psi^{\mu\alpha}_I$ splitting into a preserved part $\hat{\Psi}^{\mu\alpha}_I$ and a broken part $\tilde{\Psi}^{\mu\alpha}_I$. Similar to above we thus define a fermionic displacement operator $\mathbb{A}$ following from the broken part of the supercurrent

$$\partial_\mu \left\langle \tilde{\Psi}_I^{\mu\alpha}(x_0, x_1, x_2, x_3)\chi(z)\right\rangle = \delta(x_0)\delta(x_1)\delta(x_2)\left\langle \mathbb{A}_I^\alpha(x_3)\chi(z)\right\rangle . \tag{2.8b}$$

Similar to the supercurrent also the R-symmetry current $J_{KL}^{\mu IJ}$ is split in an unbroken part and an anti-symmetric broken part $\tilde{J}_{KL}^{\mu IJ} = \Omega_{KL}\mathrm{j}^{\mu IJ}$ with

$$\partial_\mu \left\langle \mathrm{j}^{\mu IJ}\chi(z)\right\rangle = \delta(x_0)\delta(x_1)\delta(x_2)\left\langle \mathbb{O}^{IJ}(x_3)\chi(z)\right\rangle . \tag{2.8c}$$

Equations (2.8) are the analog of Equations (5), (6), (7) in [38]. They define the full displacement multiplet [85].

Acting with the broken generators on the Wilson loop operator yields precisely the broken currents above and thus

$$\langle \mathfrak{Q}_\alpha^I W\rangle = -2\mathrm{i}\int \mathrm{d}x_3 \, \left\langle \mathbb{A}_\alpha^I W\right\rangle , \tag{2.9a}$$

$$\langle \mathfrak{P}^m W\rangle = -\mathrm{i}\int \mathrm{d}x_3 \, \langle \mathbb{D}^m W\rangle , \tag{2.9b}$$

$$\langle \mathfrak{R}^{IJ} W\rangle = -\mathrm{i}\int \mathrm{d}x_3 \, \left\langle \mathbb{O}^{IJ} W\right\rangle , \tag{2.9c}$$

where $m = 0, 1, 2$. This yields the tracelessness condition $\Omega_{IJ}\mathbb{O}^{IJ} = 0$ for the primary of the displacement multiplet. The action of the preserved supercharges on the displacement multiplet is

$$\delta\mathbb{O}^{JK} = 2\Omega^{IJ}\mathbb{A}_\alpha^K\zeta_I^\alpha - 2\Omega^{IK}\mathbb{A}_\alpha^J\zeta_I^\alpha + \Omega^{JK}\mathbb{A}_\alpha^I\zeta_I^\alpha , \tag{2.10a}$$

$$\delta\mathbb{A}_\beta^J = -4\mathrm{i}\Omega^{IJ}\left(\sigma^{3m}\right)_\alpha^{\ \gamma}\epsilon_{\gamma\beta}\mathbb{D}_m\zeta_I^\alpha - 2\mathrm{i}\epsilon_{\alpha\beta}\mathcal{D}_3\mathbb{O}^{IJ}\zeta_I^\alpha , \tag{2.10b}$$

$$\delta\mathbb{D}^m = 2\left(\sigma^{3m}\right)_\alpha^{\ \beta}\mathcal{D}_3\mathbb{A}_\beta^I\zeta_I^\alpha . \tag{2.10c}$$

The stress tensor multiplet of $\mathcal{N} = 4$ SYM is well known [87, 88]. Its field content is given in table 2.1 below where we include also the displacement multiplet for convenience.
More details on the stress tensor multiplet and its relation to $\mathcal{N} = 2$ multiplet are summarized in Appendix (C). For later convenience consider the transformation of the field ψ under the preserved supersymmetry transformations [87]

$$\begin{aligned}\delta\psi_{I\alpha}^{JK} = \mathcal{Q}_\beta^N\psi_{I\alpha}^{JK}\zeta_N^\beta &= -\frac{\mathrm{i}}{4}\partial_{\alpha\dot{\alpha}}\varphi_{IN}^{JK}\bar{\sigma}_3^{\dot{\alpha}\alpha}\zeta_\alpha^N + \varphi_{IN}^{JK}\eta_\alpha^N - \Omega^{JK}\Omega_{IP}f_{\alpha\beta}^{PN}\zeta_N^\beta \\ &- \frac{1}{3}\delta_I^J f_{\alpha\beta}^{KN}\zeta_N^\beta + \frac{1}{3}\delta_I^K f_{\alpha\beta}^{JN}\zeta_N^\beta - \rho_{IL}\epsilon_{\alpha\beta}\epsilon^{JKNO}\zeta_O^\beta \\ &- \Omega^{JK}\Omega_{IP}\Omega^{QR}J_{\alpha\dot{\alpha}QR}^{PN}\bar{\sigma}_3^{\dot{\alpha}\beta}\zeta_N^\beta - \frac{1}{3}\left(\delta_I^J J_{\alpha\dot{\alpha}ML}^{KM} - \delta_I^K J_{\alpha\dot{\alpha}ML}^{JM}\right)\bar{\sigma}_3^{\dot{\alpha}\beta}\zeta_\beta^L .\end{aligned} \tag{2.11}$$

field	field constraints	Δ	(j_1, j_2)	$\mathfrak{su}(4)_R$ rep	$\mathfrak{usp}(4)_R$ rep
${\varphi^{IJ}}_{KL}$	${\varphi^{IJ}}_{IJ} = 0$	2	$(0,0)$	[0,2,0]	[0,0]⊕[0,1]⊕[0,2]
${\psi_{I\alpha}}^{JK}$	${\psi_{I\alpha}}^{JK} = {\psi_{I\alpha}}^{[JK]}\,,\ {\psi_{I\alpha}}^{IK} = 0$	$\frac{5}{2}$	$\left(\frac{1}{2},0\right)$	[0,1,1]	[1,0]⊕[1,1]
$\bar{\psi}^I_{JK\dot{\alpha}}$	$\bar{\psi}^I_{JK\dot{\alpha}} = \bar{\psi}^I_{[JK]\dot{\alpha}}\,,\ \bar{\psi}^I_{IK\dot{\alpha}} = 0$	3	$\left(0,\frac{1}{2}\right)$	[1,1,0]	[1,0]⊕[1,1]
$f^{IJ}_{\alpha\beta}$	$f^{IJ}_{\alpha\beta} = f^{IJ}_{(\alpha\beta)} = f^{[IJ]}_{\alpha\beta}$	3	$(1,0)$	[0,1,0]	[0,0]⊕[0,1]
$\bar{f}^{IJ}_{\dot{\alpha}\dot{\beta}}$	$f^{IJ}_{\dot{\alpha}\dot{\beta}} = f^{IJ}_{(\dot{\alpha}\dot{\beta})} = f^{[IJ]}_{\dot{\alpha}\dot{\beta}}$	3	$(0,1)$	[0,1,0]	[0,0]⊕[0,1]
ρ_{IJ}	$\rho_{IJ} = \rho_{(IJ)}$	3	$(0,0)$	[0,0,2]	[2,0]
$\bar{\rho}^{IJ}$	$\bar{\rho}^{IJ} = \bar{\rho}^{(IJ)}$	3	$(0,0)$	[2,0,0]	[2,0]
$J^{IJ}_{\mu\,KL}$	$J^{IJ}_{\mu\,KL} = J^{[IJ]}_{\mu\,KL} = J^{IJ}_{\mu\,[KL]}$	3	$\left(\frac{1}{2},\frac{1}{2}\right)$	[1,0,1]	[0,1]⊕[2,0]
$\lambda_{I\alpha}$		$\frac{7}{2}$	$\left(\frac{1}{2},0\right)$	[0,0,1]	[1,0]
$\bar{\lambda}^I_{\dot{\alpha}}$		$\frac{7}{2}$	$\left(0,\frac{1}{2}\right)$	[1,0,0]	[1,0]
$\Psi^I_{\alpha\beta\dot{\alpha}}$	$\Psi^I_{\alpha\beta\dot{\alpha}} = \Psi^I_{(\alpha\beta)\dot{\alpha}}\,,\ \ \partial^{\alpha\dot{\alpha}}\Psi^I_{\alpha\beta\dot{\alpha}} = 0$	$\frac{7}{2}$	$\left(1,\frac{1}{2}\right)$	[1,0,0]	[1,0]
$\bar{\Psi}_{I\alpha\dot{\alpha}\dot{\beta}}$	$\bar{\Psi}_{I\alpha\dot{\alpha}\dot{\beta}} = \bar{\Psi}_{I\alpha(\dot{\alpha}\dot{\beta})}\,,\ \ \partial^{\dot{\alpha}\alpha}\bar{\Psi}_{I\alpha\dot{\alpha}\dot{\beta}} = 0$	$\frac{7}{2}$	$\left(\frac{1}{2},1\right)$	[0,0,1]	[1,0]
Φ		4	$(0,0)$	[0,0,0]	[0,0]
$\bar{\Phi}$		4	$(0,0)$	[0,0,0]	[0,0]
$T^{\mu\nu}$	$T^{\mu\nu} = T^{(\mu\nu)}$	4	$(1,1)$	[0,0,0]	[0,0]
$\mathbb{O}^{IJ}$	$\mathbb{O}^{IJ} = \mathbb{O}^{[IJ]}\,,\ \ \Omega_{IJ}\mathbb{O}^{IJ} = 0$	1	$(0,0)$		[0,1]
$\mathbb{\Lambda}^I_\alpha$		$\frac{3}{2}$	$\left(\frac{1}{2},0\right)$		[1,0]
$\mathbb{D}^m$		2	$\left(\frac{1}{2},\frac{1}{2}\right)$		[0,0]

Table 2.1: The fields of the stress-tensor and displacement multiplet with their respective quantum numbers, constraints and representations. This table can originally be found in [87] with the stress-tensor multiplet only.

The relevant correlation functions for our main calculation are bulk-to-defect two-point functions between operators of the two above multiplets. For the case of $\mathcal{N} = 2$ the relevant correlators are known from [38, 89]. Here we suggest the respective ones for $\mathcal{N} = 4$. This can be done as the structure of these two-point functions is equivalent to the $\mathcal{N} = 2$ ones. In Appendix E we check that all fields and correlation function can be consistently reduced to theories with lower supersymmetry. We will explain the structure of the R-symmetry index which must agree with the field properties and constraints following table 2.1. The kinematic structure and the spinor indices are equivalent to the $\mathcal{N} = 2$ case [83].

$$\left\langle f^{IJ}_{\alpha\beta}(X,0)\mathbb{O}^{KL}(0,y)\right\rangle^{(T=0)}_W = -\frac{3h}{2\pi}\frac{(X_m\sigma^m\bar{\sigma}^3\epsilon)_{\alpha\beta}}{X^3(y^2+X^2)}\times$$

$$\times\left(\Omega^{IK}\Omega^{JL} - \Omega^{IL}\Omega^{JK} - \frac{1}{2}\Omega^{IJ}\Omega^{KL}\right)\,. \tag{2.12a}$$

The indices IJ and KL are anti-symmetric while at the same time we should get zero upon contraction with Ω_{KL} which is the reason for the structure that is slightly more complicated than simply picking $\Omega^{IJ}\Omega^{KL}$. The reason we cannot have a structure like $y\left(\sigma^3\bar{\sigma}^3\epsilon\right)_{\alpha\beta} = -y\epsilon_{\alpha\beta}$ in the numerator is that the indices α and β are symmetric which is not true for the anti-symmetric epsilon tensor.

$$\left\langle J_{3\,KL}^{IJ}(X,0)\mathbb{O}^{MN}(0,y)\right\rangle_W^{(T=0)} = \frac{B}{\pi}\frac{X^2 y}{|X|^3(y^2+X^2)^2}\Omega_{KL}\times$$
$$\times\left(\Omega^{IM}\Omega^{JN} - \Omega^{IN}\Omega^{JM} - \frac{1}{2}\Omega^{IJ}\Omega^{MN}\right)\,, \tag{2.12b}$$

$$\left\langle J_{m\,KL}^{IJ}(X,0)\mathbb{O}^{MN}(0,y)\right\rangle_W^{(T=0)} = \frac{B}{2\pi}\frac{X_m(y^2-X^2)}{|X|^3(y^2+X^2)^2}\Omega_{KL}\times$$
$$\times\left(\Omega^{IM}\Omega^{JN} - \Omega^{IN}\Omega^{JM} - \frac{1}{2}\Omega^{IJ}\Omega^{MN}\right)\,. \tag{2.12c}$$

Again the above property of contracting with Ω_{MN} leads to an identical structure which is antisymmetric in the three pairs IJ, KL and MN as can be seen readily. Note that we split the current J_μ into the component J_3 along the Wilson line and the orthogonal components J_m because they have different kinematics.

$$\left\langle \psi_{I\alpha}^{\;KL}(X,0)\mathbb{A}_\beta^J(0,y)\right\rangle_W^{(T=0)} = \frac{3h}{2\pi}\frac{\left[\mathrm{i}y\epsilon_{\alpha\beta} - \left(X_m\sigma^m\bar{\sigma}^3\right)_{\alpha\beta}\right]}{X(y^2+X^2)^2}\times$$
$$\times\left(3\delta_I^J\Omega^{KL} - \delta_I^K\Omega^{JL} + \delta_I^L\Omega^{JK}\right)\,. \tag{2.12d}$$

This structure makes sure that we find anti-symmetry in the indecs K and L. Furthermore the tracelessness condition [87] is satisfied, $\left\langle \psi_{I\alpha}^{\;IL}(X,0)\mathbb{A}_\beta^J(0,y)\right\rangle_W^{(T=0)} = 0$.

$$\left\langle \rho_{IJ}(\vec{x},0)\mathbb{O}^{KL}(\vec{0},y)\right\rangle_W^{(T=0)} = 0\,. \tag{2.12e}$$

Here, lastly, we find the correlator be identically zero. This can be seen by the index structure as there is no invariant tensor that can combine with the symmetric I, J indices in a non-vanishing manner.

We want to define the following three functions $g_i(X_m, y)$ $(i = 1,2,3)$ which capture the kinematics of the above correlators

$$g_1(X,y) := \frac{\mathrm{i}y}{6\pi|X|(X^2+y^2)^2}\,,$$
$$g_2(X,y) := \frac{-1}{6\pi(X^2+y^2)^2}\,, \tag{2.13}$$

$$g_3(X,y) := \frac{1}{12\pi X^2(X^2+y^2)} \; .$$

Furthermore, define

$$\hat{X}_{\alpha\beta} := \frac{X_m}{|X|}\left(\sigma^m\bar{\sigma}^3\right)_{\alpha\beta} \; , \tag{2.14}$$

which captures the dependence on spinor indices. The correlators needed for the Ward identity we consider below thus are

$$\begin{aligned}\left\langle f^{IJ}_{\alpha\beta}(X,0)\mathbb{O}^{KL}(0,y)\right\rangle^{(T=0)}_W &= -18h\, g_3(X,y)\hat{X}_{\alpha\beta}\times\\ &\times\left(\Omega^{IK}\Omega^{JL}-\Omega^{IL}\Omega^{JK}-\frac{1}{2}\Omega^{IJ}\Omega^{KL}\right)\end{aligned} \tag{2.15a}$$

$$\begin{aligned}\left\langle J^{IJ}_{3\,KL}(X,0)\mathbb{O}^{MN}(0,y)\right\rangle^{(T=0)}_W &= 6Bg_1(X,y)\Omega_{KL}\times\\ &\times\left(\Omega^{IM}\Omega^{JN}-\Omega^{IN}\Omega^{JM}-\frac{1}{2}\Omega^{IJ}\Omega^{MN}\right) \; ,\end{aligned} \tag{2.15b}$$

$$\begin{aligned}\left\langle J^{IJ}_{m\,KL}(X,0)\mathbb{O}^{MN}(0,y)\right\rangle^{(T=0)}_W &= 6B(g_2(X,y)+g_3(X,y))\frac{X_m}{|X|}\Omega_{KL}\times\\ &\times\left(\Omega^{IM}\Omega^{JN}-\Omega^{IN}\Omega^{JM}-\frac{1}{2}\Omega^{IJ}\Omega^{MN}\right) \; ,\end{aligned} \tag{2.15c}$$

$$\begin{aligned}\left\langle \psi_{I\alpha}^{\;KL}(X,0)\mathbb{A}^J_\beta(0,y)\right\rangle^{(T=0)}_W &= 9h\left(g_1(X,y)\epsilon_{\alpha\beta}+g_2(X,y)\hat{X}_{\alpha\beta}\right)\times\\ &\times\left(3\delta^J_I\Omega^{KL}-\delta^K_I\Omega^{JL}+\delta^L_I\Omega^{JK}\right) \; .\end{aligned} \tag{2.15d}$$

2.1.2 Zero temperature Ward Identity

We want to consider the zero temperature Ward identity

$$\langle \mathcal{Q}^N_\beta\left(\psi_{I\alpha}^{\;JK}(\tau,\vec{x},0)\mathbb{O}^{LM}(0,0,y)\right)\rangle_W = 0 \; , \tag{2.16}$$

which is the $\mathcal{N}=4$ version of the Ward identity discussed in [37,38]. Before we go into the calculation let us review why this identity needs to hold. We follow the derivation of a similar Ward identity done in [90].
Therefore recall that the supercharges are linked to the time component of the supercurrent via an integral,

$$\mathcal{Q}^I_\alpha = \int \mathrm{d}^3x\hat{\Psi}^{I\,0}_\alpha \; . \tag{2.17}$$

The supercharge we consider is the one preserved by the Wilson line $\sqrt{2}\mathcal{Q} = Q + \sigma^3\bar{Q}$ and similarly the supercurrent we consider comes from the combination $\sqrt{2}\hat{\Psi} = \Psi + \sigma^3\bar{\Psi}$. This is the part of the supercurrent which remains preserved by the Wilson loop and thus

$$\partial_\mu \hat{\Psi}^{I\,\mu}_\alpha = 0\,. \tag{2.18}$$

We can define a related current by contracting the spinor index

$$j^{I\,\mu} := \hat{\Psi}^{I\,\mu}_\alpha \psi^\alpha\,, \tag{2.19}$$

where ψ^α is an a priori arbitrary spinor. For our purpose we assume that it is constant in spacetime, $\partial_\mu \psi^\alpha = 0$.
Consider any local operator $\varphi(x)$. Its operator product expansion with the supercurrent is given by

$$\hat{\Psi}^{I\,0}_\alpha(x)\varphi(0) = \ldots + \frac{x^\mu}{2\pi^2|x|^4}\left[\mathcal{Q}^I_\alpha, \varphi\right](0) + \frac{x^\mu x_{\alpha\dot\alpha}}{2\pi^2|x|^6}\left[\mathcal{S}^{I\,\dot\alpha}, \varphi\right](0) + \ldots\,, \tag{2.20}$$

with no other operators that have the same powers of x. If we now integrate the current $j^{I\,\mu}$ around the insertion point of the operator $\varphi(x)$ we find that just these two parts of the OPE can yield non-vanishing contributions to the contour integral.

$$\begin{aligned}\oint_x \mathrm{d}\Sigma_\mu\, j^{I\,\mu}(z)\varphi(x) &= \oint_x \mathrm{d}\Sigma_\mu\, \hat{\Psi}^{I\,\mu}_\alpha(z)\varphi(x)\psi^\alpha \\ &= \left[\mathcal{Q}^I_\alpha, \varphi\right](x)\psi^\alpha + \sigma^\mu_{\alpha\dot\alpha}\left[\mathcal{S}^{I\,\dot\alpha}, \varphi\right](x)\partial_\mu\psi^\alpha = \left[\mathcal{Q}^I_\alpha, \varphi\right](x)\psi^\alpha\,. \end{aligned}\tag{2.21}$$

In the last line we used our assumption that ψ^α is in fact constant.

Let us now define a general correlator[3] with $n \in \mathbb{N}$ operator insertions of generic operators φ_i at different points $x^\mu_i \neq x^\mu_j \in \mathbb{R}^{1,3}$.

$$V^{I\,\mu} := \left\langle j^{I\,\mu}(z^\mu)\varphi_1(x^\mu_1)\ldots\varphi_n(x^\mu_n)\right\rangle_W\,. \tag{2.24}$$

We want to consider this correlator away from the insertion points of the φ_k. Therefore define a subset of the spacetime which does not contain these insertions,
$\mathcal{D} = \mathbb{R}^{1,3}\backslash\{x_1, \ldots, x_n\}$. The correlator defined above then is a divergentless field and hence $\partial_\mu V^{I\,\mu} = 0$ on $\mathcal{D}$ [90]. For completeness we will proof this relation. Let us act with the

[3]Recall the definition of the expectation value

$$\langle\hat{\mathcal{O}}\rangle_W := \int [\mathrm{d}\phi_{\text{all}}]\hat{\mathcal{O}}e^{-S}W\,. \tag{2.22}$$

For later use note that the supersymmetry transformation δ was chosen in such a way that

$$\delta_{SUSY}W = 0 \quad\Leftrightarrow\quad \mathcal{Q}^I_\alpha W = 0\,, \tag{2.23}$$

namely we act with the supercharge $\mathcal{Q}^I_\alpha$ as given by Equation (2.2).

derivative where we are careful to use the definition of the correlation function.

$$
\begin{aligned}
\partial_\mu V^{I\,\mu} &= \partial_\mu \left\langle j^{I\,\mu}(z)\varphi_1(x_1)\dots\varphi_n(x_n)\right\rangle_W \\
&= \partial_\mu \int [\mathrm{d}\phi_{\text{all}}] j^{I\,\mu}(z)\varphi_1(x_1)\dots\varphi_n(x_n)e^{-S}W && \text{(2.25a)} \\
&= \int [\mathrm{d}\phi_{\text{all}}]\Big[(\partial_\mu j^{I\,\mu}(z))\varphi_1(x_1)\dots\varphi_n(x_n)e^{-S}W \\
&\qquad + j^{I\,\mu}(z)\sum_{k=1}^{n}\varphi_1(x_1)\dots(\partial_\mu\varphi_k(x_k))\dots\varphi_n(x_n)e^{-S}W && \text{(2.25b)} \\
&\qquad + j^{I\,\mu}(z)\varphi_1(x_1)\dots\varphi_n(x_n)e^{-S}(\partial_\mu W) \\
&\qquad - (\partial_\mu S) j^{I\,\mu}(z)\varphi_1(x_1)\dots\varphi_n(x_n)e^{-S}W\Big] .
\end{aligned}
$$

The last line consists of four terms. Let us consider them individually. The first has the derivative acting on the supercurrent. We established above that this is a preserved current $\partial_\mu j^{I\,\mu} = 0$. Furthermore we look at a domain $\mathcal{D}$ away from the insertion points of the φ_k and therefore $\partial_\mu\varphi_k(x_k) = 0$ for all k thus killing the second term. The third and fourth term can be treated in a very similar manner. The action S and the Wilson loop operator W both are functionals of the $\mathcal{N} = 4$ fields. They do, however, include a spacetime integral over the coordinates. Therefore they do not depend on any coordinate and thus a derivative acting on them must readily give zero. Therefore indeed

$$
\partial_\mu V^{I\,\mu} = \frac{\partial}{\partial z^\mu} V^{I\,\mu}(z) = 0\,, \quad \text{for } z \in \mathcal{D} = \mathbb{R}^{1,3}\backslash\{x_1,\dots,x_n\}\,. \tag{2.26}
$$

This relation can be integrated over a volume $V \subset \mathcal{D}$. As the integrand is identically zero inside the volume V the remaining integral continues to vanish. When expanding the volume we only find problems in crossing the insertion points of the operators, however these singular points can be excluded from the integration domain. Therefore we can extend the volume to cover all of $\mathcal{D}$. By applying Stoke's theorem instead of integrating over the divergence of the field $V^{I\,\mu}$ we can consider the surface integral over the boundary of $\mathcal{D}$,

$$
0 = \int_{\mathcal{D}} \partial_\mu V^{I\mu}(z) = \int_{\partial\mathcal{D}} \mathrm{d}S_\mu V^{I\mu}(z) = \oint_{S^3_\infty} \mathrm{d}S_\mu V^{I\mu}(z) - \sum_{k=1}^{n}\oint_{x_k} \mathrm{d}S_\mu V^{I\mu}(z)\,. \tag{2.27}
$$

Note that there is no integral which goes around the Wilson line because this is not a boundary of $\mathcal{D}$. We mentioned above that the relation $\partial_\mu V^{I\mu} = 0$ also holds on the Wilson line operator. We only get boundaries from the sphere at infinity and from the insertion

points of the operators φ_k. The conformal dimension of the current $\hat{\Psi}^{I\,\mu}_{\alpha}$ is $\Delta = 7/2$ and thus $V^{I\,\mu}$ falls off at least as $|x|^{-7}$ at infinity thus canceling the first term.
For the remaining terms we can use Equation (2.21) to finally find the result

$$\begin{aligned} 0 = \int_{\mathcal{D}} \partial_\mu V^{I\,\mu}(z) &= -\sum_{k=1}^{n} \left\langle \varphi_1(x_1)\dots[\mathcal{Q}^I_\alpha, \varphi_k(x_k)]\dots\varphi_n(x_n)\right\rangle_W \psi^\alpha \\ &= -\left\langle\left[\mathcal{Q}^I_\alpha, \varphi_1(x_1)\dots\varphi_n(x_n)\right]\right\rangle_W \psi^\alpha \,. \end{aligned} \tag{2.28}$$

This result is indeed analogous to result (A.8) in [90]. This result can be rewritten in a manner that is slightly more approachable for our further considerations

$$0 = \int_{\mathbb{R}^{1,3}} \mathrm{d}^4u \left\langle \partial_\mu \hat{\Psi}^{I\mu}_\alpha(u)\varphi_1(x_1)\dots\varphi_n(x_n)\right\rangle_W \zeta^\alpha_I = -\left\langle\delta\varphi_1(x_1)\dots\varphi_n(x_n)\right\rangle_W \,. \tag{2.29}$$

Let us now write out the identity. The supersymmetry transformation acting on the two fields above is given in the previous chapter by Equations (2.10a) and (2.11). They are furthermore derived in Appendix D. Recall

$$\begin{aligned} \delta\psi^{JK}_{I\alpha} = &-\frac{\mathrm{i}}{4}\partial_{\alpha\dot\alpha}\varphi^{JK}_{IN}\bar\sigma_3^{\dot\alpha\alpha}\zeta^N_\alpha + \varphi^{JK}_{IN}\eta^N_\alpha - \Omega^{JK}\Omega_{IP}f^{PN}_{\alpha\beta}\zeta^\beta_N - \frac{1}{3}\delta^J_I f^{KN}_{\alpha\beta}\zeta^\beta_N \\ &+ \frac{1}{3}\delta^K_I f^{JN}_{\alpha\beta}\zeta^\beta_N - \rho_{IL}\epsilon_{\alpha\beta}\epsilon^{JKNO}\zeta^\beta_O - \Omega^{JK}\Omega_{IP}\Omega^{QR}J^{PN}_{\alpha\dot\alpha QR}\bar\sigma_3^{\dot\alpha\beta}\zeta^\beta_N \\ &- \frac{1}{3}\left(\delta^J_I J^{KM}_{\alpha\dot\alpha ML} - \delta^K_I J^{JM}_{\alpha\dot\alpha ML}\right)\bar\sigma_3^{\dot\alpha\beta}\zeta^L_\beta \,, \end{aligned} \tag{2.30}$$

$$\delta\mathbb{O}^{JK} = \left(\bar\Omega^{IJ}\mathbb{A}^K_\alpha - \bar\Omega^{IK}\mathbb{A}^J_\alpha + \frac{1}{2}\bar\Omega^{JK}\mathbb{A}^I_\alpha\right)\zeta^\alpha_I \,, \tag{2.31}$$

The Ward identity then is given by a combination of the two-point functions we introduced above:

$$\begin{aligned} 0 &= \left\langle \mathcal{Q}^N_\beta \psi^{JK}_{I\alpha}\mathbb{O}^{LM}\right\rangle_W \\ &= 2\Omega^{NL}\left\langle\mathbb{A}^M_\beta\psi_{I\alpha}^{JK}\right\rangle_W - 2\Omega^{NM}\left\langle\mathbb{A}^L_\beta\psi_{I\alpha}^{JK}\right\rangle_W + \Omega^{LM}\left\langle\mathbb{A}^N_\beta\psi_{I\alpha}^{JK}\right\rangle_W - \Omega^{JK}\Omega_{IP}\left\langle f^{IP}_{\alpha\beta}\mathbb{O}^{LM}\right\rangle_W \\ &\quad - \frac{1}{3}\delta^J_I\left\langle f^{KN}_{\alpha\beta}\mathbb{O}^{LM}\right\rangle_W + \frac{1}{3}\delta^K_I\left\langle f^{JN}_{\alpha\beta}\mathbb{O}^{LM}\right\rangle_W - \Omega^{JK}\Omega_{IP}\Omega^{QR}\left\langle J^{PN}_{\mu QR}\mathbb{O}^{LM}\right\rangle_W \\ &\quad - \frac{1}{3}\left(\delta^J_I\left\langle J^{KP}_{\mu PQ}\mathbb{O}^{LM}\right\rangle_W - \delta^K_I\left\langle J^{JP}_{\mu PQ}\mathbb{O}^{LM}\right\rangle_W\right)\Omega^{QN}\left(\sigma^\mu\bar\sigma\right)_{\alpha\beta} \,. \end{aligned} \tag{2.32}$$

The correlators above are bulk-to-defect two-point functions between operators of the displacement and stress tensor multiplet. These were derived in the previous Subsection 2.1.1. Some operators which are zero were omitted.

Let us rewrite this in terms of the kinematic functions g_i (2.13). Therefore use the expressions (2.15) for the correlators, yielding

$$\boxed{(3h - B)\left[g_1(X,y)\epsilon_{\alpha\beta} + g_2(X,y)\hat{X}_{\alpha\beta} + g_3(X,y)\hat{X}_{\alpha\beta}\right]\mathfrak{C}_I^{JKLMN}\zeta_N^{\beta} = 0\ .} \tag{2.33}$$

were $\mathfrak{C}_I^{JKLMN}$ is some $\mathrm{Sp}(4)_R$ index structure:

$$\begin{aligned}\mathfrak{C}_I^{JKLMN} = &\left[\delta_I^K(2\Omega^{JL}\Omega^{MN} - 2\Omega^{JM}\Omega^{LN} + \Omega^{JN}\Omega^{LM}) - (J \leftrightarrow K)\right] \\ &+ 3\delta_I^N\Omega^{JK}\Omega^{LM} - 6\delta_I^M\Omega^{JK}\Omega^{LN} + 6\delta_I^L\Omega^{JK}\Omega^{MN}\ .\end{aligned} \tag{2.34}$$

It is anti-symmetric in J, K and in L, M, respectively. Furthermore it obeys two traceless-ness conditions $\mathfrak{C}_I^{IKLMN} = 0$ and $\Omega_{LM}\mathfrak{C}_I^{JKLMN} = 0$. We readily see that this yields

$$\boxed{B = 3h\ ,} \tag{2.35}$$

as expected.

2.2 Perturbative check

The above identity (1.1) $B = 3h$ can be checked by a perturbative calculation. Therefore we need to calculate the expectation value of the circular Wilson loop from which the bremsstrahlung can be obtained [35] and the one-point function of the stress-energy tensor. For a better understanding of the used methodology it is convenient to calculate the circular Wilson loop first. This $1/2$-BPS Wilson loop was first perturbatively calculated in [62] with a conjecture for an exact result due to a cancellation of interacting diagrams. This was later proven by [63, 64]. For our purposes we will only consider the leading order in perturbation theory.

2.2.1 Circular Wilson loop

The goal of this derivation is to reproduce the first order of the result from [62]. This will be the groundwork to obtain similar results at finite temperature.
Recall the definition of the Lagrangian (1.56) from which we can read off the bare propagators to be [62]

$$\langle A_\mu^a(x_1)A_\nu^b(x_2)\rangle = \frac{g^2\delta^{ab}\eta_{\mu\nu}}{4\pi^2(x_1 - x_2)^2}\ , \quad \langle \Phi_r^a(x_1)\Phi_s^b(x_2)\rangle = \frac{g^2\delta^{ab}\delta_{rs}}{4\pi^2(x_1 - x_2)^2}\ . \tag{2.36}$$

A note considering our normalization is in order at this point. Because the action depends only on an overall coupling constant $1/g^2$ and the terms themselves are all independent of g the counting is simplified. Every propagator in a Feynman diagram contributes with g^2 and every vertex with $1/g^2$. For the vertices it is not important whether they are three- or four-point.

The Maldacena-Wilson loop [58] for a general contour C is defined by (1.63)

$$W(C) = \frac{1}{N}\mathrm{tr}\bar{\mathcal{P}}\exp\left[\oint_C \mathrm{dt}\,(\mathrm{i}A_\mu \dot{x}^\mu + \Phi_r|\dot{x}|n^r)\right]. \tag{2.37}$$

We want to expand the exponential in the Wilson loop with a power series. This will eventually yield a perturbative expansion.

$$\begin{aligned}\langle W(C)\rangle \approx & 1 + \frac{1}{N}\bar{\mathcal{P}}\mathrm{tr}\left\langle \oint_C \mathrm{dt}_1\,(\mathrm{i}A_{\mu\,1}\dot{x}_1^\mu + \Phi_{r\,1}|\dot{x}_1|n^r)\right\rangle \\ & + \frac{1}{2N}\bar{\mathcal{P}}\mathrm{tr}\left\langle \oiint_C \mathrm{dt}_1\mathrm{dt}_2\,(\mathrm{i}A_{\mu\,1}\dot{x}_1^\mu + \Phi_{r\,1}|\dot{x}_1|n^r)\,(\mathrm{i}A_{\nu\,2}\dot{x}_2^\nu + \Phi_{s\,2}|\dot{x}_2|n^s)\right\rangle + \dots\,,\end{aligned} \tag{2.38}$$

where we used the notation $A_{\mu\,i} = A_\mu(x(\mathrm{t}_i))$, $\Phi_{r\,i} = \Phi_r(x(\mathrm{t}_i))$ and $x_i = x(\mathrm{t}_i)$. When taking the expectation value $\langle\dots\rangle$ we Wick contract the bosonic fields among themselves (or with fields from the expansion of the Euclidean action at higher order). For the leading order correction there is simply a sum of single fields which cannot be contracted. This term thus is zero[4]. For the second term we can multiply out and find

$$\begin{aligned}\langle W(C)\rangle \approx 1 + \frac{1}{2N}\mathrm{tr}(T^aT^b)\bar{\mathcal{P}}\Big\langle \oiint_C \mathrm{dt}_1\mathrm{dt}_2\Big(&|\dot{x}_1||\dot{x}_2|\overbracket{\Phi^a_{r\,1}\Phi^b_{s\,2}}n^r n^s \\ &-\dot{x}_1^\mu\dot{x}_2^\nu\overbracket{A^a_{\mu\,1}A^b_{\nu\,2}}\Big)\Big\rangle + \dots\,.\end{aligned} \tag{2.39}$$

The minus sign comes from the imaginary unit in front of the gauge field. The color charges were made explicit. We thus see that we will have a gauge propagator and a scalar propagator connecting two points on the Wilson loop. This can be pictured by the diagram in Figure 2.1. For a circular loop with radius $R \in \mathbb{R}^+$ we can parameterize the

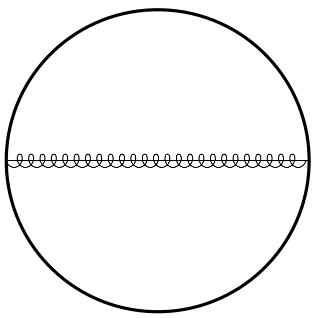


Figure 2.1: Leading order Feynman diagram of the circular Wilson loop. The bold circle is said Wilson loop. Inside it we see a propagator with the mix of curly and straight line representing the vector and scalar propagator, respectively.

[4] In fact also the color trace will be $\mathrm{Tr}T^a = 0$.

contour by (1.64)

$$C: \; x^\mu(\mathfrak{t}) = (0\,, \, R\cos(\mathfrak{t})\,, \, R\sin(\mathfrak{t})\,, \, 0) \quad , \quad \mathfrak{t} \in [0, 2\pi]\,.$$

We want to plug these boundaries into the integration over the contour. The two choices of $\mathfrak{t}$-ordering will give identical contributions. We can therefore include a factor of 2 and assume $\mathfrak{t}_1 > \mathfrak{t}_2$ yielding

$$\begin{aligned}
\langle W_{\bigcirc} \rangle &= 1 + \frac{1}{N} \mathrm{tr} \left\langle T^a T^b \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \left(|\dot{x}_1| |\dot{x}_2| \overbracket{\Phi^a_{t\,1} \Phi^b_{s\,2}} n^r n^s - \dot{x}_1^\mu \dot{x}_2^\nu \overbracket{A^a_{\mu\,1} A^b_{\nu\,2}} + \right) \right\rangle + \ldots \\
&= 1 + \frac{g^2}{N} \mathrm{tr}(T^a T^b \delta^{ab}) \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \\
&\qquad \left(|\dot{x}_1| |\dot{x}_2| \delta_{rs} n^r n^s - \eta_{\mu\nu} \dot{x}_1^\mu \dot{x}_2^\nu \right) \Delta(x_1 - x_2) + \ldots\,.
\end{aligned} \tag{2.40}$$

In the second step we plugged in the propagators (2.36)

$$\langle A^a_\mu(x_1) A^b_\nu(x_2) \rangle = g^2 \delta^{ab} \eta_{\mu\nu} \Delta(x_1 - x_2)\,, \quad \langle \Phi^a_r(x_1) \Phi^b_s(x_2) \rangle = g^2 \delta^{ab} \delta_{rs} \Delta(x_1 - x_2)\,, \tag{2.41}$$

$$\Delta(x_1 - x_2) = \frac{1}{4\pi^2 (x_1 - x_2)^2}\,.$$

Before we consider the kinematics let us focus on the other contractions. The color trace will have two identical generators yielding a quadratic contribution from the number of charges N. The S^5 unit vectors n^r will simply give one while the loop parameterizations $\dot{x}$ will be contracted

$$\mathrm{tr}(T^a T^a) = \frac{N}{2} \mathrm{tr}(\mathbb{1}) = \frac{N^2}{2}\,, \quad \delta_{rs} n^r n^s = n^2 = 1\,, \quad \eta_{\mu\nu} \dot{x}_1^\mu \dot{x}_2^\nu = \dot{x}_1 \cdot \dot{x}_2\,. \tag{2.42}$$

Note that the N^2 will combine with the prefactor ${}^{g^2}\!/_{N}$ to the 't Hooft coupling $\lambda = g^2 N$. Therefore, we indeed find that this diagram is the leading order in perturbation theory. Diagrams involvoing more propagators or vertices or loop corrections are of higher order in λ [62]. We thus find

$$\langle W_{\bigcirc} \rangle = 1 + \frac{\lambda}{2} \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \left(|\dot{x}_1| |\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2 \right) \Delta(x_1 - x_2) + \mathcal{O}(\lambda^2) \tag{2.43}$$

$$= 1 + \frac{\lambda}{2} \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \frac{|\dot{x}_1| |\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2}{4\pi^2 (x_1 - x_2)^2} + \mathcal{O}(\lambda^2)\,. \tag{2.44}$$

Let us now focus on the kinematics. Recall the parameterization (1.64) of the contour. This implies some simplifications in the above double integral. We have

$$(x(\mathfrak{t}_1) - x(\mathfrak{t}_2))^2 = 2R^2 (1 - \cos(\mathfrak{t}_1 - \mathfrak{t}_2)) = 4R^2 \sin^2 \left(\frac{\mathfrak{t}_1 - \mathfrak{t}_2}{2} \right)$$

$$\dot{x}^{\mu}(\mathfrak{t}) = (0\,,\, -R\sin(\mathfrak{t})\,,\, R\cos(\mathfrak{t})\,,\, 0) \tag{2.45}$$

$$\Rightarrow \quad |\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2 = R^2(1 - \cos(\mathfrak{t}_1 - \mathfrak{t}_2)) = 2R^2 \sin^2\left(\frac{\mathfrak{t}_1 - \mathfrak{t}_2}{2}\right)$$

$$\Rightarrow \quad \frac{|\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2}{(x_1 - x_2)^2} = \frac{1}{2}\,. \tag{2.46}$$

Plugging this in the integral over the contour becomes indeed trivial and we find

$$\langle W_{\bigcirc}\rangle = 1 + \frac{\lambda}{16\pi^2}\int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 + \mathcal{O}(\lambda^2) = 1 + \frac{\lambda}{8} + \mathcal{O}(\lambda^2)\,. \tag{2.47}$$

In [62] also the diagrams at the next order where calculated. For insertions of several bare propagators the integral at all loops remains trivial and can be summed consistently, see the discussion around Figure 1.2. Generally further diagrams can contribute at higher order. These are self-energy diagrams and contributions with internal vertices. It was shown in [62] that these diagrams cancel each other at leading order. The conjecture that this cancellation is found at all orders allowed a summation of only rainbow diagrams and the all-loop result

$$\langle W_{\bigcirc}\rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})\,. \tag{2.48}$$

This non-perturbative result agrees with the large coupling computation [25, 91]. The conjecture was later proven by [63,64]. Following [35] the bremsstrahlung can be computed from this expectation value of the circular Wilson loop.

Note that the calculation for the straight line Wilson loop at leading order is almost identical. Independently from the parameterization all Feynman diagrams have a canceling prefactor[5] and thus

$$\boxed{\langle W_{—}\rangle = 1\,.} \tag{2.49}$$

This result can also be proven non-perturbatively [63].

2.2.2 Bremsstrahlung

The bremsstrahlung of a heavy particle moving in $\mathcal{N} = 4$ SYM can be obtained from the expectation value of the circular Wilson loop [35]. In this section we review the general derivation. In a later section we will discuss corrections yielded by thermal effects. The derivation relies on two different calculations of scalar two-point functions and a relation between circular Wilson loops that conserve different amounts of supersymmetry. Consider a circular loop where the vector $\vec{n} \in S^5$ that chooses the contributing scalars depends on

[5] This cancelation is discussed in more detail in Section 5.5.

an angle θ and the parameterization of the loop. Let

$$\vec{n} = (n_1, n_2, n_3, \vec{0}) \quad \text{with } n_1 = \sin(\theta)\cos(\mathfrak{t})\,, \quad n_2 = \sin(\theta)\sin(\mathfrak{t})\,, \quad n_3 = \cos(\theta)\,, \tag{2.50}$$

where θ is some angle and $\mathfrak{t}$ parameterises the Wilson loop (1.64)

$$C: \; x^\mu(\mathfrak{t}) = (0\,, R\cos(\mathfrak{t})\,, R\sin(\mathfrak{t})\,, 0) \quad , \quad \mathfrak{t} \in [0, 2\pi]\,. \tag{2.51}$$

The Wilson loop as defined above is 1/4–BPS [35,92]. For $\theta = 0$ the vector $\vec{n}$ is independent of the parameterization and therefore this is the known 1/2-BPS circular loop computed in [62–64]. Let us compute the 1/4–BPS circular loop to leading order. Recall its definition

$$W_\theta = \frac{1}{N}\mathrm{tr}\bar{\mathcal{P}}\exp\left[\oint_C \mathrm{d}\mathfrak{t}\,(\mathrm{i}A_\mu(\mathfrak{t})\dot{x}^\mu(\mathfrak{t}) + |\dot{x}(\mathfrak{t})|\Phi_r(\mathfrak{t})n^r(\mathfrak{t},\theta))\right]\,, \tag{2.52}$$

As for the other calculation we find

$$\begin{aligned}\langle W_\theta\rangle = 1 + \frac{\mathrm{tr}(T^aT^b)}{2N}\bar{\mathcal{P}}\oiint_C \mathrm{d}\mathfrak{t}_1\mathrm{d}\mathfrak{t}_2\Big(&|\dot{x}_1||\dot{x}_2|\Delta^{ab}_{rs}(x_1-x_2)n^r(\theta,\mathfrak{t}_1)n^s(\theta,\mathfrak{t}_2)\\ &-\dot{x}_1^\mu\dot{x}_2^\nu\Delta^{ab}_{\mu\nu}(x_1-x_2)+\Big)+\mathcal{O}(\lambda^2)\,.\end{aligned} \tag{2.53}$$

We can insert the bare propagators

$$\Delta^{ab}_{\mu\nu}(x_1-x_2) = \frac{g^2\delta^{ab}\eta_{\mu\nu}}{4\pi^2(x_1-x_2)^2}\,, \quad \Delta^{ab}_{rs}(x_1-x_2) = \frac{g^2\delta^{ab}\delta_{rs}}{4\pi^2(x_1-x_2)^2}\,, \tag{2.54}$$

and thus find [92]

$$\begin{aligned}\langle W_\theta\rangle =& 1 + \frac{\lambda}{2}\int_0^{2\pi}\mathrm{d}\mathfrak{t}_1\int_0^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2\frac{R^2\sin^2(\theta)\cos(\mathfrak{t}_1-\mathfrak{t}_2)+R^2\cos^2(\theta)-R^2\cos(\mathfrak{t}_1-\mathfrak{t}_2)}{4\pi^2(2R^2-2R^2\cos(\mathfrak{t}_1-\mathfrak{t}_2))}+\mathcal{O}(\lambda^2)\\ =& 1 + \frac{\lambda}{2}\int_0^{2\pi}\mathrm{d}\mathfrak{t}_1\int_0^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2\frac{\cos^2(\theta)\,[1-\cos(\mathfrak{t}_1-\mathfrak{t}_2)]}{8\pi^2(1-\cos(\mathfrak{t}_1-\mathfrak{t}_2))}+\mathcal{O}(\lambda^2)\\ =& 1 + \frac{\lambda}{8}\cos^2(\theta)+\mathcal{O}(\lambda^2)\,.\end{aligned} \tag{2.55}$$

The loop-to-loop propagator combines to the same prefactor in the numerator for all rainbow diagrams and yields only $\mathfrak{t}$–independent integrals. This is analogous to the case studied in [62]. It can further be argued [92] that the interacting graphs cancel each other at all orders[6] and for all values of θ. In order to do this one needs to consider the contributions from Φ_1 and Φ_2 separately from the one coming from Φ_3. The former ones will have a sine as prefactors while the latter yield a cosine. Then the two can be regarded

[6]This is where the finite temperatrure calculation will differ. The self-energy contributions are not the same and the cancellation does not take place.

as the $\theta = \pi/2$ and $\theta = 0$ case, respectively. For $\theta = \pi/2$ the corresponding Wilson loop is trivial for reasons similar to the straight line while the case $\theta = 0$ is known from [62]. Therefore one finds that the θ-dependent loop can be obtained from the $1/2$-BPS circular loop by a simple redefinition of the coupling [92]

$$\langle W_\theta \rangle = \langle W_{\theta=0} \rangle \left|_{\lambda \to \lambda \cos^2(\theta)} \right. . \tag{2.56}$$

The expansion of this result in terms of small angles yields [35]

$$\frac{\langle W_\theta \rangle - \langle W_{\theta=0} \rangle}{\langle W_{\theta=0} \rangle} = -\theta^2 \lambda \partial_\lambda \log \langle W_{\theta=0} \rangle . \tag{2.57}$$

Let us now look at the left hand side by using the original definition of the WL operator. The dependence on the angle θ sits in the unit vector $\vec{n}$ of the first Wilson loop. Expanding around small angles will therefore yield factors of scalar fields coming from the exponential. At leading order in the 't Hooft coupling they sit in one- and two-point functions contracted with the n_r.

$$\begin{aligned}
&\frac{\langle W_\theta \rangle - \langle W_{\theta=0} \rangle}{\langle W_{\theta=0} \rangle} \\
&= \theta \oint_C \mathrm{d}\mathfrak{t} |\dot{x}| (\partial_\theta n^r(\mathfrak{t}, \theta)) \operatorname{tr} \langle \Phi_r(\mathfrak{t}) \rangle_W + \frac{\theta^2}{2} \oint_C \mathrm{d}\mathfrak{t} |\dot{x}| (\partial_\theta^2 n^r(\mathfrak{t}, \theta)) \operatorname{tr} \langle \Phi_r(\mathfrak{t}) \rangle_W \\
&+ \frac{\theta^2}{2} \oint_C \mathrm{d}\mathfrak{t}_1 \oint_C \mathrm{d}\mathfrak{t}_2 |\dot{x}_1| |\dot{x}_2| (\partial_\theta n^r(\mathfrak{t}_1, \theta)) (\partial_\theta n^s(\mathfrak{t}_2, \theta)) \operatorname{tr} \langle \Phi_r(\mathfrak{t}_1) \Phi_s(\mathfrak{t}_2) \rangle_W + \mathcal{O}(\theta^3) .
\end{aligned} \tag{2.58}$$

In the first line we have one-point functions along the defect. The conformal invariance is preserved and thus they are zero. For the term in the second line the two-point function will contract the n_r. As the dependence on the angle in the prefactor is already quadratic not all components of the unit vector remain relevant.

$$\begin{aligned}
\partial_\theta \vec{n} &= (\cos(\theta) \cos(\mathfrak{t}), \cos(\theta) \sin(\mathfrak{t}), -\sin(\theta), 0, 0, 0) \\
&= (\cos(\mathfrak{t}), \sin(\mathfrak{t}), 0, 0, 0, 0) + \mathcal{O}(\theta) .
\end{aligned} \tag{2.59}$$

Hence only the fields Φ_1 and Φ_2 contribute above. The contour yields $|\dot{x}_1||\dot{x}_2| = R^2$. The remaining correlator is different from the bare propagator introduced above because it is defined in the presence of the Wilson loop while the propagator is defined in the $\mathcal{N} = 4$ vacuum. For the scalar two-point function along the defect we have

$$\operatorname{tr} \langle \Phi_r(\mathfrak{t}_1) \Phi_s(\mathfrak{t}_2) \rangle_W = \operatorname{tr} \frac{\langle \Phi_r(\mathfrak{t}_1) \Phi_s(\mathfrak{t}_2) W_{\theta=0} \rangle}{\langle W_{\theta=0} \rangle} \tag{2.60}$$

$$= \frac{\operatorname{tr}\left\langle \Phi_r(\mathfrak{t}_1) e^{\int_{\mathfrak{t}_2}^{\mathfrak{t}_1} \mathrm{d}t (|\dot{x}|\vec{n}\vec{\Phi} + \mathrm{i}\dot{x}\cdot A)} \Phi_s(\mathfrak{t}_2) e^{\int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \mathrm{d}t (|\dot{x}|\vec{n}\vec{\Phi} + \mathrm{i}\dot{x}\cdot A)} \right\rangle}{\operatorname{tr}\left\langle e^{\oint \mathrm{d}t (|\dot{x}|\vec{n}\vec{\Phi} + \mathrm{i}\dot{x}\cdot A)} \right\rangle} . \tag{2.61}$$

The closed integral of the Wilson loop operator in the numerator was split into two parts connecting the insertion points of the fields. This is a correlation function on the defect where conformal symmetry is preserved. Therefore, for some constant γ it has the general form

$$\operatorname{tr}\langle \Phi_r(\mathfrak{t}_1)\Phi_s(\mathfrak{t}_2)\rangle_W = \frac{\gamma\delta_{rs}}{(x_1 - x_2)^2} = \frac{\gamma\delta_{rs}}{2R^2(1 - \cos(\mathfrak{t}_1 - \mathfrak{t}_2))} . \tag{2.62}$$

The path ordering $\bar{\mathcal{P}}$ in the above integral yields a factor of 2 which cancels the $1/2$ prefactor from the expansion. The contour and the unit vectors yield further prefactors such that finally

$$\begin{aligned}
\frac{\langle W_\theta\rangle - \langle W_{\theta=0}\rangle}{\langle W_{\theta=0}\rangle} &= \frac{\theta^2}{2}\gamma \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{2\pi} \mathrm{d}\mathfrak{t}_2 \frac{R^2\cos(\mathfrak{t}_1 - \mathfrak{t}_2)}{2R^2(1 - \cos(\mathfrak{t}_1 - \mathfrak{t}_2))} + \mathcal{O}(\theta^3) \\
&= \theta^2\gamma\frac{1}{4}\int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \left[\int_0^{\mathfrak{t}_1-\varepsilon} \mathrm{d}\mathfrak{t}_2 + \int_{\mathfrak{t}_1+\varepsilon}^{2\pi} \mathrm{d}\mathfrak{t}_2\right] \frac{\cos(\mathfrak{t}_1 - \mathfrak{t}_2)}{1 - \cos(\mathfrak{t}_1 - \mathfrak{t}_2)} + \mathcal{O}(\theta^3) \\
&= \theta^2\left(\frac{2\gamma\pi}{\varepsilon} - \pi^2\gamma\right) + \mathcal{O}(\theta^3) = -\pi^2\theta^2\gamma + \mathcal{O}(\theta^3) ,
\end{aligned} \tag{2.63}$$

in agreement with [35]. The $1/\varepsilon$ divergence can be cured by a proper renormalization and was thus removed. Above we showed that the left hand side of the last calculation is equal to the derivative of the logarithm of the circular Wilson loop. Simplifying with the result we find an expression for the scalar two-point coefficient [35]

$$\gamma = \frac{\lambda}{\pi^2}\partial_\lambda \log\langle W_\bigcirc\rangle . \tag{2.64}$$

To relate this to the bremsstrahlung we switch the setting and take a look at the anomalous dimension of a cusped WL. Therefore, consider a straight Wilson line with a cusp introduced by giving the heavy particle a push such that it's contour is altered by an angle φ. Due to the cusp the line is not equal to one but gets a logarithmic divergence

$$\langle W\rangle \sim e^{-\Gamma_{\text{cusp}}(\phi)\log(\Lambda_{IR}/\Lambda_{UV})} . \tag{2.65}$$

This divergence depends on an infrared and an ultraviolet cut-off and the prefactor $\Gamma_{\text{cusp}}(\varphi)$ is called the anomalous dimension of the cusp. Note that in $\mathcal{N} = 4$ we consider the supersymmetric Maldacena-Wilson loop (1.63) which also includes the six adjoint scalar fields. They contribute by a factor $\vec{n}\vec{\Phi}$ where $\vec{n}$ is a unit vector on the S^5. A sudden change in this vector (happening precisely at the cusp) leads to another angle ϑ with $\cos\vartheta = \vec{n}\vec{n}'$. We thus consider $\Gamma_{\text{cusp}}(\varphi,\vartheta)$. Both angles contribute to the same anomalous

dimension because they are introduced at the same point of the contour.
If both these angles vanish we know that the cusp anomaly is zero thus we can expand around small angles

$$\Gamma_{\text{cusp}}(\varphi,\vartheta) = \left(\vartheta^2 - \varphi^2\right) B \qquad \text{for} \quad \varphi,\vartheta << 1\,, \tag{2.66}$$

where B is defined as the bremsstrahlung function [35].
Consider a map from $\mathbb{R}^4$ to $\mathbb{R}\times S^3$ where the real line is precisely where the straight Wilson line sits. A cusped Wilson line thus is represented on the cylinder by two Wilson lines inserted at two different points of the cylinder with an angle $\pi - \varphi$ as figure (2.2) shows. Setting the angle $\varphi = 0$ for the moment we want to calculate the second derivative of the

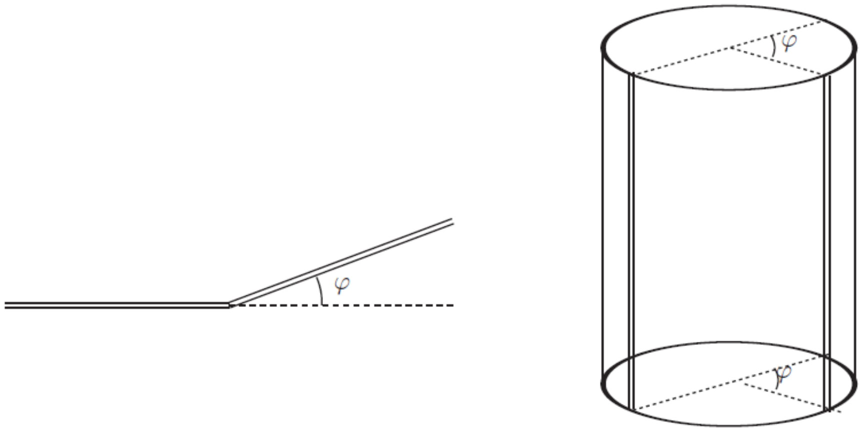


Figure 2.2: Map from $\mathbb{R}^4$ to the cylinder $\mathbb{R}\times S^3$. The map goes from polar coordinates to the cylinder yielding that the side of the cylinder is the radius of the polar coordinates. The line can be cut at the cusp which yields an angle $\pi - \varphi$ on the sphere as shown in the picture [93]. For $\varphi = 0$ the Wilson lines sit at opposite points of the cylinder, for instance the north and south pole and we rediscover the 1/2-BPS configuration without the cusp. On the cylinder the setup is essentially that of a quark anti-quark configuration and the Wilson loop can be calculated for these quarks sitting at a relative angle $\pi - \varphi$ on two points of the S^3 [35, 94, 95].

cusp anomalous dimension. This can be viewed as the energy exchange between to static quarks sitting at opposite points on the S^3 [35, 94, 95]. At leading order the cusp anomaly is thus given by the exchange of two scalars

$$\Gamma_{\text{cusp}} = \frac{\vartheta^2}{2}\int_{-\infty}^{\infty} \mathrm{d}\mathfrak{t}_1 \int_{-\infty}^{\infty} \mathrm{d}\mathfrak{t}_2 |\dot{x}(\mathfrak{t}_1)||\dot{x}(\mathfrak{t}_2)| \mathrm{tr}\langle\Phi(\mathfrak{t}_1)\Phi(\mathfrak{t}_2)\rangle_W\,. \tag{2.67}$$

Again we find the same correlator as above

$$\mathrm{tr}\left\langle\Phi(x_1)\Phi(x_2)\right\rangle_W = \frac{\gamma}{(x(\mathfrak{t}_1) - x(\mathfrak{t}_2))^2}\,. \tag{2.68}$$

The straight line can be parameterized by a hyperbolic tangent function (1.65) which is identical to [35].

$$C: \; x^\mu(\mathfrak{t}) = \left(0,\, 0,\, 0,\, \tilde{R}\tanh\left(\frac{\mathfrak{t}}{2}\right)\right) \quad , \quad \mathfrak{t} \in [-\infty, \infty] \, . \tag{2.69}$$

While other, seemingly simpler, parameterizations would be possible they might change the renormalization prescription used and for later convenience we will stick to the scheme used in [35]. The prefactor in the above integral and the two point function can simplified to

$$|\dot{x}(\mathfrak{t})| = \frac{\tilde{R}}{2}\mathrm{sech}^2\left(\frac{\mathfrak{t}}{2}\right) \, , \quad \frac{\frac{\tilde{R}}{2}\mathrm{sech}^2\left(\frac{\mathfrak{t}_1}{2}\right)\frac{\tilde{R}}{2}\mathrm{sech}^2\left(\frac{\mathfrak{t}_2}{2}\right)}{\left[\tilde{R}\tanh\left(\frac{\mathfrak{t}_1}{2}\right) - \tilde{R}\tanh\left(\frac{\mathfrak{t}_2}{2}\right)\right]^2} = \frac{1}{2\cosh\left(\mathfrak{t}_1 - \mathfrak{t}_2\right) - 2} \, . \tag{2.70}$$

Plugging this into the above integral we find

$$\begin{aligned} \Gamma_{\text{cusp}} &= \frac{\vartheta^2}{4}\gamma \int_{-\infty}^{\infty} \mathrm{d}\mathfrak{t}_1 \left[\int_{-\infty}^{\mathfrak{t}_1-\varepsilon} \mathrm{d}\mathfrak{t}_2 + \int_{\mathfrak{t}_1+\varepsilon}^{\infty} \mathrm{d}\mathfrak{t}_2\right] \frac{1}{\cosh\left(\mathfrak{t}_1 - \mathfrak{t}_2\right) - 1} \\ &= \vartheta^2\gamma \int_{-\infty}^{\infty} \mathrm{d}\mathfrak{t}_1 \left(\frac{1}{\varepsilon} - \frac{1}{2}\right) + -\vartheta^2\left(\frac{\gamma}{\varepsilon} - \frac{\gamma}{2}\right) = \vartheta^2\frac{\gamma}{2} \, . \end{aligned} \tag{2.71}$$

Again the cut-off could be dropped due to renormalization. Regarding the second integral an overall factor "length of time" was extracted. Due to the relation between partition function and energy, $Z = e^{-\Gamma_{\text{cusp}}(\text{Length of Time})}$ an additional minus sign arose [35]. Recalling that for $\varphi = 0$ we had $\Gamma_{\text{cusp}}(\vartheta) = \vartheta^2 B$ lets us conclude

$$\gamma = 2B \, , \tag{2.72}$$

and therefore we finally find a relation between the bremsstrahung and the expectation value of the circular loop [35]

$$\boxed{B = \frac{\lambda}{2\pi^2}\partial_\lambda \log\langle W_{\bigcirc}\rangle \, .} \tag{2.73}$$

Recall that at zero temperature the straight line is exactly one

$$\langle W_{\text{line}}\rangle = 1 \, . \tag{2.74}$$

The transformation of the straight line to a circle turns out to be anomalous and thus the computation of the circular Wilson loop is non-trivial. We are only interested in the leading orders at small coupling. Following [62–64] it is given by the modified Laguerre Polynomial L

$$\langle W_{\text{circle}}\rangle = \frac{1}{N}L^1_{N-1}\left(-\frac{\lambda}{4N}\right)e^{\lambda/8N} = 1 + \frac{\lambda}{8} + \mathcal{O}(\lambda^2) \, . \tag{2.75}$$

This yields an expression for the bremsstrahlung which mainly depends on two Bessel functions [35]. We expand the results in weak coupling to leading order as this is the regime in which we shall later consider to test at finite temperature

$$\boxed{B = \frac{1}{4\pi}\frac{\lambda I_2\left(\sqrt{\lambda}\right)}{I_1\left(\sqrt{\lambda}\right)} = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2)\ .} \tag{2.76}$$

We want to compare this with the 'energy at infinity' h, namely the coefficient of the stress tensor one-point function. We will do this for the straight line as here the calculations can be carried out much more easily. For later convenience we also calculate the stress tensor coefficient for the circular loop. Following [96] both coefficients are the same in $\mathcal{N} = 4$. Generally the stress tensor one-point function in the directions of the defect is given by [37, 96]

$$\langle T_{33}\rangle_{W_{\text{line}}} = \frac{h}{r^4}\ , \tag{2.77}$$

where r is the radial distance to the defect and R is the radius of the circular loop.
The relation we want to proof is (1.1) as derived in [35, 36]

$$\boxed{B = 3h\ .}$$

2.2.3 Stress tensor one-point function

In the previous subsection we computed the bremsstrahlung function B to leading order in λ. We now want to confirm that it is indeed three times the coefficient of the stress tensor one-point function. We will show this for the straight line as well as for the circular loop. The $\mathcal{N} = 4$ stress energy tensor is given by [36, 81, 82, 97, 98], recall Equation (1.62)

$$T_{\mu\nu} = \frac{1}{g^2}\left(T^V_{\mu\nu}(A) + T^F_{\mu\nu}(\lambda) + T^S_{\mu\nu}(\Phi) + T^{\text{interactions}}_{\mu\nu}(A, \lambda, \Phi)\right)\ , \tag{2.78}$$

$$T^V_{\mu\nu}(A) = \operatorname{tr}\left(-F_{\mu\rho}F_{\nu\sigma}\eta^{\rho\sigma} + \frac{1}{4}\eta_{\mu\nu}(F_{\rho\sigma})^2\right)\ ,$$

$$T^F_{\mu\nu}(\lambda) = -\frac{1}{4}\operatorname{tr}\left(\bar{\lambda}_I\gamma_\mu\partial_\nu\lambda^I - \bar{\lambda}_I\gamma_\nu\partial_\mu\lambda^I\right)\ ,$$

$$T^S_{\mu\nu}(\Phi) = \operatorname{tr}\left(-\partial_\mu\Phi^{IJ}\partial_\nu\bar{\Phi}_{IJ} + \frac{1}{2}\eta_{\mu\nu}\partial_\rho\Phi^{IJ}\partial^\rho\bar{\Phi}_{IJ} + \frac{1}{6}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right)\Phi^{IJ}\bar{\Phi}_{IJ}\right)\ .$$

The trace is taken with respect to color indices. We only wrote the kinetic terms explicitly as the interaction terms are not relevant at the leading order which we consider. The conventions of the stress tensor are the same as in [36].
Let us explicitly derive the contribution of free scalar and gauge fields. We ignore color and R-symmetry indices and also ignore the contribution which ensures tracelessness.

Therefore consider

$$\mathcal{L}_S = \frac{1}{2g^2}\eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi\ , \quad \mathcal{L}_V = \frac{1}{4g^2}\eta_{\mu\nu}\eta_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma}\ . \tag{2.79}$$

We define the stress-energy tensor as a differential of the Lagrangian with respect to the metric

$$T^{\mu\nu} := -2\frac{\partial\mathcal{L}}{\partial\eta_{\mu\nu}} + \eta^{\mu\nu}\mathcal{L}\ . \tag{2.80}$$

Then we find for the scalar by a straight forward calculation

$$T_S^{\mu\nu} = -\frac{1}{g^2}\frac{\partial}{\partial\eta_{\mu\nu}}\left(\eta_{\rho\sigma}\partial^\rho\phi\partial^\sigma\phi\right) + \frac{1}{2g^2}\eta^{\mu\nu}(\partial\phi)^2 = \frac{1}{g^2}\left[-\partial^\mu\phi\partial^\nu\phi + \frac{1}{2g^2}\eta^{\mu\nu}(\partial\phi)^2\right]\ , \tag{2.81}$$

and similarly for the vector

$$T_V^{\mu\nu} = -\frac{1}{2g^2}\frac{\partial}{\partial\eta_{\mu\nu}}\left(\eta_{\rho\sigma}\eta_{\rho'\sigma'}F^{\rho\rho'}F^{\sigma\sigma'}\right) + \frac{1}{4g^2}\eta^{\mu\nu}F^2 = \frac{1}{g^2}\left[-\eta_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}\eta^{\mu\nu}(\partial\phi)^2\right]\ . \tag{2.82}$$

This indeed reproduces the above definition. Note that the stress tensor given by [81] has a minus sign in front of the scalar stress tensor, but not for the gauge stress tensor.

Following [37] we want to compute the two-point function between the stress tensor and the straight Wilson line. This is defined by an expectation value which gets normalized by the WL operator.

$$\langle T_{\mu\nu}\rangle_{W_{—}} = \frac{\langle T_{\mu\nu}W_{—}\rangle}{\langle W_{—}\rangle} \quad \text{where } \langle W_{—}\rangle = 1\ . \tag{2.83}$$

As the expectation value of the straight line is equal to one we do not need to worry about the denominator at all. If we were to consider the circular loop instead we would have to account for loop corrections in the denominator. However at leading order in the 't Hooft coupling the result would still not change. The corrections from the denominator only contribute at the next to leading order corrections.

2.2.3.1 Leading order contribution

The exponential in the definition of the WL is expanded as before and at leading order we just get a simple insertion of the free part of the stress tensor. The free part is then contracted with scalar or gauge fields, respectively. Other diagrams and their order are discussed below.

We can therefore simply insert the free stress tensor into the expansion Equation (2.39) as given above:

$$\langle T_{\mu\nu}\rangle_{W_{—}} = \frac{1}{2N}\bar{\mathcal{P}}\mathrm{tr}\left\langle T^a T^b \oiint_{C_{—}} \mathrm{d}\mathfrak{t}_1 \mathrm{d}\mathfrak{t}_2 \left(|\dot{x}_1||\dot{x}_2|\Phi^a_{r\,1}\Phi^b_{s\,2} n^r n^s T_{\mu\nu} - \dot{x}_1^\mu \dot{x}_2^\nu A^a_{\mu\,1} A^b_{\nu\,2} T_{\mu\nu}\right)\right\rangle + \mathcal{O}(\lambda^2)\,. \tag{2.84}$$

Let us plug in the parameterization of the straight line above. Analogously to before the choice of y-ordering $\bar{\mathcal{P}}$ cancels the factor $1/2$ in front. Thus

$$\langle T_{\mu\nu}\rangle_{W_{—}} = \frac{1}{N}\mathrm{tr}\left\langle T^a T^b \int_{-\infty}^{\infty} \mathrm{d}\mathfrak{t}_1 \int_{-\infty}^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \left(\Phi^a_I(x_1)\Phi^b_J(x_2) n^I n^J T_{\mu\nu} - A^a_3(x_1) A^b_3(x_2) T_{\mu\nu}\right)\right\rangle + \mathcal{O}(\lambda^2)\,. \tag{2.85}$$

We now want to Wick contract the fields. Let us start with the vector fields. We could in principle contract $\overbracket{A^a_3(x_1)A}^b_3(x_2)$ and the fields in the stress tensor with themselves. However, this would yield a splitting of the correlators

$$\langle T_{\mu\nu} W_{—}\rangle \to \langle T_{\mu\nu}\rangle\langle W_{—}\rangle\,. \tag{2.86}$$

At zero temperature, however, the one-point function $\langle T_{\mu\nu}\rangle \equiv 0$ by conformal invariance. Therefore, such contractions can be ignored. Each gauge field from the loop needs to be contracted with one of the gauge field from the free stress tensor. Analogous for the scalars. This yields the diagram shown in Figure (2.3)

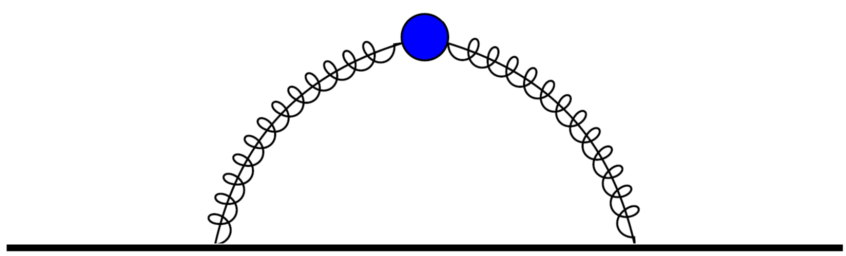


Figure 2.3: The diagram shows the one-loop Feynman diagram for the expectation value of the stress tensor in presence of the Wilson loop. The former is represented by the blue circle while the former is given through the thick line. The interacting particles are scalars or gluons which, as before, we depict by a single propagator.

2.2.3.1.1 Contractions For the contractions first focus on the vector fields. The free part of the stress tensor is given by

$$\begin{aligned} T^V_{\mu\nu} &= -F^c_{\mu\rho}F^c_{\nu\sigma}\eta^{\rho\sigma} + \frac{1}{4}\eta_{\mu\nu}(F^c_{\rho\sigma})^2 \\ &= -\partial_\mu A^c_\rho \partial_\nu A^{\rho\, c} - \partial_\rho A^c_\mu \partial^\rho A^c_\nu + \partial_\mu A^c_\rho \partial^\rho A^c_\nu + \partial_\rho A^c_\mu \partial_\nu A^{\rho\, c} \\ &\quad + \frac{\eta_{\mu\nu}}{2}\partial_{\rho_1} A^c_{\rho_2}\partial^{\rho_1} A^{\rho_2\, c} - \frac{\eta_{\mu\nu}}{2}\partial_{\rho_1} A^c_{\rho_2}\partial^{\rho_2} A^{\rho_1\, c} \,. \end{aligned} \tag{2.87}$$

Each of the terms in the above equation contains two gauge fields. Each of these can be contracted with each of the fields from the Wilson loop expansion. This yields two different possibilities for contractions and therefore twelve terms in total.

$$\begin{aligned} &: T^V_{\mu\nu} A^a_3(x_1) A^b_3(x_2) : \\ &= -(\partial_\mu A^c \rho)(\partial_\nu A^{\rho\, c}) A^a_3(x_1) A^b_3(x_2) - (\partial_\mu A^c_\rho)(\partial_\nu A^{\rho\, c}) A^a_3(x_1) A^b_3(x_2) \\ &\quad - (\partial_\rho A^c_\mu)(\partial^\rho A^c_\nu) A^a_3(x_1) A^b_3(x_2) - (\partial_\rho A^c_\mu)(\partial^\rho A^c_\nu) A^a_3(x_1) A^b_3(x_2) \\ &\quad + (\partial_\mu A^c_\rho)(\partial^\rho A^c_\nu) A^a_3(x_1) A^b_3(x_2) + (\partial_\mu A^c_\rho)(\partial^\rho A^c_\nu) A^a_3(x_1) A^b_3(x_2) \\ &\quad + (\partial_\rho A^c_\mu)(\partial_\nu A^{\rho\, c}) A^a_3(x_1) A^b_3(x_2) + (\partial_\rho A^c_\mu)(\partial_\nu A^{\rho\, c}) A^a_3(x_1) A^b_3(x_2) \\ &\quad + \frac{\eta_{\mu\nu}}{2}(\partial_{\rho_1} A^c_{\rho_2})(\partial^{\rho_1} A^{\rho_2\, c}) A^a_3(x_1) A^b_3(x_2) + \frac{\eta_{\mu\nu}}{2}(\partial_{\rho_1} A^c_{\rho_2})(\partial^{\rho_1} A^{\rho_2\, c}) A^a_3(x_1) A^b_3(x_2) \\ &\quad - \frac{\eta_{\mu\nu}}{2}(\partial_{\rho_1} A^c_{\rho_2})(\partial^{\rho_2} A^{\rho_1\, c}) A^a_3(x_1) A^b_3(x_2) - \frac{\eta_{\mu\nu}}{2}(\partial_{\rho_1} A^c_{\rho_2})(\partial^{\rho_2} A^{\rho_1\, c}) A^a_3(x_1) A^b_3(x_2) \,. \end{aligned} \tag{2.88}$$

Note that for each term in $T^V_{\mu\nu}$ there are two possible contractions written next to each other. The structure of all terms above is very similar, the difference is in the space-time indices. Define the coordinate where the stress tensor is inserted as $z \in \mathbb{R}^4$. Then the contractions will yield the following Wick contractions between WL and stress tensor fields

$$A^c_\rho A^a_3(x_i) = g^2 \delta^{ac}\eta_{3\rho}\Delta_i \qquad \text{where } \Delta_i := \Delta(z - x_i) = \frac{1}{4\pi^2(z-x_i)^2} \,. \tag{2.89}$$

From the color contractions will will always find $\delta^{ac}\delta^{bc} = \delta^{ab}$. The index of the metric may be free or contracted. If it is a free index we will keep the metric. For contracted indices we directly apply the metric and set the corresponding index to $\rho = 3$ as the metric demands. We are especially interested in the component T_{33} of the stress tensor. In this case the above simplifies

$$: T^V_{33} A^a_3(x_1) A^b_3(x_2) : = g^4 \delta^{ab}\big[-(\partial_\rho \Delta_1)(\partial^\rho \Delta_2) + (\partial_3 \Delta_1)(\partial_3 \Delta_2)\big] \,. \tag{2.90}$$

Here we simply plugged in $\mu = 3$ and $\nu = 3$ in the first term. Using a mostly plus metric we set $\eta_{33} = 1$ and then collected the terms leaving us with a simplified result.

