String-net models for pivotal bicategories and rational conformal field theories with defects

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1 Introduction

Two-dimensional conformal field theories are renowned for their richness, in the sense that through studying them, one often reveals connections between different disciplines in both physics and mathematics, at the same time deepens the understanding thereof. In this thesis, with the aim of constructing consistent systems of correlators for worldsheets with topological defects and physical boundaries in *rational* conformal field theories, we generalize and apply the string-net models for spherical fusion categories, which were first introduced in [LW05] as lattice models of topological phases and later formalized in [KJ11] as 2-dimensional skein theories.

The primary theoretical foundation of this project is three-fold: First, it is widely believed [MS89, BK01] (and proven for the cases of genus 0 and genus 1) that through a Riemann-Hilbert type correspondence, the notion of an *algebraic* modular functor provided by the Reshetikhin-Turaev 3-dimensional topological field theory for a modular fusion category $\mathcal{C}$, is equivalent to that of a *complex-analytic* modular functor for a rational chiral conformal field theory whose monodromy is governed by the very same modular fusion category $\mathcal{C}$ (e.g. as a suitable representation category of a rational chiral vertex operator algebra $\mathfrak{V}$). Second, it is demonstrated in a series of works [FRS02, FRS04a, FRS04b, FRS05, FFRS06a] from two decades ago that upon specifying the chiral theory as well as the representation theoretic data in the category $\mathcal{C}$ that decorate the (topological) worldsheets, a consistent system of correlators can be fully constructed; by the latter we mean an assignment of an *invariant* of the appropriate mapping class group in the relevant space of conformal blocks (provided by the 3d TFT) to every worldsheet that satisfies *sewing constraints.* Third, it has been established that we have the following chain of equivalences of once-extended topological field theories of top-dimension three:

$$\mathrm{SN}_{\mathcal{C}} \simeq \mathrm{TV}_{\mathcal{C}} \simeq \mathrm{RT}_{\mathcal{Z}(\mathcal{C})} \tag{1.0.1}$$

for any spherical fusion category $\mathcal{C}$, where $\mathrm{TV}_{\mathcal{C}}$ stands for the Turaev-Viro state-sum TFT for $\mathcal{C}$ and $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ stands for the Reshetikhin-Turaev surgery TFT for the Drinfeld center $\mathcal{Z}(\mathcal{C})$, the equivalence between which was proven independently and via different approaches in [Bal11] and [TV13], and $\mathrm{SN}_{\mathcal{C}}$ is the TFT extended from the string-net model for $\mathcal{C}$, whose existence and equivalence to the other two TFTs were established in [KJ11] and [Goo18]. In view of the equivalence of braided tensor categories

$$\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} \simeq \mathcal{Z}(\mathcal{C})$$

which is equivalent to the modularity of $\mathcal{C}$ [Shi19], and the fact that one can achieve the necessary procedure of "combining the left and right movers" (which is called *holomorphic factorization* [Wit92] in the literature) for the construction of correlators by taking the braided double $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ as the category of chiral data, (1.0.1) suggests that the construction of a consistent system of correlators can be carried out via the string-net model $\mathrm{SN}_{\mathcal{C}}$. The relevant backgrounds as well as a precise formulation of the tasks of establishing the chiral theory as an *open-closed modular functor* and constructing a consistent system of correlators for all worldsheets with topological defects are provided in Chapter 2, and the concrete construction of correlators in terms of string-nets is given in Chapter 5.

As it turned out, not only does the string-net approach to RCFT correlators reduce the technicality of the construction greatly compared to the *TFT approach* adopted in [FRS02, FRS04a, FRS04b, FRS05, FFRS06a] (for instance, the proof of fulfillment of the sewing constraints in the latter involves rather complicated 3-dimensinal topology), it also verifies (in the semisimple setting) the conjecture proposed in [FS21a] that internal homs, internal natural transformations and their compositions [FS21b] provide the algebraic structures that describe the *field contents* and their *operator products* in conformal field theories. These results are presented in Chapter 6. As an additional demonstration of the usefulness of the string-net construction for relating algebraic structures in the braided category $\mathcal{Z}(\mathcal{C})$ to the geometric ones provided by the decorated worldsheets, in Chapter 7 we present the proof (and a concrete formulation) of the statement that the vertical and horizontal compositions of the internal natural transformations obey a braided version of the *Eckmann-Hilton relation* satisfied by the compositions of ordinary natural transformations, which is done by concretizing the observation that the objects of internal natural transformations are *braided algebras* over the *braided colored operad* $\mathsf{WS}_{\mathcal{C}}$ of genus-0 worldsheets.

Moreover, by including line and point defects into the theory, one realizes that it is necessary to generalize the string-net models to the bicategorical setting: The collection of representation theoretic defect conditions for the worldsheets in an RCFT with fixed chiral data $\mathcal{C}$ comprises a *pivotal bicategory* $\mathcal{F}r(\mathcal{C})$ of simple special symmetric Frobenius algebras, bimodules and bimodule morphisms internal to $\mathcal{C}$, whose composition rules describe the fusion of (point and line) defects. In Chapter 8, we show that worldsheets related by the local composition rules for defects provided by the pivotal bicategory $\mathcal{F}r(\mathcal{C})$ share the same correlator. Put differently: the prescription of string-net correlators gives rise to a family of MCG-intertwiners between string-net spaces for the pivotal bicategory $\mathcal{F}r(\mathcal{C})$ which classify the worldsheets up to local relations, and the string-net spaces for the modular fusion category $\mathcal{C}$ which model the spaces of conformal blocks of the theory. We call these intertwiners *universal correlators*, alluding to the non-linear analog provided by universal bundles and classifying spaces. In this sense, it is the equivalence classes of worldsheets, which form a vector space of $\mathcal{F}r(\mathcal{C})$-colored string-nets upon fixing a boundary datum for the defect patterns, that are detectable by “observing” the correlators – we therefore refer to these equivalence classes as *quantum worldsheets*, and define the mapping class groups for them that take the local relations into account. The short proof of Theorem 8.1.1 relies on the careful formulation of the *unframed* graphical calculi and the string-net models for *strictly pivotal bicategories*, as well as the discussion of their functoriality under *rigid separable Frobenius functors*, which are given in Chapter 3 and Chapter 4. It should be noted that our formulations *do not* require any finiteness of the bicategories.

Chapter 9 concludes this thesis with an observation of theoretical interest: our sporadic constructions of string-net modular functors, field functors and universal correlators fit neatly into the framework of *double categories*: the symmetric monoidal functors of the type $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \to \mathcal{P}\mathrm{rof}_{\Bbbk}$ (which is what the term *“open-closed modular functors”* means to us in this thesis) provided by the string-net models for pivotal bicategories canonically extend to *symmetric monoidal double functors*, and the universal correlators along with the field functors comprise a *monoidal vertical transformation*, whereas various desired properties such as *factorization* of modular functors, MCG-invariance and the fulfillment of sewing constraints of the correlators correspond directly to the axioms of double functors and vertical transformations, respectively.

Relevant contribution

This thesis is based on the following preprint that has been accepted after refereeing for publication in the book series Springer Briefs in Mathematical Physics, volume 45 and that has appeared as a preprint in:

- Jürgen Fuchs, Christoph Schweigert, and Yang Yang. String-net construction of RCFT correlators. arXiv:2112.12708 [math.QA]

The publication is also listed in the bibliography at the end of the thesis and is cited as [FSY21]. The results therein were developed together with the coauthors, whose contributions I fully acknowledge.

The idea to use string-net models to understand CFT correlators in the Cardy case was proposed to the author already for his master thesis. The original proposal was substantially simplified through the insights of the author of the present thesis; in particular he realized the importance of empty string-nets in the Cardy case.

The construction of string-net correlators for general worldsheets with values in string-net spaces was developed in discussions with Jürgen Fuchs and Christoph Schweigert. All authors contributed to shaping these results. The idea to generalize disorder fields for two defects as in [FS21a] to multi-pronged bulk fields is due to the author of this thesis. The author also developed the idea that the construction of correlators factorizes through a bicategorical string-net construction based on the bicategory of defect data. Conceptual aspects, including the interpretation as quantum worldsheets, of this were sharpened in discussions with Jürgen Fuchs and Christoph Schweigert. The mathematical realization of this idea using rigid separable Frobenius functors is also due to the author. The idea of formalizing unframed graphical calculus as a symmetric monoidal functor from a category of corollas and graphs as well as that of recognizing the string-net spaces as colimits were inspired by the discussions between the author and Lukas Woike during the preparation for a separate project, but the formalisms adapted to pivotal bicategories were conceptualized and written up by the author after discussions with Jürgen Fuchs and Christoph Schweigert. The author provided after the discussions a first draft of the material published in [FSY21] that was subsequently polished by contributions from all authors. The ideas of those parts of the thesis that do not appear in [FSY21] (Chapter 3, Chapter 4, Chapter 9, as well as the proof of Theorem 8.1.1) have been discussed with Jürgen Fuchs and Christoph Schweigert, but were written up independently by the author.

Acknowledgment

I am deeply indebted to my advisor Christoph Schweigert for his excellent guidance and tireless support, especially during the difficult times when I needed patience and understanding from others the most. To my former colleagues Vincent Koppen, David Krusche, Manasa Manjunatha, Vincentas Mulevičius, Arpan Saha, and Lukas Woike I am grateful for their selfless and salvaging assistance during those times of hardship.

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I cannot express enough gratitude towards my family who supported me unconditionally, and last but not the least, to Alice, for always being there with me.

2 Backgrounds

In this chapter, we give a brief review of the categorical framework of RCFT correlators developed in [FRS02, FRS04a, FRS04b, FRS05, FFRS06a, FFS12], emphasizing the point of view that RCFT correlators are *surface defects* [KS11]. After introducing the recent insight [FS21b, FS21a] that the *field contents* are *internal homs* of suitable module categories, we give a precise formulation of the task of solving an RCFT with *topological defects* in this framework.

2.1 A brief review of the 3d-TFT approach

In a categorical approach to CFT correlation functions, the starting point is not a Lagrangian or a partial differential equation describing classical field equations, and one does not rely on any type of perturbation theory. Instead, one postulates that the correlation functions of the theory exhibit certain chiral symmetries, which means concretely that they are solutions to a collection of linear differential equations, also known as *chiral Ward identities.* For the two-dimensional conformal field theories of our interest, these symmetries can be encoded in the structure of a *conformal vertex algebra*, and the chiral Ward identities on any surface can be derived from that algebra. The solutions to these equations – which are called *conformal blocks* and contain the correlation functions we are looking for as particular elements – form vector spaces. For the models of our interest these vector spaces are finite-dimensional; when regarded as elements of these spaces, we refer to the correlation functions for brevity also as *correlators.* Typically the solutions are multi-valued, so the fundamental groups of the relevant parameter spaces (conformal structure of the surfaces and positions of insertion points), i.e. *mapping class groups* of surfaces, act on the solution spaces. The particular elements in these spaces that are correlation functions of *bulk fields* must be single-valued, and thus transform trivially under the action of the mapping class group, while correlation functions involving general *defect fields* are invariant under a subgroup of the mapping class group. In addition to the invariance under the actions of the corresponding mapping class groups, the assignment of correlators to worldsheets should be compatible with *sewing.*

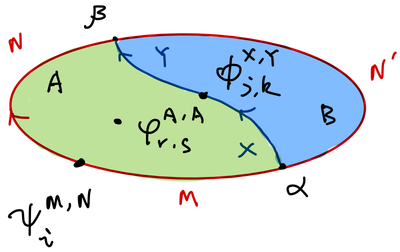
Let's fix once and for all a conformal vertex operator algebra $\mathfrak{V}$ and denote its (suitable) category of representations by $\mathcal{C}$, to be referred to as the *category of chiral data.* For a *rational* CFT – the case of interest in this thesis – the vertex operator algebra $\mathfrak{V}$ is required to be rational, meaning that $\mathcal{C}$ is a modular fusion category (i.e. a non-degenerate braided fusion category with a spherical pivotal structure). It is a widely accepted assumption, and proven for the cases of genus 0 and 1 , that in this case the *Reshetikhin–Turaev* *topological field theory*[1]

$$\mathrm{RT}_{\mathcal{C}}\colon \widehat{\mathrm{Bord}_3}^{\mathrm{or},\mathcal{C}\text{-ribbons}} \to \mathrm{Vect}_{\Bbbk}$$

provides the correct spaces of conformal blocks.

We now give a very rough summary of the so-called *TFT approach* to the construction of correlators for worldsheets with topological defects developed in [FRS02, FRS04a, FRS04b, FRS05, FFRS06a].