Before we plug in the propagators and take derivatives we will repeat the same exercise for the scalar fields. Let us first expand the stress tensor

$$\begin{aligned} T^S_{\mu\nu}(\Phi) &= -\partial_\mu \Phi^c_t \partial_\nu \Phi^{t\,c} + \frac{1}{2}\eta_{\mu\nu}(\partial_\rho \Phi^c_t)^2 + \frac{1}{6}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right)(\Phi^c_t)^2 \\ &= -\frac{2}{3}(\partial_\mu \Phi^c_t)(\partial_\nu \Phi^{t\,c}) + \frac{1}{3}\Phi^c_t(\partial_\mu\partial_\nu\Phi^{t\,c}) + \frac{1}{6}\eta_{\mu\nu}(\partial_\rho\Phi^c_t)(\partial^\rho\Phi^{t\,c}) \\ &\quad - \frac{1}{3}\eta_{\mu\nu}\Phi^c_t\left(\partial_\rho\partial^\rho\Phi^{t\,c}\right) . \end{aligned} \tag{2.91}$$

We Wick contract this with the two scalars coming from the Wilson loop $\Phi^a_r(x_1)$ and $\Phi^b_s(x_2)$. The contraction yields a propagator analogous to the vector case

$$\overbrace{\Phi^c_t\Phi^a_r}(x_i) = g^2\delta^{ac}\delta_{rt}\Delta_i \qquad \text{where } \Delta_i := \Delta(z - x_i) = \frac{1}{4\pi^2(z-x_i)^2} . \tag{2.92}$$

We will always find $\delta^{ac}\delta^{bc} = \delta^{ab}$ as in the case of the gauge field. Similarly for the R-symmetry index we find $\delta_{rt}\delta^t_s = \delta_{rs}$. As the propagator is on-shell it is easy to see that $\partial_\rho\partial^\rho\Delta_i = 0$. Furthermore, we focus once again on the component T_{33} of the stress tensor. Compared to the case of the gauge fields the spacetime indices are relatively simple and the resulting structure can be obtained readily.

$$\begin{aligned} :T^S_{33}\Phi^a_r(x_1)\Phi^b_s(x_2): = g^4\delta^{ab}\delta_{rs}\Big[&-\frac{4}{3}(\partial_3\Delta_1)(\partial_3\Delta_2) + \frac{1}{3}\Delta_1(\partial_3\partial_3\Delta_2) + \frac{1}{3}\Delta_2(\partial_3\partial_3\Delta_1) \\ &+\frac{1}{3}(\partial_\rho\Delta_1)(\partial^\rho\Delta_2)\Big] . \end{aligned} \tag{2.93}$$

2.2.3.1.2 Coefficient of the Stress Tensor With this results we can finally go back to the full calculation. Recall the expression we need to integrate. It includes a scalar and a gauge part with WL fields in a correlator with the stress tensor.

$$\begin{aligned} \langle T_{33}\rangle_{W_{—}} = \frac{1}{N}\int_{-\infty}^{\infty} \mathrm{d}\mathfrak{t}_1 \int_{-\infty}^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \mathrm{tr}\Bigg(&T^aT^b\left\langle T^S_{33}(\Phi)\Phi^a_I(x_1)\Phi^b_J(x_2)\right\rangle n^I n^J \\ &- T^aT^b\left\langle T^V_{33}(A)A^a_3(x_1)A^b_3(x_2)\right\rangle\Bigg) + \mathcal{O}(\lambda^2) . \end{aligned} \tag{2.94}$$

We can use the above results to contract the color generators with δ^{ab}. We find $\mathrm{tr}(T^aT^a) = {}^{N^2}\!/_2$ together with the ${}^1\!/_N$ from the Wilson loop, the ${}^1\!/_{g^2}$ from the definition of the stress tensor and the g^4 from the two propagators this yields a prefactor ${}^\lambda\!/_2$. For the gauge fields note that they have an imaginary prefactor coming from the Wilson loop definition and thus the sign of this contribution is negative. For the scalars we additionally find δ_{rs} and

thus can plug in the contraction of the unit vectors $n^2 = 1$.
Let us separate the scalar and the vector contributions. Using the above Equations (2.90) and (2.93) we find the integrals

$$\langle T_{33}\rangle_{W_-} = \langle T_{33}\rangle^S_{W_-} + \langle T_{33}\rangle^V_{W_-} \,, \tag{2.95}$$

$$\langle T_{33}\rangle^V_{W_-} = -\frac{\lambda}{2}\int_{-\infty}^{\infty}\mathrm{d}\mathfrak{t}_1\int_{-\infty}^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2\,\left[-(\partial_\rho\Delta_1)(\partial^\rho\Delta_2)+(\partial_3\Delta_1)(\partial_3\Delta_2)\right]\,, \tag{2.96}$$

$$\begin{aligned}\langle T_{33}\rangle^S_{W_-} = \frac{\lambda}{2}\int_{-\infty}^{\infty}\mathrm{d}\mathfrak{t}_1\int_{-\infty}^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2\,\Big[&-\frac{4}{3}(\partial_3\Delta_1)(\partial_3\Delta_2)+\frac{1}{3}\Delta_1(\partial_3\partial_3\Delta_2)\\ &+\frac{1}{3}\Delta_2(\partial_3\partial_3\Delta_1)+\frac{1}{3}(\partial_\rho\Delta_1)(\partial^\rho\Delta_2)\Big]\,.\end{aligned} \tag{2.97}$$

We want to integrate the different propagators. Recall

$$\Delta_i = \Delta(x_i - z) = \frac{1}{4\pi^2(-z_0^2+z_1^2+z_2^2+(y_i-z_3)^2)}\,. \tag{2.98}$$

Define $r = \sqrt{-z_0^2+z_1^2+z_2^2}$ as the distance between the stress tensor and the Wilson loop. We expect the final result to not depend on the coordinate z_3. This coordinate is along the defect where translational invariance is preserved. We shall keep the coordinate for the calculations and it will eventually cancel out.
Let us look more closely on the different components of these contributions. The different terms we need to integrate are

$$(\partial_3\Delta_1)(\partial_3\Delta_2) = \frac{(y_1-z_3)\,(y_2-z_3)}{4\pi^4\,(r^2+(y_1-z_3)^2)^2\,(r^2+(y_2-z_3)^2)^2}\,, \tag{2.99}$$

$$(\partial_\rho\Delta_1)(\partial^\rho\Delta_2) = \frac{r^2+(y_1-z_3)\,(y_2-z_3)}{4\pi^4\left(r^2+(y_1-z_3)^2\right)^2(r^2+(y_2-z_3)^2)^2}\,, \tag{2.100}$$

$$\partial_3\partial_3\Delta_i = -\frac{-3\,(y_i-z_3)^2+r^2}{2\pi^2\left(r^2+(y_i-z_3)^2\right)^3}\,. \tag{2.101}$$

The above functions now need to be integrated along the Wilson line. We find only one of the integrals to give a nonzero result. The two other integrals vanish due to symmetry properties of the numerator. Recall that we use the parameterization (1.65)

$$\int_{-\infty}^{\infty}\mathrm{d}\mathfrak{t}_1\int_{-\infty}^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2\,(\partial_3\Delta_1)(\partial_3\Delta_2) = 0\,, \tag{2.102}$$

$$\int_{-\infty}^{\infty}\mathrm{d}\mathfrak{t}_1\int_{-\infty}^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2\,\Delta_i(\partial_3\partial_3\Delta_j) = 0\,, \tag{2.103}$$

$$\int_{-\infty}^{\infty}\mathrm{d}\mathfrak{t}_1\int_{-\infty}^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2\,(\partial_\rho\Delta_1)(\partial^\rho\Delta_2) = \frac{1}{32\pi^2r^4}\,. \tag{2.104}$$

Due to the symmetry of the integrals only the term with r^2 in the nominator did not vanish under the integral sign. Note furthermore that as expected the result does not depend on z_3.
We can plug these results into the full calculation.

$$\langle T_{33}\rangle_W^V = \frac{\lambda}{2}\int_{-\infty}^{\infty} \mathrm{d}\mathrm{t}_1 \int_{-\infty}^{\mathrm{t}_1} \mathrm{d}\mathrm{t}_2 (\partial_\rho \Delta_1)(\partial^\rho \Delta_2) = \frac{\lambda}{64\pi^2 r^4}\,, \tag{2.105}$$

$$\langle T_{33}\rangle_W^S = \frac{\lambda}{6}\int_{-\infty}^{\infty} \mathrm{d}\mathrm{t}_1 \int_{-\infty}^{\mathrm{t}_1} \mathrm{d}\mathrm{t}_2 (\partial_\rho \Delta_1)(\partial^\rho \Delta_2) = \frac{\lambda}{192\pi^2 r^4}\,. \tag{2.106}$$

This agrees with the results (Equations (3.13)) from [36] and with (Equations (3.3) and (3.5)) from [81] found for free fields up to a factor of $1/2$. As mentioned before the difference in the sign of the scalar between our result and the one in [81] is due to a difference in the scalar part of the stress tensor, see also the discussion in [36]. The factor $1/2$ comes from the color group ($\mathrm{tr}(T^a T^a) = N^2/2$) and can be seen in Equation (4.1) of [81] for the gauge boson.
We therefore find for the full stress tensor

$$\langle T_{33}\rangle_W = \langle T_{33}\rangle_W^S + \langle T_{33}\rangle_W^V = \frac{\lambda}{48\pi^2 r^4} \qquad \Rightarrow \qquad h = \frac{\lambda}{48\pi^2}\,. \tag{2.107}$$

2.2.3.2 Stress tensor in the presence of a circular loop

For completeness we want to also calculate the stress tensor for the circular loop. Following [96] we expect to find the same expression for h. However, this does not necessarily mean that the expressions coming from the gauge field and the scalar field need to be equal. The reason for the similarity of the two calculations are the symmetries of $\mathcal{N} = 4$ SYM which cannot be expected to hold for free field theories. We parameterize the WL defect by a circle in the $[12]$−plane (1.64)

$$C:\ x^\mu(\mathrm{t}) = (0\,,\, R\cos(\mathrm{t})\,,\, R\sin(\mathrm{t})\,,\, 0) \quad , \quad \mathrm{t} \in [0, 2\pi]\,. \tag{2.108}$$

The bare propagator then is similar to before

$$\Delta(z - x(\mathrm{t}_i)) = \frac{1}{4\pi^2\left(\rho^2 + z_r^2 + R^2 - 2Rz_r\cos(\phi - \mathrm{t}_i)\right)}\,. \tag{2.109}$$

Here $z_r^2 = z_1^2 + z_2^2$ is the radial coordinate in the $[12]$−plane and ϕ is the corresponding angle. Similarly $\rho^2 = z_0^2 + z_3^2$ is the distance between the stress tensor and the plane in which the circular loop lies. Also in this $[03]$−plane an angular dependence could be included. Due to the rotational symmetry in this problem, however, the correlator does not depend on said angle and we can exclude it from our further calculations. The distance between the loop and the stress tensor is given by $r = \sqrt{\rho^2 + (z_r - R)^2}$. The calculation is analogous to the straight line calculation. We are interested in the component $\langle T_{\phi\phi}\rangle_W$ of the stress tensor which can be achieved naively by replacing $3 \to \phi$ in the derivation above. However, the calculation of the vector contribution requires some additional care. Recall for example the contribution in the stress tensor. We suppress color charges and use euclidean signature

$$
\begin{aligned}
:-(\partial_\mu A_\rho)(\partial_\nu A_\rho)A_{\sigma_1}A_{\sigma_2}\dot{x}^{\sigma_1}\dot{x}^{\sigma_2} &:= -\partial_\mu A_\rho \partial_\nu A_\rho A_{\sigma_1} A_{\sigma_2}\dot{x}^{\sigma_1}\dot{x}^{\sigma_2} - \partial_\mu A_\rho \partial_\nu A_\rho A_{\sigma_1} A_{\sigma_2}\dot{x}^{\sigma_1}\dot{x}^{\sigma_2} \\
&= -(\partial_\mu \Delta_1 \partial_\mu \Delta_2 + \partial_\mu \Delta_2 \partial_\mu \Delta_1)\delta_{\rho\sigma_1}\delta_{\rho\sigma_2}\dot{x}^{\sigma_1}(\mathsf{t}_1)\cdot \dot{x}^{\sigma_2}(\mathsf{t}_2) \\
&= -(\partial_\mu \Delta_1 \partial_\mu \Delta_2 + \partial_\mu \Delta_2 \partial_\mu \Delta_1)\dot{x}(\mathsf{t}_1)\dot{x}(\mathsf{t}_2) \qquad (2.110) \\
&= -(\partial_\mu \Delta_1 \partial_\mu \Delta_2 + \partial_\mu \Delta_2 \partial_\mu \Delta_1)R^2\cos(\mathsf{t}_1 - \mathsf{t}_2)\ .
\end{aligned}
$$

This is very similar to the contribution we also found from the Wilson loop at leading order [62]. For some of the other components we do not find only the cosine function. Instead

$$
\begin{aligned}
&:-(\partial_\rho A_\mu)(\partial_\rho A_\nu)A_{\sigma_1}A_{\sigma_2}\dot{x}^{\sigma_1}\dot{x}^{\sigma_2}: \\
&\qquad = -\partial_\rho A_\mu \partial_\rho A_\nu A_{\sigma_1} A_{\sigma_2}\dot{x}^{\sigma_1}\dot{x}^{\sigma_2} - \partial_\rho A_\mu \partial_\rho A_\nu A_{\sigma_1} A_{\sigma_2}\dot{x}^{\sigma_1}\dot{x}^{\sigma_2} \qquad (2.111) \\
&\qquad = -\partial_\rho \Delta_1 \partial_\rho \Delta_2 (\delta_{\mu\sigma_1}\delta_{\nu\sigma_2} + \delta_{\nu\sigma_1}\delta_{\mu\sigma_2})\dot{x}^{\sigma_1}(\mathsf{t}_1)\dot{x}^{\sigma_2}(\mathsf{t}_2)\ .
\end{aligned}
$$

Taking the component $T_{\mu\nu} \to T_{\phi\phi}$ of this expression means to use the sum of all components of $\dot{x}^\sigma(\mathsf{t}_i)$. More precisely we have in the $[12]-$plane

$$
\begin{aligned}
\dot{x}(\mathsf{t}_i) &= \dot{R}\vec{e}_R + R\dot{\mathsf{t}}\vec{e}_{\mathsf{t}} = R\begin{pmatrix} -\sin(\mathsf{t}) \\ \cos(\mathsf{t}) \end{pmatrix}\ , \\
\delta_{\phi\sigma_1}\delta_{\phi\sigma_2}\dot{x}^{\sigma_1}(\mathsf{t}_1)\dot{x}^{\sigma_2}(\mathsf{t}_2) &= R^2\begin{pmatrix} -\sin(\mathsf{t}_1) & \cos(\mathsf{t}_1) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -\sin(\mathsf{t}_2) \\ \cos(\mathsf{t}_2) \end{pmatrix} \qquad (2.112) \\
&= R^2(\cos(\mathsf{t}_1 - \mathsf{t}_2) - \sin(\mathsf{t}_1 + \mathsf{t}_2))\ .
\end{aligned}
$$

This finally yields

$$
:-(\partial_\rho A_\phi)(\partial_\rho A_\phi)A_{\sigma_1}A_{\sigma_2}\dot{x}^{\sigma_1}\dot{x}^{\sigma_2}: = -2R^2\partial_\rho \Delta_1 \partial_\rho \Delta_2 (\cos(\mathsf{t}_1 - \mathsf{t}_2) - \sin(\mathsf{t}_1 + \mathsf{t}_2))\ . \qquad (2.113)
$$

All remaining terms in $T^V_{\mu\nu}$ can be treated similarly. The scalar contribution is calculatedin the same manner as before. Therefore, we find for the stress tensor

$$
\langle T_{\phi\phi}\rangle_{W_-} = \langle T_{\phi\phi}\rangle^S_{W_-} + \langle T_{\phi\phi}\rangle^V_{W_-}\ , \qquad (2.114)
$$

$$\langle T_{\phi\phi}\rangle^V_{W_-} = -\frac{\lambda}{2}R^2\int_{-\infty}^{\infty}\mathrm{d}\mathfrak{t}_1\int_{-\infty}^{y_1}\mathrm{d}\mathfrak{t}_2\ \left[(-(\partial_\rho\Delta_1)(\partial^\rho\Delta_2)+(\partial_3\Delta_1)(\partial_3\Delta_2))\cos(\mathfrak{t}_1-\mathfrak{t}_2)\right.$$

$$\left.+(-2(\partial_\rho\Delta_1)(\partial^\rho\Delta_2)+3(\partial_3\Delta_1)(\partial_3\Delta_2))\sin(\mathfrak{t}_1+\mathfrak{t}_2)\right]\ , \tag{2.115}$$

$$\langle T_{33}\rangle^S_{W_-} = \frac{\lambda}{2}R^2\int_{-\infty}^{\infty}\mathrm{d}y_1\int_{-\infty}^{y_1}\mathrm{d}y_2\left[-\frac{4}{3}(\partial_3\Delta_1)(\partial_3\Delta_2)+\frac{1}{3}\Delta_1(\partial_3\partial_3\Delta_2)+\frac{1}{3}\Delta_2(\partial_3\partial_3\Delta_1)\right.$$

$$\left.+\frac{1}{3}(\partial_\rho\Delta_1)(\partial^\rho\Delta_2)\right]\ . \tag{2.116}$$

For simplicity and without loss of generality of the result we assume that the projection of the stress tensor to the polar plane is in the origin of the circular loop, $z_r = 0$. Including this in the calculations we find the following integrands from the correlator

$$(\partial_\phi\Delta_1)(\partial_\phi\Delta_2) = \frac{R^2\sin(\mathfrak{t}_1)\sin(\mathfrak{t}_2)}{4\pi^4r^8}\ , \tag{2.117}$$

$$(\partial_\rho\Delta_1)(\partial^\rho\Delta_2) = \frac{r^2+R^2\cos(\mathfrak{t}_1-\mathfrak{t}_2)-R^2}{4\pi^4r^8}\ , \tag{2.118}$$

$$\partial_\phi^2\Delta_i = -\frac{r^2-2R^2+2R^2\sin(2\mathfrak{t}_i)+2R^2\cos(2\mathfrak{t}_i)}{2\pi^2r^6}\ . \tag{2.119}$$

The stress tensor one-point function is thus given by

$$\langle T_{\phi\phi}\rangle_{W_\bigcirc} = \frac{\lambda R^4}{12\pi^2r^8}-\frac{\lambda R^4}{16\pi^2r^8}+\mathcal{O}(\lambda^2) = \frac{\lambda R^4}{48\pi^2r^8}+\mathcal{O}(\lambda^2)\ . \tag{2.120}$$

This matches the form in [96]. We therefore see that also for the circular loop we are able to reproduce the result of [36, 81]

$$h = \frac{\lambda}{48\pi^2}+\mathcal{O}(\lambda^2)\ , \quad \Rightarrow\ 3h = \frac{\lambda}{16\pi^2}+\mathcal{O}(\lambda^2)\ . \tag{2.121}$$

Above we found

$$B = \frac{\lambda}{16\pi^2}+\mathcal{O}(\lambda^2)\ . \tag{2.122}$$

Hence, we indeed reproduced the Relation (1.1), $B = 3h$ in a perturbative calculation.

2.2.3.3 Diagrams of subleading orders

So far we simply assumed that the above diagram (Figure 2.3) is the only one relevant at leading order. Although this makes sense from a calculational point of view, we want to show it more explicitly. Therefore will will look at the relevant Feynman diagrams.

There are several diagrams that we have to consider in a perturbative expansion. Recall from the Feynman rules of $\mathcal{N} = 4$ (J.2) that any propagator comes with a factor of g^2 while all three- and four-point vertices yield $^1/_{g^2}$. Due to the prefactor of the stress tensor

we also get the $1/g^2$ factor for all insertions. These are given in [99] for QCD. In our case the possible vertices include all the vertices that come from the Lagrangian but with a stress tensor insertion instead of the actual vertex. The precise Feynman rules for the different diagrams are of minor importance here and we will not write them out.
The simplest diagram has an insertion of the stress tensor which is connected to the straight line through two propagators, either scalars or vectors, see figure 2.3 above. This diagram comes with a prefactor g^2 which will combine with the N yielded by the color trace to the prefactor λ thus agreeing with the similar leading order diagram of the Wilson loop. We saw this in the above calculation for the leading order.

At subleading order there are serveral diagrams relevant. They are shown in Figure 2.4. A one-loop self-energy might be inserted to either propagators of the above diagram. Also a three-point connection between the stress tensor and the Wilson line is possible as well as a tad-pole. In principle there might also be diagrams with one connection between the

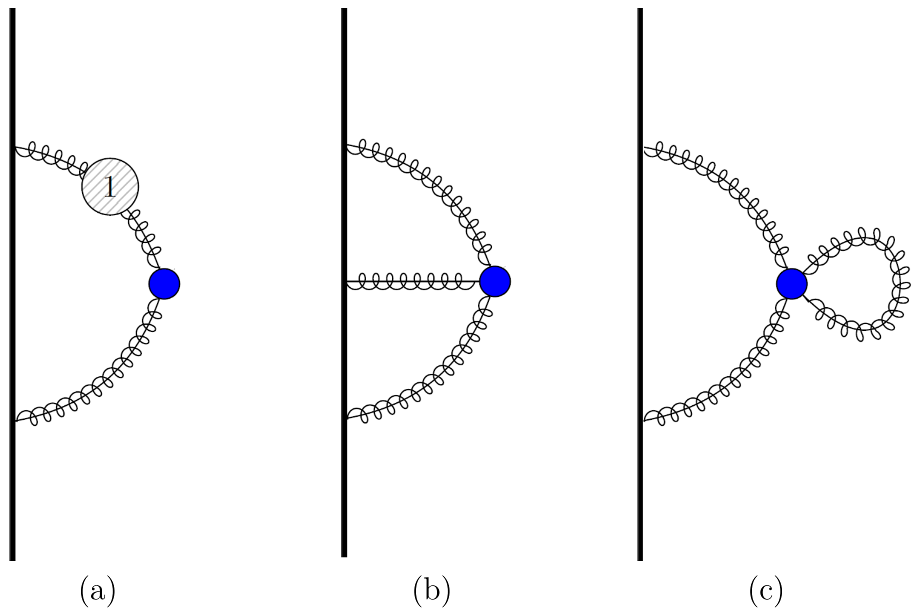
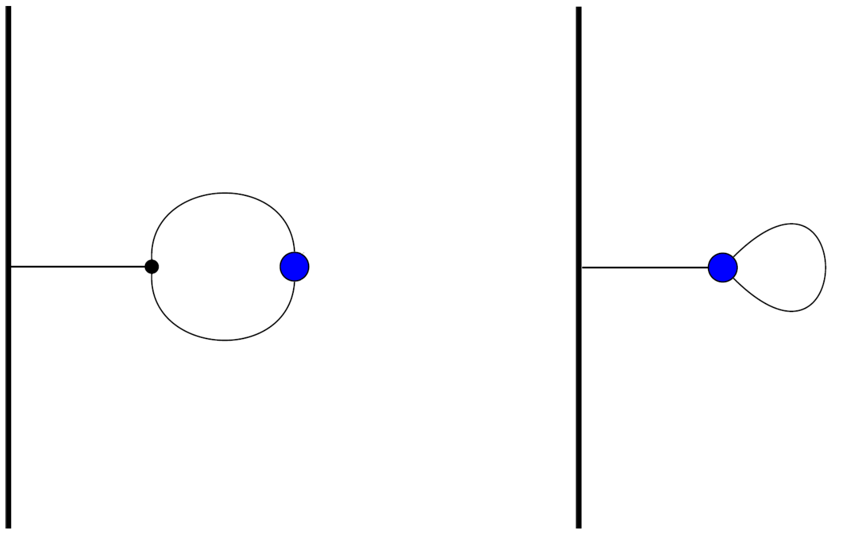


Figure 2.4: The diagrams show the two-loop ($\mathcal{O}(\lambda^2)$) Feynman diagrams for the expectation value of the stress tensor (blue circle) in presence of the Wilson loop (thick line). At second order we can have a one-loop correction to the diagram of lower order (a). Other possible diagrams are a three-point (b) or four-point vertex (c). The propagators in the four-point vertex can also be all connected to the Wilson loop operator. However, this represents a higher order diagram, namely $\mathcal{O}(\lambda^3)$.

Wilson loop and the stress tensor and a propagator between two insertion points on the Wilson line. From the Wilson line calculation, however, we know that such insertions are always zero (this holds for the straight line, not the circle).
Other diagram at low orders are the so-called lollipop diagrams. The lollipop diagrams are of the kind shown in Figure 2.5 with the displayed diagram just being one of many examples. This diagram and all others of the same kind, however, will not contribute. The



(a) (b)

Figure 2.5: In the Lollipop diagrams only one propagator is connected to the Wilson line. This can a priori be a scalar or a gluon. This one propagator is either directly connected to a vertex-like insertion of the stress tensor (a) or to a vertex coming from the action which itself connects to the stress tensor.

reason lies in the color index structure of the diagrams. Without writing their contribution explicitly this can be seen.

Every propagator carries a color charge. This charge cannot change. From the color charges in the definition of the stress tensor (1.62) it is straight forward that also the stress tensor insertion must preserve the incoming and outgoing charges. The Wilson loop operator contracts the open charge of the incoming propagator with the SU(N) generator T^a. Later a trace is taken over this group (but this is not important here).

Furthermore, the diagrams of the type depicted above have one vertex where one line comes from the Wilson line, carrying color charge a. The two outgoing lines carry charges b and c, respectively. These two lines eventually connect yielding some $\delta^{b,c}$. The vertex itself comes with a structure constant f^{abc}.

$$\mathrm{tr}\left(T^a f^{abc} \delta^{b,c}\right) = \mathrm{tr}\left(T^a f^{abb}\right) = 0 \,. \tag{2.123}$$

All in all, we therefore find the color structure to vanish because the structure constants are anti-symmetric and all diagrams of the type shown in Figure 2.5 vanish. In other words, this follows because propagators are diagonal in color space and the vertices are skew-symmetric [20].

Chapter 3

Finite temperature and zero coupling

In this chapter we consider the relation between the bremsstrahlung and the stress tensor at finite temperature while setting the coupling to zero ($g = 0$). The calculations from the previous Chapter 2 are applied to the new setting. For the Ward identity (2.16) we find thermal correction terms. Setting the coupling to zero will ensure that the new terms preserve R-symmetry and we will be able to indeed reproduce the Relation (1.1), $B = 3h$. The implications of nonzero coupling are addressed in Chapter 4.
The analytic result at $g = 0$ will be supported by a perturbative calculation. The first term in a weak coupling perturbative expansion of the Wilson loop represents the contribution coming from free fields. We will find a scheme for the parameter $\tilde{R}$ introduced in the straight line parameterization for which the relation $B = 3h$ indeed is reproduced. This scheme is dependent on the temperature T and the radius R of the circular loop. We consider a Taylor expansion around small temperatures up to order T^4 and in the high temperature limit.
The action as defined in Equation (1.56) has a prefactor $^1/_{2g^2}$. Before sending the coupling to zero we therefore renormalize the coupling dependence of the field by setting

$$A_\mu \to gA_\mu\ , \quad \lambda_I \to g\lambda_I\ , \quad \Phi_{IJ} \to g\Phi_{IJ}\ , \quad \bar{c} \to g\bar{c}\ . \tag{3.1}$$

Consequentially the kinetic terms and propagators will be independent of the coupling. The interactions are coupling dependent. The Yukawa and other three-particle interactions contribute at order g while for quartic interactions like Φ^4 we find a prefactor g^2. Now setting $g = 0$ is possible and the Lagrangian consists only of the kinetic terms

$$\begin{aligned} S = \frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, \mathrm{tr}\Big[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu A^\mu)^2 + \partial_\mu\Phi^{IJ}\partial^\mu\bar{\Phi}_{IJ} + \mathrm{i}\bar{\lambda}^I\sigma^\mu\partial_\mu\lambda_I + \\ + \mathrm{i}\lambda_I\bar{\sigma}^\mu\partial_\mu\bar{\lambda}^I\Big] + \mathcal{O}(g)\ , \end{aligned} \tag{3.2}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
The Wilson loop operator will likewise obtain a coupling dependence in the exponent. In

a strict zero coupling regime the Wilson loop will thus always be one. We will, however, allow for a coupling between the bosonic operators and the Wilson loop at leading order (yielding diagrams of order $\lambda = g^2 N$). This order is independent from interactions between the particles of $\mathcal{N} = 4$ while still allowing for a computation of the bremsstrahlung and the stress tensor one-point function. In other words, we allow for the appearance of the diagram 1.1a only.

3.1 Thermal Broken Ward Identity

In this section we want to study how the Ward identity is corrected at finite temperature. The breaking of (2.16) arises mainly from the fermionic boundary conditions on the compactified time dimension. Recall that the anti-commutation properties of fermionic fields together with the definition of the thermodynamic ensemble partition function yield anti-periodic boundary conditions for fermions on the thermal circle [49, 50].
For completeness let us show this explicitly. Therefore, consider a fermionic field $\psi(\tau)$ where we only explicitly write the dependence on the temporal component. For two such fields we have the anti-commutation property $\psi(\tau_1)\psi(\tau_2) = -\psi(\tau_2)\psi(\tau_1)$. The partition function is defined as the trace of the density matrix $\mathcal{Z} = \operatorname{tr}\rho = \operatorname{tr} e^{-\beta H}$ and hence (assuming $\tau_1 > \tau_2$) [47, 100, 101]

$$\begin{aligned}
\langle\psi(\tau_1)\psi(\tau_2)\rangle &= \operatorname{tr}\left[\psi(\tau_1)\psi(\tau_2)e^{-\beta H}\right] = \operatorname{tr}\left[e^{-\beta H}\psi(\tau_1)e^{\beta H}e^{-\beta H}\psi(\tau_2)\right] \\
&= \operatorname{tr}\left[\psi(\tau_1+\beta)e^{-\beta H}\psi(\tau_2)\right] = -\operatorname{tr}\left[\psi(\tau_1+\beta)\psi(\tau_2)e^{-\beta H}\right] \\
&= -\langle\psi(\tau_1+\beta)\psi(\tau_2)\rangle \ .
\end{aligned} \tag{3.3}$$

These anti-periodic boundary conditions for the fermions will break the supersymmetry of the theory [53] and in consequence also the Ward identity (2.16). To see this we make the periodic and anti-periodic boundary conditions explicit in a redefinition of the fields. Then the BWI can be obtained consistently.
The thermal theory is obtained by Wick rotating before the compactificaion of the time dimension. The coordinate along the time dimension therefore is

$$\tau = \mathrm{i}x_0 \ . \tag{3.4}$$

Likewise, also the the o-th spinor matrix is Wick-rotated such that we find the Euclidean $\sigma_E^\mu = (\mathrm{i}\mathbb{1}, \sigma^i)$ and $\bar{\sigma}_E^\mu = (\mathrm{i}\mathbb{1}, -\sigma^i)$. All relevant relations are given in Appendix A.

To consider how the superalgebra breaks let us introduce an intermediate step. When the time dimension is compactified on a circle S^1_β the boundary conditions are not a priori fixed. As the calculation (3.3) clarified they get determined by the thermodynamic ensemble. For this calculation we do not introduce such a thermodynamic ensemble. Instead we simply compactify the time dimension similar to a Kaluza-Klein (KK) reduction and assume all boundary conditions of both, bosonic and fermionic fields, to be periodic. This is a reduction purely with respect to the spacetime which a priori will not yield a thermal theory. In such a setting supersymmetry can remain unbroken.

Note that the introduction of the WL operator already reduced the preserved superalgebra to $\mathfrak{osp}(4|4)$ [83]. For details on the full algebra see Appendix A, the breaking of the algebra is shown in detail in Appendix B while the commutation relations of the $\mathfrak{osp}(4|4)$ subalgebra are given in Equation (3.34) below.

We want to see how the superalgebra $\mathfrak{osp}(4|4)$ is effected upon compactification of the time dimension on a circle. The symmetries of the theory yield conserved Noether currents. If these currents remain conserved in the new setting we can infer that the respective symmetry is unbroken. Consider the supercurrent $\hat{\Psi}^{I\mu}_{\alpha}$ as the main important example. It is related to the Poincaré supercharges $\mathcal{Q}^I_{\alpha}$ and the conformal supercharges $\mathcal{S}^I_{\dot{\alpha}}$, see Equations (B.7) and (B.8) for the definitions. They are preserved after the insertion of the straight Wilson line defect. Meanwhile it is rather intuitive for the R-symmetry generator to not be affected by compactified time at all[1]. Similarly the rotations in the [12]-plane, M_{12}, must be preserved as said plane is neither affected by the WL nor by the compactified time. Conversely the rotations M_{01} and M_{02} involving the time dimension are broken. The WL already lead to a breaking of any rotation M_{3m} including the x_3 direction where the line is placed.

Let us consider the supercurrent $\hat{\Psi}^{I\mu}_{\alpha}$ which is the Noether current of the supercharges. It is the preserved part of the field $\Psi^I_{\alpha\beta\dot{\alpha}}$ in the stress tensor multiplet 2.1. For our considerations it is not important how it looks like precisely. If supersymmetry is unbroken the current is preserved,

$$\partial_\mu \hat{\Psi}^{I\mu}_{\alpha} = 0 \quad \text{and} \quad \hat{\Psi}^{I\mu}_{\alpha}\sigma^{\beta\dot{\beta}}_{\mu} = 0 \,. \tag{3.5}$$

Consider the related current

$$j^\mu := \zeta^I_\alpha(x_\mu)\Omega_{IJ}\epsilon^{\alpha\beta}\hat{\Psi}^{J\mu}_{\beta} \,. \tag{3.6}$$

This current is obtained by contracting the supercurrent with a superconformal Killing spinor ζ^I_α which can depend a priori on the spacetime coordinates. The newly introduced current is assumed to be preserved, $\partial_\mu j^\mu = 0$. This restricts the Killing spinor. The most general solution to make the current conserved is [90]

$$\zeta^I_\alpha(x_\mu) = \eta^I_\alpha + x_\mu\sigma^\mu_{\alpha\dot{\alpha}}\bar{\nu}^{I\dot{\alpha}} \,. \tag{3.7}$$

[1]We will eventually show that finite temperature also induces a breaking of the R-symmetry. This is considered in details below.

We introduced the constant left and right spinors η^I_α and $\bar{\nu}^{I\dot{\alpha}}$, respectively. The Poincaré and conformal supercharge can be constructed from the supercurrent by an integration

$$\mathcal{Q}^I_\alpha \sim \int \mathrm{d}^3x \; \hat{\Psi}^{I0}_\alpha \,, \quad \bar{\mathcal{S}}^{I\dot{\alpha}} \sim \int \mathrm{d}^3x \; x^\mu \bar{\sigma}^{\dot{\alpha}\alpha}_\mu \hat{\Psi}^{I0}_\alpha \,. \tag{3.8}$$

For the Killing spinor this implies that the choice $\eta^I \neq 0$ and $\bar{\nu}^I = 0$ is associated to the Poincaré supercharge $\mathcal{Q}$ while the converse choice $\eta^I = 0$ and $\bar{\nu}^I \neq 0$ yields the conformal supercharge $\mathcal{S}$ [90].
When the time dimension is compactified on a circle S^1_β we assume periodic boundary conditions for all bosonic and fermionic fields as stated earlier. Therefore the above superconformal Killing spinors must satisfy the periodic KMS condition [49, 50]

$$\zeta^I_\alpha(x_\mu) = \zeta^I_\alpha(x_0, x_i) = \zeta^I_\alpha(x_0 + \beta, x_i) \,. \tag{3.9}$$

The spinors η^I and $\bar{\nu}^I$ are constant an hence the spacetime dependence is only in the prefactor of $\bar{\nu}^I$. Shifting $x_0 \to x_0 + \beta$ yields an additional factor which needs to vanish for the KMS condition to hold,

$$\beta \sigma^0_{\alpha\dot{\alpha}} \bar{\nu}^{I\dot{\alpha}} = 0 \,. \tag{3.10}$$

Therefore the conformal supercharges S must be broken. The above equation can only be satisfied if $\bar{\nu}^I \equiv 0$ which, as stated above, corresponds to considering the Poincaré supercharges $\mathcal{Q}$ only. Therefore conformal invariance is broken by the introduction of the compactified time with finite radius β. For identical reasons finite temperature will neither be conformally invariant [54].
In principle this clearly shows the breaking of the conformal part of the algebra as expected. Due to the broken Lorentz invariance it is, however, possible to define a different Killing spinor which preserves the KMS condition. Such a Killing spinor is given by a construction with Heaviside theta functions,

$$\zeta^I_\alpha(\tau, \vec{x}) = \eta^I_\alpha + x_i \sigma^i_{\alpha\dot{\alpha}} \bar{\nu}^{I\dot{\alpha}} + f_\beta(\tau) \sigma^0_{\alpha\dot{\alpha}} \bar{\nu}^{I\dot{\alpha}} \,, \tag{3.11}$$

$$f_\beta(\tau) = \sum_{m=-\infty}^{\infty} (\tau - m\beta) \Theta(\tau - m\beta) \Theta((m+1)\beta - \tau) \,.$$

This superconformal Killing spinor clearly obeys the KMS periodicity condition $\zeta^I_\alpha(\tau, x_i) = \zeta^I_\alpha(\tau + \beta, x_i)$ due to the introduced step-function $f_\beta(\tau)$ which is piece-wise linear and periodic. Note also that at $\tau = \beta m \in \beta \times \mathbb{Z}$ the functions is not smooth. The function jumps back to zero as an effect of the Heaviside theta functions.
The Killing spinor equation $\partial_\mu j^\mu = 0$ is not necessarily preserved with the newly defined

Killing spinor. In fact we will find a breaking of the conformal supercharges.

$$\partial_\mu j^\mu = \partial_\mu \zeta^I_\alpha(x_\mu)\hat{\Psi}^{\alpha\mu}_I = \partial_\tau \zeta^I_\alpha(\tau,\vec{x})\hat{\Psi}^{\alpha 0}_I + \partial_i \zeta^I_\alpha(\tau,\vec{x})\hat{\Psi}^{\alpha i}_I$$
$$= (\sigma_i)_{\alpha\dot{\alpha}}\hat{\Psi}^{i\alpha}_I \bar{\nu}^{I\dot{\alpha}} + (\partial_\tau f_\beta(\tau))\sigma^0_{\alpha\dot{\alpha}}\hat{\Psi}^{\alpha 0}_I \bar{\nu}^{I\dot{\alpha}} \,. \tag{3.12}$$

The derivative acting on the supercurrent $\hat{\Psi}^\mu_I$ and on the spinors η^I and $\bar{\nu}^I$ vanishes as before. Therefore we only find a non-trivial contribution from the action of the derivatives on the chainsaw function $f_\beta(\tau)$ and a term with the Pauli matrices. We write the sums out explicitly. If the derivative acts on the linear term, we get a sum over theta functions only. This will ultimately yield 1 and cancel in combination with the first term in the above equation. For the derivatives acting on the Heaviside theta functions we will find Dirac delta functions,

$$\sum_{m=-\infty}^{\infty}\left[\Theta(\tau - m\beta)\Theta((m+1)\beta-\tau)\right] = 1 \quad \text{and} \tag{3.13}$$

$$\sum_{m=-\infty}^{\infty}(\tau - m\beta)\partial_\tau\left[\Theta(\tau - m\beta)\Theta((m+1)\beta-\tau)\right]$$
$$= \sum_{m=-\infty}^{\infty}(\tau - m\beta)\left[\delta(\tau - m\beta) - \delta((m+1)\beta-\tau)\right]$$
$$= -\sum_{m=-\infty}^{\infty}\beta\delta(\tau - m\beta)\,. \tag{3.14}$$

This gives all in all

$$\partial_\mu \zeta^I_\alpha(x_\mu)J^{\alpha\mu}_I = \sigma^i_{\alpha\dot{\alpha}}\bar{\nu}^{I\dot{\alpha}}\hat{\Psi}^{\alpha i}_I + \sigma^0_{\alpha\dot{\alpha}}\bar{\nu}^{I\dot{\alpha}}\hat{\Psi}^{\alpha 0}_I - \sum_{m=-\infty}^{\infty}\beta\delta(\tau - m\beta)\sigma^0_{\alpha\dot{\alpha}}\bar{\nu}^{I\dot{\alpha}}J^{\alpha 0}_I\,. \tag{3.15}$$

The first two terms combine and we can use $\hat{\Psi}^{I\mu}_\alpha\sigma^{\beta\dot{\beta}}_\mu = 0$ to eliminate it. Thus we are left with the final expression

$$\partial_\mu j^\mu = \partial_\mu \zeta^I_\alpha(x_\mu)J^{\alpha\mu}_I = \sum_{m=-\infty}^{\infty}\beta\delta(\tau - m\beta)\sigma^0_{\alpha\dot{\alpha}}\bar{\nu}^{I\dot{\alpha}}\hat{\Psi}^{\alpha 0}_I\,. \tag{3.16}$$

This yields that the generators S^I_α are broken whenever $\tau/\beta \in \mathbb{Z}$. Therefore the full superconformal group cannot be preserved.

This calculation has a further important implication. While the conformal supercharge is broken, the Poincare supercharges remain unbroken. Therefore the Ward identity (2.16) still holds for compactified time. This has major importance when we consider the thermal corrections of the Ward identity.

3.1.1 Thermal action

In the previous considerations we compactified the time dimension and showed that supersymmetry can be preserved in such a setting. This is no longer the case for finite temperature. Our calculation in Equation (3.3) shows that fermions get anti-periodic boundary condition from the KMS conditions [47, 100]. Let us consider the thermal action at zero coupling. Following our definition (1.56) we have in the thermal setting

$$\tilde{S} = \frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\,\mathrm{tr}\left[\frac{1}{2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \left(\partial_\mu \tilde{A}^\mu\right)^2 + \partial_\mu\tilde{\Phi}^{IJ}\partial^\mu\tilde{\bar{\Phi}}_{IJ} + \mathrm{i}\tilde{\bar{\lambda}}^I\sigma^\mu\partial_\mu\tilde{\lambda}_I + \right.$$
$$\left. + \mathrm{i}\tilde{\lambda}_I\bar{\sigma}^\mu\partial_\mu\tilde{\bar{\lambda}}^I\right] + \mathcal{O}(g)\;. \tag{3.17}$$

As we are considering the free case $g = 0$ here only the kinetic terms are present. All fields were renamed with a tilde for later convenience. We want to make the boundary conditions explicit by giving the fields a phase factor. Thus we will rewrite the action with identical terms consisting solely of periodic fields and hence preserving supersymmetry. The phase factor will yield further terms explicitly breaking the symmetries. As the gauge field is real it cannot acquire any phase factor. The ghost fields, although being Grassmann variables, can be assumed to be periodic in order to ensure BRST invariance. This is an allowed transformations as the ghost fields are unphysical. We can thus define the phase of the ghosts to be zero [53]. The fermionic fields, however, will get a phase to ensure that they do get anti-periodic boundary conditions. As there are no interactions in the case at hand the scalar fields do not necessarily need a phase as they are already periodic by construction. Therefore we replace

$$\tilde{A}_\mu(\tau,\vec{x}) \to A_\mu(\tau,\vec{x})\;,\quad \tilde{\Phi}_{IJ}(\tau,\vec{x}) \to \Phi_{IJ}(\tau,\vec{x})\;,\quad \tilde{\bar{c}}(\tau,\vec{x}) \to \bar{c}(\tau,\vec{x})\;,$$
$$\tilde{\lambda}_I(\tau,\vec{x}) \to e^{\mathrm{i}\alpha\tau}\lambda_I(\tau,\vec{x}) \qquad \text{with} \quad \alpha = \frac{\pi}{\beta}(2n+1)\;,\quad n \in \mathbb{Z}\;. \tag{3.18}$$

All fields without the tilde are periodic. The α are fermionic phases yielding anti-periodic boundary conditions for τ. They are real coefficients and the conjugated fields $\bar{\lambda}^I$ consequently have a phase with a negative sign in the exponent. The action thus will get corrections from the kinetic terms while the phases cancel each other. We find

$$\tilde{S} = \frac{1}{2}\mathrm{tr}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu A^\mu)^2 + D_\mu\Phi^{IJ}\tilde{D}^\mu\bar{\Phi}_{IJ} + \mathrm{i}\bar{\lambda}^I\sigma^\mu D_\mu\lambda_I + \right.$$
$$\left. + \mathrm{i}\lambda_I\bar{\sigma}^\mu\bar{D}_\mu\bar{\lambda}^I - \alpha\bar{\lambda}^I\sigma^0\lambda_I + \alpha\lambda_I\bar{\sigma}^0\bar{\lambda}^I\right] + \mathcal{O}(g)$$
$$= S + \alpha\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3x\,\mathrm{tr}\left[\lambda_I\bar{\sigma}^0\bar{\lambda}^I\right] = S + \alpha\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3x\,\mathrm{tr}\mathcal{M}_\lambda\;. \tag{3.19}$$

The kinetic terms thus get reproduced depending only on the periodic fields. Additionally a new term is introduced for the spinor fields. An interpretation of the new term is that we introduced a covariant derivative

$$\nabla_\mu = \partial_\mu + \mathrm{i}\alpha\delta_{\mu,0} \qquad \text{with} \quad \alpha = \frac{\pi}{\beta}(2n+1)\ , \quad n \in \mathbb{Z}\ . \tag{3.20}$$

This covariant derivative is the gravitational coupling of the spinor to the background S^1_β. It is suggestive of an U(1)-type coupling where the spinors have a charge α while all other fields are not charged.
The original action S depends only on periodic fields and preserves supersymmetry unlike the new term. This new mass-like term suggest the preservation of $\mathfrak{su}(4)_R$ symmetry. This is only true for the free theory ($g = 0$) we consider here. For non-zero coupling the conditions yielded by the Yukawa interactions contradict a choice where every spinor in the tuple λ_I get the same phase and consequentially R-symmetry is broken. This is studied in detail in Chapter 4.
The transformations (3.18) act not only on the fields in the action but also transform the measure of the path integral yielding a Jacobian. This can potentially yield a non-vanishing quantum anomaly for the thermal theory [6]. In Appendix F we show that the above transformation is in fact anomaly-free.

3.1.2 Broken Ward Identity

Let us now go back to the Ward identity. Even though supersymmetry is manifestly broken through the introduction of a fermionic mass term, we are allowed to introduce an operator $\hat{\Psi}^{I\mu}_\alpha$ defined identically as the field content of the supercurrent. The main difference to note here is that this 'supercurrent' is no longer a Noether current of the full action. In a similar manner, supercharges $\mathcal{Q}$ can still be defined in the thermal setting. They transform bosonic operators to fermionic ones, but are not a symmetry of the theory. Let us consider the action of a supersymmetry on the action with $\delta = \mathcal{Q}^I_\alpha \zeta^\alpha_I$

$$\begin{aligned}\delta S = \delta S_{\mathcal{N}=4} + \delta S_m &= \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, \frac{1}{2}\left[\partial_\mu \hat{\Psi}^{I\mu}_\alpha \zeta^\alpha_I + \alpha \mathcal{Q}^I_\alpha \mathcal{M}_\lambda \zeta^\alpha_I\right] \\ &= \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, \frac{1}{2}\left[\partial_\mu \hat{\Psi}^{I\mu}_\alpha + \alpha \mathcal{A}^I_\alpha\right]\zeta^\alpha_I\ .\end{aligned} \tag{3.21}$$

We defined a new operator $\mathcal{A}^I_\alpha = \mathcal{Q}^I_\alpha \mathcal{M}_\lambda$ which represents the thermal corrections at zero coupling. Recall the Ward identity derived in Section 2.1.2 from the supercurrent $\hat{\Psi}^{I\mu}_\alpha$. At zero temperature we found (2.29)

$$\int \mathrm{d}^4u \left\langle \partial_\mu \hat{\Psi}^{I\mu}_\alpha(u)\varphi_1(x_1)\dots\varphi_n(x_n)\right\rangle_W = -\left\langle \mathcal{Q}^I_\alpha \varphi_1(x_1)\dots\varphi_n(x_n)\right\rangle_W\ . \tag{3.22}$$

The left hand side of this equation was shown to be zero hence yielding the Ward identity we used in Equation (2.16). In the thermal case not only the spacetime changes. We saw above that we need to consider a mass correction for the fermions as well. The respective terms must hence be included to the Ward identity.

$$-\left\langle \mathcal{Q}^I_\alpha \left(\varphi_1(x_1)\dots\varphi_n(x_n)\right)\right\rangle_W = \int_{\delta(S^1_\beta\times\mathbb{R}^3)} \mathrm{d}\Sigma_\mu \left\langle \hat{\Psi}^{I\,\mu}_\alpha(u_0,\vec{u})\varphi_1(x_1)\dots\varphi_n(x_n)\right\rangle_W$$
$$+\alpha\int_0^\beta \mathrm{d}u_0 \int \mathrm{d}^3u \left\langle \mathcal{A}^I_\alpha(u_0,\vec{u})\varphi_1(x_1)\dots\varphi_n(x_n)\right\rangle_W \,. \tag{3.23}$$

The right hand side of this identity has two terms. The first one was also present in the zero temperature case. It is derived using Stoke's theorem and is a boundary integral of the thermal manifold. Due to all fields being periodic we can show that this term also will not contribute in the thermal setting at zero coupling. The additional term is the one breaking the Ward identity and it is generally non-zero.
Consider the term with the original supercurrent. We need to be careful with the boundaries of this integral. The full space is $S^1_\beta \times \mathbb{R}^3$. Then

$$\int_{\delta(S^1_\beta\times\mathbb{R}^3)} \mathrm{d}\Sigma_\mu \left\langle \hat{\Psi}^{I\,\mu}_\alpha(u_0,\vec{u})\varphi_1(x_1)\dots\varphi_n(x_n)\right\rangle_W$$
$$= \int_0^\beta \mathrm{d}u_0 \int_{S^2_\infty} \mathrm{d}\vec{\Sigma} \left\langle \vec{\hat{\Psi}}^I_\alpha(u_0,\vec{u})\varphi_1\dots\varphi_n\right\rangle_W \tag{3.24}$$
$$+\,\alpha\int \mathrm{d}^3u \left\langle \left[\hat{\Psi}^{I\,0}_\alpha(u_0,\vec{u})\right]^\beta_{u_0=0}\varphi_1\dots\varphi_n\right\rangle_W \,.$$

Let us first consider the last term. The main reason behind the introduction of the mass term correction above was to keep all fields periodic. The field $\hat{\Psi}$ hence is a combination of these periodic elementary fields and therefore periodic on the thermal circle. The respective term thus vanishes

$$\left[\hat{\Psi}^{I\,0}_\alpha(u_0,\vec{u})\right]^\beta_{u_0=0} = \hat{\Psi}^{I\,0}_\alpha(\beta,\vec{u}) - \hat{\Psi}^{I\,0}_\alpha(0,\vec{u}) = 0 \,. \tag{3.25}$$

The remaining term involves an integral over a sphere at infinity. In the zero temperature case one can argue using the conformal dimension of the field that it needs to fall off with a certain power law at infinity thus not giving any contribution [90]. While a similar approach might naively work in the thermal case as well, the conformal symmetry and hence the notion of conformal dimensions is, strictly speaking, no longer valid. We therefore provide another argument.
As we are in a limit where the radial component $|\vec{u}| \to \infty$ we can use the thermal

clustering [54]. The field $\hat{\Psi}$ in the correlator is split into a product of one-point functions

$$\lim_{|\vec{u}|\to\infty}\left\langle\hat{\vec{\Psi}}^I_\alpha(u_0,\vec{u})\varphi_1\dots\varphi_n\right\rangle_W = \left\langle\hat{\vec{\Psi}}^I_\alpha(u_0,\vec{u})\right\rangle_W \langle\varphi_1\rangle_W\dots\langle\varphi_n\rangle_W + \mathcal{O}\left(e^{-m|\vec{u}|}\right)\,. \tag{3.26}$$

As the thermal mass is positive by a *folk theorem* the corrections fall of exponentially fast and we can henceforth ignore them. The leading term involves a one-point function of the $\hat{\vec{\Psi}}^i_\alpha$ which is in the $[1,0] = \mathbf{4}$ representation of the $\mathfrak{usp}(4)_R$ symmetry, see table 2.1. This representation is charged and thus yields that the one-point function must be zero. Furthermore, $\hat{\vec{\Psi}}^i_\alpha$ is not a symmetric traceless tensor with even spin which would be required for a non-vanishing one-point function at finite temperature [54]. Therefore the terms from the spacetime boundary indeed cancel and we finally find the general thermal broken Ward identity

$$-\left\langle\mathcal{Q}^I_\alpha\left(\varphi_1\dots\varphi_n\right)\right\rangle_W = \alpha\int_0^\beta \mathrm{d}u_0\int \mathrm{d}^3u\left\langle\mathcal{A}^I_\alpha(u_0,\vec{u})\varphi_1\dots\varphi_n\right\rangle_W\,. \tag{3.27}$$

Above we considered a Ward identity between the first descendant ψ from the stress tensor multiplet and the diplacement primary $\mathcal{O}$. The respective BWI thus is

$$-\left\langle\mathcal{Q}^N_\beta\left({\psi_{I\alpha}}^{JK}\mathbb{O}^{LM}\right)\right\rangle_{W,T} = \alpha\int_0^\beta \mathrm{d}u_0\int \mathrm{d}^3u\,\left\langle{\psi_{I\alpha}}^{KL}\mathbb{O}^{MN}\mathcal{A}^J_\beta(u_0,\vec{u})\right\rangle_{W,T}\,. \tag{3.28}$$

The integration over the insertion points of the correction $\mathcal{A}$ is non-trivial. It is thus useful to simplify the correlator by taking an operator product expansion between the two operators ${\psi_{I\alpha}}^{KL}$ and $\mathbb{O}^{MN}$. Following [54] such an OPE can be used in a finite temperature setting provided that the operators are within a certain radius of convergence. This will be the focus of the following chapter.

3.1.3 Operator Product Expansion

Operator product expansions are a useful tool in any conformal field theory, especially in the computation of multipoint correlation functions. They allow to express an n−point function by a sum over $(n-1)$−point functions which are generally easier to compute. Generally OPEs can be applied for two operators which share a sphere in spacetime in which no other operator is inserted.

In a conformal field theory with a defect two kinds of OPEs are relevant. One might be interested in a product of either two bulk or two defect operators, respectively. For these cases the operator product expansions of two scalar fields read

$$\mathcal{O}_1(X)\mathcal{O}_2(0) = \sum_{\mathcal{O}_3}\frac{c_{123}}{|X|^{\Delta_1+\Delta_2-\Delta_3}}\mathcal{O}_3(X) + \dots\,. \tag{3.29a}$$

$$\hat{\mathbb{O}}_1(y)\hat{\mathbb{O}}_2(0) = \sum_{\hat{\mathbb{O}}_3}\frac{\hat{c}_{123}}{|y|^{\hat{\Delta}_1+\hat{\Delta}_2-\hat{\Delta}_3}}\hat{\mathbb{O}}_3(y) + \dots\,. \tag{3.29b}$$

Here X are the bulk coordinates while y is on the defect. The c_{ijk} are the CFT structure constants. Here and in many subsequent equations the dots $(\ldots)$ represent regular terms which we will ignore in our calculations. This is a valid limit if $X, y << 1$.
Note, however, that bulk operators can also be expanded in terms of defect operators provided they are close enough to the defect [89]

$$\mathcal{O}(X,y) = \sum_{\hat{\mathbb{O}}} \frac{b_{\mathcal{O}\hat{\mathbb{O}}}}{|X|^{\Delta-\hat{\Delta}}} \hat{\mathbb{O}}(y) + \ldots \, . \tag{3.30}$$

This expansions underlines that a bulk operator which is very close to the defect can be interpreted as an excitation of the defect [89].
In the equations given thus far we suppressed any indices. In the more general case the OPE is restricted by these indices. While in principle any operator can appear in the OPE, the overall indices need to match between the two sides. If this cannot be achieved by including dependencies on spacetime coordinates or invariant tensor structures the respective operator cannot appear in the OPE.
Although finite temperature breaks conformal invariance it is still possible to apply the OPEs in a thermal setting. The OPE is valid in case the operators lie in a sphere with flat interior. Demanding that $x^\mu x_\mu = X^2 + y^2 < \beta^2$ this is in fact the case in the thermal setting [54]. See also Figure 1.3 with more details on the OPE in finite temperature given in the introduction, Subsection 1.2.1.

3.1.3.1 Representation theory

Before we consider the details of the operator product expansions it is useful to review some relevant aspect of 1d Representation theory [85] .
We originally study $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with superconformal algebra $\mathfrak{psu}(2,2|4)$. It has 16 Poincaré supercharges Q^I_α, $\bar{Q}^{\dot\alpha}_I$ and the conformal supercharges $S_{I\alpha}$, $\bar{S}^{I\dot\alpha}$ where $I = 1,\ldots,4$ is an index of the $\mathfrak{su}(4)_R$ R-symmetry and $\alpha, \dot\alpha = 1,2$ are spinor indices of $\mathfrak{su}(2)_\alpha$ and $\mathfrak{su}(2)_{\dot\alpha}$, respectively. The bosonic subalgebra consists of the conformal $\mathfrak{su}(2,2)$ and the aforementioned R-symmetry $\mathfrak{su}(4)_R$. The conformal generators are the translations P_μ, the rotations $M_{\mu\nu}$, the dilatation Δ and the special conformal transformations K_μ. The R-symmetry is generated by $R^I{}_J$. For details on commutation relations of the unbroken algebra see Appendix A.
The Wilson loop operator is introduced according to the definition (1.63). For the purpose of this subsection we restrict ourselves to the straight line Wilson loop

$$W_{\text{line}} = \frac{1}{N} \text{tr} \overleftarrow{P} \exp\left[\mathrm{i} \int \mathrm{dt}\, (A_3 - \mathrm{i} n_r \Phi^r) \right] . \tag{3.31}$$

Due to the introduction of the $\frac{1}{2}$–BPS Wilson loop defect the superconformal algebra $\mathfrak{psu}(2,2|4)$ breaks to the subalgebra $\mathfrak{osp}(4^*|4)$. This preserved subalgebra has an $\mathfrak{su}(2)_{\text{rot}} \oplus$

$\mathfrak{usp}(4)_R$ R-symmetry. Generic representations are labeled by $[j]^{(R_1,R_2)}_\Delta$ where $j \in \mathbb{N}$ is the spin quantum number of $\mathfrak{su}(2)_{\text{rot}}$ while R_1, R_2 are the Dynkin labels of the $\mathfrak{usp}(4)_R$ representations. There is a special isomorphism for this algebra, $\mathfrak{usp}(4) \simeq \mathfrak{so}(5)$. Therefore, we clarify our definition $(1,0) = \mathbf{4}$ as the $\mathfrak{usp}(4)_R$ fundamental while $(0,1) = \mathbf{5}$. Off the original 16 Poincaré supercharges 8 are preserved and likewise 8 conformal supercharges are preserved. Those are

$$\mathcal{Q}^I_\alpha = \frac{1}{\sqrt{2}}\left(Q^I_\alpha + \Omega^{IJ}\sigma^3_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha}_J\right) \ , \quad \mathcal{S}^I_\alpha = -\frac{1}{\sqrt{2}}\left(\Omega^{IJ}S_{J\alpha} - \sigma^3_{\alpha\dot\alpha}\bar{S}^{I\dot\alpha}\right) \ . \tag{3.32}$$

The symplectic metric Ω^{IJ} is the $\mathfrak{usp}(4)$ singlet we introduced in (1.66). Furthermore, the conformal symmetry along the Wilson line ($\Delta,\ P_3,\ K_3$) and rotations M_{mn} orthogonal to the defect (with $m,n = 0,1,2$) are preserved. The preserved R-symmetry generator is the symmetric [83]

$$\mathcal{R}^{IJ} = \frac{1}{2}\left(R^I{}_K\Omega^{KJ} + R^J{}_K\Omega^{KI}\right) \ . \tag{3.33}$$

The $\mathfrak{osp}(4^*|4)$ algebra is thus given by

$$\begin{aligned}
&[\Delta, P_3] = -P_3 \ , \quad [\Delta, K_3] = +K_3 \ , \quad [P_3, K_3] = 2\Delta \ , \\
&[M_{mn}, \Delta] = 0 \ , \quad [M_{mn}, P_3] = 0 \ , \quad [M_{mn}, K_3] = 0 \ , \\
&\left[P_3, \mathcal{Q}^I_\alpha\right] = 0 \ , \quad \left[K_3, \mathcal{S}^I_\alpha\right] = 0 \ , \\
&\left[\Delta, \mathcal{Q}^I_\alpha\right] = -\frac{1}{2}\mathcal{Q}^I_\alpha \ , \quad \left[\Delta, \mathcal{S}^I_\alpha\right] = -\frac{1}{2}\mathcal{S}^I_\alpha \ , \\
&\left[M_{mn}, \mathcal{Q}^I_\alpha\right] = \mathrm{i}(\sigma_{mn})_\alpha{}^\beta\mathcal{Q}^I_\beta \ , \quad \left[M_{mn}, \mathcal{S}^I_\alpha\right] = \mathrm{i}(\sigma_{mn})_\alpha{}^\beta\mathcal{S}^I_\beta \ , \\
&\left[\mathcal{R}^{IJ}, \mathcal{Q}^K_\alpha\right] = \Omega^{KI}\mathcal{Q}^J_\alpha + \Omega^{KJ}\mathcal{Q}^I_\alpha \ , \quad \left[\mathcal{R}^{IJ}, \mathcal{S}^K_\alpha\right] = -\Omega^{KI}\mathcal{S}^J_\alpha - \Omega^{KJ}\mathcal{Q}^I_\alpha \ , \\
&\left\{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\right\} = 2\Omega^{IJ}P_3\epsilon_{\alpha\beta} \ , \quad \left\{\mathcal{S}^I_\alpha, \mathcal{S}^J_\beta\right\} = 2\Omega^{IJ}K_3\epsilon_{\alpha\beta} \ , \\
&\left\{\mathcal{Q}^I_\alpha, \mathcal{S}^J_\beta\right\} = \Omega^{IJ}\left(\sigma^m\bar\sigma^n\right)_{\alpha\beta}M_{mn} + \Omega^{IJ}\epsilon_{\alpha\beta}\Delta + \frac{3}{2}\epsilon_{\alpha\beta}\mathcal{R}^{IJ} \ .
\end{aligned} \tag{3.34}$$

Note that the supercharges transform in the fundamental representation of $\mathfrak{su}(2)_{\text{rot}} \oplus \mathfrak{usp}(4)_R$. The original algebra of $\mathcal{N} = 4$ SYM without the defect is given in Appendix A while the broken algebra given above is derived in detail in Appendix B.

Let us consider a generic long multiplet $\mathcal{L} : [j]^{(R_1,R_2)}_\Delta$ labeled by its primary state. It is generated by the action of the 8 supercharges $\mathcal{Q}^I_\alpha$. By definition all 8 supercharges $\mathcal{S}^I_\alpha$ annihilate the primary state. Short multiplets are obtained if the primary is also annihilated by a certain combination of supercharges. These shortening conditions are summarized in table 3.1. Most of the short multiplets can be recombined into long ones at

Multiplet	Unitarity bound	Shortening condition	BPS
$\mathcal{L} : [j]_\Delta^{(R_1,R_2)}$	$\Delta > R_1 + R_2 + \frac{j}{2} + 1$	-	-
$\mathcal{A}_1 : [j]_\Delta^{(R_1,R_2)}$	$\Delta = R_1 + R_2 + \frac{j}{2} + 1$	$\mathcal{Q}_2^1$	$\frac{1}{8}$
$\mathcal{A}_1 : [j]_\Delta^{(0,R_2)}$	$\Delta = R_2 + \frac{j}{2} + 1$	$\mathcal{Q}_2^1\,,\ \mathcal{Q}_2^2$	$\frac{1}{4}$
$\mathcal{A}_1 : [j]_\Delta^{(0,0)}$	$\Delta = \frac{j}{2} + 1$	$\mathcal{Q}_2^I$	$\frac{1}{2}$
$\mathcal{A}_2 : [0]_\Delta^{(R_1,R_2)}$	$\Delta = R_1 + R_2 + 1$	$\mathcal{Q}_2^1\mathcal{Q}_1^1$	$\frac{1}{8}$
$\mathcal{A}_2 : [0]_\Delta^{(0,R_2)}$	$\Delta = R_2 + 1$	$\mathcal{Q}_2^1\mathcal{Q}_1^1\,,\ \mathcal{Q}_2^2\mathcal{Q}_1^1 + \mathcal{Q}_2^1\mathcal{Q}_1^2$	$\frac{1}{4}$
$\mathcal{A}_2 : [0]_\Delta^{(0,0)}$	$\Delta = 1$	$\mathcal{Q}_2^I\mathcal{Q}_1^1 + \mathcal{Q}_2^1\mathcal{Q}_1^I$	$\frac{1}{2}$
$\mathcal{B}_1 : [0]_\Delta^{(R_1,R_2)}$	$\Delta = R_1 + R_2$	$\mathcal{Q}_\alpha^1$	$\frac{1}{4}$
$\mathcal{B}_1 : [0]_\Delta^{(0,R_2)}$	$\Delta = R_2$	$\mathcal{Q}_\alpha^1\,,\ \mathcal{Q}_\alpha^2$	$\frac{1}{2}$

Table 3.1: Long and short multiplets of $\mathfrak{osp}(4^*|4)$.

threshold $\Delta_* = R_1 + R_2 + \frac{j}{2} + 1$ following the recombination rules

$$\mathcal{L} : [j]_{\Delta\to\Delta_*}^{(R_1,R_2)} = \mathcal{A}_1 : [j]_{\Delta_*}^{(R_1,R_2)} \oplus \mathcal{A}_1 : [j-1]_{\Delta_*+\frac{1}{2}}^{(R_1+1,R_2)}\,, \tag{3.35}$$

$$\mathcal{L} : [0]_{\Delta\to\Delta_*}^{(R_1,R_2)} = \mathcal{A}_2 : [0]_{\Delta_*}^{(R_1,R_2)} \oplus \mathcal{B}_1 : [0]_{\Delta_*+1}^{(R_1+2,R_2)}\,. \tag{3.36}$$

From this it follows readily that the multiplets $\mathcal{B}_1 : [0]_\Delta^{(R_1,R_2)}$ with $R_1 \leq 1$ are absolutely protected.

3.1.3.2 Bulk OPE

We want to simplify the correlator we obtained from the broken Ward identity,

$$\langle {\psi_{I\alpha}}^{JK}(X_m, 0)\mathbb{O}^{LM}(0, y)\mathcal{A}_\beta^N(u_m, u_3)\rangle_W\,.$$

Let us look at the bulk operator ${\psi_{I\alpha}}^{JK}(X_m, 0)$ and write it as defect excitations in the limit close to the defect, $X_m \to 0$

$${\psi_{I\alpha}}^{JK}(X_m, 0) = \sum_{\hat{\mathbb{O}}} \frac{b_{\psi\hat{\mathbb{O}}}}{X^{\frac{5}{2}-\hat{\Delta}}} \left[C \cdot \hat{\mathbb{O}}(0)\right]_{I\alpha}^{JK} + \ldots\,, \quad \text{for } X \to 0\,. \tag{3.37}$$

Here C is a placeholder for any invariant tensor necessary to match all indices. Just like the two-point functions in the above ward identity this OPE is valid for the operators close to each other. We therefore only focus on the **divergent terms in the limit** $|X| << 1$, the terms that blow up. Due to the convergence of the OPE the regular terms can be omitted. This can only be guaranteed if we allow solely defect operators with $\hat{\Delta} < 5/2$. Following the above discussion this OPE is valid for finite temperature. The reason is that for $y < \beta$ the thermal OPE condition $\tau^2 + x^2 + y^2 < \beta$ is obeyed [54].

Let us discuss which defect operators satisfy the constraints. There are several multiplets on the defect which have primaries and descendants with conformal dimension $\hat{\Delta} < 5/2$, see the general multiplets in table 3.1 in Subsection 3.1.3.1 [85].
For a better understanding let us consider only the displacement multiplet for the moment. It consists of the three fields $\mathbb{D}^m, \mathbb{\Lambda}^I_\alpha$ and $\mathbb{O}^{IJ}$ defined through Equations (2.8). The terms coming from the displacement multiplet thus are

$$\psi_{I\alpha}{}^{JK}(|X|_m, 0) \supset \frac{b_{\psi\mathbb{D}}}{|X|^{\frac{1}{2}}}\mathbb{D}^m\Omega^{JK}C_{Im\alpha} + \frac{b_{\psi\mathbb{\Lambda}}}{|X|}\left(\delta_I^K\mathbb{\Lambda}^J_\alpha - \delta_I^J\mathbb{\Lambda}^K_\alpha + 3\Omega_{IO}\Omega^{JK}\mathbb{\Lambda}^O_\alpha\right)$$
$$+ \frac{b_{\psi\mathbb{O}}}{|X|^{\frac{3}{2}}}\left(\mathbb{O}^{JK}C_{I\alpha} + (\text{further structures with } I, J, K)\right) . \tag{3.38}$$

In the above equation, we are missing fitting structures for the identity and the operators $\mathbb{D}$ and $\mathbb{O}$. Especially the open index α cannot be reconstructed by any known structures. We furthermore cannot have any $\mathfrak{usp}(4)_R$ structure with one or three open indices[2]. Therefore the operators $\mathbb{D}$ and $\mathbb{O}$ cannot appear in the OPE. There neither is an invariant tensor with one spinor index, nor one one with a single $\mathrm{Sp}(4)_R$ symmetry index. This observation is crucial to identify further operators which cannot contribute. Concretely we need to have $\hat{\Delta} < 5/2$, $\hat{j} = 1, 3, \ldots$ and matching $\mathfrak{usp}(4)_R$ indices.
Let us now look at the general case and consider all multiplets that possibly can contribute. We need to check which multiplets are in the preserved $\mathfrak{osp}(4^*|4)$ algebra. Following [85] (compare also table 3.1) the relevant short multiplets (including the primary field) are

$$\mathcal{A}_1 : \ [j]_\Delta^{(R_1,R_2)} , \quad \Delta = \frac{1}{2}j + R_1 + R_2 + 1 , \tag{3.39}$$

$$\mathcal{A}_2 : \ [0]_\Delta^{(R_1,R_2)} , \quad \Delta = R_1 + R_2 + 1 , \tag{3.40}$$

$$\mathcal{B}_1 : \ [0]_\Delta^{(R_1,R_2)} , \quad \Delta = R_1 + R_2 . \tag{3.41}$$

Here j is the spin of the preserved $\mathfrak{su}(2)$ while the R_i are the Dynkin labels of the R-symmetry representation, see paragraph 3.1.3.1 for details.
Let us consider spinor indices. The operator $\psi_{I\alpha}^{KL}$ has one spinor index and thus $j = 1$. There are no invariant tensors with odd spinor indices and therefore any operator contributing to the OPE must have an odd number of spinor indices. Odd spins of a representation yield half-integer scaling dimensions and we therefore have the constraint

$$\Delta < \frac{5}{2} , \quad \Delta \in \frac{\mathbb{Z}}{2}\backslash\mathbb{Z} \quad \Rightarrow \quad \Delta_{\text{primary}} \leq \frac{3}{2} \quad \Rightarrow \quad \Delta = \frac{1}{2} \text{ or } \Delta = \frac{3}{2} . \tag{3.42}$$

[2]To be precise only even numbers of open indices can work because we construct from the symplectic metric Ω_{IJ}.

This means with the above arguments we can exclude multiplets whose primary has dimension 2. These operators cannot have the correct spinor index structure and their descendants will have too high conformal dimensions. The only conformal dimension allowed for operators contributing to the OPE thus are $1/2$ and $3/2$. These can be the dimensions of primary or descendant operators. Consequentially, all multiplets which might give a contribution to the OPE are

$$\mathcal{A}_1 : [1]^{(0,0)}_{\Delta=3/2} , \quad \mathcal{A}_2 : [0]^{(0,0)}_{\Delta=1} , \quad \mathcal{B}_1 : [0]^{(0,0)}_{\Delta=0} , \quad \mathcal{B}_1 : [0]^{(0,1)}_{\Delta=1} , \quad \mathcal{B}_1 : [0]^{(1,0)}_{\Delta=1} . \tag{3.43}$$

Let us go through these multiplets one at a time.
The primary of $\boldsymbol{\mathcal{A}_1 : [1]^{(0,0)}_{\Delta=3/2}}$ is in a singlet of $\mathfrak{usp}(4)_R$ and thus cannot appear in the OPE. For all descendants the other quantum numbers clearly will not match. First descendants are $[2]^{(R_1,R_2)}_{\Delta=2}$. Independently of the R-symmetry charges these descendants have even spin and thus two spinor indices. This cannot reproduce the single spinor index we need for the OPE. Higher descendants already will have a too large conformal dimension. The identity $\boldsymbol{\mathcal{B}_1 : [0]^{(0,0)}_{\Delta=0}}$ will not contribute. The multiplet $\boldsymbol{\mathcal{B}_1 : [0]^{(1,0)}_{\Delta=1}}$ has as first descendant $[1]^{(0,1)}_{\Delta=3/2}$ which is in the **5** representation of $\mathfrak{usp}(4)_R$ and thus does not contribute. The first mentioned displacement multiplet $\boldsymbol{\mathcal{B}_1 : [0]^{(0,1)}_{\Delta=1}}$ yields a contribution from the operator $\mathbb{\Lambda}^I_\alpha$. The multiplet $\boldsymbol{\mathcal{A}_1 : [0]^{(0,0)}_{\Delta=1}}$ is the Konishi multiplet $\mathcal{K}$ on the defect has a descendant $[1]^{(1,0)}_{\Delta=3/2}$ which is an operator $\mathbb{K}^I_\alpha$ with matching indices and thus appears in the OPE. The descendant of the Konishi operator is called $\mathbb{K}$ to ensure a clean notation. Higher descendants with $\Delta \geq 2$ will either have an even number of spinor indices or a conformal dimension that is too high to match our condition.

Both of the fields λ and $\mathbb{K}$ are in a representation $[1]^{(1,0)}_{\Delta=3/2}$. Let us consider the primaries of the respective multiplets $\mathcal{B}_1 : [0]^{(0,1)}_{\Delta=1}$ and $\boldsymbol{\mathcal{A}_1 : [0]^{(0,0)}_{\Delta=1}}$. The displacement scalar contains the scalar fields which are not coupled to the Wilson loop

$$\mathbb{O}^{IJ} = \mathrm{i}\Phi^r \left(\Sigma^{IJ}_r - n_s \Sigma^{IJ}_s n_r\right) . \tag{3.44}$$

In the literature [39, 102–105] a preferred direction is often chosen for the Wilson loop coupling to the scalars by setting $n_r \Phi^r = \Phi^6$ in (1.63). In this case we can write the displacement scalar

$$\mathbb{O}' \sim \Phi^a , \tag{3.45}$$

with $a = 1, \ldots, 5$ the R-symmetry indices of the bulk scalars. The prime was introduced to distinguish the operator from the one we defined in our calculations. The primary of the Konishi multiplet in this setting is

$$\mathcal{K}' \sim \Phi^6 . \tag{3.46}$$

In our general notation the Konishi primary is

$$\mathcal{K}^{IJ}_{\mathcal{K}} = \mathrm{i} n_s \Sigma^{IJ}_s n_r \Phi^r \ . \tag{3.47}$$

The Konishi operator is in an unprotected multiplet $\mathcal{A}_2 : \ [0]^{(0,0)}_{\Delta=1}$ which can combine with a multiplet $\mathcal{B}_1 : \ [0]^{(2,0)}_{\Delta=2}$ to the long multiplet $\mathcal{L} : \ [0]^{(0,0)}_{\Delta\to 1}$. The relation for the recombination rules are given in Equation (3.35). Therefore the Konishi multiplet can get an anomalous dimension. It has been computed in strong and weak coupling [106, 107]

$$\Delta_{\mathcal{K}} = 1 + \gamma_{\mathcal{K}} = 1 + \frac{\lambda}{4\pi^2} - \frac{\lambda^2}{16\pi^4} + \left(-\frac{56}{45}\pi^4 + 128\right)\frac{\lambda^3}{4096\pi^6} + \mathcal{O}(\lambda^4) \ , \tag{3.48a}$$

$$\Delta_{\mathcal{K}} = 1 + \gamma_{\mathcal{K}} = 2 - \frac{5}{\sqrt{\lambda}} + \frac{295}{24}\frac{1}{\lambda} - \frac{305}{16}\frac{1}{\lambda^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \ . \tag{3.48b}$$

The displacement multiplet is, in contrast, absolutely protected [85]. Note that we found the constraint $\Delta < 5/2$ for operators appearing in the OPE. As $\gamma_{\mathcal{K}} < 1$ for all values of the coupling the Konishi fermion does indeed appear in the OPE. We can furthermore write the displacement and Konishi operator in terms of the original spinor fields λ^I_α and $\bar{\lambda}^{\dot{\alpha}}_I$ projected onto the defect[3] by the limit $X \to 0$

$$\mathbb{A}^I_\alpha = \lambda^I_\alpha|_{X\to 0} - \Omega^{IJ}\sigma^3_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_J|_{X\to 0} \ , \quad \mathbb{K}^I_\alpha = \lambda^I_\alpha|_{X\to 0} + \Omega^{IJ}\sigma^3_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_J|_{X\to 0} \ . \tag{3.50}$$

This clearly shows the separation into fields orthogonal and longitudinal to the defect.