Given a worldsheet with *physical boundaries*, *(topological) line and point defects*, as well as *field insertions*, for instance

$\mathfrak{S} =$  . (2.1.1)

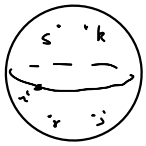
It is part of the construction to determine the labels of the decorations, that is, to determine a sensible set of *boundary conditions*, as well as that of *defect conditions*, and *degeneracy spaces* for the field insertions. It turns out that these labels are representation theoretic. As an illustration, we spell out all the labels for the worldsheet $\mathfrak{S}$ displayed in (2.1.1):

- The labels for the two *bulk phases* A and B are *simple, special symmetric Frobenius algebras* (see Section 5.1 for a brief review) in $\mathcal{C}$.
- The *boundary conditions* M and N are given by *right* A*-modules in* $\mathcal{C}$*,* and the boundary condition $N^{'}$ is given by a *right* B*-module.*
- The *defect conditions* X and Y for the *line defects* are given by A*-*B*-bimodules.*
- The *defect conditions* α and β for the *point defects* are give by *module morphisms*: $\alpha \in \mathrm{mod}^{\mathcal{C}}\text{-}B(N', M \otimes_A X)$, $\beta \in \mathrm{mod}^{\mathcal{C}}\text{-}B(N \otimes_A Y, N')$.
- The *boundary field insertion* $\psi_i^{M,N} \in \mathrm{mod}^{\mathcal{C}}\text{-}A(i \otimes M, N)$, where the *chiral label* i is a simple object in $\mathcal{C}$.
- The *defect field insertion* $\phi^{X,Y}_{j,k} \in A\text{-}\mathrm{mod}^{\mathcal{C}}\text{-}B(j \otimes^- X \otimes^+ k, Y)$, where the *chiral labels* j and k are simple objects in $\mathcal{C}$ and $j \otimes^- X \otimes^+ k$ is an A-B-bimodule with underlying object $j \otimes X \otimes k$ and specific A- and B-actions defined with the help of the braiding of $\mathcal{C}$ (for details see e.g. [FRS02] or [Run10, Definition 6.7]).
- The bulk field insertion $\varphi^{A}_{r,s} \in A\text{-}\mathrm{mod}^{\mathcal{C}}\text{-}A(r \otimes^- A \otimes^+ s, A)$, where the *chiral labels* r *and* s are simple objects in $\mathcal{C}$.

[1] We view the Reshetikhin–Turaev TFT for the modular fusion category $\mathcal{C}$ as a *symmetric monoidal functor* from (a *central extension* of) the category of closed oriented surfaces with $\mathcal{C}$-colored marked points and 3-dimensional bordisms with embedded $\mathcal{C}$-colored framed string diagrams to the category of vector spaces and linear maps. We refer the readers to [Tur16] for the details therein.

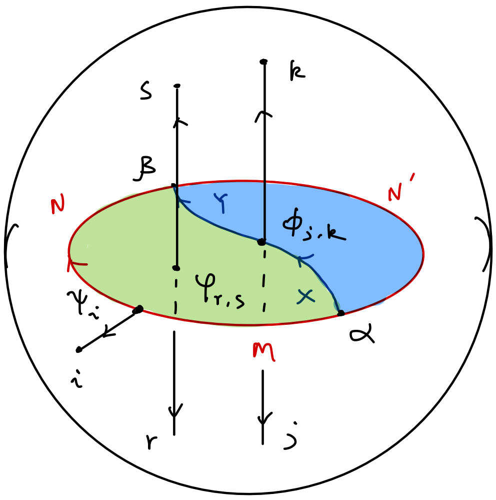
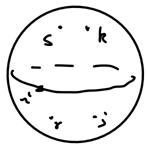
Note that the boundary conditions being *right* modules is a consequence of the chosen convention for the orientation of the physical boundaries, see Definition 2.3.3.

In order to account for the *combination of left and right movers* in the *full field theory* (in this thesis we focus on the situations where the monodromies of left- and right movers are controlled by the same modular fusion category $\mathcal{C}$), one takes the *holomorphic double* $\widehat{\Sigma}_{\mathfrak{S}}$ of the worldsheet $\mathfrak{S}$, i.e. a *closed oriented surface with marked points* obtained by taking two copies of the surface $\Sigma_{\mathfrak{S}}$ underlying the worldsheet (forgetting the defects) and identifying their boundaries, with the marked points colored by the chiral labels of the field insertions. For the worldsheet $\mathfrak{S}$ given by (2.1.1), we have

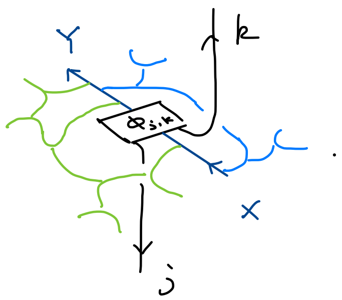
$\widehat{\Sigma}_{\mathfrak{S}} =$  .

The procedure of taking holomorphic doubles of the worldsheets is called *holomorphic factorization.*

The relevant space of conformal blocks is the vector space $\mathrm{RT}_{\mathcal{C}}(\widehat{\Sigma}_{\mathfrak{S}})$ assigned to the marked surface $\widehat{\Sigma}_{\mathfrak{S}}$ by the TFT and the correlator for the worldsheet $\mathfrak{S}$ is an element thereof. We have the following expression for the correlator via evaluating the TFT on a specific bordism from the empty manifold $\emptyset$ to the holomorphic double $\widehat{\Sigma}_{\mathfrak{S}}$:

$\mathrm{Cor}(\mathfrak{S}) = \mathrm{RT}_{\mathcal{C}}($  $)(1) \in \mathrm{RT}_{\mathcal{C}}($  $).$ (2.1.2)

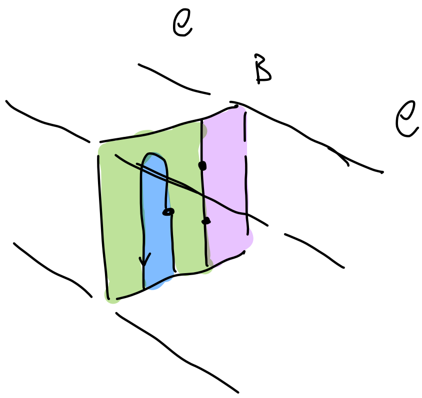
Here the colored 2-dimensional regions are actually networks of string diagrams colored with the corresponding Frobenius algebras and their structure morphisms (products, coproducts, units, and counits) along with a choice of *fine triangulations* on both regions, and the coupons at the sites of the field insertions are given by the corresponding module morphisms. For instance, if we zoom in on the defect field insertion $\phi_{j,k} \equiv \phi_{j,k}^{X,Y}$, we see



For any *mapping class group element* of the worldsheet $\mathfrak{S}$, that is, any isotopy class of diffeomorphisms from $\Sigma_{\mathfrak{S}}$ to itself *preserving the defects and field insertion points*, the canonical action on the vector space $\mathrm{RT}_{\mathcal{C}}(\widehat{\Sigma}_{\mathfrak{S}})$ thereof merely modifies the triangulations of the 2d regions, therefore preserves the element $\mathrm{Cor}(\mathfrak{S})$ due to that the axioms of a special symmetric Frobenius algebra accounts for the 2d Pachner moves.

2.2 RCFT correlators as surface defects

An important insight of [KS11] is that the correlators obtained via the TFT approach (more precisely, the network of Frobenius lines) should be considered as *surface defects* (along with the line and point defects as we include them in the worldsheets) separating two *3d bulk theories* of Reshetikhin-Turaev type labeled by their *categories of Wilson lines,* both of which being $\mathcal{C}$. The collections of such 2-, 1-, and 0-dimensional defects make up the objects, 1-morphisms and 2-morphisms[2] of a *pivotal bicategory* $\mathcal{B}$ (see Chapter 3 for a review of pivotal bicategories and the graphical calculus thereof), respectively, where the *horizontal* composition of 1-morphisms is induced by the fusion of parallel line defects, while the *vertical* and *horizontal* compositions of the 2-morphisms are given by fusing point defects *along a line defect*, and fusing point defects *across two parallel line defects*, respectively.



[2]Notice that the dimension of a defect is Poincaré dual to its categorical level.

Let us denote by $\mathcal{C}$-$\mathcal{M}$od the bicategory of *finitely semisimple left $\mathcal{C}$-module categories, module functors and module natural transformations*. This bicategory is in fact pivotal [Sch13a, Theorem 4.5.1]. In [FSV13], the authors proposed the ansatz

$$\mathcal{B} = \mathcal{C}\text{-}\mathcal{M}\mathrm{od} \tag{2.2.1}$$

and gave their heuristic justification thereof based on the analysis of the local interactions between the defects and the bulk Wilson lines. We now give a *partial* account of their argument in the following.

We start off with *folding* the 3-manifold along the surface region in which the lower dimensional defects are located, thereby reducing the setting to that of bulk-versus-boundary. Reflected over the surface, the braidings of the bulk Wilson lines get reversed, therefore we get a new bulk theory whose category of Wilson lines is the *Deligne product* $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$. Now pick an arbitrary object $a \in \mathcal{B}$ and focus on its *endomorphism category* $\mathcal{W}_a := \mathrm{End}_{\mathcal{B}}(a)$. The horizontal composition of $\mathcal{B}$ endows $\mathcal{W}_a$ with the structures of a monoidal category.

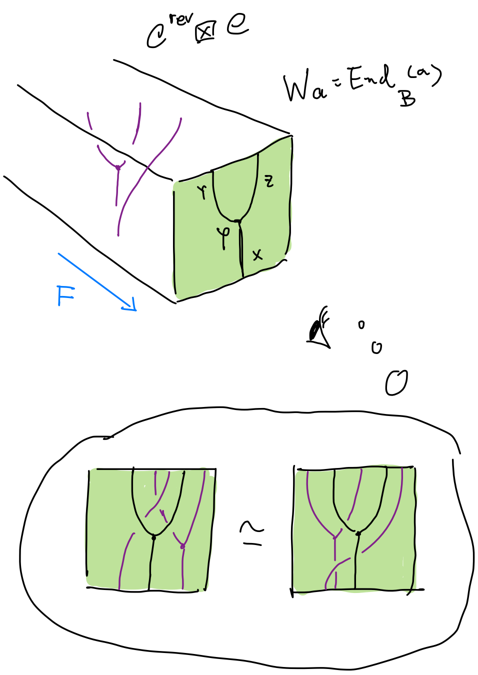


Figure 2.2.1: Different ways (from the left v. from the right) of fusing a line defect on the boundary with the image of a Wilson line produce canonically isomorphic objects in the monoidal category $\mathcal{W}_a = \mathrm{End}_{\mathcal{B}}(a)$. On the other hand, the image of any point defects on Wilson line in the bulk under the projection can freely pass through line defects on the boundary. Therefore, the monoidal functor $F\colon \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} \to \mathcal{W}_a$ factors through the forgetful functor $\mathcal{Z}(\mathcal{W}_a) \to \mathcal{W}_a$, i.e. it is *central*. Note that we have taken into account the fact that the category $\mathcal{C}$ of bulk Wilson lines is braided, hence the different relative positions of the images of purple lines.

Now, for any diagram of Wilson lines in the bulk and any diagram of line and point defects (separating the phase a with itself) on the boundary, by projecting the diagrams in the bulk onto the boundary (see Figure 2.2.1), we get a combined picture of the two, where the intersecting parts manifest as *half-braidings* (see (5.0.3) for our convention for half-braidings, and note that apart from Figure 2.2.1 we use an *over crossing* of a strand labeled with $Y \in \mathcal{Z}(\mathcal{C})$ to represent the half-braiding of Y). Due to the topological nature of the Wilson lines, the relative position between the two types of diagrams is unessential. Assuming the maximum compatibility with the *fusion* of bulk Wilson lines, it follows that the act of projecting bulk Wilson lines onto the boundary can be described by a *central functor* $F\colon \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} \to \mathcal{W}_a$. This means we have the following commutative diagram of *strong monoidal* functors:

$$\begin{tikzcd} & \mathcal{Z}(\mathcal{W}_a) \arrow[d] \\ \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} \arrow[ur, dashed, "\widetilde{F}"] \arrow[r, "F"'] & \mathcal{W}_a \end{tikzcd} \tag{2.2.2}$$

where the unnamed vertical arrow stands for the forgetful functor of the *Drinfeld center* $\mathcal{Z}(\mathcal{W}_a)$ and $\widetilde{F}$ is in addition *braided.* If we further assume that the projection is *holographic* and *saturated*, i.e. it preserves *all* information of the bulk Wilson lines and conversely, *any* information of the Drinfeld center $\mathcal{Z}(\mathcal{W}_a)$ can be obtained from looking at the projection of some Wilson lines from the bulk, then the functor

$$\widetilde{F}\colon \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{W}_a)$$

is required to be an *equivalence of ribbon categories.* Such $\widetilde{F}$ is called a *Witt trivialization* of the ribbon category $\mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$.