Then we find for the OPE

$$\begin{aligned}\psi_{I\alpha}^{\ JK}(|X|_m, 0) = {}& \frac{b_{\psi\mathbb{A}}}{|X|}\left(\delta^K_I \mathbb{A}^J_\alpha - \delta^J_I \mathbb{A}^K_\alpha + 3\Omega_{IO}\Omega^{JK}\mathbb{A}^O_\alpha\right) \\ & + \frac{b_{\psi\mathbb{K}}}{|X|^{1-\gamma_{\mathcal{K}}}}\left(\delta^K_I \mathbb{K}^J_{i\alpha} - \delta^J_I \mathbb{K}^K_{i\alpha} + 3\Omega_{IO}\Omega^{JK}\mathbb{K}^O_{i\alpha}\right) + \dots \ .\end{aligned} \tag{3.51}$$

The operator in the Broken Ward Identity (3.28) therefore simplifies to

$$\begin{aligned}\langle \psi_\alpha(X_m, 0)\mathbb{O}(0, y)\mathcal{A}_\beta(u_m, u_3)\rangle_W = {}& \frac{b_{\psi\mathbb{A}}}{|X|}\langle \mathbb{A}_\alpha(0)\mathbb{O}(0, y)\mathcal{A}_\beta(u_m, u_3)\rangle_W \\ & + \frac{b_{\psi\mathbb{K}}}{|X|^{1-\gamma_{\mathcal{K}}}}\langle \mathbb{K}_\alpha(0)\mathbb{O}(0, y)\mathcal{A}_\beta(u_m, u_3)\rangle_W + \dots \ ,\end{aligned} \tag{3.52}$$

[3] The action of the preserved supercharges on the two operators is as follows

$$\mathcal{Q}^I_\alpha \mathbb{A}^J_\beta = -4\mathrm{i}\Omega^{IJ}\left(\sigma^{3m}\right)_\alpha^{\ \gamma}\epsilon_{\gamma\beta}\mathbb{D}_m - 2\mathrm{i}\epsilon_{\alpha\beta}\mathcal{D}_3\mathbb{O}^{IJ} \tag{3.49a}$$

$$\mathcal{Q}^I_\alpha \mathbb{K}^J_\beta = -2\left(\sigma^{mn}\right)_{\alpha\beta}F_{mn} - 8\left[\Phi^{LM}\Omega_{LM}, \Phi^{IJ}\right]\epsilon_{\alpha\beta} + 8\mathrm{i}\left(\sigma^m\bar{\sigma}^3\right)_{\alpha\beta}D_m\Phi^{IJ} - 2\mathrm{i}\left(\sigma^3\bar{\sigma}^\mu\right)_{\alpha\beta}D_\mu\Phi^{IJ} \tag{3.49b}$$

where it is important to note that $m, n = 0, 1, 2$ denote directions orthogonal to the defect.

where for simplicity we omitted the $\mathfrak{usp}(4)R$-symmetry structures.
Let us check that the above OPE satisfies the constraints on ${\psi_{I\alpha}}^{KL}$ as given in table 2.1. The above expression is manifestly antisymmetric in the indices K and L. Also the tracelessness condition is obeyed as can be checked readily:

$$
\begin{aligned}
0 = {\psi_{I\alpha}}^{IK}(|X|_m, 0) &= \frac{b_{\psi\mathbb{\Lambda}}}{|X|}\left(\delta_I^K \mathbb{\Lambda}_\alpha^I - \delta_I^I \mathbb{\Lambda}_\alpha^K + 3\Omega_{IO}\Omega^I K \mathbb{\Lambda}_\alpha^O\right) + [\mathbb{\Lambda} \to \mathbb{K}_\gamma] + \dots \\
&= \frac{b_{\psi\mathbb{\Lambda}}}{|X|}\left(\mathbb{\Lambda}_\alpha^K - 4\mathbb{\Lambda}_\alpha^K + 3\mathbb{\Lambda}_\alpha^K\right) + [\mathbb{\Lambda} \to \mathbb{K}_\gamma] + \dots = 0 + \dots .
\end{aligned} \tag{3.53}
$$

3.1.3.3 Defect OPE

In the previous section we expressed the bulk operator ψ in terms of the defect operator $\mathbb{\Lambda}$ and the operator $\mathbb{K}$. Thus we can now use the defect OPE between each of these operators and $\mathbb{O}$. The constraints we will get from the limit of the operators close to each other will be very similar to before as also the quantum numbers are largely identical. The main difference to the previous OPE we will find is coming from the anomalous dimension of the Konishi multiplet. This will allow for operators with higher dimension which did not appear before.
With this calculation we are in fact projecting the operators ψ and $\mathbb{O}$ directly on top of each other. This means we are in fact calculating an OPE between a bulk and a defect operator by expanding it in two subsequent OPEs. Therefore we in fact do not have to expect any inconsistencies from the convergence and the finiteness of the tails. For details on OPE convergence see [108].
First consider the OPE between $\mathbb{\Lambda}$ and $\mathbb{O}$. From the above formula we find

$$
\mathbb{\Lambda}_\alpha^K(0)\mathbb{O}^{LM}(y) = \sum_{\hat{\mathcal{O}}} \frac{\hat{c}_{\mathbb{\Lambda}\mathbb{O}\hat{\mathcal{O}}}}{y^{\frac{5}{2}-\hat{\Delta}}}\left[C \cdot \hat{\mathcal{O}}(y)\right]_\alpha^{KLM} + \dots , \quad \text{for } y \to 0 . \tag{3.54}
$$

As before the limit $y \to 0$ ensures that the product is within the radius of convergence $\tau^2 + x^2 + y^2 < \beta$ of the thermal OPE [54]. The power on the spacetime coordinate is $5/2 - \hat{\Delta}$ as in the first OPE yielding the same constraint on dimensions of fields in the OPE. The $\mathfrak{usp}(4)_R$ representation of the operator product, following table 2.1,

$$
[0,1] \otimes [1,0] = [1,0] \oplus [1,1] , \tag{3.55}
$$

which is precisely the representation of ψ on the defect. Therefore both operator product expansions are identical, only $\mathbb{\Lambda}$ and $\mathbb{K}$ are in the OPE

$$
\begin{aligned}
\mathbb{\Lambda}_\alpha^K(0)\mathbb{O}^{LM}(y) = &\frac{\hat{c}_{\mathbb{\Lambda}\mathbb{O}\mathbb{\Lambda}}}{y}\left(\Omega^{KL}\mathbb{\Lambda}_\alpha^M(y) - \Omega^{KM}\mathbb{\Lambda}_\alpha^L(y) + \frac{1}{2}\Omega^{LM}\mathbb{\Lambda}_\alpha^K(y)\right) \\
&+ \sum_i \frac{\hat{c}_{\mathbb{\Lambda}\mathbb{O}\mathbb{K}}}{y^{1-\gamma_i}}\left(\Omega^{KL}\mathbb{K}_{i\alpha}^M(y) - \Omega^{KM}\mathbb{K}_{i\alpha}^L(y) + \frac{1}{2}\Omega^{LM}\mathbb{K}_{i\alpha}^K(y)\right) + \dots .
\end{aligned} \tag{3.56}
$$

With this expression we indeed find the constraints from table 2.1

$$
\begin{aligned}
0 &= \Omega_{LM}\mathbb{A}^K_\alpha(0)\mathbb{O}^{LM}(y) \\
&= \frac{\hat{c}_{\mathbb{A}\mathbb{O}\mathbb{A}}}{y}\left(\Omega_{LM}\Omega^{KL}\mathbb{A}^M_\alpha(y) - \Omega_{LM}\Omega^{KM}\mathbb{A}^L_\alpha(y) + \frac{1}{2}\Omega_{LM}\Omega^{LM}\mathbb{A}^K_\alpha(y)\right) + [\mathbb{A} \leftrightarrow \mathbb{K}_\gamma] + \ldots \\
&= \frac{\hat{c}_{\mathbb{A}\mathbb{O}\mathbb{A}}}{y}\left(-\mathbb{A}^K_\alpha(y) - \mathbb{A}^K_\alpha(y) + \frac{4}{2}\mathbb{A}^K_\alpha(y)\right) + [\mathbb{A} \leftrightarrow \mathbb{K}_\gamma] + \ldots = 0 + \ldots\,. \qquad (3.57)
\end{aligned}
$$

In a similar way we can find the OPE between $\mathbb{O}$ and $\mathbb{K}_i$. The anomalous dimension $\gamma_{\mathbb{K}}$ of $\mathbb{K}$ is positive and smaller than 1, see Equation (3.48). Thus we find

$$
\mathbb{K}^K_\alpha(0)\mathbb{O}^{LM}(y) = \sum_{\hat{\mathcal{O}}} \frac{\hat{c}_{\mathbb{A}\mathbb{O}\hat{\mathcal{O}}}}{y^{\frac{5}{2}+\gamma_{\mathcal{K}}-\hat{\Delta}}}\left[C\cdot\hat{\mathcal{O}}(y)\right]^{KLM}_\alpha + \ldots\,, \quad \text{for } y \to 0\,. \qquad (3.58)
$$

We therefore see that the constraint for this OPE is $\hat{\Delta} < \frac{5}{2} + \gamma_{\mathcal{K}}$. The operators with $\mathbb{A}$ and $\mathbb{K}$ will thus contribute to the OPE analogous to the above case. With the same argument as made around Equation (3.42) only operators with odd dimensions may contribute. Differently to the above OPE, here also operators from short multiplets with $\Delta = 5/2$ can contribute. Higher conformal dimensions are not allowed because $\gamma_{\mathcal{K}} < 1$. Let us gather all multiplets which might have contributions from table 3.1 [85].

$$
\begin{aligned}
&\mathcal{A}_1: [1]^{(0,0)}_{\Delta=3/2}\,,\quad \mathcal{A}_2: [0]^{(0,0)}_{\Delta=1}\,,\quad \mathcal{B}_1: [0]^{(0,0)}_{\Delta=0}\,,\quad \mathcal{B}_1: [0]^{(0,1)}_{\Delta=1}\,,\quad \mathcal{B}_1: [0]^{(1,0)}_{\Delta=1}\,,\\
&\mathcal{A}_1: [2]^{(0,0)}_{\Delta=2}\,,\quad \mathcal{A}_1: [1]^{(1,0)}_{\Delta=5/2}\,,\quad \mathcal{A}_1: [1]^{(0,1)}_{\Delta=5/2}\,,\quad \mathcal{A}_2: [0]^{(1,0)}_{\Delta=2}\,,\quad \mathcal{A}_2: [0]^{(0,1)}_{\Delta=2}\,,\\
&\mathcal{B}_1: [0]^{(0,2)}_{\Delta=2}\,,\quad \mathcal{B}_1: [0]^{(1,1)}_{\Delta=2}\,,\quad \mathcal{B}_1: [0]^{(2,0)}_{\Delta=2}\,.
\end{aligned} \qquad (3.59)
$$

From these multiplets we find several representations of operators which might have the correct index structures. We list all primaries and descendants which might appear at $\Delta = 5/2$ and have matching spin $j = 1, 3$.

$$
\begin{aligned}
&[j]^{(1,0)}_{\Delta=5/2}\,,\quad [j]^{(2,0)}_{\Delta=5/2}\,,\quad [j]^{(3,0)}_{\Delta=5/2}\,,\quad [j]^{(0,1)}_{\Delta=5/2}\,,\quad [j]^{(0,2)}_{\Delta=5/2}\,,\\
&[j]^{(1,1)}_{\Delta=5/2}\,,\quad [1]^{(2,1)}_{\Delta=5/2}\,,\quad [1]^{(1,2)}_{\Delta=5/2}\,.
\end{aligned} \qquad (3.60)
$$

Most of these representations correspond to operators that are part of several of the above multiplets. All of them are in a multiplet which can eventually recombine with another short multiplet to a long one. As mentioned before, to correctly connect spinor indices with invariant tensors we need to have half integer spin operators (yielding an odd number of spinor indices) which boils down to operators with $j = 1, 3$.

Not all of these representations can contribute to the final result. Before we go into details of matching the indices on both sides of the OPE, we take a step forward. The resulting operator will be in a two-point function with an operator in the $(1,0)$ representation of $\mathfrak{usp}(4)_R$. Furthermore the R-symmetry group has a subalgebra $\mathfrak{usp}(4)_R \supset \mathfrak{su}(2) \otimes \mathfrak{u}(1)$ [109]. Only if the combined representation have a charge 0 associated to the U(1) the corresponding two-point function can be non-zero. Therefore we only have potential contributions from

$$[j]^{(1,0)}_{\Delta={}^5\!/_2}\,, \quad [j]^{(3,0)}_{\Delta={}^5\!/_2}\,, \quad [j]^{(1,1)}_{\Delta={}^5\!/_2}\,, \quad [1]^{(1,2)}_{\Delta={}^5\!/_2}\,. \tag{3.61}$$

where $j = 1, 3$.

The operators in the fundamental representation **(1,0)** have one R-symmetry index and are similar to all representations above.

The representation **(3,0)** has dimension 20. This comes from one free index and two anti-symmetric ones yielding an operator $\mathbb{D}^{IJK} = \mathbb{D}^{I[JK]}$. Such an operator has $4 \times 6 = 24$ independent components. Together with the four trace-like conditions $\Omega_{JK}\mathbb{D}^{IJK} = 0$ we indeed find 20 independent components.

The representation **(1,1)** has two independent indices and hence dimension 16. Operators from this represenntation cannot match the number of indices on the left-hand side of the OPE.

Finally, operators in the **(1,2)** representation have three indices where two of them are symmetric. The dimension thus is 10×4=40. Although the number of free indices matches, such operators cannot contribute. We need to have anti-symmetry in two indices from the operator $\mathbb{O}^{IJ}$ in the OPE and symmetry in two of the three indices from the representation. Such a combination is impossible with non-zero coefficients and hence this representation also does not contribute.

Let us now discuss spinor indices. Consider an operator in the $[3]^{(1,0)}_{\Delta={}^5\!/_2}$ representation. We can call this operator $\mathbb{C}^I_{\alpha\dot{\beta}\beta}$. In the OPE we just have a single open spinor index and therefore need to contract. This is usually done with $x^\mu \bar{\sigma}_\mu^{\beta\dot{\beta}}$. The OPE we consider, however, is localised on the defect and thus the only x_μ that may contribute is the coordinate y. Therefore the relevant contraction is $y\left(\sigma_3\bar{\sigma}_3\right)^{\alpha\beta} = y\epsilon^{\alpha\beta}$ which is anti-symmetric. Contracted with the symmetric operator this yields zero.

The operators which can additionally contribute in the OPE thus are

$$[1]^{(1,0)}_{\Delta=\frac{5}{2}} \to \mathbb{\Psi}^N_{j\,\alpha}\,, \quad [1]^{(3,0)}_{\Delta=\frac{5}{2}} \to \hat{\mathbb{\Psi}}^{LMN}_{j\,\alpha}\,. \tag{3.62}$$

Each of these operators can come from several multiplets and hence we gave them an additional index j which we sum over in the OPE. Similarly these operators might have anomalous dimensions γ_j. The full OPE between $\mathbb{K}$ and $\mathbb{O}$ in the limit $y \to 0$ is then given by

$$
\begin{aligned}
\mathbb{K}_\alpha^L(0)\mathbb{O}^{MN}(y) = & \frac{\hat{c}_{\mathbb{A}\mathbb{O}\mathbb{K}}}{y^{1+\gamma_\mathbb{K}}}\left(\Omega^{LM}\mathbb{A}_\alpha^N(y) - \Omega^{LN}\mathbb{A}_\alpha^M(y) + \frac{1}{2}\Omega^{MN}\mathbb{A}_\alpha^L(y)\right) \\
& + \frac{\hat{c}_{\mathbb{K}\mathbb{O}\mathbb{K}}}{y}\left(\Omega^{LM}\mathbb{K}_\alpha^N(y) - \Omega^{LN}\mathbb{K}_\alpha^M(y) + \frac{1}{2}\Omega^{MN}\mathbb{K}_\alpha^L(y)\right) \\
& + \sum_j \frac{\hat{c}_{\mathbb{K}\mathbb{O}\Psi}}{y^{\gamma_\mathbb{K}-\gamma_j}}\left(\Omega^{LM}\Psi_{j\,\alpha}^N(y) - \Omega^{LN}\Psi_{j\,\alpha}^M(y) + \frac{1}{2}\Omega^{MN}\Psi_{j\,\alpha}^L(y)\right) \\
& + \sum_j \frac{\hat{c}_{\mathbb{K}\mathbb{O}\hat{\Psi}}}{y^{\gamma_\mathbb{K}-\gamma_j}}\hat{\Psi}_{j\,\alpha}^{LMN} + \ldots\,.
\end{aligned}
\tag{3.63}
$$

As mentioned multiple times before we focus on the divergent terms only, sending all other terms to the tails as they are not relevant in the limit $y \to 0$. This yields that we can further constrain the anomalous dimension of the newly introduced contributions to

$$
0 \le \gamma_j < \gamma_\mathbb{K} < 1\,. \tag{3.64}
$$

3.1.4 Thermal correction

The zero temperature Ward identity (2.16) is broken at finite temperature and zero coupling to (3.28). We are able to simplify the three-point thermal correlator to a sum over two-point functions. Therefore we can use the Operator Product Expansions (3.51), (3.56), (3.63). Still there is a spacetime integral over the insertions of the thermal mass operator $\mathcal{A}$ which needs to be computed. At first glance it seems difficult to constrain these integrals as complex thermal effects have to be taken care of and although the two-point function in the zero temperature limit could be optained from [89] the extension to finite temperature is non-trivial. As we would want an expression for all values of T a Taylor expansion is not feasible and the introduction of the temperature as a scale parameter allows for a dependence on the massless combination $T|u_\mu|$ as the argument of any arbitrary function. It is, however, possible to further constrain the two-point functions of the thermal corrections after using the OPE. They are correlators of two fermionic operators each carrying one spinor and one or three R-symmetry indices. We want to make the index structure of these correlators explicit. Here we profitfromthe preservation of R-symmetry in the $g = 0$ case. Turning on a coupling the R-symmetry group will eventually break. Here, the $\mathfrak{usp}(4)_R$ symmetry group preserved by the defect comes equipped with a symplectic metric Ω^{IJ}. This singlet can be used to raise and lower R-symmetry indices [83]. Therefore the correlators will naturally depend on it. Let us now look at the spinor indices. We have a correlator of two fermionic operators $\langle \mathbb{A}_\alpha \mathcal{A}_\beta \rangle_W$ which has two left spinor indices. There is only one possible general combination for the numerator then. Namely

$$
\langle \mathbb{A}_\alpha^I(0)\mathcal{A}_\beta^J(u_\mu)\rangle_W \propto \left[u_\mu \sigma^\mu \bar{\sigma}^3\right]_{\alpha\beta}\,. \tag{3.65}
$$

This makes sense as for example the operator $\langle\psi_\alpha \mathbb{A}_\beta\rangle_W$ has an identical spinor structure [38, 89] and it is the only structure allowed here. One might argue that the epsilon tensor $\epsilon_{\alpha\beta}$ should also be in the above, however it is in fact already there because $(\sigma^3\bar{\sigma}^3)_{\alpha\beta} = \epsilon_{\alpha\beta}$. Further note that $\sigma^m\bar{\sigma}^3 = -\sigma^3\bar{\sigma}^m$ for $m = 0, 1, 2$ and therefore the above is the only combination we need to care about. Any more complex structure would eventually reduce to the contraction (3.65). Following [38, 89] and in comparison to the correlator $\langle\psi^I_\alpha \mathbb{A}^J_\beta\rangle_W$ we can fix

$$\left\langle \mathbb{A}^I_\alpha(0)\mathcal{A}^J_\beta(u_\mu)\right\rangle_W = u_\mu(\sigma^\mu\bar{\sigma}^3)_{\alpha\beta}\Omega^{IJ} f_{\mathbb{A}}(u_\mu^2)\,, \tag{3.66}$$

$$\begin{aligned}
\left\langle \mathbb{K}^I_\alpha(0)\mathcal{A}^J_\beta(u_\mu)\right\rangle_W &= \Omega^{IJ} u_\mu(\sigma^\mu\bar{\sigma}^3)_{\alpha\beta} f_{\mathbb{K}}(u_\mu^2)\,,\\
\left\langle \Psi^I_{j\,\alpha}(0)\mathcal{A}^J_\beta(u_\mu)\right\rangle_W &= \Omega^{IJ} u_\mu(\sigma^\mu\bar{\sigma}^3)_{\alpha\beta} f_{\Psi}(u_\mu^2)\,,\\
\left\langle \hat{\Psi}^{LMN}_{j\,\alpha}(0)\mathcal{A}^J_\beta(u_\mu)\right\rangle_W &= u_\mu(\sigma^\mu\bar{\sigma}^3)_{\alpha\beta}\left(\Omega^{LM}\Omega^{NJ} - \Omega^{LN}\Omega^{MJ} + \frac{1}{2}\Omega^{MN}\Omega^{LJ}\right) f_{\hat{\Psi}\mathcal{A}}(u_\mu^2)\,.
\end{aligned} \tag{3.67}$$

The last correlator is anti-symmetric in M, N and satisfies the trace condition $\Omega_{MN}\hat{\Psi}^{LMN}_{j\,\alpha}$. These two conditions are met by the above combination of the symplectic metric Ω_{IJ}.

The R-symmetry indices can then be simplified by using the OPEs (3.51), (3.56) and (3.63). The resulting structure $\mathfrak{C}_I{}^{JKLMN}$ is the same as the one from the original zero temperature Ward identity, Equation (2.34). The thermal Ward identity (recall the zero temperature Ward identity 2.33) including corrections can be simplified to

$$\begin{aligned}
&(3h - B)\lim_{X,y\to 0}\left[g^{(1)}_{T\neq 0}(X,y)\epsilon_{\alpha\beta} + g^{(2)}_{T\neq 0}(X,y)\hat{X}_{\alpha\beta} + g^{(3)}_{T\neq 0}(X,y)\hat{X}_{\alpha\beta}\right]\mathfrak{C}_I{}^{JKLMN}\\
&= (\sigma^\mu\bar{\sigma}^3)_{\alpha\beta}\Bigg[m\left(\frac{b_{\psi\mathbb{A}}\hat{c}_{\mathbb{A}\mathbb{O}\mathbb{A}}}{|X|\,y} + \frac{b_{\psi\mathbb{K}}\hat{c}_{\mathbb{K}\mathbb{O}\mathbb{A}}}{|X|^{1-\gamma_{\mathbb{K}}}\,y^{1+\gamma_{\mathbb{K}}}}\right)\int_0^\beta \mathrm{d}u_0\int \mathrm{d}^3u\, u_\mu\, f_{\mathbb{A}}(u_\mu^2)\\
&+ \alpha\left(\frac{b_{\psi\mathbb{A}}\hat{c}_{\mathbb{A}\mathbb{O}\mathbb{K}}}{|X|\,y^{1-\gamma_{\mathbb{K}}}}` + \frac{b_{\psi\mathbb{K}}\hat{c}_{\mathbb{K}\mathbb{O}\mathbb{K}}}{|X|^{1-\gamma_{\mathbb{K}}}\,y}\right)\int_0^\beta \mathrm{d}u_0\int \mathrm{d}^3u\, u_\mu\, f_{\mathbb{K}}(u_\mu^2)\\
&+ \alpha\sum_j \frac{b_{\psi\mathbb{K}}\hat{c}_{\mathbb{K}\mathbb{O}\Psi}}{|X|^{1-\gamma_{\mathbb{K}}}\,y^{\gamma_{\mathbb{K}}-\gamma_j}}\int_0^\beta \mathrm{d}u_0\int \mathrm{d}^3u\, u_\mu\, f_{\Psi}(u_\mu^2)\\
&+ \alpha\sum_j \frac{b_{\psi\mathbb{K}}\hat{c}_{\mathbb{K}\mathbb{O}\hat{\Psi}}}{|X|^{1-\gamma_{\mathbb{K}}}\,y^{\gamma_{\mathbb{K}}-\gamma_j}}\int_0^\beta \mathrm{d}u_0\int \mathrm{d}^3u\, u_\mu\, f_{\hat{\Psi}}(u_\mu^2)\Bigg]\mathfrak{C}_I{}^{JKLMN} + \ldots\,.
\end{aligned} \tag{3.68}$$

This furthermore clearly shows that the result of the integrals is a function of β only. The functions f are quadratic in the u_μ. This can be seen straight forwardly at zero temperature as the functions f will be mainly denominators which depend on the squared coordinates. In the thermal case the temperature introduced allows for more general terms in principle. On the other hand, the temperature does not allow for new combinations of a single coordinate. Especially for the coordinates u_1 and u_2 rotational symmetry is preserved hence allowing only for a dependence on $u_1^2 + u_2^2$. Therefore we are interested in the following integral. After the integration is carried out it will only depend on the temperature

$$\int_0^\beta \mathrm{d}u_0 \int \mathrm{d}^3u \left[u_0\sigma^0\bar{\sigma}^3 + u_1\sigma^1\bar{\sigma}^3 + u_2\sigma^2\bar{\sigma}^3 + u_3\sigma^3\bar{\sigma}^3\right]_{\alpha\beta} f_{\mathbb{\Lambda}}(u_m^2, u_3^2)\ . \tag{3.69}$$

The spatial integrations over the full $\mathbb{R}^3$ are symmetric and therefore cancel the anti-symmetric integrands. Only the term with u_0 can have a non-vanishing contribution. Thus define

$$\mathfrak{I}^{\mathbb{\Lambda}}(\beta) = \int_0^\beta \mathrm{d}u_0 \int \mathrm{d}^3u\, u_0\, f_{\mathbb{\Lambda}}(u_0^2, u^2, u_3^2)\ . \tag{3.70}$$

The above integral then reads

$$\begin{aligned}\int_0^\beta \mathrm{d}u_0 \int \mathrm{d}^3u\, u_\mu(\sigma^\mu\bar{\sigma}^3)_{\alpha\beta} f_{\mathbb{\Lambda}\mathcal{A}}(u_\mu^2) &= \int_0^\beta \mathrm{d}u_0 \int \mathrm{d}^3\vec{u}\ u_0(\sigma^0\bar{\sigma}^3)_{\alpha\beta} f_{\mathbb{\Lambda}\mathcal{A}}(u_0^2, u_i^2) \\ &=: (\sigma^0\bar{\sigma}^3)_{\alpha\beta}\mathfrak{I}_{\mathbb{\Lambda}\mathcal{A}}(\beta)\ . \end{aligned}\tag{3.71}$$

The arguments presented here for $\mathfrak{I}^{\mathbb{\Lambda}}$ hold analogously for all other integrals and we can always take out a factor $u_0(\sigma^0\bar{\sigma}^3)_{\alpha\beta}$. Therefore we finally have the correction to the zero temperature Ward identity (2.33). We can then write the new thermal Ward identity (3.68) as

$$\begin{aligned}(3h - B) \lim_{X,y\to 0} &\left[g_1(X,y)\epsilon_{\alpha\beta} + g_2(X,y)\hat{X}_{\alpha\beta} + g_3(X,y)\hat{X}_{\alpha\beta}\right]^{(T\neq 0)} \\ &= \alpha(\sigma^0\bar{\sigma}^3)_{\alpha\beta} G(X,y,\beta) \end{aligned}\tag{3.72}$$

$$\begin{aligned} G(X,y,\beta) = \Bigg[&\left(\frac{b_{\psi\mathbb{\Lambda}}\hat{c}_{\mathbb{\Lambda}\mathbb{O}\mathbb{\Lambda}}}{|X|\,y} + \frac{b_{\psi\mathbb{K}}\hat{c}_{\mathbb{K}\mathbb{O}\mathbb{\Lambda}}}{|X|^{1-\gamma_{\mathbb{K}}}\,y^{1+\gamma_{\mathbb{K}}}}\right)\mathfrak{I}^{\mathbb{\Lambda}}(\beta) + \sum_j \frac{b_{\psi\mathbb{K}}\hat{c}_{\mathbb{K}\mathbb{O}\hat{\mathbb{\Psi}}}}{|X|^{1-\gamma_{\mathbb{K}}}\,y^{\gamma_{\mathbb{K}}-\gamma_j}}\mathfrak{I}^{\mathbb{\Psi}_j}(\beta) \\ &+ \left(\frac{b_{\psi\mathbb{\Lambda}}\hat{c}_{\mathbb{\Lambda}\mathbb{O}\mathbb{K}}}{|X|\,y^{1-\gamma_{\mathbb{K}}}} + \frac{b_{\psi\mathbb{K}}\hat{c}_{\mathbb{K}\mathbb{O}\mathbb{K}}}{|X|^{1-\gamma_{\mathbb{K}}}\,y}\right)\mathfrak{I}^{\mathbb{K}_j}(\beta) + \sum_j \frac{b_{\psi\mathbb{K}}\hat{c}_{\mathbb{K}\mathbb{O}\hat{\mathbb{\Psi}}}}{|X|^{1-\gamma_{\mathbb{K}}}\,y^{\gamma_{\mathbb{K}}-\gamma_j}}\mathfrak{I}^{\hat{\mathbb{\Psi}}_j}(\beta)\Bigg]\ , \end{aligned}$$

Recall the original definition of the kinematic functions (2.13)

$$\begin{aligned} g_1^{(T=0)}(X,y) &:= \tfrac{\mathrm{i}y}{6\pi|X|(X^2+y^2)^2}\ , \\ g_2^{(T=0)}(X,y) &:= \tfrac{-1}{6\pi(X^2+y^2)^2}\ , \\ g_2^{(T=0)}(X,y) &:= \tfrac{1}{12\pi X^2(X^2+y^2)}\ . \end{aligned}\tag{3.73}$$

and the spinor structure

$$\hat{X}_{\alpha\beta} := \frac{X_m}{|X|}\left(\sigma^m\bar{\sigma}^3\right)_{\alpha\beta} . \tag{3.74}$$

Furthermore note that if we would have introduced the method of images it might have changed the integral. This is by applying method of images to the u_0 in particular. However, this change could only yield a different prefactor for the integrals and is not able to change the result (3.72) as a whole.

Consider that the index structure of the whole Ward identity can be captured by the following matrix.

$$C_\mu\left(\sigma^\mu\bar{\sigma}^3\right)_{\alpha\beta} = C_\mu\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{3\,\dot{\alpha}\gamma}\epsilon_{\gamma\beta} = -\begin{pmatrix} C_1 - \mathrm{i}C_2 & C_0 - C_3 \\ C_0 + C_3 & C_1 + \mathrm{i}C_2 \end{pmatrix}_{\alpha\beta} . \tag{3.75}$$

where the C_μ are any prefactors. However, due to the integration the only allowed structure for the thermal correction was $C_0\left(\sigma^0\bar{\sigma}^3\right)_{\alpha\beta}$. Therefore **the components in the diagonal of the above matrix must vanish independently from the thermal correction**. This is one reason for our result $B = 3h$.
To make this more precise let us contract the Ward identity with the matrix combination

$$\epsilon^{\beta\gamma}\sigma^1_{\gamma\dot{\gamma}}\bar{\sigma}^{3\,\dot{\gamma}\alpha} . \tag{3.76}$$

Note that this yields

$$\left(\sigma^\mu\bar{\sigma}^3\right)_{\alpha\beta}\left(\sigma^1\bar{\sigma}^3\right)^{\beta\alpha} = -2\delta^\mu_1 . \tag{3.77}$$

and thus for the Ward identity (3.72)

$$(3h - B)\lim_{X,y\to 0}\frac{X_1}{|X|}\left[g^{(2)}_{T\neq 0}(X,y) + g^{(3)}_{T\neq 0}(X,y)\right] = 0 \tag{3.78}$$

$$\Rightarrow \qquad \boxed{\mathbf{B = 3h}} . \tag{3.79}$$

In conclusion we showed that the exact relation between the bremsstrahlung of a moving particle in $\mathcal{N} = 4$ SYM and the coefficient of the stress tensor holds at finite temperature in the OPE limit.

3.2 Perturbative check

In the previous part we provided an algebraic proof for the relation

$$B = 3h\,, \tag{3.80}$$

between the bremsstrahlung and the stress tensor coefficient at finite temperature and at zero coupling. In this chapter we will provide a "perturbative" check for the above relation. We explicitly see that in perturbation theory there exists scheme which allows for $B = 3h$. We find such a scheme in an expansion around small temperatures and in a high temperature limit. This suggests that there also exists a respective scheme for any arbitrary temperature.
Strictly speaking in a theory with zero coupling the Wilson loop expectation value must always be one and the bremsstrahlung consequentially zero. For the purpose of this chapter we keep the coupling in the Wilson loop operator. We are thus allowing for scalars and vector fields to couple to the Wilson loop. This will yield a contribution as presented in Figure 1.1a at leading order in a perturbative expansion. At the first subleading order we would find a contribution as in Figure 1.1b. The diagrams of Figures 1.1c and 1.1d contain vertices from the action which are not allowed at zero coupling. However, these two diagrams are also contributing at the first subleading order in a weak coupling expansion. Henceforth our ansatz is only consistent at leading order in λ which is thus what we will consider in this chapter. We gather all higher order corrections coming from Figures 1.1b, 1.1c and 1.1d as well as even higher orders writing

$$\langle W\rangle = 1 + \lambda\langle W\rangle^{(1)} + \langle W\rangle^{\text{interactions}}\,. \tag{3.81}$$

The "zero coupling" we consider is thus defined as setting all terms of the last piece to zero, $\langle W\rangle^{\text{interactions}} = 0$.
In Chapter 2.2 we provided a perturbative proof for the relation $B = 3h$ at zero temperature. In this chapter we will repeat all the steps while discussing corrections due to thermal effects. The thermal effects then come from the insertion of one bare thermal propagator. While all steps then are a priori straight foreward, some of the integral we need to calculate cannot be obtained from the general expression of the finite temperature. Instead we will expand in both, high and low temperature where appropriate.

Before we consider the circular Wilson loop and the bremsstrahlung, consider thermal effects on the stress tensor. Due to conformal invariance, one-point functions in conformal field theories are always identically zero[4]. This can be seen because there is no relative scale on which the correlator could possibly depend on. The introduction of finite temperature breaks the conformal invariance and introduces such a scale. It can be shown [54] that

[4]Only the identity is an exception to this.

only symmetric traceless tensors with even spin J can aquire non-zero one-point functions which are of the form

$$\langle \mathcal{O}^{\mu_1 \dots \mu_J} \rangle = b_{\mathcal{O}} T^{\Delta_{\mathcal{O}}} \left(\delta_0^{\mu_1} \dots \delta_0^{\mu_J} - \text{traces} \right) \, . \tag{3.82}$$

The stress tensor $T_{\mu\nu}$ is such a symmetric traceless operator. Splitting up the correlator between stress tensor and Wilson line into parts which are connected and disconnected we find

$$\langle T_{33} \rangle_W = \frac{\langle T_{33} W \rangle_{\text{connected}}}{\langle W \rangle} + \langle T_{33} \rangle \, . \tag{3.83}$$

Let us look more deeply into the structure of the stress energy tensor expectation value. In the case without a Wilson loop insertion and zero temperature we have a one-point function in full $\mathcal{N} = 4$ which is zero

$$\langle T_{\mu\nu} \rangle (r) = 0 \, , \tag{3.84}$$

while in the thermal case without defect the result was derived by [54], see Equation (3.82)

$$\langle T_{\mu\nu} \rangle_T (r) = b_T \, T^4 \left(\delta_\mu^0 \delta_\nu^0 - \frac{1}{4} \eta_{\mu\nu} \right) \, . \tag{3.85}$$

Let us derive the coefficient b_T from the theory. The energy density E is given by the time component of the stress tensor [54]

$$E = -\langle T_{00} \rangle = -(1 - \frac{1}{4}) b_T = -\frac{3}{4} b_T \, . \tag{3.86}$$

The energy E must be positive and thus $b_T < 0$. This derivation can be compared to the free energy density which we consider as a thermodynamic ensemble equation

$$F = E - TS = E + T \frac{dF}{dT} = f T^4 \, . \tag{3.87}$$

The last equality follows from dimensional analysis and f is an a priori unknown coefficient. We introduce it for convenience of the calculation. Combining the last two equations yields that the stress stensor coefficient can be obtained from the free energy [54]

$$f = \frac{b_T}{4} < 0 \qquad \Rightarrow \qquad b_T = \frac{4F}{T^4} \, . \tag{3.88}$$

Therefore we should focus on the computation of the free energy of $\mathcal{N} = 4$ SYM. It can be derived from usual statistical physics. Recall the definitions of the partition function and free energy [47, 101]

$$\mathcal{Z} = \int [\mathrm{d}\phi] \exp\left(- \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^3 x \, \mathcal{L} \right) , \quad F = -T \log(\mathcal{Z}) \, . \tag{3.89}$$

In small coupling perturbation theory the partition function $\mathcal{Z}$ is computed by vacuum bubbles. The leading order one is simply a single propagator connected with itself [47, 53]. The free energy for free scalars, free dirac fermions and QCD are derived in [101].

$$F^{\text{free scalar}} = \quad T \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \log \left[1 - \exp\left(- \frac{|k|}{T} \right) \right] = - \quad \frac{\pi^2}{90} T^4 \, , \tag{3.90}$$

$$F^{\text{free fermion}} = -2T \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \log \left[1 + \exp\left(- \frac{|k|}{T} \right) \right] = - \frac{7}{4} \frac{\pi^2}{90} T^4 \, , \tag{3.91}$$

$$F^{\text{QCD}} = - \frac{\pi^2}{90} T^4 \left[2(N_c^2 - 1) + \frac{7}{4} N_c N_f \right] \, . \tag{3.92}$$

For the fermions we include a minus sign due to the fermionic loop and use Fermi-Dirac statistics instead of the Bose-Einstein statistic used in the scalar case. The additional factor of 2 for the fermions is yielded by the two components of the Weyl spinor. Note that this factor 2 is different in [101] where Dirac fermions are considered which have four components. The QCD results includes the number of fermions (flavor charges) N_f and the number of color charges N_c.

We consider $\mathcal{N} = 4$ SYM which consists of 6 real scalars, 4 fermions and a gauge field each in the adjoint representation and $N_c = N$ color charges. We use the large N limit with $N^2 - 1 \sim N^2$. Therefore we find in agreement with [71]

$$F^{\mathcal{N}=4} = - \frac{\pi^2}{90} T^4 \left[2N^2 + \frac{7}{4} 4N^2 + 6N^2 \right] = - \frac{15\pi^2}{90} N^2 T^4 = - \frac{\pi^2}{6} N^2 T^4 \, . \tag{3.93}$$

Thus using $b_T = {}^{4F}/_{T^4}$

$$b_T^{\mathcal{N}=4} = - \frac{2\pi^2}{3} N^2 \, . \tag{3.94}$$

We should compare this to the zero temperature result of the stress tensor one-point function in the presence of the Wilson loop,

$$\langle T_{33} \rangle_W = \frac{g^2 N}{48 \pi^2 r^4} \, . \tag{3.95}$$

In the large N limit which we are considering, the thermal coefficient is thus of higher order than the zero temperature expactation value. We can assume that the thermal one-point function is not relevant for the interaction energy h. The coefficient b_T is a constant parameterizing the interaction between the stress tensor and the thermal vacuum while for h instead the interaction with the Wilson line defect is considered. Therefore, we will exclude b_T from further calculations and consider

$$h = r^4 \frac{\langle T_{33} W \rangle_{\text{connected}}}{\langle W \rangle} =: r^4 \langle T_{33} \rangle'_{W,\beta} \ . \tag{3.96}$$

In an abuse of notation let us drop the prime in all following steps and assume that due to the argument given here we do not consider disconnected diagrams for the calculation of the stress-tensor.

Bare Thermal Propagator

As mentioned above the main difference in the thermal theory compared to zero temperature will arise from the propagator. This is especially true for the zero coupling case we consider here. Thermal propagators are usually [47, 53, 100] defined in momentum space for either the real time or the imaginary time formalism. We first focus on the Fourier transformation of the real time propagator and show that it is equivalent to the Fourier transform of the imaginary time one.
Introducing finite temperature to a quantum field theory means to compactify the time component on a circle with the radius being the inverse temperature $\beta = 1/T$. Calculations can be done in the imaginary time (Matsubara) formalism where the periodicity of the time component is satisfied by introducing discrete frequencies $p_0 \to \mathrm{i}\omega_n,\ n \in \mathbb{Z}$. To obtain kinetic behaviors the propagators need to be analytically continued. Alternatively one can also introduce the real time (Kheldysh) formalism keeping the kinetic behavior throughout the process. However the real time formalism leads to a doubling of degrees of freedom. If the system under consideration is in a thermal equilibrium both formalism are equivalent. For a review of these formalisms see the textbooks [47, 53, 100] or the lecture notes [101]. As the calculations are more hands-on and we have defined the Wilson loop to be orthogonal to the time cicle, we will mostly work in the Matsubara formalism. Let us however start by considering the Keldysh formalism and show for the bare thermal propagator that they are equal in both formalisms.
As mentioned above the real time formalism leads to a doubling of the degrees of freedom. For all the fields the real time formalism yields a splitting $\varphi \to \begin{Bmatrix} \varphi^+ \\ \varphi^- \end{Bmatrix}$ where $\varphi^\pm$ are independent fields. Similarly the new Lagrangian becomes $\mathcal{L}(\varphi^\pm) = \mathcal{L}(\varphi^+) - \mathcal{L}(\varphi^-)$. The propagator of a simple massless scalar field thus has a matrix structure with four components

$$\Delta^{++} = \frac{1}{p^2 + i\epsilon} + 2\pi i n_B(|p_0|)\delta(p^2) \ , \tag{3.97a}$$

$$\Delta^{+-} = 2\pi i \left[\Theta(-p_0) + n_B(|p_0|)\right] \delta(p^2) \ , \tag{3.97b}$$

$$\Delta^{-+} = 2\pi i \left[\Theta(p_0) + n_B(|p_0|)\right] \delta(p^2) \ , \tag{3.97c}$$

$$\Delta^{--} = \frac{-1}{p^2 - i\epsilon} + 2\pi i n_B(|p_0|)\delta(p^2)\ . \tag{3.97d}$$

$n_B(p) = \left(e^{\beta p} - 1\right)^{-1}$ is the Bose-Einstein distribution which for Fermions gets replaced by the Fermi-Dirac distribution $n_F(p) = -\left(e^{\beta p} + 1\right)^{-1}$. We denote the respective fermionic propagators by $\tilde{\Delta}^{\pm\pm}$. The propagators for the fields of $\mathcal{N} = 4$ SYM become in Feynman gauge

$$\begin{aligned}
\langle A^a_\mu A^b_\nu \rangle^{\pm\pm}(p) &= \delta^{ab}\left(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}\right)\Delta^{\pm\pm}\ , \quad \langle \bar{c}^a c^b \rangle^{\pm\pm}(p) = \delta^{ab}\Delta^{\pm\pm}\ ,\\
\langle \Phi^a_I \Phi^b_J \rangle^{\pm\pm}(p) &= \delta^{ab}\delta_{IJ}\Delta^{\pm\pm}\ , \quad \langle \lambda^a_I \bar{\lambda}^b_J \rangle^{\pm\pm}(p) = -2i\delta^{ab}\delta_{IJ}\sigma^\mu p_\mu \tilde{\Delta}^{\pm\pm}\ .
\end{aligned} \tag{3.98}$$

It is worth noting at this point that the finite temperature part of all four components of the propagator is the same and that they satisfy the constraint $\Delta^{++} + \Delta^{--} = \Delta^{+-} + \Delta^{-+}$. As the Lagrangian factorizes into parts with plus and parts with minus presign it is easy to obtain the vertex rules from the ones given in Equations (J.2). Any of these vertices will be present twice where once all legs carry a plus sign and the other time all carry a minus sign. The former ones stay equal while the latter ones obtain an addition minus sign due to the one in front of the Lagrangian.

The Wilson loop will similarly also get two different contributions. It now reads

$$W(C) = \frac{1}{N}\mathrm{tr}\overleftarrow{P}\exp\left[\frac{g}{2}\oint_C \mathrm{dt}\left(iA^+_\mu \dot{x}^\mu + \Phi^+_I|\dot{x}|\Theta^I\right) + \frac{g}{2}\oint_C \mathrm{dt}\left(iA^-_\mu \dot{x}^\mu + \Phi^-_I|\dot{x}|\Theta^I\right)\right]\ . \tag{3.99}$$

From the above discussion it is clear that the diagrammatic expansion will be the same as in Figure 2.1 at leading order, however, now every vertex (internal and the ones directly on the loop) can be labeled by either $''+''$ or $''-''$ and we need to sum all possibilities. When multiplying out the expansion we find all four propagators $\Delta^{\pm\pm}$ contributing. In fact the zero temperature propagator $^1/_{4\pi^2(x_1-x_2)^2}$ gets replaced by the real time propagators while the rest of the calculations stays the same.

$$\begin{aligned}
\langle W \rangle_\beta = 1 + \frac{\lambda}{8}\int_0^{2\pi}\mathrm{dt}_1\int_0^{\mathrm{t}_1}\mathrm{dt}_2\left(|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2\right)[\Delta^{++}(x_1-x_2)+\\
+\Delta^{--}(x_1-x_2) + \Delta^{+-}(x_1-x_2) + \Delta^{-+}(x_1-x_2)] + \mathcal{O}(\lambda^2)
\end{aligned} \tag{3.100}$$

$$\begin{aligned}
= 1 + \frac{\lambda}{4}\int_0^{2\pi}\mathrm{dt}_1\int_0^{\mathrm{t}_1}\mathrm{dt}_2\left(|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2\right)\times\\
[\Delta^{++}(x_1-x_2) + \Delta^{--}(x_1-x_2)] + \mathcal{O}(\lambda^2)\ .
\end{aligned} \tag{3.101}$$

In the last step we used the above relation $\Delta^{++} + \Delta^{--} = \Delta^{+-} + \Delta^{-+}$. In Equations (3.97) these propagators are given in momentum space. Therefore we will calculate the Fourier transform to coordinate space. Therefore consider that using $\lim_{\varepsilon\to 0}\left(\frac{1}{p^2+i\varepsilon}-\frac{1}{p^2-i\varepsilon}\right)=\delta(p^2)$:

$$\Delta^{++}+\Delta^{--}=-2\pi\mathrm{i}\,\delta(p^2)\,(1+2n_B(|p_0|))=-2\pi\mathrm{i}\,\delta(p^2)\coth\left(\frac{|p_0|}{2T}\right). \tag{3.102}$$

Noting that $\Theta(-p_0)+\Theta(p_0)=1$ it is easy to see that $\Delta^{+-}+\Delta^{-+}$ will yield the same expression.

We can now start with the Fourier transformation. The delta function makes the integral over the time component trivial

$$(\Delta^{++}+\Delta^{--})(x)=-2\pi\mathrm{i}\int\frac{\mathrm{d}^4p_M\;e^{\mathrm{i}p\cdot x}}{(2\pi)^4}\delta(p^2)\coth\left(\frac{|p_0|}{2T}\right) \tag{3.103}$$

$$=2\pi\int\frac{\mathrm{d}^4p_E\;e^{p_0x_0-\mathrm{i}\vec{p}\vec{x}}}{(2\pi)^4}\delta(p^2)\coth\left(\frac{|p_0|}{2T}\right) \tag{3.104}$$

$$=\int\frac{\mathrm{d}^3p\;e^{-\mathrm{i}\vec{p}\vec{x}}}{(2\pi)^3}\frac{1}{2p}\coth\left(\frac{p}{2T}\right)\left[e^{px_0}+e^{-px_0}\right]\,. \tag{3.105}$$

In the first step we Wick rotated from Minkowski to Euclidean space as indicated by the subscripts M and E, respectively. In this expression $\pm 1/2p$ is the residue of $1/p_0^2-p^2$ at $p_0=\pm p$ which comes from the delta function. As the hyperbolic cotangent function is even we can write the above as

$$(\Delta^{++}+\Delta^{--})(x)=\int\frac{\mathrm{d}^3p\;e^{-\mathrm{i}\vec{p}\vec{x}}}{(2\pi)^3}\mathrm{Res}_{p_0=\pm p}\left[\frac{1}{p_0^2-p^2}\right]\coth\left(\frac{p_0}{2T}\right)e^{p_0x_0}\Big|_{p_0=\pm p}\,. \tag{3.106}$$

This expression is precisely twice the result of a frequency sum according to the imaginary time formalism. Thus we see that the calculation in real and imaginary time is indeed equal. The additional factor of 2 is canceled by the $1/2$ in the exponential of the Wilson loop for real time.

We can then go backwards and rewrite the full original sum which in this case would be

$$(\Delta^{++}+\Delta^{--})(x)=2T\sum_{n=-\infty}^{\infty}e^{i\omega_nx_0}\int\frac{\mathrm{d}^3p\;e^{-\mathrm{i}\vec{p}\vec{x}}}{(2\pi)^3}\frac{-1}{(i\omega_n)^2-p^2}\,, \tag{3.107}$$

where $\omega_n=2\pi i\,nT$ for bosons. As the Wilson loop only couples to bosonic fields, we exclude the fermionic case here. Generally this would lead to different statistics and hyperbolic cotangent functions being replaced by hyperbolic tangent functions. First consider now a more general sum over a function depending on ω_n which will allow us to simplify the above calculation [54]:

$$\sum_{n\in\mathbb{Z}}f(\omega_n)e^{\mathrm{i}\omega_nx_0}=\sum_{n\in\mathbb{Z}}\int\mathrm{d}\omega\;\delta(\omega-\omega_n)f(\omega)e^{\mathrm{i}\omega x_0} \tag{3.108}$$

$$= \sum_{m\in\mathbb{Z}} \int \mathrm{d}\omega \left[\delta(\widetilde{\omega - \omega_n})\right](2\pi m)\, f(\omega) e^{\mathrm{i}\omega x_0} = \frac{1}{T}\sum_{m\in\mathbb{Z}} \int \frac{\mathrm{d}\omega}{2\pi} f(\omega) e^{i\omega(x_0 + m/T)} \tag{3.109}$$

$$= \frac{1}{T}\sum_{m\in\mathbb{Z}} \left[\widetilde{f(\omega)}\right](x_0 + {}^m\!/T)\ . \tag{3.110}$$

From the first to the second line the Poisson resummation formula was used. The factor of 2π in the argument arose because of the different prescription of Fourier transformation. In our case $f(\omega) = {}^1\!/_{\omega^2+p^2}$ and thus in total we need to compute the four dimensional Fourier transform of ${}^1\!/_{P^2}$ which is given by ${}^1\!/_{4\pi^2 x^2}$. Therefore we obtain an expression which can be summed[5]:

$$\boxed{\begin{aligned}\Delta^{T\neq 0} &= \frac{\Delta^{++} + \Delta^{--}}{2} = \frac{1}{4\pi^2} \sum_{m=-\infty}^{\infty} \frac{1}{(x_0 + {}^m\!/T)^2 + |\vec{x}|^2} \\ &= \frac{T}{8\pi|\vec{x}|}\left[\coth\left(\pi T(|\vec{x}| + \mathrm{i}x_0)\right) + \coth\left(\pi T(|\vec{x}| - \mathrm{i}x_0)\right)\right]\ .\end{aligned}} \tag{3.111}$$

This propagator matches the result from [110, 111] and satisfies two important properties. In the zero temperature limit it yields the correct propagator and it is periodic in the inverse temperature β, obeying the Kudo-Martin-Schwinger (KMS) Relation [49, 50]

$$\lim_{T\to 0} \Delta^{T\neq 0} = \frac{1}{4\pi^2\, x^2}\ , \qquad \Delta^{T\neq 0}(x_0) = \Delta^{T\neq 0}\left(x_0 + \frac{1}{T}\right)\ . \tag{3.112}$$

The above result (3.111) is the bare thermal propagator which we shall use throughout the rest of our calculations.

3.2.1 Circular Wilson loop

Let us consider first the expectation value of the circular Wilson loop. We expect it to be related to the bremsstrahlung in an equation very similar to (2.73). Finite temperature effect at leading order will enter through the propagators. The expansion of the WL is equivalent to the zero temperature case,

$$\langle W_{\bigcirc}\rangle = 1 + \frac{\lambda}{2}\int_0^{2\pi} \mathrm{d}\mathrm{t}_1 \int_0^{\mathrm{t}_1} \mathrm{d}\mathrm{t}_2\, (|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2)\, \Delta^{(T\neq 0)}(x_1 - x_2) + \mathcal{O}(\lambda^2)\ , \tag{3.113}$$

with $|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2 = 2R^2 \sin^2\left(\frac{\mathrm{t}_1-\mathrm{t}_2}{2}\right)$. Where from now on we stick to the imaginary time formalism as in this case we do not need to worry about the different fields and as shown above there is no relevant effect on the calculations. The bare propagator now is the finite temperature one derived in Equation (3.111) [110, 111]

$$\Delta^{(T)}(\vec{x}, x_0) = \frac{T}{8\pi|\vec{x}|}\left[\coth\left(\pi T(|\vec{x}| + ix_0)\right) + \coth\left(\pi T(|\vec{x}| - ix_0)\right)\right]\ . \tag{3.114}$$

[5] We include a factor ${}^1\!/_2$ because of the similar factor in the Wilson loop.

As the integral cannot be carried out with this more complex hypergeometric function we have to expand with respect to the temperature. Plugging in the parameterization (1.64) we find

$$\langle W_{\bigcirc} \rangle = 1 + \frac{\lambda(RT)}{8\pi} \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \sin\left(\frac{\mathfrak{t}_1 - \mathfrak{t}_2}{2}\right) \coth\left(2\pi RT \sin\left(\frac{\mathfrak{t}_1 - \mathfrak{t}_2}{2}\right)\right) + \mathcal{O}(\lambda^2) \,. \tag{3.115}$$

We can calculate this integral by a power series expansion in the parameter RT. This will generally yield a power series of terms with $\sin\left(\frac{\mathfrak{t}_1-\mathfrak{t}_2}{2}\right)^n$ where $n \in \mathbb{N}$. For every n these terms can be integrated. In general every such integral will yield a hypergeometric function.

Note also that it makes no sense to expand separately around a certain temperature or radius as the Wilson loop depends only on their product and thus only their respective behavior is of importance. We will start by assuming $RT \to 0$ and later go to $RT \to \infty$. Moreover, we will find the dependence on the combination RT also for the higher orders. This is because $\mathcal{N} = 4$ is a conformal theory at zero temperature and the circular Wilson loop is thus independent of the radius. When switching to finite temperature also the radius of the circular loop becomes a relevant scale. However, the Wilson loop is a dimensionless object and can thus only depend on the dimensionless combination RT.

For small values of RT with some $c \in \mathbb{R}$ we find

$$\coth(cRT) = \frac{1}{cRT} + \frac{cRT}{3} - \frac{c^3(RT)^3}{45} + \mathcal{O}((RT)^5) \,. \tag{3.116}$$

Plugging this expansion into the above integral, we are able to evaluate it for each order of RT and find

$$\langle W_{\bigcirc} \rangle = 1 + \frac{\lambda}{8} + \lambda\frac{(RT)^2\pi^2}{12} - \lambda\frac{(RT)^4\pi^4}{60} + \mathcal{O}((RT)^6) + \mathcal{O}(\lambda^2) \,. \tag{3.117}$$

Note that we reproduced the $T = 0$ term $\langle W_{(a)} \rangle\big|_{T=0} = 1/8$ first calculated by Erickson, Semenoff and Zarembo [62] which is a sanity check of our calculation. Increasing the computation time we are able to evaluate $\langle W_2 \rangle$ up to any order around $RT = 0$. To see the precision of this approximation we can compare it with sample points that are numerically integrated. Figure 3.1 shows this for $RT \leq 0.7$ with the plots of the expansion up to order $\mathcal{O}((RT)^{10})$ and $\mathcal{O}((RT)^{20})$. We can see that both curves nicely fit the sample points for $RT < 0.3$, the higher order one even until $RT < 0.4$ and diverge from it beyond that point.

For large RT, however, the hyperbolic cotangent function grows exponentially fast. Thus the Taylor expansion of the hyperbolic cotangent at $1/x_0 = 0$ (correspondig to $x \to \infty$) up

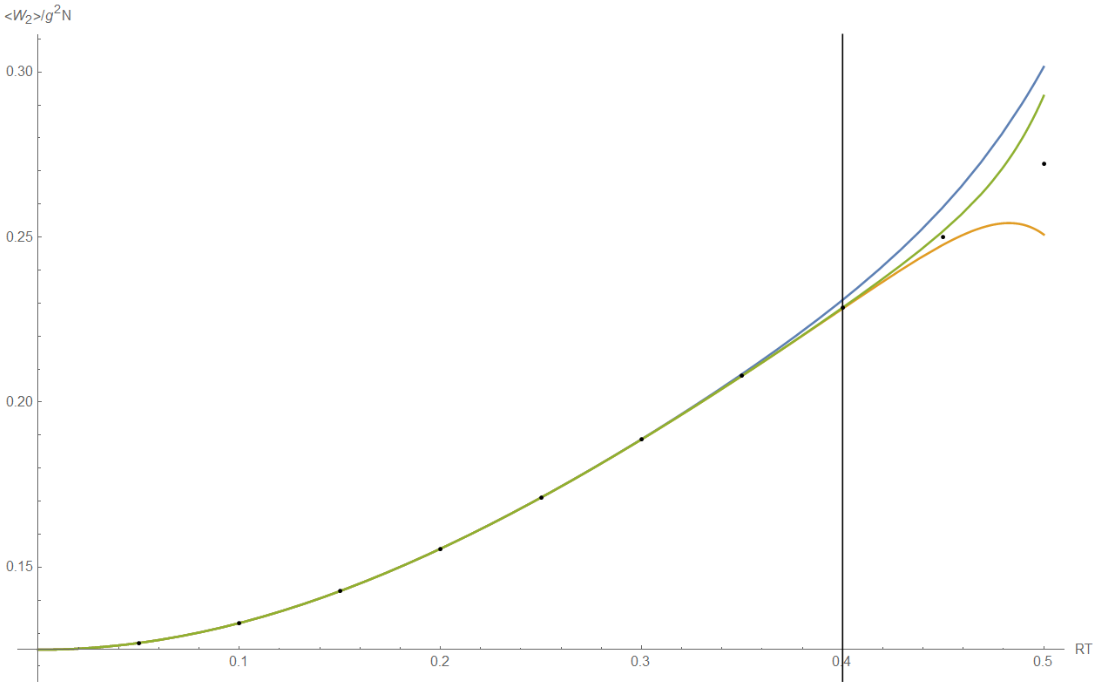


Figure 3.1: Plot of ${\langle W_{\bigcirc}\rangle^{(1)}}/{\lambda}$ for RT between 0 and 0.7. The blue line is the expansion up to order $\mathcal{O}((RT)^{10})$, while the orange one includes all terms of order $\mathcal{O}((RT)^{20})$ while the green line is the next higher order for comparison. In the region where two or three expansions almost overlap only the green line is visible. The black dots represent the numerically integrated values.

to any order n is constant,

$$\text{Taylor}\left[\coth\left(\frac{1}{x_0}\right), x_0 = 0, n\right] = 1 + \mathcal{O}(x^{n+1}) \quad \forall n \in \mathbb{N}\,. \tag{3.118}$$

This is due to the fact that all derivatives of the function vanish close to infinity. To understand how this comes about more clearly, look at the function $\exp({-1}/{x})$ which is very close to the hyperbolic cotangent with argument ${1}/{x}$. In the limit $x \to 0$ this function vanishes. It's first derivative is readily computed to be $({1}/{x^2})\exp({-1}/{x})$ which also vanishes in the limit given. From here it is not hard to see that also all other derivatives will vanish in the limit $x \to 0$ as the exponential function approaches zero faster than any power-like divergence blows up.

For the derivatives of the hyperbolic cotangent the same behavior is obtained although it is not as trivial as for the exponential function. Therefore the only consistent way to work with the hyperbolic cotangent for large RT is to take $\coth(x \to \infty) \approx 1$. We will see in the following steps that this result is a very good prediction for what we find numerically. Inserting this into the full expression (3.115) we find

$$\begin{aligned}\langle W_{(a)}\rangle_\beta\big|_{RT\to\infty} &= \frac{g^2NRT}{8\pi}\int_0^{2\pi}\mathrm{d}\mathfrak{t}_1\int_0^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2\sin\left(\frac{\mathfrak{t}_1-\mathfrak{t}_2}{2}\right) + \mathcal{O}\left((RT)^n\right)\\ &= \frac{g^2N}{2}RT + \mathcal{O}\left((RT)^n\right)\ ,\ \forall n \in \mathbb{N}\,. \end{aligned} \tag{3.119}$$

Thus we predict to find a linear behavior as $RT \to \infty$. The plot in Figure 3.2 shows how

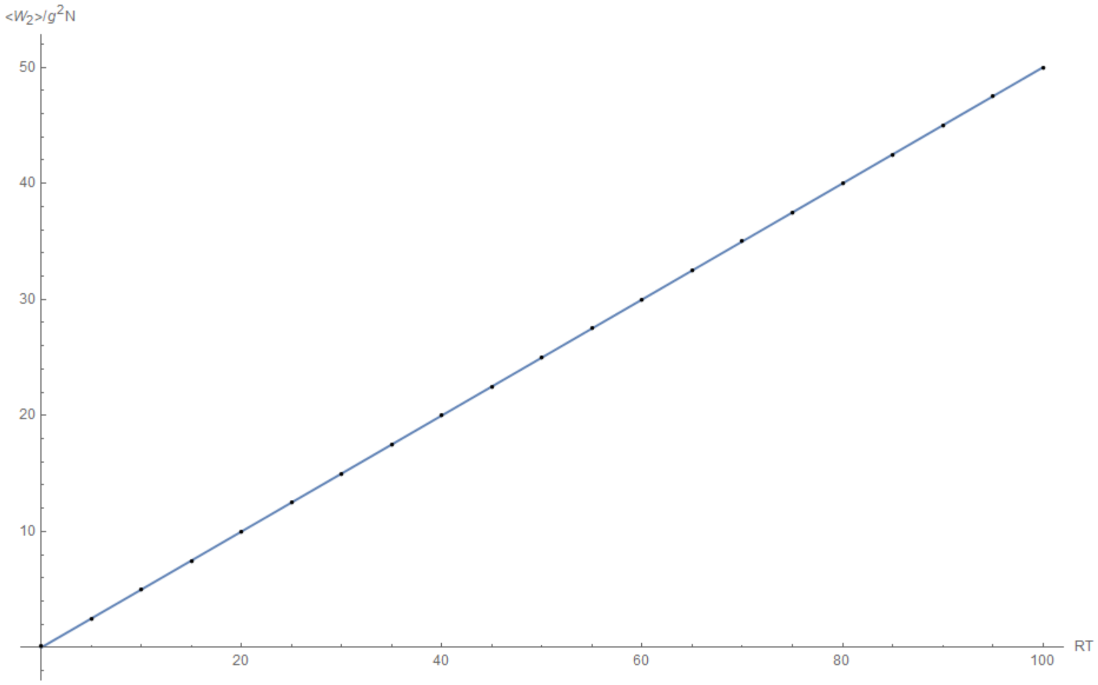


Figure 3.2: Plot of $\langle W_{(a)}\rangle_\beta / g^2 N$ for RT between 0 and 100. The blue line is the linear function $4\pi\, RT$ while the dots indicate several numerical values.

the linear function fits together with the numerical points already for $RT \gtrsim 20$.

All in all we find at leading orders in the two limits for the temperature

$$\langle W_{\bigcirc}\rangle^{(RT\to\infty)} = 1 + \frac{\lambda}{2}\, RT + \mathcal{O}\left(\lambda^2, \frac{\lambda}{(RT)^n}\right) \quad , \qquad \forall n \in \mathbb{N}$$

$$\langle W_{\bigcirc}\rangle^{(RT\to 0)} = 1 + \frac{\lambda}{8} + \lambda\frac{(RT)^2\pi^2}{12} - \lambda\frac{(RT)^4\pi^4}{60} + \mathcal{O}(\lambda^2, \lambda(RT)^6)\,. \tag{3.120}$$

Note that here we consider the limit always for the dimensionless quantity RT on which the Wilson loop naturally depends.
Meanwhile up to this order the straight line remains

$$\langle W_{-}\rangle^{(T)} = 1 + \mathcal{O}(\lambda^2)\,. \tag{3.121}$$

because of the same cancellation as above. This will only change for the self-energy insertions. For more details on the thermal calculation of the straight line consider chapter 5.5. Higher order corrections to the circular WL at finite temperature are calculated in 5.4

3.2.2 Small temperature expansion

3.2.2.1 Bremsstrahlung

Let us go through the derivation of the bremsstrahlung function again and point out where finite temperature effects need to be included.
Therefore start with the 1/4-BPS circular Wilson loop given in the same way as in the zero temperature case:

$$\vec{n} = (n_1, n_2, n_3, 0, 0, 0) \qquad \text{with} \quad n_1 = \sin(\theta)\cos(\mathfrak{t}) \,, \quad n_2 = \sin(\theta)\sin(\mathfrak{t}) \,, \quad n_3 = \cos(\theta)$$

$$C:\; x^\mu(\mathfrak{t}) = (0\,,\, R\cos(\mathfrak{t})\,,\, R\sin(\mathfrak{t})\,,\, 0) \quad , \quad \mathfrak{t} \in [0, 2\pi]\,.$$

The leading order expansion has only a single propagator insertion. While the propagator gets thermal corrections in the manner discussed above, the prefactors at this order are not affected

$$\langle W_\theta \rangle = 1 + \frac{\lambda}{2}\cos^2(\theta) \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 R^2 \left[1 - \cos(\mathfrak{t}_1 - \mathfrak{t}_2)\right] \Delta(x_1 - x_2) + \mathcal{O}(\lambda^2)\,. \tag{3.122}$$

This again is the same contribution as for the 1/2-BPS Wilson loop up to a redefinition $\lambda \to \lambda\cos^2(\theta)$ of the coupling. Note, however, that this only holds at leading order. For the subleading order the gauge and scalar will combine with different prefactors, especially considering the self-energy diagram, and therefore no common $\cos^2(\theta)$ can be pulled out. Hence we find

$$\langle W_\theta \rangle = \langle W_{\theta=0} \rangle \,|_{\lambda \to \lambda\cos^2(\theta)} + \mathcal{O}(\lambda^2)\,. \tag{3.123}$$

and therefore

$$\frac{\langle W_\theta \rangle - \langle W_{\theta=0} \rangle}{\langle W_{\theta=0} \rangle} = -\theta^2 \lambda \partial_\lambda \log \langle W_{\theta=0} \rangle + \mathcal{O}(\lambda^2)\,. \tag{3.124}$$

Once again we then expand the left hand side around small angles θ. This will yield one-point and two-point functions of the scalar on the Wilson line. Recall

$$\begin{aligned} \frac{\langle W_\theta \rangle - \langle W_{\theta=0} \rangle}{\langle W_{\theta=0} \rangle} &= \theta \oint_C \mathrm{d}\mathfrak{t} |\dot{x}| (\partial_\theta n^r(\mathfrak{t}, \theta)) \operatorname{tr} \langle \Phi_r(\mathfrak{t}) \rangle_W + \frac{\theta^2}{2} \oint_C \mathrm{d}\mathfrak{t} |\dot{x}| (\partial_\theta^2 n^r(\mathfrak{t}, \theta)) \operatorname{tr} \langle \Phi_r(\mathfrak{t}) \rangle_W \\ &+ \frac{\theta^2}{2} \oiint_C \mathrm{d}\mathfrak{t}_1 \mathrm{d}\mathfrak{t}_2 |\dot{x}_1||\dot{x}_2| (\partial_\theta n^r(\mathfrak{t}_1))(\partial_\theta n^s(\mathfrak{t}_2)) \operatorname{tr} \langle \Phi_r(\mathfrak{t}_1) \Phi_s(\mathfrak{t}_2) \rangle_W \\ &+ \mathcal{O}(\theta^3)\,. \end{aligned} \tag{3.125}$$

A priori the one-point functions do not have to be zero as we are in a finite temperature setting [54]. However, the fields we are interested in carry a nonzero charge with respect to the R-symmetry and thus still have a vanishing one-point function,

$$\operatorname{tr}\langle\Phi(\mathfrak{t})_r\rangle_W = 0\,. \tag{3.126}$$

Similar to our calculation in Section 3.2 the thermal two-point function of scalars is obtained by the method of images applied to the time component.

$$\begin{aligned}\operatorname{tr}\langle\Phi_r(x)\Phi_s(0)\rangle_W &= \sum_{m=-\infty}^{\infty}\frac{\gamma\delta_{rs}}{\left(x_0+\frac{m}{T}\right)^2+|\vec{x}|^2}\\ &= \frac{\pi\gamma T}{2|\vec{x}|}\delta_{rs}\left[\coth\left(\pi T(|\vec{x}|+\mathrm{i}x_0)\right)+\coth\left(\pi T(|\vec{x}|-\mathrm{i}x_0)\right)\right]\,. \end{aligned}\tag{3.127}$$

Hence for our parameterization (1.64) we find the correlator

$$\operatorname{tr}\langle\Phi_r(x(\mathfrak{t}_1))\Phi_s(x(\mathfrak{t}_2))\rangle_W = \pi\gamma\delta_{rs}T\frac{\coth\left[\pi T\sqrt{2R^2-2R^2\cos(\mathfrak{t}_1-\mathfrak{t}_2)}\right]}{\sqrt{2R^2-2R^2\cos(\mathfrak{t}_1-\mathfrak{t}_2)}}\,. \tag{3.128}$$

The hyperbolic cotangent function with a sine function as an argument could not be integrated analytically. Note also that a full convergent series expansion of the hyperbolic cotangent is only known for $\coth(x)$ with $0<|x|<\pi$. This would yield that an expansion in RT is only valid for $RT<1/2$. We instead consider a Taylor expansion at leading order for small and high temperature:

$$\operatorname{tr}\langle\Phi_r(x)\Phi_s(0)\rangle_W|_{T\to 0} = \frac{\gamma}{x^2}+\frac{\pi^2}{3}\gamma T^2-\frac{\pi^4}{45}\gamma T^4x^4+\mathcal{O}(T^6)\,, \tag{3.129}$$

$$\operatorname{tr}\langle\Phi_r(x)\Phi_s(0)\rangle_W|_{T\to\infty} = \frac{T\pi\gamma}{|\vec{x}|}\,. \tag{3.130}$$

The high temperature limit thus yields a correlator with an x-dependence one would expect to see in three dimensions. This is intuitively explained because the limit $T\to\infty$ sets the radius of the circle on which we compactified the time dimension to zero. This is essentially a Kaluza-Klein reduction. We will therefore address this limit in a later section.

For the small temperature limit the leading term was computed in the zero temperature case. We thus find

$$\begin{aligned}\frac{\langle W_\theta\rangle-\langle W_{\theta=0}\rangle}{\langle W_{\theta=0}\rangle} &= \theta^2\Bigg[-\pi^2\gamma+\frac{1}{2}\int_0^{2\pi}\mathrm{d}\mathfrak{t}_1\int_0^{2\pi}\mathrm{d}\mathfrak{t}_2R^2\cos(\mathfrak{t}_1-\mathfrak{t}_2)\Bigg[\frac{\pi^2}{3}\gamma T^2\\ &\qquad -\frac{\pi^4}{45}\gamma(RT)^4(2-2\cos(\mathfrak{t}_1-\mathfrak{t}_2))^2\Bigg]\Bigg]+\mathcal{O}(\theta^3,\theta^2T^6)\end{aligned}\tag{3.131}$$

$$= -\pi^2\theta^2\gamma+\frac{4\pi^6}{45}\theta^2\gamma(RT)^4+\mathcal{O}(\theta^3,\theta^2T^6)\,. \tag{3.132}$$

Plugging this into the original expressions yields

$$\gamma = \frac{\lambda}{\pi^2 - \frac{4\pi^6}{45}(RT)^4 + \mathcal{O}(T^6)} \partial_\lambda \log\langle W_\bigcirc \rangle \ . \tag{3.133}$$

Let us now go to the calculation of the cusp anomalous dimension at finite temperature. We find again that the leading correction in the cusp angles is the bremsstrahlung function

$$\Gamma_{\text{cusp}}(\varphi, \vartheta) = \left(\vartheta^2 - \varphi^2\right) B \qquad \text{for} \quad \varphi, \vartheta << 1 \ . \tag{3.134}$$

The anomalous dimension is then computed again as the scalar correlator in the presence of a straight line.

$$\Gamma_{\text{cusp}} = \frac{\vartheta^2}{2} \int_{-\infty}^{\infty} \mathrm{d}y_1 \int_{-\infty}^{\infty} \mathrm{d}y_2 |\dot{x}(y_1)||\dot{x}(y_2)| \mathrm{tr}\langle \Phi(y_1)\Phi(y_2)\rangle_W \ . \tag{3.135}$$

However, we again use the two-point function obtained by method of images

$$\mathrm{tr}\left\langle \Phi_r(x)\Phi_s(0)\right\rangle_W = \frac{\gamma\delta_{rs}\pi T}{2|\vec{x}|}\left[\coth\left(\pi T(|\vec{x}| + \mathrm{i}x_0)\right) + \coth\left(\pi T(|\vec{x}| - \mathrm{i}x_0)\right)\right] \tag{3.136}$$

$$= \frac{\gamma}{x^2} + \frac{\pi^2}{3}\gamma T^2 - \frac{\pi^4}{45}\gamma T^4 x^2 + \mathcal{O}(T^6) \ . \tag{3.137}$$

given the parameterization of the straight line (1.65)

$$C: \ x^\mu(\mathfrak{t}) = \left(0,\, 0,\, 0,\, \tilde{R}\tanh\left(\frac{\mathfrak{t}}{2}\right)\right) \quad , \quad \mathfrak{t} \in [-\infty, \infty] \ . \tag{3.138}$$

this yields

$$\begin{aligned}\Gamma_{\text{cusp}} &= \vartheta^2\frac{\gamma}{2} + \frac{\vartheta^2}{2}\gamma \int_{-\infty}^{\infty}\mathrm{d}\mathfrak{t}_1 \left[\int_{-\infty}^{\mathfrak{t}_1-\epsilon}\mathrm{d}\mathfrak{t}_2 + \int_{\mathfrak{t}_1+\epsilon}^{\infty}\mathrm{d}\mathfrak{t}_2\right]\frac{\tilde{R}^2}{4}\mathrm{sech}^2\left(\frac{\mathfrak{t}_1}{2}\right)\mathrm{sech}^2\left(\frac{\mathfrak{t}_2}{2}\right)\times \\ &\quad \times\left[\frac{\pi^2}{3}\gamma T^2 - \frac{\pi^4}{45}\gamma\tilde{R}^2T^4\left(\tanh\left(\frac{\mathfrak{t}_1}{2}\right) - \tanh\left(\frac{\mathfrak{t}_2}{2}\right)\right)^2\right] + \mathcal{O}(T^6) \qquad (3.139) \\ &= \vartheta^2\gamma\left[\frac{1}{2} + \frac{2\pi^2}{3}(\tilde{R}T)^2 - \frac{4\pi^4}{135}(\tilde{R}T)^4\right] + \mathcal{O}(T^6) \ . \qquad (3.140)\end{aligned}$$

Here we see why the introduction of the scale parameter $\tilde{R}$ was necessary. While the dependence vanishes directly in the conformal case, a dependence on the parameter is obtained for the thermal theory. Similar to the circular WL the dependence is in fact in terms of the dimensionless combination $\tilde{R}T$. Thus the bremsstrahlung function is given by

$$B = \gamma\left[\frac{1}{2} + \frac{\pi^4}{6}(\tilde{R}T)^2 - \frac{\pi^6}{45}(\tilde{R}T)^4\right] + \mathcal{O}(T^6) \ . \tag{3.141}$$

Plugging in the result for γ from the circular loop yields a formula similar to (2.73)

$$B = \frac{\frac{1}{2} + \frac{2\pi^2}{3}(\tilde{R}T)^2 - \frac{4\pi^4}{135}(\tilde{R}T)^4 + \mathcal{O}(T^6)}{\pi^2 - \frac{4\pi^6}{45}(RT)^4 + \mathcal{O}(T^6)} \lambda\partial_\lambda \log\langle W_\bigcirc\rangle \tag{3.142}$$

$$= \left(\frac{1}{2\pi^2} + \frac{2}{3}(\tilde{R}T)^2 + \frac{2\pi^2}{45}(RT)^4 - \frac{4\pi^2}{135}(\tilde{R}T)^4\right)\lambda\partial_\lambda \log\langle W_\bigcirc\rangle + \mathcal{O}(T^6)\,. \tag{3.143}$$

Recall the expectation value of the circular loop (3.120)

$$\langle W_\bigcirc\rangle^{(RT\to 0)} = 1 + \frac{\lambda}{8} + \lambda\frac{(RT)^2\pi^2}{12} - \lambda\frac{(RT)^4\pi^4}{60} + \mathcal{O}(\lambda^2,\, \lambda(RT)^6)\,. \tag{3.144}$$

From this we can derive the thermal corrections to the bremsstrahlung

$$\begin{aligned} B^{(RT\to 0)} = {}& \frac{\lambda}{16\pi^2} + \frac{\lambda}{24}\left(2(\tilde{R}T)^2 + (RT)^2\right) \\ & + \frac{\lambda\pi^2\left(60(\tilde{R}T)^2(RT)^2 - 4(\tilde{R}T)^4 - 3(RT)^4\right)}{1080} + \mathcal{O}(\lambda^2,\, \lambda(RT)^6)\,. \end{aligned} \tag{3.145}$$

3.2.2.2 Stress tensor

The bremsstrahlung above shall be compared to the coefficient of the one-point function of the stress tensor. The relation between the circle and the line for $T = 0$ $\mathcal{N} = 4$ SYM namely that one goes into the other might not hold any longer at finite temperature. We therefore repeat the above calculation for the circular loop interacting with the stress tensor. We consider the small temperature expansion including the first two corrections. The calculations are identical for the zero temperature case. For convenience we will not include the lengthy expressions we have to consider after taking derivatives but we focus on the final result. We find

$$\langle T_{\phi\phi} W_\bigcirc\rangle^V_{T\to 0} = -\frac{\lambda R^4}{16\pi^2 r^8} - \frac{\pi^2\lambda(RT)^4}{360 r^4} + \mathcal{O}(T^6)\,, \tag{3.146}$$

$$\begin{aligned} \langle T_{\phi\phi} W_\bigcirc\rangle^S_{T\to 0} = {}& \frac{\lambda R^4}{12\pi^2 r^8} + \frac{\lambda(RT)^2\left(r^2 - 2R^2\right)}{36 r^6} \\ & - \frac{\pi^2\lambda R^2 T^4\left(r^4 - 2r^2\left(R^2 + 3z_0^2\right) + 4R^2 z_0^2\right)}{270 r^6} + \mathcal{O}(T^6)\,. \end{aligned} \tag{3.147}$$

The full stress tensor is then given by

$$\langle T_{\phi\phi}\rangle'_{W,\beta} = \frac{\langle T_{\phi\phi} W_\bigcirc\rangle^S_{T\to 0} + \langle T_{\phi\phi} W_\bigcirc\rangle^V_{T\to 0}}{\langle W_\bigcirc\rangle^{(T\to 0)}} \tag{3.148}$$

$$= +\frac{\lambda R^4}{48\pi^2 r^8} - \frac{\lambda(RT)^2\left(r^2 - 2R^2\right)}{36 r^6} + \frac{\lambda\pi^2 T^4 R^2(4r^2 - 11R^2)}{1080 r^4}$$

$$+\frac{\lambda\pi^2 R^2 T^4 z_0^2 \left(2R^2 - 3r^2\right)}{135 r^6} + \mathcal{O}((RT)^6) \tag{3.149}$$

To make this result more comparable let us set the stress tensor in the center of the circular loop by setting $r \to R$. Also note that $\langle T_{\phi\phi}\rangle_W = {}^{h}\!/_{r^4}$ and we therefore find

$$3h = 3R^4 \langle T_{\phi\phi}\rangle_W = \frac{\lambda}{16\pi^2} + \frac{\lambda(RT)^2}{12} - \frac{7\lambda\pi^2(RT)^4}{360} + \mathcal{O}(\lambda^2, \lambda(RT)^6)\ . \tag{3.150}$$

This should be compared to the result for the circular Wilson loop

$$\begin{aligned} B^{(RT\to 0)} = &\frac{\lambda}{16\pi^2} + \frac{\lambda}{24}\left(2(\tilde{R}T)^2 + (RT)^2\right) \\ &+ \frac{\lambda\pi^2\left(60(\tilde{R}T)^2(RT)^2 - 4(\tilde{R}T)^4 - 3(RT)^4\right)}{1080} + \mathcal{O}(\lambda^2, \lambda(RT)^6)\ . \end{aligned} \tag{3.151}$$

The coefficient $\tilde{R}$ of the straight line is a priori an arbitrary constant. It can depend on the radius of the circular loop as well as on the temperature. By setting

$$\tilde{R} \to R\left[\frac{1}{\sqrt{2}} - \frac{47\pi^2(RT)^2}{90\sqrt{2}} + \frac{611\pi^4(RT)^4}{3240\sqrt{2}} + \mathcal{O}((RT)^6)\right]\ , \tag{3.152}$$

we can indeed reproduce the above relation

$$B = 3h\ , \tag{3.153}$$

and therefore provide a cross-check for the fact that it still holds at finite temperature. More precisely we found a scheme for the coefficient $\tilde{R}$ reproducing the relation between bremsstrahlung and stress tensor in a small temperature expansion.
We will now turn to the high temperature limit. We will find a new scheme for $\tilde{R}$ at large temperatures.