Let's halt our review of the arguments presented in [FSV13] and appreciate that already at this point, non-trivial constraints have been imposed on the bicategory $\mathcal{B}$ of the surface defects. For instance, combining with the fact that the ribbon fusion category $\mathcal{C}$ is *modular*, which is equivalent to the statement that the canonical braided functor $\Xi_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ is a *braided equivalence* (see (5.0.4)), it follows that the Drinfeld center $\mathcal{Z}(\mathcal{W}_a)$ of the endomorphism category $\mathcal{W}_a \coloneqq \mathrm{End}_{\mathcal{B}}(a)$ is braided equivalent to $\mathcal{Z}(\mathcal{C})$, i.e. the monoidal categories $\mathcal{W}_a$ and $\mathcal{C}$ are Morita equivalent, for an arbitrary object $a \in \mathcal{B}$. This constraint is fulfilled by any full sub-bicategories of $\mathcal{C}$-$\mathcal{M}$od, since (see e.g. [EGNO15, Corollary 7.16.2] for a more general statement) for any finitely semisimple $\mathcal{C}$-module category $\mathcal{M}$, we have a canonical braided equivalence $\mathcal{Z}(\mathrm{End}_{\mathcal{C}\text{-}\mathcal{M}\mathrm{od}}(\mathcal{M})) \simeq \mathcal{Z}(\mathcal{C})$.

Under a mild extra assumption (see [KMRS21, Section 3.3]), we see that the identity of $\mathcal{B}$ should be narrowed down to the *full* sub-bicategory[3] $\mathcal{C}$-$\mathcal{M}\mathrm{od}^{\mathrm{tr}} \subset \mathcal{C}$-$\mathcal{M}$od, whose objects are finitely semisimple left $\mathcal{C}$-module categories *that admits module traces.* Ultimately, the legitimacy of $\mathcal{C}$-$\mathcal{M}\mathrm{od}^{\mathrm{tr}}$ as a bicategory $\mathcal{B}$ of surface defects separating two bulk theories of Reshetikhin–Turaev type with label $\mathcal{C}$ is confirmed by the concrete construction of such TFT with defects given in [KMRS21] which provides a link between the model

[3]Note that by taking the full sub-bicategory $\mathcal{C}$-$\mathcal{M}\mathrm{od}^{\mathrm{tr}} \subset \mathcal{C}$-$\mathcal{M}$od, we are keeping all the module functors rather than just the isometries.

independent analysis in [FSV13] and the general description of defect TFT as a symmetric monoidal functor developed in [CRS19].

On the other hand, according to our description of the defect conditions for the RCFT worldsheets, the bicategory $\mathcal{B}$ should contain the bicategory $\mathcal{F}r(\mathcal{C})$ of *simple special symmetric Frobenius algebras, bimodules, and bimodule morphisms* as a full sub-bicategory. Indeed, according to the following theorem, the bicategory $\mathcal{C}\text{-}\mathcal{M}\mathrm{od}^{\mathrm{tr}}$ is canonically biequivalent to $\widehat{\mathcal{F}r}(\mathcal{C})$ of *all* special symmetric Frobenius algebras in $\mathcal{C}$.

Theorem 2.2.1 ([Sch13b]). *Let $\mathcal{C}$ be a pivotal fusion category. The pseudofunctor $\widehat{\mathcal{F}r}(\mathcal{C}) \to \mathcal{C}\text{-}\mathcal{M}\mathrm{od}^{\mathrm{tr}}$ given by*

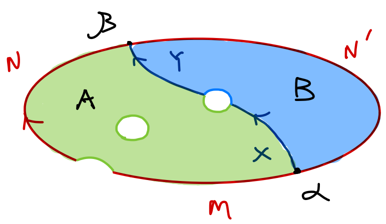
- $A \mapsto \mathrm{mod}^{\mathcal{C}}\text{-}A$
- $X \mapsto - \otimes_A X$
- $\varphi \mapsto - \otimes_A \varphi$

is an equivalence of bicategories. (Recall that the canonical left $\mathcal{C}$-module structure on $\mathrm{mod}^{\mathcal{C}}\text{-}A$ *is given by tensoring with objects in $\mathcal{C}$ on the left.)*

2.3 Field contents and worldsheets with sewing boundaries

It is advantageous, both methodologically and conceptually, to combine all the field insertions of *a given type* (with all possible chiral labels) into a single object, seen as the space of all fields of the sort. Such an object naturally carries an action of an appropriate VOA, which is either the chiral algebra $\mathfrak{V}$ if the fields are inserted on a physical boundary, or $\mathfrak{V} \otimes \mathfrak{V}$ if the fields are inserted in the bulk or on line defects. Thus the objects associated to the boundary field insertions, which we call *boundary field contents*, are objects in $\mathcal{C}$ and the objects associated to the bulk/defect field insertions, which are called *bulk/defect field contents*, are objects in the Drinfeld center $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$.

Let's illustrate this by a concrete example: consider the worldsheet with field insertions $\mathfrak{S}$ as in (2.1.1). By removing a little disk encircling each field insertion, we obtain a worldsheet $\mathcal{S}$ *with sewing boundaries*:

$\mathcal{S} =$  . (2.3.1)

In this example, we have three sewing boundaries of different types: one separating two physical boundaries with boundary conditions M and N, one separating two line defects with defect conditions X and Y, and the remaining one surrounded by the bulk phase A. The associated field contents (in this construction) are:

- *boundary field content* $\mathbb{B}^{M,N} := \bigoplus_{i \in \mathcal{I}(\mathcal{C})} \mathrm{mod}^{\mathcal{C}}\text{-}A(i \otimes M, N) \otimes_{\mathbb{k}} i \in \mathcal{C}$,

- *defect field content* $\mathbb{D}^{X,Y} := \bigoplus_{i,j\in\mathcal{I}(\mathcal{C})} A\text{-mod}^{\mathcal{C}}\text{-}B(i \otimes^- X \otimes^+ j, Y) \otimes_{\Bbbk} \Xi_{\mathcal{C}}(i \boxtimes j) \in \mathcal{Z}(\mathcal{C})$,
- *bulk field content* $\mathbb{D}^{A,A} := \bigoplus_{i,j\in\mathcal{I}(\mathcal{C})} A\text{-mod}^{\mathcal{C}}\text{-}A(i \otimes^- A \otimes^+ j, A) \otimes_{\Bbbk} \Xi_{\mathcal{C}}(i \boxtimes j) \in \mathcal{Z}(\mathcal{C})$.

Here $\mathcal{I}(\mathcal{C})$ is a set of representatives for the isomorphism classes of simple objects of $\mathcal{C}$ that contains the tensor unit $\mathbb{1} \in \mathcal{C}$.

One arrives at these prescriptions by an analysis of the requirement of independence on the chosen triangulation. The resulting explicit expressions may seem unenlightening at first. It turns out, however, that all of them are instances of the notion of *internal homs* in module categories. Recall that for a left module category $\mathcal{M}$ over a tensor category $\mathcal{A}$ and a pair of objects $m, n \in \mathcal{M}$ therein, the internal hom $\underline{\mathrm{Hom}}_{\mathcal{M}}(m,n) \in \mathcal{A}$, if exists, is an object in $\mathcal{A}$ defined (up to unique isomorphisms) by the adjunction

$$\mathcal{M}(c \triangleright m, n) \cong \mathcal{A}(c, \underline{\mathrm{Hom}}_{\mathcal{M}}(m,n)).$$

When the module category $\mathcal{M}$ and the tensor category $\mathcal{A}$ are both finitely semisimple, all internal homs exist. Applying this to the left $\mathcal{C}$-module category $\mathrm{mod}^{\mathcal{C}}\text{-}A$, we find:

$$\begin{aligned} \mathbb{B}^{M,N} &= \bigoplus_{i\in\mathcal{I}(\mathcal{C})} \mathrm{mod}^{\mathcal{C}}\text{-}A(i \otimes M, N) \otimes_{\Bbbk} i \\ &= \int^{c\in\mathcal{C}} \mathrm{mod}^{\mathcal{C}}\text{-}A(c \otimes M, N) \otimes_{\Bbbk} c \\ &\cong \int^{c\in\mathcal{C}} \mathcal{C}(c, \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M,N)) \otimes_{\Bbbk} c \\ &= \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M,N) \in \mathcal{C}. \end{aligned} \tag{2.3.2}$$

Here we first converted the direct sum over all simples into a coend as allowed by the finitely-semisimplicity of $\mathcal{C}$, then used the existence and definition of the internal hom, and finally arrived at the last line by a generalization of the co-Yoneda lemma (see [FSS20, Proposition 2.7]). It should be mentioned that for any $M \in \mathrm{mod}^{\mathcal{C}}\text{-}A$ with non-zero dimension, the boundary field content $\mathbb{B}^{M,M} = \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M,M)$ is a simple special symmetric Frobenius algebra in $\mathcal{C}$ that is *Morita equivalent* to A. Moreover, $\mathbb{B}^{M,M}$ is *haploid* (another term used for this is *"connected"*), i.e. $\mathcal{C}(\mathbb{1}, \mathbb{B}^{M,M}) = \Bbbk$, if and only if M is a simple module.

Remark 2.3.1. In fact, for the $\mathcal{C}$-module category $\mathrm{mod}^{\mathcal{C}}\text{-}A$, we can express the object $\underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M,N)$ in more concrete terms: from the canonical natural isomorphism $\mathrm{mod}^{\mathcal{C}}\text{-}A(c \otimes M, N) \cong \mathcal{C}(c, N \otimes_A M^{\vee})$, we see that, up to a unique isomorphism, $\underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M,N) = N \otimes_A M^{\vee}$.

The recognition of bulk and defect field contents as internal homs is less straight forward and crucially relies on the theory of *internal natural transformations* developed in [FS21b].

Let us denote

$$G^X := - \otimes_A X \qquad \text{and} \qquad G^Y := - \otimes_A Y, \tag{2.3.3}$$

where $X, Y \in A\text{-mod}^{\mathcal{C}}\text{-}B$. They are left $\mathcal{C}$-module functors from $\mathrm{mod}^{\mathcal{C}}\text{-}A$ to $\mathrm{mod}^{\mathcal{C}}\text{-}B$, i.e.

$$G^X, G^Y \in [\mathrm{mod}^{\mathcal{C}}\text{-}A, \mathrm{mod}^{\mathcal{C}}\text{-}B] \equiv \mathcal{C}\text{-}\mathcal{M}\mathrm{od}^{\mathrm{tr}}(\mathrm{mod}^{\mathcal{C}}\text{-}A, \mathrm{mod}^{\mathcal{C}}\text{-}B).$$

The hom-category $[\mathrm{mod}^{\mathcal{C}}\text{-}A, \mathrm{mod}^{\mathcal{C}}\text{-}B]$ is finitely semisimple and has a canonical left $\mathcal{Z}(\mathcal{C})$-module structure induced by the half-braidings. As a consequence, the internal homs for this $\mathcal{Z}(\mathcal{C})$-module category exists. We denote for each pair $G, G' \in [\mathrm{mod}^{\mathcal{C}}\text{-}A, \mathrm{mod}^{\mathcal{C}}\text{-}B]$,

$$\underline{\mathrm{Nat}}(G, G') := \underline{\mathrm{Hom}}_{[\mathrm{mod}^{\mathcal{C}}\text{-}A, \mathrm{mod}^{\mathcal{C}}\text{-}B]}(G, G') \in \mathcal{Z}(\mathcal{C}),$$

and call it the *object of internal natural transformations from G to G'.*

Using the biequivalence from Theorem 2.2.1, we now recognize the defect field content $\mathbb{D}^{X,Y}$ as $\underline{\mathrm{Nat}}(G^X, G^Y)$:

$$\begin{aligned}\mathbb{D}^{X,Y} &= \bigoplus_{i,j \in \mathcal{I}(\mathcal{C})} A\text{-}\mathrm{mod}^{\mathcal{C}}\text{-}B(i \otimes^- X \otimes^+ j, Y) \otimes_{\Bbbk} \Xi_{\mathcal{C}}(i \boxtimes j) \\ &\cong \int^{z \in \mathcal{Z}(\mathcal{C})} [\mathrm{mod}^{\mathcal{C}}\text{-}A, \mathrm{mod}^{\mathcal{C}}\text{-}B](z \triangleright G^X, G^Y) \otimes_{\Bbbk} z \\ &\cong \int^{z \in \mathcal{Z}(\mathcal{C})} \mathcal{Z}(\mathcal{C})(z, \underline{\mathrm{Nat}}(G^X, G^Y)) \otimes_{\Bbbk} z \\ &= \underline{\mathrm{Nat}}(G^X, G^Y) \in \mathcal{Z}(\mathcal{C}).\end{aligned} \tag{2.3.4}$$

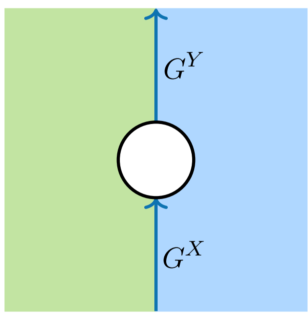
Just like for conventional set of natural transformations, we have the following expression of $\underline{\mathrm{Nat}}(G^X, G^Y)$ as an end [FS21b, Theorem 18]:

$$\underline{\mathrm{Nat}}(G^X, G^Y) = \int_{M \in \mathrm{mod}^{\mathcal{C}}\text{-}A} \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}B}(G^X(M), G^Y(M)) \in \mathcal{Z}(\mathcal{C}).$$

Here we have made the canonical half-braiding $\gamma_{\underline{\mathrm{Nat}}(G^X,G^Y)} \equiv \gamma_{\mathbb{D}^{X,Y}}$ obtained from the universal coaction of the central comonad of $\mathcal{C}$ (for details see [FS21b, Lemma 13]) for the moment implicit, see (6.1.2) for the explicit formula. Combining this with the finitely-semisimplicity of $\mathrm{mod}^{\mathcal{C}}\text{-}A$ and Remark 2.3.1, we have

$$\mathbb{D}^{X,Y} = \Big(\bigoplus_{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} m \otimes_A Y \otimes_B X^\vee \otimes_A m^\vee, \gamma_{\mathbb{D}^{X,Y}}\Big) \in \mathcal{Z}(\mathcal{C}). \tag{2.3.5}$$

Remark 2.3.2. It's worth pointing out that taking the Poincaré dual of the picture (after we simplify the sewing circle to a vertex)



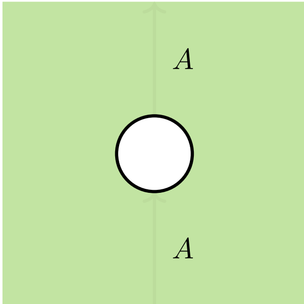
(2.3.6)

gives us

$$\begin{tikzcd} \mathrm{mod}^{\mathcal{C}}\text{-}A \arrow[r, bend left, "G^Y"] \arrow[r, bend right, "G^X"'] \arrow[r, phantom, "\Uparrow"] & \mathrm{mod}^{\mathcal{C}}\text{-}B \end{tikzcd} \tag{2.3.7}$$

The reminiscence of this picture with the standard depiction of natural transformations in terms of pasting diagrams fits well with the choice of terminology for $\underline{\mathrm{Nat}}$.