3.2.3 High temperature limit

In the limit $T \to \infty$ we again find a scheme for $\tilde{R}$ that reproduces $B = 3h$. We start by discussing how the high temperature limit is best implemented in our calculations.
In finite temperature we compactify the time dimension on an S^1 with radius $\beta = {}^{1}\!/_{T}$. Taking the high temperature limit $T \to \infty$ correponds to making the radius infinitesimal $\beta \to 0$. This corresponds to a Kaluza-Klein reduction of the four dimensional theory to three dimensions. Following the standard procedure for KK-reduction for the scalar fields we find a tower of massive scalar fields with $M_n = nT$ where $n \in \mathbb{N}$. In the high temperature limit we can integrate out all massive fields and simply keep the massless field $\Phi(x) = \Phi^{(n=0)}(x)$ $(x \in \mathbb{R}^3)$. Similar the gauge fields A_μ split into the three-dimensional vector fields A_i and an additional scalar A_0. As the field A_0 will not couple to any field in

the Wilson loop, we can ignore it for all further considerations. Similarly we also ignore the fermionic fields.
When considering the action let us for a moment ignore that we are at zero coupling. We consider the original action (1.56) with the coupling as an overall prefactor. The action then schematically has the form

$$S = \int \mathrm{d}^3 x \mathcal{L} = \frac{1}{2g^2} \int \mathrm{d}^3 x \mathrm{tr} \left[\frac{1}{2T} F^2 + (\partial\Phi)^2 + \dots \right] . \tag{3.154}$$

It is useful to analyze the mass dimensions of the action. For a dimensionless action the Lagrangian needs to have mass dimension $[\mathcal{L}] = 3$. The three-dimensional scalars have dimension $[\Phi] = {}^{d-2}/_{2}|_{d=3} = {}^{1}/_{2}$ and therefore $[(\partial\Phi)^2] = 3$. Contrary to the scalars, the mass dimension of the gauge fields is independ of the space-time dimension. For all values of d we have $[A_i] = 1$ and therefore $[F^2] = 4$. We thus see that the kinetic term of the scalars reproduces the correct dimension while the vector fields do not. Usually this is fixed by making the Yang-Mills coupling g dimensionfull. Here the coupling g is a prefactor to all fields in the Lagrangian alike and assuming its dimension to fit one term of the Lagrangian will sabotage another term. Therefore we need to add an additional dimensionfull parameter. This can only be the scale T as there is no other possible scale in the theory. This is why we introduced the respective prefactor in the above equation. Going back to zero coupling we henceforth find the high temperature action

$$\begin{aligned} S_{T\to\infty} = \frac{1}{2} \int \mathrm{d}^3 x \, \mathrm{tr} \Big[& \frac{1}{2T} F_{\mu\nu} F^{\mu\nu} + \frac{1}{T} (\partial_\mu A^\mu)^2 + \partial_\mu \Phi^{IJ} \partial^\mu \bar{\Phi}_{IJ} + \mathrm{i} \bar{\lambda}^I \sigma^\mu \partial_\mu \lambda_I + \\ & + \mathrm{i} \lambda_I \bar{\sigma}^\mu \partial_\mu \bar{\lambda}^I \Big] + \mathcal{O}(g) , \end{aligned} \tag{3.155}$$

The stress tensor has the same dimensions as the Lagrangian. We find

$$T_{ij} = -\frac{1}{T} F^a_{ik} F^a_{jk} - \partial_i \Phi^{r\,a} \partial_j \Phi^a_r + \mathcal{O}(g) . \tag{3.156}$$

It is important to note that the stress tensor now has mass dimension $[T_{ij}] = 3$ yielding that also the dependence on the distance to the defect is expected to contribute at cubic order

$$\langle T_{ij} \rangle_W \sim \frac{h}{r^3} . \tag{3.157}$$

For the Wilson loop it is important to recall that we have a parameterization $x(\mathfrak{t})$ where $\mathfrak{t}$ is a dimensionless parameter by construction. Keeping this in mind we need to ensure that the argument of the exponential function is dimensionless. We find

$$W(C) = \frac{1}{N} \mathrm{tr} \overleftarrow{P} \exp \left[g \oint_C \mathrm{d}\mathfrak{t} \left(\mathrm{i} A_\mu \dot{x}^\mu + \sqrt{T} \Phi_r |\dot{x}| \Theta^r \right) \right] , \tag{3.158}$$

We need to include the temperature as a factor for the scalar to match the mass dimensions. Note that it would be possible to make the unit vector $n_r \in S^5$ dimensionfull. However, it would have to scale with $\sqrt{T}$ in this case thus finally yielding an identical result.

We then find the propagators to be

$$\langle A^a_\mu(x_1) A^b_\nu(x_2)\rangle = \frac{T\delta^{ab}\eta_{\mu\nu}}{4\pi\sqrt{(x_1-x_2)^2}} \,, \quad \langle \Phi^a_r(x_1)\Phi^b_s(x_2)\rangle = \frac{\delta^{ab}\delta_{rs}}{4\pi\sqrt{(x_1-x_2)^2}} \,. \tag{3.159}$$

This means that for Wilson loop calculations we in fact have the same propagators $^T\!/_x$ as one would predict in a more naive approach to high temperature. The combination $[^T\!/_x] = 2$ has the correct dimension for a gauge propagator. For the scalar propagator note that the temperature factor will at later calculations be included from the expansion of the Wilson loop (which is what we use in all calculations) and therefore also fits in the dimensional analysis.
The two-point function in the presence of the Wilson loop is then analogously given by

$$\mathrm{tr}\langle \Phi_r(x)\Phi_s(0)\rangle_W = \frac{\gamma\pi}{x} \,. \tag{3.160}$$

We thus find including a factor T from the Wilson loop

$$\left.\frac{\langle W_\theta\rangle - \langle W_{\theta=0}\rangle}{\langle W_{\theta=0}\rangle}\right|_{T\to\infty} = \frac{\theta^2}{2}\int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{2\pi} \mathrm{d}\mathfrak{t}_2 \frac{\pi\gamma RT\cos(\mathfrak{t}_1-\mathfrak{t}_2)}{\sqrt{2-2\cos(\mathfrak{t}_1-\mathfrak{t}_2)}} \tag{3.161}$$

$$= -4\pi^2\theta^2\gamma RT\log\left(\frac{\epsilon}{4}\right) - 8\pi^2\theta^2\gamma RT = -8\pi^2\theta^2\gamma RT \,. \tag{3.162}$$

As in the zero temperature case the divergence can be taken care of by a renormalization. This yields for the coefficient

$$\gamma_{T\to\infty} = \frac{\lambda}{8\pi^2 RT}\partial_\lambda \log\langle W_\bigcirc\rangle \,. \tag{3.163}$$

Let us now go to the calculation of the cusp anomalous dimension at finite temperature. We find again that

$$\Gamma_{\mathrm{cusp}}(\varphi,\vartheta) = \vartheta^2 B \qquad \text{for} \quad \varphi = 0, \vartheta << 1 \,, \tag{3.164}$$

and the anomalous dimension is computed via

$$\Gamma_{\mathrm{cusp}} = \frac{\vartheta^2}{2}T\int_{-\infty}^{\infty}\mathrm{d}\mathfrak{t}_1\int_{-\infty}^{\infty}\mathrm{d}\mathfrak{t}_2 |\dot{x}(\mathfrak{t}_1)||\dot{x}(\mathfrak{t}_2)|\mathrm{tr}\langle\Phi(\mathfrak{t}_1)\Phi(\mathfrak{t}_2)\rangle_{W,\beta\to 0} \,, \tag{3.165}$$

However, we again use the two-point function obtained by method of images and expand around high temperature. Given the parameterization of the straight line (1.65) this yields

$$\Gamma_{\text{cusp}} = -\frac{\vartheta^2}{2}\frac{\pi}{4}\gamma\tilde{R}T\left[\int_{-\infty}^{-\epsilon}\mathrm{d}\mathfrak{t} + \int_{\epsilon}^{\infty}\mathrm{d}\mathfrak{t}_2\right]\int_{-\infty}^{\infty}\mathrm{d}\mathfrak{t}_1\frac{\operatorname{sech}^2\left(\frac{\mathfrak{t}_1}{2}\right)\operatorname{sech}^2\left(\frac{\mathfrak{t}_2}{2}\right)}{\sqrt{\left(\tanh\left(\frac{\mathfrak{t}_1}{2}\right) - \tanh\left(\frac{\mathfrak{t}_2}{2}\right)\right)^2}} \tag{3.166}$$

$$= 2\vartheta^2\pi\gamma\tilde{R}T\log(2)\ . \tag{3.167}$$

This would mean that the bremsstrahlung in the $T \to \infty$ limit is

$$B = \frac{\lambda\log(2)}{4\pi}\frac{\tilde{R}}{R}\partial_\lambda\log\langle W_{\bigcirc}\rangle\ . \tag{3.168}$$

Recall that

$$\langle W_{\bigcirc}\rangle^{(RT\to\infty)} = 1 + \frac{\lambda}{2}RT + \mathcal{O}\left(\lambda^2, \frac{\lambda}{(RT)^n}\right) \quad , \qquad \forall n \in \mathbb{N}\ . \tag{3.169}$$

From this we can derive the thermal corrections to the bremsstrahlung

$$B = \frac{\lambda\log(2)}{8\pi}\frac{\tilde{R}}{R}(RT) + \mathcal{O}\left(\lambda^2, \frac{\lambda}{(RT)^n}\right) \quad , \qquad \forall n \in \mathbb{N}\ . \tag{3.170}$$

For the calculation of the stress tensor we can count powers of T. Let us start with the easier scalar case. Here we find a power of T from the Wilson loop and no further powers from propagators. For the gauge fields the propagators each yield a temperature factor, hence T^2. However, one of the temperature factors cancels with the $1/T$ from the definition of the stress tensor.
Plugging in the integrals with the $1/x$ propagator we find

$$\langle T_{\phi\phi}W_{\bigcirc}\rangle^V = -\frac{\lambda R^4 T}{256 r^6} \quad , \qquad \langle T_{\phi\phi}W_{\bigcirc}\rangle^S = \frac{\lambda R^2 T(2R^2 - r^2)}{192 r^6}\ . \tag{3.171}$$

The full stress tensor is then given by

$$\langle T_{\phi\phi}\rangle_W = \frac{\langle T_{\phi\phi}W_{\bigcirc}\rangle^S_{\beta\to 0} + \langle T_{\phi\phi}W_{\bigcirc}\rangle^V_{\beta\to 0}}{\langle W_{\bigcirc}\rangle_{\beta\to 0}} = \frac{\lambda R^2 T\left(5R^2 - 4r^2\right)}{768 r^6} + \mathcal{O}(\lambda^2)\ . \tag{3.172}$$

Here we ignored the one-point coefficient from [54], because we are considering effectively a zero temperature 3d theory with a distance parameter β which is an artifact of the KK-reduction. Setting $r \to R$ as above and recalling that in the KK-reduced theory we have $\langle T_{\phi\phi}\rangle_W = h/R^3$ we find

$$3h = \frac{\lambda(RT)}{256} + \mathcal{O}\left(\lambda^2\right)\ , \quad B = \frac{\lambda\log(2)}{8\pi}(\tilde{R}T)\ . \tag{3.173}$$

As in the small temperature limit we can find a convenient replacement for the line parameter $\tilde{R}$:

$$\tilde{R} \to \frac{\pi}{32\log(2)} R \qquad \text{for} \quad T \to \infty \,. \tag{3.174}$$

In this scheme we indeed find the relation

$$B = 3h \tag{3.175}$$

to hold.
To conclude, we showed similar to the previous subsection that there exists a scheme for $\tilde{R}$ in which the Relation (1.1) holds. We explicitly found a scheme for small temperature (3.152) and high temperature (3.174). Therefore we assume that such a scheme exists for any temperature T.

Chapter 4

Finite temperature and non-zero coupling

The calculation in the previous Chapter 3 has been considered without interactions by setting the coupling $g = 0$. In this chapter we consider how interactions affect the thermal Broken Ward Identity (3.27) and the resulting relation $B = 3h$. I will present our current understanding of the interactive theory as well as give an outline for the follow up steps which are necessary for a deeper understanding. This describes our ongoing work.
Effects on the perturbative calculation from interactions at finite temperature similarly to Section 3.2 are considered in the next Chapter 5 where we consider the circular Wilson loop at order λ^2. These include the leading corrections yielded by interactions in the action such as self-energies and vertex diagrams.

We will follow two different ansätze here. One allows a splitting of the action into a term preserving supersymmetry and another consisting of mass terms similar to (3.19)

$$S_{4d \text{ th. } \mathcal{N}=4} = S_{\text{SUSY}} + S_{\text{mass-terms}} \,. \tag{4.1}$$

While such a splitting is very helpful in the derivation of a thermal BWI like (3.27) the respective ansatz breaks R-symmetry. More precisely we consider field redefinitions with phase transformations as used in the previous chapter. They will be restricted by the Yukawa interactions and yield a non-trivial breaking of the R-symmetry. The upshot of this approach is that it is straight forward to isolate one part of the Lagrangian which preserves supersymmetry and we can thus consider the BWI in a very similar manner to before. However, the way R-symmetry is broken is ambiguous. There seems to be a freedom of choice preserving different amounts of R-symmetry. We suggest that with a dynamical calculation including the phases this issue might be solved. We suspect that by considering the self-energy and the comparison of thermal masses we are able to find restrictions on the phases determining the preserved R-symmetry. As Appendix H shows we in fact find an inconsistency. The thermal mass of the theory with the phase

redefinitions cannot be matched with the known thermal mass of thermal $\mathcal{N}=4$ SYM. For completeness we still present this ansatz here. An alternative ansatz which in turn preserves R-symmetry is thus presented in Appendix I. While being relatively novel this ansatz provides a consistent way of defining thermal masses and we are able to obtain a BWI similar to (3.27) which preserves $B=3h$.
What is more, we introduce the different periodicity conditions by employing a Fourier series representation of the fields. From the KK-like reduction of the thermal circle we will find a tower of massive modes. This then becomes a three-dimensional theory of infinitely many fields. On the one hand, contrarily to the previous ansatz, this approach will manifestly preserve R-symmetry,

$$S_{4d\ \mathrm{th.}\mathcal{N}=4} = S_{\mathrm{massive}\ 3d\,R\mathrm{-sym.}}\,. \tag{4.2}$$

On the other hand, it is not clear whether a part of the Lagrangian can be identified that preserves supersymmetry. Most importantly, we again consider Yukawa interactions. We find the interaction between different modes. This might spoil the preservation of supersymmetry. Upon the application of supersymmetry transformations we expect to find some constraints which are consistent with the constraints of the first ansatz.

Let us stress here that these two approaches have as of now not been carried out to the fullest. We present how the respective ansatz is applied to $\mathcal{N}=4$ SYM at finite temperature and discuss all steps necessary to validate them. Furthermore we will indicate possible implications on the thermal BWI. Ideally we hope to find that it can be applied as in the last sections, see Equation (3.27).

4.1 Field redefinitions in 4d

Let us discuss here how the field redefinitions with phase transformations (3.18) and the thermal Broken Ward Identity are affected by interactions. The field redefinitions provide a setting where a part of the Lagrangian can be identified that preserves supersymmetry as in (4.1). We will find, however, that the Yukawa terms yield conditions on the phases of the fermionic fields introduced in the previous chapter. Additionally we find that each scalar field Φ_{IJ} will required a unique phase $\hat{\alpha}_{IJ}$ determined by a combination of the fermionic phases α_I. These phases are different for each spinor in the tuple λ_I which yields a breaking of the R-symmetry. We will consider in detail how this affects the Ward identity and possible ways to preserve the Relation (1.1).

4.1.1 Yukawa interaction

Compared to the previous chapter we now allow for all values of g and can thus use the action as originally given in Equation (1.56)

$$\tilde{S} = \frac{1}{2g^2}\mathrm{tr}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\frac{1}{2}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \left(\partial_\mu \tilde{A}^\mu\right)^2 + \tilde{D}_\mu\tilde{\Phi}^{IJ}\tilde{\bar{D}}^\mu\tilde{\bar{\Phi}}_{IJ} + \mathrm{i}\tilde{\bar{\lambda}}^I\sigma^\mu\tilde{D}_\mu\tilde{\lambda}_I + \right. \\ +\mathrm{i}\tilde{\lambda}_I\bar{\sigma}^\mu\tilde{\bar{D}}_\mu\tilde{\bar{\lambda}}^I + \mathrm{i}\tilde{\bar{\Phi}}_{IJ}\left\{\tilde{\lambda}^I,\tilde{\lambda}^J\right\} - \mathrm{i}\tilde{\Phi}^{IJ}\left\{\tilde{\bar{\lambda}}_I,\tilde{\bar{\lambda}}_J\right\} + \\ \left. +\frac{1}{2}\left[\tilde{\Phi}^{IJ},\tilde{\Phi}^{KL}\right]\left[\tilde{\bar{\Phi}}_{IJ}\tilde{\bar{\Phi}}_{KL}\right] + \partial_\mu\tilde{\bar{c}}\tilde{D}^\mu\tilde{c}\right] . \tag{4.3}$$

As before we write all fields with a tilde and we assign phases to them such that we can write down a new action depending only on periodic fields in the finite temperature setting. We will derive step-by-step which fields need to obtain a phase dependence. In each consecutive step we try to introduce as few phases as possible, however, keeping in mind new restrictions we found.

As argued in Subsection 3.1.1 the gauge and ghost field do not require a phase. The spinor fields meanwhile must get a phase factor which ensures the anti-periodic boundary conditions on the thermal circle. For the scalars let us first assume that as in the zero coupling case they do not get a phase factor. In this case we would again have the transformations of Equation (3.18)

$$\tilde{A}_\mu(\tau,\vec{x}) \to A_\mu(\tau,\vec{x}) , \quad \tilde{\Phi}_{IJ}(\tau,\vec{x}) \to \Phi_{IJ}(\tau,\vec{x}) , \quad \tilde{\bar{c}}(\tau,\vec{x}) \to \bar{c}(\tau,\vec{x}) ,$$

$$\tilde{\lambda}_I(\tau,\vec{x}) \to e^{\mathrm{i}\alpha\tau}\lambda_I(\tau,\vec{x}) \qquad \text{with} \quad \alpha = \frac{\pi}{\beta}(2n+1) , \quad n \in \mathbb{Z} .$$

All fields without tilde are periodic in $\tau \to \tau + \beta$. This yields identical mass terms for the fermions as we found above. All kinetic terms are then preserved. Consider, however, the Yukawa interaction

$$\mathrm{i}\tilde{\bar{\Phi}}_{IJ}\left\{\tilde{\lambda}^I,\tilde{\lambda}^J\right\} \to \mathrm{i}\bar{\Phi}_{IJ}\left\{e^{\mathrm{i}\alpha\tau}\lambda^I, e^{\mathrm{i}\alpha\tau}\lambda^J\right\} = \mathrm{i}e^{2\mathrm{i}\alpha\tau}\bar{\Phi}_{IJ}\left\{\lambda^I,\lambda^J\right\} . \tag{4.4}$$

By keeping the scalar fields without a phase we thus find that an overall phase factor would appear in the action. To avoid that we introduce a phase for the scalar fields as well

$$\tilde{\Phi}_{IJ}(\tau,\vec{x}) \to e^{\mathrm{i}\hat{\alpha}\tau}\Phi_{IJ}(\vec{x}) \qquad \text{with} \quad \hat{\alpha} = \frac{2n\pi}{\beta} , \quad n \in \mathbb{Z} . \tag{4.5}$$

This new phase is even to ensure periodicity of the scalar field as demanded by the KMS conditions. From the above Yukawa interaction we would find a relation between the bosonic and the fermionic phases, $\hat{\alpha} = 2\alpha$.

Let us, however, consider the scalar fields more closely. The matrix Φ_{IJ} is a convenient way of manifestly writing the action in a manner reflecting the self-dual R-symmetry representation the scalars are in. In fact we have that [112]

$$\Phi_{IJ} = \begin{pmatrix} 0 & \phi_1 & \phi_2 & \phi_3 \\ -\phi_1 & 0 & \bar{\phi}_3 & -\bar{\phi}_2 \\ -\phi_3 & -\bar{\phi}_3 & 0 & \bar{\phi}_1 \\ -\phi_3 & \bar{\phi}_2 & -\bar{\phi}_1 & 0 \end{pmatrix}_{IJ} \tag{4.6}$$

Hence, assigning an overall phase to Φ_{IJ} would mean to assign the same phase to each of the three complex scalars ϕ_i as well as their conjugates $\bar{\phi}_i$. Therefore, we cannot simply assign an overall phase. Instead the following transformations can be suggested,

$$\begin{aligned} &\tilde{A}_\mu(\tau,\vec{x}) \to A_\mu(\tau,\vec{x}) \,, \quad \tilde{\bar{c}}(\tau,\vec{x}) \to \bar{c}(\tau,\vec{x}) \,, \\ &\tilde{\phi}_i(\tau,\vec{x}) \to e^{\mathrm{i}\hat{\alpha}\tau}\phi_i(\tau,\vec{x}) \qquad \text{with} \quad \hat{\alpha} = \frac{\pi}{\beta}(2m) \,, \quad m \in \mathbb{Z} \,, \\ &\tilde{\lambda}_I(\tau,\vec{x}) \to e^{\mathrm{i}\alpha\tau}\lambda_I(\tau,\vec{x}) \qquad \text{with} \quad \alpha = \frac{\pi}{\beta}(2n+1) \,, \quad n \in \mathbb{Z} \,. \end{aligned} \tag{4.7}$$

The index $i = 1,2,3$ labels the three complex scalars.
We turn once more to the Yukawa interaction. Let us expand the sum over R-symmetry indices by using explicitly the matrix representation of Φ_{IJ},

$$\begin{aligned} \tilde{\Phi}^{IJ}\left\{\tilde{\bar{\lambda}}_I, \tilde{\bar{\lambda}}_J\right\} &= \tilde{\Phi}^{12}\left\{\tilde{\bar{\lambda}}_1, \tilde{\bar{\lambda}}_2\right\} + \tilde{\Phi}^{43}\left\{\tilde{\bar{\lambda}}_4, \tilde{\bar{\lambda}}_3\right\} + \dots \\ &= \tilde{\phi}_1\left\{\tilde{\bar{\lambda}}_1, \tilde{\bar{\lambda}}_2\right\} + \tilde{\bar{\phi}}_1\left\{\tilde{\bar{\lambda}}_4, \tilde{\bar{\lambda}}_3\right\} + \dots \\ &\to e^{-\mathrm{i}(2\alpha-\hat{\alpha})\tau}\phi_1\left\{\bar{\lambda}_1, \bar{\lambda}_2\right\} + e^{-\mathrm{i}(2\alpha+\hat{\alpha})\tau}\bar{\phi}_1\left\{\bar{\lambda}_4, \bar{\lambda}_3\right\} + \dots \end{aligned} \tag{4.8}$$

The dots represent additional terms in the sum over R-symmetry indices. In the last step we used the above phase transformations. The kind of interactions we find here, however does not allow for a consistent choice of α and $\hat{\alpha}$ that makes the Yukawa interaction independent of the phases.
Conclusively, we need to break the R-symmetry and introduce different phases to each scalar and each fermion. This can be done by by considering the phase transformations

$$\begin{aligned} &\tilde{A}_\mu(\tau,\vec{x}) \to A_\mu(\tau,\vec{x}) \,, \quad \tilde{\bar{c}}(\tau,\vec{x}) \to \bar{c}(\tau,\vec{x}) \,, \\ &\tilde{\phi}_i(\tau,\vec{x}) \to e^{\mathrm{i}\hat{\alpha}_i\tau}\phi_i(\tau,\vec{x}) \qquad \text{with} \quad \hat{\alpha}_i = \frac{\pi}{\beta}(2m_i) \,, \quad m_i \in \mathbb{Z} \,, \\ &\tilde{\lambda}_I(\tau,\vec{x}) \to e^{\mathrm{i}\alpha_I\tau}\lambda_I(\tau,\vec{x}) \qquad \text{with} \quad \alpha_I = \frac{\pi}{\beta}(2n_I+1) \,, \quad n_I \in \mathbb{Z} \,. \end{aligned} \tag{4.9}$$

The indices i and I above are not summed. The considerations above suggest that the Yukawa interaction will restrict the choice for these phases. In order to understand these restrictions it is convenient to work in a notation using the old R-symmetry. Therefore introduce phases to the self-dual scalars by writing

$$\tilde{\Phi}_{IJ}(\tau,\vec{x}) \to e^{\mathrm{i}\tilde{\alpha}_{IJ}\tau}\Phi_{IJ}(\tau,\vec{x}) \qquad \text{with} \quad \tilde{\alpha}_{IJ} = \frac{\pi}{\beta}(2m_{IJ}) \ , \quad m_{IJ} \in \mathbb{Z} \tag{4.10}$$

Explicitly this yields

$$\tilde{\Phi}_{IJ} \to \begin{pmatrix} 0 & e^{\mathrm{i}\tilde{\alpha}_{12}\tau}\phi_1 & e^{\mathrm{i}\tilde{\alpha}_{13}\tau}\phi_2 & e^{\mathrm{i}\tilde{\alpha}_{14}\tau}\phi_3 \\ -e^{\mathrm{i}\tilde{\alpha}_{21}\tau}\phi_1 & 0 & e^{\mathrm{i}\tilde{\alpha}_{23}\tau}\bar{\phi}_3 & -e^{\mathrm{i}\tilde{\alpha}_{24}\tau}\bar{\phi}_2 \\ -e^{\mathrm{i}\tilde{\alpha}_{31}\tau}\phi_3 & -e^{\mathrm{i}\tilde{\alpha}_{32}\tau}\bar{\phi}_3 & 0 & e^{\mathrm{i}\tilde{\alpha}_{34}\tau}\bar{\phi}_1 \\ -e^{\mathrm{i}\tilde{\alpha}_{41}\tau}\phi_3 & e^{\mathrm{i}\tilde{\alpha}_{42}\tau}\bar{\phi}_2 & -e^{\mathrm{i}\tilde{\alpha}_{43}\tau}\bar{\phi}_1 & 0 \end{pmatrix}_{IJ} . \tag{4.11}$$

Consider the constraints on the scalar coefficients $\tilde{\alpha}_{IJ}$. By looking at the above matrix we see that they are symmetric $\tilde{\alpha}_{IJ} = \tilde{\alpha}_{JI}$ and the diagonal components are $\tilde{\alpha}_{II} = 0$. Furthermore for each of the three scalar fields the phase of the conjugate ones must come with an opposite sign yielding the constraints

$$\hat{\alpha}_1 = \tilde{\alpha}_{12} = -\tilde{\alpha}_{34} \ , \quad \hat{\alpha}_2 = \tilde{\alpha}_{13} = -\tilde{\alpha}_{24} \ , \quad \hat{\alpha}_3 = \tilde{\alpha}_{14} = -\tilde{\alpha}_{23} \ . \tag{4.12}$$

The final constraints furthermore follow from the transformations (4.9). The Yukawa interaction then relates the fermionic and bosonic phases to each other

$$\begin{aligned} &\mathrm{i}\bar{\tilde{\Phi}}_{IJ}\left\{\tilde{\lambda}^I,\tilde{\lambda}^J\right\} \to \mathrm{i}e^{\mathrm{i}(\tilde{\alpha}_{IJ}-\alpha_I-\alpha_J)\tau}\bar{\Phi}_{IJ}\left\{\lambda^I,\lambda^J\right\} \\ \Rightarrow \qquad &\tilde{\alpha}_{IJ} = \alpha_I + \alpha_J \ . \end{aligned} \tag{4.13}$$

Combining these conditions we find a final relation for the fermionic coefficients

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \ . \tag{4.14}$$

This condition shows that the $\mathfrak{su}(4)_R$ symmetry cannot be preserved because it is not possible to have equal phases for the four spinors. The constraint (4.14) can also be applied to the bosonic phases. We then find the following transformations

$$\begin{aligned} &\tilde{A}_\mu(\tau,\vec{x}) \to A_\mu(\tau,\vec{x}) \ , \quad \tilde{\bar{c}}(\tau,\vec{x}) \to \bar{c}(\tau,\vec{x}) \ , \\ &\tilde{\phi}_1(\tau,\vec{x}) \to e^{\mathrm{i}(\alpha_1+\alpha_2)\tau}\phi_1(\tau,\vec{x}) \ , \quad \tilde{\phi}_2(\tau,\vec{x}) \to e^{\mathrm{i}(\alpha_1+\alpha_3)\tau}\phi_2(\tau,\vec{x}) \ , \\ &\tilde{\phi}_3(\tau,\vec{x}) \to e^{-\mathrm{i}(\alpha_2+\alpha_3)\tau}\phi_3(\tau,\vec{x}) \ , \\ &\tilde{\lambda}_1(\tau,\vec{x}) \to e^{\mathrm{i}\alpha_1\tau}\lambda_1(\tau,\vec{x}) \ , \quad \tilde{\lambda}_2(\tau,\vec{x}) \to e^{\mathrm{i}\alpha_2\tau}\lambda_2(\tau,\vec{x}) \ , \end{aligned} \tag{4.15}$$

$$\tilde{\lambda}_3(\tau,\vec{x}) \to e^{\mathrm{i}\alpha_3\tau}\lambda_3(\tau,\vec{x}) , \quad \tilde{\lambda}_4(\tau,\vec{x}) \to e^{-\mathrm{i}(\alpha_1+\alpha_2+\alpha_3)\tau}\lambda_4(\tau,\vec{x}) ,$$

$$\text{with} \quad \alpha_1 = \frac{\pi}{\beta}(2n_1+1) , \quad \alpha_2 = \frac{\pi}{\beta}(2n_2+1) , \quad \alpha_3 = \frac{\pi}{\beta}(2n_3+1) , \qquad n_1,n_2,n_3 \in \mathbb{Z} .$$

Note that all fields in satisfy their respective KMS conditions. The bosonic phases are even yielding periodic boundary conditions. The fermionic phases are odd and thus manifest their anti-periodicity on the compactified time dimension. Note that we can find special choices for the α_I which partly preserve some part of the original R-symmetry. These will be considered in Section 4.1.3 below. In Appendix G we show that the above transformations are anomaly-free.
We will now turn to the effects on the action. As in the zero coupling case studied above we will find mass terms for the scalars and spinors depending on the phases.

4.1.2 Mass terms

The derivatives in the kinetic terms of the action will act on the phases as well as on the fields. Therefore, we will find new mass-like terms similar to Equation (3.19). We only need to consider the kinetic terms of scalars and spinors as the gauge and ghost fields do not obtain a phase factor. Therefore consider

$$\tilde{S} = \frac{1}{2g^2}\mathrm{tr}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[4\tilde{D}_\mu\tilde{\phi}^i\tilde{\bar{D}}^\mu\tilde{\bar{\phi}}_i + \mathrm{i}\tilde{\bar{\lambda}}^I\sigma^\mu\tilde{D}_\mu\tilde{\lambda}_I + \mathrm{i}\tilde{\lambda}_I\bar{\sigma}^\mu\tilde{\bar{D}}_\mu\tilde{\bar{\lambda}}^I\right] + \dots .$$

Compared to the original definition of the Lagrangian we used the kinetic term for the three complex scalars explicitly. The dots denote terms that will not be affected by the phase transformations.
Let us start by considering the spinors

$$\begin{aligned}\mathrm{i}\tilde{\bar{\lambda}}^I\sigma^\mu\tilde{D}_\mu\tilde{\lambda}_I + \mathrm{i}\tilde{\lambda}_I\bar{\sigma}^\mu\tilde{\bar{D}}_\mu\tilde{\bar{\lambda}}^I &\to \mathrm{i}\bar{\lambda}^I\sigma^\mu D_\mu\lambda_I + \mathrm{i}\lambda_I\bar{\sigma}^\mu\bar{D}_\mu\bar{\lambda}^I - \bar{\lambda}^I\sigma^0\alpha_I\lambda_I + \lambda_I\bar{\sigma}^0\alpha_I\bar{\lambda}^I \\ &= \mathrm{i}\bar{\lambda}^I\sigma^\mu D_\mu\lambda_I + \mathrm{i}\lambda_I\bar{\sigma}^\mu\bar{D}_\mu\bar{\lambda}^I - 2\alpha_I\bar{\lambda}^I\sigma^0\lambda_I . \end{aligned} \tag{4.16}$$

These mass terms for the fermions are in fact identical to the ones we obtained at zero couplig. For the scalars let us focus on one of the complex scalars for convenience

$$\begin{aligned}\tilde{D}_\mu\tilde{\phi}^1\tilde{\bar{D}}^\mu\tilde{\bar{\phi}}_1 &\to D_\mu\phi^1\bar{D}^\mu\bar{\phi}_1 + \mathrm{i}(\alpha_1+\alpha_2)\phi^1\bar{D}^0\bar{\phi}_1 - (\alpha_1+\alpha_2)D_0\phi^1\bar{\phi}_1 + (\alpha_1+\alpha_2)^2\phi^1\bar{\phi}_1 \\ &= D_\mu\phi^1\bar{D}^\mu\bar{\phi}_1 + \mathrm{i}(\alpha_1+\alpha_2)\left(\phi^1\partial^0\bar{\phi}_1 - \partial_0\phi^1\bar{\phi}_1\right) \\ &\quad + (\alpha_1+\alpha_2)\left(\phi^1[A^0,\bar{\phi}_1] + [A_0,\phi^1]\bar{\phi}_1\right) + (\alpha_1+\alpha_2)^2\phi^1\bar{\phi}_1 \end{aligned} \tag{4.17}$$

Consider the terms linear in the phases. We can make the $\mathrm{SU}(N)$ indices explicit and include the trace over the gauge group by using $\phi = \phi^a T^a$ with T^a being the generators of the gauge group SU(N). This is only interesting for the term with the gauge interaction. It becomes

$$\begin{aligned}\operatorname{tr}\left(\phi_1[A_0,\bar\phi_1]+[A_0,\phi_1]\bar\phi_1\right) &= A^a\phi_1^b\bar\phi_1^c\operatorname{tr}\left(T^b[T^a,T^c]+[T^a,T^b]T^c\right)\\ &= A^a\phi_1^b\bar\phi_1^c\operatorname{tr}\left(T^bT^aT^c-T^bT^cT^a+T^aT^bT^c-T^bT^aT^c\right)=0\,. \qquad (4.18)\end{aligned}$$

The first and last term canceled readily and the remaining two chancel by cyclicity of the trace. The correction terms for the scalars ϕ_2 and ϕ_3 are obtained similarly with dependence on the respective phases.
For both, scalars and spinors, we see that we reproduce the original kinetic term and additional terms with phases as prefactors. Then we can rewrite the action as the original one, Equation (1.56), and additional terms

$$\begin{aligned}\tilde S = S + \frac{1}{2g^2}\operatorname{tr}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3x\Big[&-2\alpha_1\bar\lambda^1\bar\sigma^0\lambda_1-2\alpha_2\bar\lambda^2\bar\sigma^0\lambda_2-2\alpha_3\bar\lambda^3\bar\sigma^0\lambda_3\\ &+2(\alpha_1+\alpha_2+\alpha_3)\bar\lambda^4\bar\sigma^0\lambda_4+4\mathrm{i}(\alpha_1+\alpha_2)\left(\phi^1\partial^0\bar\phi_1-\partial_0\phi^1\bar\phi_1\right) \qquad (4.19)\\ &+4\mathrm{i}(\alpha_1+\alpha_3)\left(\phi^2\partial^0\bar\phi_2-\partial_0\phi^2\bar\phi_2\right)-4\mathrm{i}(\alpha_2+\alpha_3)\left(\phi^3\partial^0\bar\phi_3-\partial_0\phi^3\bar\phi_3\right)\\ &+4(\alpha_1+\alpha_2)^2\phi^1\bar\phi_1+4(\alpha_1+\alpha_3)^2\phi^2\bar\phi_2+4(\alpha_2+\alpha_3)^2\phi^3\bar\phi_3\Big]\,,\end{aligned}$$

$$\text{with}\quad \alpha_1=\frac{\pi}{\beta}(2n_1+1)\,,\quad \alpha_2=\frac{\pi}{\beta}(2n_2+1)\,,\quad \alpha_3=\frac{\pi}{\beta}(2n_3+1)\,,\qquad n_1,n_2,n_3\in\mathbb{Z}\,.$$

The original action S depends only on periodic fields and preserves supersymmetry unlike the new terms. This is because S is the original action of $\mathcal{N}=4$ SYM in terms of only periodic fields on the thermal manifold $S^1_\beta\times\mathbb{R}^3$. As the periodicity is identical for all fields the supersymmetry transformations are unchanged and supersymmetry is preserved. For the entire action $\tilde S$ it is broken by the additional mass terms[1].

When looking at the new action (4.19) after the introduction of the phases note that we can write it as the old action with a set of new covariant derivatives. Focus first on the spinor fields. Here the covariant derivative becomes

$$D_\mu=\partial_\mu+\mathrm{i}A_\mu\to\nabla_\mu=\partial_\mu+\mathrm{i}A_\mu+\mathrm{i}\alpha_I\delta_{\mu,0} \qquad (4.20)$$

Due to the broken R-symmetry this would yield, however, a different covariant derivative for each of the fields. Similarly also each of the three complex scalar fields would obtain

[1] See also our discussion at the beginning of Chapter 3

Field	Phase	No R-sym.	SU(2)⊗SU(2)	SU(3)⊗U(1)
λ_1	$\alpha_1 = \pi T(2n_1+1)$	$3\pi T\ (n_1=1)$	$\pi T\ (n_1=0)$	$-3\pi T\ (n_1=-2)$
λ_2	$\alpha_2 = \pi T(2n_2+1)$	$5\pi T\ (n_2=2)$	$\pi T\ (n_2=0)$	$\pi T\ (n_2=0)$
λ_3	$\alpha_3 = \pi T(2n_3+1)$	$7\pi T\ (n_3=3)$	$-\pi T\ (n_3=-1)$	$\pi T\ (n_3=0)$
λ_4	$-\alpha_1-\alpha_2-\alpha_3$	$-15\pi T$	$-\pi T$	πT
ϕ_1	$\alpha_1+\alpha_2$	$8\pi T$	$2\pi T$	$-2\pi T$
ϕ_2	$\alpha_1+\alpha_3$	$10\pi T$	0	$-2\pi T$
ϕ_3	$-\alpha_2-\alpha_3$	$-12\pi T$	0	$-2\pi T$

Table 4.1: In this table we present possible choices for the phases which preserve a different amount of R-symmetry. The first choice is one where no R-symmetry is preserved. The other two choices preserve a particularly large amount of R-symmetry that will be suggestive of different $\mathcal{N}=1$ and $\mathcal{N}=2$ multiplets, respectively. Those are considered in more detail below. Note that these are not actual supersymmetric multiplets as bosons and fermions have different masses.

its own new covariant derivative depending on the phase, for example

$$\nabla_\mu = \partial_\mu + \mathrm{i}A_\mu + \mathrm{i}(\alpha_1+\alpha_2)\delta_{\mu,0}\,, \tag{4.21}$$

for the field ϕ_1.

These covariant derivatives can be interpreted as the coupling of the fields to the background gravity. Namely they couple to the compactified S^1_β where the phases determine the respective couplings.

4.1.3 R-symmetry breaking patterns

The above consideration suggest that the R-symmetry is broken in $\mathcal{N}=4$ at nonzero coupling while in Section 3.1 of the previous chapter we saw that the R-symmetry is preserved at zero coupling. In this section we look more closely at this breaking. Table 4.1 shows that by conveniently choosing the integers that are associated to the phases it is possible to preserve some amount of R-symmetry. Considering only the classical action as we did above the choice of these phases seems arbitrary. Following [113, 114] it might be possible to find constraints on how and which amount of R-symmetry is preserved. Therefore we will consider a dynamical calculation in Appendix H.

This consideration will also have an impact on the Ward identity calculation. We showed that $B=3h$ at zero coupling using a Ward identity and its thermal corrections. In fact, when constraining the two-point functions after taking the OPE (see Subsection 3.1.4, Equation (3.66)) we explicitly used the $\mathfrak{usp}(4)_R$ symmetry preserved by the Wilson line. Therefore we hope to find a choice which preserves as much R-symmetry as possible. Ideally this allows us to constrain the two-point functions of the thermal BWI similarly to before and still obtain information on the relation between bremsstrahlung and stress tensor.

What is next, we focus on possible subgroups of the original $\mathrm{SU}(4)_R$ as presented in Table 4.1 preserving a large amount of R-symmetry.

4.1.3.1 Preserved subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

It is convenient to consider a different choice where two pairs of two fermionic phases are equal

$$\alpha_1 = \alpha_2 = -\alpha_3 = -\alpha_4 \ . \tag{4.22}$$

For example we can choose $n_1 = n_2 = 0$ and $n_3 = -1$ yielding $\hat{\alpha}_1 = \hat{\alpha}_2 = \pi T$ and $\hat{\alpha}_3 = \hat{\alpha}_4 = -\pi T$ such that the three scalar phases become

$$\hat{\alpha}_1 = 2\pi T \ , \quad \hat{\alpha}_2 = \hat{\alpha}_3 = 0 \ . \tag{4.23}$$

In this case we see that also the scalar fields are separated with one having a different phase than the two others. This suggests that the two spinors λ_1 and λ_2 form a $\mathcal{N} = 2$ vector multiplet together with the complex scalar ϕ_1 and the gauge field. Of course supersymmetry is in fact broken by the introduction of the phases. This can also be seen by the masses which are not equal for bosons and fermions[2]. However, the identical masses for the spinors are suggestive of the respective supermultiplet. Similarly the remaining two fermions λ_3 and λ_4 and scalars ϕ_2 and ϕ_3 suggest an $\mathcal{N} = 2$ hypermultiplet. The R-symmetry is reduced to a maximal subalgebra [115]

$$\mathfrak{su}(4) \to \mathfrak{su}(2) \oplus \mathfrak{su}(2) \ . \tag{4.24}$$

This preserved symmetry can be made explicit in the action. We focus on the new mass terms. Recall that with the above definitions we can write

$$\begin{aligned}\tilde{S} \to S + \frac{1}{2g^2}\mathrm{tr}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \Big[&\frac{2\pi}{\beta}\left(\bar{\lambda}^1\sigma^0\lambda_1 + \bar{\lambda}^2\sigma^0\lambda_2 - \bar{\lambda}^3\sigma^0\lambda_3 - \bar{\lambda}^4\sigma^0\lambda_4\right) + \\ &+\frac{16\pi^2}{\beta^2}\phi_1\bar{\phi}_1 + \frac{8\pi \mathrm{i}}{\beta}\left(\partial_0\phi_1\bar{\phi}_1 - \phi_1\partial_0\bar{\phi}_1\right)\Big] \ .\end{aligned} \tag{4.25}$$

We do not see terms for the scalars ϕ_2 and ϕ_3 because we made a convenient choice that set their masses to zero. The above equation otherwise would become more confusing while still preserving the same amount of symmetry.
It is now convenient to define two projectors

$$P_1 := \frac{\mathbb{1} + \gamma_5}{2} \ , \quad P_2 := \frac{\mathbb{1} - \gamma_5}{2} \ , \quad \gamma_5 = \mathrm{diag}(-1,-1,1,1) \ . \tag{4.26}$$

[2] For the vector multiplet masses moreover would have to be zero to ensure supersymmetry.

The above projectors clearly satisfy

$$P_1^2 = P_1 \ , \quad P_2^2 = P_2 \ , \quad P_1 + P_2 = \mathbb{1} \ , \quad P_1 \cdot P_2 = 0 \ . \tag{4.27}$$

This separation is almost identical to the separation of 4-component Dirac spinors to 2-component Weyl spinors. The same matrices are used but instead of the spinor indices they now affect R-symmetry indices spliting the index $I = 1, \dots, 4$ to $i = 1,2$ and $\bar{\imath} = 1,2$. The scalars and spinors thus separate into two independent $\mathfrak{su}(2)$'s as follows

$$P_1\lambda_I = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \\ 0 \end{pmatrix} \ , \quad P_1\Phi_{IJ}P_1 + P_2\Phi_{IJ}P_2 = \begin{pmatrix} 0 & \phi_1 & 0 & 0 \\ -\phi_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\phi}_1 \\ 0 & 0 & -\bar{\phi}_1 & 0 \end{pmatrix} \ , \tag{4.28}$$

$$P_2\lambda_I = \begin{pmatrix} 0 \\ 0 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \ , \quad P_1\Phi_{IJ}P_2 + P_2\Phi_{IJ}P_1 = \begin{pmatrix} 0 & 0 & \phi_2 & \phi_3 \\ 0 & 0 & \bar{\phi}_3 & -\bar{\phi}_2 \\ -\phi_2 & -\bar{\phi}_3 & 0 & 0 \\ -\phi_3 & \bar{\phi}_2 & 0 & 0 \end{pmatrix} \ . \tag{4.29}$$

We can thus separate the matrices into independent $\mathfrak{su}(2)_i$ and $\mathfrak{su}(2)_{\bar{\imath}}$ ones. The algebra $\mathfrak{su}(2)_i \oplus \mathfrak{su}(2)_{\bar{\imath}}$ is a maximal subalgebra of the original $\mathfrak{su}(4)_R$ [115]. We can define two new scalar matrices

$$\Phi_{ij} = \begin{pmatrix} 0 & \phi_1 \\ -\phi_1 & 0 \end{pmatrix}_{ij} \ , \quad \Phi_{\bar{\imath}\bar{\jmath}} = \begin{pmatrix} \phi_2 & \phi_3 \\ \bar{\phi}_3 & \bar{\phi}_2 \end{pmatrix}_{\bar{\imath}\bar{\jmath}} \qquad \Rightarrow \qquad \Phi_{IJ} = \begin{pmatrix} \Phi_{ij} & \Phi_{\bar{\imath}\bar{\jmath}} \\ -\bar{\Phi}^{\bar{\imath}\bar{\jmath}} & -\bar{\Phi}^{ij} \end{pmatrix} \ . \tag{4.30}$$

Note that the original matrix satisfies the self-duality condition [112] and is anti-symmetric

$$\bar{\Phi}^{IJ} = \frac{1}{2}\epsilon^{IJKL}\Phi_{KL} \ , \quad \Phi_{IJ} = -\Phi_{JI} \ . \tag{4.31}$$

Compare this to the new matrices we found above. The scalars Φ_{ij} inherit the anti-symmetry while the self-duality is lost in the reduction

$$\Phi_{ij} = -\Phi_{ji} \ . \tag{4.32}$$

Note, however, that the matrix Φ_{ij} depends only on one complex scalar field ϕ_1. Therefore it is in fact convenient to write the splitting of the scalars as

$$\sigma := \phi_1 \ , \quad \phi_{\bar{\imath}} := \begin{pmatrix} \phi_2 \\ \phi_3 \end{pmatrix} \tag{4.33}$$

Similarly the fermions can be split into

$$\lambda_i = (\lambda_1, \lambda_2)_i \ , \quad \lambda_{\bar{\imath}} = (\lambda_3, \lambda_4)_{\bar{\imath}} \tag{4.34}$$

Using this notation the action can be rewritten in form manifestly showing the $\mathfrak{su}(2)_i \oplus \mathfrak{su}(2)_{\bar{\imath}}$ symmetry. We focus on the mass-like terms

$$\begin{aligned}\tilde{S} \to S + \frac{1}{2g^2}\mathrm{tr}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \Big[&\frac{2\pi}{\beta}\left(\bar{\lambda}^i\sigma^0\lambda_i - \bar{\lambda}^{\bar{\imath}}\sigma^0\lambda_{\bar{\imath}}\right) + \frac{8\pi^2}{\beta^2}\sigma\bar{\sigma} \\ &+ \frac{4\pi\mathrm{i}}{\beta}\left(\partial_0\sigma\bar{\sigma} - \sigma\partial_0\bar{\sigma}\right)\Big] \ .\end{aligned} \tag{4.35}$$

As two of the scalar masses are zero in our choice the fields $\phi_{\bar{\imath}}$ only appear in the original action S.

Wilson loop defect

What is more, when introducing the Wilson loop operator, the R-symmetry is broken due to the interaction of scalar fields and WL. We saw above that this breaks $\mathfrak{su}(4)_R \to \mathfrak{usp}(4)_R$. For the case at hand the $\mathfrak{su}(4)_R$ symmetry is broken in a way that is suggestive of an $\mathcal{N} = 2$ hyper and an $\mathcal{N} = 2$ vector multiplet. Only the scalar field of the vector multiplet, $\phi_1 = \sigma$ in our case, can couple to the Wilson loop[3]. The action reduced to $\mathcal{N} = 2$ multiplets is invariant under $\mathrm{SU}(2)_i \otimes \mathrm{SU}(2)_{\bar{\imath}} \otimes \mathrm{U}(1)$ [79]. The scalar $\phi_1 = \sigma$ is in a singlet with respect to both SU(2)'s but charged under the U(1). When coupled to the Wilson loop the U(1) is thus broken [38]. Further note the isomorphism $\mathfrak{su}(2) \simeq \mathfrak{usp}(2)$ [115] which thus induces a breaking of the R-symmetry as

$$\mathfrak{usp}(4)_R \to \mathfrak{usp}(2)_i \oplus \mathfrak{usp}(2)_{\bar{\imath}} \ , \tag{4.36}$$

where Usp(2)⊗USp(2) is a maximal subgroup of Usp(4) [115]. This can be made explicit by choosing $n_r = \delta_{r,6}$. We can then write down the matrix Ω_{IJ} and see a splitting intos the two $\mathfrak{usp}(2)$'s. In our convention this yields

$$\Omega_{IJ} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \Rightarrow \quad \Omega_{ij} = \epsilon_{ij} \ , \quad \Omega_{\bar{\imath}\bar{\jmath}} = -\epsilon_{\bar{\imath}\bar{\jmath}} \ , \quad \Omega_{i\bar{\jmath}} = 0 \ . \tag{4.37}$$

We can then straight forwardly restrict to be only in $\mathfrak{usp}(2)_i$ by choosing the appropriate indices. By similarity all arguments presented in the following also hold for a restriction to

[3]It is the current state of the art that a coupling of a scalar field coming from a hypermultipet to the Wilson loop is not allowed.

$\mathfrak{usp}(2)_{\bar{\imath}}$. The OPE is not affected by the reduction of the R-symmetry and it is sufficient to apply the reduction to the new operators. Consider for instance that after applying the OPEs the thermal correction will depend on the correlators (3.66). For example

$$\left\langle \mathbb{A}^I_\alpha(0)\mathcal{A}^J_\beta(u_\mu)\right\rangle_W = u_\mu(\sigma^\mu\bar{\sigma}^3)_{\alpha\beta}\Omega^{IJ} f_{\mathbb{A}\mathcal{A}}(u^2_\mu) \,. \tag{4.38}$$

We restrict the fields to be in the $\mathfrak{usp}(2)_i$. Then

$$\left\langle \mathbb{A}^i_\alpha(0)\mathcal{A}^i_\beta(u_\mu)\right\rangle_W = u_\mu(\sigma^\mu\bar{\sigma}^3)_{\alpha\beta}\epsilon^{ij} f_{\mathbb{A}\mathcal{A}}(u^2_\mu) \,, \tag{4.39}$$

and similarly for all other operators appearing in the OPE. Furthermore the coefficient ${\mathfrak{C}_I}^{JKLMN}$ is likewise reduced to ${\mathfrak{C}_i}^{jklmn}$. This index structure was defined in Equation(2.34).

What is more, the phases of the scalar fields introduce additional mass terms. These must then consequentially appear in the OPE as do the fermionic mass terms through the operator $\mathcal{A}$. This yields further corrections which could potentially break the relation $B = 3h$ in this setting.

Let us thus review the process that lead to the final form of the Ward identity at zero coupling. We first considered the original terms of the Ward identity at $T = 0$, (2.33). They where given through kinematic functions g_i defined in Equation (2.13), some spinor coefficients and the index structure ${\mathfrak{C}_I}^{JKLMN}$. The latter one was discussed above, it can conveniently be reduced to the $\mathfrak{usp}(2)_i$ subalgebra. The kinematic function can be thermalized and the spinor structure is neither affected by finite temperature nor by the broken R-symmetry.
In a subsequent step we considered a thermal contribution to the BWI (3.28), namely

$$\left\langle {\psi_{I\alpha}}^{KL}\mathbb{O}^{MN}\mathcal{Q}^J_\beta\mathcal{M}_\lambda(u_0,\vec{u})\right\rangle_{W,T} \,. \tag{4.40}$$

The operator $\mathcal{M}_\lambda$ is the newly introduced mass term for the fermions. We then considered an OPE. We assumed the two operators $\psi_{I\alpha}^{KL}$ and $\mathbb{O}^{MN}$ such that we could use the expansion despite being in a finite temperature setting [54]. This lead to several two-point functions. These can be reduced in the case of broken R-symmetry as given above.
Lastly we considered that the Ward identity has an integral over the insertion points of the operator $\mathcal{M}_\lambda$. By restricting the spacetime dependence of the correlators the integral yielded a unique dependence on spinor indices. This was sufficient to ensure the relation $B = 3h$.
When we consider also scalar masses, two new mass terms

$$\mathcal{M}_\phi = \frac{8\pi^2}{\beta^2}\Phi_{ij}\bar{\Phi}^{ij} \quad \text{and} \quad \tilde{\mathcal{M}_\phi} = \frac{4\pi\mathrm{i}}{\beta}\left(\partial_0\Phi_{ij}\bar{\Phi}^{ij} - \Phi_{ij}\partial_0\bar{\Phi}^{ij}\right) \,, \tag{4.41}$$

are introduced to the Ward identity. Similar to the fermions this will yield the contribution of a correlator such that the BWI (3.28) will become

$$\begin{aligned} -\left\langle \mathcal{Q}^n_\beta \left(\psi_{i\alpha}^{\ jk} \mathbb{O}^{lm} \right) \right\rangle_{W,T} = \int_0^\beta \mathrm{d}u_0 \int \mathrm{d}^3 u \ \Big[&\left\langle \psi_{i\alpha}^{\ kl} \mathbb{O}^{mn} \mathcal{Q}^j_\beta \mathcal{M}_\lambda(u_0, \vec{u}) \right\rangle_{W,T} \\ &+ \left\langle \psi_{i\alpha}^{\ kl} \mathbb{O}^{mn} \mathcal{Q}^j_\beta \mathcal{M}_\phi(u_0, \vec{u}) \right\rangle_{W,T} \\ &+ \left\langle \psi_{i\alpha}^{\ kl} \mathbb{O}^{mn} \mathcal{Q}^j_\beta \tilde{\mathcal{M}}_\phi(u_0, \vec{u}) \right\rangle_{W,T} \Big] \ . \end{aligned} \tag{4.42}$$

As the OPE was not taken with respect to the mass operator, we can use the same OPE for this new contribution. Therefore, we also can use the same restrictions on the yielded two-point functions as before. This is possible as the mass operators $\mathcal{M}_\phi$ and $\tilde{\mathcal{M}}_\phi$ are singlets with respect to the spacetime-, spinor- and R-symmetry. Combined with the acting supercharge $\mathcal{Q}^i_\alpha$ it is then in the same representation as $\mathcal{A}^i_\alpha$, just possibly with a different conformal dimension. The dimension, however, was not relevant for the argument. To see this consider Equation (3.66). Restricting ourselves to Λ^i_α we now find

$$\left\langle \mathbb{A}^i_\alpha(0) \mathcal{Q}^j_\beta \mathcal{M}_\lambda(u_\mu) \right\rangle_W = u_\mu (\sigma^\mu \bar{\sigma}^3)_{\alpha\beta} \epsilon^{ij} f_{\mathbb{A}\mathcal{M}_\lambda}(u_\mu^2) \ , \tag{4.43}$$

$$\left\langle \mathbb{A}^i_\alpha(0) \mathcal{Q}^j_\beta \mathcal{M}_\phi(u_\mu) \right\rangle_W = u_\mu (\sigma^\mu \bar{\sigma}^3)_{\alpha\beta} \epsilon^{ij} g_{\mathbb{A}\mathcal{M}_\phi}(u_\mu^2) \tag{4.44}$$

$$\left\langle \mathbb{A}^i_\alpha(0) \mathcal{Q}^j_\beta \tilde{\mathcal{M}}_\phi(u_\mu) \right\rangle_W \ , = u_\mu (\sigma^\mu \bar{\sigma}^3)_{\alpha\beta} \epsilon^{ij} \tilde{g}_{\mathbb{A}\tilde{\mathcal{M}}_\phi}(u_\mu^2) \ . \tag{4.45}$$

The new functions g and $\tilde{g}$ have different conformal dimensions as the original f and thus yield different powers of β when we integrate over them. This yields an equation similar to (3.72) with different integral functions $\mathcal{J}_{\mathbb{A}\mathcal{M}_\lambda}(\beta)$, $\mathcal{J}_{\mathbb{A}\mathcal{M}_\phi}(\beta)$ and $\mathcal{J}_{\mathbb{A}\tilde{\mathcal{M}}_\phi}(\beta)$. Their result differs in the prefactor and the power of β. For the argument we make, however, these functions are clearly not relevant. Henceforth, the relation $B = 3h$ is again obtained.
Conclusively, it is very important to note that this relies on the projection to the preserved $\mathfrak{usp}(2)_i$ which allows for all correlators to depend on the singlet matrix Ω_{ij}. We cannot assume that this argumentation also works for a different breaking of the R-symmetry. One example of a different breaking is considered in the next subsection. We will focus on the case where all bosons and three of the four fermions have the same mass. This will be suggestive of a triplet of $\mathcal{N} = 1$ chiral multiplets, similarly to the considerations made in the beginning of this subsection.

4.1.3.2 Preserved subalgebra $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$

The combination of phases discussed above is one convenient choice that preserves a large amount of R-symmetry. Another possibility is to preserve an $\mathfrak{su}(3)$ symmetry that interchanges three of the spinors and the three complex scalars. It is obtained by the choice

$$\alpha_2 = \alpha_3 = \alpha_4 = -3\alpha_1 \ . \tag{4.46}$$

For convenience let us specify $\hat{\alpha}_2 = \hat{\alpha}_3 = \hat{\alpha}_4 = \pi T$ and $\hat{\alpha}_1 = -3\pi T$. The bosonic phases then are

$$\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_3 = -2\pi T \ , \tag{4.47}$$

and thus all scalar fields obtain the same phase. This splitting is very suggestive of $\mathcal{N} = 1$ supermultipets. The fermion λ_1 with $\hat{\alpha}_1 = -3\pi T$ is associated to a vector multiplet while the remaining spinors together with the three scalars form a triplet of chiral multiplets. Again these phases suggest mass terms for the respective fields and as they are different for bosons and fermions supersymmetry is clearly broken as expected at finite temperature [53]. The breaking of the R-symmetry algebra consequentially is

$$\mathfrak{su}(4) \to \mathfrak{su}(3) \oplus \mathfrak{u}(1) \ . \tag{4.48}$$

Note that the combination SU(3)⊗U(1) is a maximal subgroup of the original SU(4) group [115]. In the previous subsection we introduced projectors that restricted the $\mathfrak{su}(4)_R$ invariant structures to the respective preserved subgroups. By similarity we suggest

$$\hat{P}_1 = \mathrm{diag}(1,0,0,0) \ , \quad \hat{P}_2 = \mathrm{diag}(0,1,1,1) \ . \tag{4.49}$$

It is easy to check that these matrices are indeed projectors as they satisfy the relations (4.27). Let us consider how these matrices act on the scalar and spinor fields of the Lagrangian. We find

$$\hat{P}_1\lambda_I = \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ , \quad \hat{P}_1\Phi_{IJ}\hat{P}_1 = 0 \ , \quad \hat{P}_2\Phi_{IJ}\hat{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\phi}_3 & -\bar{\phi}_2 \\ 0 & -\bar{\phi}_3 & 0 & \bar{\phi}_1 \\ 0 & \bar{\phi}_2 & -\bar{\phi}_1 & 0 \end{pmatrix} \ , \tag{4.50}$$

$$\hat{P}_2\lambda_I = \begin{pmatrix} 0 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \ , \quad \hat{P}_1\Phi_{IJ}\hat{P}_2 = \begin{pmatrix} 0 & \phi_1 & \phi_2 & \phi_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \quad \hat{P}_2\Phi_{IJ}\hat{P}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\phi_1 & 0 & 0 & 0 \\ -\phi_2 & 0 & 0 & 0 \\ -\phi_3 & 0 & 0 & 0 \end{pmatrix} \ . \tag{4.51}$$

We see that this does not create a clear structure of scalar matrices as we had in the above case. Let us thus first look at the Yukawa interactions

$$\mathcal{L}_Y = -\mathrm{i}\Phi^{IJ}\{\bar{\lambda}_I, \bar{\lambda}_J\} + \mathrm{h.c.} = -\mathrm{i}\{\bar{\lambda}_I(\hat{P}_1 + \hat{P}_2)^2\Phi^{IJ}, (\hat{P}_1 + \hat{P}_2)^2\bar{\lambda}_J\} + \mathrm{h.c.}$$

$$= -\mathrm{i}\{(\bar\lambda_I\hat P_1)(\hat P_1\Phi^{IJ}\hat P_1),(\hat P_1\bar\lambda_J)\} - \mathrm{i}\{(\bar\lambda_I\hat P_2)(\hat P_2\Phi^{IJ}\hat P_2),(\hat P_2\bar\lambda_J)\}$$

$$- \mathrm{i}\{(\bar\lambda_I\hat P_1)(\hat P_1\Phi^{IJ}\hat P_2),(\hat P_2\bar\lambda_J)\} - \mathrm{i}\{(\bar\lambda_I\hat P_2)(\hat P_2\Phi^{IJ}\hat P_1),(\hat P_1\bar\lambda_J)\} + \text{h.c.}$$

$$= -\mathrm{i}\left\{\begin{pmatrix}\bar\lambda_1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}\bar\lambda_1\\0\\0\\0\end{pmatrix}\right\} - \mathrm{i}\left\{\begin{pmatrix}0\\\bar\lambda_2\\\bar\lambda_3\\\bar\lambda_4\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&\bar\phi_3&-\bar\phi_2\\0&-\bar\phi_3&0&\bar\phi_1\\0&\bar\phi_2&-\bar\phi_1&0\end{pmatrix},\begin{pmatrix}0\\\bar\lambda_2\\\bar\lambda_3\\\bar\lambda_4\end{pmatrix}\right\}$$

$$- \mathrm{i}\left\{\begin{pmatrix}\bar\lambda_1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0&\phi_1&\phi_2&\phi_3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\begin{pmatrix}0\\\bar\lambda_2\\\bar\lambda_3\\\bar\lambda_4\end{pmatrix}\right\} - \mathrm{i}\left\{\begin{pmatrix}0\\\bar\lambda_2\\\bar\lambda_3\\\bar\lambda_4\end{pmatrix}\begin{pmatrix}0&0&0&0\\-\phi_1&0&0&0\\-\phi_2&0&0&0\\-\phi_3&0&0&0\end{pmatrix},\begin{pmatrix}\bar\lambda_1\\0\\0\\0\end{pmatrix}\right\} + \text{h.c.}$$

$$= \mathrm{i}\epsilon^{ijk}\left\{\bar\psi_i\bar\phi_j,\bar\psi_k\right\} - \mathrm{i}\left\{\bar\lambda\phi^i,\bar\psi_i\right\} + \mathrm{i}\left\{\bar\psi_i\phi^i,\bar\lambda\right\} + \text{h.c.}\ . \tag{4.52}$$

Following this consideration it is convenient to introduce a new notation respecting the $\mathfrak{su}(3)\oplus\mathfrak{u}(1)$ symmetry

$$\lambda_1 \to \lambda\ , \quad \begin{pmatrix}\lambda_2\\\lambda_3\\\lambda_4\end{pmatrix} \to \begin{pmatrix}\psi_1\\\psi_2\\\psi_3\end{pmatrix} =: \psi_i\ , \quad \phi_i := \begin{pmatrix}\phi_1\\\phi_2\\\phi_3\end{pmatrix}\ , \quad i = 1,2,3\ . \tag{4.53}$$

The new index $i = 1,2,3$ reflects the $\mathfrak{su}(3)$ invariance of the respective vectors. Writing the new mass terms of the action in a manner reflecting this invariance yields

$$\tilde S = S + \frac{1}{2g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\,\mathrm{tr}\Big[-\frac{2\pi}{\beta}\bar\psi^i\bar\sigma^0\psi_i + \frac{6\pi}{\beta}\bar\lambda\bar\sigma^0\lambda + \frac{16\pi^2}{\beta^2}\phi^i\bar\phi_i$$
$$- \frac{8\pi\mathrm{i}}{\beta}\left(\phi^i\partial^0\bar\phi_i - \partial_0\phi^i\bar\phi_i\right)\Big]\ . \tag{4.54}$$

The newly arising terms can in fact be identified as specific operators. The mass term of the scalars by itself preserves the $\mathfrak{su}(4)_R$ invariance, the breaking can only be seen for the spinor fields. Therefore the pure mass term of the scalars can be identified as the primary Konishi operator [116]

$$\mathcal{K}_1 = \frac{1}{2}\mathrm{tr}\phi^i\bar\phi_i\ . \tag{4.55}$$

The R-symmetry group has 15 generators, three of which can be diagonalized simultaneously. Most importantly one of them is $t_{15} = \mathrm{diag}(1,1,1,-3)$. The spinors and scalars have charges associated to this generators. These are precisely proportional to the mass terms in the above equation [117, 118]. This yields that the remaining terms are in fact a component of the R-symmetry current

$$J_0^{15} = \frac{1}{\sqrt{6}}\mathrm{tr}\bar{\psi}^i\bar{\sigma}^0\psi_i - \frac{3}{\sqrt{6}}\mathrm{tr}\bar{\lambda}\bar{\sigma}^0\lambda + \frac{2}{\sqrt{6}}\mathrm{tr}\left(\phi^i\partial^0\bar{\phi}_i - \partial_0\phi^i\bar{\phi}_i\right) . \tag{4.56}$$

Therefore we can write the mass terms of the action conveniently as

$$\tilde{S} = S + \frac{1}{2g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\frac{32\pi^2}{\beta^2}\mathcal{K}_1 - \frac{2\sqrt{6}\pi\mathrm{i}}{\beta}J_0^{15}\right] . \tag{4.57}$$

Wilson loop defect

As in the previous case the coupling of the scalar fields to the Wilson loop should be discussed. They are in a fundamental of $\mathfrak{su}(3)$, however, this is not associated to a supersymmetry. Therefore we can split the triplet into a doublet and a single scalar[4]

$$\phi_i = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \to \begin{pmatrix} \sigma \\ \phi_{\bar{\imath}} \end{pmatrix} , \quad \bar{\imath} = 1, 2 . \tag{4.58}$$

The scalar doublet $\phi_{\bar{\imath}}$ can then again be interpreted as coming from an $\mathcal{N} = 2$ hypermultiplet. Due to the scalar and fermionic phases being equal, respectively, this is as in the breaking discussed previously. We therefore know that this will yield a $\mathfrak{su}(2)_{\bar{\imath}} \simeq \mathfrak{usp}(2)_{\bar{\imath}}$ preserved by the Wilson line. The complex scalar $\phi_1 = \sigma$ again preserves a $\mathfrak{u}(1)_\sigma$ symmetry. It can be considered as coming from an $\mathcal{N} = 2$ vector multiplet and hence couple to the Wilson loop. The difference to the previous consideration, however, is that the spinors of the vector multiplet obtained different masses and we therefore interpreted them as $\mathcal{N} = 1$ vector and chiral multiplets, respectively. Hence the $\mathfrak{su}(2)_i$ symmetry obtained previously is not present in the case at hand.

The $\mathfrak{u}(1)_\sigma$ symmetry associated to the complex scalar σ is broken when the Wilson loop is introduced. We can thus again write the Matrix (4.40)

$$\Omega_{IJ} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{4.59}$$

The full algebra preserved after the Wilson loop insertion is

$$\mathfrak{su}(3) \oplus \mathfrak{u}(1) \to \mathfrak{su}(2)_{\bar{\imath}} \oplus \mathfrak{u}(1)_\sigma \oplus \mathfrak{u}(1) \to \mathfrak{usp}(2)_{\bar{\imath}} \oplus \mathfrak{u}(1) . \tag{4.60}$$

Restricting to operators that are in the leftover $\mathfrak{u}(1)$ corresponds to taking the original $\mathfrak{usp}(4)_R$ index $I \equiv 1$. Comparing this to the symplectic Ω_{IJ} we see that the respective component is zero because it is skew-symmetric. We then would find for example

$$\left\langle \mathbb{A}^I_\alpha(0) \mathcal{A}^J_\beta(u_\mu) \right\rangle_W \to \left\langle \mathbb{A}^1_\alpha(0) \mathcal{A}^1_\beta(u_\mu) \right\rangle_W \propto \Omega^{11} = 0 . \tag{4.61}$$

This would be the case for all operators in the Ward identity. This implies that we are not able to use the Ward identity calculation introduced in 3.1 to relate the bremsstrahlung and the stress tensor.

If we restrict all operators to be in the $\mathfrak{usp}(2)_{\bar{\imath}}$ representation we can follow the argument

[4]There is choice that we make here corresponding to a rotational symmetry between all three complex scalars.

starting at equation (4.40) analogously and obtain that $B = 3h$ is still preserved.

What is next, we consider a different ansatz. By using a Fourier series representation of the time dimension the theory is reduced to three dimensions. In this case we are able to show that R-symmetry can be preserved. For the beta function this yields that the expected result is to find that the phases go to zero hence preserving R-symmetry.

4.2 Fourier series to 3d

In the previous section we chose an ansatz of field redefinitions allowing us to write the action in terms of periodic fields only. This yielded a convenient splitting of the action like

$$S_{\text{th. }\mathcal{N}=4} = S_{\text{SUSY}} + S_{\text{new}} \,. \tag{4.62}$$

The part S_{SUSY} is supersymmetric and in the above case in fact has an identical Lagrangian to the $\mathcal{N} = 4$ theory at zero temperature. As this part depended only on periodic fields it preserves supersymmetry. The new term of the action is the one breaking supersymmetry. We discussed above at length that this ansatz, however, yields an ambiguity concerning the R-symmetry. Therefore we suggest an alternative approach here.
The compactification of the time dimension is analogous to a Kaluza-Klein reduction as we already saw in Section 3.2.3. Following [119] we write the four-dimensional theory as a tower of KK modes with masses depending on the temperature. The modes are introduced by a Fourier series. In this section, we will introduce these modes and discuss their effects on the manner in which we can write the Lagrangian as well as the possibility to preserve R-symmetry. We focus on the scalar and spinor fields only as gauge and ghost fields are not relevant for the considerations at interest.
The compactification of the scalars is straight forward. We use a simple Fourier series. For simplicity start by considering a single scalar

$$\phi(\tau, \vec{x}) = \sum_n e^{i\hat{\omega}_n \tau} \phi_n(\vec{x}) \qquad \text{with} \quad \hat{\omega}_n = \frac{\pi}{\beta}(2n) \,. \tag{4.63}$$

The extension to a set of scalars ϕ_i is straight forward. $\mathcal{N} = 4$ SYM theory is manifestly $\mathrm{SU}(4)_R$ invariant and the scalars are conveniently written in a self-dual matrix Φ^{IJ}. The Fourier series for this matrix is written as

$$\Phi^{IJ}(\tau, \vec{x}) = \sum_n e^{\mathrm{i}\hat{\omega}_n \tau} \Phi^{IJ}_n(\vec{x}) := \sum_n e^{\mathrm{i}\hat{\omega}_n \tau} \begin{pmatrix} 0 & \phi^1_n & \phi^2_n & \phi^3_n \\ -\phi^1_n & 0 & \bar{\phi}^3_{-n} & -\bar{\phi}^2_{-n} \\ -\phi^2_n & -\bar{\phi}^3_{-n} & 0 & \bar{\phi}^1_{-n} \\ -\phi^3_n & \bar{\phi}^2_{-n} & -\bar{\phi}^1_{-n} & 0 \end{pmatrix}^{IJ} \,. \tag{4.64}$$

From this matrix we see that complex conjugation is consistent. The conjugate scalars indeed get an Fourier phases $\hat{\omega}_n$ with an opposite sign. Furthermore, the self-duality is preserved and continues to hold for each of the modes

$$\Phi_n^{IJ} = \frac{1}{2}\epsilon^{IJKL}\bar{\Phi}_{-n\,KL}\ . \tag{4.65}$$

This is a first indication that R-symmetry is preserved by the KK-modes. To fully understand this we will below consider the Yukawa terms.
Before this, we turn to the spinors. Similarly to above we find [119]

$$\lambda^{I\alpha}(\tau,\vec{x}) = \sum_n e^{\mathrm{i}\omega_n\tau}\lambda_n^{I\alpha}(\vec{x}) \qquad \text{with} \quad \omega_n = \frac{\pi}{\beta}(2n+1)\ . \tag{4.66}$$

What is next, we consider the kinetic terms for bosons and spinors. We will show that these can be combined in a set of periodic fields with a correction term similar to what we found in the zero coupling calculation 3.1. This will thus link this approach to the previous one.

4.2.1 Kinetic terms

The Fourier representation of the compactification can be plugged in the kinetic terms of the action. This is in fact where we expect all correction terms to come from. The reason being the derivatives acting. We find

$$\begin{aligned}
S_{\text{kin.}} &= \frac{1}{2g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \mathrm{tr}\left[\partial_\mu\Phi^{IJ}\partial^\mu\bar{\Phi}_{IJ} + \mathrm{i}\bar{\lambda}^I\sigma^\mu\partial_\mu\lambda_I + \mathrm{i}\lambda_I\bar{\sigma}^\mu\partial_\mu\bar{\lambda}^I\right] \\
&= \frac{1}{2g^2}\sum_{n,m}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, e^{\mathrm{i}(\hat{\omega}_n-\hat{\omega}_m)\tau}\mathrm{tr}\left[\hat{\omega_n}\hat{\omega}_m\Phi_n^{IJ}\bar{\Phi}_{m\,IJ} + \partial_i\Phi_n^{IJ}\partial^i\bar{\Phi}_{m\,IJ}\right. \\
&\qquad\left. -\omega_m\bar{\lambda}_n^I\sigma^0\lambda_{m\,I} + \mathrm{i}\bar{\lambda}_n^I\sigma^i\partial_i\lambda_{m\,I} + \omega_m\lambda_{n\,I}\bar{\sigma}^0\bar{\lambda}_m^I + \mathrm{i}\lambda_{n\,I}\bar{\sigma}^i\partial_i\bar{\lambda}_m^I\right] \\
&= \frac{T}{2g^2}\sum_n\int \mathrm{d}^3x\,\mathrm{tr}\left[+\partial_i\Phi_n^{IJ}\partial^i\bar{\Phi}_{n\,IJ} + \mathrm{i}\bar{\lambda}_n^I\sigma^i\partial_i\lambda_{n\,I} + \mathrm{i}\lambda_{n\,I}\bar{\sigma}^i\partial_i\bar{\lambda}_n^I\right. \\
&\qquad\left. +\hat{\omega_n}^2\Phi_n^{IJ}\bar{\Phi}_{n\,IJ} - 2\omega_n\bar{\lambda}_n^I\sigma^0\lambda_{n\,I}\right]
\end{aligned} \tag{4.67}$$

The final action thus contains indeed what appears as a three-dimensional theory. This theory has a tower of infinitely many massive scalar[5] and massive spinor fields. To find the final equation we solved the integration over τ which yielded Kronecker deltas for the frequencies. Note that these fields can be recombined to periodic fields. For instance note

[5] The scalars Φ_0^{IJ} are massless.

that we can write

$$\omega_n = \hat{\omega}_n + \frac{\pi}{\beta} \,. \tag{4.68}$$

The fermionic frequencies can be separated in such a way. Then doing all the steps backwards we can recreate a four-dimensional theory depending only on periodic fields

$$S_{\text{kin.}} = S_{\text{kin.}}^{(\text{periodic})} + \frac{1}{2g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, \mathrm{tr}\left[\frac{2\pi}{\beta}\lambda_I\bar{\sigma}^0\bar{\lambda}^I\right] \tag{4.69}$$

This is (up to a field redefinition in terms of the coupling) exactly the action in (3.19) with $\alpha = {}^{\pi}\!/\!_{\beta}$. Therefore we find that the difference between this approach and the one considered in the previous one must be yielded by interactive terms. For this it is further notable that we did implicitly fix a choice for α in our calculation. As the sums are infinite we would find a similar result by taking out $\alpha = {}^{3\pi}\!/\!_{\beta}$ as the mass. This freedom is reflected in the freedom for α in (3.18). Similarly one can argue that also mass terms similar to (4.17) can be obtained. If we restrict to the $g = 0$ case, however, we would be free to chose the scalar phases. Implicitly we chose the zero mode in our above calculation which does not yield a mass term.
This freedom of choosing is different here than above. We have no interactions which break the R-symmetry and hence all choices preserve symmetry in exactly the same manner and our choice truly does not matter.

4.2.2 Interaction

We can infer from the previous considerations that the interactions are what should mostly interest us. Therefore consider the Yukawa terms

$$\begin{aligned}
S_{\text{Yukawa}} &= \frac{\mathrm{i}}{2g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \mathrm{tr}\ \Phi^{IJ}\{\bar{\lambda}_I, \bar{\lambda}_J\} + h.c. \\
&= \frac{\mathrm{i}}{2g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \sum_{n,m,k} e^{\mathrm{i}(\hat{\omega}_k - \omega_n - \omega_m)\tau}\mathrm{tr}\ \Phi_k^{IJ}\{\bar{\lambda}_{n\,I}, \bar{\lambda}_{m\,J}\} + h.c. \\
&= \frac{\mathrm{i}}{2g^2}T\int \mathrm{d}^3x \sum_{n,m} \mathrm{tr}\ \Phi^{IJ}_{m+n+1}\{\bar{\lambda}_{n\,I}, \bar{\lambda}_{m\,J}\} + h.c.\,.
\end{aligned} \tag{4.70}$$

The Yukawa interaction thus preserves R-symmetry and so will the quartic scalar interactions. This is especially interesting when comparing with the previous approach where we saw that the Yukawa terms break R-symmetry. Such symmetry breakings are analyzed in [113]. While $\mathcal{N} = 4$ SYM theory was not studied in the cited paper, the general structure suggests that one should not expect R-symmetry to break in $\mathcal{N} = 4$ SYM at finite temperature. This is all the more a reason to suspect that the beta function will show a running of the coefficients towards a fixed point that does actually preserve R-symmetry.

What is more, an interesting observation can be made in the above result. Consider the phases in the exponent

$$\hat{\omega}_k - \omega_n - \omega_m = \frac{\pi}{\beta}(2k - (2n+1) - (2m+1)) = \frac{2\pi}{\beta}(k - n - m - 1) \tag{4.71}$$

Setting this to zero in the Kronecker deltas which emerge from the dτ integral we find that $k = m + n + 1$. The shift by one integer is a consequence from the fermionic frequencies. Let us trace back how this affects the possibility of recreating the original theory with only periodic fields. We will find the additional shift in the interaction. This can also not simply be absorbed in the infinite sum. To see this consider the scalar only by assuming we made the fermions periodic. This yields

$$\sum_k e^{\frac{(2k-2)\mathrm{i}\pi}{\beta}\tau}\Phi_k^{IJ} = \sum_k e^{\mathrm{i}\hat{\omega}_{k-1}\tau}\Phi_k^{IJ} = \sum_k e^{\mathrm{i}\hat{\omega}_k\tau}\Phi_{k+1}^{IJ} \neq \Phi^{IJ}\ . \tag{4.72}$$

Hence it is not feasible to recombine the Yukawa terms into a four-dimensional theory depending only on periodic fields using this approach. This reflects, in a yet to be untwisted way, the observation that we found a breaking of the R-symmetry in the above considerations.
Let us try to phrase this a bit differently. The reason we did not find a consistent way to preserve the R-symmetry of $\mathcal{N} = 4$ when introducing the phases (4.9) is a consequence of the way the Yukawa interactions of the theory are built. This can be seen from the Fourier series as we see an interaction between different modes. Naively one would expect to find an interaction between the same modes only. These two observations are closely linked to each other. Unfortunately the true nature of this connection remains mysterious to us.

For completeness regarding our arguments, let us consider the quartic scalar interactions as well. We have

$$\begin{aligned}
S_{\phi^4} &= \frac{1}{4g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \mathrm{tr}\left[\Phi^{IJ}, \Phi^{KL}\right]\left[\bar{\Phi}_{IJ}\bar{\Phi}_{KL}\right] \\
&= \frac{1}{4g^2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \sum_{m,n,k,l} e^{\mathrm{i}(\hat{\omega}_m + \hat{\omega}_n - \hat{\omega}_k - \hat{\omega}_l)\tau}\mathrm{tr}\left[\Phi_m^{IJ}, \Phi_n^{KL}\right]\left[\bar{\Phi}_{k\,IJ}\bar{\Phi}_{l\,KL}\right] \\
&= \frac{1}{4g^2}\int \mathrm{d}^3x \sum_{m,n,k} \mathrm{tr}\left[\Phi_m^{IJ}, \Phi_n^{KL}\right]\left[\bar{\Phi}_{k\,IJ}\bar{\Phi}_{m+n-k\,KL}\right]
\end{aligned} \tag{4.73}$$

We hence find an expected mixing of phases. The striking difference to the Yukawa interactions is that there is no shift involved.

4.2.3 Supersymmetry

What is more, we have not discussed how and if the above three-dimensional theory preserves supersymmetry. If we can find a part of the action that preserves some three-dimensional supersymmetry, we can construct a thermal BWI with the additional terms similar to (3.27) and thus hopefully recreate the relation $B = 3h$ in an analogous way as we did in the zero coupling case.

If we ignore finite temperature for a moment we would find identical modes for Bosonic and Fermionic fields in the Kaluza-Klein reduction. With the radius (corresponding to β in the thermal case) of the S^1 taken to zero we would only find the massless zero modes. This reduction preserves supersymmetries of the four-dimensional theories. In fact, an $\mathcal{N} = 8$ theory is obtained from $\mathcal{N} = 4$ SYM [120]. This scales down to theories with a lower amount of supersymmetry. 4d $\mathcal{N} = 2$ theories are reduced to 3d $\mathcal{N} = 4$ theories [121] and 4d $\mathcal{N} = 1$ theories are reduced to 3d $\mathcal{N} = 2$ theories [122]. We therefore have to consider how the temperature affects the reduction.

Finite temperature yields the KMS periodicity conditions. Therefore we find only even modes for the bosons and odd modes for the fermions. Furthermore, we do not consider any high temperature limit and thus the radius β of the compactified circle is not send to zero. Hence all massive modes and their respective mass terms are kept. We have shown above that the KMS conditions ultimately yield an interaction between different modes. We considered this for the Yukawa interaction in (4.70).

Let us thus consider a chiral multiplet consisting of only one spinor and one scalar field. We try to suggest supersymmetry transformations which are indeed consistent with the interaction of different modes. Albeit providing a consistent cancellation between the purely kinetic term and a cancelation between the Yukawa and the ϕ^4 interaction, we see that supersymmetry in fact cannot be preserved with the interacting modes.

Consider the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{chiral}} &= \mathcal{L}_{\text{kin.}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\phi^4} \\ &= -\partial_\mu \phi_n \partial^\mu \phi_n + \mathrm{i}\lambda_n \not{\partial} \lambda_n + \phi_m \lambda_n \lambda_{m-n-1} - \frac{1}{4}\phi_m \phi_n \phi_k \phi_{m-n+k} \,. \end{aligned} \tag{4.74}$$

Repeated indices m, n are summed over all modes.

Supersymmetry is introduced by the action of the supercharge Q on the fields. We propose the following transformations

$$Q\phi_n = \lambda_n \,, \quad Q\lambda_n = -\mathrm{i}\not{\partial}\phi_n + \frac{1}{2}\phi_k \phi_{n+k+1} \,. \tag{4.75}$$

Again, the index k above is summed over all integers. Let us now look separately at the supersymmetry transformation of the different parts of the Lagrangian. Note that modes can be shifted in a convenient manner. We have for example

$$\sum_{m,n} \phi_m \lambda_{m+n} = \sum_{m,n} \phi_{m-n} \lambda_m \ , \tag{4.76}$$

which will help us in some of the considerations. For instance the supercharge acting on a product of identical fields with different charges is written readily as the action on one of the fields with a respective prefactor. Therefore we find for the kinetic term

$$\begin{aligned} Q\mathcal{L}_{\text{kin.}} &= -2\partial_i (Q\phi_n)\partial^i \phi_n + 2\mathrm{i}\lambda_n \partial\!\!\!/ (Q\lambda_n) + \partial_\mu(\ldots) \\ &= -2\partial_i \lambda_n \partial^i \phi_n + 2\lambda_n \partial\!\!\!/ \partial\!\!\!/ \phi_n + \mathrm{i}\lambda_n \partial\!\!\!/ (\phi_k \phi_{n+k+1}) + \partial_\mu(\ldots) \\ &= 2\lambda_n \partial^2 \phi_n - 2\lambda_n \partial^2 \phi_n + \mathrm{i}\lambda_n \phi_k \partial\!\!\!/ \phi_{n+k+1} + \mathrm{i}\lambda_n \phi_{n+k+1} \partial\!\!\!/ \phi_k + \partial_\mu(\ldots) \ . \end{aligned} \tag{4.77}$$

We see that the purely kinetic part cancels readily and we only obtain an interaction-like term. We took out a total derivative which is not relevant to supersymmetry preservation. We now turn to the Yukawa interaction

$$\begin{aligned} Q\mathcal{L}_{\text{Yukawa}} &= (Q\phi_m)\lambda_n \lambda_{m-n-1} + 2\phi_m (Q\lambda_n)\lambda_{m-n-1} \\ &= \lambda_m \lambda_n \lambda_{m-n-1} - 2\mathrm{i}\phi_m \lambda_{m-n-1} \partial\!\!\!/ \phi_n + \phi_m \lambda_{m-n-1} \phi_k \phi_{n+k+1} \\ &= \lambda_m \lambda_n \lambda_{m-n-1} - 2\mathrm{i}\lambda_n \phi_k \partial\!\!\!/ \phi_{-n+k-1} + \phi_m \lambda_n \phi_k \phi_{m-n+k} \ . \end{aligned} \tag{4.78}$$

Note that the term with the four spinors will cancel independently by a Fierz identity [13]. For the quartic interaction we find

$$Q\mathcal{L}_{\phi^4} = -\phi_m (Q\phi_n)\phi_k \phi_{m-n+k} = -\phi_m \lambda_n \phi_k \phi_{m-n+k} \ . \tag{4.79}$$

Let us therefore collect all nonzero terms

$$Q\mathcal{L}_{\text{kin.}} = \mathrm{i}\lambda_n \phi_k \partial\!\!\!/ \phi_{-n+k-1} + \mathrm{i}\lambda_n \phi_k \partial\!\!\!/ \phi_{n+k+1} + \partial_\mu(\ldots) \ , \tag{4.80}$$

$$Q\mathcal{L}_{\text{Yukawa}} = -2\mathrm{i}\lambda_n \phi_k \partial\!\!\!/ \phi_{-n+k-1} + \phi_m \lambda_n \phi_k \phi_{m-n+k} \ , \tag{4.81}$$

$$Q\mathcal{L}_{\phi^4} = -\phi_m \lambda_n \phi_k \phi_{m-n+k} \ . \tag{4.82}$$

The term from ϕ^4 cancels the second term from the Yukawa as they have the same phases. Consider the remaining terms. The second term of the kinetic ones has the same modes as the remaining Yukawa term. The leftover term of the kinetic, however, cannot be canceled. Here is where the problematic of the mixing modes comes into play. The two scalar fields are distinct and we cannot combine them readily. Where there no modes involved, the two terms of the kinetic Lagrangian would combine into a single term with prefactor. We would have

$$Q\mathcal{L}_{\text{kin.}} = \mathrm{i}\lambda\phi\partial\!\!\!/\phi + \mathrm{i}\lambda\phi\partial\!\!\!/\phi + \partial_\mu(\dots) \qquad = 2\mathrm{i}\lambda\phi\partial\!\!\!/\phi + \partial_\mu(\dots)\ , \tag{4.83}$$

$$Q\mathcal{L}_{\text{Yukawa}} = -2\mathrm{i}\lambda\phi\partial\!\!\!/\phi + \phi\lambda\phi\phi \qquad = -2\mathrm{i}\lambda\phi\partial\!\!\!/\phi + \lambda\phi^3\ , \tag{4.84}$$

$$Q\mathcal{L}_{\phi^4} = -\phi\lambda\phi\phi \qquad = -\lambda\phi^3\ . \tag{4.85}$$

In this case we would obtain supersymmetry. The modes, however, make this impossible. This clearly shows that due to the mixing of the modes, supersymmetry cannot be preserved. Further note that the cancellation of all other terms clearly showed that we need to have the supersymmetry transformations as suggested above to cancel the ϕ^4 term and the kinetic terms.

Supersymmetry is broken by a specific term. Considering again the supersymmetry transformation of the kinetic term note that we can shift indices to find

$$\begin{aligned} Q\mathcal{L}_{\text{kin.}} &= \mathrm{i}\lambda_n\phi_k\partial\!\!\!/\phi_{-n+k-1} + \mathrm{i}\lambda_n\phi_{-n+k-1}\partial\!\!\!/\phi_k + \partial_\mu(\dots) \\ &= \mathrm{i}\lambda_n\phi_k\partial\!\!\!/\phi_{-n+k-1} - \mathrm{i}\lambda_n\phi_k\partial\!\!\!/\phi_{-n+k-1} - \mathrm{i}\phi_k\phi_{-n+k-1}\partial\!\!\!/\lambda_n + \partial_\mu(\dots) \\ &= -\mathrm{i}\phi_k\phi_{-n+k-1}\partial\!\!\!/\lambda_n + \partial_\mu(\dots) \end{aligned} \tag{4.86}$$

In the second line we used partial integration. We can thus combine all terms and find

$$\begin{aligned} Q\mathcal{L}_{\text{chiral}} &= -\mathrm{i}\phi_k\phi_{-n+k-1}\partial\!\!\!/\lambda_n - 2\mathrm{i}\lambda_n\phi_k\partial\!\!\!/\phi_{-n+k-1} + \partial_\mu(\dots) \\ &= -\frac{\mathrm{i}}{3}Q\left(\phi_k\phi_{-n+k-1}\partial\!\!\!/\phi_n\right) - \frac{2\mathrm{i}}{3}Q\left(\phi_n\phi_k\partial\!\!\!/\phi_{-n+k-1}\right) + \partial_\mu(\dots) \\ &= -\mathrm{i}Q\left(\phi_n\phi_k\partial\!\!\!/\phi_{-n+k-1}\right) + \partial_\mu(\dots) \end{aligned} \tag{4.87}$$

The full action can then by hand be written with a term that preserves supersymmetry

$$\begin{aligned} \mathcal{L}_{\text{chiral}} &= -\partial_\mu\phi_n\partial^\mu\phi_n + \mathrm{i}\lambda_n\partial\!\!\!/\lambda_n + \phi_m\lambda_n\lambda_{m-n-1} - \frac{1}{4}\phi_m\phi_n\phi_k\phi_{m-n+k} \\ &\quad + \mathrm{i}\phi_n\phi_k\partial\!\!\!/\phi_{-n+k-1} - \mathrm{i}\phi_n\phi_k\partial\!\!\!/\phi_{-n+k-1} \\ &= \mathcal{L}_{\text{SUSY}} - \mathrm{i}\phi_n\phi_k\partial\!\!\!/\phi_{-n+k-1}\ . \end{aligned} \tag{4.88}$$

Note that throughout these calculations we did not consider the mass terms of the phases. As the fermionic and bosonic mass terms differ they also break supersymmetry.
Naively we thus wrote the action satisfying (4.1). Therefore a BWI similar to (3.27) can be derived allowing us to relate the bremsstrahlung to the stress tensor. The difference

here, however, are the different modes. We would have to derive the BWI of the above action taking into account the interactions between the different modes. Furthermore, we note that the supersymmetric Lagrangian is very different from the zero temperature one. We had to include an additional term to the Lagrangian which would change the present supercurrent. Also, the above considerations should be explicitly calculate for the full $\mathcal{N} = 4$ SYM theory.

4.3 Outlook

Let us summarize the outcome of this chapter. We found that the Yukawa terms are what does not allow for a clean redefinition of the action as a supersymmetric term depending on periodic fields only plus mass corrections, see (4.1)

$$S_{4d\mathrm{th}.\mathcal{N}=4} = S_{\mathrm{SUSY}} + S_{\text{mass-terms}}\,. \tag{4.89}$$

This is what was a key to deriving and using the BWI in Chapter 3. We compared two approaches.
Firstly we employed a field redefinition making the fermions periodic by taking out a phase. This approach is analogous to the one used at zero coupling in Section 3.1. The Yukawa interaction then yielded similar phases for the bosons and finally a breaking of R-symmetry. While at $g = 0$ the choice of phases had no impact on our final result, here we would find that the symmetry breaks into different preserved subgroups. Therefore an explicit choice for the phases turned out to be very relevant to the interacting theory. The idea to fix such a choice consistently is to consider a dynamic calculation. This is inspired by [113]. By computing the self-energy we expect to find some restriction. Our advances in the self-energy calculation are presented in Appendix H. There we start by considering the scalar self-energy and eventually find that this ansatz is seemingly inconsistent. As of now we are uncertain regarding the source of said inconsistency as the derivation given in this chapter is straight forward. We suspect that it is possible to employ the phase redefinition ansatz if appropriate changes are made. Those changes have to, for instance, preserve R-symmetry and reproduce a consistent thermal mass.
While the first approach had the upshot of manifestly preserving supersymmetry (modulo the mass terms) the R-symmetry is seemingly broken. We therefore employ an additional different ansatz. We considered the fields of $\mathcal{N} = 4$ SYM by a dimensional reduction inspired by [119]. From the KK-modes we found a tower of massive fields. The kinetic terms allowed for us to reproduce our findings at zero coupling. This can be interpreted as a consistency check for Section 3.1. The Yukawa interaction, however, created an ambiguity. By plugging in the phase expansion we found that, unlike the quartic scalar interaction, there is an interaction between different Fourier modes. We showed that this interaction breaks supersymmetry with a new breaking term. Furthermore, to further

employ calculations with this theory another issue needs to be adressed. The KK reduction also introduced an infinite tower of massive gauge fields. To allow for these we would have to introduce a Higgs-mechanism for each of the fields making loop calculations considerably more difficult.

In conclusion the two approaches both have an ambiguity created by the Yukawa interaction. One seemingly breaks R-symmetry, the other breaks supersymmetry. Whatsoever, each ansatz for itself follows a self-consistent logic. With the considerations outlined we hope to be able to find that both sides eventually merge into a common result. If this result, as desired, has a SUSY-preserving part and another part with (mass-like) breaking terms (4.1) we would be able to reproduce the thermal BWI (3.27). This might lead to a calculation similar to the one at zero coupling showing that $B = 3h$ also in the interacting finite temperature theory.

In Appendix I we thus suppose a different approach which does not rely on an explicit field redefinition. Ideally we will find that this ansatz for the action is the consistent combination of the two approaches presented in this chapter. We write down the most general Lagrangian with periodic fields. This includes also all operators of small mass dimension which then get prefactors of the temperature. In this way the mass term for the scalars and fermions are readily recreated. In a dynamical calculation of self-energies we find (at leading order) that we get constraints on newly introduced mass-terms. Therefore this approach is very promising and would allow us to use the BWI in the same manner as in Section 3.1. Although the arguments presented in I strongly suggest that $B = 3h$ holds as an exact relation at finite temperature it is relevant to mention that the presented consideration are to some point work in progress and we therefore do not claim to definitely know that the relation holds.

What is more, this relation between bremsstrahlung and stress tensor can also be studied in a weak coupling perturbative approach. In the following chapter we provide a starting point for such calculations by considering the circular and straight line Wilson loop at order λ^2, that is including self-energy corrections and vertex diagrams.

Chapter 5

Higher order corrections to thermal Wilson loops

The result obtained thus far have been at zero coupling. In the preceding chapter we discussed the effects of interaction terms on the Broken Ward Identity. Most importantly we found a breaking of R-symmetry. The relation between the bremsstrahlung and the stress tensor can, however, also be considered following the perturbative argument of Section 3.2. For any steps in the presented argumentation it was vital to know an expression for the circular Wilson loop.

Therefore, in this chapter, we turn to the perturbative calculation of the the Maldacena-Wilson loop in SU(N) $\mathcal{N}=4$ SYM theory [57,58] at finite temperature. More precisely we study the corrections of order λ^2 by going to a non-zero coupling. This will yield vertices from the action in the weak coupling perturbative expansion. We will not consider the bremsstrahlung explicitly here. Instead, we simply compute the Wilson loop expectation value. We will find that the thermal corrections at this order yield non-trivial terms which do not allow for a straight foreward derivation as presented in 2.2.2. Consider for instance the result (2.55). The calculation in this chapter, especially concering the self-energy diagram will make clear that such an explicit relation cannot be expected at order λ^2.

As the interactive diagrams are not trivially obtained we require some preliminary considerations. Recall the definition of the Wilson loop

$$W(C) = \frac{1}{N}\mathrm{tr}\overleftarrow{\mathcal{P}}\exp\left[\oint_C \mathrm{d}\mathfrak{t}\left(\mathrm{i}A_\mu(x(\mathfrak{t}))\dot{x}^\mu(\mathfrak{t}) + \Phi_r(x(\mathfrak{t}))|\dot{x}(\mathfrak{t})|n^r\right)\right] \tag{5.1}$$

$$= \frac{1}{N}\mathrm{tr}\overleftarrow{\mathcal{P}}\exp\left[\oint_C \mathrm{d}\mathfrak{t}\left(\mathrm{i}A_\mu\dot{x}^\mu + \Phi_r|\dot{x}|n^r\right)\right], \tag{5.2}$$

where as before $A_\mu = A_\mu^a T^a$ is the gauge field and $\Phi_r = \Phi_r^a T^a$ ($r = 1,\ldots,6$) are the six scalars. We denote the generators of the SU(N) gauge group by T^a as mentioned above. n^r is a six-dimensional unit vector ($n^2 = 1$) on the S^5 on which the 10 dimensional $\mathcal{N}=1$ theory was compactified to get $\mathcal{N}=4$ in $4d$. $\overleftarrow{\mathcal{P}}$ is the $\mathfrak{t}$-path-ordering.

The integral in the exponential function runs along a (closed) contour C. We are interested in two different paths. We considered the circular loop with radius R

$$C_{\bigcirc} : \mathrm{t} \mapsto x(\mathrm{t}) = (0, R\sin(\mathrm{t}), R\cos(\mathrm{t}), 0) \ , \quad \mathrm{t} \in [0, 2\pi] \ , \tag{5.3}$$

and the straight line of length $L \to \infty$

$$C_{\_} : \mathrm{t} \mapsto x(\mathrm{t}) = (0,0,0,\mathrm{t}) \ , \quad \mathrm{t} \in \left[-\frac{L}{2}, \frac{L}{2}\right] \ . \tag{5.4}$$

Generally we have $x^\mu = (x_0, \vec{x}) \in \mathbb{R}^{1,3}$. Zero temperature results for these are known from [62–64]. The leading order thermal correction was considered in the previous chapter. For the circular loop we found an expansion in small and high temperatures (3.120) while the straight line remains to be equally one due to vanishing prefactors.
When including higher orders, further effects need to be accounted for. The Wilson loop gets three different corrections at order λ^2. The respective diagrams are gathered in Figure 1.1. The single propagator insertion 1.1a is the only diagram contributing at order λ. This diagram will be called $\langle W_{(a)} \rangle_\beta$ in the following. From the Feynman rules we saw that the propagator yields a factor $g^2\delta^{ab}$ and the color trace $\mathrm{tr}\, T^a T^a = {}^{N^2}\!/_2$ combines with the ${}^1\!/_N$ to give a total prefactor of ${}^{g^2 N}\!/_2 = {}^\lambda\!/_2$.
Such a combination will in fact occur for all the diagrams and we will note everything in terms of the 't Hooft coupling λ in the following whenever possible. When we turn to calculating the diagrams explicitly we shall see that this prescription can indeed be found. The insertion of two parallel propagators 1.1b conversely is then proportional to λ^2 and called $\langle W_{(b)} \rangle_\beta$. The self-energy insertion 1.1c has only one-loop insertions at leading order which contribute at order λ^0 such that with the two external legs we find again λ^2. We will call this diagram $\langle W_{(c)} \rangle_\beta$.The insertion of the three-vertex interaction shall be called 1.1d. From the Feynman rules it follows that the vertex contributes as λ^{-1}. Hence, the coupling dependence of one of the three propagators is canceled and we are at order λ^2 as expected.
One could suspect that there should be a diagram with two crossed propagators. However, such a diagram is non-planar and thus subleading in the large N limit. Also the insertion of a four-point vertex does not contribute at this order because the four propagators and one vertex yield $\lambda^{4-1} = \lambda^3 \ll \lambda^2$. The full Wilson loop is then given by

$$\langle W \rangle = 1 + \underbrace{\langle W_{(a)} \rangle_\beta}_{\mathcal{O}(\lambda)} + \underbrace{\left[\langle W_{(b)} \rangle_\beta + \langle W_{(c)} \rangle_\beta + \langle W_{(d)} \rangle_\beta\right]}_{\mathcal{O}(\lambda^2)} + \mathcal{O}(\lambda^{5/2}) \ . \tag{5.5}$$

Note that the next subleading order is not λ^3 as naively expected. This is due to the infrared divergences which appear at finite temperature [47, 100].
The calculation of the diagrams at order λ^2 is significantly more complicated compared to the leading order. While the rainbow diagram 1.1b is relatively straight forward, the

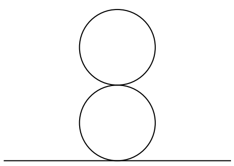
self-energy insertion 1.1c and the vertex 1.1d are challenging. The vertex calculation will mostly be calculationally heavy with integrals that require a numeric integration. For the self-energy at $T \neq 0$ one finds [47, 53, 100] that even the leading orders have integrals which cannot be solved analytically. We therefore will need to consider a limit before considering their Fourier transform from momentum to coordinate space.

In this chapter we will consider a couple of preliminary calculations for this self-energy diagram. Firstly we will explain corrections of higher order and show that in fact we will expect the next order to be $\lambda^{5/2}$. The gauge propagators in general might have a non-trivial dependence on spacetime indices, especially when self-energy corrections are included. How this affects the WL calculation will be regarded in the second section. Finally we will consider the scalar and gauge self-energy. We calculate in a Feynman diagram approach and by using a thermal averaging process will be able to Fourier transform the resulting propagators to momentum space. Equipped with these preliminary results we are then able to calculate the next-to-leading order contributions to the circular and straight line Wilson loop at weak coupling.

5.1 Corrections beyond λ^2

Let us consider higher order contributions to the perturbative Wilson loop calculation. Naively one would expect the next subleading order to come with λ^3, however some of the diagrams in the finite temperature approximation will have infrared divergences [47, 100]. These divergences can be resummed to a finite contribution which turns out to be of order $\lambda^{5/2}$. We will not calculate these corrections explicitly but rather explain how the dependence on λ^3 is yielded.

Therefore consider a simple self-energy in ϕ^4 theory. There are two self-energy diagrams we naively expect to contribute at order λ^2, the one with two bubbles and the sunset diagram. We name the former $\Sigma_2^{(1)}$ where the subscript indicates the naive order in perturbation theory and the superscript distinguishes it from the sunset diagram.

$\Sigma_2^{(1)} =$ 

$$= 96\lambda^2 \int \frac{\mathrm{d}^3q}{(2\pi)^3} \int \frac{\mathrm{d}^3k}{(2\pi)^3} T^2 \sum_{m,l=-\infty}^{\infty} \frac{1}{\left[(\mathrm{i}\omega_m)^2 - q^2\right]^2} \frac{1}{(\mathrm{i}\omega_l)^2 - k^2} \,. \tag{5.6}$$

We clearly see that the two sums and integrals may be separated. The integral over the one propagator ($\vec{k}$ and ω_l) is the one-loop self-energy integral which is proportional to the thermal mass $m^2 = \lambda T^2$. We will call this one-loop self-energy $\Sigma_1 = 12m^2$. Hence

$$\Sigma_2^{(1)} = 8\lambda\Sigma_1 T \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \sum_{m=-\infty}^{\infty} \frac{1}{(\mathrm{i}\omega_l)^2 - k^2} \,. \tag{5.7}$$

Let us look at the zero mode which is proportional to

$$\lambda \int \mathrm{d}q \, \frac{q^2}{q^4} = \lambda \int \mathrm{d}q \, \frac{1}{q^2} \,. \tag{5.8}$$

This integral is IR divergent. However, this is not the only IR divergent diagram proportional to λ. In fact only diagrams of the form in Figure 5.1 will contribute. The diagram

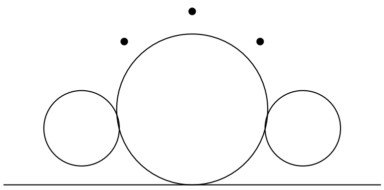


Figure 5.1: Self energy corrections with N bubbles. On the tadpole correcting the propagator N more tadpoles are attached.

consists of a single loop correction attatched to the propagator. This loop itself carries several one-loop corrections. For an N loop diagram there are $N-1$ of the small loop corresponding to Σ_1, where $N \geq 2$.
Following Kapusta's book [47] and defining the bare propagator $\Delta_0^{-1}(\omega_n, \vec{p}) = \omega_n^2 + |\vec{p}|^2$, we may sum up all such diagrams to be

$$\begin{aligned}\Sigma_{3/2} &= \lambda \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{N=1}^{\infty} (-\Sigma_1)^{N-1} \Delta_0^N(\omega_n, \vec{p}) \\ &= -\lambda \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\omega_n^2 + |\vec{p}|^2 + \Sigma_1} \,. \end{aligned} \tag{5.9}$$

To find the correct order in the coupling we can replace $\Sigma_1 = 12\lambda T^2$ and calculate the zero mode

$$\Sigma_{3/2}^{(n=0)} = -\lambda \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{|\vec{p}|^2 + 12\lambda T^2} = \Sigma_1^{(T=0)} - \frac{6\sqrt{3}}{\pi} T^2 \lambda^{3/2} \,. \tag{5.10}$$

The first term above is the temperature independent term which is of no importance as its ultraviolet divergence can be renormalized. The second term, however, clearly shows the next order in perturbation theory is $\lambda^{3/2}$. Note also that in agreement with [47] the sign of the new subleading term is negative. In the case of a three-particle vertex we also find diagrams depending on $\log(\lambda)$, which, however, are of even higher order. For the scalar self-energies they would start to contribute at order $\lambda^2 \log(\lambda)$ yielding an appearence at order $\lambda^3 \log(\lambda)$ for the Wilson loop calculation.

Conclusively, the next higher order we expect in the thermal perturbative expansion of Wilson loops in weak coupling is $\lambda^{5/2}$.

5.2 Theorem on gauge propagators and closed Wilson loops

In the perturbative expansion of the Wilson loop we will consider the insertion of scalar and gauge propagators including low order Feynman diagram corrections. While the structure of the scalar propagator is straight forward, the gauge propagator has two spacetime indices yielding a nontrivial structure especially at finite temperature. We thus want to write down the gauge propagator in the most general and most covariant form possible, considering that at finite temperature Lorentz invariance is broken and we therefore get a separation between transverse and longitudinal components [47, 100]. We then want to see which part of it can contribute to the Wilson loop.
The momentum space gauge propagator in any R_ξ gauge is generally given by [47, 100, 123, 124]

$$\Delta_{\mu\nu} = \frac{\hat{P}_{\mu\nu}}{P^2 - \Pi_T} + \frac{\hat{Q}_{\mu\nu}}{P^2 - \Pi_L} + \xi \frac{P_\mu P_\nu}{P^4} \ . \tag{5.11}$$

with the transverse and longitudinal projectors $\hat{P}_{\mu\nu}$ and $\hat{Q}_{\mu\nu}$, respectively. They are given by

$$\hat{P}_{\mu\nu} = -\delta_{\mu\nu} + u_\mu u_\nu + \frac{(P_\mu - p_0 u_\mu)(P_\nu - p_0 u_\nu)}{|\vec{p}|^2} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2} \end{pmatrix} \quad \text{and} \tag{5.12a}$$

$$\hat{Q}_{\mu\nu} = -\frac{\left(|\vec{p}|^2 u_\mu + p_0 P_\mu - p_0^2 u_\mu\right)\left(|\vec{p}|^2 u_\nu + p_0 P_\nu - p_0^2 u_\nu\right)}{P^2 |\vec{p}|^2} = -\delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} - \hat{P}_{\mu\nu} \ . \tag{5.12b}$$

$u_\mu = \delta_{\mu,0}$ is the unit vector along the time direction, sometimes called "mean velocity of the heat bath". See [47,53,100] for further explanation. Also, as mentioned above, $P^\mu = (p_0, \vec{p})$. The transverse and longitudinal self-energies Π_T and Π_L are a priori arbitrary functions. They can be obtained from the vacuum polarization tensor $\Pi_{\mu\nu}$ as defined in Section 5.3 below (Equations 5.41). Generally the transverse polarization Π_T is related to the transverse projector $\hat{P}_{\mu\nu}$ and likewise the longitudinal polarization Π_L is related to the longitudinal projector $\hat{Q}_{\mu\nu}$ as can be seen from the above definition of $\Delta_{\mu\nu}$.
We can show that the terms with projectors $P_\mu P_\nu$ and $p_i p_j$ will not contribute to a closed Wilson loop. Thus we also do not need take care of the gauge part. We state this as a theorem:

Theorem 1.
Any smooth closed Wilson loop parametrized by $x(\mathfrak{t})$, $\mathfrak{t} \in \mathcal{I} \subset \mathbb{R}$ *does not receive contributions*

from any propagator term with index structure $P_\mu P_\nu$ or $p_i p_j$ at leading order in the 't Hooft coupling.

To plug these terms of the propagator into the Wilson loop they need to be Fourier transformed. The idea of the proof is that P_μ can be pulled out of the Fourier transform as a derivative. Using the prefactor $\dot{x}_1^\mu \dot{x}_2^\nu$ these can be combined to form derivatives with respect to the contour parameters $\mathfrak{t}_i$. Partial integration can then be used to show that only boundary terms contribute. These are the beginning and end-points of the parametrized contour and for a closed contour these terms thus vanish. We will proof the theorem in full detail below.
Note that the straight line Wilson loop as parameterized above is not closed. However, it would also not be gauge invariant. We therefore assume that we can identify the points at $+\infty$ and $-\infty$ with each other to thus close the loop[1] [63]. Then we can indeed use the above theorem also for the straight line.
The contribution of the propagator in the Wilson loop comes from

$$\dot{x}_1^\mu \dot{x}_2^\nu \Delta_{\mu\nu}(x_1 - x_2)\,. \tag{5.13}$$

Due to the broken Lorentz invariance it is most useful to consider

$$\dot{x}^\mu(\mathfrak{t}) = \begin{pmatrix} a(\mathfrak{t}) \\ \vec{b}(\mathfrak{t}) \end{pmatrix}^\mu\,. \tag{5.14}$$

where the unorthodox convention $\dot{x}^\mu = (a, \vec{b})$ is introduced for later convenience. Theorem (1) shows that terms with $P_\mu P_\nu$ structure can contribute only with boundary terms in case the loop is not closed. The same holds for $p_i p_j$ analogously.
Let us look at how $\dot{x}^\mu(\mathfrak{t}_1)\dot{x}^\nu(\mathfrak{t}_2)$ contracts with the two projectors. We ignore the fact that the projectors are in momentum space as the momentum space dependence will eventually be Fourier transformed. For the transverse case we obtain only the spatial components

$$\dot{x}^\mu(\mathfrak{t}_1)\dot{x}^\nu(\mathfrak{t}_2)\hat{P}_{\mu\nu} = \vec{b}(\mathfrak{t}_1)\vec{b}(\mathfrak{t}_2) - \frac{[\vec{b}(\mathfrak{t}_1)\vec{p}]\,[\vec{b}(\mathfrak{t}_1)\vec{p}]}{|\vec{p}|^2} = \vec{b}(\mathfrak{t}_1)\vec{b}(\mathfrak{t}_2)\,. \tag{5.15}$$

As argued above the second term does not contribute. For the longitudinal contribution we find the opposite, only the time component contributes:

$$\begin{aligned}\dot{x}^\mu(\mathfrak{t}_1)\dot{x}^\nu(\mathfrak{t}_2)\hat{Q}_{\mu\nu} &= \dot{x}^\mu(\mathfrak{t}_1)\dot{x}_\mu(\mathfrak{t}_2) - \frac{[\dot{x}^\mu(\mathfrak{t}_1)P_\mu]\,[\dot{x}^\nu(\mathfrak{t}_1)P_\nu]}{P^2} - \vec{b}(\mathfrak{t}_1)\vec{b}(\mathfrak{t}_2) + \frac{[\vec{b}(\mathfrak{t}_1)\vec{p}]\,[\vec{b}(\mathfrak{t}_1)\vec{p}]}{|\vec{p}|^2} \\ &= a(\mathfrak{t}_1)a(\mathfrak{t}_2) + \vec{b}(\mathfrak{t}_1)\vec{b}(\mathfrak{t}_2) - \vec{b}(\mathfrak{t}_1)\vec{b}(\mathfrak{t}_2) = a(\mathfrak{t}_1)a(\mathfrak{t}_2)\,. \end{aligned} \tag{5.16}$$

[1] An alternative way to get gauge invariance would be to calculate the Wilson line sandwiched between two fields.

Again we canceled structures with $P_\mu P_\nu$. As mentioned above this result is independent of the momentum. We can clearly make one crucial observation from this.

Corollary 1.
Non-constant Wilson loop contours in the 0*-direction (time) allow only for longitudinal contributions.*
Non-constant Wilson loop contours in the i*-directions (space) allow only for transverse contributions.*

Wilson loops that include both, transverse and longitudinal terms thus must come from contours that are in the 0− and also in the i-directions. Further note that constants in the parameterization of the contour are of no importance as the contributions depend only on $\dot{x}^\mu(\mathfrak{t}) = {}^{\partial x^\mu(\mathfrak{t})}/_{\partial \mathfrak{t}}$.
The theorem 1 implies that for closed Wilson loops in the spatial directions the gauge-propagator to consider is

$$\Delta_{\mu\nu} = \frac{\delta_{\mu\nu}}{P^2 - \Pi_T} + \ldots = \delta^i_\mu \delta^j_\nu \frac{\delta_{ij}}{P^2 - \Pi_T} + \ldots . \tag{5.17}$$

where the dots represent all terms which do not contribute. Similarly this is also what contributes for the straight line.

Let us now proof theorem 1 in detail.

Proof.
Recall the transverse and longitudinal projectors (5.12),

$$P_{\mu\nu} = \eta_{\mu\nu} - u_\mu u_\nu + \frac{(P_\mu - \omega u_\mu)(P_\nu - \omega u_\nu)}{\omega^2 - P^2} \quad \text{and} \tag{5.18a}$$

$$Q_{\mu\nu} = -\frac{\left([\omega^2 - P^2]u_\mu + \omega P_\mu - \omega^2 u_\mu\right)\left([\omega^2 - P^2]u_\nu + \omega P_\nu - \omega^2 u_\nu\right)}{P^2[\omega^2 - P^2]} , \tag{5.18b}$$

respectively. Thus we can write the gauge propagator as

$$\Delta_{\mu\nu}(P) = \eta_{\mu\nu}\Delta_1(P) + P_\mu P_\nu \Delta_2(P) + u_\mu u_\nu \Delta_3(P) + (P_\mu u_\nu + u_\mu P_\nu)\Delta_4(P) . \tag{5.19}$$

The contribution of the gauge propagator to the Wilson loop at leading order is

$$\langle W \rangle_1 = \frac{\lambda}{2} \oint_I \mathrm{d}\vec{\mathfrak{t}}\; \dot{x}_1^\mu \dot{x}_2^\nu \, \mathcal{F}\left[\Delta_{\mu\nu}(P)\right](x_1 - x_2) , \tag{5.20}$$

where $\mathcal{F}$ denotes the Fourier transform. The first two terms of $\Delta_{\mu\nu}(P)$ above may contribute to the Wilson loop following corollary 1. The last term is odd in P_μ and hence necessarily cancels in the Fourier transform which is naturally even.

We thus only need to show that $P_\mu P_\nu \Delta_2(P)$ does not contribute to a closed Wilson loop, this term is denoted by $\widehat{\langle W\rangle}$. The idea is to extract the $P^\mu P^\nu$ dependence from the Fourier transformation and include it as an external derivative acting on the propagator. This step is valid if the function $\Delta_2(P)$ is continuous differentiable and if $\Delta_2(P)$ as well as its derivative are integrable. Remembering that the propagators get an additional $+i\varepsilon$ in the denominator to move poles away from the real axis which is usually omitted, one clearly sees that for real P the stated assumptions are met.
The only possible pole one would expect is at $P = 0$ which the $+i\varepsilon$ shifts to a non-divergent result. One needs to take the limit $\varepsilon \to 0$ in the end and to be most precise we define the propagator which has this cut-off as Δ_ε[2]. Then the $P^\mu P^\nu$ can be taken out of the Fourier transform as derivatives.

$$\begin{aligned}\mathcal{F}\left[P_\mu P_\nu \Delta_\varepsilon(P)\right](x) &= \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\mathcal{F}\left[\Delta_\varepsilon(P)\right](x) = \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\mathcal{F}\left[\Delta_\varepsilon(P)\right](x) \\ &=: \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\tilde{\Delta}_\varepsilon(x)\,. \end{aligned} \tag{5.21}$$

We want to turn this expression into a total derivative by combining with $\dot{x}_2^\nu = \partial x_2^\nu / \partial \mathfrak{t}_2$, however, we need to be careful with x_1 and x_2 appearing in the calculation. The propagator depends on $x_1 - x_2$ and therefore we also want every other factor to have the same dependence. Therefore we write

$$\frac{\partial x_2^\nu}{\partial \mathfrak{t}_2} = -\frac{\partial(x_1^\nu - x_2^\nu)}{\partial \mathfrak{t}_2} + \frac{\partial x_1^\nu}{\partial \mathfrak{t}_2} = -\frac{\partial(x_1^\nu - x_2^\nu)}{\partial \mathfrak{t}_2} + \dot{x}_1^\nu \delta(\mathfrak{t}_1 - \mathfrak{t}_2)\,. \tag{5.22}$$

The Dirac delta function in the last term will set the insertion points of the propagators into the Wilson loop to the same point. This will cause a divergence which is well known in the literature and can be regularized by a mass renormalization of the particle forming the Wilson loop [125–127][3]. We implicitly do this renormalization and can thus assume the two insertion points never to meet, writing

$$\frac{\partial x_2^\nu}{\partial \mathfrak{t}_2} = -\frac{\partial(x_1^\nu - x_2^\nu)}{\partial \mathfrak{t}_2}. \tag{5.23}$$

Define the interval on which the Wilson loop is parametrized as $\mathcal{I} = [a, b]$ with $x(a) = x(b)$ as the loop is closed. Then the gauge contribution becomes

$$\begin{aligned}\widehat{\langle W\rangle} =& \frac{\lambda}{2}\int_a^b \mathrm{d}\mathfrak{t}_1 \int_a^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2\, \dot{x}_1^\mu \dot{x}_2^\nu \tilde{\Delta}_{\mu\nu}(x_1 - x_2) \\ =& \frac{\lambda}{2}\int_a^b \mathrm{d}\mathfrak{t}_1 \int_a^b \mathrm{d}\mathfrak{t}_2\, \Theta(\mathfrak{t}_1 - \mathfrak{t}_2)\dot{x}_1^\mu \dot{x}_2^\nu \tilde{\Delta}_{\mu\nu}(x_1 - x_2)\,. \end{aligned} \tag{5.24}$$

[2]Another way to circumvent the divergence at $P = 0$ is to define the radial integration in the Fourier transform as $\lim\limits_{\Lambda_{IR}\to 0}\int_{\Lambda_{IR}}^\infty \mathrm{d}P$. As our result does not depend on the type of regularization we use, we will stick to the above notation for both.

[3]For a review see [67].

$\tilde{\Delta}_{\mu\nu}$ is the Fourier transform of the momentum space propagator as defined above. This implies that in the Wilson loop the part with $P_\mu P_\nu$ is given by

$$\begin{aligned}
\widehat{\langle W\rangle} =& \frac{\lambda}{2}\int_a^b \mathrm{d}\mathrm{t}_1 \int_a^b \mathrm{d}\mathrm{t}_2\, \Theta(\mathrm{t}_1-\mathrm{t}_2)\dot{x}_1^\mu \dot{x}_2^\nu \left(\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\tilde{\Delta}_{2,\varepsilon}\right)(x_1-x_2) \\
=& -\frac{\lambda}{2}\int_a^b \mathrm{d}\mathrm{t}_1 \int_a^b \mathrm{d}\mathrm{t}_2\, \Theta(\mathrm{t}_1-\mathrm{t}_2-\epsilon)\frac{\partial}{\partial \mathrm{t}_1}\frac{\partial}{\partial \mathrm{t}_2}\tilde{\Delta}_{2,\varepsilon}(x_1-x_2) \qquad (5.25)\\
=& -\frac{\lambda}{2}\int_a^b \mathrm{d}\mathrm{t}_1 \left(\left[\Theta(\mathrm{t}_1-\mathrm{t}_2)\frac{\partial}{\partial \mathrm{t}_1}\tilde{\Delta}_{2,\varepsilon}(x_1-x_2)\right]_{\mathrm{t}_2=a}^b - \right.\\
& \left. -\frac{\lambda}{2}\int_a^b \mathrm{d}\mathrm{t}_2\, \delta(\mathrm{t}_1-\mathrm{t}_2)\frac{\partial}{\partial \mathrm{t}_1}\tilde{\Delta}_{2,\varepsilon}(x_1-x_2)\right) .
\end{aligned}$$

In the last step we partially integrated by moving the derivative with respect to t_2. The t_2 integral in the second term sets $x_1 = x_2$ and thus the t_1 derivative acts on a constant, thus vanishing

$$\int_a^b \mathrm{d}\mathrm{t}_2\, \delta(\mathrm{t}_1-\mathrm{t}_2)\frac{\partial}{\partial \mathrm{t}_1}\tilde{\Delta}_{2,\varepsilon}(x(\mathrm{t}_1)-x(\mathrm{t}_2)) = \frac{\partial}{\partial \mathrm{t}_1}\tilde{\Delta}_{2,\varepsilon}(0) = 0 . \qquad (5.26)$$

For the remaining piece we can plug in the boundaries. $\Theta(\mathrm{t}_1 - a)$ does not change the integration boundaries while $\Theta(\mathrm{t}_1 - b)$ sets the lower boundary to b. Again partial integration can be used to solve these integrals such that we only get boundary terms.

$$\begin{aligned}
\widehat{\langle W\rangle} &= \frac{\lambda}{2}\int_a^b \mathrm{d}\mathrm{t}_1\, \frac{\partial}{\partial \mathrm{t}_1}\tilde{\Delta}_{2,\varepsilon}(x(\mathrm{t}_1)-x(a)) - \frac{\lambda}{2}\int_b^b \mathrm{d}\mathrm{t}_1\, \frac{\partial}{\partial \mathrm{t}_1}\tilde{\Delta}_{2,\varepsilon}(x(\mathrm{t}_1)-x(b)) \\
&= \frac{\lambda}{2}\left[\tilde{\Delta}_{2,\varepsilon}(x(b)-x(a)) - \tilde{\Delta}_{2,\varepsilon}(0)\right] . \qquad (5.27)
\end{aligned}$$

Using $x(a) = x(b)$ yields

$$\widehat{\langle W\rangle} = \frac{\lambda}{2}\left[\tilde{\Delta}_{2,\varepsilon}(0) - \tilde{\Delta}_{2,\varepsilon}(0)\right] = 0 . \qquad (5.28)$$

This completes the proof. ■

5.3 Thermal self-energy of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

One key ingredient we need for the order λ^2 calculation of the WLs are the self-energy and vacuum polarization of the scalar and vector field, respectively. They will enter through diagrams in the second order correction as explained in the introduction, see Figure 1.1c. Unlike in the zero temperature case [62] the insertion of the self-energy will not cancel against the insertion of a vertex (Figure 1.1d).

Thermal self-energies are known quantities in textbooks on thermal QFTs [47, 53, 100], however, their computation usually involves a nontrivial integration over hyperbolic functions that cannot be solved analytically. One known approach to thermal theories is the Hot Thermal Loop expansion [119]. Internal and external momenta are separated in soft $p \sim \sqrt{\lambda}T$ and hard $p \sim T$ momenta. While In this section we apply the two above theories to a concrete example. Let us consider $\mathcal{N} = 4$ SYM theory at a finite temperature T. We calculate the scalar self-energy and the vacuum polarization without focusing on the presence of the Wilson loops into which they will be inserted in later.

5.3.1 Vacuum polarization and scalar self-energy

We want to obtain the vacuum polarization and scalar self-energy starting from the Lagrangian, focusing only on the $T \neq 0$ terms as the Wilson loop at zero temperature is well known [62, 63, 67].
Loop integrals at zero temperature can be calculated using the known methods presented for example in [128]. For loop integrals at finite temperature we will use the imaginary time formalism [47, 53, 100]. This yields that frequency sums must be evaluated. The relevant frequencies are $\hat{\omega}_n = 2n\pi\beta$ for Bosonic and $\omega_n = (2n+1)\pi\beta$ for Fermionic loops, respectively. Recall that we introduced the inverse temperature $\beta = {}^1\!/\!_T$. The frequency sums are then given by the hyperbolic cotangent (Bosons) or hyperbolic tangent (Fermions) functions times the residues of the propagators [47, 53, 100]. It is then easy to separate the zero temperature contribution by introducing the Bose-Einstein and Fermi-Dirac statistics

$$\coth\left(\frac{|p_0|}{2T}\right) = 1 + 2n_B(|p_0|) = 1 + \frac{2}{e^{|p_0|/T} - 1}\,, \tag{5.29}$$

$$\tanh\left(\frac{|p_0|}{2T}\right) = 1 - 2n_F(|p_0|) = 1 - \frac{2}{e^{|p_0|/T} + 1}\,. \tag{5.30}$$

The more precise formula for Bosons reads

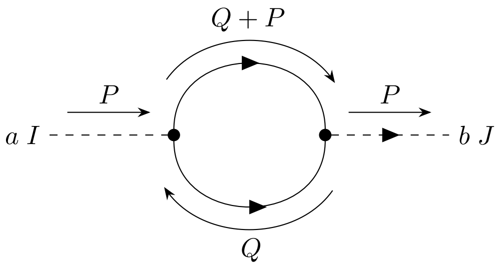
$$T\sum_{n\in\mathbb{Z}} f(p_0 = i\hat{\omega}_n) = -\frac{1}{2}\sum_{\tilde{p}=\text{poles of } f} \text{Res}_{p_0=\tilde{p}}\left[f(p_0)\right]\coth\left(\frac{\tilde{p}}{2T}\right)\,. \tag{5.31}$$

For Fermions the hyperbolic cotangent is replaced by a hyperbolic tangent because of the different statistics of Matsubara frequencies.

5.3.1.1 Scalar Self-Energy

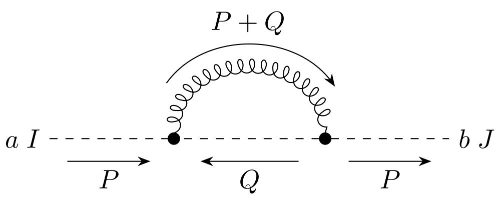
Following the Lagrangian (1.56) and the Feynman rules (J.2) we find four diagrams for the scalar self-energy:

$(\Sigma_{1,f})^{ab}_{IJ} := g^2$



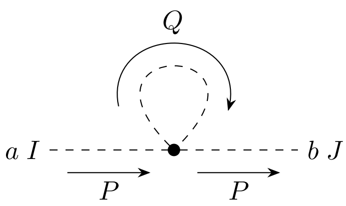
$$= 4\lambda\delta^{ab}\delta_{IJ} \sumint_Q \left[\frac{2}{Q^2} - \frac{P^2}{Q^2(Q+P)^2}\right] , \tag{5.32a}$$

$(\Sigma_{1,sv})^{ab}_{IJ} := g^2$



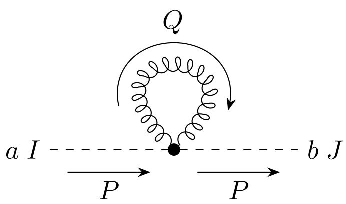
$$= 2\lambda\delta^{ab}\delta_{IJ} \sumint_Q \left[\frac{1}{2Q^2} + \frac{P^2}{Q^2(Q+P)^2}\right] , \tag{5.32b}$$

$(\Sigma_{1,s})^{ab}_{IJ} := g^2$



$$= -5\lambda\delta^{ab}\delta_{IJ} \sumint_Q \frac{1}{Q^2} , \tag{5.32c}$$

$(\Sigma_{1,v})^{ab}_{IJ} := g^2$



$$= -4\lambda\delta^{ab}\delta_{IJ} \sumint_Q \frac{1}{Q^2} . \tag{5.32d}$$

The additional g^2 was added due to the discussion in 3.2.1. Moreover, here we used the notation $P = (p_0, \vec{p})$, $Q = (q_0, \vec{q})$ and

$$\sumint_Q := T \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^{d-1}q}{(2\pi)^{d-1}} \xrightarrow{T=0} \int \frac{\mathrm{d}^d Q}{(2\pi)^d} . \tag{5.33}$$

The full scalar self-energy is then given by the sum of all four diagrams

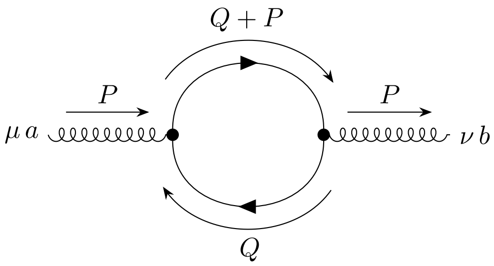
$$(\Sigma_1)^{ab}_{IJ} = (\Sigma_{1,f})^{ab}_{IJ} + (\Sigma_{1,sv})^{ab}_{IJ} + (\Sigma_{1,s})^{ab}_{IJ} + (\Sigma_{1,v})^{ab}_{IJ} , \tag{5.34}$$

however, we need to pay attention to the different statistics for Bosons and Fermions when adding the different contributions. Because of this some terms which cancel at zero temperature due to a relative minus sign will give non-vanishing contributions in the thermal case yielding for example the thermal mass [47, 53, 100].
Note also that we did not yet include the external propagators of the diagram proportional to $\left({}^{g^2}\!/_{P^2}\right)^2$ which will, together with the N from the Wilson loop exponential yield a prefactor of $g^4N^2 = \lambda^2$ as is expected for this order of perturbation.

5.3.1.2 Vacuum Polarization

For the vacuum polarization of the vector field we find

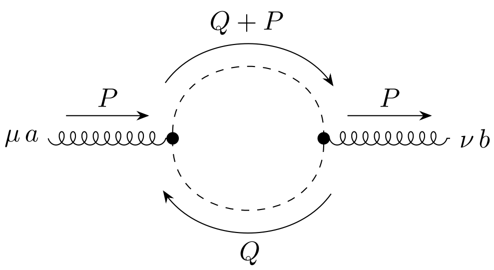
$(\Pi_{1,f})^{ab}_{\mu\nu} := g^2$



$$= -4\lambda\delta^{ab}\left(P^2\eta_{\mu\nu} - P_\mu P_\nu\right) \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2} - 16\lambda\delta^{ab} \sum\!\!\!\!\!\!\!\int_Q \frac{Q_\mu Q_\nu}{Q^2(Q+P)^2} \qquad (5.35a)$$

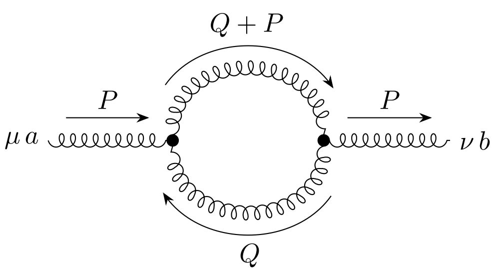
$$+ 4\lambda\delta^{ab}P_\mu P_\nu \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2} + 8\lambda\delta^{ab} \sum\!\!\!\!\!\!\!\int_Q \frac{\eta_{\mu\nu}}{Q^2} \,,$$

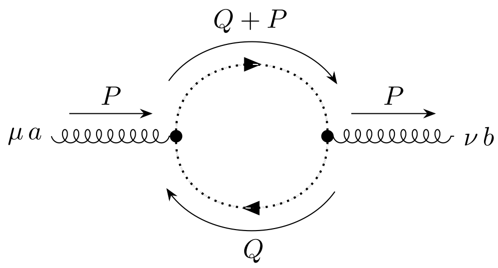
$(\Pi_{1,s})^{ab}_{\mu\nu} := g^2$



$$= 12\lambda\delta^{ab} \sum\!\!\!\!\!\!\!\int_Q \frac{Q_\mu Q_\nu}{Q^2(Q+P)^2} - 3\lambda\delta^{ab}P_\mu P_\nu \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2} \,, \qquad (5.35b)$$

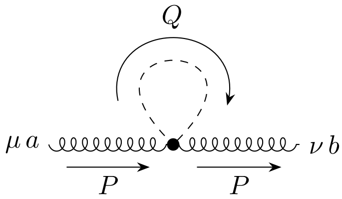
$(\Pi_{1,vg})^{ab}_{\mu\nu} := g^2$

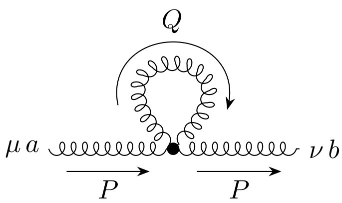


$+ g^2$ 

$$= 2\lambda\delta^{ab}\left(P^2\eta_{\mu\nu} - P_\mu P_\nu\right) \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2} + 4\lambda\delta^{ab} \sum\!\!\!\!\!\!\int_Q \frac{Q_\mu Q_\nu}{Q^2(Q+P)^2} \tag{5.35c}$$

$$- \lambda\delta^{ab} P_\mu P_\nu \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2} \,,$$

$(\Pi_{1,ts})^{ab}_{\mu\nu} := g^2$  $= -6\lambda\delta^{ab} \sum\!\!\!\!\!\!\int_Q \frac{\eta_{\mu\nu}}{Q^2}$ (5.35d)

$(\Pi_{1,tv})^{ab}_{\mu\nu} := g^2$  $= -2\lambda\delta^{ab} \sum\!\!\!\!\!\!\int_Q \frac{\eta_{\mu\nu}}{Q^2}$. (5.35e)

The full vacuum polarization at one loop is given by

$$\Pi^{ab}_{1\,\mu\nu} = (\Pi_{1,f})^{ab}_{\mu\nu} + (\Pi_{1,s})^{ab}_{\mu\nu} + (\Pi_{1,vg})^{ab}_{\mu\nu} + (\Pi_{1,ts})^{ab}_{\mu\nu} + (\Pi_{1,tv})^{ab}_{\mu\nu} \,. \tag{5.36}$$

Mind again fermionic and bosonic statistics. Although the ghosts are anti-commuting Grassmann fields we set their statistic to be bosonic following [53, 129]. Therefore, only $(\Pi_{1,f})^{ab}_{\mu\nu}$ is a fermionic integral.

5.3.1.3 Contribution From The Divergence

The full results for the scalar self-energy and the vacuum polarization are

$$\Sigma = -8\lambda \sum\!\!\!\!\!\!\int_Q (\delta_B - \delta_F)\frac{1}{Q^2} + 2\lambda P^2 \sum\!\!\!\!\!\!\int_Q (\delta_B - 2\delta_F)\frac{1}{Q^2(Q+P)^2} \,, \tag{5.37a}$$

$$\Pi_{\mu\nu} = -8\lambda\eta_{\mu\nu} \sum\!\!\!\!\!\!\int_Q (\delta_B - \delta_F)\frac{1}{Q^2} + 2\lambda\left(P^2\eta_{\mu\nu} - P_\mu P_\nu\right) \sum\!\!\!\!\!\!\int_Q (\delta_B - 2\delta_F)\frac{1}{Q^2(Q+P)^2} \tag{5.37b}$$

$$- 4\lambda P_\mu P_\nu \sum\!\!\!\!\!\!\int_Q (\delta_B - \delta_F)\frac{1}{Q^2(Q+P)^2} + 16\lambda \sum\!\!\!\!\!\!\int_Q (\delta_B - \delta_F)\frac{Q_\mu Q_\nu}{Q^2(Q+P)^2} \,.$$

Again, δ_B indicates Bosonic Matsubara frequencies $\hat{\omega}_m = 2\pi m T$ while δ_F indicates Fermionic Matsubara frequencies $\omega_m = \pi(2m+1)T$ with $m \in \mathbb{Z}$. These frequencies are the ones associated to the loop momentum $Q^2 = (i\omega_m)^2 - |\vec{q}|^2$ where we defined furthermore

$$\oint\!\!\!\!\!\!\sum_Q = T \sum_{m=-\infty}^{\infty} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \; . \tag{5.38}$$

These integrals are evaluated using expansions in high and low temperatures. At exactly zero temperature we can simply add the above results and find, including also the external propagators explicitly [62]

$$\Sigma^{ab}_{1\,IJ} = -12 g^4 N \delta_{IJ} \delta^{ab} \frac{\Gamma(\varepsilon)}{3!(4\pi)^2} \frac{1}{P^2} \; , \tag{5.39a}$$

$$\Pi^{ab}_{1\,\mu\nu} = -12 g^4 N \delta_{IJ} \delta^{ab} \frac{\Gamma(\varepsilon)}{3!(4\pi)^2} \frac{\eta_{\mu\nu} - {}^{P_\mu P_\nu}\!/_{P^2}}{P^2} \; . \tag{5.39b}$$

We used dimensional regularization in $d = 4 - 2\varepsilon$ dimensions to compute

$$\int \frac{\mathrm{d}^{4-2\varepsilon} Q}{(2\pi)^{4-2\varepsilon}} \frac{1}{Q^2 (Q+P)^2} = \frac{\Gamma(\varepsilon)}{(4\pi)^2} \; . \tag{5.40}$$

For the calculation of the Wilson loop it is important that we have the same prefactor for the scalar and vector fields, such that we can combine their respective contributions as we did for the ladder diagrams. The above result reproduced the well known one from Erickson, Semenoff and Zarembo [62]. We will not go into details of how the divergence for $\varepsilon \to 0$ is renormalized and the Wilson loop computed as this is done in the cited paper.

5.3.1.4 Longitudinal And Transverse Parts

From the general vacuum polarization the longitudinal and transverse part can be obtained via

$$\Pi_L = -\frac{P^2}{|\vec{p}|^2} u^\mu u^\nu \Pi_{\mu\nu} \; , \tag{5.41a}$$

$$\Pi_T = -\frac{1}{2}\Pi_L + \frac{1}{2}\eta^{\mu\nu}\Pi_{\mu\nu} \; , \tag{5.41b}$$

respectively [100, 123, 124]. This yields the two expressions

$$\Pi_L = -\frac{P^2}{|\vec{p}|^2} \Bigg[\lambda T^2 + 2\lambda(P^2 - 6p_0^2) \oint\!\!\!\!\!\!\sum_Q \frac{\delta_B}{Q^2(Q+P)^2} + 4\lambda(P^2 - 2p_0^2) \oint\!\!\!\!\!\!\sum_Q \frac{\delta_F}{Q^2(Q+P)^2}$$
$$+ 16\lambda \oint\!\!\!\!\!\!\sum_Q (\delta_B - \delta_F) \frac{q_0^2}{Q^2(Q+P)^2} \Bigg] \tag{5.42a}$$

$$\Pi_T = -\frac{1}{2}\Pi_L + \lambda T^2 + \lambda P^2 \sum\!\!\!\!\!\!\!\int_Q \frac{\delta_B - 4\delta_F}{Q^2(Q+P)^2} \,. \tag{5.42b}$$

5.3.2 Frequency sum and angular integral

To obtain a final expression for the self energies we need to evaluate the frequency sum and an integral over the three-momentum. Without expansion it is possible to sum all Matsubara frequencies and to evaluate the angular part of the integral. The result of this calculation for the polarizations is known [123] and therefore we will here go through the calculation for the scalar self-energy.
Recall

$$\Sigma = -8\lambda \sum\!\!\!\!\!\!\!\int_Q (\delta_B - \delta_F)\frac{1}{Q^2} + 2\lambda\, P^2 \sum\!\!\!\!\!\!\!\int_Q (\delta_B - 2\delta_F)\frac{1}{Q^2(Q+P)^2} \,. \tag{5.43}$$

with $P^2 = p_0^2 - |\vec{p}|^2$ and $Q^2 = -\omega_n^2 - |\vec{q}|^2$. The Matsubara frequency sums are obtained by using residues as follows

$$T\sum_{n\in\mathbb{Z}} f(q_0 = i\hat{\omega}_n)\delta_B = -\frac{1}{2}\sum_{\tilde{q}=\text{poles of } f} \mathrm{Res}_{q_0=\tilde{q}}\left[f(q_0)\right] \times \coth\left(\frac{|\tilde{q}|}{2T}\right) , \tag{5.44a}$$

$$T\sum_{n\in\mathbb{Z}} f(q_0 = i\omega_n)\delta_F = -\frac{1}{2}\sum_{\tilde{q}=\text{poles of } f} \mathrm{Res}_{q_0=\tilde{q}}\left[f(q_0)\right] \times \tanh\left(\frac{|\tilde{q}|}{2T}\right) . \tag{5.44b}$$

Using this we find

$$\Sigma = 4\lambda \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{1}{q}\left[4\mathrm{csch}\left(\frac{q}{T}\right) - \frac{P^2\left(p_0^2 + q^2 - |\vec{p}-\vec{q}|^2\right)\tanh\left(\frac{q}{2T}\right)\left(\coth^2\left(\frac{q}{2T}\right) - 2\right)}{-2p_0^2\left(q^2 + |\vec{p}-\vec{q}|^2\right) + p_0^4 + \left(q^2 - |\vec{p}-\vec{q}|^2\right)^2}\right] . \tag{5.45}$$

In the next step we turn to the angular integral by using spherical coordinates $\mathrm{d}^3 q = r^2\sin(\theta)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$ and $(\vec{p}-\vec{q})^2 = p^2 + q^2 - 2pq\cos(\theta)$. As the integral over ϕ is trivial it simply yields a factor 2π while after integrating over θ we find

$$\begin{aligned}\Sigma = \frac{\lambda}{4\pi^2}\int_0^\infty \mathrm{d}q \Bigg[32q\mathrm{csch}\left(\frac{q}{T}\right) + \frac{P^2}{p}\left(\coth\left(\frac{q}{T}\right) - 3\mathrm{csch}\left(\frac{q}{T}\right) - 1\right) \times \\ \log\left(\frac{(p-p_0-2q)(p+p_0-2q)}{(p-p_0+2q)(p+p_0+2q)}\right)\Bigg] .\end{aligned} \tag{5.46a}$$

Similarly the longitudinal and transverse polarization can be obtained. The calculation is known [123] and we find

$$\begin{aligned}\Pi_T = \frac{\lambda}{8\pi^2 p^3}\int_0^\infty \mathrm{d}q \left(\coth\left(\frac{q}{T}\right) - 1\right)\left[16qpe^{q/T}\left(p^2p_0^2\right) + \left(p^2 - p_0^2\right)\right. \\ \left(\left(e^{q/T}\left(2(p_0-2q)^2 + p^2\right) - p^2\right)\log\left(1 - \frac{2p}{2q+p-p_0}\right)\right.\end{aligned}$$

$$+\left(p^2 - e^{q/T}\left(2(2q+p_0)^2+p^2\right)\right)\log\left(\frac{2p}{2q-p+p_0}+1\right)$$

$$+16qp_0e^{q/T}\log\left(\frac{p+p_0}{p_0-p}\right)\Bigg)\Bigg] \tag{5.46b}$$

$$\Pi_L = \frac{\lambda}{8\pi^2p^3}(p^2-p_0^2)\int_0^\infty \mathrm{d}q\left(\coth\left(\frac{q}{T}\right)-1\right)\times$$

$$\times\Bigg[\left(e^{q/T}\left(3p^2-4(p_0-2q)^2\right)-p^2\right)\log\left(1-\frac{2p}{2q+p-p_0}\right)$$

$$+\left(e^{q/T}\left(4(2q+p_0)^2-3p^2\right)+p^2\right)\log\left(\frac{2p}{2q-p+p_0}+1\right)$$

$$-32qp_0e^{q/T}\log\left(\frac{p+p_0}{p_0-p}\right)+32qpe^{q/T}\Bigg]\,. \tag{5.46c}$$

These integrals over q cannot be solved without expanding. Therefore we need to consider different limits.

5.3.3 High and small temperature limits

Let us assume that $T\to\infty$ and that further $P << T$. This is in fact the same assumption also made for the Hard Thermal Loop approximation [119] where one sets $P \propto \sqrt{\lambda}T$. For the high temperature limit first rescale $q \to Tq$ and in the next step expand around high temperature. This yields

$$\Sigma^{(T\to\infty)}(p_0,p) = \frac{8\lambda T^2}{\pi^2}\int_0^\infty \mathrm{d}q\, q\, \operatorname{csch}(q)$$

$$-\frac{\lambda}{2\pi^q}\int_0^\infty \frac{\mathrm{d}q}{q}(p^2-p_0^2)\tanh\left(\frac{q}{2}\right)\left(\coth\left(\frac{q}{2}\right)-1\right)\left(\coth\left(\frac{q}{2}\right)+2\right)\,, \tag{5.47a}$$

$$\Pi_T^{(T\to\infty)}(p_0,p) = -\frac{2T^2}{\pi^2p^3}\int_0^\infty \mathrm{d}q\left(e^q q\lambda p_0(\coth(q)-1)\left((p_0^2-p^2)\log\left(\frac{p+p_0}{p_0-p}\right)-2pp_0\right)\right)$$

$$-\frac{\lambda(p^2-p_0^2)}{12\pi^2}\int_0^\infty \frac{\mathrm{d}q}{q}\,(5e^q-3)\,(\coth(q)-1)+\mathcal{O}\left(\frac{1}{T^2}\right)\,, \tag{5.47b}$$

$$\Pi_L^{(T\to\infty)}(p_0,p) = \frac{4\lambda T^2}{\pi^2p^3}\int_0^\infty \mathrm{d}q\, e^q q(\coth(q)-1)(p^2-p_0^2)\left(2p-p_0\log\left(\frac{p+p_0}{p_0-p}\right)\right)$$

$$-\frac{\lambda}{12\pi^2}(p^2-p_0^2)\int_0^\infty \frac{\mathrm{d}q}{q}\,(5e^q-3)\,(\coth(q)-1)+\mathcal{O}\left(\frac{1}{T^2}\right)\,. \tag{5.47c}$$

For all three cases the first integral is readily evaluated. The second integral, however, suffers from an IR divergence which we will regularize by a cut-off ε_q. Furthermore the constant part can only be computed analytically. All in all we thus find

$$
\begin{aligned}
\Sigma^{(T\to\infty)}(p_0,p) &= 2\lambda T^2 + 2\lambda\left(p_0^2-p^2\right)\left(\frac{1}{2\pi^2\varepsilon_q}-\frac{\log(\varepsilon_q)}{4\pi^2}+c_1\right)+\mathcal{O}\left(\frac{1}{T^2}\right)\,,\\
\Pi_T^{(T\to\infty)}(p_0,p) &= \frac{\lambda T^2 p_0}{2p^3}\left(\left(p_0^2-p^2\right)\log\left(\frac{p+p_0}{p_0-p}\right)-2pp_0\right)\\
&\quad +\lambda\left(p_0^2-p^2\right)\left(-\frac{1}{6\pi^2\varepsilon_q}+\frac{\log(\varepsilon_q)}{4\pi^2}+c_2\right)+\mathcal{O}\left(\frac{1}{T^2}\right)\,, \qquad (5.48)\\
\Pi_L^{(T\to\infty)}(p_0,p) &= \frac{\lambda T^2\left(p_0^2-p^2\right)}{p^3}\left(2p-p_0\log\left(\frac{p+p_0}{p_0-p}\right)\right)\\
&\quad +\lambda(p^2-p_0^2)\left(-\frac{1}{6\pi^2\varepsilon_q}+\frac{\log(\varepsilon_q)}{4\pi^2}+c_2\right)+\mathcal{O}\left(\frac{1}{T^2}\right)\,.
\end{aligned}
$$

The leading order does indeed agree with the HTL limit in [123]. The constants came from numeric integration with

$$
c_1 \simeq 0.07659513875211767\,,\quad c_2 \simeq 0.014887402805171021 \qquad (5.49)
$$

Assuming that the temperature is smaller than any other scale is very similar and in this case the remaining integral is readily evaluated in all cases such that we find

$$
\begin{aligned}
\Sigma^{(T\to0)}(p_0,p) &= \Sigma^{(T=0)}(p_0,p)+\frac{2\lambda T^2}{3}-\frac{44\pi^2\lambda T^4(p^2+3p_0^2)}{45(p^2-p_0^2)^2}+\mathcal{O}\left(T^6\right)\,,\\
\Pi_T^{(T\to0)}(p_0,p) &= \Pi_T^{(T=0)}(p_0,p)+\frac{\lambda T^2}{3}-\frac{4\pi^2\lambda T^4\left(13p^2+9p_0^2\right)}{45(p^2-p_0^2)^2}+\mathcal{O}\left(T^6\right)\,, \qquad (5.50)\\
\Pi_L^{(T\to0)}(p_0,p) &= \Pi_L^{(T=0)}(p_0,p)+\frac{\lambda T^2}{3}+\frac{4\pi^2\lambda T^4\left(17p^2-9p_0^2\right)}{45\left(p^2-p_0^2\right)^2}+\mathcal{O}\left(T^6\right)\,.
\end{aligned}
$$

Once again, we do not write out the $T=0$ result (5.39) explicitly as it is known from [62].

5.3.4 Fourier transform of the corrected propagator

The most straight forward but naive calculation is to take the self-energy corrections (5.48) and (5.50) as they are and plug them into the Wilson loops. This involves a 4-dimensional Fourier transform and the nested integral along the loop contour. Due to the involved structure of the self-energy which needed to be expanded we use one further simplification before considering the Fourier transform.

Let us move to the calculation of the straight Wilson line which we define to be entirely in the spatial $x_3=z$ direction as an example. The point at which the straight line attaches to the thermal circle should in principle be kept arbitrary. Therefore we will average over all points $x_0 \in [0, 1/T]$ thus taking the **thermal average**. Let $f(\hat{\omega}_n, \vec{p})$ be the insertion to the Wilson loop in momentum space and $\tilde{f}(x_0, \vec{x})$ its Fourier transform according to Equation (5.51). From this the thermal average is defined as

$$\begin{aligned} \tilde{f}_{\text{th. av.}}(\vec{x}) &= \frac{\int_0^{1/T} \mathrm{d}x_0\, \tilde{f}(x_0, \vec{x})}{\int_0^{1/T} \mathrm{d}x_0\, 1} = T \int_0^{1/T} \mathrm{d}x_0 \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f(\hat{\omega}_n, \vec{p}) e^{i\hat{\omega}_n x_0 - i\vec{p}\vec{x}} \\ &= T \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f(\hat{\omega}_n = 0, \vec{p}) e^{-i\vec{p}\vec{x}} \,. \end{aligned} \tag{5.51}$$

We used that

$$T \int_0^{1/T} \mathrm{d}x_0\, e^{\mathrm{i}\hat{\omega}_n x_0} = -i \frac{-1 + e^{2\mathrm{i}\pi n}}{2\pi n} = 0 \quad \forall\, n \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad \lim_{n \to 0} \left(-\mathrm{i} \frac{-1 + e^{2i\pi n}}{2\pi n} \right) = 1 \,. \tag{5.52}$$

Therefore, by taking the thermal average we pick up only the zero mode in the self-energies and do a three dimensional Fourier transform *with prefactor*[4] T. Note, however, that this smearing cannot be used for the Polyakov loop which lies on the thermal circle.

When considering Wilson loops we can use the smearing introduced in the above Equation (5.51). By taking the thermal average we effectively set $p_0 = 0$ which thus simplifies the self-energies and vacuum polarization we consider. They are given for the high and low temperature limit by Equations (5.48) and (5.50), respectively. They read

$$\begin{aligned} \Sigma^{(T\to\infty)}(0, p) &= 2\lambda T^2 - 2\lambda p^2 \left(\frac{1}{2\pi^2 \varepsilon_q} - \frac{\log(\varepsilon_q)}{4\pi^2} + c_1 \right) + \mathcal{O}\left(\frac{1}{T^2} \right) , \\ \Pi_T^{(T\to\infty)}(0, p) &= \lambda p^2 \left(-\frac{1}{6\pi^2 \varepsilon_q} + \frac{\log(\varepsilon_q)}{4\pi^2} + c_2 \right) + \mathcal{O}\left(\frac{1}{T^2} \right) , \\ \Pi_L^{(T\to\infty)}(0, p) &= 2\lambda T^2 + \lambda p^2 \left(-\frac{1}{6\pi^2 \varepsilon_q} + \frac{\log(\varepsilon_q)}{4\pi^2} + c_2 \right) + \mathcal{O}\left(\frac{1}{T^2} \right) . \end{aligned} \tag{5.53}$$

and

$$\begin{aligned} \Sigma^{(T\to0)}(0, p) &= \Sigma^{(T=0)}(0, p) + \frac{2\lambda T^2}{3} - \frac{44\pi^2 \lambda T^4}{45 p^2} + \mathcal{O}\left(T^6\right) , \\ \Pi_T^{(T\to0)}(p0, p) &= \Pi_T^{(T=0)}(0, p) + \frac{\lambda T^2}{3} - \frac{52\pi^2 \lambda T^4}{45 p^2} + \mathcal{O}\left(T^6\right) , \\ \Pi_L^{(T\to0)}(0, p) &= \Pi_L^{(T=0)}(0, p) + \frac{\lambda T^2}{3} + \frac{68\pi^2 \lambda T^4}{45 p^2} + \mathcal{O}\left(T^6\right) . \end{aligned} \tag{5.54}$$

[4]The fact that this prefactor is present can for example be seen by dimensional analysis.

The remaining Fourier transform is then rather straight forward. Using Equations (5.51) we are interested for each of these self-energies in the propagators

$$\Delta_\Pi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\Pi(p_0=0,p)}{p^4} e^{\mathrm{i}\,p\cdot x} \,. \tag{5.55}$$

The Fourier transformations for a massless propagator in three dimensions[5] is [62, 128]

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{\mathrm{i}\,p\cdot x}}{(p^2)^a} = \frac{\Gamma\left(\frac{3}{2}-a\right)}{4^a \pi^{3/2}\Gamma(a)} \frac{1}{(x^2)^{3/2-a}} \,. \tag{5.56}$$

We have three different powers of interest given through the above self-energies, namely we need

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{\mathrm{i}\,p\cdot x}}{(p^2)^1} = \frac{1}{4\pi|x|}\,, \quad \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{\mathrm{i}\,p\cdot x}}{(p^2)^2} = -\frac{|x|}{8\pi}\,, \quad \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{e^{\mathrm{i}\,p\cdot x}}{(p^2)^3} = \frac{|x|^3}{96\pi}\,. \tag{5.57}$$

Note the negative sign which arises in the second integral. This will be the leading behavior in the high and the low temperature limit alike. Due to this integral the overall sign of the result will be negative. Comparing with the general formula above this sign comes from $\Gamma(-1/2) = -2\sqrt{\pi}$.