Bulk fields should be thought of as a special case of defect fields, where the defect conditions are all *trivial*, i.e. given by the algebra itself, viewed as a bimodule. In the example of ours, we have



.

Since the functor $G^A = - \otimes_A A$ is canonically isomorphic to the identity functor $\mathrm{id}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}$, the bulk field content is indeed the *full center* of A:

$$\mathbb{D}^{A,A} = \underline{\mathrm{Nat}}(\mathrm{id}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}, \mathrm{id}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}) = Z(A) \in \mathcal{Z}(\mathcal{C}),$$

which has a canonical structure of a commutative symmetric Frobenius algebra in the modular fusion category $\mathcal{Z}(\mathcal{C})$ that is canonically isomorphic to the one investigated in [KR08].

After discussing the heuristics, we now formulate the precise notion of a worldsheet (with sewing boundaries). This will proceed in two steps: first we describe the purely geometric features, then complement with the algebraic data. In this thesis, we work with topological manifolds to avoid bureaucracy. After all, the category of two-dimensional oriented smooth manifolds is equivalent to that of two-dimensional oriented topological manifolds.

Definition 2.3.3. An *unlabeled worldsheet* $\breve{\mathcal{S}}$ is an oriented compact surface (with possibly non-empty boundary) equipped with a collection of closed submanifolds of dimension 0, 1, and 2 – to be referred to as *0-cells*, *1-cells* and *2-cells*, respectively – satisfying the following conditions:

- The number of cells is finite.
- The set of 0-cells is precisely the set of boundaries of all 1-cells, the union of the 1-cells is the union of the boundaries of all 2-cells, and the union of the 2-cells is $\breve{\mathcal{S}}$.
- The intersection of any pair of 1-cells is contained in the set of 0-cells, and the intersection of any pair of 2-cells is contained in the set of 1-cells.
- The interior of every 1-cell is either contained in the interior of $\breve{\mathcal{S}}$ or in the boundary $\partial\breve{\mathcal{S}}$.
- Every 1-cell in the interior of $\breve{\mathcal{S}}$ is oriented as a 1-manifold.

- There are two types of *boundary 1-cells*, i.e. 1-cells in $\partial\check{\mathcal{S}}$: oriented and unoriented. The orientation of an oriented boundary 1-cell is opposite to the one induced by orientation of $\check{\mathcal{S}}$.

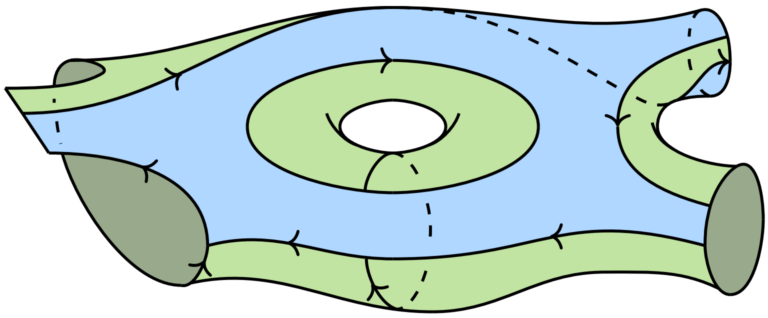
- A 0-cell in $\partial\check{\mathcal{S}}$ at which two unoriented boundary 1-cells meet is in addition met by precisely one oriented 1-cell, whose interior is contained in the interior of $\check{\mathcal{S}}$.

- A 0-cell in $\partial\check{\mathcal{S}}$ at which an unoriented and an oriented boundary 1-cell meet, is not met by any further 1-cell.

- Every connected component of $\partial\check{\mathcal{S}}$ contains at least one 0-cell.

The stratification prescribed in Definition 2.3.3 is intended to capture the notions of topological defects, physical boundaries and sewing boundaries. In detail: the 1-cells in the interior of an unlabeled worldsheet $\check{\mathcal{S}}$ are called *line defects*; the oriented 1-cells in $\partial\check{\mathcal{S}}$ are called *physical boundaries*; 0-cells in the interior and on the boundary at which two physical boundaries meet are called *point defects* or *defect junctions*; the unoriented 1-cells in $\partial\check{\mathcal{S}}$ are called *sewing boundaries.* We refer to the connected components of the boundary of $\check{\mathcal{S}}$ as *geometric boundary circles* and denote the set of these by $\pi_0(\partial\check{\mathcal{S}})$. If all 1-cells in a geometric boundary circle c are sewing boundaries, then c is called a *sewing circle.* If a geometric boundary circle c contains at least one physical boundary, then each connected component of the complement of the interior of the union of all physical boundaries in c is called a *sewing interval.* A geometric boundary circle can contain any finite number of sewing intervals.

Example 2.3.4. The following picture shows an example of an unlabeled worldsheet of genus 1 and with three geometric boundary circles, among which two of them are sewing circles and the remaining one contains a sewing interval.



(2.3.8)

Here for clarity we have dyed the two 2-cells with different colors.

Having the terminologies at hand, we can now formulate the notion of a worldsheet whose strata are colored with $\mathcal{F}r(\mathcal{C})$.

Definition 2.3.5. A worldsheet $\mathcal{S}$ is an unlabeled worldsheet $\check{\mathcal{S}}$ together with the following assignments of labels to the strata (aka cells) of $\check{\mathcal{S}}$:

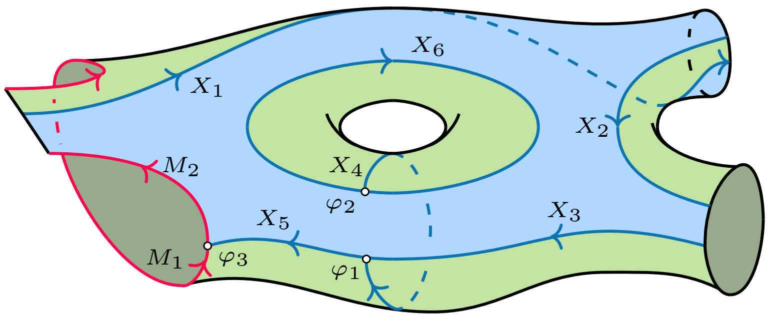
- to any 2-cell of $\check{\mathcal{S}}$ a simple special symmetric Frobenius algebra in $\mathcal{C}$ called its *phase*;

- to any line defect an A_{l}-A_{r}-bimodule called its *defect condition*, where A_{l} and A_{r} are the labels for the adjacent 2-cells to the left and to the right[4] of it, respectively;
- to any physical boundary a right A-module called its *boundary condition*, where A is the label for the adjacent 2-cell;
- to any point defect v in the interior of $\breve{\mathcal{S}}$ a bimodule morphism (called its *defect condition*) in a space H_v, which is determined by the defect conditions of the line defects that meet at v;
- to any point defect w on the boundary of $\breve{\mathcal{S}}$ a module morphism (also called its *defect condition*) in a space H_w determined by the conditions of the line defects and the physical boundaries meeting at w.

The sewing boundaries, as well as their end points are not labeled and the spaces H_v and H_w will be specified in (2.3.10) and (2.3.11).

Before specifying the spaces of point defect conditions, let us first illustrate the already somewhat lengthy description of a worldsheet by an example.

Example 2.3.6. The worldsheet



(2.3.9)

with the underlying unlabeled worldsheet (2.3.8) has:

- two 2-cells, colored with green and blue, with respective phases $A, B \in \mathcal{F}r(\mathcal{C})$;
- six line defects with respective defect conditions $X_1, X_2, X_3, X_5 \in A\text{-mod}^{\mathcal{C}}\text{-}B$, $X_4 \in A\text{-mod}^{\mathcal{C}}\text{-}A$, $X_6 \in B\text{-mod}^{\mathcal{C}}\text{-}A$;
- two physical boundaries with respective boundary conditions $M_1 \in \text{mod}^{\mathcal{C}}\text{-}A$ and $M_2 \in \text{mod}^{\mathcal{C}}\text{-}B$;
- three point defects with defect conditions φ_1, φ_2 and φ_3 (in the corresponding morphism spaces that will be given in Example 2.3.7 below).

Concerning the space of defect conditions H_v assigned to a point defect v in the interior of a worldsheet, we first note that the orientation of the worldsheet furnishes a *cyclic order* of the line defects[5] incident to v in such a way that this cyclic order is *clockwise* if

[4] Here the perspective is given by an imaginary observer standing on the line defect facing towards the direction of the defect, whose "up" pointing towards the direction given by the orientation of the surface according to the *right-hand rule*.

[5] Here they should be thought of as *half-edges*.

the orientation of the surface is counter-clockwise. Suppose that for some selected linear order *compatible with this cyclic order*, the defect lines meeting at v are labeled clockwise by bimodules X_i for $i = 1, 2, \ldots, n$. We write $X_i^\vee$ for the bimodule dual to X_i, and set

$$X^\epsilon := \begin{cases} X & \text{for } \epsilon = +, \\ X^\vee & \text{for } \epsilon = -. \end{cases}$$

Then $X_i^{\epsilon_i}$ is an A_i-A_{i+1}-bimodule for $i < n$ and $X_n^{\epsilon_n}$ an A_n-A_1-bimodule, for Frobenius algebras A_j the corresponding phases, where $\epsilon_i = +$ if the line defect labeled by X_i is *oriented away* from v, while $\epsilon_i = -$ if it is *oriented towards* v. For the chosen linear order, the space H_v is now defined to be

$$H_v := A_1\text{-mod}^{\mathcal{C}}\text{-}A_1(A_1, X_1^{\epsilon_1} \otimes_{A_2} \cdots \otimes_{A_n} X_n^{\epsilon_n}). \tag{2.3.10}$$

If a different choice of linear order of the defect lines incident at v is made, the same prescription gives instead the space

$$A_j\text{-mod}^{\mathcal{C}}\text{-}A_j(A_j, X_j^{\epsilon_j} \otimes_{A_{j+1}} \cdots \otimes_{A_{j-1}} X_{j-1}^{\epsilon_{j-1}})$$

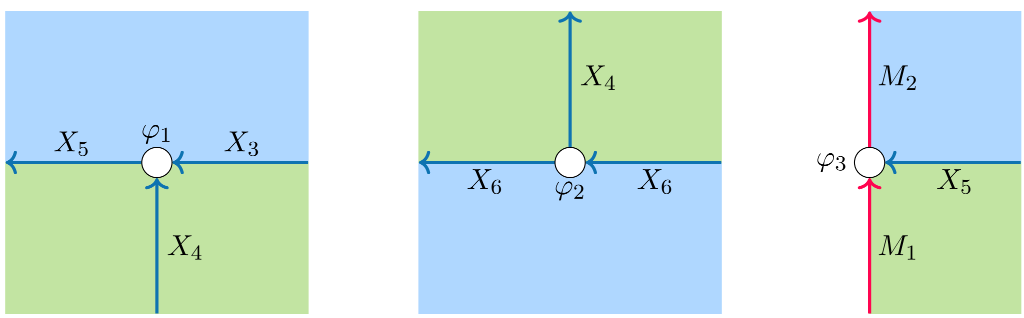
for some $j \in \{2, 3, \ldots, n\}$ (with labels counted modulo n). Owing to pivotality of the bicategory $\mathcal{F}r(\mathcal{C})$, this space is *canonically* isomorphic to H_v as defined in (2.3.10). Accordingly, the choice of linear order is immaterial. This will be articulated more formally in Section 3.2.

For a point defect located at w in the boundary $\partial\check{\mathcal{S}}$, there is directly a linear order on the set of (half-edges of) physical boundaries and line defects incident to w. If the physical boundary oriented towards w is labeled by a right A_{n+1}-module M and the physical boundary oriented away from w is labeled by a right A_1-module N, then the space assigned to w is defined to be

$$H_w := \text{mod}^{\mathcal{C}}\text{-}A_{n+1}(M, N \otimes_{A_1} X_1^{\epsilon_1} \otimes_{A_2} \cdots \otimes_{A_n} X_n^{\epsilon_n}), \tag{2.3.11}$$

with the same convention for ϵ_i's as before. In particular, in case $n = 0$, so that only the two physical boundaries meet at w, we deal with the space $\text{mod}^{\mathcal{C}}\text{-}A_1(M, N)$.

Example 2.3.7. For the worldsheet shown in Example 2.3.6 the situations near the three point defects appear as follows:



Then for a suitable choice of linear order the prescription (2.3.10) amounts to

$$\varphi_1 \in A\text{-mod}^{\mathcal{C}}\text{-}A(A, X_5 \otimes_B X_3^\vee \otimes_A X_4^\vee) \quad \text{and} \quad \varphi_2 \in B\text{-mod}^{\mathcal{C}}\text{-}B(B, X_6 \otimes_A X_4 \otimes_A X_6^\vee),$$

while (2.3.11) gives $\varphi_3 \in \text{mod}^{\mathcal{C}}\text{-}A(M_1, M_2 \otimes_B X_5^\vee)$.

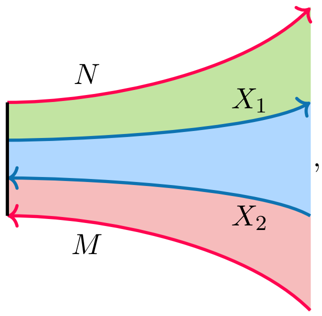
We end this section by giving a detailed discussion of field contents associated to the general types of sewing circles and intervals appearing in the worldsheets that are allowed by Definition 2.3.5.

Let us start with the boundary field contents. Consider a sewing interval r, we associate to it an object $\mathbb{B}_r \in \mathcal{C}$ as follows. Denote the set of 0-cells in r at which line defects end by O_r. On this set there is a linear order inherited from the prescription for the orientation of physical boundaries. Assume that the physical boundary oriented towards one of the end points of r is labeled by a right A_{n+1}-module M and the physical boundary oriented away from the other end point of r is labeled by a right A_1-module N, and that the defect lines that meet the 0-cells in O_r are labeled clockwise by bimodules X_i for $i = 1, 2, \ldots, n = |O_r|$. Define $\epsilon_i \in \{+, -\}$ in a way analogous to that in (2.3.10) and (2.3.11), i.e. set $\epsilon_i = +$ if the defect line labeled by X_i is oriented away from a 0-cell in O_r and $\epsilon_i = -$ otherwise. Then the *boundary field content for the sewing interval* r is

$$\begin{aligned} \mathbb{B}_r &\coloneqq \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A_{n+1}}(M, G^{X_n^{\epsilon_n}} \circ \cdots \circ G^{X_1^{\epsilon_1}}(N)) \\ &= N \otimes_{A_1} X_1^{\epsilon_1} \otimes_{A_2} \cdots \otimes_{A_n} X_n^{\epsilon_n} \otimes_{A_{n+1}} M^\vee \in \mathcal{C}. \end{aligned} \tag{2.3.12}$$

In particular, in case $n = 0$ we have $\mathbb{B}_r = \mathbb{B}^{M,N}$.

Example 2.3.8. For the sewing interval



the boundary field content is

$$\mathbb{B}_r = \underline{\mathrm{Hom}}_{\mathrm{mod}\text{-}A_3}(M, G^{X_2^\vee} \circ G^{X_1}(N)) = N \otimes_{A_1} X_1 \otimes_{A_2} X_2^\vee \otimes_{A_3} M^\vee \in \mathcal{C}.$$

Finally, let's look at defect field contents, which include, as already mentioned, bulk field contents as special cases. Recall that a geometric boundary circle c is called a sewing circle if it does not contain any physical boundaries. Denote by O_c the finite set of 0-cells contained in c, which inherits from the orientation of the worldsheet a cyclic order that agrees with the induced orientation of the sewing circle c. Unlike the space of defect conditions H_v of a point defect living in the interior of a worldsheet, the defect *field content* $\mathbb{D}_c \in \mathcal{Z}(\mathcal{C})$ *for the sewing circle* c does depend on the choice of linear order on O_c (that is compatible with the cyclic order): even though different choices thereof result in isomorphic objects in $\mathcal{Z}(\mathcal{C})$, the isomorphisms are not canonical. Let's say a linear order is chosen and the line defects ending at the 0-cells in O_c are labeled by bimodules X_i for $i = 1, 2, \ldots, n = |O_c|$, numbered according to the chosen linear order, with each corresponding $X_j^{\epsilon_j}$ an A_j-A_{j+1}-bimodule (where $A_{n+1} = A_1$). Then

$$\mathbb{D}_c \coloneqq \underline{\mathrm{Nat}}(\mathrm{id}_{\mathrm{mod}^{\mathcal{C}}\text{-}A_1}, G^{X_n^{\epsilon_n}} \circ \cdots \circ G^{X_1^{\epsilon_1}})$$

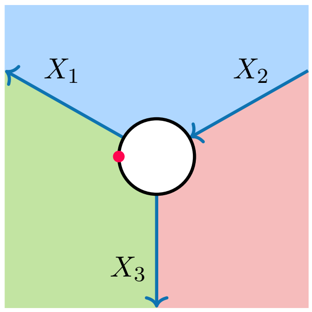
$$= \int_{M \in \mathrm{mod}^{\mathcal{C}}\text{-}A_1} \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A_1}(M, M \otimes_{A_1} X_1^{\epsilon_1} \otimes_{A_2} \cdots \otimes_{A_n} X_n^{\epsilon_n})$$
$$= \Big(\bigoplus_{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A_1)} m \otimes_{A_1} X_1^{\epsilon_1} \otimes_{A_2} \cdots \otimes_{A_n} X_n^{\epsilon_n} \otimes_{A_1} m^\vee, \gamma_{\mathbb{D}_c} \Big) \in \mathcal{Z}(\mathcal{C}). \tag{2.3.13}$$

Note that for the situation in Remark 2.3.2, we have $n = 2$ and O_c is implicitly ordered such that $X_1 = Y$ and $X_2 = X$, while $\epsilon_1 = +$ and $\epsilon_2 = -$. In this case, we have a canonical isomorphism

$$\mathbb{D}_c = \underline{\mathrm{Nat}}(\mathrm{id}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}, G^{X^\vee} \circ G^Y) \cong \underline{\mathrm{Nat}}(G^X, G^Y) = \mathbb{D}^{X,Y}.$$

We end this section by another example of a defect field content.

Example 2.3.9. Consider the following sewing circle parameterized by the standard circle $S^1 \subset \mathbb{C}$:

$c =$  ,

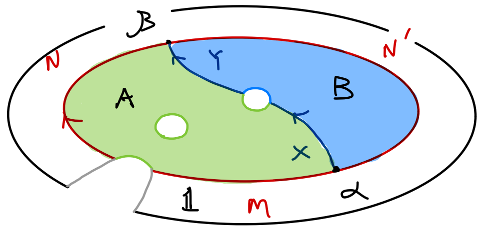
where the canonical orientation of the standard circle S^1 (conventionally chosen to be counterclockwise) is reversed under the parametrization, and the image of the distinguished point $-1 \in S^1$ is marked by a red dot. This parametrization allows us to pick a linear order on O_c, as we will always do hereafter, by choosing the starting point to be the 0-cell immediately before the red dot (which is the endpoint of the line defect labeled by X_1 in this case). Under this choice, we have

$$\mathbb{D}_c = \underline{\mathrm{Nat}}(\mathrm{id}_{\mathrm{mod}^{\mathcal{C}}\text{-}A_1}, G^{X_3} \circ G^{X_2^\vee} \circ G^{X_1})$$
$$= \Big(\bigoplus_{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A_1)} m \otimes_{A_1} X_1 \otimes_{A_2} X_2^\vee \otimes_{A_3} X_3 \otimes_{A_1} m^\vee, \gamma_{\mathbb{D}_c} \Big) \in \mathcal{Z}(\mathcal{C}).$$

2.4 Complemented worldsheets and ambient bordisms

An important fact about the spaces of conformal blocks is that they only depend on the topology and the types of sewing boundaries of the worldsheets for which they are defined. Therefore, it is necessary to provide a suitable notion of the *underlying surface* for a worldsheet with sewing boundaries.

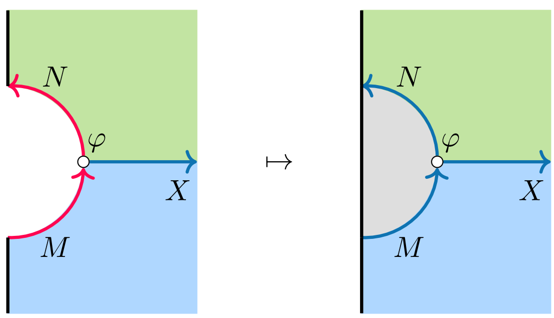
It turns out to be natural for the constructions in this thesis to equip every worldsheet with an embedding into a surface that is homeomorphic to the worldsheet itself. Before giving a complete formulation, let us return to the expression of the correlator in (2.1.2) for some insights. As we already mentioned, correlators should be viewed as surface defects separating two bulk TFTs of Reshetikhin-Turaev type with categories of Wilson lines both given by the modular fusion category $\mathcal{C}$. If one looks at (2.1.2) carefully enough, they will notice that there are in fact *transparent surface defects* (i.e. those that are labeled by the trivial Frobenius algebra $\mathbb{1}$ or equivalently, the trivial $\mathcal{C}$-module category $\mathcal{C} = \mathrm{mod}^{\mathcal{C}}\text{-}\mathbb{1}$) filling the gap between the physical boundaries of $\mathcal{S}$ and the holomorphic double $\widehat{\Sigma}_{\mathfrak{S}}$. After being detached from the field insertions, the continuum of surface defects looks like:



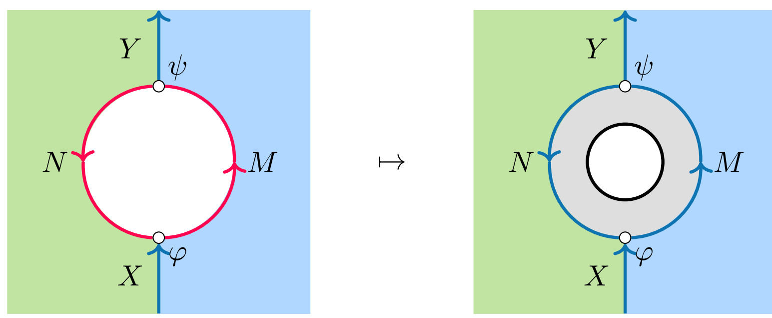
(2.4.1)

The above picture differs from (2.3.1) in that it carries an extra 2-cell, glued along the union of physical boundaries, that is labeled by the trivial Frobenius algebra $\mathbb{1}$. We call 2-cells like this *transparent 2-cells*. Consequently, we should now view each physical boundary as a line defect separating the trivial phase from the corresponding (possibly) non-trivial phase. This is consistent with the fact that the category of right modules $\mathrm{mod}^{\mathcal{C}}\text{-}A$ is canonically equivalent to the category of bimodules $\mathbb{1}\text{-}\mathrm{mod}^{\mathcal{C}}\text{-}A$.

In general, if a geometric boundary circle c contains sewing boundaries, we attach to each connected component of the union of physical boundaries in c a transparent 2-cell that is homeomorphic to a *disk*, as depicted below:



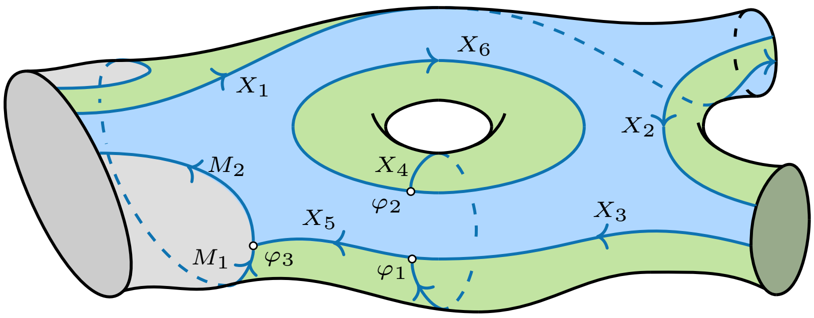
On the other hand, if c is a *pure physical boundary*, i.e. it does not contain any sewing boundaries, we then attach to it a transparent 2-cell that is homeomorphic to a *cylinder*, for instance:



.