From this we can write down a propagator for all the self-energies above. Note that we have to add an additional factor of T coming from the thermal average (5.51).

$$\begin{aligned}
\Delta_\Sigma^{(T\to\infty)}(x) &= -\frac{\lambda T^3}{4\pi}|x| - \frac{\lambda T}{2\pi|x|}\left(\frac{1}{2\pi^2\varepsilon_q} - \frac{\log(\varepsilon_q)}{4\pi^2} + c_1\right) + \mathcal{O}\left(\frac{1}{T}\right)\,, \\
\Delta_{\Pi_T}^{(T\to\infty)}(x) &= \frac{\lambda T}{4\pi|x|}\left(-\frac{1}{6\pi^2\varepsilon_q} + \frac{\log(\varepsilon_q)}{4\pi^2} + c_2\right) + \mathcal{O}\left(\frac{1}{T}\right)\,, \\
\Delta_{\Pi_L}^{(T\to\infty)}(x) &= -\frac{\lambda T^3}{4\pi}|x| + \frac{\lambda T}{4\pi|x|}\left(-\frac{1}{6\pi^2\varepsilon_q} + \frac{\log(\varepsilon_q)}{4\pi^2} + c_2\right) + \mathcal{O}\left(\frac{1}{T}\right)\,,
\end{aligned} \tag{5.58}$$

$$\begin{aligned}
\Delta_\Sigma^{(T\to 0)}(0,p) &= \Delta_\Sigma^{(T=0)}(0,p) - \frac{\lambda T^3}{12\pi}|x| - \frac{11\pi\lambda T^5}{1080}|x|^3 + \mathcal{O}\left(T^7\right)\,, \\
\Delta_{\Pi_T}^{(T\to 0)}(p0,p) &= \Delta_{\Pi_T}^{(T=0)}(0,p) - \frac{\lambda T^3}{24\pi}|x| - \frac{13\pi\lambda T^5}{1080}|x|^3 + \mathcal{O}\left(T^7\right)\,, \\
\Delta_{\Pi_L}^{(T\to 0)}(0,p) &= \Delta_{\Pi_L}^{(T=0)}(0,p) - \frac{\lambda T^3}{24\pi}|x| + \frac{17\pi\lambda T^4}{1080}|x|^3 + \mathcal{O}\left(T^7\right)\,.
\end{aligned} \tag{5.59}$$

As mentioned above, all these correlators (except for Δ_{Π_T}) have a leading order contribution proportional to $-T^3|x|$ with the negative sign coming from the above Fourier transform.

[5] The temporal dimension is not relevant anymore due to the thermal average.

These propagators can thus be used in the calculation of the self-energy insertions of the Wilson loops we consider below.

5.4 The circular Wilson loop

Given the results of the previous sections it is now straight foreward to calculate the circular Maldacena-Wilson loop at order λ^2 in the weak coupling expansion. The contour of the circular loop with radius $R \in \mathbb{R}^+$ is parameterized (**??**) by

$$C: \; x^\mu(\mathfrak{t}) = (0\,,\, R\cos(\mathfrak{t})\,,\, R\sin(\mathfrak{t})\,,\, 0) \quad , \quad \mathfrak{t} \in [0, 2\pi]\,.$$

As derived in the previous chapter, we can calculate the Wilson loop perturbatively by expanding the exponential. The first terms of this expansion are given in Equations(2.38)

$$\begin{aligned}\langle W(C)\rangle \approx 1 &+ \frac{1}{2N}\overleftarrow{\mathcal{P}}\mathrm{tr}\left\langle \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \,(\mathrm{i}A_{\mu\,1}\dot{x}_1^\mu + \Phi_{r\,1}|\dot{x}_1|n^r)\,(\mathrm{i}A_{\nu\,2}\dot{x}_2^\nu + \Phi_{s\,2}|\dot{x}_2|n^s)\right\rangle \\ &+ \frac{1}{3!N}\overleftarrow{\mathcal{P}}\mathrm{tr}\left\langle \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \int_0^{\mathfrak{t}_2} \mathrm{d}\mathfrak{t}_3 \,(\mathrm{i}A_{\mu\,1}\dot{x}_1^\mu + \Phi_{r\,1}|\dot{x}_1|n^r)\times \right. \\ &\left. (\mathrm{i}A_{\nu\,2}\dot{x}_2^\nu + \Phi_{s\,2}|\dot{x}_2|n^s)\,(\mathrm{i}A_{\rho 3}\dot{x}_3^\rho + \Phi_{t\,3}|\dot{x}_3|n^t)\right\rangle + \dots\,. \end{aligned} \tag{5.60}$$

where we used the notation $A_{\mu\,i} = A_\mu(x(\mathfrak{t}_i))$, $\Phi_{I\,i} = \Phi_r(x(\mathfrak{t}_i))$ and $x_i = x(\mathfrak{t}_i)$. When taking the expectation value $\langle \dots \rangle$ we Wick contract the bosonic fields among themselves or with fields from the expansion of the Euclidean action. We work in perturbation theory around small 't Hooft coupling $\lambda = g^2 N$ as well as in the large N limit[6]. This means that we only need to consider planar diagrams of low loop order. As derived in chapter 3.2.1 the above expansion of the exponential is indeed also an expansion in weak coupling λ. The leading order diagrams following from this expansion were shown in Figure 1.1 for the circular loop.

The first order diagram has been calculated in the above Section 3.2.1. We saw that at this order corrections to the zero temperature result [62] arose from the corrected bare thermal propagator [111] and we obtained the result in an expansion around small and high temperature. The result was found to entirely depend on the dimensionless combination RT between the temperature and the radius of the circular Wilson loop. Finally we obtained (3.120):

$$\langle W_{\bigcirc}\rangle^{(RT\to\infty)} = 1 + \frac{\lambda}{2}RT + \mathcal{O}\left(\lambda^2, \frac{\lambda}{(RT)^n}\right) \quad , \qquad \forall n \in \mathbb{N}$$

[6]More precisely we take $N \to \infty$ and $g \to 0$ while keeping $\lambda = g^2 N \ll 1$ fixed.

$$\langle W_{\bigcirc}\rangle^{(RT\to 0)} = 1 + \frac{\lambda}{8} + \lambda\frac{(RT)^2\pi^2}{12} - \lambda\frac{(RT)^4\pi^4}{60} + \mathcal{O}(\lambda^2,\, \lambda(RT)^6)\,.$$

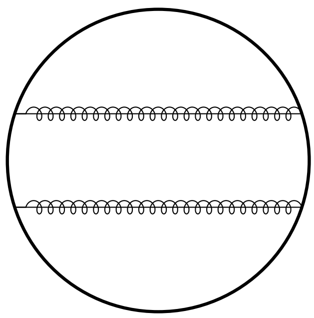
In this chapter we focus on the corrections of the next order in the weak coupling expansion, λ^2. The diagrams in Figure 1.1b, 1.1c and 1.1d show the three contributions we need to consider.

The rainbow diagram, the insertion of two bare propagators, is relatively easy and follows straight forwardly from the calculation of a single insertion at lower order. For the self energy we need to insert the corrected propagator at one-loop order. This propagator was derived in a thermal average in Section 5.3 from the self-energies in momentum space. Again we thus only need to insert the parameterization of the WL and integrate its contour. The vertex insertion features an additional spacetime integral over the insertion point of the vertex which can be chosen arbitrarily. Due to the more complex structure of this insertion and the additional integral even in the thermal limits we were only able to obtain numeric results.

5.4.1 Rainbow diagram

The rainbow diagram in Figure 1.1b,

$$\langle W_{(b)}\rangle_\beta =$$



, (5.61)

follows straight from the one at lower order, Figure 1.1a. For the single propagator insertion we considered a loop to loop propagator

$$(|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2)\,\Delta^{(T\neq 0)}(x_1 - x_2) = \sin\left(\frac{\mathfrak{t}_1 - \mathfrak{t}_2}{2}\right)\coth\left(2\pi RT\sin\left(\frac{\mathfrak{t}_1 - \mathfrak{t}_2}{2}\right)\right)\,, \tag{5.62}$$

which was then integrated along the parameterization (??) with $\mathfrak{t}_1 > \mathfrak{t}_2$. The hyperbolic cotangent function had to be expanded around small and high values of the dimensionless parameter RT on which the circular WL naturally depends. Finally the result (3.120) was obtained after integrating along the contour.

The rainbow diagram of order λ^2 similarly has two insertions of the above loop-to-loop propagator. We need to take into account all possibilities for contractions. Therefore assume for the insertion points $\mathfrak{t}_1 > \mathfrak{t}_2 > \mathfrak{t}_3 > \mathfrak{t}_4$. Then the points 12 and 34 can be connected but similarly also the connection 41 and 23 are possible. As we consider a closed loop the insertion points on opposite ends are still next to each other. Contrarily a connection 13 and 24 is not relevant as it would lead to a subleading diagram in the large N expansion. Inserting these two possibilities we find

$$\begin{aligned}
\langle W_{(b)}\rangle_\beta =& \frac{\lambda^2}{4}\int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \int_0^{\mathfrak{t}_2} \mathrm{d}\mathfrak{t}_3 \int_0^{\mathfrak{t}_3} \mathrm{d}\mathfrak{t}_4 \times \\
&\times \Bigg[\left[\left(|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2\right)\left(|\dot{x}_3||\dot{x}_4| - \dot{x}_3\cdot\dot{x}_4\right)\right] \Delta^{(T\neq 0)}(x_1-x_2)\Delta^{(T\neq 0)}(x_3-x_4) + \\
&+ \left[\left(|\dot{x}_1||\dot{x}_4| - \dot{x}_1\cdot\dot{x}_4\right)\left(|\dot{x}_2||\dot{x}_3| - \dot{x}_2\cdot\dot{x}_3\right)\right] \Delta^{(T\neq 0)}(x_1-x_4)\Delta^{(T\neq 0)}(x_2-x_3) \Bigg] \\
=& \frac{(RT)^2\lambda^2}{64\pi^2}\int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \int_0^{\mathfrak{t}_2} \mathrm{d}\mathfrak{t}_3 \int_0^{\mathfrak{t}_3} \mathrm{d}\mathfrak{t}_4 \times \qquad (5.63) \\
&\left[\sin\left(\frac{\mathfrak{t}_{12}}{2}\right)\coth\left(2\pi RT \sin\left(\frac{\mathfrak{t}_{12}}{2}\right)\right) \sin\left(\frac{\mathfrak{t}_{34}}{2}\right)\coth\left(2\pi RT \sin\left(\frac{\mathfrak{t}_{34}}{2}\right)\right) + \right. \\
&\left. + \sin\left(\frac{\mathfrak{t}_{14}}{2}\right)\coth\left(2\pi RT \sin\left(\frac{\mathfrak{t}_{14}}{2}\right)\right) \sin\left(\frac{\mathfrak{t}_{23}}{2}\right)\coth\left(2\pi RT \sin\left(\frac{\mathfrak{t}_{23}}{2}\right)\right) \right] ,
\end{aligned}$$

where $\mathfrak{t}_{ij} = ploop_i - \mathfrak{t}_j$. As in the one loop case we will expand the dimensionless combination of temperature and radius of the circular loop RT before evaluating the contour integrations. At small RT we expand around 0 and integrate term by term. For $RT \to \infty$ we set the hyperbolic cotangent to 1 and then evaluate the integral. We will compare both of these results with the numerically integrated values.

The hyperbolic cotangent function is expanded according to Equation (3.116) around $RT = 0$. The terms can be multiplied out between the two hyperbolic functions yielding different powers of π at the same order in RT. Integrating over the contour then yields the final result

$$\langle W_{(b)}\rangle_{RT\to 0} = \lambda^2 \left[\frac{1}{192} + (RT)^2\left(\frac{\pi^2}{144} - \frac{1}{96}\right) + (RT)^4\left(\frac{\pi^4}{1080} - \frac{7\pi^2}{1152}\right) + \mathcal{O}\left((RT)^6\right)\right] . \quad (5.64)$$

Figure 5.2 shows the result of the expansion compared with the numerical value. We see a very nice fitting for $RT < 0.4$ which could in principle be improved by including also higher orders in the expansion. Again one could attempt to look for a general expression for arbitrary powers of RT. As mentioned in Section 3.2.1 the known series expansion of the hyperbolic cotangent has a certain radius of convergence yielding that only for $RT < 1/2$ the expansion would be valid. Figure 5.2, however, shows that in this regime good results can be obtained already by a simple perturbative expansion.

For large RT we again take the hyperbolic cosine to be one as explained above, see Equation (3.118). Then the $\mathfrak{t}$ integrals can be carried out readily and we find a quadratic

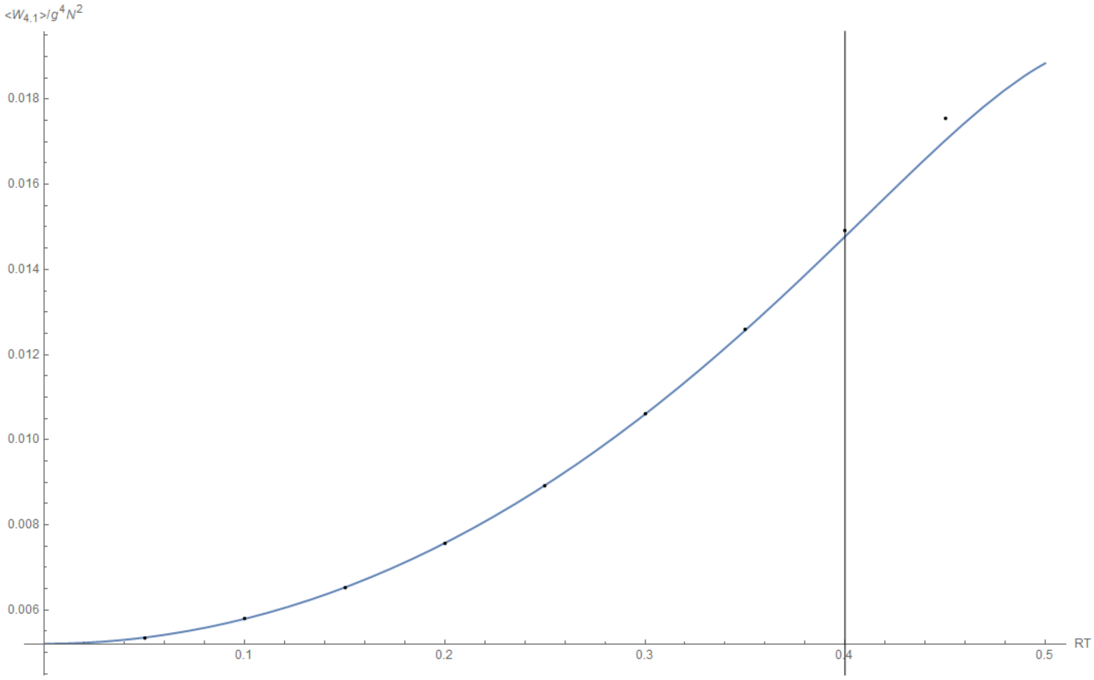


Figure 5.2: Plot of $\langle W_{(b)}\rangle_\beta / g^4 N^2$ for RT between 0 and 0.7. The blue line is the expansion up to order $\mathcal{O}((RT)^6)$. The black dots represent the numerically integrated values. We again added the vertical line at $RT = 0.4$ to indicate where the perturbation seemingly breaks down.

dependence on the dimensionless combination RT,

$$\langle W_{(b)}\rangle_{RT\to\infty} = \frac{\lambda^2}{2^4}(RT)^2 + \mathcal{O}\left((RT)^n\right) \ , \forall n \in \mathbb{N} \, . \tag{5.65}$$

We thus expect to see a quadratic behavior which can in fact be seen to fit the numerical values. This is shown in Figure 3.2. This is an interesting observation. In the $RT \to \infty$ limit each propagator comes with RT to the power one. Therefore the rainbow diagrams in the high temperature limit are expected to precisely yield a dependence of $(\lambda RT)^n$ at order n in the small coupling expansion. While this observation certainly is interesting we will see below that the self-energy and vertex diagram at quadratic order in the weak coupling yield a cubic dependence on the dimensionless combination RT. Therefore the rainbow diagram is actually subleading in said limit.

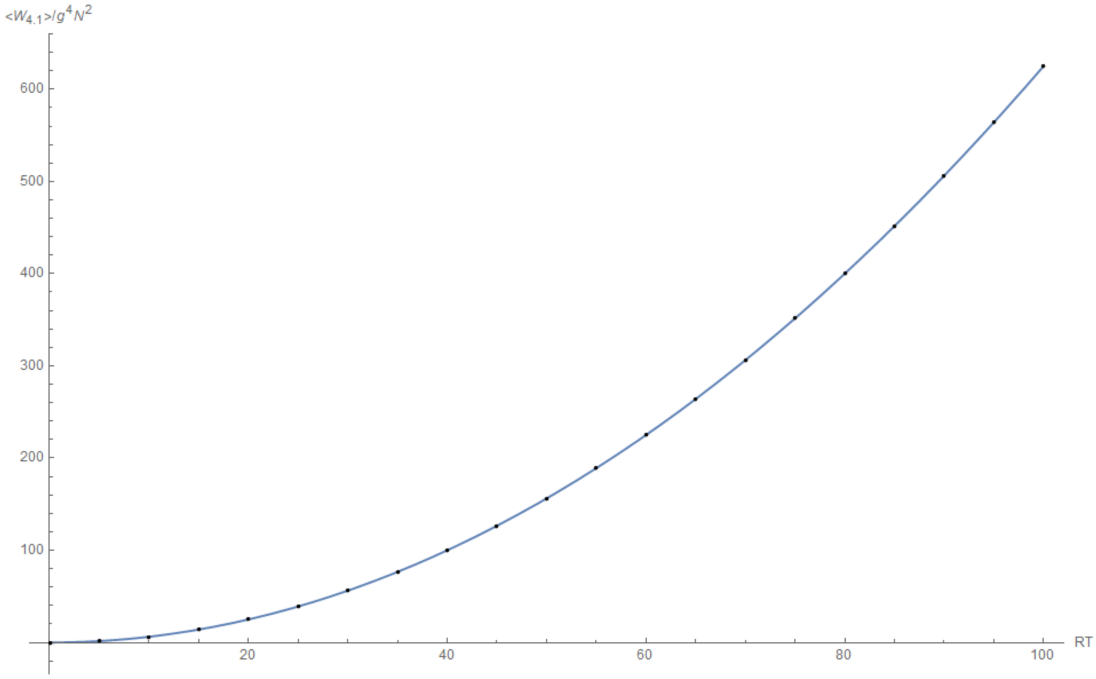
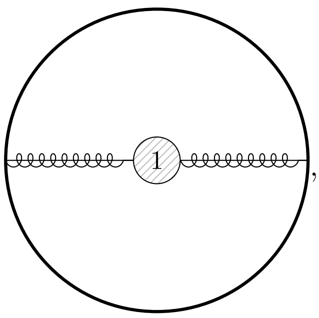


Figure 5.3: Plot of $\langle W_{(b)}\rangle_\beta / g^2 N$ for RT between 0 and 100. The blue line is the quadratic function $4\pi^2\,(RT)^2$ while the dots indicate several numerical values.

5.4.2 Self-Energy Insertion

Let us consider the self-energy diagram Figure 1.1c,

$$\langle W_{(c)}\rangle_\beta = \quad , \tag{5.66}$$



containing the insertion of a single propagators at one loop order. The calculation of the self-energy and vacuum polarization was considered in chapter 5.3 where also the Fourier transform of the corrected propagator was obtained. Recall that from the definition of the WL we find

$$\langle W_{(c)}\rangle_\beta = \frac{N}{2}\int_0^{2\pi} \mathrm{dt}_1 \int_0^{\mathrm{t}_1} \mathrm{dt}_2 \left(|\dot{x}_1||\dot{x}_2| n^r n^s \langle \Phi^a_{r\,1}\Phi^b_{s\,2}\rangle_1 - \dot{x}_1^\mu \dot{x}_2^\nu \langle A^a_{\mu\,1} A^b_{\nu\,2}\rangle_1\right) , \tag{5.67}$$

where the correlators as written denote the one-loop corrections to the bare propagators. Due to theorem 1 we only need to consider the scalar propagator and the transverse parts of the vector bosons. The propagators we thus consider are given in Equations (5.58)

$$\Delta_\Sigma^{(T\to\infty)}(x) = -\frac{\lambda T^3}{4\pi}|x| - \frac{\lambda T}{2\pi|x|}\left(\frac{1}{2\pi^2\varepsilon_q} - \frac{\log(\varepsilon_q)}{4\pi^2} + c_1\right) + \mathcal{O}\left(\frac{1}{T}\right) ,$$

$$\Delta_{\Pi_T}^{(T\to\infty)}(x) = \frac{\lambda T}{4\pi|x|}\left(-\frac{1}{6\pi^2\varepsilon_q} + \frac{\log(\varepsilon_q)}{4\pi^2} + c_2\right) + \mathcal{O}\left(\frac{1}{T}\right) ,$$

$$\Delta_{\Sigma}^{(T\to 0)}(x) = \Delta_{\Sigma}^{(T=0)}(x) - \frac{\lambda T^3}{12\pi}|x| - \frac{11\pi\lambda T^5}{1080}|x|^3 + \mathcal{O}\left(T^7\right) ,$$

$$\Delta_{\Pi_T}^{(T\to 0)}(x) = \Delta_{\Pi_T}^{(T=0)}(x) - \frac{\lambda T^3}{24\pi}|x| - \frac{13\pi\lambda T^5}{1080}|x|^3 + \mathcal{O}\left(T^7\right) .$$

The zero temperature part is known from [62, 64, 130] and cancels against the vertex. We will hence ignore it in the following calculation. Further recall that we used a thermal average (5.51) for these propagators.

In the limit $RT \to 0$ the parameterzation (1.64) can be inserted with $x \to x(\mathfrak{t}_1) - x(\mathfrak{t}_2)$ and as usual we satisfy the path ordering by setting $\mathfrak{t}_1 > \mathfrak{t}_2$. Then the contour integral at each order is straight forward

$$\begin{aligned}\langle W_{(c)}\rangle_\beta =& \lambda^2 T^3 \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \Bigg[R^2\left(-\frac{R\sin\left(\frac{\mathfrak{t}_1-\mathfrak{t}_2}{2}\right)}{2\pi} - \frac{11\pi T^2 R^3 \sin\left(\frac{\mathfrak{t}_1-\mathfrak{t}_2}{2}\right)}{1080}\right) \\ & - R^2\cos(\mathfrak{t}_1-\mathfrak{t}_2)\left(-\frac{R\sin\left(\frac{\mathfrak{t}_1-\mathfrak{t}_2}{2}\right)}{12\pi} - \frac{13\pi T^2 R^3 \sin\left(\frac{\mathfrak{t}_1-\mathfrak{t}_2}{2}\right)}{1080}\right) + \mathcal{O}\left(T^7\right)\Bigg] \\ =& -\frac{19\lambda^2 (RT)^3}{9} - \frac{752\pi^2 (RT)^5}{2025} + \mathcal{O}\left(\lambda^2 (RT)^7\right) . \end{aligned} \tag{5.68}$$

Compared to the above result (5.64) this diagram is subleading in the limit considered.

The calculation for the limit $RT \to \infty$ is mostly very similar. However, the contour integration yields a divergence when the insertion points of the correlators coincide. This is handled by the introduction of a small cut-off parameter $\varepsilon_{\mathfrak{t}}$

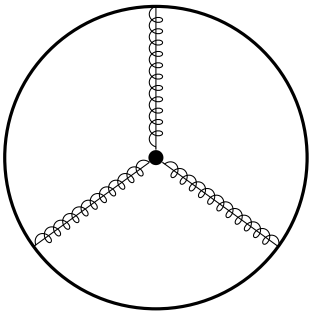
$$\begin{aligned}\langle W_{(c)}\rangle_\beta =& \lambda^2 T \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \Bigg[-R^2\Bigg(T^2 \frac{R\sin\left(\frac{\mathfrak{t}_1-\mathfrak{t}_2}{2}\right)}{2\pi} \\ & - \frac{1}{4\pi R\sin\left(\frac{\mathfrak{t}_1-\mathfrak{t}_2}{2}\right)}\left(\frac{1}{2\pi^2\varepsilon_q} - \frac{\log(\varepsilon_q)}{4\pi^2} + c_1\right)\Bigg) \\ & - \frac{R^2\cos(\mathfrak{t}_1-\mathfrak{t}_2)}{8\pi R\sin\left(\frac{\mathfrak{t}_1-\mathfrak{t}_2}{2}\right)}\left(-\frac{1}{6\pi^2\varepsilon_q} + \frac{\log(\varepsilon_q)}{4\pi^2} + c_2\right) + \mathcal{O}\left(\frac{1}{T^2}\right)\Bigg] \\ =& -2\lambda^2 (RT)^3 + \frac{1}{2}\lambda^2 RT \Bigg[\frac{\log(\varepsilon_q)\left(\log\left(\frac{4}{\varepsilon_{\mathfrak{t}}}\right) - 2\right)}{4\pi^2} + \frac{10\log\left(\frac{4}{\varepsilon_{\mathfrak{t}}}\right) + 4}{12\pi^2\varepsilon_q} \end{aligned}$$

$$+2c_1 \log\left(\frac{4}{\varepsilon_{\mathfrak{t}}}\right) - c_2\left(\log\left(\frac{\varepsilon_{\mathfrak{t}}}{4}\right) + 2\right)\Bigg] + \mathcal{O}\left(\frac{1}{T^2}\right) . \tag{5.69}$$

This result depends on the two cut-off parameters ε_q and $\varepsilon_{\mathfrak{t}}$ coming from the loop integration and the contour insertion points, respectively. The leading term does, however not depend on these parameters and as mentioned above we see that the self-energy insertion is indeed of higher order in RT than the rainbow diagram (5.65).

5.4.3 Vertex Diagram

The contribution from the vertex diagram, Figure (1.1d) has three bare propagator insertions which are connected in a vertex in the bulk

$\langle W_{(d)}\rangle_\beta =$  . (5.70)

The insertion point w of the vertex in the bulk is arbitrary and thus we need to integrate it over the spacetime. As only bare thermal propagators are considered the general structure does not change from the conformal result which is given in [62]. Let us briefly discuss how to obtain the vertex expression. There are two possible three point vertices whose operators can be contracted with the Wilson loop expansion

$$\frac{\mathrm{i}^3}{3!}\oint \mathrm{d}\vec{\mathfrak{t}}\dot{x}_1^\mu \dot{x}_2^\nu \dot{x}_3^\rho \left\langle \mathrm{tr}\mathcal{P}\left[A_1^\mu A_2^\nu A_3^\rho\right]\left(-\int \mathrm{d}^4 w f^{abc}\partial^\sigma A_\lambda^a(w) A_\sigma^b(w) A_\lambda^c(w)\right)\right\rangle ,$$

$$\frac{\mathrm{i}}{2!1!}\oint \mathrm{d}\vec{\mathfrak{t}}|\dot{x}_1|\dot{x}_2^\mu|\dot{x}_3| \left\langle \mathrm{tr}\mathcal{P}\left[\phi_1 A_2^\mu \phi_3\right]\left(-\int \mathrm{d}^4 w f^{abc}\partial^\nu \phi^a(w) A_\nu^b(w)\phi^c(w)\right)\right\rangle . \tag{5.71}$$

There is always one contraction with a field depending on $\mathfrak{t}_i$ and one depending on w. The open $\dot{x}_i$ whos A_i is contracted with the $A(w)$ that contracts the partial derivative will lead to terms of the form $\dot{x}_i\partial_{x_j}$. The remaining prefactors then form $(|\dot{x}_i||\dot{x}_j| - \dot{x}_i\cdot\dot{x}_j)$. The path ordering $\mathcal{P}$ and the different possibilities of contracting then cancel the prefactors and yield the introduction of $\varepsilon(\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3) = 1$ for $\mathfrak{t}_1 > \mathfrak{t}_2 > \mathfrak{t}_3$. The function ε we defined is an anti-symmetric path ordering symbol.

$$\langle W_{(d)}\rangle_\beta = -\frac{\lambda^2}{4}\oint \mathrm{d}\mathfrak{t}_1\mathrm{d}\mathfrak{t}_2\mathrm{d}\mathfrak{t}_3\varepsilon(\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3)\left(|\dot{x}_1||\dot{x}_3| - \dot{x}_1\cdot\dot{x}_3\right)\times$$

$$\times\,\dot{x}_2\cdot\frac{\partial}{\partial x_1}\int \mathrm{d}^4 w\; \Delta(x_1-w)\Delta(x_2-w)\Delta(x_3-w) . \tag{5.72}$$

$\Delta(x)$ is the finite temperature propagator which we derived in Equation (3.111). For the vertex it is important to recall the full expression

$$\Delta^{T\neq 0}\left[x = \begin{pmatrix} x_0 \\ \vec{x} \end{pmatrix}\right] = \frac{T}{8\pi|\vec{x}|}\left[\coth\left(\pi T(|\vec{x}| + \mathrm{i}x_0)\right) + \coth\left(\pi T(|\vec{x}| - \mathrm{i}x_0)\right)\right] . \tag{5.73}$$

The vertex depends on operators $\Delta(x - w)$ where $w = (w_0, w_1, w_2, w_3)$ is the vertex coordinate over which we integrate and $x = (0, x_1, x_2, 0)$ it the insertion point along the WL contour. Therefore the full propagator reads

$$\Delta(x - w) = \frac{T}{(4\pi)^3}\left(\frac{\coth\left[\pi T\left(\sqrt{(x_1 - w_1)^2 + (x_2 - w_2)^2 + w_3^2} + iw_0\right)\right]}{\sqrt{(x_1 - w_1)^2 + (x_2 - w_2)^2 + w_3^2}} + \frac{\coth\left[\pi T\left(\sqrt{(x_1 - w_1)^2 + (x_2 - w_2)^2 + w_3^2} - iw_0\right)\right]}{\sqrt{(x_1 - w_1)^2 + (x_2 - w_2)^2 + w_3^2}}\right) . \tag{5.74}$$

Analogous to all above calculations we are not aware of a possibility to integrate the hyperbolic cotangent without expanding. Usually the expansion was around the dimensionless combination RT between temperature and radius of the circular loop. Let us make this dependence explicit. Therefore replace $x_i \to R\,x_i$ for $i \in \{1, 2, 3\}$ and the new x_i will then parameterize a unit circle. Furthermore it is useful to substitute $w \to Rw$. The above propagator thus becomes

$$\Delta(x - w) = \frac{T}{(4\pi)^3 R}\left(\frac{\coth\left[\pi RT\left(\sqrt{(x_1 - w_1)^2 + (x_2 - w_2)^2 + w_3^2} + iw_0\right)\right]}{\sqrt{(x_1 - w_1)^2 + (x_2 - w_2)^2 + w_3^2}} + \frac{\coth\left[\pi RT\left(\sqrt{(x_1 - w_1)^2 + (x_2 - w_2)^2 + w_3^2} - iw_0\right)\right]}{\sqrt{(x_1 - w_1)^2 + (x_2 - w_2)^2 + w_3^2}}\right) . \tag{5.75}$$

The prefactor proportional to T/R will come three times from the three propagators. Additionally the contour prefactors and the integral measure yield a total R^6 such that finally we will have, before integrating, $\langle W_{(d)}\rangle_\beta \propto (RT)^3 \coth^3(RT(\ldots))$ meaning that an expansion around some value of the radius will always match the expansion around that temperature as we expected.

5.4.3.1 Small radius & temperature

Changing $\mathfrak{t}_2 \leftrightarrow \mathfrak{t}_3$ with respect to Equation (5.72) for the vertex and setting $\mathfrak{t}_1 > \mathfrak{t}_2 > \mathfrak{t}_3$ we can write the vertex diagram in the following form

$$\langle W_{(d)}\rangle_\beta = \frac{\lambda^2}{4}\int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \int_0^{\mathfrak{t}_2} \mathrm{d}\mathfrak{t}_3 \left(|\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2\right) \times$$

$$\times \dot{x}_3 \cdot \frac{\partial}{\partial x_1} \int \mathrm{d}^4 w \; R^2 \Delta(x_1 - w) R^2 \Delta(x_2 - w) R^2 \Delta(x_3 - w) \,. \tag{5.76}$$

The propagators $R^2\Delta(x_i - w)$ depend on the combination RT as derived above. We thus can consider the expansion

$$\begin{aligned} R^2 \Delta^{T\neq 0}(x) &= \frac{RT}{8\pi|\vec{x}|} \left[\coth\left(\pi RT(|\vec{x}| + \mathrm{i}x_0)\right) + \coth\left(\pi T(|\vec{x}| - \mathrm{i}x_0)\right)\right] \\ &= \frac{1}{4\pi x^2} + \frac{R^2T^2}{12} + \mathcal{O}\left((RT)^4\right) \,. \end{aligned} \tag{5.77}$$

We use this expansion for all three propagators and combine the expansion keeping only the leading order in RT. Keeping also expression of the order $(RT)^4$ would yield more complex expressions. We did not keep them as wee see that even the result for order $(RT)^2$ can only be obtained by a numerical integration. The vertex diagram (5.72) thus becomes

$$\begin{aligned} \langle W_{(d)} \rangle_\beta = -\frac{\lambda^2}{4} \int_0^{2\pi} \mathrm{dt}_1 \int_0^{\mathrm{t}_1} \mathrm{dt}_2 \int_0^{\mathrm{t}_2} \mathrm{dt}_3 \left(|\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2\right) \dot{x}_3 \cdot \frac{\partial}{\partial x_1} \times \\ \times \int \mathrm{d}^4 w \left[\frac{1}{(4\pi^2)^3 (x_1 - w)^2 (x_2 - w)^2 (x_3 - w)^2} + \right. \\ \left. + \frac{T^2 R^2 \left[(x_1 - w)^2 + (x_2 - w)^2 + (x_3 - w)^2\right]}{12 (4\pi^2)^2 (x_1 - w)^2 (x_2 - w)^2 (x_3 - w)^2} + \mathcal{O}(RT)^4 \right] \,. \end{aligned} \tag{5.78}$$

The first fraction gives precisely the zero temperature term which is treated in [62]. It eventually can be cancelled against the self-energy insertion Figure 1.1c. The focus will therefore be on the next order term. Here we see three very similar terms, however, the first one will vanish because after canceling numerator and denominator it is independent of x_1. Let us therefore focus on the last term as the second term follows by simply taking $x_2 \leftrightarrow x_3$.

For this term we will first calculate the integral over w by introducing Feynman parameters in the known way. Note that, since we parameterize unit circles with the x_i we can set $x_i^2 = 1$. Then the integral we need to compute is

$$\int \mathrm{d}^d w \frac{1}{(x_1 - w)^2 (x_2 - w)^2} = \int_0^1 \mathrm{d}\alpha \int \mathrm{d}^d w \frac{1}{(w^2 + \Delta)^2} = \int_0^1 \mathrm{d}\alpha \frac{(2\pi)^{2-\varepsilon} \Gamma(\varepsilon)}{2\Delta^\varepsilon} \,, \tag{5.79}$$

with $\Delta = 2(1 - \alpha + \alpha^2) + 2\alpha(1 - \alpha) x_1 \cdot x_2$. We used dimensional regularization in $d = 4 - 2\varepsilon$ dimensions to compute the logarithmically divergent integral. This yielded the gamma function $\Gamma(\varepsilon)$ which has a simple pole at zero. For the moment we keep the gamma function as it will eventually vanish when taking the derivative with respect to x_1. Before calculating further let us include also the operator $\dot{x}_3 \cdot \partial/\partial x_1$ yielding

$$\begin{aligned}
\dot{x}_3 \cdot \frac{\partial}{\partial x_1} \int_0^1 \mathrm{d}\alpha \frac{(2\pi)^{2-\varepsilon}\Gamma(\varepsilon)}{2\Delta^{\varepsilon}} &= \int_0^1 \mathrm{d}\alpha \frac{(2\pi)^{2-\varepsilon}\Gamma(\varepsilon)(-2\varepsilon)\dot{x}_3 \cdot x_2 \alpha(1-\alpha)}{2\Delta^{1+\varepsilon}} \\
&= -\int_0^1 \mathrm{d}\alpha \frac{(2\pi)^2 \dot{x}_3 \cdot x_2 \alpha(1-\alpha)}{2(1-\alpha+\alpha^2) + 2\alpha(1-\alpha) x_1 \cdot x_2} + \mathcal{O}(\varepsilon) \\
&= \frac{(2\pi)^2}{2} \dot{x}_3 \cdot x_2 \left(\frac{1}{1 - x_1 \cdot x_2} + \frac{4 \arctan\left(\sqrt{\frac{1-x_1\cdot x_2}{3-x_1\cdot x_2}}\right)}{\sqrt{(3+x_1\cdot x_2)(1-x_1\cdot x_2)^3}} \right) .
\end{aligned} \tag{5.80}$$

After acting with the operator $\dot{x}_3 \cdot \partial/\partial x_1$ we found a factor ε in the numerator which canceled the divergence $\varepsilon\Gamma(\varepsilon) = 1 + \mathcal{O}(\varepsilon)$. Recall that there is also a similar term like this which needs to be added with the replacement $x_2 \leftrightarrow x_3$, however $\dot{x}_3$ must remain unchanged as we included it only later as an overall operator.
Plugging in our parameterization $x_i = (0, \cos(\mathfrak{t}_i), \sin(\mathfrak{t}_i), 0)$ we see that $x_i \cdot x_j = \cos(\mathfrak{t}_i - \mathfrak{t}_j)$ and $\dot{x}_i \cdot x_j = \sin(\mathfrak{t}_i - \mathfrak{t}_j)$ yielding that $\dot{x}_3 \cdot x_3 = \sin(0) = 0$ and thus the contribution from replacing $x_2 \leftrightarrow x_3$ in the above equation vanishes. The first integral over $\mathfrak{t}_3$ is readily evaluated

$$\begin{aligned}
\langle W_{(d)} \rangle_\beta &= \frac{\lambda^2}{4} (RT)^2 \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \int_0^{\mathfrak{t}_2} \mathrm{d}\mathfrak{t}_3 \left(|\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2 \right) \times \\
&\quad \times \frac{(2\pi)^2}{2} \dot{x}_3 \cdot x_2 \left(\frac{1}{1 - x_1 \cdot x_2} + \frac{4 \arctan\left(\sqrt{\frac{1-x_1\cdot x_2}{3-x_1\cdot x_2}}\right)}{\sqrt{(3+x_1\cdot x_2)(1-x_1\cdot x_3)^2}} \right) \\
&= -\frac{\lambda^2\pi^2}{2} (RT)^2 \int_0^{2\pi} \mathrm{d}\mathfrak{t}_1 \int_0^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 \; (1 - \cos(\mathfrak{t}_2)) \times \\
&\quad \times \left(1 + \frac{4 \arctan\left(\sqrt{\frac{1-\cos(\mathfrak{t}_{12})}{3-\cos(\mathfrak{t}_{12})}}\right)}{\sqrt{(3+\cos(\mathfrak{t}_{12}))(1-\cos(\mathfrak{t}_{12}))}} \right) .
\end{aligned} \tag{5.81}$$

The second part of this integration cannot be done analytically. However we see that it does not depend on any variable and thus we can get a numeric result.

$$\langle W_{(d)} \rangle_\beta \approx -\frac{\lambda^2\pi^2}{2} (RT)^2 \left(2\pi + 24.94 \right) \approx -220.51 \lambda^2 R^2 T^2 . \tag{5.82}$$

Comparing this result with the ones from the rainbow diagrams (5.64) and the self-energy insertion (5.68) we see that as the rainbow diagram it is at leading order in the small RT expansion while the self-energy insertion is subleading.

5.4.3.2 High temperature

Instead of small radius we can also consider the limit $RT \to \infty$. As $\coth(RT \to \infty) = 1$ to all orders in Taylor expansion, see Equation (3.118), we have to consider a seemingly simple expression in the high temperature limit

$$\langle W_{(d)} \rangle_{R\to\infty} = \frac{\lambda^2}{4}(RT)^3 \int_0^{2\pi} \mathrm{dt}_1 \int_0^{\mathrm{t}_1} \mathrm{dt}_2 \int_0^{\mathrm{t}_2} \mathrm{dt}_3 \left(|\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2\right) \dot{x}_3 \cdot \frac{\partial}{\partial x_1} \times$$
$$\times \int \mathrm{d}^4 w \frac{1}{(8\pi)^3 (x_1 - w)(x_2 - w)(x_3 - w)} + \mathcal{O}\left((RT)^n\right) \,, \forall n \in \mathbb{N} \,. \qquad (5.83)$$

Note that we already substituted $x \to Rx$ and $w \to Rx$ as discussed above to determine the dependence on RT. To evaluate the integral over w (the second line in the above expression) we again introduce Feynman parameters and shift the w integration to find

$$\begin{aligned}\int \mathrm{d}^4 w \prod_{i=1}^{3} \frac{1}{(x_i - w)} &= \int_0^1 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma \int \mathrm{d}^4 w \frac{\delta(\alpha+\beta+\gamma-1)}{\left[\alpha(x_1 - w) + \beta(x_2 - w) + \gamma(x_3 - w)\right]^{3/2}} \\ &= \int_0^1 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma \int \mathrm{d}^4 w \frac{\delta(\alpha+\beta+\gamma-1)}{\left[w^2 + \Delta\right]^{3/2}} \qquad (5.84) \\ &= -4\pi \int_0^1 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma \delta(\alpha+\beta+\gamma-1)\sqrt{\Delta} \,.\end{aligned}$$

where Δ depends on the Feynman parameters and insertion points. Plugging this into the above equation for the vertex insertion we find

$$\langle W_{(d)} \rangle_{RT\to\infty} = -\pi \frac{\lambda^2}{2}(RT) \int_0^{2\pi} \mathrm{dt}_1 \int_0^{\mathrm{t}_1} \mathrm{dt}_2 \int_0^{\mathrm{t}_2} \mathrm{dt}_3 \left(|\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2\right) \times$$
$$\times \int_0^1 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma \delta(\alpha+\beta+\gamma-1) \frac{\dot{x}_3 \cdot \Delta'}{\sqrt{\Delta}} + \mathcal{O}\left((RT)^n\right) \,, \forall n \in \mathbb{N} \,, \qquad (5.85)$$

$$\Delta = \alpha(1-\alpha)x_1^2 + \beta(1-\beta)x_2^2 + \gamma(1-\gamma)x_3^2 - \alpha\beta x_1 x_2 - \alpha\gamma x_1 x_3 - \beta\gamma x_2 x_3 \,,$$

$$\Delta' = \frac{\partial \Delta}{\partial x_1} = 2\alpha(1-\alpha)x_1 - \alpha\beta x_2 - \alpha\gamma x_3 \,.$$

We can plug in the parametrization of the circle as given in (**??**), most importantly $x_i \cdot x_j = \cos(\mathrm{t}_i - \mathrm{t}_j)$ and $\dot{x}_i \cdot x_j = \sin(\mathrm{t}_i - \mathrm{t}_j)$, to find

$$\langle W_{(d)} \rangle_{R\to\infty} = \pi \frac{\lambda^2}{2} R^3 T^3 \int_0^{2\pi} \mathrm{dt}_1 \int_0^{\mathrm{t}_1} \mathrm{dt}_2 \int_0^{\mathrm{t}_2} \mathrm{dt}_3 \left(1 - \cos(\mathrm{t}_1 - \mathrm{t}_2)\right) \int_0^1 \mathrm{d}\alpha \mathrm{d}\beta \mathrm{d}\gamma \times \qquad (5.86)$$
$$\times \frac{\delta(\alpha+\beta+\gamma-1)\left[2\alpha(1-\alpha)\sin(\mathrm{t}_1 - \mathrm{t}_3) - \alpha\beta\sin(\mathrm{t}_2 - \mathrm{t}_3)\right]}{\sqrt{\alpha(1-\alpha) + \beta(1-\beta) + \gamma(1-\gamma) - \alpha\beta\cos(\mathrm{t}_1 - \mathrm{t}_2) - \alpha\gamma\cos(\mathrm{t}_1 - \mathrm{t}_3) - \beta\gamma\cos(\mathrm{t}_2 - \mathrm{t}_3)}}$$
$$+ \mathcal{O}\left((RT)^n\right) \,, \forall n \in \mathbb{N} \,.$$

After the integral over the first Feynman parameters we again find an expression which we need to integrate numerically. Finally we then find

$$\langle W_{(d)}\rangle_{R\to\infty} \approx -3.13187\pi\frac{\lambda^2}{2}R^3T^3 + \mathcal{O}\left((RT)^n\right) \approx -4.91952\,\lambda^2\,R^3T^3\,. \tag{5.87}$$

This result can again be compared with the calculations of the rainbow diagram (5.65) and the self-energy insertion (5.69). The vertex can be seen to contribute at the same order in RT while the rainbow diagram is in fact subleading.

5.4.4 Full Thermal Circular Wilson Loop

We are now ready to put all the above results together. Actual results can only be obtained when taking the limit $RT \to 0$ or $RT \to \infty$, in some cases also a numerical calculation was necessary. The results for the single propagator insertion were obtained in Equation (3.120). The second order results for small thermal corrections are derived in Equation (5.64) for the rainbows, Equation (5.68) for the self-energy and Equation (5.82) for the vertex. The respective results in the high temperature limit can be found in Equations 5.65, 5.69 and 5.87. All of these results can be combined to the final expression for the thermal circular Wilson loop

$$\begin{aligned}
\langle W\rangle_\beta &= 1 + \underbrace{\langle W_{(a)}\rangle_\beta}_{\mathcal{O}(\lambda)} + \underbrace{\left[\langle W_{(b)}\rangle_\beta + \langle W_{(c)}\rangle_\beta + \langle W_{(d)}\rangle_\beta\right]}_{\mathcal{O}(\lambda^2)} + \mathcal{O}(\lambda^{5/2})\,,\\
\langle W_{(a)}\rangle_{RT\to 0} &= \frac{\lambda}{8} + \lambda\frac{(RT)^2\pi^2}{12} - \lambda\frac{(RT)^4\pi^4}{60} + \mathcal{O}(\lambda(RT)^6)\,,\\
\langle W_{(b)}\rangle_{RT\to 0} &= \frac{\lambda^2}{192} + \lambda^2(RT)^2\left(\frac{\pi^2}{144} - \frac{1}{96}\right) + \mathcal{O}\left(\lambda^2(RT)^4\right)\,,\\
\langle W_{(c)}\rangle_{RT\to 0} &= -\frac{19\lambda^2(RT)^3}{9} + \mathcal{O}\left(\lambda^2(RT)^5\right)\,,\\
\langle W_{(d)}\rangle_{RT\to 0} &\approx -220.51\lambda^2R^2T^2 + \mathcal{O}\left(\lambda^2(RT)^4\right)\,,
\end{aligned} \tag{5.88}$$

$$\begin{aligned}
\langle W_{(a)}\rangle_{RT\to\infty} &= \frac{\lambda}{2}\,RT + \mathcal{O}\left(\frac{\lambda}{(RT)^n}\right)\,,\ \forall n\in\mathbb{N}\,,\\
\langle W_{(b)}\rangle_{RT\to\infty} &= \frac{\lambda^2}{2^4}\,(RT)^2 + \mathcal{O}\left(\frac{\lambda^2}{(RT)^n}\right)\,,\ \forall n\in\mathbb{N}\,,\\
\langle W_{(c)}\rangle_{RT\to\infty} &= -2\lambda^2(RT)^3 + \frac{1}{2}\lambda^2RT\left[\frac{\log(\varepsilon_q)\left(\log\left(\frac{4}{\varepsilon_t}\right)-2\right)}{4\pi^2}\right.
\end{aligned}$$

$$+\frac{10\log\left(\frac{4}{\varepsilon_t}\right)+4}{12\pi^2\varepsilon_q}+2c_1\log\left(\frac{4}{\varepsilon_t}\right)-c_2\left(\log\left(\frac{\varepsilon_t}{4}\right)+2\right)\Bigg]+\mathcal{O}\left(\frac{\lambda^2}{RT}\right)\,,$$

$$\langle W_{(d)}\rangle_{RT\to\infty}\approx -4.91952\,\lambda^2\,R^3T^3+\mathcal{O}\left(\frac{\lambda^2}{(RT)^n}\right)\,,\ \forall n\in\mathbb{N}\,.$$

5.4.5 Effective coupling

Recall from Equation (2.59) that the 1/4-BPS circular Wilson loop can obtained from the one preserving half of the symmetries by a redefinition of the coupling [92]. We want to look at similar relations for the thermal Wilson loop. The coupling λ_{eff} is expected to depend on the parameters RT in an expansion.

$$\left\langle W_{\bigcirc}^{(T=0)}\right\rangle(\lambda\to\lambda_{\text{eff}})=\left\langle W_{\bigcirc}^{(T)}\right\rangle\Big|_{RT\to 0}\,. \tag{5.89}$$

Naturally this replacement can only be obtained for small thermal corrections. In the high temperature limit we are disconnected from the original zero temperature result for the circular loop [62]. A zero temperature limit from which the conformal result can be obtained does not make sense in a high temperature limit.
Therefore recall the results up to order λ^2 for the circular Wilson loop at temperature zero and in the small thermal expansion (5.88)

$$\begin{aligned}\left\langle W_{\bigcirc}^{(T=0)}\right\rangle =&1+\frac{\lambda}{8}+\frac{\lambda^2}{192}+\mathcal{O}\left(\lambda^3\right)\,,\\ \left\langle W_{\bigcirc}^{(T)}\right\rangle\Big|_{RT\to 0} =&1+\frac{\lambda}{8}+\lambda\frac{(RT)^2\pi^2}{12}-\lambda\frac{(RT)^4\pi^4}{60}+\frac{\lambda^2}{96}+\frac{\lambda^2(RT)^2}{192}-\frac{19\lambda^2(RT)^3}{9}\\ &+\frac{\pi^2\lambda^2(RT)^2}{144}-\frac{\lambda^2(RT)^2}{96}-\lambda^2(RT)^2v+\mathcal{O}\left(\lambda(RT)^6,\lambda^2(RT)^4,\lambda^{5/2}\right)\,.\end{aligned}$$

Ignoring higher powers in RT we obtain

$$\begin{aligned}\lambda_{\text{eff}}^{(RT\to 0)}&=\lambda\left(1+\frac{2\pi^2(RT)^2}{3}-\frac{2\pi^4(RT)^4}{15}\right)\\ &\quad-\lambda^2(RT)^2\left(\frac{1}{12}+8v\right)+\mathcal{O}\left(\lambda(RT)^6,\lambda^2(RT)^4,\lambda^{5/2}\right)\,,\\ v&\approx -220.51\,.\end{aligned} \tag{5.90}$$

5.5 The straight line Wilson loop

Let us again recall the definition of the Maldacena Wilson Loop of $\mathcal{N} = 4$ SYM theory as given in [24, 54]

$$W(C) = \frac{1}{N}\mathrm{tr}\overleftarrow{\mathcal{P}}\exp\left[\oint_C \mathrm{d}\mathfrak{t}\,(\mathrm{i}A_\mu \dot{x}^\mu + \Phi_r|\dot{x}|n^r)\right] , \tag{5.91}$$

In this chapter we consider a straight line insertion. By assuming that the points at $\pm\infty$ along the straight line can be identified this setup is a closed WL and hence a gauge invariant observable. The contour of the straight line in the x_3 spatial direction can be generally written as

$$C : x^\mu(\mathfrak{t}) = (0,0,0,f(\mathfrak{t})) \quad , \quad \mathfrak{t}\in\mathcal{I} , \tag{5.92}$$

with some interval $\mathcal{I}\subset\mathbb{R}$ as proposed in [63]. To form a proper straight line $f(\mathfrak{t})$ needs to be differentiable and monotonically increasing $f'(\mathfrak{t}) \geq 0\,\forall\mathfrak{t}\in\mathcal{I}$. Furthermore, if we set the interval to be $\mathcal{I} = (a,b)$ for some $a < b$, then $f(\mathfrak{t}\searrow a) = -\infty$ and $f(\mathfrak{t}\nearrow b) = \infty$[7]. Figure 5.4 shows the leading order corrections to the straight line. At leading order ($\mathcal{O}(\lambda)$) there is only the insertion of a bare propagator (Figure 5.4a) while at order λ^2 there are a priori three diagrams contributing. These are essentially the same diagrams we found for the straight line, see Figure 1.1. We will go through all these diagrams one by one and will see that most of them will actually have a vanishing contribution. The thermal corrections to the straight line WL are only obtained from self-energy insertions as the supersymmetry is manifestly broken in the expansions we consider. Note, however, that the thermal mass $m_\infty^2 = \lambda T^2$ is in fact equal for all fields [123, 131].

The straight line is parameterized by $x^\mu(\mathfrak{t}) = (0,0,0,f(\mathfrak{t}))$ which for all monotonically increasing[8] differentiable functions f yields that the combination between scalar and vector propagator in fact gives zero readily,

$$|\dot{x}(\mathfrak{t}_i)||\dot{x}(\mathfrak{t}_j)| - \dot{x}(\mathfrak{t}_i)\cdot\dot{x}(\mathfrak{t}_j) = |f'(\mathfrak{t}_i)||f'(\mathfrak{t}_j)| - f'(\mathfrak{t}_i)f'(\mathfrak{t}_j) = 0 . \tag{5.93}$$

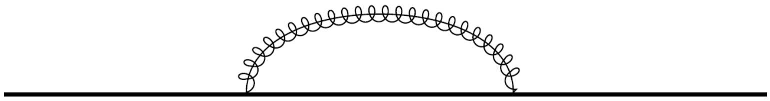
The expression $(|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2)$ appears in most of the Wilson loop diagrams. While the bare propagator gets thermal correction it does not change this structure.
Consider for instance the leading order insertion of a single propagator

$$\langle W_a\rangle = \frac{\lambda}{2}\int_a^b \mathrm{d}\mathfrak{t}_1 \int_a^{\mathfrak{t}_1} \mathrm{d}\mathfrak{t}_2 (|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2)\Delta^{T\neq 0}(x_{12}) = 0 , \tag{5.94a}$$

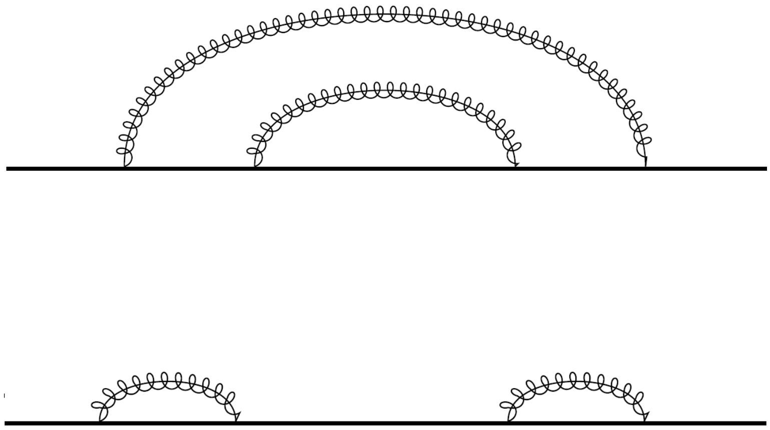
[7] The function f can also be monotonically decreasing with $f'(\mathfrak{t}) \leq 0\,\forall\mathfrak{t}\in\mathcal{I}$, $f(\mathfrak{t}\searrow a) = \infty$ and $f(\mathfrak{t}\nearrow b) = -\infty$. In this case we would, however, prefer to work with the function $\tilde{f}(\mathfrak{t}) = f(a+b-\mathfrak{t})$ defined on the same interval which now is monotonically increasing.

[8] Or monotonically decreasing, see footnote 7.

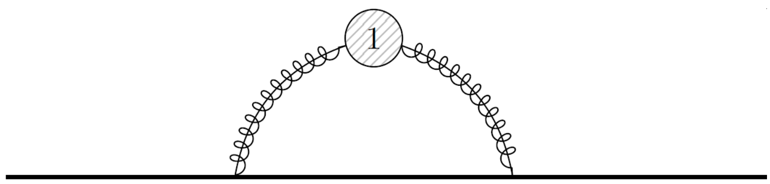
Figure 5.4: The diagrams show insertions into the straight Wilson loop at low order. The thicker line is the circular loop of the heavy particle. The curly and straight line respectively represent a gauge and a scalar Boson inserted. We most sum over all possible combinations allowed by the present vertices.



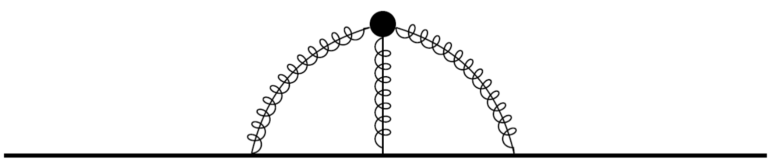
(a) Single propagator insertion, $\langle W_{(a)} \rangle$. This is the only diagram of order λ.



(b) Ladder diagrams: Insertion of two single propagators,$\langle W_{(b)} \rangle$. These diagrams contribute at order λ^2.



(c) Self-energy insertion, $\langle W_{(c)} \rangle$. The hatched circle with the 1 stands for any possible one-loop insertion. This diagram contributes at order λ^2.



(d) Vertex insertion, $\langle W \rangle$. This diagram contributes at order λ^2.

where the part responsible for the vanishing was marked in red. Similarly for the ladder diagram at order λ^2 and the vertex diagram we find the above prefactor

$$\langle W_b \rangle = \frac{\lambda^2}{4} \int_a^{bi} \mathrm{d}t_1 \int_a^{t_1} \mathrm{d}t_2 \int_a^{t_2} \mathrm{d}t_3 \int_a^{t_3} \mathrm{d}t_4 \times \tag{5.94b}$$

$$\times \Bigg[\left[\left(|\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2 \right) \left(|\dot{x}_3||\dot{x}_4| - \dot{x}_3 \cdot \dot{x}_4 \right) \right] \Delta^{(T\neq 0)}(x_{12}) \Delta^{(T\neq 0)}(x_{34}) +$$

$$+ \left[\left(|\dot{x}_1||\dot{x}_4| - \dot{x}_1 \cdot \dot{x}_4 \right) \left(|\dot{x}_2||\dot{x}_3| - \dot{x}_2 \cdot \dot{x}_3 \right) \right] \Delta^{(T\neq 0)}(x_{14}) \Delta^{(T\neq 0)}(x_{23}) \Bigg]$$

$$= 0 \,, \tag{5.94c}$$

$$\langle W_d \rangle = \frac{\lambda^2}{4} \int_a^{b} \mathrm{d}t_1 \int_{-a}^{t_1} \mathrm{d}t_2 \int_{-a}^{t_2} \mathrm{d}t_3 \left(|\dot{x}_1||\dot{x}_2| - \dot{x}_1 \cdot \dot{x}_2 \right) \times$$

$$\times \dot{x}_3 \cdot \frac{\partial}{\partial x_1} \int \mathrm{d}^4 w \, \Delta(x_1 - w) \Delta(x_2 - w) \Delta(x_3 - w) = 0 \,, \tag{5.94d}$$

respectively. Only the self-energy diagram will have a non-vanishing contribution coming from the difference between the scalar self-energy and the vacuum polarization of the gauge fields,

$$\langle W_c \rangle = \frac{\lambda}{2} \int_a^{b} \mathrm{d}t_1 \int_a^{t_1} \mathrm{d}t_2 \left[|\dot{x}_1||\dot{x}_2| \mathcal{F}\left[\frac{\Sigma(P)}{P^4} \right](x_{12}) - \dot{x}_1^{\mu} \dot{x}_2^{\nu} \mathcal{F}\left[\frac{\Pi^{\mu\nu}(P)}{P^4} \right](x_{12}) \right]$$

$$= \frac{\lambda}{2} \int_a^{b} \mathrm{d}t_1 \int_a^{t_1} \mathrm{d}t_2 \mathcal{F}\left[\frac{\Sigma(P) - \Pi_T(P)}{P^4} \right](x_{12}) \,. \tag{5.94e}$$

The calligraphic $\mathcal{F}$ denotes the Fourier transform, $\Sigma(P)$ is the scalar self-energy and $\Pi^{\mu\nu}(P)$ the vacuum polarization of the gauge field. They were calculated and Fourier transformed in the above chapter 5.3. The contribution above only came from self energy diagrams. Therefore this is the only diagram we need to consider in our calculations. Note that we already substituted in the transverse polarization via theorem 1.

We parameterize the Wilson loop by a linear function. This is the easiest choice for the function $f(t)$ satisfying all of the above requirements,

$$x^{\mu}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ t \end{pmatrix} \,, \qquad t \in \left[-\frac{L}{2}, \frac{L}{2} \right] \quad \text{for} \quad L \to \infty \,. \tag{5.95}$$

We know the Fourier transform of the self-energy diagrams from Equation (5.58) in a limit of large and small temperature where the thermal average (5.51) was used,

$$\Delta_{\Sigma}^{(T\to\infty)}(x) - \Delta_{\Pi_T}^{(T\to\infty)}(x) = -\frac{\lambda T^3}{4\pi}|x| - \frac{\lambda T}{4\pi|x|}\left(\frac{25}{6\pi^2\varepsilon_q} - \frac{3\log(\varepsilon_q)}{4\pi^2} + 2c_1 - c_2\right) + \mathcal{O}\left(\frac{1}{T}\right) ,$$

$$\Delta_{\Sigma}^{(T\to 0)}(x) - \Delta_{\Pi_T}^{(T\to 0)}(x) = -\frac{\lambda T^3}{24\pi}|x| - \frac{\pi\lambda T^5}{540}|x|^3 + \mathcal{O}\left(T^7\right) .$$

The zero temperature terms are known to cancel each other [63]. The integral we then need to evaluate has the following form

$$\langle W_c\rangle = \frac{\lambda}{2}L^2\int_{-1/2}^{1/2}\mathrm{d}\mathfrak{t}_1\int_{-1/2}^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2\,\left(\Delta_\Sigma(L\mathfrak{t}_1 - L\mathfrak{t}_2) - \Delta_{\Pi_T}(L\mathfrak{t}_1 - L\mathfrak{t}_2)\right) . \tag{5.96}$$

where we substituted $\mathfrak{t}_i \to L\mathfrak{t}_i$ which changes the above propagators to

$$L^2\left((\Delta_\Sigma - \Delta_{\Pi_T})^{(T\to\infty)}(Lx)\right) = -\frac{\lambda(TL)^3}{4\pi}|x| - \frac{\lambda(TL)}{4\pi|x|}\left(\frac{25}{6\pi^2\varepsilon_q} - \frac{3\log(\varepsilon_q)}{4\pi^2} + 2c_1 - c_2\right) + \mathcal{O}\left(\frac{1}{T}\right) , \tag{5.97}$$

$$L^2\left((\Delta_\Sigma - \Delta_{\Pi_T})^{(T\to 0)}(Lx)\right) = -\frac{\lambda(TL)^3}{24\pi}|x| - \frac{\pi\lambda(TL)^5}{540}|x|^3 + \mathcal{O}\left(T^7\right) . \tag{5.98}$$

Therefore we clearly see that the Wilson loop will only depend on the dimensionless combination TL. Henceforth the limit $T\to 0$ is of no relevance for this calculation. As clearly $L\to\infty$ we would have a limit $\lim_{T\to 0}\left(f(TL)|_{L\to\infty}\right)$ which cannot be subject to a rigorous expansion. Therefore we only consider the limit $LT\to\infty$. Here

$$\begin{aligned}\langle W_{—}\rangle_{LT\to\infty} &= 1 + \frac{\lambda}{2}L^2\int_{-1/2}^{1/2}\mathrm{d}\mathfrak{t}_1\int_{-1/2}^{\mathfrak{t}_1}\mathrm{d}\mathfrak{t}_2\left((\Delta_\Sigma - \Delta_{\Pi_T})_{\Pi_T}^{(T\to\infty)}(L\mathfrak{t}_1 - L\mathfrak{t}_2)\right)\\ &= 1 - \frac{\lambda^2(LT)^3}{24\pi} - \frac{\lambda^2(LT)}{2\pi}\left(\log\left(\varepsilon_L\right) + 1\right)\times\\ &\quad\times\left(\frac{25}{6\pi^2\varepsilon_q} - \frac{3\log(\varepsilon_q)}{4\pi^2} + 2c_1 - c_2\right) + \mathcal{O}\left(\frac{\lambda^2}{LT}, \lambda^{5/2}, \lambda^2\log(\lambda)\right) .\end{aligned} \tag{5.99}$$

Again we introduced a dimensionless cut-off ε_L to regularize the divergence that occurs when the insertion points are equal. In analogy to the calculation of the straight line (5.88) the leading order is cubic in the temperature and indipendent of any cut-off parameters we had to introduce throughout the calculations.

Chapter 6

Conclusion

Supersymmetric and superconformal Yang-Mills theories have been studied with great interest throughout the last decades. The high amount of symmetries allowed for the introduction of various tools such as integrability, localization, holography and bootstrap which aimed at tackling problems from different sides. Using these it was possible to derive a number of exact results. Defects in superconformal theories partly break the symmetry group, however, in recent considerations some further exact relation could be obtained in these theories. In particular line defects in the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory have been studied with great interest. The introduction of a finite temperature is similar to a boundary defect and it was discovered that techniques of the conformal bootstrap are applicable to thermal theories under certain conditions.

In this thesis the implications of a line defect placed in thermal $\mathcal{N} = 4$ supersymmetric Yang-Mills theory have been studied by considering the bremsstrahlung function, the energy loss of a heavy particle (WL). This bremsstrahlung B can be related to the coefficient of the stress tensor's one-point correlation function, h. While the Relation (1.1),

$$B = 3h \,, \tag{6.1}$$

is well known at zero temperature we have shown in this thesis that said exact relation continues to hold in the thermal setting at zero coupling.

The proof of this relation was done in two different manners. Algebraically the relation $B = 3h$ can be proven by a supersymmetric Ward identity, (2.16). However, thermal effects introduce new additional terms to the identity which potentially yield corrections to the exact relation. We introduced a phase $\alpha \propto 2n + 1$ which allowed us to express the action in terms of fields that have periodic boundary conditions on the thermal circle only. This anomaly-free field redefinition yields new mass-terms in the Lagrangian which break supersymmetry. Henceforth the above Ward identity needs to be corrected and becomes a thermal Broken Ward Identity (3.28). We expressed the resulting correlator using an operator product expansion borrowing tools of the thermal and defect bootstrap. In this

OPE limit the correlation function can be constraint and compared to the original Ward identity. Due to an integral of the correlator some spinor components could be shown to cancel and therefore we were able to show the conservation of the relation between bremsstrahlung and stress tensor at zero coupling.
In a second step we provided a perturbative check of the result. The bremsstrahlung function can be obtained from the expectation value of a circular Wilson loop. The leading order in a weak coupling expansion has controllable thermal corrections to the known formulas in the zero temperature theory. In this perturbative calculation we allowed only for the fields of $\mathcal{N} = 4$ to couple to the Wilson loop but not for interactions coming from the action. Therefore this calculation is comparable with the algebraic approach at zero coupling. Likewise the one-point function of the stress tensor can be computed using Feynman diagrams at leading order in the thermal theory. For both calculations an expansion around high and low values of the temperature proved convenient. Consequentially we were able to compare the two leading order thermal results in two different thermal expansions. We found that a regularization scheme can be found in both limits which reproduces $B = 3h$. The scheme determined an a priori arbitrary parameter that needed to be introduced for the straight line WL in a dimensional analysis. This length parameter has an expression in terms of the radius of the circular WL and the temperature. As we found a matching scheme in a small temperature expansion, Equation (3.152), as well as for the high temperature limit, Equation (3.174), it is reasonable to assume that a similar scheme can be obtained for any value of the temperature.

What is more, we studied the implications of interactions to the algebraic argument. Especially the Yukawa-type coupling between spinors and scalars introduced new constraints on the way the thermal theory is introduced. We studied two different approaches.
Firstly, we introduced phases as in the zero coupling case. However, the Yukawa interactions yielded constraints. The scalar fields need to obtain phase factors as well. Also each of the complex scalars and each of the spinors get an individual phase prefactor, Equation (4.15). The four fermionic phases are related by Equation (4.14) showing that their sum is zero. They cannot all be equal. Therefore we found that R-symmetry is broken. We discussed the implications of this breaking. While it seems possible that R-symmetry is not entirely broken this depends on the choice of some coefficients. While not being able to further restrict this ambiguity we discussed that with a dynamical calculation of the self-energy more constraints might be found. Our current understanding of this calculation was presented in Appendix H where we showed that this ansatz suffers from an inconsistency with known results. From this we inferred that this approach is somewhat flawed and we discuss the most likely sources of the appearing inconsistencies. In consequence, a new ansatz is developed in Appendix I suggesting a more consistent approach. We argue that in this case the result from the BWI can again be obtained.
Secondly, we used a KK dimensional reduction to three dimensions. We considered the

whole tower of emerging massive modes and showed that R-symmetry can in fact be preserved.

Overall our considerations indicate that it is indeed possible to rewrite the action in a convenient way, see (4.1). We would have a sum of a part of the action which preserves supersymmetry and a part with corrections. Everything depends only on periodic fields. The first part preserves supersymmetry and obeys a zero temperature Ward identity similar to (2.16). When the "new" term, the parts breaking supersymmetry, are included we thus find a thermal BWI. The thermal corrections are introduced through a term which introduces an interaction with newly defined operators. Those are yielded by the supersymmetry transformation of the SUSY-breaking part of the action. Once these operator is known we can use an OPE and hopefully constrain the correlator as in the zero coupling case. Should this be possible, we might be able to show that the Relation (6.1) continues to hold in the interacting theory.
Looking forward, the aim must be to better understand the different approaches we employed to understand the interacting thermal theory. Our hope is that they can be condensed straight forwardly into a single consistent ansatz which yields a thermal BWI in the form expected and we are then able to generally proof the relation $B = 3h$ holds at finite temperature.

Last, but not least, we considered a brute force perturbative calculation in the non-zero coupling case. We considered the circular and straight line Wilson loop at subleading order in a weak coupling expansion. We had to restrict ourselves partly to numeric calculations. Moreover, it proved convenient to consider a thermal average. This underlines our observation of the arising difficulties in the non-zero coupling case.

All in all, we can conclude that we were able to proof the relation $B = 3h$ between the bremsstrahlung and the stress-energy tensor to hold for $\mathcal{N} = 4$ SYM theory at finite temperature and zero coupling. This is an interesting result as it shows how exact results from the theory at zero coupling might be applicable to thermal theories as well. We hope that further research will proof that this relation can be extended to the interacting theory as well. Considering the interacting case we further reviewed our current understanding which indicates a high probability for the relation to continue to hold. The most promising approach is to write down a general consistent Lagrangian including all periodic fields from which the thermal BWI are again obtained. Further investigations in this direction are very captivating due to the possibility of finding a first exact result for a finite temperature theory.

Appendix

Appendix A

Group theory & algebras definitions

In this appendix we review the superalgebra of the full $\mathcal{N} = 4$ SYM theory without any defect insertion as well as definitions for generators and matrices used in group theory considerations. For Minkowski space we use a metric in the mostly plus convention $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ unless stated otherwise.

Pauli matrices

Let us start with the Pauli matrices. In Minkowski space they are defined in the usual manner

$$\sigma^0_M = \bar{\sigma}^0_M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma^1_M = -\bar{\sigma}^1_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{A.1a}$$

$$\sigma^2_M = -\bar{\sigma}^2_M = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad , \quad \sigma^3_M = -\bar{\sigma}^3_M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.1b}$$

This means essentially $\sigma^\mu_M = (\mathbb{1}, \sigma^i)$, $\bar{\sigma}^\mu_M = (\mathbb{1}, -\sigma^i)$. For the calculations at nonzero temperature we first Wick rotate and then compactify the time dimension on a circle. Consequentially we need to use Euclidean Pauli matrices. This affects only the 0-th component such that we find

$$\sigma^\mu_E = (\mathrm{i}\mathbb{1}, \sigma^i) \quad \text{and} \quad \bar{\sigma}^\mu_E = (\mathrm{i}\mathbb{1}, -\sigma^i) . \tag{A.2}$$

The relations we discuss in this chapter hold for both, the Minkowski and the Euclidean Pauli matrices alike such that we leave out the subscript in the considerations to follow. Only when the importance is actually significant we will specify which matrices we mean. Then the sigma matrices with two indices are a combination of these, namely

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{4} \left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu \right) . \tag{A.3}$$

From this equation it is clear that these double-index sigmas are anti-symmetric, $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$. The relations with bars then follow analogously.
Following [6,13,132] with $\epsilon^{12} = +1$ we have the relations

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{A.4a}$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{\mu\,\dot{\beta}\beta} , \qquad \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}} . \tag{A.4b}$$

between the barred and unbarred sigma matrices. We introduced $\mathfrak{su}(2)_\alpha$ and $\mathfrak{su}(2)_{\dot{\alpha}}$ spinor notation with $\alpha, \beta = 1, 2$ and $\dot{\alpha}, \dot{\beta} = 1, 2$, respectively.
For arbitrary spinors χ^α and ζ^α this yields [13,132]

$$\chi\zeta = \zeta\chi , \quad \chi\sigma^\mu\bar{\zeta} = -\bar{\zeta}\bar{\sigma}^\mu\chi . \tag{A.4c}$$

Furthermore for products of the Pauli matrices we find

$$\left(\sigma^\mu_M\bar{\sigma}^\nu_M + \sigma^\nu_M\bar{\sigma}^\mu_M\right)_\alpha^{\ \beta} = 2\eta^{\mu\nu}\delta_\alpha^{\ \beta} , \tag{A.4d}$$

$$\left(\sigma^\mu_E\bar{\sigma}^\nu_E + \sigma^\nu_E\bar{\sigma}^\mu_E\right)_\alpha^{\ \beta} = 2\delta^{\mu\nu}\delta_\alpha^{\ \beta} , \tag{A.4e}$$

$$\left(\bar{\sigma}^\mu_M\sigma^\nu_M + \bar{\sigma}^\nu_M\sigma^\mu_M\right)^{\dot{\alpha}}_{\ \dot{\beta}} = 2\eta^{\mu\nu}\delta^{\dot{\alpha}}_{\ \dot{\beta}} , \tag{A.4f}$$

$$\left(\bar{\sigma}^\mu_E\sigma^\nu_E + \bar{\sigma}^\nu_E\sigma^\mu_E\right)^{\dot{\alpha}}_{\ \dot{\beta}} = 2\delta^{\mu\nu}\delta^{\dot{\alpha}}_{\ \dot{\beta}} , \tag{A.4g}$$

$$\bar{\sigma}^\mu_M\sigma^\nu_M\bar{\sigma}^\rho_M = -\eta^{\mu\nu}\bar{\sigma}^\rho_M - \eta^{\nu\rho}\bar{\sigma}^\mu_M + \eta^{\mu\rho}\bar{\sigma}^\nu_M + i\epsilon^{\mu\nu\rho\kappa}\bar{\sigma}_{M\kappa} , \tag{A.4h}$$

$$\bar{\sigma}^\mu_E\sigma^\nu_E\bar{\sigma}^\rho_E = -\delta^{\mu\nu}\bar{\sigma}^\rho_E - \delta^{\nu\rho}\bar{\sigma}^\mu_E + \delta^{\mu\rho}\bar{\sigma}^\nu_E + i\epsilon^{\mu\nu\rho\kappa}\bar{\sigma}_{E\kappa} , \tag{A.4i}$$

$$\sigma^\mu_M\bar{\sigma}^\nu_M\sigma^\rho_M = -\eta^{\mu\nu}\sigma^\rho_M - \eta^{\nu\rho}\sigma^\mu_M + \eta^{\mu\rho}\sigma^\nu_M - i\epsilon^{\mu\nu\rho\kappa}\sigma_{M\kappa} , \tag{A.4j}$$

$$\sigma^\mu_E\bar{\sigma}^\nu_E\sigma^\rho_E = -\delta^{\mu\nu}\sigma^\rho_E - \delta^{\nu\rho}\sigma^\mu_E + \delta^{\mu\rho}\sigma^\nu_E - i\epsilon^{\mu\nu\rho\kappa}\sigma_{E\kappa} . \tag{A.4k}$$

For traces of the Pauli matrices we find

$$\mathrm{tr}\sigma^{\mu\nu} = 0 , \tag{A.5a}$$

$$\mathrm{tr}\sigma^{\mu\nu}_M\sigma^{\rho\sigma}_M = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} + \epsilon^{\mu\nu\rho\sigma}\right) , \tag{A.5b}$$

$$\mathrm{tr}\sigma^{\mu\nu}_E\sigma^{\rho\sigma}_E = \frac{1}{2}\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} + \epsilon^{\mu\nu\rho\sigma}\right) , \tag{A.5c}$$

$$\mathrm{tr}\bar{\sigma}^{\mu\nu}_M\bar{\sigma}^{\rho\sigma}_M = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho} - \epsilon^{\mu\nu\rho\sigma}\right) , \tag{A.5d}$$

$$\mathrm{tr}\bar{\sigma}_E^{\mu\nu}\bar{\sigma}_E^{\rho\sigma} = \frac{1}{2}\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} - \epsilon^{\mu\nu\rho\sigma}\right) \ . \tag{A.5e}$$

Superconformal algebra

The superalgebra of the full $\mathcal{N} = 4$ theory is $\mathfrak{usp}(2,2|4)$. It consists of the conformal algebra $\mathfrak{su}(2,2)$, an extension of the Lorentz algebra. The generators are the rotations $M_{\mu\nu}$, the translations $P_{\alpha\dot{\alpha}}$, the dilatation Δ and special conformal transformations $K^{\dot{\alpha}\alpha}$. The fermionic sector consists of the known $16+16$ supercharges. These are the 16 Poincaré superchagres Q^I_α and $\bar{Q}_{I\dot{\alpha}}$ together with another 16 conformal supercharges S^I_α and $\bar{S}^{I\dot{\alpha}}$. The spinor indices $\alpha,\beta = 1,2$ and $\dot{\alpha},\dot{\beta} = 1,2$ denote left and right spinors, respectively. The index $I = 1,\ldots,4$ labels the different supercharges and can be rotated with the anti-hermitian generators $R^I{}_J$ of the $\mathfrak{su}(4)_R$ R-symmetry algebra. By construction these generators are traceless, $R^I{}_I = 0$. The supersymmetry algebra is given by

$$\left[P_\mu, Q^I_\alpha\right] = 0 = \left[P_\mu, \bar{Q}^{I\dot{\alpha}}\right] \ , \tag{A.6a}$$

$$\left[L_{\mu\nu}, Q^I_\alpha\right] = \mathrm{i}\left(\sigma_{\mu\nu}\right)_\alpha^{\ \beta} Q^I_\beta \ , \tag{A.6b}$$

$$\left[L_{\mu\nu}, \bar{Q}^{\dot{\alpha}}_I\right] = \mathrm{i}\left(\bar{\sigma}_{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}} \bar{Q}^{\dot{\beta}}_I \ , \tag{A.6c}$$

$$\left\{Q^I_\alpha, \bar{Q}^{\dot{\alpha}}_J\right\} = 2\delta^I_J \sigma^\mu_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}} P_\mu \ , \tag{A.6d}$$

$$\left\{Q^I_\alpha, Q^J_\beta\right\} = 0 = \left\{\bar{Q}^{\dot{\alpha}}_I, \bar{Q}^{\dot{\beta}}_J\right\} \ , \tag{A.6e}$$

$$\left[R^I{}_J, Q^K_\alpha\right] = \delta_J{}^K Q^I_\alpha - \frac{1}{4}\delta_J{}^I Q^K_\alpha \quad , \qquad \left[R^I{}_J, \bar{Q}^{\dot{\alpha}}_K\right] = -\delta^I{}_K \bar{Q}^{\dot{\alpha}}_J + \frac{1}{4}\delta^I{}_J \bar{Q}^{\dot{\alpha}}_K \ , \tag{A.6f}$$

$$\left[R^I{}_J, S_{K\alpha}\right] = -\delta^I{}_K S_{J\alpha} + \frac{1}{4}\delta^I{}_J S_{K\alpha} \quad , \qquad \left[R^I{}_J, \bar{S}^{K\dot{\alpha}}\right] = \delta_J{}^K \bar{S}^{I\dot{\alpha}} - \frac{1}{4}\delta_J{}^I \bar{S}^{K\dot{\alpha}} \ , \ . \tag{A.6g}$$

$$\left\{Q^I_\alpha, S_{J\beta}\right\} = \delta^I{}_J \left(\sigma^\mu\bar{\sigma}^\nu\right)_\alpha^{\ \gamma}\epsilon_{\gamma\beta} L_{\mu\nu} + \epsilon_{\alpha\beta}\left(\frac{3}{2}R^I{}_J + \delta^I{}_J\Delta\right) \ , \tag{A.6h}$$

$$\left\{\bar{Q}^{\dot{\alpha}}_I, \bar{S}^{J\dot{\beta}}\right\} = \delta_I{}^J \left(\bar{\sigma}^\mu\sigma^\nu\right)^{\dot{\alpha}}_{\ \dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\beta}} L_{\mu\nu} + \epsilon^{\dot{\alpha}\dot{\beta}}\left(\frac{3}{2}R^J{}_I - \delta^J{}_I\Delta\right) \ , \tag{A.6i}$$

$$\left\{Q^I_\alpha, \bar{S}^{J\dot{\alpha}}\right\} = 0 = \left\{S^I_\alpha, \bar{Q}^{J\dot{\alpha}}\right\} \ . \tag{A.6j}$$

Clebsch-Gordan coefficients between SO(6) and SU(4)

We use the Clebsch-Gordan coefficients given in [80] (see also [133, 134] for alternative derivations) to convert indices between the $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$ R-symmetry algebras. The generators of $\mathfrak{so}(6)$ are given through the six matrices

$$\Gamma^r = \gamma_5 \otimes \begin{pmatrix} 0 & \Sigma^r \\ \bar{\Sigma}^r & 0 \end{pmatrix} , \quad r = 1, \ldots, 6 , \tag{A.7}$$

where $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma^3 \otimes \sigma^0$ and the Σ and $\bar{\Sigma}$ are the Clebsch-Gordon coefficients. They are given by

$$\Sigma^1 = -\bar{\Sigma}^1 = -\sigma^2 \otimes \sigma^3 \quad , \quad \Sigma^2 = -\bar{\Sigma}^2 = \sigma^2 \otimes \sigma^1 \quad , \quad \Sigma^3 = -\bar{\Sigma}^3 = -\sigma^0 \otimes \sigma^2 , \tag{A.8a}$$

$$\Sigma^4 = \bar{\Sigma}^4 = -i\sigma^1 \otimes \sigma^2 \quad , \quad \Sigma^5 = \bar{\Sigma}^5 = -i\sigma^2 \otimes \sigma^0 \quad , \quad \Sigma^6 = \bar{\Sigma}^6 = i\sigma^3 \otimes \sigma^2 . \tag{A.8b}$$

These coefficients satisfy

$$\Sigma^r_{IJ} = -\Sigma^r_{JI} \quad , \quad \bar{\Sigma}^{rIJ} = -\bar{\Sigma}^{rJI} , \tag{A.9a}$$

$$\Sigma^r_{IJ} = -\left(\bar{\Sigma}^{rJI}\right)^\dagger , \tag{A.9b}$$

$$\mathrm{tr}\left(\Sigma^r \bar{\Sigma}^s\right) = -4\delta^{rs} . \tag{A.9c}$$

Let $n^r \in S^5$ be a unit vector then

$$n_r n_s \Sigma^r_{IJ} \bar{\Sigma}^{sKL} \delta^J_L = n^2 \delta^K_I = \delta^K_I . \tag{A.9d}$$

The scalar fields Φ^r in the SO(6) representation are then related to the ones Φ^{AB} in the SU(4) one by

$$\Phi^r = \Sigma^r_{IJ} \Phi^{IJ} . \tag{A.10}$$

Appendix B

Superconformal algebra preserved by the straight Wilson line

In this appendix we explicitly derive that the superalgebra conserved by the Wilson line defect is indeed $\mathfrak{osp}(4|4)$. We first focus on the supercharges preserved by the Wilson line. We then consider their commutation relations. By assuming the algebra to be closed the other preserved and broken generators can be identified exactly. The unbroken $\mathfrak{psu}(2,2|4)$ algebra of the full theory is discussed in Appendix A.