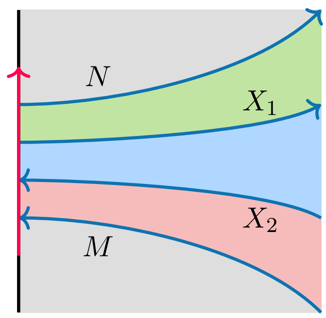
In this way, we obtain from a worldsheet $\mathcal{S}$ a new stratified surface $\widetilde{\mathcal{S}}$, to be referred to as the *complemented worldsheet* of $\mathcal{S}$. We call the underlying surface of $\widetilde{\mathcal{S}}$ (forgetting all the strata along with their labels), denoted by $\Sigma_{\mathcal{S}}$, the *ambient surface* of $\mathcal{S}$. The terminology is inspired by the canonical embedding $\mathcal{S} \hookrightarrow \Sigma_{\mathcal{S}}$. Strictly speaking, neither the complemented worldsheet $\widetilde{\mathcal{S}}$ nor the ambient surface $\Sigma_{\mathcal{S}}$ is uniquely determined by our prescription. However, given any pair of such constructions, there exists a unique homeomorphism (up to isotopies) between the ambient surfaces that is compatible with the embeddings.

Example 2.4.1. For the worldsheet in Example 2.3.6, we illustrate its complemented worldsheet as:



(2.4.2)

Just by looking at a complemented worldsheet itself, one cannot always tell whether it came from a worldsheet that contains sewing intervals. For instance, the complemented worldsheet in (2.4.2) might as well have come from a worldsheet with three sewing circles. To keep tract of the sewing intervals, we equip the ambient surface $\Sigma_{\mathcal{S}}$ the structure of an *open-closed bordism*: we parameterize (we call an *embedding* $\ell_1 \hookrightarrow \ell_2$ of the 1-manifold ℓ_1 into the 1-manifold ℓ_2 a *parametrization of* ℓ_2 *by* ℓ_1) the image of each sewing circle (under the embedding $\mathcal{S} \hookrightarrow \Sigma_{\mathcal{S}}$) in $\Sigma_{\mathcal{S}}$ by the standard circle $S^1 \subset \mathbb{C}$; and for each sewing interval r, we first choose a closed interval I_r in $\partial\Sigma_{\mathcal{S}}$ whose interior contains (aside from portions of the boundaries of some transparent 2-cells) only the image of r, then parameterize it with the standard interval $I = [0, 1] \subset \mathbb{R}$. As an example:

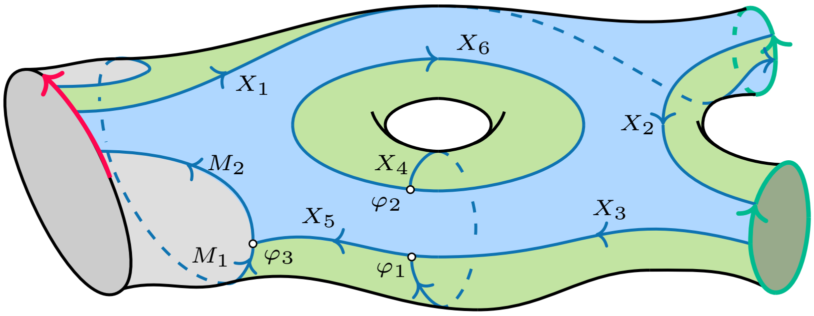


(2.4.3)

Here the image of I under the parametrization is depicted by an red arrow pointing from the image of 1 towards the image of 0. Let us always equip the standard circle with the orientation going counterclockwise and the standard interval with the one going from 1 to 0. A parametrization either preserves or reverses the orientation of the standard circle/interval, where the orientation of the boundary $\partial\Sigma_{\mathcal{S}}$ is induced by that of $\Sigma_{\mathcal{S}}$ extending the orientation of $\mathcal{S}$. We call the image of an orientation preserving parametrization *out-going* circle/interval and that of an orientation reversing one *in-going* circle/interval. For instance, the red interval in (2.4.3) is in-going, if the partially shown ambient surface is oriented counterclockwise.

An ambient surface $\Sigma_{\mathcal{S}}$ equipped with a parametrization for every sewing circle and sewing interval of the original worldsheet $\mathcal{S}$ is called an *ambient bordism*. This is an additional structure and for a given worldsheet there can be several of such. However, we use the same symbol for an ambient bordism and its underlying ambient surface leaving the parametrization implicit.

Example 2.4.2. The ambient surface in Example 2.4.1 can be promoted to an ambient bordism as follows:



(2.4.4)

It has one in-going interval, denoted by the red arrow on the left, and two out-going circles, denoted by the two oriented green circles on the right.

2.5 Boundary data and sewing

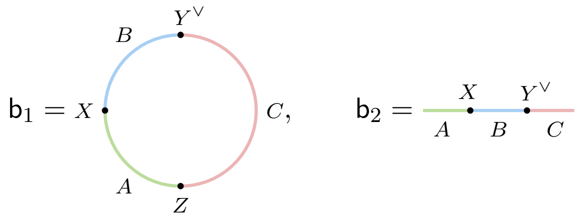
We need the following notion of a boundary datum.

Definition 2.5.1. An $\mathcal{F}r(\mathcal{C})$-*boundary datum* b on a compact oriented 1-manifold ℓ with possibly non-empty boundary is a collection of following data:

- a finite set $O_\ell \subset \operatorname{int}\ell$ of points in the *interior* of the 1-manifold ℓ;
- a coloring of the connected components of the complement of O_ℓ with the objects in $\mathcal{F}r(\mathcal{C})$, i.e. a map $\mathsf{C}^0_\mathsf{b}\colon \pi_0(\ell \setminus O_\ell) \to \operatorname{obj}\mathcal{F}r(\mathcal{C})$;
- a coloring of O_ℓ with the 1-morphisms in $\mathcal{F}r(\mathcal{C})$, i.e. a map $\mathsf{C}^1_\mathsf{b}\colon O_\ell \to \text{1-mor}\,\mathcal{F}r(\mathcal{C})$, such that for $p \in O_\ell$ and $a_\mathrm{l}, a_\mathrm{r} \in \pi_0(\ell \setminus O_\ell)$, where a_l and a_r are the components on the left and right hand side of p, respectively, with the convention that the orientation of ℓ points *from the right to the left*, the color of p is a 1-morphism in $\mathcal{F}r(\mathcal{C})$ from the color of a_l to that of a_r, i.e. $\mathsf{C}^1_\mathsf{b}(p) \in \mathcal{F}r(\mathcal{C})(\mathsf{C}^0_\mathsf{b}(a_\mathrm{l}), \mathsf{C}^0_\mathsf{b}(a_\mathrm{r})) = \mathsf{C}^0_\mathsf{b}(a_\mathrm{l})\text{-mod}^{\mathcal{C}}\text{-}\mathsf{C}^0_\mathsf{b}(a_\mathrm{r})$.

We will refer to an $\mathcal{F}r(\mathcal{C})$-boundary datum simply as a *boundary datum* most of the time from now on.

Example 2.5.2. Let $A, B, C \in \mathcal{F}r(\mathcal{C})$ and $X \in A\text{-mod}^{\mathcal{C}}\text{-}B$, $Y \in C\text{-mod}^{\mathcal{C}}\text{-}B$, $Z \in C\text{-mod}^{\mathcal{C}}\text{-}A$, then



are boundary data on the standard circle S^1 and the standard interval I, respectively.

Given an ambient bordism $\Sigma_{\mathcal{S}}$ of a worldsheet $\mathcal{S}$, the information of the sewing circles and sewing intervals of $\mathcal{S}$ can be encoded in the boundary data *pulled back* along the parametrization maps. Let ℓ be a finite disjoint union of standard circles and intervals. Under a parametrization $\ell \hookrightarrow \partial\Sigma_{\mathcal{S}}$, the image of ℓ covers a union of sewing circles and sewing intervals. We can pull back the 0-cells, which provides the finite set $O_\ell \subset \operatorname{int}\ell$, and the coloring for $\pi_0(\ell \setminus O_\ell)$. More care should be taken for the coloring of O_ℓ due to the orientation:

1. If ℓ is *out-going*, i.e. the parametrization map *preserves* the orientation of ℓ, then we define the coloring to be

$$\begin{aligned} \mathsf{C}^1_{\mathsf{b}} \colon O_\ell &\to 1\text{-mor}\,\mathcal{F}r(\mathcal{C}) \\ p &\mapsto X_p^{\epsilon_p}, \end{aligned} \tag{2.5.1}$$

   where X_p is the label of the line defect ending at the image of p, and $\epsilon_p \in \{+, -\}$ is set to be $+$ if the line defect is oriented *towards* the image of p and $-$ otherwise.

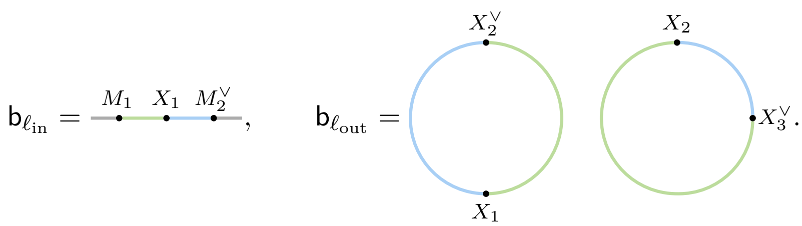
2. On the other hand, if ℓ is *in-going,* i.e. the parametrization *reverses* the orientation of ℓ, then we define the coloring to be

$$\begin{aligned} \mathsf{C}^1_{\mathsf{b}} \colon O_\ell &\to 1\text{-mor}\,\mathcal{F}r(\mathcal{C}) \\ p &\mapsto X_p^{-\epsilon_p}, \end{aligned} \tag{2.5.2}$$

   with the same convention for ϵ_p but be aware of the minus sign in front of it.

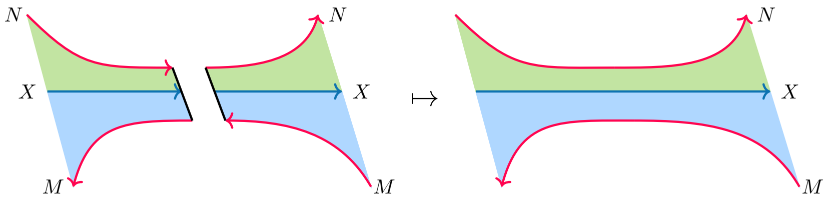
Denote by b_ℓ the boundary datum on ℓ obtained this way.

Example 2.5.3. Let the ambient bordism be the one in Example 2.4.2, we have $\ell_{\text{in}} = I$ parameterizing the in-going interval and $\ell_{\text{out}} = S^1 \sqcup S^1$ parameterizing two out going circles. The corresponding boundary data are

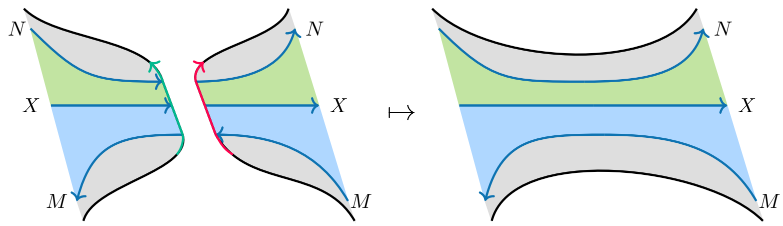


Notice that a boundary datum obtained by the parametrization of any sewing interval always has its outermost 1-cells colored with the trivial Frobenius algebra $\mathbb{1}$.

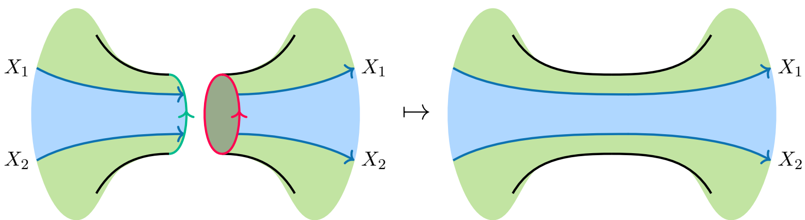
We would like to implement the sewing of worldsheets along *matching sewing boundaries* by sewing their appropriate choices of ambient bordisms *along matching parametrizations*. Let $\ell_{\text{out}} = \ell_{\text{in}}$ be a disjoint union of standard circles and intervals parameterizing subsets of *out-going* and *in-going* sewing boundaries of some worldsheet $\mathcal{S}$, respectively, with their respective images not necessarily laying in the same connected component of the ambient surface. The parametrizations are called *matching* if we have $\mathsf{b}_{\ell_{\text{out}}} = \mathsf{b}_{\ell_{\text{in}}}$. In this case, the corresponding sewing boundaries of the worldsheet $\mathcal{S}$ matches as well and sewing the corresponding ambient bordism $\Sigma_{\mathcal{S}}$ along the parametrizations by ℓ_{out} and ℓ_{in} implements the sewing of $\mathcal{S}$ along the corresponding matching sewing boundaries. For instance, the sewing of matching sewing intervals depicted by



is implemented by the following sewing of ambient bordism along matching parametrizations:



Here the out-going and in-going intervals are denoted by a green and a red arrow, respectively. Note that it is necessary to require the image of the standard interval to cover a range that is slightly larger than the sewing interval under a parametrization, so that the act of taking complemented worldsheet is compatible with sewing, i.e. $\cup_\ell \widetilde{\mathcal{S}} = \widetilde{\cup_\ell \mathcal{S}}$ up to isotopies[6]. In comparison, it is more straightforward to sew along sewing circles, since no extra transparent 2-cells are involved:



[6]Unlike in 2-dimensional topological field theories, the topology of the regions with trivial phase matters in a 2d CFT. This is essentially due to the fact that the central monad for a non-trivial tensor category acts non-trivially, which ensures the non-trivial dependence of spaces of conformal blocks on the topology of the surfaces.