Conserved supercharges

Recall that the straight line is placed in the x_3 direction. It is given in Equation (1.63)

$$W = \mathrm{tr}\overleftarrow{\mathcal{P}}\exp\left[\mathrm{i}\int \mathrm{d}\tau\mathcal{A}\right] \quad , \qquad \mathcal{A} = A_3 - \mathrm{i}n_r\Phi^r \, ,$$

where $n_r \in S^5$ and Φ^r are the six scalars. Consider the supersymmetry transformations of the Bosonic fields [135][1]

$$\delta_\zeta A_\mu = \mathrm{i}\left(\bar{\zeta}_{\dot{\alpha}I}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\lambda_\alpha^I - \bar{\lambda}_{I\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\zeta_\alpha^I\right) \, , \tag{B.1a}$$

$$\delta_\zeta\Phi^r = \Sigma^r{}_{IJ}\zeta_\alpha^I\epsilon^{\alpha\beta}\lambda_\beta^J + \bar{\Sigma}^{rIJ}\bar{\zeta}_{I\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{J\dot{\beta}} \, . \tag{B.1b}$$

In the following we derive conditions on the spinor variables ζ and $\bar{\zeta}$ by demanding the WL operator to preserve supersymmetry. Therefore consider the action of the supercharges on $\mathcal{A}$ containing the gauge and scalar field:

$$\begin{aligned} 0 = \delta_\zeta\mathcal{A} &= \delta_\zeta A_3 - \mathrm{i}n_R\delta_\zeta\Phi^R \qquad\qquad (\mathrm{B.2}) \\ &= \mathrm{i}\bar{\zeta}_{\dot{\alpha}I}\bar{\sigma}_3^{\dot{\alpha}\alpha}\lambda_\alpha^I - \mathrm{i}\bar{\lambda}_{I\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\zeta_\alpha^I - \mathrm{i}n_r\Sigma^r{}_{IJ}\zeta_\alpha^I\epsilon^{\alpha\beta}\lambda_\beta^J - \mathrm{i}n_r\bar{\Sigma}^{rIJ}\bar{\zeta}_{I\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{J\dot{\beta}} \end{aligned}$$

[1] Mind the slightly different conventions.

$$= -\mathrm{i}\left(\bar{\zeta}_{I\dot{\alpha}}\bar{\sigma}_3^{\dot{\alpha}\alpha} - n_r\Sigma^r{}_{IJ}\zeta^J_\beta\epsilon^{\beta\alpha}\right)\lambda^I_\alpha - \mathrm{i}\bar{\lambda}_{I\dot{\alpha}}\left(\bar{\sigma}_3^{\dot{\alpha}\alpha}\zeta^I_\alpha - n_r\bar{\Sigma}^{rIJ}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\zeta}_{J\dot{\beta}}\right) .$$

Here we used that $\bar{\zeta}\bar{\sigma}_\mu\lambda = -\lambda\sigma_\mu\bar{\zeta}$ [13, 132]. The conditions for $\delta_\zeta\mathcal{A} = 0$ thus are

$$\bar{\zeta}_{I\dot{\alpha}}\bar{\sigma}_3^{\dot{\alpha}\alpha} = -\Omega_{IJ}\epsilon^{\alpha\beta}\zeta^J_\beta \quad \text{and} \quad \bar{\sigma}_3^{\dot{\alpha}\alpha}\zeta^I_\alpha = \Omega^{IJ}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\zeta}_{J\dot{\beta}} . \tag{B.3}$$

We included the symplectic matrices Ω_{IJ} introduced in (1.66). The above calculation yields the following condition for the killing spinors ζ

$$\bar{\zeta}^{\dot{\alpha}}_I = \Omega_{IJ}\bar{\sigma}^{3\dot{\alpha}\alpha}\zeta^J_\alpha \quad \text{and} \quad \zeta^I_\alpha = \Omega^{IJ}\sigma^3_{\alpha\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}_J . \tag{B.4}$$

Both conditions are in fact identical and we can therefore restrict ourselves to the second one. The superconformal killing spinor ζ can be seperated into a sum of the supersymmetric killing spinor η (corresponding to the Poincaré supercharges Q) and its conformal counterpart ν (corresponding to the conformal supercharges S). More precisely

$$\zeta^I_\alpha = \eta^I_\alpha + x_\mu\sigma^\mu_{\alpha\dot{\alpha}}\bar{\nu}^{I\dot{\alpha}} \quad \text{and} \quad \bar{\zeta}^{\dot{\alpha}}_I = \bar{\eta}^{\dot{\alpha}}_I + x_\mu\bar{\sigma}^{\mu\dot{\alpha}\alpha}\nu_{I\alpha} . \tag{B.5}$$

Plugging this into the above condition yields

$$\begin{aligned} \eta^I_\alpha + x_\mu\sigma^\mu_{\alpha\dot{\alpha}}\bar{\nu}^{I\dot{\alpha}} &= \Omega^{IJ}\sigma^3_{\alpha\dot{\alpha}}\left(\bar{\eta}^{\dot{\alpha}}_J + x_\mu\bar{\sigma}^{\mu\dot{\alpha}\beta}\nu_{J\beta}\right) \\ \Leftrightarrow \quad & \begin{cases} \eta^I_\alpha &= \Omega^{IJ}\sigma^3_{\alpha\dot{\alpha}}\bar{\eta}^{\dot{\alpha}}_J \\ \Omega^{IJ}\nu_{J\alpha} &= -\sigma^3_{\alpha\dot{\alpha}}\bar{\nu}^{I\dot{\alpha}} . \end{cases} \end{aligned} \tag{B.6}$$

Therefore the preserved supercharges are a combination of the supercharges Q and their conjugates $\bar{Q}$. The factor between them can be read off from the above equation. The resulting preserved supercharges are similar to the ones in $\mathcal{N} = 2$ [38]. Note that for convenience we include a normalization factor. The conserved superchagres for this line thus are

$$\mathcal{Q}^I_\alpha = \frac{1}{\sqrt{2}}Q^I_\alpha + \frac{1}{\sqrt{2}}\Omega^{IJ}\sigma^3_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}_J , \tag{B.7}$$

$$\mathcal{S}^I_\alpha = -\frac{1}{\sqrt{2}}\Omega^{IJ}S_{J\alpha} + \frac{1}{\sqrt{2}}\sigma^3_{\alpha\dot{\alpha}}\bar{S}^{J\dot{\alpha}} . \tag{B.8}$$

The broken supercharges are given in a similar way. They are the orthogonal charges to the above ones and obtained by a sign change for the second term,

$$\mathfrak{Q}^I_\alpha = \frac{1}{\sqrt{2}}Q^I_\alpha - \frac{1}{\sqrt{2}}\Omega^{IJ}\sigma^3_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}_J , \tag{B.9}$$

$$\mathfrak{S}^I_\alpha = \frac{1}{\sqrt{2}}\Omega^{IJ}S_{J\alpha} + \frac{1}{\sqrt{2}}\sigma^3_{\alpha\dot{\alpha}}\bar{S}^{J\dot{\alpha}} . \tag{B.10}$$

Commutation relations

Putting the straight Wilson line as a defect in the theory breaks the Lorentz generators $L_{\mu 3}$ and all translations $P_{\mu\neq 3}$ as well as the special conformal transformations $K_{\mu\neq 3}$. We claim that with the charges defined above form an algebra which is still closed. This is indeed the case for $\mathcal{N}=2$ where the remaining conformal group is $\mathfrak{osp}(4^*|2)$ [38]. In the $\mathcal{N}=4$ case we shall find an $\mathfrak{osp}(4^*|4)$. The original supersymmetry algebra is given in Appendix A with the commutator of two supercharges given in Equation (A.6d). When commuting the preserved $\mathcal{Q}$ found above the result should only depend on the leftover translation P_3.

$$\begin{aligned}\left\{\mathcal{Q}^I_\alpha,\mathcal{Q}^K_\beta\right\} &= \frac{1}{2}\left\{Q^I_\alpha+\Omega^{IJ}\sigma^3_{\alpha\dot\alpha}\bar Q^{\dot\alpha}_J, Q^K_\beta+\Omega^{KL}\sigma^3_{\beta\dot\beta}\bar Q^{\dot\beta}_L\right\}\\ &= \frac{1}{2}\Omega^{KL}\sigma^3_{\beta\dot\beta}\left\{Q^I_\alpha,\bar Q^{\dot\beta}_L\right\}+\frac{1}{2}\Omega^{IJ}\sigma^3_{\alpha\dot\alpha}\left\{Q^K_\beta,\bar Q^{\dot\alpha}_J\right\}\\ &= \Omega^{KL}\delta^I{}_L\sigma^3_{\beta\dot\beta}\sigma^\mu_{\alpha\dot\gamma}\epsilon^{\dot\gamma\dot\beta}P_\mu+\Omega^{IJ}\delta^C{}_J\sigma^3_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\gamma}\epsilon^{\dot\gamma\dot\alpha}P_\mu\\ &= \Omega^{IK}\left(\sigma^3\bar\sigma^\mu+\bar\sigma^3\sigma^\mu\right)_\alpha{}^\gamma\epsilon_{\gamma\beta}P_\mu\\ &= 2\Omega^{IK}P_3\epsilon_{\alpha\beta}\,.\end{aligned}\tag{B.11}$$

We used the fact that the Ω^{IK} anti-commute, see (1.60). Considering the commutator between a broken and an unbroken supercharge we expect the result to depend on the broken translation generators P_m $(m=0,1,2)$.

$$\begin{aligned}\left\{\mathcal{Q}^I_\alpha,\mathfrak{Q}^K_\beta\right\} &= \frac{1}{2}\left\{Q^I_\alpha+\Omega^{IJ}\sigma^3_{\alpha\dot\alpha}\bar Q^{\dot\alpha}_J, Q^K_\beta-\Omega^{KL}\sigma^3_{\beta\dot\beta}\bar Q^{\dot\beta}_L\right\}\\ &= \frac{1}{2}\Omega^{IJ}\sigma^3_{\alpha\dot\alpha}\left\{Q^K_\beta,\bar Q^{\dot\alpha}_J\right\}-\frac{1}{2}\Omega^{KL}\sigma^3_{\beta\dot\beta}\left\{Q^I_\alpha,\bar Q^{\dot\beta}_L\right\}\\ &= \Omega^{IJ}\delta^C{}_J\sigma^3_{\alpha\dot\alpha}\sigma^\mu_{\beta\dot\gamma}\epsilon^{\dot\gamma\dot\alpha}P_\mu-\Omega^{KL}\delta^I{}_L\sigma^3_{\beta\dot\beta}\sigma^\mu_{\alpha\dot\gamma}\epsilon^{\dot\gamma\dot\beta}P_\mu\\ &= \Omega^{IK}\left(\sigma^3\bar\sigma^\mu-\bar\sigma^3\sigma^\mu\right)_\alpha{}^\gamma\epsilon_{\gamma\beta}P_\mu=-4\mathrm{i}\Omega^{IK}\left(\sigma^{3\mu}\right)_\alpha{}^\gamma\epsilon_{\gamma\beta}P_\mu\\ &= 2\Omega^{IK}\begin{pmatrix}P_0 & -P_1+\mathrm{i}P_2\\ P_1+\mathrm{i}P_2 & -P_0\end{pmatrix}_{\alpha\beta}\,.\end{aligned}\tag{B.12}$$

With an identical computation we can see that the commutator of the preserved conformal supercharges yields the special conformal transformation along the defect K_3 which we expect to be preserved,

$$\left\{\mathcal{S}^I_\alpha,\mathcal{S}^K_\beta\right\}=2\Omega^{IK}K_3\epsilon_{\alpha\beta}\,.\tag{B.13}$$

Similarly the special conformal transformations K_m are broken.
We thus have to consider the R-symmetry charges, the Lorentz

$$\begin{aligned}
\left\{\mathcal{Q}^I_\alpha, \mathcal{S}^K_\beta\right\} &= \frac{1}{2}\left\{Q^I_\alpha + \Omega^{IJ}\sigma^3_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha}_J, -\Omega^{KL}S_{L\beta} + \sigma^3_{\beta\dot\beta}\bar{S}^{K\dot\beta}\right\} \\
&= -\frac{1}{2}\Omega^{KL}\left\{Q^I_\alpha, S_{L\beta}\right\} + \frac{1}{2}\Omega^{IJ}\sigma^3_{\alpha\dot\alpha}\sigma^3_{\beta\dot\beta}\left\{\bar{Q}^{\dot\alpha}_J, \bar{S}^{K\dot\beta}\right\} \\
&= -\frac{1}{2}\Omega^{KL}\delta^I{}_L\left(\sigma^\mu\bar\sigma^\nu\right)_\alpha{}^\gamma\epsilon_{\gamma\beta}M_{\mu\nu} - \frac{1}{2}\Omega^{KL}\epsilon_{\alpha\beta}\left(\frac{3}{2}R^I{}_L + \delta^I{}_L\Delta\right) \\
&\quad + \frac{1}{2}\Omega^{IJ}\sigma^3_{\alpha\dot\alpha}\sigma^3_{\beta\dot\beta}\left[\delta^K_J\left(\bar\sigma^\mu\sigma^\nu\right)^{\dot\alpha}{}_{\dot\gamma}\epsilon^{\dot\gamma\dot\beta}M_{\mu\nu} + \epsilon^{\dot\alpha\dot\beta}\left(\frac{3}{2}R^K{}_J - \delta^K{}_J\Delta\right)\right] \\
&= \frac{1}{2}\Omega^{IK}\left[\sigma^\mu\bar\sigma^\nu - \sigma^3\bar\sigma^\mu\sigma^\nu\bar\sigma^3\right]_\alpha{}^\gamma\epsilon_{\gamma\beta}L_{\mu\nu} + \Omega^{IK}\epsilon_{\alpha\beta}\Delta + \frac{3}{4}\epsilon_{\alpha\beta}\left(R^K{}_J\Omega^{IJ} + R^I{}_J\Omega^{JK}\right) \\
&= \Omega^{IJ}\left(\sigma^m\bar\sigma^n\right)_{\alpha\beta}M_{mn} + \Omega^{IJ}\epsilon_{\alpha\beta}\Delta + \frac{3}{2}\epsilon_{\alpha\beta}\mathcal{R}^{IJ}\,.
\end{aligned} \tag{B.14}$$

This is indeed independent of the broken $L_{3\mu}$. Note furthermore that we defined

$$\mathcal{R}^{IK} = \frac{1}{2}R^I{}_J\Omega^{JK} + \frac{1}{2}R^K{}_J\Omega^{JI}\,. \tag{B.15}$$

This symmetric combination of the $\mathrm{SU}(4)_R$ generators are the ten generators of $\mathrm{Sp}(4)$ [83]. Likewise the five broken generators are

$$\mathfrak{R}^{IK} = \frac{1}{2}R^I{}_J\Omega^{JK} - \frac{1}{2}R^K{}_J\Omega^{JI}\,. \tag{B.16}$$

This can be seen by commuting the unbroken supercharge with the broken $\mathfrak{S}^K_\beta$

$$\left\{\mathcal{Q}^I_\alpha, \mathfrak{S}^K_\beta\right\} = \Omega^{IJ}\left(\sigma^3\bar\sigma^m\right)_{\alpha\beta}M_{3m} + \frac{3}{2}\epsilon_{\alpha\beta}\mathfrak{R}^{IJ}\,. \tag{B.17}$$

This depends indeed on the three broken rotations M_{3m} and thus $\mathfrak{S}_{I\alpha}$ must indeed be excluded from the algebra.
Therefore, in conclusion, we see that the algebra $\mathfrak{osp}(4|4)$ is indeed closed and preserved in the presence of the Wilson line defect.

Appendix C

Mulitplets

In this appendix we gather some multiplets of $\mathcal{N} = 4$ and $\mathcal{N} = 2$ SYM theories. The aim is to explicitly understand how the $\mathcal{N} = 4$ stress tensor multiplet can be obtained from several $\mathcal{N} = 2$ multiplest [112].
The $\mathcal{N} = 4$ stress tensor multiplet $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,1,0]_{0,0}}$ as given in [87] with our index notation is

Δ									
2					φ_{rs}				
$\frac{5}{2}$				$\psi_{Ir\alpha}$		$\overline{\psi}^{I}{}_{r\dot{\alpha}}$			
3			$f_{r\alpha\beta}$, ρ_{IJ}		$J_{rs\alpha\dot{\alpha}}$		$\overline{f}_{r\dot{\alpha}\dot{\beta}}$, $\overline{\rho}^{IJ}$		
$\frac{7}{2}$		$\lambda_{I\alpha}$		$\Psi^{I}_{\alpha\beta\dot{\alpha}}$		$\overline{\Psi}_{I\alpha\dot{\alpha}\dot{\beta}}$		$\overline{\lambda}^{I}_{\dot{\alpha}}$	
4	Φ				$T_{\alpha\beta\dot{\alpha}\dot{\beta}}$				$\overline{\Phi}$
Y	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2

It has dimension 256 The samemultiplet can be written using $\mathfrak{su}(4)_R \otimes \mathfrak{su}(2)_\alpha \otimes \mathfrak{su}(2)_{dot\alpha}$ representations [88].

Δ									
2					$[0,2,0]_{(0,0)}$				
$5/2$				$[0,1,1]_{(1/2,0)}$		$[1,1,0]_{(0,1/2)}$			
3			$[0,1,0]_{(1,0)}$ $[0,0,2]_{(0,0)}$		$[1,0,1]_{(1/2,1/2)}$		$[2,0,0]_{(0,0)}$	$[0,1,0]_{(0,1)}$	
$7/2$		$[0,0,1]_{(1/2,0)}$		$[1,0,0]_{(1,1/2)}$		$[0,0,1]_{(1/2,1)}$		$[1,0,0]_{(0,1/2)}$	
4	$[0,0,0]_{(0,0)}$				$[0,0,0]_{(1,1)}$ $-[1,0,1]_{(0,0)}$				$[0,0,0]_{(0,0)}$
$9/2$				$-[1,0,0]_{(1/2,0)}$		$-[0,0,1]_{(0,1/2)}$			
5					$-[0,0,0]_{(1/2,1/2)}$				
Y	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2

Let us continue with $\mathcal{N}=2$ multiplets. These are representations as given in [88, 112]. Let us start with the multiplet $\mathcal{E}_2$ which has dimension 16

Δ					
2	$0_{(0,0)}$				
$\frac{5}{2}$		$\frac{1}{2}_{(0,\frac{1}{2})}$			
3			$0_{(0,1)}, 1_{(0,0)}$		
$\frac{7}{2}$				$\frac{1}{2}_{(0,\frac{1}{2})}$	
4					$0_{(0,0)}$
r	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0

The multiplet $\overline{\mathcal{E}}_2$ follows similarly.
The stress tensor multiplet in $\mathcal{N}=4$ is $\hat{\mathcal{C}}_{0(0,0)}$ and has dimension 48

Δ					
2			$0_{(0,0)}$		
$\frac{5}{2}$		$\frac{1}{2}_{(\frac{1}{2},0)}$		$\frac{1}{2}_{(0,\frac{1}{2})}$	
3	$0_{(1,0)}$		$1_{(\frac{1}{2},\frac{1}{2})}, 0_{(\frac{1}{2},\frac{1}{2})}$		$0_{(0,1)}$
$\frac{7}{2}$		$\frac{1}{2}_{(1,\frac{1}{2})}$		$\frac{1}{2}_{(\frac{1}{2},1)}$	
4			$0_{(1,1)}$ $-0_{(0,0)}, -1_{(0,0)}$		
$\frac{9}{2}$		$-\frac{1}{2}_{(\frac{1}{2},0)}$		$-\frac{1}{2}_{(0,\frac{1}{2})}$	
5			$-0_{(\frac{1}{2},\frac{1}{2})}$		
r	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1

Next we have the multiplet $\hat{\mathcal{B}}_1$ which also has dimension 16

Δ					
2			$1_{(0,0)}$		
$\frac{5}{2}$		$\frac{1}{2}_{(\frac{1}{2},0)}$		$\frac{1}{2}_{(0,\frac{1}{2})}$	
3	$0_{(0,0)}$		$0_{(\frac{1}{2},\frac{1}{2})}$		$0_{(0,0)}$
$\frac{7}{2}$		$-\frac{1}{2}_{(\frac{1}{2},0)}$		$-\frac{1}{2}_{(0,\frac{1}{2})}$	
4			$-0_{(0,0)}$		
r	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1

Last we have the multiplet $\mathcal{D}_{\frac{1}{2}(0,0)}$ with dimension 16 and its analogous counterpart $\overline{\mathcal{D}}_{\frac{1}{2}(0,0)}$

Δ						
2			$\frac{1}{2}_{(0,0)}$			
$\frac{5}{2}$		$0_{(\frac{1}{2},0)}$		$1_{(0,\frac{1}{2})}$, $0_{(0,\pm\frac{1}{2})}$		
3	$-\frac{1}{2}_{(0,0)}$		$\frac{1}{2}_{(\frac{1}{2},\frac{1}{2})}$, $-\frac{1}{2}_{(\frac{1}{2},\pm\frac{1}{2})}$		$\frac{1}{2}_{(0,1)}$, $\pm\frac{1}{2}_{(0,0)}$	
$\frac{7}{2}$		$0_{(0,\frac{1}{2})}$, $-1_{(0,\pm\frac{1}{2})}$		$0_{(\frac{1}{2},1)}$, $0_{(\frac{1}{2},0)}$ $-1_{(\frac{1}{2},0)}$		$0_{(0,\frac{1}{2})}$
4			$-\frac{1}{2}_{(0,1)}$, $-\frac{1}{2}_{(0,0)}$ $-\frac{3}{2}_{(0,0)}$		$-\frac{1}{2}_{(\frac{1}{2},\frac{1}{2})}$	
$\frac{9}{2}$				$-2_{(0,\frac{1}{2})}$		
r	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$

These $\mathcal{N} = 2$ muliplets can be combined into the $\mathcal{N} = 4$ stress tensor multiplet

$$\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,2,0]_{(0,0)}} \simeq 3\hat{\mathcal{B}}_1 \oplus \mathcal{E}_{2(0,0)} \oplus \overline{\mathcal{E}}_{2(0,0)} \oplus \hat{\mathcal{C}}_{0(0,0)} \oplus 2\mathcal{D}_{\frac{1}{2}(0,0)} \oplus 2\overline{\mathcal{D}}_{\frac{1}{2}(0,0)} \ . \tag{C.1}$$

As a consistency check we can see that the dimensions match

$$256 = 3 \times 16 + 16 + 16 + 48 + 2 \times 32 + 2 \times 32 \ . \tag{C.2}$$

Appendix D

Some details on supersymmetry transformations

For the superconformal Ward identity (2.16) we need to know the supersymmetry transformation of the field ${\psi_{I\alpha}}^{JK}$ of the stress tensor multiplet as well as of the displacement primary fields $\mathbb{O}^{IJ}$.

The SUSY transformation of ${\psi_{I\alpha}}^{JK}$ is given in a $\mathfrak{so}(6)$ index notation in [87]. in this appendix we derive the transformation using $\mathfrak{su}(4)$ indices only.

$$\begin{aligned}\delta_{SUSY}\psi_{r\alpha} &= \mathrm{i}\partial_{\alpha\dot{\alpha}}\varphi_{rs}\bar{\gamma}_s\bar{\varepsilon}^{\dot{\alpha}} + 4\varphi_{rs}\bar{\gamma}_s\eta_\alpha - f_{r\alpha\beta}\varepsilon^\beta - \frac{1}{6}f_{s\alpha\beta}\bar{\gamma}_r\gamma_s\varepsilon^\beta \\ &\quad + \rho\epsilon_{\alpha\beta}\varepsilon^\beta + J_{rs\alpha\dot{\alpha}}\bar{\gamma}_s\bar{\varepsilon}^{\dot{\alpha}} + \frac{1}{6}J_{st\alpha\dot{\alpha}}\bar{\gamma}_r\gamma_s\bar{\gamma}_t\bar{\varepsilon}^{\dot{\alpha}}\ . \end{aligned} \tag{D.1}$$

Note already that the indices for the contribution with ρ are not matching. Therefore we will add a matrix γ_r here to restore the missing index. Furthermore, this representation supresses the SU(4) indices. Let us restore these.

$$\begin{aligned}\delta_{SUSY}\psi_{Ir\alpha} &= \mathrm{i}\partial_{\alpha\dot{\alpha}}\varphi_{rs}\bar{\gamma}_{sIJ}\bar{\varepsilon}^{J\dot{\alpha}} + 4\varphi_{rs}\bar{\gamma}_{sIJ}\eta^J_\alpha - f_{r\alpha\beta}\varepsilon^\beta_I - \frac{1}{6}f_{s\alpha\beta}\bar{\gamma}_{rIJ}\gamma^{JK}_s\varepsilon^\beta_K \\ &\quad + \rho_{IJ}\epsilon_{\alpha\beta}\gamma^{JK}_r\varepsilon^\beta_K + J_{rs\alpha\dot{\alpha}}\bar{\gamma}_{sIJ}\bar{\varepsilon}^{J\dot{\alpha}} + \frac{1}{6}J_{st\alpha\dot{\alpha}}\bar{\gamma}_{rIJ}\gamma^{JK}_s\bar{\gamma}_{tKL}\bar{\varepsilon}^{L\dot{\alpha}}\ . \end{aligned} \tag{D.2}$$

For convenience let us contract all terms with γ^{MN}_r

$$\begin{aligned}\delta\psi_{Ir\alpha}\gamma^{MN}_r &= \mathrm{i}\partial_{\alpha\dot{\alpha}}\varphi_{rs}\bar{\gamma}_{sIJ}\gamma^{MN}_r\bar{\varepsilon}^{J\dot{\alpha}} + 4\varphi_{rs}\bar{\gamma}_{sIJ}\gamma^{MN}_r\eta^J_\alpha - f_{r\alpha\beta}\gamma^{MN}_r\varepsilon^\beta_I \\ &\quad - \frac{1}{6}f_{s\alpha\beta}\bar{\gamma}_{rIJ}\gamma^{MN}_r\gamma^{JK}_s\varepsilon^\beta_K + \rho_{IJ}\epsilon_{\alpha\beta}\gamma^{JK}_r\gamma^{MN}_r\varepsilon^\beta_K + J_{rs\alpha\dot{\alpha}}\gamma^{MN}_r\bar{\gamma}_{sIJ}\bar{\varepsilon}^{J\dot{\alpha}} \\ &\quad + \frac{1}{6}J_{st\alpha\dot{\alpha}}\bar{\gamma}_{rIJ}\gamma^{MN}_r\gamma^{JK}_s\bar{\gamma}_{tKL}\bar{\varepsilon}^{L\dot{\alpha}}\ . \end{aligned} \tag{D.3}$$

This is now fully consistent in all indices. In the next step we want to get rid of the SO(6) indices[1] Note that the gamma matrices contract to deltas $\gamma_{rIJ}\bar{\gamma}_r^{KL} = 2\delta_I^K\delta_J^L - 2\delta_I^L\delta_J^K$.

$$\begin{aligned}\delta\psi_{I\alpha}^{JK}\bar{\gamma}_{rJK}\gamma_r^{MN} = {} & \mathrm{i}\partial_{\alpha\dot\alpha}\varphi_{IJ}^{MN}\bar{\varepsilon}^{J\dot\alpha} + 4\varphi_{IJ}^{MN}\eta_\alpha^J - 4f_{\alpha\beta}^{MN}\varepsilon_I^\beta - \frac{4}{3}\delta_I^M f_{\alpha\beta}^{NK}\varepsilon_K^\beta + \frac{4}{3}\delta_I^N f_{\alpha\beta}^{MK}\varepsilon_K^\beta \\ & - 4\rho_{IJ}\epsilon_{\alpha\beta}\epsilon^{JKMN}\varepsilon_K^\beta + J_{\alpha\dot\alpha IJ}^{MN}\bar{\varepsilon}^{J\dot\alpha} + \frac{1}{3}\left(\delta_I^M J_{\alpha\dot\alpha KL}^{NK} - \delta_I^N J_{\alpha\dot\alpha KL}^{MK}\right)\bar{\varepsilon}^{L\dot\alpha}\,, \end{aligned} \tag{D.5}$$

$$\begin{aligned}\delta\psi_{I\alpha}^{JK} = {} & \frac{\mathrm{i}}{4}\partial_{\alpha\dot\alpha}\varphi_{IL}^{JK}\bar{\varepsilon}^{L\dot\alpha} + \varphi_{IL}^{JK}\eta_\alpha^L - f_{\alpha\beta}^{JK}\varepsilon_I^\beta - \frac{1}{3}\delta_I^J f_{\alpha\beta}^{KL}\varepsilon_L^\beta + \frac{1}{3}\delta_I^K f_{\alpha\beta}^{JL}\varepsilon_L^\beta \\ & - \rho_{IL}\epsilon_{\alpha\beta}\epsilon^{JKLM}\varepsilon_M^\beta + J_{\alpha\dot\alpha IL}^{JK}\bar{\varepsilon}^{L\dot\alpha} + \frac{1}{3}\left(\delta_I^J J_{\alpha\dot\alpha ML}^{KM} - \delta_I^K J_{\alpha\dot\alpha ML}^{JM}\right)\bar{\varepsilon}^{L\dot\alpha}\,. \end{aligned} \tag{D.6}$$

Note that we found for the Killing spinor the constraint $\bar{\epsilon}^{I\dot\alpha} = -\bar{\sigma}_3^{\dot\alpha\alpha}\epsilon_\alpha^I$. Then we find

$$\begin{aligned}\delta\psi_{I\alpha}^{JK} = {} & -\frac{\mathrm{i}}{4}\partial_{\alpha\dot\alpha}\varphi_{IN}^{JK}\bar{\sigma}_3^{\dot\alpha\alpha}\epsilon_\alpha^N + \varphi_{IN}^{JK}\eta_\alpha^N - f_{\alpha\beta}^{JK}\varepsilon_I^\beta - \frac{1}{3}\delta_I^J f_{\alpha\beta}^{KN}\varepsilon_N^\beta + \frac{1}{3}\delta_I^K f_{\alpha\beta}^{JN}\varepsilon_N^\beta \\ & - \rho_{IL}\epsilon_{\alpha\beta}\epsilon^{JKNO}\varepsilon_O^\beta - J_{\alpha\dot\alpha IL}^{JK}\bar{\sigma}_3^{\dot\alpha\beta}\epsilon_\beta^L - \frac{1}{3}\left(\delta_I^J J_{\alpha\dot\alpha ML}^{KM} - \delta_I^K J_{\alpha\dot\alpha ML}^{JM}\right)\bar{\sigma}_3^{\dot\alpha\beta}\epsilon_\beta^L\,. \end{aligned} \tag{D.7}$$

Then we readily see that the Ward identity (2.16) indeed yields

$$\begin{aligned}0 = \left\langle\delta\psi_{I\alpha}^{JK}\mathbb{O}^{LM}\right\rangle_W = {} & 2\Omega^{NL}\left\langle\mathbb{A}_\beta^M\psi_{I\alpha}^{JK}\right\rangle_W\epsilon_N^\beta - 2\Omega^{NM}\left\langle\mathbb{A}_\beta^L\psi_{I\alpha}^{JK}\right\rangle_W\epsilon_N^\beta + \Omega^{LM}\left\langle\mathbb{A}_\beta^N\psi_{I\alpha}^{JK}\right\rangle_W\epsilon_N^\beta \\ & - \frac{\mathrm{i}}{4}\partial_{\alpha\dot\alpha}\left\langle\varphi_{IN}^{JK}\mathbb{O}^{LM}\right\rangle_W\bar{\sigma}_3^{\dot\alpha\alpha}\epsilon_\alpha^N + \left\langle\varphi_{IN}^{JK}\mathbb{O}^{LM}\right\rangle_W\eta_\alpha^N - \left\langle f_{\alpha\beta}^{JK}\mathbb{O}^{LM}\right\rangle_W\varepsilon_I^\beta \\ & - \frac{1}{3}\delta_I^J\left\langle f_{\alpha\beta}^{KN}\mathbb{O}^{LM}\right\rangle_W\varepsilon_N^\beta + \frac{1}{3}\delta_I^K\left\langle f_{\alpha\beta}^{JN}\mathbb{O}^{LM}\right\rangle_W\varepsilon_N^\beta \\ & - \left\langle\rho_{IL}\mathbb{O}^{LM}\right\rangle_W\epsilon_{\alpha\beta}\epsilon^{JKNO}\varepsilon_O^\beta - \left\langle J_{\alpha\dot\alpha IN}^{JK}\mathbb{O}^{LM}\right\rangle_W\bar{\sigma}_3^{\dot\alpha\beta}\epsilon_\beta^N \\ & - \frac{1}{3}\left(\delta_I^J\left\langle J_{\alpha\dot\alpha PN}^{KP}\mathbb{O}^{LM}\right\rangle_W - \delta_I^K\left\langle J_{\alpha\dot\alpha PN}^{JN}\mathbb{O}^{LM}\right\rangle_W\right)\bar{\sigma}_3^{\dot\alpha\beta}\epsilon_\beta^N\,. \end{aligned} \tag{D.8}$$

Now consider the displacement multiplet. Before we apply the supersymmetry transforma-

[1]Recall that

$$\psi_{Ir\alpha} = \psi_{I\alpha}^{\ \ MN}\bar{\gamma}_{rMN}\,, \tag{D.4a}$$

$$f_{\alpha\beta}^{IJ} = \frac{1}{4}f_{r\alpha\beta}\gamma_r^{IJ}\,, \quad f_{r\alpha\beta} = f_{\alpha\beta}^{IJ}\bar{\gamma}_{rIJ}\,, \tag{D.4b}$$

$$J_{rs\alpha\dot\alpha} = J_{\alpha\dot\alpha}^{IJKL}\bar{\gamma}_{rIJ}\bar{\gamma}_{sKL}\,, \tag{D.4c}$$

$$\varphi^{IJ}{}_{KL} = \varphi_{rs}\gamma_r^{IJ}\bar{\gamma}_{sKL}\,. \tag{D.4d}$$

tion we derive the primary $\mathbb{O}$ in terms of the scalar fields. This can be done by commuting the broken R-symmetry generator with the Wilson line. Consider first that for $R^I{}_J$

$$[R^I{}_J, \mathcal{A}] = [R^I{}_J, A_3 - \mathrm{i}n_r\Phi^r] = -\mathrm{i}\Omega_{KL}[R^I{}_J, \Phi^{KL}] \tag{D.9}$$

$$= -\frac{\mathrm{i}}{2}\Omega_{KL}\left(\delta^K{}_J\Phi^{IL} - \delta^L{}_J\Phi^{IK}\right) + \frac{\mathrm{i}}{2}\Omega_{KL}\epsilon^{IKLM}\bar{\Phi}_{MJ} \tag{D.10}$$

$$= \frac{-\mathrm{i}}{2}\left(\Omega_{JL}\Phi^{IL} - \Omega_{KJ}\Phi^{IK}\right) + \mathrm{i}\Omega^{IM}\bar{\Phi}_{MJ} \tag{D.11}$$

$$= \mathrm{i}\Phi^{IK}\Omega_{KJ} + \mathrm{i}\Omega^{IM}\bar{\Phi}_{MJ}\ . \tag{D.12}$$

The broken R-symmetry generator is the anti-symmetric part of the original generators. We thus find

$$[\mathfrak{R}^{IJ}, \mathcal{A}] = \frac{1}{2}[R^I{}_M, \mathcal{A}]\Omega^{MJ} - \frac{1}{2}[R^J{}_M, \mathcal{A}]\Omega^{MI} \tag{D.13}$$

$$= \frac{\mathrm{i}}{2}\Phi^{IK}\Omega_{KM}\Omega^{MJ} + \frac{\mathrm{i}}{2}\Omega^{IN}\bar{\Phi}_{NM}\Omega^{MJ} - \frac{\mathrm{i}}{2}\Phi^{JK}\Omega_{KM}\Omega^{MI} - \frac{\mathrm{i}}{2}\Omega^{JN}\bar{\Phi}_{NM}\Omega^{MI} \tag{D.14}$$

$$= -\frac{\mathrm{i}}{2}\Phi^{IJ} + \frac{\mathrm{i}}{2}\Phi^{JI} + \mathrm{i}\Omega^{IK}\bar{\Phi}_{KL}\Omega^{LJ} \tag{D.15}$$

$$= -\mathrm{i}\Phi^{IJ} + \mathrm{i}\Omega^{JK}\bar{\Phi}_{KL}\Omega^{LJ} = -2\mathrm{i}\Phi^{IJ} + \frac{\mathrm{i}}{2}\Omega^{IJ}\Omega_{KL}\Phi^{KL}\ . \tag{D.16}$$

The operator $\mathbb{O}$ can be read off from this following Equation (2.9c)

$$\mathbb{O}^{IJ} = \Phi^{IJ} - \Omega^{IK}\bar{\Phi}_{KL}\Omega^{LJ} = 2\Phi^{IJ} - \frac{1}{2}\Omega^{IJ}\Omega_{KL}\Phi^{KL}\ . \tag{D.17}$$

The antisymmetric $\mathbb{O}^{IJ}$ obeys the trace condition $\Omega_{IJ} = 0$ has five components. These are the five scalars that are not coupled to the WL through the unit vector n_r. The sixth field is the Konishi primary operator [116].
Let us now consider how the preserved supercharge acts on this operator

$$\begin{aligned}
\delta_\zeta\mathbb{O}^{IJ} &= \delta_\zeta\Phi^{IJ} - \Omega^{IK}\delta_\zeta\bar{\Phi}_{KL}\Omega^{LJ} \\
&= \frac{1}{2}\left(\lambda^{I\alpha}\zeta^J_\alpha - \lambda^{J\alpha}\zeta^I_\alpha + \epsilon^{IJKL}\bar{\zeta}_{K\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_L\right) \\
&\quad - \frac{1}{2}\Omega^{IK}\left(\bar{\lambda}^{\dot{\alpha}}_K\bar{\zeta}_{L\dot{\alpha}} - \bar{\lambda}^{\dot{\alpha}}_L\bar{\zeta}_{K\dot{\alpha}} + \epsilon_{KLMN}\zeta^{M\alpha}\lambda^N_\alpha\right)\Omega^{LJ} \\
&= \frac{1}{2}\left(\lambda^{I\alpha}\zeta^J_\alpha - \lambda^{J\alpha}\zeta^I_\alpha + \epsilon^{IJKL}\Omega_{KM}\zeta^M_\alpha\epsilon^{\alpha\beta}\sigma^3_{\beta\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_L\right) \\
&\quad - \frac{1}{2}\Omega^{IK}\left(\Omega_{LM}\zeta^M_\alpha\epsilon^{\alpha\beta}\sigma^3_{\beta\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_K - \Omega_{KM}\zeta^M_\alpha\epsilon^{\alpha\beta}\sigma^3_{\beta\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_L + \epsilon_{KLMN}\zeta^{M\alpha}\lambda^N_\alpha\right)\Omega^{LJ}
\end{aligned} \tag{D.18}$$

$$
\begin{aligned}
&= \left(\lambda^I_\alpha \zeta^{J\alpha} - \lambda^J_\alpha \zeta^{I\alpha} + \epsilon^{IJKL}\Omega_{KM}\zeta^M_\alpha \epsilon^{\alpha\beta}\sigma^3_{\beta\dot\alpha}\bar\lambda^{\dot\alpha}_L\right) \\
&\quad - \frac{1}{2}\left(\zeta^{J\alpha}\Omega^{IK}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_K - \zeta^{I\alpha}\Omega^{JL}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_L + \Omega^{IK}\epsilon_{KLMN}\zeta^{M\alpha}\lambda^N_\alpha \Omega^{LJ}\right) \\
&= -\frac{1}{2}\zeta^{I\alpha}\left(\lambda^J_\alpha - \Omega^{JK}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_K\right) + \frac{1}{2}\zeta^{J\alpha}\left(\lambda^I_\alpha - \Omega^{IK}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_K\right) \\
&\quad - \frac{1}{2}\zeta^{M\alpha}\left(\Omega^{IK}\epsilon_{KLMN}\lambda^N_\alpha\Omega^{LJ} - \epsilon^{IJKL}\Omega_{KM}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_L\right) \; .
\end{aligned}
$$

Let us first rewrite the totally antisymmetric tensor as

$$
\begin{aligned}
\epsilon_{IJKL} &= -\Omega_{IJ}\Omega_{KL} + \Omega_{IK}\Omega_{JL} - \Omega_{IL}\Omega_{JK} \\
\Rightarrow \qquad \zeta^{M\alpha}\Omega^{IK}\epsilon_{KLMN}\lambda^N_\alpha\Omega^{LJ} &= \zeta^{M\alpha}\lambda^N_\alpha\left(-\Omega^{IK}\Omega_{KN}\Omega_{LM}\Omega^{LJ} + \Omega^{IK}\Omega_{KL}\Omega_{MN}\Omega^{LJ}\right. \\
&\qquad \left. + \Omega^{IK}\Omega_{KM}\Omega_{LN}\Omega^{LJ}\right) \\
&= \zeta^{M\alpha}\lambda^N_\alpha\left(\delta^I_N\delta^J_M + \Omega^{IJ}\Omega_{MN} - \delta^I_M\delta^J_N\right) \qquad \text{(D.19)} \\
&= \zeta^{J\alpha}\lambda^I_\alpha - \zeta^{I\alpha}\lambda^J_\alpha + \Omega^{IJ}\zeta^\alpha_M\lambda^M_\alpha \; . \\
\text{And} \qquad \zeta^{M\alpha}\epsilon^{IJKL}\Omega_{KM}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_L &= \zeta^{M\alpha}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_L\left(-\Omega^{IL}\Omega^{JK}\Omega_{JM} - \Omega^{IJ}\Omega^{KL}\Omega_{KM}\right. \\
&\qquad \left. + \Omega^{IK}\Omega^{JL}\Omega_{KM}\right) \\
&= \zeta^{M\alpha}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_L\left(\Omega^{IL}\delta^J_M + \Omega^{IJ}\delta^L_M - \Omega^{JL}\delta^I_M\right) \qquad \text{(D.20)} \\
&= \zeta^{J\alpha}\Omega^{IL}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_L - \zeta^{I\alpha}\Omega^{JL}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_L \\
&\qquad + \Omega^{IJ}\zeta^\alpha_M\Omega^{ML}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_L \; .
\end{aligned}
$$

From this we finally can combine all the expressions to the displacement operators $\mathbb{\Lambda}^I_\alpha = \lambda^I_\alpha - \Omega^{IJ}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_J$ and thus find

$$
\delta_\zeta \mathbb{O}^{IJ} = -\zeta^{J\alpha}\mathbb{\Lambda}^I_\alpha + \zeta^{I\alpha}\mathbb{\Lambda}^J_\alpha - \frac{1}{2}\Omega^{IJ}\zeta^\alpha_K\mathbb{\Lambda}^K_\alpha \; . \qquad \text{(D.21)}
$$

Appendix E

Reduction to $\mathcal{N} = 2$ SYM

The correlators including the displacement and stress tensor multiplet are known for the zero temperature case in $\mathcal{N} = 2$ from [38]. We should be able to obtain these from the $\mathcal{N} = 4$ correlators above. This is a consistency check for the correlators (2.12). In order to do that assume $n_r = \delta_{r,3}$. We make this choice because then we find

$$\Omega_{IJ} = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \quad \Rightarrow \quad \Omega_{ab} = -\mathrm{i}\epsilon_{ab}\ , \quad \Omega^{ab} = \mathrm{i}\epsilon^{ab} \quad \text{for } a, b = 1, 2\ . \tag{E.1}$$

Note furthermore that the field Φ^{IJ} in terms of the three complex scalars is given by

$$\Phi^{IJ} = \begin{pmatrix} 0 & \phi_1 & \phi_2 & \phi_3 \\ -\phi_1 & 0 & \bar{\phi}_3 & -\bar{\phi}_2 \\ -\phi_2 & -\bar{\phi}_3 & 0 & \bar{\phi}_1 \\ -\phi_3 & \bar{\phi}_2 & -\bar{\phi}_1 & 0 \end{pmatrix}\ . \tag{E.2}$$

We choose the scalar field of $\mathcal{N} = 2$ to be $\phi = \phi_1 = \Phi^{12}$. Therefore we are generally interested in the components with $I, J = 1, 2$ (or $I, J = 3, 4$ for the complex conjugate). We call these indices $a, b, \ldots = 1, 2$. Let us check that we indeed find the results of [38] step by step. Note that contrary to the cited paper we define the Wilson line to be in the x_3 not the x_0 direction.

Displacement Multiplet

Let us start with the operators in the displacement multiplet. We have the following fields:

$$\mathbb{O}^{IJ} = \Phi^{IJ} - \Omega^{IK}\bar{\Phi}_{KL}\Omega^{LJ}\ , \tag{E.3a}$$

$$\mathbb{\Lambda}^I_\alpha = \lambda^I_\alpha - \Omega^{IJ}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_J \,, \tag{E.3b}$$

$$\mathbb{D}^m = -\mathrm{i}F^{3m} - \frac{1}{2}\Omega_{IJ}D^m\Phi^{IJ} \,. \tag{E.3c}$$

Including the above Ω_{IJ} we thus find

$$\mathbb{O} = -\mathbb{O}^{12} = -\Phi^{12} + \Omega^{1K}\bar\Phi_{KL}\Omega^{L2} = \bar\phi - \phi \,, \tag{E.4a}$$

$$\mathbb{\Lambda}^a_\alpha = \lambda^a_\alpha - \mathrm{i}\epsilon^{ab}\sigma^3_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_b \,, \tag{E.4b}$$

$$\begin{aligned}\mathbb{D}^m &= -\mathrm{i}F^{3m} - \frac{1}{2}\Omega_{IJ}D^m\Phi^{IJ} = -\mathrm{i}F^{3m} - \frac{\mathrm{i}}{2}D^m\left(\Phi^{12} - \Phi^{21} + \Phi^{34} - \Phi^{43}\right) \\ &= -\mathrm{i}\left(F^{3m} + D^m\phi + D^m\bar\phi\right) \,. \end{aligned}\tag{E.4c}$$

This fits the expressions in [38]. For consistency consider the two-point functions

$$\left\langle \mathbb{O}^{IJ}\mathbb{O}^{KL}\right\rangle_W = \frac{\gamma}{3\tau^2}\left(\Omega^{IK}\Omega^{JL} - \Omega^{IL}\Omega^{JK} - \frac{1}{2}\Omega^{IJ}\Omega^{KL}\right) \,, \tag{E.5a}$$

$$\left\langle \mathbb{\Lambda}^I_\alpha\mathbb{\Lambda}^J_\beta\right\rangle_W = \frac{2\mathrm{i}\gamma}{3\tau^3}\Omega^{IJ}\epsilon_{\alpha\beta} \,, \tag{E.5b}$$

$$\left\langle \mathbb{D}^m\mathbb{D}^n\right\rangle_W = \frac{\gamma\delta^{mn}}{\tau^4} \,. \tag{E.5c}$$

Reducing as above we find

$$\left\langle \mathbb{O}\mathbb{O}\right\rangle_W = \frac{\gamma}{3\tau^2}\left(\Omega^{11}\Omega^{22} - \Omega^{12}\Omega^{21} - \frac{1}{2}\Omega^{12}\Omega^{12}\right) = \frac{\gamma}{6\tau^2} \,, \tag{E.6a}$$

$$\left\langle \mathbb{\Lambda}^a_\alpha\mathbb{\Lambda}^b_\beta\right\rangle_W = -\frac{2\gamma}{3\tau^3}\epsilon^{ab}\epsilon_{\alpha\beta} \,, \tag{E.6b}$$

$$\left\langle \mathbb{D}^m\mathbb{D}^n\right\rangle_W = \frac{\gamma\delta^{mn}}{\tau^4} \,. \tag{E.6c}$$

Thus we find that for $\gamma = 12B$ the correlators indeed match.

Stress Tensor Multiplet

For the stress tensors multiplet we are only interested in some of the fields in the $\mathcal{N} = 4$ multiplet. These are

$$\rho_{IJ} \to O_2 = \rho_{12} \,, \tag{E.7a}$$

$$f^{IJ}_{\alpha\gamma} \to H_\alpha{}^\beta = f^{12}_{\alpha\gamma}\epsilon^{\gamma\beta} \,, \tag{E.7b}$$

$$J^{IJ}_{\mu\,KL} \to j_\mu = J^{12}_{\mu\,34} + J^{34}_{\mu\,34} \,, \tag{E.7c}$$

$$\psi_{I\alpha}^{\ KL} \to \chi_\alpha^a = \Omega^{ab}\psi_{b\alpha}^{\ 34} \ , \tag{E.7d}$$

$$T_{\mu\nu} \to T_{\mu\nu} \ , \tag{E.7e}$$

$$\mathbb{O}^{IJ} \to \mathbb{O} = \mathbb{O}^{12} \ , \tag{E.7f}$$

$$\mathbb{A}_\alpha^I \to \mathbb{A}_\alpha^a \ . \tag{E.7g}$$

Here we defined the fields with the same letters as in [38].

Bulk-to-Defect Functions

Let us turn to the correlators which are actually included in the Ward identity. These are for $\mathcal{N} = 4$

$$\begin{aligned}\left\langle f_{\alpha\beta}^{IJ}(\vec{x},0)\mathbb{O}^{KL}(\vec{0},y)\right\rangle_W^{(T=0)} &= -\frac{3h}{2\pi}\frac{\sum_{m=0}^2\left(x_m\sigma^m\bar{\sigma}^3\right)_{\alpha\beta}}{x^3(y^2+x^2)} \\ &\quad\times\left(\Omega^{IK}\Omega^{JL} - \Omega^{IL}\Omega^{JK} - \frac{1}{2}\Omega^{IJ}\Omega^{KL}\right) \ , \end{aligned}\tag{E.8a}$$

$$\begin{aligned}\left\langle J_{3\ KL}^{IJ}(\vec{x},0)\mathbb{O}^{MN}(\vec{0},y)\right\rangle_W^{(T=0)} &= \frac{By}{\pi}\frac{1}{\sqrt{x^2}(y^2+x^2)^2}\Omega_{KL} \\ &\quad\times\left(\Omega^{IM}\Omega^{JN} - \Omega^{IN}\Omega^{JM} - \frac{1}{2}\Omega^{IJ}\Omega^{MN}\right) \ , \end{aligned}\tag{E.8b}$$

$$\begin{aligned}\left\langle J_{m\ KL}^{IJ}(\vec{x},0)\mathbb{O}^{MN}(\vec{0},y)\right\rangle_W^{(T=0)} &= \frac{B}{2\pi}\frac{x_m(y^2-x^2)}{x^3(y^2+x^2)^2}\Omega_{KL} \\ &\quad\times\left(\Omega^{IM}\Omega^{JN} - \Omega^{IN}\Omega^{JM} - \frac{1}{2}\Omega^{IJ}\Omega^{MN}\right) \ , \end{aligned}\tag{E.8c}$$

$$\begin{aligned}\left\langle \psi_{I\alpha}^{\ KL}(\vec{x},0)\mathbb{A}_\beta^J(\vec{0},y)\right\rangle_W^{(T=0)} &= \frac{h}{2\pi}\frac{\left[\mathrm{i}y\epsilon_{\alpha\beta} - \sum_{m=0}^2\left(x_m\sigma^m\bar{\sigma}^3\right)_{\alpha\beta}\right]}{} \\ &\quad\times\left(3\delta_I^J\Omega^{KL} - \delta_I^K\Omega^{JL} + \delta_I^L\Omega^{JK}\right)x(y^2+x^2)^2 \ , \end{aligned}\tag{E.8d}$$

$$\left\langle \rho^{IJ}(\vec{x},0)\mathbb{O}^{KL}(\vec{0},y)\right\rangle_W^{(T=0)} = 0 \ . \tag{E.8e}$$

This would yield for the $\mathcal{N} = 2$ case

$$\begin{aligned}\left\langle H_\alpha^{\ \beta}(\vec{x},0)\mathbb{O}(\vec{0},y)\right\rangle_W^{(T=0)} &= -\frac{3h}{2\pi}\left(\Omega^{11}\Omega^{22} - \Omega^{12}\Omega^{21} - \frac{1}{2}\Omega^{12}\Omega\Omega^{12}\right)\frac{\sum_{m=0}^2\left(x_m\sigma^m\bar{\sigma}^3\right)_\alpha^{\ \beta}}{x^3(y^2+x^2)} \\ &= -\frac{3h}{4\pi}\frac{\sum_{m=0}^2\left(x_m\sigma^m\bar{\sigma}^3\right)_\alpha^{\ \beta}}{x^3(y^2+x^2)} \ , \end{aligned}\tag{E.9a}$$

$$
\begin{aligned}
\left\langle j_3(\vec{x},0)\mathbb{O}(\vec{0},y)\right\rangle_W^{(T=0)} &= \frac{B}{\pi}\frac{\mathrm{i}y}{\sqrt{x^2}(y^2+x^2)^2}\Bigg[\Omega_{12}\left(\Omega^{31}\Omega^{42}-\Omega^{32}\Omega^{41}-\frac{1}{2}\Omega^{34}\Omega^{12}\right)\\
&\qquad +\Omega_{34}\left(\Omega^{31}\Omega^{42}-\Omega^{32}\Omega^{41}-\frac{1}{2}\Omega^{34}\Omega^{12}\right)\Bigg]\\
&= \frac{B}{\pi}\frac{y}{\sqrt{x^2}(y^2+x^2)^2}\ , \qquad\qquad \text{(E.9b)}\\
\left\langle j_m(\vec{x},0)\mathbb{O}(\vec{0},y)\right\rangle_W^{(T=0)} &= \frac{B}{2\pi}\frac{x_m(y^2-x^2)}{x^3(y^2+x^2)^2}\ , \qquad m=0,1,2\ , \qquad\qquad \text{(E.9c)}\\
\left\langle \chi^a_\alpha(\vec{x},0)\Lambda^b_\beta(\vec{0},y)\right\rangle_W^{(T=0)} &= -\frac{h}{2\pi}\Omega^{ac}\left(3\delta^b_c\Omega^{34}-\delta^3_c\Omega^{b4}+\delta^4_c\Omega^{b3}\right)\frac{\left[\mathrm{i}y\epsilon_{\alpha\beta}-\sum_{m=0}^2\left(x_m\sigma^m\bar{\sigma}^3\right)_{\alpha\beta}\right]}{x(y^2+x^2)^2}\\
&= \frac{3h}{2\pi}\epsilon^{ab}\frac{\left[\mathrm{i}y\epsilon_{\alpha\beta}-\sum_{m=0}^2\left(x_m\sigma^m\bar{\sigma}^3\right)_{\alpha\beta}\right]}{x(y^2+x^2)^2}\ , \qquad\qquad \text{(E.9d)}\\
\left\langle O_2(\vec{x},0)\mathbb{O}(\vec{0},y)\right\rangle_W^{(T=0)} &= 0\ . \qquad\qquad \text{(E.9e)}
\end{aligned}
$$

Appendix F

Anomaly calculation at zero coupling

In this appendix we calculate the anomaly of the phase transformations at zero coupling, (3.18). We closely follow the calculation of the anomaly as presented in Terning [6]. The spinor fields are transformed with a phase factor. This field redefinition must be accounted for in the path integral measure thus yielding a Jacobean which generally might be nonzero. We will show that the transformations (3.18) are anomaly-free in the free theory. As all interactions are turned off, we will furthermore see no effects of the gauge fields as one might expect for such a transformation.

For the purpose of this appendix we ignore R-symmetry indices. In the free theory the spinors of the tuple λ_I can get an overall phase factor thus preserving R-symmetry. The anomaly can then be calculated independently for each of the fields λ_I. It is therefore convenient to focus on a single spinor field λ. If the transformation of this single field is anomaly-free the free $\mathcal{N} = 4$ theory is also anomaly-free consequentially. From the Lagrangian only the spinor part is relevant for our considerations

$$\mathcal{L}_\lambda = \mathrm{i}\bar{\lambda}\bar{\sigma}^\mu \partial_\mu \lambda \ . \tag{F.1}$$

We then Wick rotate and compactify the time dimension. Recall that this yields Euclidean Pauli matrices $\sigma_E^\mu = (\mathrm{i}\mathbb{1}, \sigma^i)$ and $\bar{\sigma}_E^\mu = (-\mathrm{i}\mathbb{1}, -\sigma^i)$. Therefore we also have

$$\sigma_E^{\mu\nu} = \frac{\mathrm{i}}{4}\left(\sigma_E^\mu \bar{\sigma}_E^\nu - \sigma_E^\nu \bar{\sigma}_E^\mu\right) \qquad \text{and} \qquad \sigma_E^\mu \bar{\sigma}_E^\nu + \sigma_E^\nu \bar{\sigma}_E^\mu = -2\delta^{\mu\nu} \ . \tag{F.2}$$

For better readability we drop the subscript indicating Euclidean Pauli matrices in the following. To account for the periodicity conditions (KMS) we redefine the spinor fields with an explicit phase

$$\lambda \to \tilde{\lambda} = e^{\mathrm{i}\alpha\tau}\lambda \qquad \text{with} \quad \alpha = \frac{(2n+1)\pi}{\beta} \ , \quad \text{for } n \in \mathbb{Z} \ . \tag{F.3}$$

The fields with the tilde are the new fields respecting the thermal KMS conditions which are made explicit through the phases. The fields λ are periodic. The Lagrangian thus becomes

$$\mathcal{L}_\lambda \to \mathcal{L}_\lambda + \mathrm{i}\alpha\bar{\lambda}\bar{\sigma}^0\lambda \tag{F.4}$$

The original Lagrangian $\mathcal{L}_\lambda$ is the one depending on periodic fields only and thus preserves supersymmetry. The additional mass-like term breaks supersymmetry. The full quantum theory is given by the path integral over the exponentiated action. Therefore, we need to consider the Jacobean of the above transformation. This is where possibly anomalies might arise. We will follow the anomaly calculation as presented in [6][1].

Let us first consider which differential operators act as we will need them to regularize in a later step. For the fermionic fields we find

$$\frac{\mathrm{i}}{2}\tilde{\bar{\lambda}}\bar{\sigma}^\mu\partial_\mu\tilde{\lambda} = \frac{\mathrm{i}}{2}e^{-\mathrm{i}\alpha\tau}\bar{\lambda}\bar{\sigma}^\mu\partial_\mu e^{\mathrm{i}\alpha\tau}\lambda = \frac{\mathrm{i}}{2}\bar{\lambda}\left(\bar{\sigma}^\mu\partial_\mu + \mathrm{i}\bar{\sigma}^0\alpha\right)\lambda\,. \tag{F.5}$$

the calculation follows analogously for $\bar{\lambda}$. This leads to the definition of the slashed differential operators similar to the covariant derivative mentioned above, see Equation (3.20),

$$\not{\nabla} = \mathrm{i}\left(\bar{\sigma}^\mu\partial_\mu + \mathrm{i}\bar{\sigma}^0\alpha\right)\,, \qquad \bar{\not{\nabla}} = \mathrm{i}\left(\sigma^\mu\partial_\mu - \mathrm{i}\sigma^0\alpha\right)\,. \tag{F.6}$$

Following [6] it will prove convenient to build an operator which does not depend on a single Pauli Matrix. We therefore define

$$D_\lambda^2 = \bar{\not{D}}\not{D} = \mathrm{i}\left(\sigma^\mu\partial_\mu - \mathrm{i}\sigma^0\alpha\right)\mathrm{i}\left(\bar{\sigma}^\nu\partial_\nu + \mathrm{i}\bar{\sigma}^0\alpha\right) = \partial_\mu\partial^\mu + 4\sigma^{0i}\alpha\partial_i + \alpha^2\,, \tag{F.7}$$

$$\bar{D}_\lambda^2 = \not{D}\bar{\not{D}} = \partial_\mu\partial^\mu - 4\bar{\sigma}^{0i}\alpha\partial_i + \alpha^2\,. \tag{F.8}$$

We used that $\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = -2\delta^{\mu\nu}$ in Euclidean signature and the fact that the derivatives commute.

For completeness we check that the differential operators defined above are hermitian and have negative eigenvalues. We work in a basis with fermionic eigenfunctions $\chi_n(x)$ and $\eta_n(x)$. The completeness properties are

$$\sum_n \bar{\chi}_n(x)\chi_n(y) = \delta(x-y)\,, \quad \sum_n \bar{\eta}_n(x)\eta_n(y) = \delta(x-y)\,, \tag{F.9}$$

$$-\mathrm{i}\,\mathrm{tr}\int_{\mathcal{M}_\beta}\mathrm{d}^4x\bar{\chi}_n(x)\bar{\sigma}^0\chi_m(x) = \delta_{mn}\,, \quad -\mathrm{i}\,\mathrm{tr}\int_{\mathcal{M}_\beta}\mathrm{d}^4x\bar{\eta}_n(x)\bar{\sigma}^0\eta_m(x) = \delta_{mn}\,. \tag{F.10}$$

[1]Concretely Section 7.2.

We included a unity matrix by writing $\mathbb{1} = -\mathrm{i}\sigma^0$ to contract right and left spinors. We now focus on the operator D_λ^2 showing that it is hermitian by considering

$$\begin{aligned}
\left\langle \chi | D_\lambda^2 \eta \right\rangle &= -\mathrm{i}\,\mathrm{tr}\int_{\mathcal{M}_\beta} \mathrm{d}^4x \bar{\chi}_n(x)\bar{\sigma}^0 D_\lambda^2 \eta_n(x) \\
&= -\mathrm{i}\,\mathrm{tr}\int_{\mathcal{M}_\beta} \mathrm{d}^4x \bar{\chi}_n(x)\bar{\sigma}^0 \left(\partial_\mu\partial^\mu + 4\sigma^{0i}\alpha\partial_i + \alpha^2\right)\eta_n(x) \\
&= -\mathrm{i}\,\mathrm{tr}\int_{\mathcal{M}_\beta} \mathrm{d}^4x \overline{\left(\partial_\mu\partial^\mu + 4\sigma^{0i}\alpha\partial_i + \alpha^2\right)\chi_n(x)}\bar{\sigma}^0\eta_n(x) = \left\langle D_\lambda^2\chi | \eta\right\rangle \,. \qquad \text{(F.11)}
\end{aligned}$$

Negativity is shown by setting $\eta = \chi$

$$\begin{aligned}
\left\langle \chi | D_\lambda^2 \chi \right\rangle &= -\mathrm{itr}\int_{\mathcal{M}_\beta} \mathrm{d}^4x \bar{\chi}_n(x)\bar{\sigma}^0 D_\lambda^2 \chi_n(x) \\
&= \mathrm{itr}\int_{\mathcal{M}_\beta} \mathrm{d}^4x \bar{\chi}_n(x)\left(\bar{\sigma}^\mu\overleftarrow{\partial}_\mu - \mathrm{i}\bar{\sigma}^0\alpha\right)\sigma^0\left(\bar{\sigma}^\nu\partial_\nu + \mathrm{i}\bar{\sigma}^0\alpha\right)\chi_n(x) \\
&= -\mathrm{tr}\int_{\mathcal{M}_\beta} \mathrm{d}^4x \left|\left(\bar{\sigma}^\nu\partial_\nu + \mathrm{i}\bar{\sigma}^0\alpha\right)\chi_n(x)\right|^2 \leq 0 \,. \qquad \text{(F.12)}
\end{aligned}$$

The calculation for $\bar{D}_\lambda^2$ is analogous.
Equipped with this we can define the above introduced functions χ and η as eigenfunctions of the respective differential operators with eigenvalues $\mathfrak{c}_n$

$$D_\lambda^2\chi_n(x) = -\mathfrak{c}_n^2\chi_n(x)\,, \quad D_\lambda^2\eta_n(x) = -\mathfrak{c}_n^2\eta_n(x)\,. \qquad \text{(F.13)}$$

The spinor fields can thus be expanded in terms of those eigenfunctions

$$\lambda(x) = \sum_n a_n\chi_n(x)\,, \quad \bar{\lambda}(x) = \sum_n b_n\eta_n(x)\,. \qquad \text{(F.14)}$$

The coefficients a_n and b_n are Grassmann variables. Let us now write our phase factor as a transformation of these coefficients. Therefore consider the field λ. It transforms as follows

$$\lambda(x) = \sum_n a_n\chi_n(x) \to \tilde{\lambda}(x) = e^{\mathrm{i}\alpha\tau}\lambda(x) = \sum_n a_n'\chi_n(x) \qquad \text{(F.15)}$$

The redefined field is expanded in the same basis but with different coefficients. The new primed coefficients can be obtained from the original ones due to the completeness relation

$$\begin{aligned}
&\sum_n a_n'\chi_n(x) = \sum_n a_n e^{\mathrm{i}\alpha\tau}\chi_n(x) \\
&\quad\Rightarrow \quad a_n' = C_{nm}a_m\,, \quad C_{nm} = \int_{\mathcal{M}_\beta}\mathrm{d}^4x e^{\mathrm{i}\alpha\tau}\bar{\chi}_n(x)\chi_m(x)\,. \qquad \text{(F.16)}
\end{aligned}$$

In an analogous way we define the transformation matrices for the conjugate spinor

$$b'_n = \bar{C}_{nm} b_m \ , \quad \bar{C}_{nm} = \int_{\mathcal{M}_\beta} \mathrm{d}^4 x e^{-\mathrm{i}\alpha\tau} \bar{\eta}_n(x) \eta_m(x) \ . \tag{F.17}$$

The Jacobian is then given by the inverse determinant of the transformation matrices [6],

$$\mathrm{Jac}_\lambda = \det\left(C\bar{C}\right)^{-1} = \exp\left(-\mathrm{i}\int_{\mathcal{M}_\beta} \mathrm{d}^4 x\, \alpha A(x)\right) , \tag{F.18}$$

$$A(x) = \mathrm{tr}\sum_n (\bar{\chi}_n(x)\chi_n(x) - \bar{\eta}_n(x)\eta_n(x)) \ . \tag{F.19}$$

The function $A(x)$ can be evaluated by introducing a regulator function R which suppresses the highest eigenvalues. The regulator is chosen in such a way that its value and all its derivatives vanish at $x = \infty$. Furthermore it is regularized to unity at zero, $R(0) = 1$. An example for such a function is e^{-x}. Using the completeness relations we can write

$$A(x) = \lim_{\Lambda\to\infty}\lim_{y\to x} \mathrm{tr}\left[R\left(\frac{-D_\lambda^2}{\Lambda^2}\right) - R\left(\frac{-\bar{D}_\lambda^2}{\Lambda^2}\right)\right]\delta(x-y) \ . \tag{F.20}$$

We use the Fourier expansion of the Dirac delta function allowing us to express the differential operator in terms of Fourier momenta and Matsubara frequencies $\omega_n = {}^{n\pi}\!/\!_\beta$

$$\begin{aligned} A(x) = \lim_{\Lambda\to\infty} T\sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{tr}\Bigg[R\left(\frac{-(p^2+\omega_n^2+4\sigma^{0i}\hat{\alpha}p_i-\alpha^2)}{\Lambda^2}\right) \\ -R\left(\frac{-(p^2+\omega_n^2-4\bar{\sigma}^{0i}\hat{\alpha}p_i-\alpha^2)}{\Lambda^2}\right)\Bigg] \ . \end{aligned} \tag{F.21}$$

We can use a formal Taylor expansion of the regulator functions R. Expanding around $4\sigma^{0i}p_i = 0$ yields the exact formula

$$\begin{aligned} A(x) &= \lim_{\Lambda\to\infty} T\,\mathrm{tr}\sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{j=0}^{\infty} \frac{\hat{\alpha}^j}{j!}\left[\left(\frac{4\sigma^{0i}p_i}{\Lambda^2}\right)^j - \left(\frac{-4\bar{\sigma}^{0i}p_i}{\Lambda^2}\right)^j\right] R^{(j)}\left(\frac{-(p^2+\omega_n^2-\alpha^2)}{\Lambda^2}\right) \\ &= \lim_{\Lambda\to\infty} T\,\mathrm{tr}\sum_n \int \frac{\mathrm{d}^3 \hat{p}}{(2\pi)^3} \sum_{j=0}^{\infty} \frac{\hat{\alpha}^j\Lambda^3}{j!}\left[\left(\frac{4\sigma^{0i}\hat{p}_i}{\Lambda}\right)^j - \left(\frac{-4\bar{\sigma}^{0i}\hat{p}_i}{\Lambda}\right)^j\right] R^{(j)}\left(-p^2 - \frac{\omega_n^2-\alpha^2}{\Lambda^2}\right) \end{aligned}$$

In the second step we rescaled $p \to \hat{p}\Lambda$. This Taylor expansion is not a perturbative expansion. We need to consider all terms of the infinite sum, however the cut-off parameter Λ allows us to ignore certain higher order terms. The limit $\Lambda \to \infty$ in fact cancels all terms with $j \geq 4$. These must eventually converge to zero when the limit is taken. The term with $j = 0$ vanishes readily due to the intermediate minus sign. The odd terms with $j = 1,3$

vanish due to the even integration over $\mathrm{d}^3\hat{p}$. Therefore only the $j = 2$ term can contribute

$$A(x) = \lim_{\Lambda\to\infty} \mathrm{tr} T \sum_n \int \frac{\mathrm{d}^3\hat{p}}{(2\pi)^3} \frac{16\alpha^2\Lambda}{2} \left[\left(\sigma^{0i}\hat{p}_i\right)^2 - \left(\bar{\sigma}^{0i}\hat{p}_i\right)^2\right] R^{(2)} \left(-p^2 - \frac{\omega_n^2 - \alpha^2}{\Lambda^2}\right) . \quad \text{(F.22)}$$

Let us now look at the spinor traces. We need

$$\begin{aligned} \mathrm{tr}\sigma^{\mu\nu}\sigma^{\rho\sigma} &= \frac{1}{2}\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} + \epsilon^{\mu\nu\rho\sigma}\right) \qquad \Rightarrow \qquad \mathrm{tr}\sigma^{0i}\sigma^{0j} = \frac{1}{2}\delta^{ij} , \\ \mathrm{tr}\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma} &= \frac{1}{2}\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} - \epsilon^{\mu\nu\rho\sigma}\right) \qquad \Rightarrow \qquad \mathrm{tr}\bar{\sigma}^{0i}\bar{\sigma}^{0j} = \frac{1}{2}\delta^{ij} . \end{aligned} \quad \text{(F.23)}$$

Plugging this back into the anomaly calculation we find

$$A(x) = \lim_{\Lambda\to\infty} \mathrm{tr} T \sum_n \int \frac{\mathrm{d}^3\hat{p}}{(2\pi)^3} 2\alpha^2\Lambda \left[\hat{p}^2 - \hat{p}^2\right] R^{(2)} \left(-p^2 - \frac{\omega_n^2 - \alpha^2}{\Lambda^2}\right) = 0 . \quad \text{(F.24)}$$

Conclusively, there is no quantum anomaly arising from the transformations of the spinors (3.18) in the free ($g = 0$) $\mathcal{N} = 4$ SYM theory.

Appendix G

Thermal quantum anomaly at nonzero coupling

Similar to the previous Appendix F we consider the anomaly of the phase transformations (4.9). Here we consider the interacting theory which most importantly yields that also gauge interactions need to be included.
For the calculation of the quantum anomaly we again closely follow the derivation of Terning [6]. In a first step we show that a phase transformation acting on a single free scalar field is anomaly free. Using this result and the calculation of Appendix F we then consider $\mathcal{N} = 4$ showing that it is indeed anomaly-free.