2.6 Formulation of the task

We are now in the position to give a clear formulation of the task of solving a rational conformal field theory with given category of chiral data $\mathcal{C}$, which is assumed to be *small* for convenience[7], for all worldsheet with possibly non-empty topological defects and physical boundaries. Let's start with a precise definition of an *open-closed modular functor.*

Definition 2.6.1. An *open-closed modular functor* is a symmetric monoidal pseudofunctor

$$\mathrm{Bl}\colon \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \to \mathcal{P}\mathrm{rof}_{\Bbbk}$$

from the symmetric monoidal bicategory of two-dimensional open-closed bordisms to the symmetric monoidal bicategory of $\Bbbk$-linear profunctors.

Let us unwrap this definition. First, the bicategory $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ of *2-dimensional oriented open-closed bordisms* is the symmetric monoidal bicategory with:

- objects: finite disjoint unions of the standard closed interval $I = [0,1] \subset \mathbb{R}$ with the orientation going *from* 1 *to* 0, and the standard circle $S^1 \subset \mathbb{C}$ with the orientation going *counterclockwise*;

- 1-morphisms: for objects α and β, a 1-morphism[8] $\Sigma\colon \alpha \nrightarrow \beta$, called an *open-closed bordism from α to β* has an underlying compact oriented surface Σ with an *orientation reversing* embedding $\alpha \hookrightarrow \partial\Sigma$, called the *in-going* parametrization, and an *orientation preserving* embedding $\beta \hookrightarrow \partial\Sigma$, called the *out-going* parametrization, such that the images of the two are disjoint (and their union is not necessarily the whole boundary);

- 2-morphisms: for 1-morphisms $\Sigma, \Sigma'\colon \alpha \nrightarrow \beta$, a 2-morphism $\xi\colon \Sigma \Rightarrow \Sigma'$ is an *isotopy class* of homeomorphisms from Σ to Σ', with every homeomorphism $\xi \ni x\colon \Sigma \to \Sigma'$ therein compatible with the parametrizations, i.e. the diagram

$$\begin{tikzcd} & \partial\Sigma \arrow[dd, "x|_{\partial\Sigma}"] & \\ \alpha \arrow[ur, hook] \arrow[dr, hook] & & \beta \arrow[ul, hook'] \arrow[dl, hook'] \\ & \partial\Sigma' & \end{tikzcd}$$

  commutes[9];

- vertical composition: is induced by the composition of homeomorphisms;

[7]Note that $\mathcal{C}$ is already *essentially small* due to its finitely-semisimplicity.

[8]We use a marked arrow here because the bicategory $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ (as well as $\mathcal{P}\mathrm{rof}_{\Bbbk}$, and we regard a bicategory as a double category with only trivial vertical 1-morphisms) can be canonically embedded in a double category where the images of the 1-morphisms of $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ are horizontal 1-morphisms (also called *proarrows*).

[9]Due to the compatibility with the parametrizations, the homeomorphism $x\colon \Sigma \to \Sigma'$ necessarily preserves orientation.

- horizontal composition: for a composable pair of 1-morphisms $\alpha \xrightarrow{\Sigma} \beta \xrightarrow{\Sigma'} \gamma$, the horizontal composite is given by the bordism $\Sigma \cdot \Sigma' := \Sigma \cup_\beta \Sigma' \colon \alpha \nrightarrow \beta$ obtained from sewing along the parametrizations by β; the horizontal composition for 2-morphisms is induced thereby;

- monoidal structure given by disjoint union, with the usual symmetric braiding.

While the bicategory $\mathcal{P}\mathrm{rof}_\Bbbk$ of $\Bbbk$-linear profunctors[10] is the symmetric monoidal bicategory with:

- objects: small categories that are enriched in the cocomplete category $\mathrm{Vect}_\Bbbk$ of all $\Bbbk$-vector spaces;

- 1-morphisms: for objects A and B, a 1-morphism $P\colon A \nrightarrow B$, called a $\Bbbk$-*linear profunctor from A to B*, is a $\Bbbk$-linear functor $P\colon A^{\mathrm{op}} \times B \to \mathrm{Vect}_\Bbbk$;

- 2-morphisms: for 1-morphisms $P, Q\colon A \nrightarrow B$, a 2-morphism $\varphi\colon P \Rightarrow Q$ is a natural transformation of the underlying functors;

- vertical composition: given by the vertical composition of natural transformations;

- horizontal composition: for a composable pair of 1-morphisms $A \xrightarrow{P} B \xrightarrow{Q} C$, the horizontal composite is given by the coend[11]

$$P \cdot Q := \int^{b \in B} P(-, b) \otimes_\Bbbk Q(b, \sim) \colon A \nrightarrow C;$$

  the horizontal composition for 2-morphisms is induced thereby; (Note that those coends exist because all the domain categories are small and the target categories are cocomplete, see [Ric20, Proposition 4.5.3].)

- monoidal structure given by the Cartesian product, with the obvious symmetric braiding.

Empowered by the coherence theorem for bicategories [Gur13, Chapter 2], in the sequel we tacitly strictify each of the bicategories $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ and $\mathcal{P}\mathrm{rof}_\Bbbk$, i.e. consider them as (strict) 2-categories. We then also take the pseudofunctor Bl as strict, and thus require strict preservation of the horizontal composition. Therefore, for any composable pair of 1-morphisms $\alpha \xrightarrow{\Sigma} \beta \xrightarrow{\Sigma'} \gamma$ in the source bicategory $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$, we have

$$\begin{aligned} \mathrm{Bl}(\Sigma \cdot \Sigma') &= \mathrm{Bl}(\Sigma) \cdot \mathrm{Bl}(\Sigma') \\ &= \int^{y \in \mathrm{Bl}(\beta)} \mathrm{Bl}(\Sigma; -, y) \otimes_\Bbbk \mathrm{Bl}(\Sigma'; y, \sim) \colon \mathrm{Bl}(\alpha) \nrightarrow \mathrm{Bl}(\gamma), \end{aligned} \tag{2.6.1}$$

[10] Note that we've made a different choice for the bicategory of $\Bbbk$-linear profunctors in this thesis than that in [FSY21], which contains the latter as a sub-bicategory.

[11] Strictly speaking, the composite is only defined up to a *contractible* choice of coend, i.e. different choices of coend along with the canonical isomorphisms between them for each composite form a *contractible groupoid*, which is just as good as being well-defined.

where we used the abbreviation $\mathrm{Bl}(\Sigma; -, \sim) \equiv \mathrm{Bl}(\Sigma)(-, \sim)$. As a coend, (2.6.1) comes with a universal dinatural family of natural transformations, whose component at $(x_0, y_0, z_0) \in \mathrm{Bl}(\alpha) \times \mathrm{Bl}(\beta) \times \mathrm{Bl}(\gamma)$ is given by a linear map

$$s^{\Sigma,\Sigma'}_{x_0,y_0,z_0} \colon \mathrm{Bl}(\Sigma; x_0, y_0) \otimes_{\Bbbk} \mathrm{Bl}(\Sigma'; y_0, z_0) \to \mathrm{Bl}(\Sigma \cdot \Sigma'; x_0, z_0), \tag{2.6.2}$$

which will be later thought of as the *induced sewing map for the conformal blocks.*

Moreover, for any 1-morphism Σ in $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$, the profunctor $\mathrm{Bl}(\Sigma)$ naturally carries an action of the mapping class group $\mathrm{Map}(\Sigma)$ of the bordism[12] Σ, i.e. the group of 2-endomorphisms of Σ, by evaluating the pseudofunctor Bl on the 2-morphisms in $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$.

We need a specific open-closed modular functor $\mathrm{Bl}_{\mathcal{C}}$ to model the spaces of conformal blocks for our RCFT. Such an open-closed modular functor needs to satisfy certain constraints: first of all, the categories $\mathrm{Bl}_{\mathcal{C}}(I)$ and $\mathrm{Bl}_{\mathcal{C}}(S^1)$ obtained by evaluating $\mathrm{Bl}_{\mathcal{C}}$ on the standard 1-manifolds should be canonically equivalent to the category of chiral data $\mathcal{C}$ and its Drinfeld center $\mathcal{Z}(\mathcal{C})$, respectively; secondly, when restricted to the *closed sector*, i.e. the sub-bicategory $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_2$ of $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ of *closed* bordisms, the modular functor should be equivalent to the modular functor induced by the once-extended topological field theory $\mathrm{TV}_{\mathcal{C}}$ of Turaev-Viro type for $\mathcal{C}$, or equivalently [TV13, TV17, Bal11] the one of Reshetikhin-Turaev type for $\mathcal{Z}(\mathcal{C})$. We formulate these requirements in the following problem that will be solved in this thesis (Section 4.7):

Problem 2.6.2. Construct an open-closed modular functor $\mathrm{Bl}_{\mathcal{C}} \colon \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \to \mathcal{P}\mathrm{rof}_{\Bbbk}$, equipped with

- canonical equivalences of $\Bbbk$-linear categories

$$\Phi_I \colon \mathrm{Bl}_{\mathcal{C}}(I) \xrightarrow{\simeq} \mathcal{C} \tag{2.6.3}$$

  and

$$\Phi_{S^1} \colon \mathrm{Bl}_{\mathcal{C}}(S^1) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C}); \tag{2.6.4}$$

- a canonical equivalence of (closed) modular functors

$$\mathrm{Bl}_{\mathcal{C},\mathrm{c}} \overset{\simeq}{\Longrightarrow} \mathrm{TV}_{\mathcal{C},\epsilon\text{-}2\text{-}1} \colon \mathcal{B}\mathrm{ord}^{\mathrm{or}}_2 \to \mathcal{P}\mathrm{rof}_{\Bbbk}, \tag{2.6.5}$$

  where $\mathrm{Bl}_{\mathcal{C},\mathrm{c}}$ and $\mathrm{TV}_{\mathcal{C},\epsilon\text{-}2\text{-}1}$ are given by the restrictions of symmetric monoidal pseudofunctors[13]:

$$\mathrm{Bl}_{\mathcal{C},\mathrm{c}} = (\mathcal{B}\mathrm{ord}^{\mathrm{or}}_2 \hookrightarrow \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \xrightarrow{\mathrm{Bl}_{\mathcal{C}}} \mathcal{P}\mathrm{rof}_{\Bbbk});$$
$$\mathrm{TV}_{\mathcal{C},\epsilon\text{-}2\text{-}1} = (\mathcal{B}\mathrm{ord}^{\mathrm{or}}_2 \hookrightarrow \mathrm{Bord}^{\mathrm{or}}_{3\text{-}2\text{-}1} \xrightarrow{\mathrm{TV}_{\mathcal{C}}} \mathcal{P}\mathrm{rof}_{\Bbbk}).$$

[12]Later on, we will be interested in the mapping class group $\mathrm{Map}(\mathcal{S})$ of a worldsheet $\mathcal{S}$, which is a subgroup of the mapping class group $\mathrm{Map}(\Sigma_{\mathcal{S}})$ of its ambient bordism $\Sigma_{\mathcal{S}}$.

[13]Here $\mathrm{Bord}^{\mathrm{or}}_{321}$ is the symmetric monoidal bicategory of closed oriented 1-manifolds, 2-bordisms and *equivalence classes of 3-bordisms with corners.* The symmetric monoidal bicategory $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_2$ is embedded therein via taking every isotopy class of homeomorphisms to the *mapping cylinder* of any of its representatives.

The next ingredient we need is a family of *field maps.* Let $\mathcal{S}$ be an arbitrary worldsheet and $\Sigma_{\mathcal{S}}\colon \ell_{\mathrm{in}} \nrightarrow \ell_{\mathrm{out}}$ an ambient bordism thereof. Recall that by the construction introduced in Section 2.5, we obtain boundary data $\mathsf{b}_{\ell_{\mathrm{in}}}$ and $\mathsf{b}_{\ell_{\mathrm{out}}}$ on the 1-manifolds $\ell_{\mathrm{in}} = \bigsqcup_{i=1}^{p_{\mathrm{in}}} I \sqcup \bigsqcup_{j=1}^{q_{\mathrm{in}}} S^1$ and $\ell_{\mathrm{out}} = \bigsqcup_{i=1}^{p_{\mathrm{out}}} I \sqcup \bigsqcup_{j=1}^{q_{\mathrm{out}}} S^1$, respectively.