G.1 Thermal anomaly for a free scalar

The calculations for the free scalar are analogous to the free spinor. We start with an action

$$S_\phi = -\int \mathrm{d}^4x \partial^\mu \phi^* \partial_\mu \phi \,. \tag{G.1}$$

We then Wick rotate and compactify the time dimension as above and introduce phases to the scalar fields

$$\phi \to \tilde{\phi} = e^{\mathrm{i}\hat{\alpha}\tau}\phi \qquad \text{where} \quad \hat{\alpha} = \frac{(2m)\pi}{\beta}\,, \quad \text{for } m \in \mathbb{Z}\,. \tag{G.2}$$

The scalars with an without tilde have to be periodic to satisfy the KMS conditions. Forconvenience let us again consider which differential operators act. For the scalar field we have

$$-\frac{1}{2}\tilde{\phi}^*\partial_\mu\partial^\mu\tilde{\phi} = -\frac{1}{2}e^{-\mathrm{i}\hat{\alpha}\tau}\phi^*\partial_\mu\partial^\mu e^{\mathrm{i}\hat{\alpha}\tau}\phi = -\frac{1}{2}\phi^*\left(\partial_\mu\partial^\mu + 2\mathrm{i}\hat{\alpha}\partial_\tau - \hat{\alpha}^2\right)\phi$$

$$\Rightarrow \qquad D_\phi^2 = \partial_\mu\partial^\mu + 2\mathrm{i}\hat{\alpha}\partial_\tau - \hat{\alpha}^2\,. \tag{G.3}$$

The calculation for ϕ^* is analogous. Again we check that these differential operators are hermitian and negative. Therefore define a basis of eigenfunctions $f_n(x)$, $g_n(x)$ with completeness properties

$$\sum_n f_n^*(x)f_n(y) = \delta(x-y)\ , \quad \sum_n g_n^*(x)g_n(y) = \delta(x-y)\ , \tag{G.4}$$

$$\int_{\mathcal{M}_\beta} \mathrm{d}^4x f_n^*(x)f_m(x) = \delta_{mn}\ , \quad \int_{\mathcal{M}_\beta} \mathrm{d}^4x g_n^*(x)g_m(x) = \delta_{mn}\ . \tag{G.5}$$

Hermitianity is then shown by considering

$$\begin{aligned}\left\langle f|D_\phi^2 g\right\rangle &= \int_{\mathcal{M}_\beta} \mathrm{d}^4x\, f_n^*(x)D_\phi^2 g_n(x) = \int_{\mathcal{M}_\beta} \mathrm{d}^4x\, f_n^*(x)\left(\partial_\mu\partial^\mu + 2\mathrm{i}\hat{\alpha}\partial_\tau - \hat{\alpha}^2\right)g_n(x)\\ &= \int_{\mathcal{M}_\beta} \mathrm{d}^4x\left(\partial_\mu\partial^\mu - 2\mathrm{i}\hat{\alpha}\partial_\tau - \hat{\alpha}^2\right)f_n^*(x)g_n(x)\\ &= \int_{\mathcal{M}_\beta} \mathrm{d}^4x\overline{\left(\partial_\mu\partial^\mu + 2\mathrm{i}\hat{\alpha}\partial_\tau - \hat{\alpha}^2\right)f_n(x)}g_n(x) = \left\langle D_\phi^2 f|g\right\rangle\ .\end{aligned} \tag{G.6}$$

Setting $f = g$ we find that the operator is negative definite

$$\begin{aligned}\left\langle f|D_\phi^2 f\right\rangle &= \int_{\mathcal{M}_\beta} \mathrm{d}^4x f_n^*(x)\left(\partial_\mu\partial^\mu + 2\mathrm{i}\hat{\alpha}\partial_\tau - \hat{\alpha}^2\right)f_n(x)\\ &= \int_{\mathcal{M}_\beta} \mathrm{d}^4x\left(-\partial^\mu + \mathrm{i}\hat{\alpha}\delta^{\mu 0}\right)f_n^*(x)\left(\partial_\mu + \mathrm{i}\hat{\alpha}\delta_{\mu 0}\right)f_n(x)\\ &= -\int_{\mathcal{M}_\beta} \mathrm{d}^4x\left|\left(\partial_\mu + \mathrm{i}\hat{\alpha}\delta_{\mu 0}\right)f_n(x)\right|^2 \le 0\ .\end{aligned} \tag{G.7}$$

Conclusively D_ϕ^2 is indeed hermitian and negative. By an analogous calculation the same holds for $\bar{D}_{\bar{\phi}}^2$. Once again f and g are now defined as eigenfunctions of the respective differential operators

$$D_\phi^2 f_n(x) = -\hat{\mathfrak{c}}_n^2 f_n(x)\ , \quad \bar{D}_{\bar{\phi}}^2 g_n(x) = -\hat{\mathfrak{c}}_n^2 g_n(x)\ . \tag{G.8}$$

The scalars can thus be expanded in terms of those eigenfunctions

$$\phi(x) = \sum_n a_n f_n(x)\ , \quad \bar{\phi}(x) = \sum_n b_n g_n(x)\ . \tag{G.9}$$

The coefficients a_n and b_n are now $\mathbb{C}$-numbers.
Let us now write our phase factor as a transformation of these coefficients

$$\phi(x) = \sum_n a_n f_n(x) \to e^{\mathrm{i}\hat{\alpha}\tau}\phi(x) = \sum_n a_n' f_n(x) \tag{G.10}$$

The redefined field is expanded in the same basis but with different coefficients. The new primed coefficients can be obtained from the original ones due to the completeness relation

$$\sum_n a'_n f_n(x) = \sum_n a_n e^{\mathrm{i}\hat{\alpha}\tau} f_n(x)$$

$$\Rightarrow \quad a'_n = C_{nm} a_m \; , \quad C_{nm} = \int_{\mathcal{M}_\beta} \mathrm{d}^4 x e^{\mathrm{i}\hat{\alpha}\tau} f^*_n(x) f_m(x) \; . \tag{G.11}$$

In an analogous way we define the transformation matrices

$$b'_n = \bar{C}_{nm} b_m \; , \quad \bar{C}_{nm} = \int_{\mathcal{M}_\beta} \mathrm{d}^4 x e^{-\mathrm{i}\hat{\alpha}\tau} g^*_n(x) g_m(x) \; . \tag{G.12}$$

The Jacobian is then again given by the inverse determinant of the transformation matrices [6],

$$\mathrm{Jac}_\phi = \det\left(C\bar{C}\right)^{-1} = \exp\left(-\mathrm{i}\int_{\mathcal{M}_\beta} \mathrm{d}^4 x\, \hat{\alpha}\hat{A}(x)\right) \; , \tag{G.13}$$

$$\hat{A}(x) = \sum_n \left(f^*_n(x) f_n(x) - g^*_n(x) g_n(x)\right) \; . \tag{G.14}$$

We introduce the same regulator function R which suppresses the highest eigenvalues as above. Using the completeness relations we can write

$$\begin{aligned}
\hat{A}(x) &= \lim_{\Lambda\to\infty} \sum_n \left[f^*_n(x) R\left(\frac{\mathfrak{c}^2_n}{\Lambda^2}\right) f_n(x) - g^*_n(x) R\left(\frac{\hat{\mathfrak{c}}^2}{\Lambda^2}\right) g_n(x)\right] \\
&= \lim_{\Lambda\to\infty} \sum_n \left[f^*_n(x) R\left(\frac{-D^2_\phi}{\Lambda^2}\right) f_n(x) - g^*_n(x) R\left(\frac{-\bar{D}^2_{\bar{\phi}}}{\Lambda^2}\right) g_n(x)\right] \\
&= \lim_{\Lambda\to\infty} \lim_{y\to x} \left[R\left(\frac{-D^2_\phi}{\Lambda^2}\right) - R\left(\frac{-\bar{D}^2_{\bar{\phi}}}{\Lambda^2}\right)\right] \delta(x-y) \; .
\end{aligned} \tag{G.15}$$

We can again use the Fourier representation of the delta function. The anomaly can thus be rewritten as

$$\begin{aligned}
\hat{A}(x) = \lim_{\Lambda\to\infty} T \sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \Bigg[& R\left(\frac{-(p^2 + \omega_n^2 + 2\mathrm{i}\hat{\alpha}\omega_n - \hat{\alpha}^2)}{\Lambda^2}\right) \\
& - R\left(\frac{-(p^2 + \omega_n^2 - 2\mathrm{i}\hat{\alpha}\omega_n - \hat{\alpha}^2)}{\Lambda^2}\right)\Bigg]
\end{aligned} \tag{G.16}$$

Let us now write down a formal power series expansion of R around ${}^{2\mathrm{i}\hat{\alpha}\omega_n}/_{\Lambda^2} \to 0$. All even term cancel readily as $(2\mathrm{i}\hat{\alpha}\omega_n)^{2j} = (-2\mathrm{i}\hat{\alpha}\omega_n)^{2j}$ for integers j. Therefore

$$\hat{A}(x) = \lim_{\Lambda\to\infty} T \sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{j=0}^{\infty} \frac{-2}{(2j+1)!} \left(\frac{2\mathrm{i}\hat{\alpha}\omega_n}{\Lambda^2}\right)^{2j+1} R^{(2j+1)}\left(\frac{-(p^2 + \omega_n^2 - \hat{\alpha}^2)}{\Lambda^2}\right) \; . \tag{G.17}$$

Recalling that $\omega_n \propto n$ it is easy to see that the above expression is zero. The sum runs over all integers n, positive and negative. However, the power series consists only over terms with odd powers and hence is uneven in n. Note that this is the same symmetry argument Terning uses to cancel all odd term in his calculation [6]. Only here instead of a Fourier integral we have a fourier series in the 0-th component. Henceforth

$$\hat{A}(x) = 0\,, \tag{G.18}$$

and the transformation of free Bosons is indeed anomaly-free.

G.2 Thermal anomaly in $\mathcal{N} = 4$ SYM

The above anomaly calculation provides a basis for the anomaly calculation of $\mathcal{N} = 4$ SYM theory. In principle all steps can be adapted. We only need to be careful about two changes which arise. $\mathcal{N} = 4$ is an interacting theory yielding that the partial derivative ∂_μ has to be promoted to a covariant derivative including the gauge field

$$D_\mu = \partial_\mu + \mathrm{i}A_\mu\,, \quad \bar{D}_\mu = \partial_\mu - \mathrm{i}A_\mu\,. \tag{G.19}$$

Additionally, we need to be careful about R-symmetry indices. Each scalar field and each fermionic field comes with a distinct phase, see Equation (4.9). For the purpose of calculating the anomaly they can be considered independently. It is not relevant that we have R-symmetry indices of $\mathfrak{su}(4)_R$ which eventually gets broken. We can keep these indices arbitrary and sum in a final step. Therefore, for the purposes of this chapter we do not assume that repeated R-symmetry indices are summed.

Consider the action of the differential operators on the scalar fields analogous to before

$$\begin{aligned}
-\frac{1}{2}\tilde{\bar{\Phi}}_{IJ} D_\mu D^\mu \tilde{\Phi}^{IJ} &= -\frac{1}{2} e^{-\mathrm{i}\hat{\alpha}_{IJ}\tau}\bar{\Phi}_{IJ} D_\mu D^\mu e^{\mathrm{i}\hat{\alpha}_{IJ}\tau}\Phi^{IJ} = -\frac{1}{2} e^{-\mathrm{i}\hat{\alpha}_{IJ}\tau}\bar{\Phi}_{IJ}\left(\partial_\mu + \mathrm{i}A_\mu\right)\left(\partial^\mu + \mathrm{i}A^\mu\right) e^{\mathrm{i}\hat{\alpha}_{IJ}\tau}\Phi^{IJ} \\
&= -\frac{1}{2} e^{-\mathrm{i}\hat{\alpha}_{IJ}\tau}\bar{\Phi}_{IJ}\left(\partial_\mu\partial^\mu + \mathrm{i}(\partial_\mu A^\mu) + 2\mathrm{i}A^\mu\partial_\mu + \frac{1}{2}\{A_\mu, A^\mu\}\right) e^{\mathrm{i}\hat{\alpha}_{IJ}\tau}\Phi^{IJ} \\
&= -\frac{1}{2}\bar{\Phi}_{IJ}\left(\partial_\mu\partial^\mu + \mathrm{i}(\partial_\mu A^\mu) + 2\mathrm{i}A^\mu\partial_\mu + \frac{1}{2}\{A_\mu, A^\mu\} + 2\mathrm{i}\hat{\alpha}_{IJ}(\partial_\tau + \mathrm{i}A_0) - \hat{\alpha}_{IJ}^2\right)\Phi^{IJ} \\
&= -\frac{1}{2}\bar{\Phi}_{IJ}\left(D_\mu D^\mu + 2\mathrm{i}\hat{\alpha}_{IJ} D_\tau - \hat{\alpha}_{IJ}^2\right)\Phi^{IJ}\,.
\end{aligned} \tag{G.20}$$

Similarly we find for the fermions

$$\begin{aligned}
D_\lambda^2 = \bar{\not{D}}\not{D} &= \mathrm{i}\left(\sigma^\mu\partial_\mu - \mathrm{i}\sigma^\mu A_\mu - \mathrm{i}\sigma^0\alpha_I\right)\mathrm{i}\left(\bar{\sigma}^\nu\partial_\nu + \mathrm{i}\bar{\sigma}^\nu A_\nu + \mathrm{i}\bar{\sigma}^0\alpha_I\right) \\
&= \partial_\mu\partial^\mu + \sigma^\mu\bar{\sigma}^\nu A_\mu A_\nu - \mathrm{i}\sigma^\mu\bar{\sigma}^\nu(\partial_\mu A_\nu) + 2\sigma^{\mu\nu}A_{[\mu}\partial_{\nu]} + 4\sigma^{0i}\alpha_I\partial_i + 2A_0\alpha_I + \alpha_I^2
\end{aligned} \tag{G.21}$$

$$= \partial_\mu \partial^\mu + \frac{1}{2}\{A_\mu, A_\nu\} - \sigma^{\mu\nu} F_{\mu\nu} + 2\sigma^{\mu\nu} A_{[\mu}\partial_{\nu]} + \mathrm{i}\partial \cdot A + 4\sigma^{0i}\alpha_I \partial_i + 2A_0\alpha_I + \alpha_I^2 \,. \tag{G.22}$$

Let us collect the differential operators we are thus interested in. They are

$$\begin{aligned} D_\Phi^2 &= \partial_\mu \partial^\mu + \mathrm{i}(\partial_\mu A^\mu) + 2\mathrm{i}A^\mu \partial_\mu + \frac{1}{2}\{A_\mu, A^\mu\} + 2\mathrm{i}\hat{\alpha}_{IJ}(\partial_\tau + \mathrm{i}A_0) - \hat{\alpha}_{IJ}^2 \\ &=: \partial^2 + A^2 - \hat{\alpha}_{IJ}^2 + 2\hat{\alpha}_{IJ}A_0 + \hat{\Delta} \end{aligned} \tag{G.23}$$

$$\begin{aligned} \bar{D}_\Phi^2 &= \partial_\mu \partial^\mu - \mathrm{i}(\partial_\mu A^\mu) - 2\mathrm{i}A^\mu \partial_\mu + \frac{1}{2}\{A_\mu, A^\mu\} - 2\mathrm{i}\hat{\alpha}_{IJ}(\partial_\tau + \mathrm{i}A_0) - \hat{\alpha}_{IJ}^2 \\ &=: \partial^2 + A^2 - \alpha_{IJ}^2 + 2\hat{\alpha}_{IJ}A_0 + \hat{\bar{\Delta}} \end{aligned} \tag{G.24}$$

$$\begin{aligned} D_\lambda^2 &= \partial_\mu \partial^\mu + \frac{1}{2}\{A_\mu, A^\mu\} - \sigma^{\mu\nu} F_{\mu\nu} + \mathrm{i}(\partial \cdot A) + 2\sigma^{\mu\nu} A_{[\mu}\partial_{\nu]} + 4\sigma^{0i}\alpha_I \partial_i + 2A_0\alpha_I + \alpha_I^2 \\ &=: \partial^2 + A^2 + \alpha_I^2 + 2A_0\alpha_I + \Delta \end{aligned} \tag{G.25}$$

$$\begin{aligned} \bar{D}_\lambda^2 &= \partial_\mu \partial^\mu + \frac{1}{2}\{A_\mu, A^\mu\} + \bar{\sigma}^{\mu\nu} F_{\mu\nu} - \mathrm{i}(\partial \cdot A) - 2\bar{\sigma}^{\mu\nu} A_{[\mu}\partial_{\nu]} - 4\bar{\sigma}^{0i}\alpha_I \partial_i + 2A_0\alpha_I + \alpha_I^2 \\ &=: \partial^2 + A^2 + \alpha_I^2 + 2A_0\alpha_I + \bar{\Delta} \,. \end{aligned} \tag{G.26}$$

We defined $A^2 = \frac{1}{2}\{A_\mu, A^\mu\}$. The terms which differ between the derivative without and with bar are called Δ and $\bar{\Delta}$, respectively, for the fermions. For the Bosons we include use $\hat{\Delta}$ and $\hat{\bar{\Delta}}$, respectively. These operators are slighly different for each of the fields Φ_{IJ} and for each λ_I due to the different phases that appear. In consequence we will compute the anomaly for each I, J independently. Further note that operators with a hat generally refer to the bosonic versions. The derivation on how the anomaly is computed in a basis of eigenfunctions is identical to the case of free fields,

$$\mathrm{Jac}_{\mathcal{N}=4} = \exp\left(-\mathrm{i}\int_0^\beta \mathrm{d}\tau \int \frac{\mathrm{d}^3x}{(2\pi)^3}\, \tau \left(\sum_{I,J} \hat{\alpha}_{IJ}\hat{A}_{IJ}(x) + \sum_{I=1}^4 \alpha_I A_I(x)\right)\right) , \tag{G.27}$$

$$A_{IJ}(x) = \mathrm{tr}\sum_n (\bar{f}_{n\,IJ}(x) f_{n\,IJ}(x) - \bar{g}_{n\,IJ}(x) g_{n\,IJ}(x)) \,, \tag{G.28}$$

$$\hat{A}_I(x) = \mathrm{tr}\sum_n (\bar{\chi}_{n\,IJ}(x)\chi_{n\,IJ}(x) - \bar{\eta}_{n\,IJ}(x)\eta_{n\,IJ}(x)) \,. \tag{G.29}$$

The traces here are taken with respect to the color group $\mathrm{SU}(N)$ and - for the fermionic case only - over spinor matrices.

The regulator function $R(z)$ can be introduced in the same manner as in the calculations for the free fields. Likewise we write the resulting Dirac delta function in the Fourier representation and take a formal power series expansion of R around the respective $\Delta = 0$.

In the bosonic case this yields substituting again $p \to \hat{p}\Lambda$

$$\hat{A}_{IJ}(x) = \lim_{\Lambda\to\infty} T \sum_n \int \frac{\mathrm{d}^3\hat{p}}{(2\pi)^3} \sum_{j=0}^{\infty} \frac{\Lambda^3}{j!} \mathrm{tr} \left[\left(\frac{\hat{\Delta}}{\Lambda^2} \right)^j - \left(\frac{\hat{\bar{\Delta}}}{\Lambda^2} \right)^j \right] \times$$
$$\times R^{(j)} \left(-p^2 - \frac{\omega_n^2 - A^2 - \hat{\alpha}_{IJ}^2 + 2\hat{\alpha}_{IJ} A_0}{\Lambda^2} \right) . \tag{G.30}$$

The trace, as mentioned above, is taken with respect to the SU(N) color group. The $j = 0$ term vanishes readily. In the limit $\Lambda \to \infty$ the highest term in $\hat{\Delta}$ has $\mathcal{O}(\Lambda)$ and hence all terms with $j \geq 4$ also vanish. We then only need to consider $j = 1, 2, 3$. It is convenient to choose a gauge $\partial \cdot A = 0$ as the final result cannot depend on the gauge choice. Then we find

$$\hat{A}_{IJ}(x) = \lim_{\Lambda\to\infty} T \sum_n \int \frac{\mathrm{d}^3\hat{p}}{(2\pi)^3} \sum_{j=1}^{3} \frac{\Lambda^3}{j!} \mathrm{tr} R^{(j)} \left(-p^2 - \frac{\omega_n^2 - A^2 - \hat{\alpha}_{IJ}^2 + 2\hat{\alpha}_{IJ} A_0}{\Lambda^2} \right) \times \tag{G.31}$$
$$\times \left[\left(\frac{2\mathrm{i}\hat{\alpha}_{IJ}\omega_n + 2\mathrm{i}A_0\omega_n + 2\mathrm{i}\Lambda A_i p^i}{\Lambda^2} \right)^j - \left(\frac{-2\mathrm{i}\hat{\alpha}_{IJ}\omega_n - 2\mathrm{i}A_0\omega_n - 2\mathrm{i}\Lambda A_i p^i}{\Lambda^2} \right)^j \right] = 0 .$$

The term for $j = 2$ is zero readily while for $j = 1, 3$ the even sum over all integers is taken over an odd function of n thus vanishing or the three-momentum integral vanishes due to an odd power of p_i in the expansion. Recall that we used an identical argument to show that the bosonic transfomrations are anomaly-free in the free model.

In the fermionic case we assume the trace to be acting on the color group and on the Pauli matrices. We find

$$\begin{aligned}
A_I(x) &= \lim_{\Lambda\to\infty} \mathrm{tr} T \sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{j=1}^{3} \frac{1}{j!} \left[\left(\frac{\Delta}{\Lambda^2} \right)^j - \left(\frac{\bar{\Delta}}{\Lambda^2} \right)^j \right] R^{(j)} \left(- \frac{p^2 + \omega_n^2 - A^2 - \alpha_I^2 + 2A_0\alpha_I}{\Lambda^2} \right) \\
&= \lim_{\Lambda\to\infty} \mathrm{tr} T \sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_{j=1}^{3} \frac{1}{j!} \left[\left(\frac{-\sigma^{\mu\nu} F_{\mu\nu} + 2\sigma^{\mu\nu} A_{[\mu} P_{\nu]} + 4\sigma^{0i}\alpha_I p_i}{\Lambda^2} \right)^j \right. \\
&\quad \left. - \left(\frac{+\sigma^{\mu\nu} F_{\mu\nu} - 2\sigma^{\mu\nu} A_{[\mu} P_{\nu]} - 4\sigma^{0i}\alpha_I p_i}{\Lambda^2} \right)^j \right] R^{(j)} \left(- \frac{p^2 + \omega_n^2 - A^2 - \alpha_I^2 + 2A_0\alpha_I}{\Lambda^2} \right) \\
&= \lim_{\Lambda\to\infty} \mathrm{tr} T \sum_n \int \frac{\mathrm{d}^3 \hat{p}}{(2\pi)^3} \sum_{j=1}^{3} \frac{\Lambda^3}{j!} R^{(j)} \left(-p^2 + \frac{-\omega_n^2 + A^2 + \alpha_I^2 + 2A_0\alpha_I}{\Lambda^2} \right) \Bigg[\\
&\quad \left(\frac{-\sigma^{\mu\nu} F_{\mu\nu} + 2\sigma^{i0} \left(A_i\omega_n - \Lambda A_0 \hat{p}_i \right) + 2\Lambda\sigma^{ij} A_{[i}\hat{p}_{j]} + 4\sigma^{0i}\alpha_I \Lambda \hat{p}_i}{\Lambda^2} \right)^j
\end{aligned} \tag{G.32}$$

$$-\left(\frac{\bar{\sigma}^{\mu\nu}F_{\mu\nu} + 2\bar{\sigma}^{i0}\left(A_i\omega_n - \Lambda A_0\hat{p}_i\right) + 2\Lambda\bar{\sigma}^{ij}A_{[i}\hat{p}_{j]} - 4\bar{\sigma}^{0i}\alpha_I\Lambda\hat{p}_i}{\Lambda^2}\right)^j\Bigg] .$$

In the last step we substituted $p \to \Lambda\hat{p}$. Any terms with $j \geq 4$ vanish readily in the $\Lambda \to \infty$ limit. Following [6] $\mathrm{tr}\sigma^{\mu\nu} = 0$. Therefore only terms with $j = 2, 3$ contribute. From the free model we know that for $j = 2$ we will have to consider contractions

$$\mathrm{tr}\sigma^{\mu\nu}\sigma^{\rho\sigma} = \frac{1}{2}\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} + \epsilon^{\mu\nu\rho\sigma}\right) \tag{G.33}$$

$$\mathrm{tr}\bar{\sigma}^{\mu\nu}\bar{\sigma}^{\rho\sigma} = \frac{1}{2}\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} - \epsilon^{\mu\nu\rho\sigma}\right) . \tag{G.34}$$

Here, only the epsilons can lead to terms which do not cancel each other due to symmetry between the two terms we substract. Furthermore terms linear in p_i will readily cancel in the integration and analogously all linear terms in $P_0 = \omega_n$ cancel against the sum. For simplicity we ignore the dependence on Λ and consider 3-momenta p_i with $P_\mu = (\omega_n, p_i)$. Hence for the $j = 2$ term we find a contribution

$$\begin{aligned}
\mathrm{tr}\Delta^2 &= \mathrm{tr}\left[-\sigma^{\mu\nu}F_{\mu\nu} + 2\sigma^{\mu\nu}A_{[\mu}P_{\nu]} + 4\sigma^{0i}\alpha_I p_i\right]^2 \\
&= \mathrm{tr}\Bigg[\sigma^{\mu\nu}\sigma^{\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - \sigma^{\mu\nu}\sigma^{\rho\sigma}F_{\mu\nu}A_{[\rho}\hat{P}_{\sigma]} - \sigma^{\mu\nu}\sigma^{\rho\sigma}A_{[\mu}\hat{P}_{\nu]}F_{\rho\sigma} \\
&\qquad - 4\sigma^{\mu\nu}\sigma^{0j}F_{\mu\nu}\alpha_I p_j - 4\sigma^{0i}\sigma^{\rho\sigma}\alpha_I p_i F_{\rho\sigma} + 4\sigma^{\mu\nu}\sigma^{\rho\sigma}A_{[\mu}\hat{P}_{\nu]}A_{[\rho}\hat{P}_{\sigma]} \\
&\qquad + 8\sigma^{\mu\nu}\sigma^{0j}A_{[\mu}\hat{P}_{\nu]}\alpha_I p_j + 8\sigma^{0i}\sigma^{\rho\sigma}\alpha_I p_i A_{[\rho}\hat{P}_{\sigma]} + 16\sigma^{0i}\sigma^{0j}\alpha_I\alpha_I p_i p_j\Bigg] \qquad \text{(G.35)} \\
&= \mathrm{tr}\left(\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + 4\epsilon^{\mu\nu\rho\sigma}A_{[\mu}\hat{P}_{\nu]}A_{[\rho}\hat{P}_{\sigma]}\right) + \ldots \\
&= \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \ldots . \qquad \text{(G.36)}
\end{aligned}$$

The dots represent terms which eventually will cancel against respective terms of $\bar{\Delta}$. The last term in the second line with the anti-symmetriced indices also canceled. Either the term depends on a linear momentum canceling against the single integral or the two indices of the momenta are identical thus canceling against the totally anti-symmetric epsilon. Similarly we have

$$\mathrm{tr}\bar{\Delta}^2 = \mathrm{tr}\left[\bar{\sigma}^{\mu\nu}F_{\mu\nu} - 2\bar{\sigma}^{\mu\nu}A_{[\mu}P_{\nu]} - 4\bar{\sigma}^{0i}\alpha_I p_i\right]^2 = -\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \ldots . \tag{G.37}$$

We then find

$$A_I(x) = \lim_{\Lambda\to\infty}\mathrm{tr}T\sum_n\int\frac{\mathrm{d}^3\hat{p}}{(2\pi)^3}\frac{\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}}{2\Lambda}\,R^{(2)}\left(-\hat{p}^2 - \frac{\omega_n^2 - A^2 - \alpha_I^2}{\Lambda^2}\right) . \tag{G.38}$$

The first term is the instanton term and yields the difference between chiral and anti-chrial spinors in the action, $n_\lambda - n_{\bar{\lambda}}$ [6]. Due to the thermal theory, the limit $\Lambda \to \infty$ eventually cancels the instanton contribution. This difference arose because instead of the integral over 4-momenta we only integrated spatial 3-momenta and had a frequency sum. This leads to Λ^3 in the numerator instead of the Λ^4 dependence found in the zero temperature case. This is the reason for the instanton cancellation in the finite temperature theory.
Now let us look at the $j = 3$ term. Consider again the dependence on Λ. Due to the pre-factor Λ^3 all terms from the expansion in j that go like $\Lambda^{\geq 4}$ in the denominator can be ignored as they are eventually zero in the limit we consider. In turn this yields that only terms with a dependence on p_i can give non-zero results. On the other hand, each of these terms will be uneven in some $\hat{p}_i$ and hence cancel against the symmetric integral. Therefore no terms with $j = 3$ contribute.
Conclusively also the fermionic transformations introduced (3.18) are anomaly-free and we find that the Jacobean of the phase transformations (4.9) is

$$\mathrm{Jac}_{\mathcal{N}=4} = \mathbb{1} \,. \tag{G.39}$$

Appendix H

Self-energy from phase redefinitions

When considering $\mathcal{N} = 4$ SYM at finite temperature we found it convenient to write the action in terms of periodic fields only. By doing this the terms breaking supersymmetry can be identified readily. In Section 4.1 we considered these phase transformations and found further conditions on the phases from the Yukawa interactions. This yielded the transformations in (4.15). It is straight forward to see that R-symmetry is broken by these phases. The dependence on three different integers creates an ambiguity. A priori these integers can take any value. For some specific choices different amounts of R-symmetry can be preserved. Some examples are reviewed in Table 4.1.

In this appendix we consider restriction on these choices by a dynamic calculation of self-energies. We present the respective considerations to the best of our understanding. We emphasize, however, that the steps in this appendix are novel and have not yet been fully verified.

The scalar self-energy for $\mathcal{N} = 4$ is known at leading order in a weak coupling expansion [123, 131]. In the Hard Thermal Loop (HTL) limit it yields the thermal mass $m_\infty^2 = \lambda T^2$ which is identical for scalars, vectors and spinors. Looking at our ansatz with only periodic fields and correction terms, we should be able to obtain an identical result for the self-energy. We will eventually show that the conditions imposed by the phase redefinitions yield an inconsistency. It turns out to not be possible to find the expected result. Therefore, a different approach is then proposed in Appendix I.

Although the ansatz with the phase definitions was introduced in straight forward manner we find said inconsistency. This strongly indicates that at least one of the steps we took or present here is not fully consistent. At the current state of the art it is difficult to identify this problematic step precisely.

Consider a perturbative expansion around the 't-Hooft coupling $\lambda = g^2 N$. The counting in Feynman diagrams is simplified by the replacements in (3.1)

$$A_\mu \to g A_\mu\,, \quad \lambda_I \to g\lambda_I\,, \quad \Phi_{IJ} \to g\Phi_{IJ}\,, \quad \bar{c} \to g\bar{c}\,. \tag{H.1}$$

Then all propagators are independent of the coupling while all three-point vertices are proportional to g and four-point vertices are proportional to g^2. Note that when comparing to the Feynman rules presented in Appendix J below, the old definitions were used. Considering the phases we use the field redefinitions of Equation (4.9). This yielded the mass terms presented in (4.19)

$$\begin{aligned}\mathcal{L}_{\text{mass}} = \operatorname{tr}\Big[&- \pi T(2n_1+1)\bar{\lambda}^1\bar{\sigma}^0\lambda_1 - \pi T(2n_2+1)\bar{\lambda}^2\bar{\sigma}^0\lambda_2 - \pi T(2n_3+1)\bar{\lambda}^3\bar{\sigma}^0\lambda_3 \\ &- \pi T(2n_4+1)\bar{\lambda}^4\bar{\sigma}^0\lambda_4 + 4\mathrm{i}\pi T\hat{n}_1\left(\phi^1\partial^0\bar{\phi}_1 - \partial_0\phi^1\bar{\phi}_1\right) \\ &+ 4\mathrm{i}\pi T\hat{n}_2\left(\phi^2\partial^0\bar{\phi}_2 - \partial_0\phi^2\bar{\phi}_2\right) + 4\mathrm{i}\pi T\hat{n}_3\left(\phi^3\partial^0\bar{\phi}_3 - \partial_0\phi^3\bar{\phi}_3\right) \\ &+ 8\pi^2T^2\hat{n}_1^2\phi^1\bar{\phi}_1 + 8\pi^2T^2\hat{n}_2^2\phi^2\bar{\phi}_2 + 8\pi^2T^2\hat{n}_3^2\phi^3\bar{\phi}_3\Big] , \\ &n_1, n_2, n_3, n_4, \hat{n}_1, \hat{n}_2, \hat{n}_3 \in \mathbb{Z} .\end{aligned} \tag{H.2}$$

The restrictions (4.14) yield that

$$n_1+n_2+n_3+n_4 = -2 , \quad \hat{n}_1 = n_1+n_2+1 , \quad \hat{n}_2 = n_1+n_3+1 , \quad \hat{n}_3 = n_1+n_4+1 . \tag{H.3}$$

Then the transformations are in fact the ones presented in (4.15). We kept all phases independent as this allows to write the following equations more conveniently. The phases were replaced by their expression in terms of temperature and integers to make the dependence on T explicit.

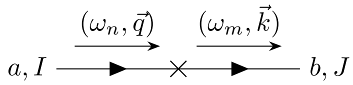
H.1 Feynman rules for new mass terms

The new terms of the action are mass-like depending on the different phases. They yield new building blocks for the Feynman diagram calculation. For both scalars and fermions we use the convention of a cross along the propagator to indicate the insertion of the respective propagator . For the Feynman rule consider for example

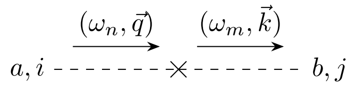
$$\begin{aligned}&4\mathrm{i}\pi T\hat{n}_1\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3x\left(\phi_1^a\partial^0\bar{\phi}_1^a - \partial_0\phi_1^a\bar{\phi}_1^a\right) \\ &= 4\mathrm{i}\pi T\hat{n}_1\delta^{ab}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3xT^2\sum_{m,n}\int \mathrm{d}^3q\int \mathrm{d}^3k\times \\ &\quad\times\left(e^{\mathrm{i}\hat{\omega}_n\tau+\mathrm{i}\vec{q}\vec{x}}\partial^0e^{-\mathrm{i}\hat{\omega}_m\tau-\mathrm{i}\vec{k}\vec{x}} - e^{-\mathrm{i}\hat{\omega}_m\tau-\mathrm{i}\vec{k}\vec{x}}\partial_0e^{\mathrm{i}\hat{\omega}_n\tau+\mathrm{i}\vec{q}\vec{x}}\right)\phi_1^a(q,n)\bar{\phi}_1^b(k,m) \\ &= 8\pi T\hat{n}_1\delta^{ab}(2\pi)^3\sum_{m,n}\int \mathrm{d}^3q\int \mathrm{d}^3k\delta_{m,n}\delta(\vec{q}-\vec{k})\hat{\omega}_m\phi_1^a(q,n)\bar{\phi}_1^b(k,m)\end{aligned} \tag{H.4}$$

$$= T^2 \sum_{m,n} \int \mathrm{d}^3 q \int \mathrm{d}^3 k \left[(2\pi)^3 16\pi^2 T^2 \hat{n}_1 m \delta^{ab} \delta_{m,n} \delta(\vec{q} - \vec{k})\right] \phi_1^a(q,n) \bar{\phi}_1^b(k,m) \, .$$

From this the Feynman rule can be read off. Overall we find



$$= -(2\pi)^3 2\pi T (2n_I + 1) \delta^{ab} \delta^I_J \delta_{m,n} \delta(\vec{q} - \vec{k}) \, , \tag{H.5}$$



$$= (2\pi)^3 16\pi^2 T^2 \left(\hat{n}_i^2 + \hat{n}_i m\right) \delta^{ab} \delta^i_j \delta_{m,n} \delta(\vec{q} - \vec{k}) \, . \tag{H.6}$$

For a convenient phrasing we will refer to these diagrams as ”phase insertion” from now on.
Note that these insertions are of order $g^0 T^2$ and $g^0 T^1$ for bosons and fermions, respectively. Henceforth, in a perturbative expansion these ”vertices” do not increase the order of the coupling. An expansion around small temperatures then seems suggestive although we will see below that the above insertions yield IR divergences and hence a resummation is in any case need.

H.2 Leading order self-energy from original $\mathcal{N} = 4$ SYM

Consider first the terms that are equivalently obtained in a zero temperature theory. This meant that we exclude the phase insertions. The relevant self-energy are derived for the gauge and the scalar in Section 5.3.1. The final result for a theory at finite temperature is given in (5.37)

$$\Sigma = -8\lambda \sum\!\!\!\!\!\!\int_Q (\delta_B - \delta_F) \frac{1}{Q^2} + 2\lambda P^2 \sum\!\!\!\!\!\!\int_Q (\delta_B - 2\delta_F) \frac{1}{Q^2 (Q+P)^2} \, , \tag{H.7}$$

$$\begin{aligned} \Pi_{\mu\nu} = & -8\lambda \eta_{\mu\nu} \sum\!\!\!\!\!\!\int_Q (\delta_B - \delta_F) \frac{1}{Q^2} + 2\lambda \left(P^2 \eta_{\mu\nu} - P_\mu P_\nu\right) \sum\!\!\!\!\!\!\int_Q (\delta_B - 2\delta_F) \frac{1}{Q^2 (Q+P)^2} \\ & - 4\lambda P_\mu P_\nu \sum\!\!\!\!\!\!\int_Q (\delta_B - \delta_F) \frac{1}{Q^2 (Q+P)^2} + 16\lambda \sum\!\!\!\!\!\!\int_Q (\delta_B - \delta_F) \frac{Q_\mu Q_\nu}{Q^2 (Q+P)^2} \, . \end{aligned} \tag{H.8}$$

Note that these results where derived considering that the fermions are anti-periodic. In the above equations δ_B thus indicated the need to substitute in even (periodic) frequencies while odd (anti-periodic) frequencies are indicated by δ_F. The periodicity condition of the fermions is now absorbed in the phase leading to the new correction terms. For all calculations we hence only consider periodic fermions. Therefore $\delta_B = \delta_F$ and the self-energies simplify to

$$\Sigma = -2\lambda\, P^2 \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2}\ , \tag{H.9}$$

$$\Pi_{\mu\nu} = -2\lambda\left(P^2\eta_{\mu\nu} - P_\mu P_\nu\right) \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2}\ . \tag{H.10}$$

The self-energy of the fermion is given in [20] and shown to depend on the same integral

$$\Sigma_\lambda^{\alpha\dot\alpha} \propto \lambda \not{P}^{\alpha\dot\alpha} \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2}\ . \tag{H.11}$$

Consider the remaining integral. We split it into two pieces

$$\sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2} = \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2}\bigg|_{T=0} + \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2}\bigg|_{T\neq 0} \tag{H.12}$$

The first term in the above is the zero temperature term. It is know for example from [62]

$$\sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2}\bigg|_{T=0} = \int \frac{\mathrm{d}^{4-2\varepsilon}Q}{(2\pi)^{4-2\varepsilon}} \frac{1}{Q^2(Q+P)^2} = \frac{\Gamma(\varepsilon)}{(4\pi)^2}\ . \tag{H.13}$$

We used dimensional regularization in $d = 4-2\varepsilon$ dimensions to determine the divergence explicitly.
We are then left with the thermal contribution of the remaining integral. In thermal field theories one usually uses a Hard Thermal Loop (HTL) approximation. It is straight forward to also employ this approximation here. Note that in thermal $\mathcal{N}=4$ there are no known phase transitions. The constraints in the HTL approximation are thus the same as the ones for the general theory and it is legitimate to pursue this ansatz. Therefore we assume that the external momenta are soft $p \sim gT$. As is known, the leading contribution to the integral will come from hard internal momenta $q \sim T$ [47, 100, 119]. Therefore we can conveniently expand around $p/q << 1$. Following [100] this yields that all dependence on external momenta in the numerator can be neglected. In this case we find

$$\Sigma = -2\lambda\, P^2 \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2} \to 0\ , \tag{H.14}$$

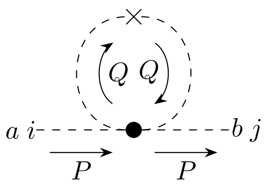
$$\Pi_{\mu\nu} = -2\lambda\left(P^2\eta_{\mu\nu} - P_\mu P_\nu\right) \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2} \to 0\ , \tag{H.15}$$

$$\Sigma_\lambda^{\alpha\dot\alpha} \propto \lambda \not{P}^{\alpha\dot\alpha} \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2} \to 0\ . \tag{H.16}$$

Hence the terms coming from the zero temperature theory can all be neglected in the HTL approach. In other words, the terms yielding thermal masses are those with a quadratic UV divergence at zero temperature which the above integrals do not posses.
What is next, we will consider the phase insertion to the scalar self-energy. We eventually find that the scalar fields get a mass due to an IR resummation. This is in fact analogous to the resummation needed in general thermal theories. We discussed the main effects for ϕ^4 theory in Section 5.1.

H.3 IR resummation

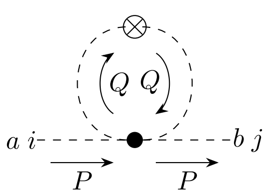
We need to further consider the self-energy diagrams with the internal propagators dressed with phase transitions. Let us first consider the scalar tadpole as it is the most straight forward example. This means including the new mass terms (H.5). We consider here the scalar tadpole diagram which contributes to the self energy. The mass insertion is denoted by a cross. We have at order T^2



$$= 16\pi^2 T^2 \lambda \delta^{ab} \delta_{ij} T \sum_{m=-\infty}^{\infty} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \sum_{k=1}^{3} \frac{\left(\hat{n}_k^2 + \hat{n}_k m\right)}{\left((2\pi \mathrm{i} m T)^2 - q^2\right)^2} \tag{H.17}$$

$$= 16\pi^2 T^2 \lambda (\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2) \delta^{ab} \delta_{ij} T \sum_{m=-\infty}^{\infty} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{\left((2\pi \mathrm{i} m T)^2 - q^2\right)^2} \, . \tag{H.18}$$

The piece anti-symmetric in m readily is evaluated to zero. Compare the remaining piece with the expression in (5.7). The sum and integral are precisely the same and we find an infrared divergence for the zero mode. The zero mode namely behaves as $1/q^2$ creating the IR divergence. Therefore a resummation is needed. All diagrams that contribute have the form of the one in Figure 5.1 but with crosses (phase insertions) instead of additional bubbles. The resummed version then is (the circle around the cross indicating the resummed result)



$$= \lambda \delta^{ab} \delta_{ij} T \sum_{m=-\infty}^{\infty} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \sum_{N=1}^{\infty} \frac{M^{N-1}}{\left(\omega_m^2 - q^2\right)^N}$$

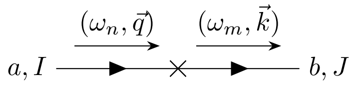
$$= \lambda \delta^{ab} \delta_{ij} T \sum_{m=-\infty}^{\infty} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{\omega_m^2 - q^2 + M^2} \, , \tag{H.19}$$

$$
\begin{aligned}
M^2 &= 16\pi^2 T^2 (\hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2) \\
&= 16\pi^2 \left(5n_1^2 + n_2^2 + n_3^2 + (n_1 + n_2 + n_3 + 2)^{\,2} - 1\right) \,.
\end{aligned}
$$

In the last step we used the constraints for the $\hat{n}_i$ and the n_I as given above in (H.3). The resummation was, as indicated, carried out in analogy to (5.9). The difference here, however, is that the new mass M is not dependent on the coupling and therefore we cannot simply expand around it. Recall that in the usual thermal IR resummation such an expansion yields diagrams of order $\lambda^{3/2}$ in the perturbative expansion. In fact our calculation shows that we have to consider all scalars in the diagrams we compute as being massive with the mass M.

What is more, let us consider the fermionic loop. Also here similar insertions must be considered potentially leading to the same kind of IR divergence. Note, however, that as the node in (H.5) is inserted to an internal propagator all indices are summed. This yielded the sum $\sum_{k=1}^{3}\left(\hat{n}_k^2 + \hat{n}_k m\right)$ in case of the scalar above. For the fermions we find schematically

$$\sum_{I=1}^{4}$$



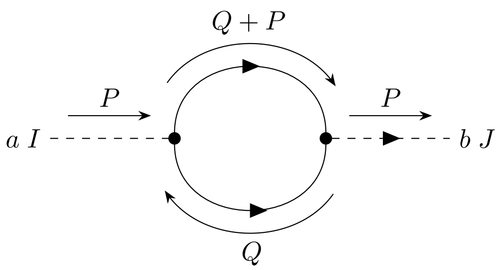
$$
\begin{aligned}
&= -\sum_{I=1}^{4} (2\pi)^3 N \alpha_I \delta^{ab} \delta^I_J \delta_{m,n} \delta(\vec{q} - \vec{k}) \\
&= -(2\pi)^3 2N \underbrace{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}_{=0} \delta^{ab} \delta^I_J \delta_{m,n} \delta(\vec{q} - \vec{k}) = 0 \,. \qquad \text{(H.20)}
\end{aligned}
$$

In the last step the Constraint (4.14) coming from the constraints on the phases was used. Therefore we see that there are no relevant mass terms for the fermions. Only the scalars become massive.

H.4 Self-energy with massive scalars

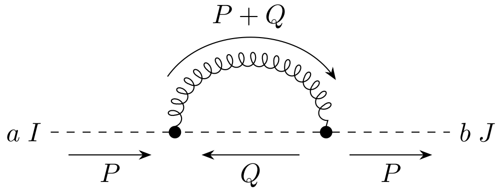
We can then reconsider the scalar self-energy. In the following all scalar propagators are implicitly assumed to be massive with $M^2 = 16\pi^2 \left(5n_1^2 + n_2^2 + n_3^2 + (n_1 + n_2 + n_3 + 2)^{\,2} - 1\right)$, $n_i \in \mathbb{Z}$. The diagrams are the same as in (5.32). The mass is included readily

$$(\Sigma_{1,f})^{ab}_{IJ} := g^2$$



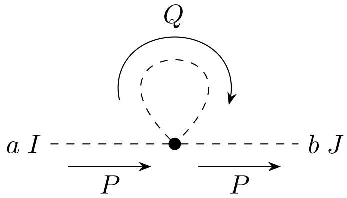
$$= 4\lambda\delta^{ab}\delta_{IJ} \sum\!\!\!\!\!\!\!\int_Q \left[\frac{2}{Q^2} - \frac{P^2}{Q^2(Q+P)^2}\right] , \tag{H.21a}$$

$$(\Sigma_{1,sv})^{ab}_{IJ} := g^2$$



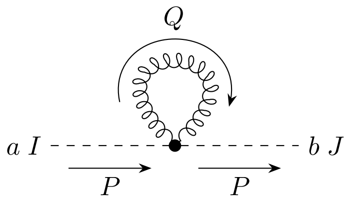
$$= 2\lambda\delta^{ab}\delta_{IJ} \sum\!\!\!\!\!\!\!\int_Q \left[\frac{1}{2Q^2+2M^2} + \frac{P^2}{(Q^2+M^2)\,(Q+P)^2}\right] , \tag{H.21b}$$

$$(\Sigma_{1,s})^{ab}_{IJ} := g^2$$



$$= -5\lambda\delta^{ab}\delta_{IJ} \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2+M^2} , \tag{H.21c}$$

$$(\Sigma_{1,v})^{ab}_{IJ} := g^2$$



$$= -4\lambda\delta^{ab}\delta_{IJ} \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2} . \tag{H.21d}$$

As bosonic and fermionic frequencies are even we can combine all diagrams to

$$\begin{aligned}(\Sigma_1)^{ab}_{IJ} = \lambda\delta^{ab}\delta_{IJ}\Bigg[&-4P^2 \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(Q+P)^2} + 4\sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2} \\ &+2P^2 \sum\!\!\!\!\!\!\!\int_Q \frac{1}{(Q^2+M^2)\,(Q+P)^2} - 4\sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2+M^2}\Bigg] \\ \to \lambda\delta^{ab}\delta_{IJ}&\left[4\sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2} - 4\sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2+M^2}\right] .\end{aligned} \tag{H.22}$$

In the second line we took the HTL limit. Consider the remaining integral [47]

$$\sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2+M^2} = \frac{1}{(4\pi)^2}\left[\Lambda^2 - M^2\ln\left(\frac{\Lambda^2}{M^2}\right)\right] + \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}q\, \frac{q^2}{\sqrt{q^2+M^2}}\,\frac{1}{e^{\beta\sqrt{q^2+M^2}}-1} . \tag{H.23}$$

We introduced a cut-off Λ to regularize the quadratic and logarithmic UV divergence. Therefore we find the final result for the HTL limit

$$(\Sigma_1)^{ab}_{IJ} = \lambda\delta^{ab}\delta_{IJ}\left[\frac{M^2}{(2\pi)^2}\ln\left(\frac{\Lambda^2}{M^2}\right) + \frac{T^2}{3} - 4I(M)\right] . \tag{H.24}$$

$$I(M) := \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}q \, \frac{q^2}{\sqrt{q^2 + M^2}} \frac{1}{e^{\beta\sqrt{q^2+M^2}} - 1} \, . \tag{H.25}$$

The UV divergent term can be renormalized by a mass counter-term and we ignore it for the following argument.
This result can be compared to known finite temperature corrections. If the anti-periodic boundary conditions of the fermions are kept in the Lagrangian, supersymmetry is fully broken [129]. The computation of the thermal masses, however, showed that all particles, scalars, gauges and fermions, obtain an identical thermal mass $m_\infty^2 = \lambda T^2$ [123, 131]. Demanding consistency of this result with our calculation yields

$$(\Sigma_1)_{IJ}^{ab} = \lambda T^2 \delta^{ab} \delta_{IJ} \, . \tag{H.26}$$

Therefore we need to consider the integral $I(M)$ more closely. We expect to find $I(M) = -T^2/6$. In the form the integral is written in now we cannot solve it. We therefore substitute

$$\tilde{q}^2 T^2 := q^2 + T^2 \tilde{M}^2 \, , \quad T^2 \tilde{M}^2 = M^2 \, , \qquad \Rightarrow \quad \mathrm{d}q = T \frac{\tilde{q} \, \mathrm{d}\tilde{q}}{\sqrt{\tilde{q}^2 - \tilde{M}^2}} \, . \tag{H.27}$$

The integral thus reads

$$I(M) = \frac{T^2}{2\pi^2} \int_{\tilde{M}}^\infty \mathrm{d}\tilde{q} \, \frac{\sqrt{\tilde{q}^2 - \tilde{M}^2}}{e^{\tilde{q}} - 1} \tag{H.28}$$

As we are integrating a smooth function the resulting expression $I(M)$ will also be smooth. Furthermore consider the limit of lagre M. Using the original definition (H.25) it is easy to convince ourselves that

$$\lim_{M \to \infty} I(M) = 0 \, . \tag{H.29}$$

Furthermore a numerical evaluation for some arbitrary M shows that

$$I(M) \geq 0 \, . \tag{H.30}$$

In figure H.1 we show some numerical results for arbitrary M underlining the above two arguments. Conclusively there is clearly still an ambiguity in this approach. Note that we used it exploiting the Fourier representation in an attempt to identify a supersymmetry preserving part of the Lagrangian. When considering the full Fourier series we, however, found an interaction between different modes, see Section 4.2. Such an interaction was ignored in this approach which might henceforth explain the inconsistency we found.

All in all, this dynamical calculation suggests that there is in fact a flaw in the manner we define the field redefinitions with phases and that we are, in fact, not allowed to do

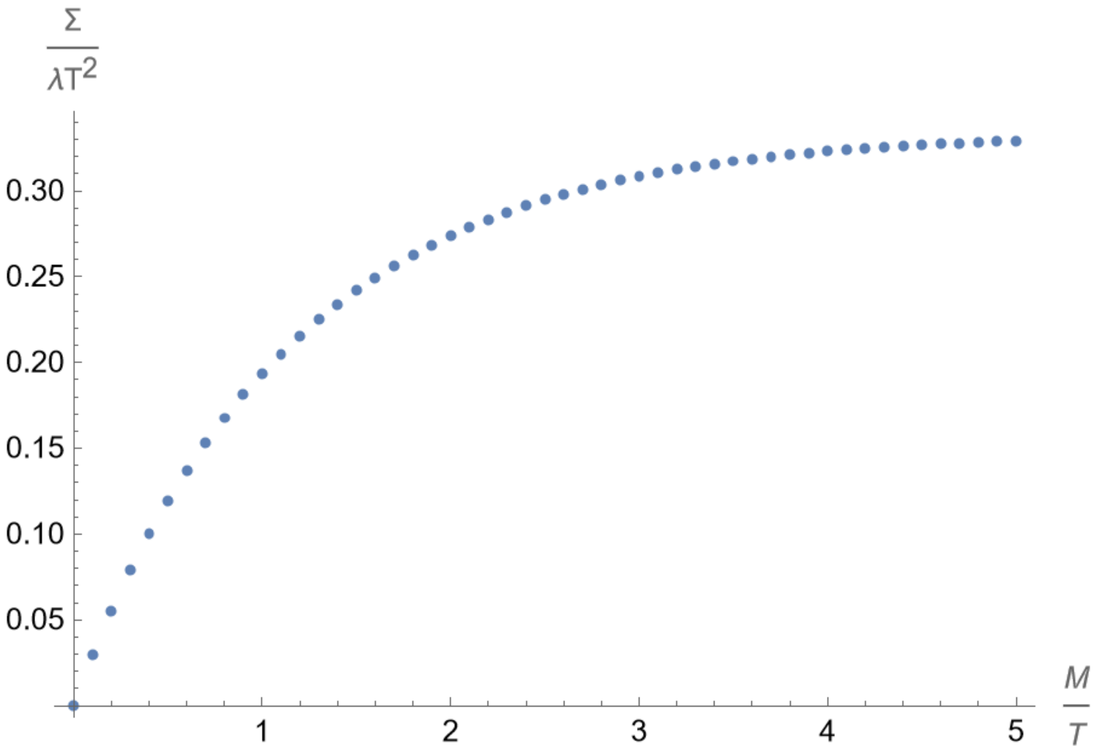


Figure H.1: Plot of Σ_1 depending on M between 0 and 5. The integral $I(M)$ was solved numerically. The results are clearly all positive and approach a value of $\frac{1}{3}\lambda T^2$ as expected. It is further clear that they cannot the expected result $\Sigma_1 = \lambda T^2$

this in the way we did. One important restriction yielded by the Yukawa-interactions was that the fermions do not obtain a resummed mass as the bosons do, as the sum of all fermionic phases is set to zero. without this restriction it might be possible to find a consistent result in the thermal mass calculation.
In the following Appendix I we suggest another approach which does not suffer from the problem we found here. We will show that generally R-symmetry is preserved for thermal $\mathcal{N} = 4$ SYM and from this derive a general Lagrangian.

In this Appendix we will finally find the Action (I.8). This action depends only on periodic fields and we show that it can be made consistent with the thermal mass of $\mathcal{N} = 4$ SYM. This suggests that we can derive a thermal BWI for this theory similar to equation (3.27). In Subsection 4.1.3.1 starting around Equation (4.40) we discuss the implications of additional operators to the Ward identity. These arguments can likewise be applied to the new Lagrangian we found here. This then suggests that the Relation (1.1), $B = 3h$ indeed holds for any coupling g in thermal $\mathcal{N} = 4$ SYM theory. This then is the first exact thermal result.

Appendix I

Implicit field redefinitions

In the previous Appendix H we considered a dynamical calculation for the phase redefinitions introduced in Section 4.1. We demanded consistency for the computation of thermal masses which is naturally considered in a HTL approximation. We were able to show, however, that the known thermal masses of $\mathcal{N} = 4$ SYM cannot be obtained in the suggested approach. Therefore, we concluded that the ansatz was flawed. Inspired by the results we found, we suggest a different ansatz here. We are in fact able to show that the $\mathrm{SU}(4)_R$ symmetry of $\mathcal{N} = 4$ is unbroken at finite temperature. We then assume that we are able to write the Lagrangian in terms of only periodic fields with breaking terms, see (4.1). In this appendix we derive these corrections from scratch.
The obvious ambiguity with this ansatz is that we will assume that the anti-periodic spinors are written in a periodic manner. We do not give these field redefinitions of the spinors, and assume that possible anomalies or constraints arising are either not present or can be ignored. This then yields that action of $\mathcal{N} = 4$ SYM but depending only on periodic field plus correction terms.

I.1 Ward identities & unbroken R-symmetry

We start by considering Ward identities similar to the ones studied in Sections 2.1 and 3.1. For any symmetry of the theory at zero temperature we find [90]

$$Q\langle\varphi_1 \dots \varphi_n\rangle = \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x \partial_\mu \langle J^\mu(x)\varphi_1 \dots \varphi_n\rangle = 0 \,. \tag{I.1}$$

In the above equation Q is the charge associated to the symmetry and J^μ its current. The φ_i denote any operators in the theory. We ignored R-symmetry and spinor indices in the above equation. They may or may not be present depending on the symmetry we consider. We do not make further assumptions on it.
We now consider finite temperature. First and foremost one assumption must be made. Symmetries can only be softly broken at finite temperature. A soft breaking of symmetries is then visible in the Ward identities by yielded breaking terms. We saw one example

of this in Section 3.1 where the breaking of the SUSY Ward identity was studied in a non-interacting setting. We hence exclude all symmetries that transform the coordinates. The breaking terms can be seen when using Stoke's theorm on the above equation. The integral will now be over the thermal manifold $S^1_\beta \times \mathbb{R}^3$,

$$\begin{aligned} Q\langle\varphi_1\dots\varphi_n\rangle_\beta &= \int_{S^1_\beta\times\mathbb{R}^3} \mathrm{d}^4x\partial_\mu\langle J^\mu(\tau,\vec{x})\varphi_1\dots\varphi_n\rangle_\beta \\ &= \int_{\delta\left(S^1_\beta\times\mathbb{R}^3\right)} \mathrm{d}\Sigma_\mu\langle J^\mu(\tau,\vec{x})\varphi_1\dots\varphi_n\rangle_\beta \\ &= \int_0^\beta \mathrm{d}\tau \lim_{\vec{x}\to\infty} n_i\langle J^i(\tau,\vec{x})\rangle_\beta\langle\varphi_1\dots\varphi_n\rangle_\beta \\ &\quad + \int_{\mathbb{R}^3}\mathrm{d}^3x\langle\left(J^0(\beta,\vec{x})-J^0(0,\vec{x})\right)\varphi_1\dots\varphi_n\rangle_\beta\ , \end{aligned} \tag{I.2}$$

with n_i, $i=1,2,3$ a unit vector in the spatial direction. For the first term in the last line we used clustering at large distances $\vec{x}\to\infty$ [54]. We assume that the one-point function of any spatial component of a current is zero

$$\langle J^i(\tau,\vec{x})\rangle_\beta = 0 \tag{I.3}$$

in any thermal theory. This follows because the spatial components must obey the residual 3d Lorentz symmetry, see also [54]. Then we only need to consider the second term. Here, the boundary conditions yielded by finite temperature can be used

$$J^0(\beta,\vec{x})-J^0(0,\vec{x}) = \begin{cases} 0 & \text{for bosonic symmetries}\ , \\ 2J^0(\beta,\vec{x}) & \text{for fermionic symmetries}\ . \end{cases} \tag{I.4}$$

Note that when considering bosonic symmetries the yielded current is also bosonic and hence obeys periodic boundary conditions. Fermionic currents similarly obtain anti-periodic boundary conditions.

As the gauge and R-symmetry are bosonic they are thus unbroken by finite temperature. This underlines the result we found when considering the reduction to three dimensions by a Fourier series, Section 4.2. It also further puts into question the first approach using explicit phase prefactors, Section 4.1. As stated previously this is probably linked to the interaction between different modes. With a more careful analysis it might be possible to re-interpret the Fourier modes as phase transitions for the full theory. We saw that this is indeed possible when the coupling is set to zero.

I.2 Periodic Lagrangian

The above consideration showed how bosonic symmetries are preserved at finite temperature. This is, however, not true for the fermionic supersymmetry. The supersymmetry Ward identity (2.16) is thus broken. For consistency, we want to write the full action of $\mathcal{N} = 4$ SYM at finite temperature in terms of periodic fields with possible additional breaking terms as in (4.1). This will clearly then yield the breaking terms of the supersymmetric Ward identity at finite temperature. We suggest the following approach:
Let us assume that there is a consistent way of making the fermions periodic and deriving corrections. This will yield

$$\mathcal{L}^{T\neq 0}_{\mathcal{N}=4} = \mathcal{L}^{T\neq 0}_{\mathcal{N}=4\,\text{periodic}} + \mathcal{L}_{\text{new, periodic}} \,. \tag{I.5}$$

Consider the new term (also depending on periodic fields) conceptually. We need to write down all relevant operators which are allowed by the preserved symmetries and further make sure that mass dimensions are consistent. The new terms can generally have various conformal dimensions Δ. The dimension of the Lagrangian is $[\mathcal{L}] = 4$. Deformations with $\Delta > 4$ are irrelevant. They would contribute with a prefactor $1/T^{\Delta-4}$ which blows up in zero temperature limit. This is forbidden by the samll temperature limit using our assumption of soft breakings. It is therefore consistent to discard these transformations. Hence only deformations $\Delta < 4$ can contribute. They can be considered one by one. The unbroken gauge and R-symmetry restrict the possible choices. All deformations need to be singlets as is the Lagrangian.
At $\Delta = 0$ only the identity operator exists. As there is no field dependence the spacetime integral in the action readily yields a volume of spacetime factor. Following [137] the prefactor of the identity is the free energy F. For $\Delta = 1$ we are not able to construct a singlet operator which could contribute. Due to the low conformal dimension only a single field could be added which would not be gauge invariant. Also single scalar fields and spinors are not R-symmetry singlets.
The primary Konishi operator is [116]

$$\mathcal{K}_1 = \frac{1}{2}\mathrm{tr}\phi^i\bar{\phi}_i \,. \tag{I.6}$$

It has mass dimension $[\mathcal{K}] = 2$ and hence is a possible contribution at the next order. The spinor fields have a conformal dimension of $[\lambda] = 3/2$ and hence are only relevant at the next order. Considering gauge fields, it turns out impossible to include a term which does not break the gauge symmetry.
For $\Delta = 3$ we find Chern-Simons terms of the gauge fields. One might suspect to find derivatives acting on the Konishi operator. However, the partial derivative will yield a total derivative term inside the spacetime integral. The covariant part including the gauge field is similar to the consideration in (4.18) and can be shown to eventually cancel. All in

all we thus write

$$S^{T\neq 0}_{\mathcal{N}=4} = S^{T\neq 0}_{\mathcal{N}=4\,\text{periodic}} + F\,T^4\,\text{Vol}\left(S^1_\beta \times \mathbb{R}^3\right)$$
$$+ \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \text{tr}\left[m_\phi^2 \mathcal{K}_1 + m_\lambda \lambda^I \sigma^0 \bar{\lambda}_I + aT\epsilon^{ijk}\left(A_i A_j A_k + \frac{2}{3} A_i \partial_j A_k\right)\right] . \quad \text{(I.7)}$$

The coefficient a is real and a priori arbitrary and so are the masses $m_\phi, m_\lambda \propto T$. The first two terms in the second line give masses to the scalars and fermions, respectively. A priori these masses are free parameters. We assume that the Chern-Simons term can be shown to cancel. The volume of the thermal manifold diverges and we can consider it renormalized thus omitting it. Hence

$$S^{T\neq 0}_{\mathcal{N}=4} = S^{T\neq 0}_{\mathcal{N}=4\,\text{periodic}} + \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \text{tr}\left[m_\phi^2 \mathcal{K} + m_\lambda \lambda^I \sigma^0 \bar{\lambda}_I\right] . \quad \text{(I.8)}$$

This action is now build in a manner that preserves R- and gauge symmetry. However, it will clearly break supersymmetry and insofar it is as expected. Further note that the term $\lambda^I \sigma^0 \bar{\lambda}_I$ is precisely the SUSY breaking term we discovered in Section 3.1 in the consideration at zero coupling.
The above is thus a consistent suggestion for a Lagrangian which preserves the symmetries we showed are preserved. In contrast, it manifestly breaks supersymmetry. Also Lorentz symmetry is broken from four to three dimensions. This can be seen by the explicit σ^0 which appears in the fermionic mass term.

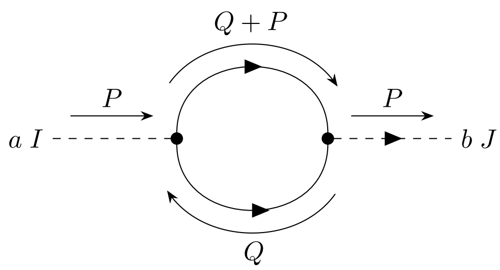
I.3 Scalar self-energy

The above action depends on two newly introduced mass coefficients

$$m_\phi = \tilde{m}_\phi T \quad \text{and} \quad m_\lambda = \tilde{m}_\lambda T \ . \quad \text{(I.9)}$$

They are a priori arbitrary, however, we again demand that the thermal mass $m^2_{\text{th.}} = \lambda T^2$ can be obtained consistently. We therefore go back to the calculation of self-energy as in the previous Appendix H.4. As previously, all fields are assumed periodic and we should demand consistency in the computation of the scalar self-energy. The relevant Feynman diagrams are given in Equation (H.21). The difference in the case at hand are the masses for the fermionic particles. Hence we find

$$(\Sigma_{1,f})^{ab}_{IJ} := g^2$$



$$= 4\lambda\delta^{ab}\delta_{IJ} \sum\!\!\!\!\!\!\int_Q \left[\frac{2}{Q^2 + m_\lambda^2} - \frac{P^2}{\left(Q^2 + m_\lambda^2\right)\left((Q+P)^2 + m_\lambda^2\right)} \right] , \tag{I.10}$$

and all other diagrams are the same with $M^2 \to m_\phi^2$. Thus the full diagram is, in the HTL approximation, given by

$$\begin{aligned}(\Sigma_1)^{ab}_{IJ} &= \lambda\delta^{ab}\delta_{IJ} \left[-4 \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2} - 4 \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2 + m_\phi^2} + 8 \sum\!\!\!\!\!\!\int_Q \frac{1}{Q^2 + m_\lambda^2} \right] \\ &= -4\lambda\delta^{ab}\delta_{IJ} \left[I(0) + I(m_\phi) - 2I(m_\lambda) \right] . \end{aligned} \tag{I.11}$$

We introduced an integral similar to the Equations (H.23), (H.24) and (H.25) in the previous chapter. With an analogous substitution this yields

$$I(m) = \sum\!\!\!\!\!\!\int_Q \left.\frac{1}{Q^2 + m^2}\right|_{\text{ren.}} = \frac{T^2}{2\pi^2} \int_{m/T}^{\infty} \mathrm{d}\tilde{q}\, \frac{\sqrt{\tilde{q}^2 - \frac{m^2}{T^2}}}{e^{\tilde{q}} - 1} . \tag{I.12}$$

We implicitly renormalized the UV divergence. As discussed above the integral is positive and converges to zero for high m. Note for the remaining term that the temperature dependence is entirely in the prefactor. We have that $m \propto T$ and hence the combination $\tilde{m} = {}^{m}\!/_{T}$ does not depend on the temperature.
Using that $I(0) = {}^{T^2}\!/_{12}$ we find in the HTL limit

$$(\Sigma_1)^{ab}_{IJ} = \lambda T^2 \delta^{ab}\delta_{IJ} \left[-\frac{1}{3} - \frac{2}{\pi^2} \int_{\tilde{m}_\phi}^{\infty} \mathrm{d}q\, \frac{\sqrt{q^2 - \tilde{m}_\phi^2}}{e^q - 1} + \frac{4}{\pi^2} \int_{\tilde{m}_\lambda}^{\infty} \mathrm{d}q\, \frac{\sqrt{q^2 - \tilde{m}_\lambda^2}}{e^q - 1} \right] . \tag{I.13}$$

The thermal mass of $\mathcal{N} = 4$ is [123, 131] $m_{\text{th.}}^2 = \lambda T^2$. Therefore we need to find solutions in terms of $\tilde{m}_\phi$ and $\tilde{m}_\lambda$ to the equation

$$-\frac{1}{3} - \frac{2}{\pi^2} \int_{\tilde{m}_\phi}^{\infty} \mathrm{d}q\, \frac{\sqrt{q^2 - \tilde{m}_\phi^2}}{e^q - 1} + \frac{4}{\pi^2} \int_{\tilde{m}_\lambda}^{\infty} \mathrm{d}q\, \frac{\sqrt{q^2 - \tilde{m}_\lambda^2}}{e^q - 1} = 1 . \tag{I.14}$$

In principle this equation will yield a relation between the two masses. Figure I.1 shows this relation through numeric results. The values for the new masses are thus also generally

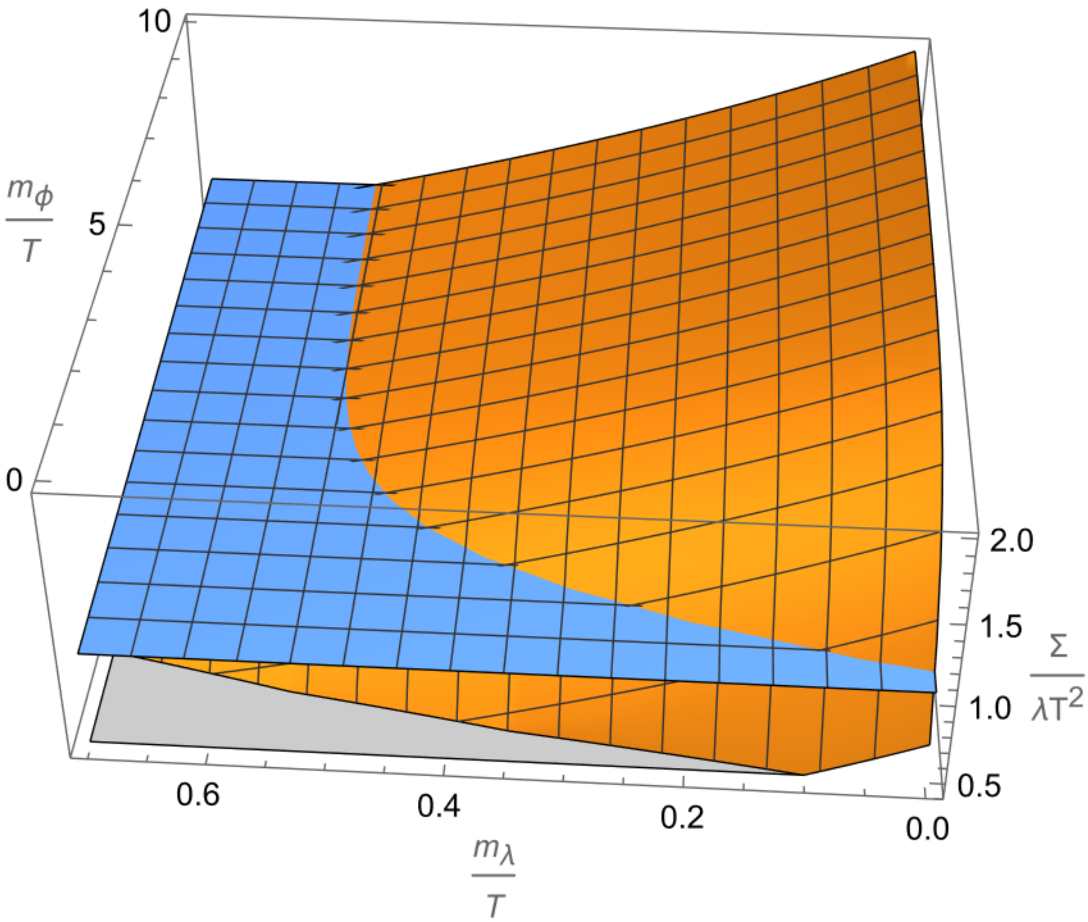


Figure I.1: Plot of Σ_1 depending on m_λ between 0 and 0.6 and m_ϕ between 0 and 10. The result Σ is plotted in orange while we included the plane $\Sigma = 1$ in blue. This is the result we expect to find following [123, 131]. The intersection between the two planes are the allowed values when demanding consistency. This clearly shows that there is some relation between the two masses. The integrals $I(m)$ were solved numerically.

constrained. Taking only the first four digits we find

$$m_\phi \gtrsim 0.3205T\,, \quad m_\lambda \lesssim 0.5315T\,. \tag{I.15}$$

Employing more numerical results it is possible to further restrict these masses. Employing a numerical calculation it is further feasible to fit a function to the relation between the two masses.

This suggest that our ansatz is indeed consistent. It might further be possible by similar consistency calculations to fully determind the masses m_ϕ^2 and m_λ^2.

With this result it is possible to consider the BWI similar to the considerations in Section 3.1. Starting from Equation (4.40) we argue why the new terms in the BWI will not change the final result. With the Lagrangian as given above we indeed obtain the Relation (1.1), $B = 3h$.

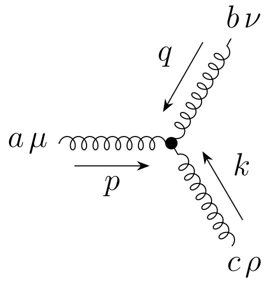
Appendix J

Feynman Rules of $\mathcal{N} = 4$ SYM

We defined our conventions for the Lagrangian in (1.56). The Feynman rules can be read off readily from it. We will discuss the momentum space Feynman rules at zero temperature. Thermal effects include the discretisation on the Matsubara circle and are taken in account when computing the self-energies. For details see [47].
Following the overall dependence on the Yang-Mills coupling, $1/g^2$, every vertex scales with g^{-2} and every propagator with g^2 thus making any diagrammatic expansion and power counting rather intuitive. The bare propagators are

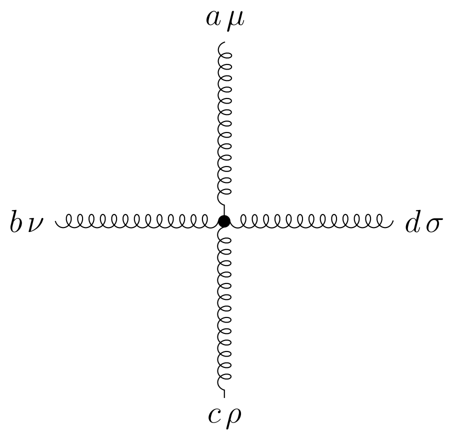
$$\langle A^a_\mu A^b_\nu \rangle(p) = g^2 \frac{\delta^{ab}}{p^2}\left(\eta_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right) , \quad \langle \bar{c}^a c^b \rangle(p) = g^2 \frac{\delta^{ab}}{p^2} ,$$
$$\langle \Phi^a_r \Phi^b_s \rangle(p) = g^2 \frac{\delta^{ab}\delta_{rs}}{p^2} , \quad \langle \lambda^a \bar{\lambda}^b \rangle(p) = -2ig^2 \frac{\delta^{ab}\gamma^\mu p_\mu}{p^2} . \tag{J.1}$$

Note that our calculations are carried out in the Feynman gauge $\xi = 1$ and the gauge propagator thus simplifies and only has a metric prefactor.
For the vertices we find [1] rules

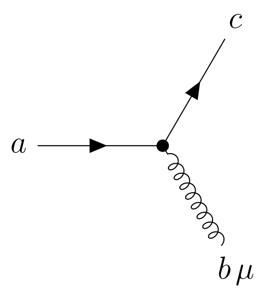


$$= \frac{-i}{g^2} f^{abc} \left[\eta_{\nu\rho}(q-k)_\mu + \eta_{\mu\rho}(k-p)_\nu + \eta_{\mu\nu}(p-q)_\rho\right] , \tag{J.2a}$$

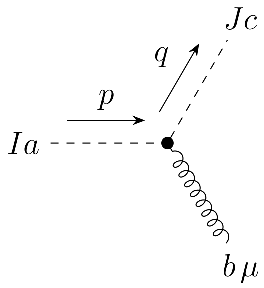
[1] These and all other Feynman diagrams where created with TikZ-Feynman [138] unless indicated differently.



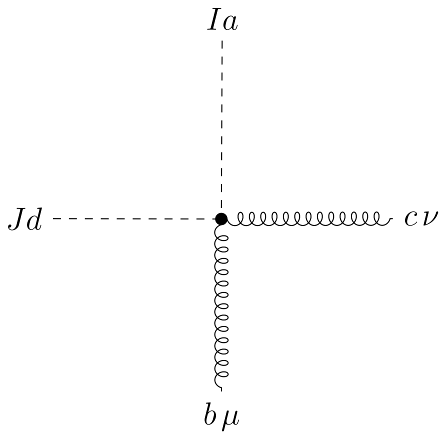
$$= \frac{1}{g^2}\Big[f^{abe} f^{cde} \left(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}\right) + f^{ace} f^{dbe} \left(\eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right) + f^{ade} f^{bce} \left(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma}\right) \Big] , \tag{J.2b}$$



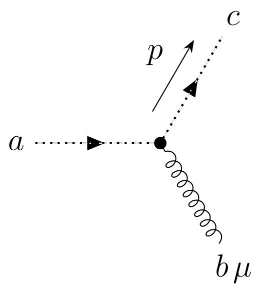
$$= \frac{i f^{abc}}{2g^2}\gamma_\mu , \tag{J.2c}$$



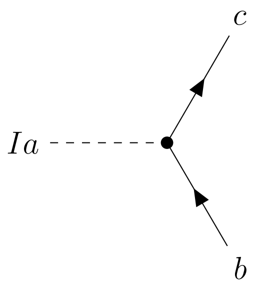
$$= \frac{-i}{g^2} f^{abc} \delta_{IJ} (p+q)_\mu , \tag{J.2d}$$



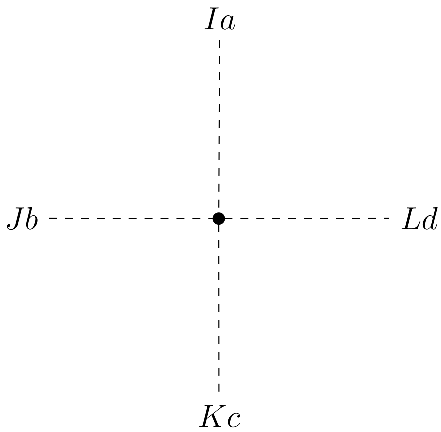
$$= \frac{1}{g^2}\delta_{IJ}\eta_{\mu\nu}\left(f^{abe}f^{cde} + f^{ace}f^{bde}\right) , \tag{J.2e}$$



$$= \frac{-if^{abc}}{g^2} p_\mu \; , \tag{J.2f}$$



$$= \frac{i}{2g^2} f^{abc} \Gamma^I \; , \tag{J.2g}$$



$$= \frac{1}{g^2} \Big[f^{abe} f^{cde} \left(\delta_{IK} \delta_{JL} - \delta_{IL} \delta_{JK} \right) + f^{ace} f^{dbe} \left(\delta_{IL} \delta_{JK} - \delta_{IJ} \delta_{KL} \right) + f^{ade} f^{bce} \left(\delta_{IJ} \delta_{KL} - \delta_{IK} \delta_{JL} \right) \Big] \, . \tag{J.2h}$$

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Eidesstattliche Versicherung / Declaration on oath

Hiermit versichere ich an Eides statt, die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt zu haben.

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