Problem 2.6.3. Construct for every 1-manifold $\ell \in \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ a *field map*, i.e. an assignment[14]

$$\mathbb{F}_\ell\colon \{\mathcal{F}r(\mathcal{C})\text{-boundary data on } \ell\} \to \mathrm{obj}\,\mathrm{Bl}_{\mathcal{C}}(\ell),$$

such that

$$\mathbb{F}_{\ell_{\mathrm{in}}}(\mathsf{b}_{\ell_{\mathrm{in}}}) \overset{\Phi_{\ell_{\mathrm{in}}}}{\longmapsto} \mathbb{F}^{\mathcal{S}}_{\mathrm{in}} \in \prod_{i=1}^{p_{\mathrm{in}}} \mathcal{C} \times \prod_{j=1}^{q_{\mathrm{in}}} \mathcal{Z}(\mathcal{C}) \tag{2.6.6}$$

and

$$\mathbb{F}_{\ell_{\mathrm{out}}}(\mathsf{b}_{\ell_{\mathrm{out}}}) \overset{\Phi_{\ell_{\mathrm{out}}}}{\longmapsto} \mathbb{F}^{\mathcal{S}}_{\mathrm{out}} \in \prod_{i=1}^{p_{\mathrm{out}}} \mathcal{C} \times \prod_{j=1}^{q_{\mathrm{out}}} \mathcal{Z}(\mathcal{C}) \tag{2.6.7}$$

for every worldsheet $\mathcal{S}$, where $\mathbb{F}^{\mathcal{S}}_{\mathrm{in}}$ and $\mathbb{F}^{\mathcal{S}}_{\mathrm{out}}$ are the combinations of field contents associated to the in-going and out-going sewing boundaries of the worldsheet $\mathcal{S}$ and we define for every 1-manifold $\ell = \bigsqcup_{i=1}^{p} I \sqcup \bigsqcup_{j=1}^{q} S^1$ an equivalence of categories

$$\Phi_\ell\colon \mathrm{Bl}_{\mathcal{C}}(\ell) = \prod_{i=1}^{p} \mathrm{Bl}_{\mathcal{C}}(I) \times \prod_{j=1}^{q} \mathrm{Bl}_{\mathcal{C}}(S^1) \xrightarrow{\simeq} \prod_{i=1}^{p} \mathcal{C} \times \prod_{j=1}^{q} \mathcal{Z}(\mathcal{C}) \tag{2.6.8}$$

obtained from combining p copies of (2.6.3) and q copies of (2.6.4).

Example 2.6.4. Let $\mathsf{b}_{\ell_{\mathrm{in}}}$ and $\mathsf{b}_{\mathrm{out}}$ be as in Example 2.5.3. Then the requirement for the field maps amounts to

$$\Phi_I \circ \mathbb{F}_I(\mathsf{b}_{\ell_{\mathrm{in}}}) = \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}B}(M_2, M_1 \otimes_A X_1) \in \mathcal{C}$$

and

$$\begin{aligned}\Phi_{S^1\times S^1} \circ \mathbb{F}_{S^1\times S^1}(\mathsf{b}_{\ell_{\mathrm{out}}}) &= \underline{\mathrm{Nat}}(\mathrm{id}_{\mathrm{mod}^{\mathcal{C}}\text{-}B}, G^{X_1} \circ G^{X_2^\vee}) \times \underline{\mathrm{Nat}}(\mathrm{id}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}, G^{X_3^\vee} \circ G^{X_2}) \\ &= \underline{\mathrm{Nat}}(G^{X_1^\vee}, G^{X_2^\vee}) \times \underline{\mathrm{Nat}}(G^{X_3}, G^{X_2}) \in \mathcal{Z}(\mathcal{C}) \times \mathcal{Z}(\mathcal{C}).\end{aligned}$$

With the open-closed modular functor $\mathrm{Bl}_{\mathcal{C}}$ and the field maps $\{\mathbb{F}_\ell\}_{\ell \in \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}}$ at hand, we obtain for each worldsheet $\mathcal{S}$ with ambient bordism $\Sigma_{\mathcal{S}}\colon \ell \nrightarrow \ell'$, a vector space with $\mathrm{Map}(\Sigma_{\mathcal{S}})$-action

$$\mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) := \mathrm{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathbb{F}_\ell(\mathsf{b}_\ell), \mathbb{F}_{\ell'}(\mathsf{b}_{\ell'})), \tag{2.6.9}$$

which we take as the *space of conformal blocks for the worldsheet $\mathcal{S}$.*

[14] For every 1-manifold $\ell \in \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$, the field map $\mathbb{F}_\ell$ will be extended to an actual functor later, see Section 5.2.

Remark 2.6.5. Our prescription for the vector space $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$ depends on the choice of the ambient worldsheet $\Sigma_{\mathcal{S}}$ a priori. However, with a fixed in-going and out-going type, there is between any two choices of ambient bordism $\Sigma_{\mathcal{S}}, \Sigma'_{\mathcal{S}} \colon \ell \nrightarrow \ell'$ a unique invertible 2-morphism in the *double category* $\mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$, which will be defined in Section 9.1, that is compatible with the canonical embeddings $\mathcal{S} \hookrightarrow \Sigma_{\mathcal{S}}$ and $\mathcal{S} \hookrightarrow \Sigma'_{\mathcal{S}}$. This induces, when provided that the pseudofunctor $\mathrm{Bl}_{\mathcal{C}}$ can be naturally upgraded to a *double functor* $\mathbb{B}\mathrm{l}_{\mathcal{C}} \colon \mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \to \mathbb{P}\mathrm{rof}_{\Bbbk}$, a unique isomorphism between each pair of vector spaces $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$ resulted from different choices of ambient bordisms. Then we only need to ensure that given a choice of *ambient surface* $\Sigma_{\mathcal{S}}$, different choices of parametrization result in canonically isomorphic spaces of conformal blocks. Such spaces constructed in this thesis using the string-net model turn out to be in fact *parametrization-independent*.

Before finally formulating the problem of finding correlators for all worldsheets, we need to define the notion of the *mapping class group* of a *worldsheet with chosen ambient bordism*. Recall that a line (resp. point) defect is *transparent* if its defect condition is a *trivial* bimodule (resp. *trivial* bimodule endomorphism).

Definition 2.6.6. The *mapping class group* $\mathrm{Map}(\mathcal{S})$ of a worldsheet $\mathcal{S}$ with chosen ambient bordism $\Sigma_{\mathcal{S}} \colon \ell \nrightarrow \ell'$, is a subgroup of $\mathrm{Map}(\Sigma_{\mathcal{S}}) = \mathrm{End}_{\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}(\ell,\ell')}(\Sigma_{\mathcal{S}})$ that consists of the isotopy classes of homeomorphisms from $\Sigma_{\mathcal{S}}$ to itself, which are already required to preserve the parametrizations hence also the orientation of $\Sigma_{\mathcal{S}}$, that fix the *non-transparent* line and point defects as well.

From now on, it is understood that we have chosen for every worldsheet an ambient bordism in a way that is compatible with sewing.

Problem 2.6.7. Construct a *consistent system of correlators for all worldsheets* in an RCFT with category of chiral data $\mathcal{C}$, i.e. an assignment

$$\mathcal{S} \mapsto \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \in \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}),$$

with the vector $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ called the *correlator* for the worldsheet $\mathcal{S}$, such that:

- $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \in \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) = \mathrm{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathbb{F}_{\ell}(\mathsf{b}_{\ell}), \mathbb{F}_{\ell'}(\mathsf{b}_{\ell'}))$ is $\mathrm{Map}(\mathcal{S})$-invariant with respect to the canonical $\mathrm{Map}(\Sigma_{\mathcal{S}})$-action on $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S})$;
- the assignment satisfies the *sewing constraints*: for every composable pair

  $$\ell \xrightarrow{\Sigma_{\mathcal{S}}} \ell' \xrightarrow{\Sigma_{\mathcal{S}'}} \ell''$$

  with matching parametrizations by ℓ', we have

  $$\begin{aligned} s^{\Sigma_{\mathcal{S}},\Sigma_{\mathcal{S}'}}_{\mathbb{F}_{\ell}\mathsf{b}_{\ell},\mathbb{F}_{\ell'}\mathsf{b}_{\ell'},\mathbb{F}_{\ell''}\mathsf{b}_{\ell''}} \colon \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) \otimes_{\Bbbk} \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}') &\to \mathrm{Bl}_{\mathcal{C}}(\mathcal{S} \cup_{\ell'} \mathcal{S}') \\ \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \otimes_{\Bbbk} \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}') &\mapsto \mathrm{Cor}_{\mathcal{C}}(\mathcal{S} \cup_{\ell'} \mathcal{S}'), \end{aligned} \tag{2.6.10}$$

  i.e. the induced sewing maps for conformal blocks, see (2.6.2), *send correlators to correlators.*

3 Graphical calculus for pivotal bicategories

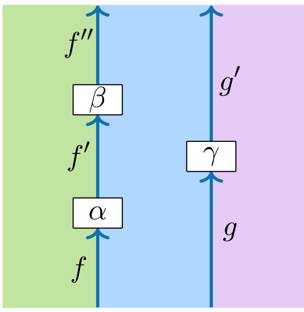
3.1 A review of string diagrams for bicategories

Recall that a *bicategory* is a category *weakly* enriched in the symmetric monoidal 2-category $\mathcal{C}\mathrm{at}$ of small categories, functors and natural transformation, with monoidal product given by the Cartesian product. This in particular means that given any bicategory $\mathcal{B}$ and objects $a, b \in \mathcal{B}$ therein, there is a *hom-category* $\mathcal{B}(a, b)$. In this and the next chapters, we assume that these hom-categories are themselves enriched in the category $\mathrm{Vect}_{\Bbbk}$ of (not necessarily finite dimensional) $\Bbbk$-vector spaces and linear maps.

It is common to use *pasting diagrams* to express composites of 2-morphisms therein. For example, for objects $a, b, c \in \mathcal{B}$, 1-morphisms $f, f', f'' \in \mathcal{B}(a, b)$, $g, g' \in \mathcal{B}(b, c)$ and 2-morphisms $\alpha\colon f \Rightarrow f'$, $\beta\colon f' \Rightarrow f''$, $\gamma\colon g \Rightarrow g'$, the following pasting diagram expresses the composite $(\beta \circ \alpha) \cdot \gamma\colon f \cdot g \Rightarrow f'' \cdot g'$:

$$\begin{tikzcd}
a \arrow[r, "f'" description] \arrow[r, bend left=60, "f''"] \arrow[r, bend right=60, "f"'] \arrow[r, phantom, shift left=3ex, "\Uparrow\beta"] \arrow[r, phantom, shift right=3ex, "\Uparrow\alpha"] & b \arrow[r, bend left=60, "g'"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\Uparrow\gamma"] & c
\end{tikzcd}$$

Note that in this thesis, we mostly use the *diagrammatic order* for horizontal compositions. By taking the Poincaré dual of the pasting diagram, we obtain the *string diagram* on the *standard square* $I \times I$ expressing the same composite:

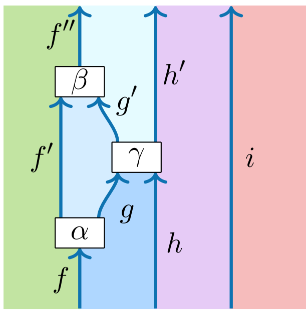


Here we have drawn the vertices of the string diagram as rectangular *coupons* and labeled the regions corresponding to different objects with different colors, as is standard in the literature.

Both pasting diagrams and string diagrams can express more complicated composites, for instance the pasting diagram

$$
\begin{tikzcd}
 & b'' & & \\
a \arrow[ur, "f''"] \arrow[r, "f'"'] \arrow[dr, "f"'] & b' \arrow[u, "g'"'] \arrow[d, "g"] \arrow[ul, phantom, "\Leftarrow\beta"] \arrow[dl, phantom, "\Rightarrow\alpha"] & c \arrow[ul, "h'"'] \arrow[r, "i"'] \arrow[l, phantom, "\Uparrow\gamma"] & d \\
 & b \arrow[ur, "h"'] & &
\end{tikzcd}
$$

and the string diagram

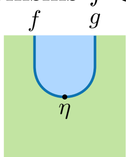


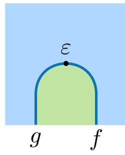
are supposed to express the same composite of certain 2-morphisms in $\mathcal{B}$. Except that to make sense of either of the diagrams, one first needs to decide for each layer of horizontal composite of 1-morphisms a *bracketing* which includes a choice of insertions of *identity* 1-morphisms. However, thanks to the coherence theorem for bicategories, see e.g. [JY21, Section 3.6], there exists between each pair of bracketed horizontal composites of the same composable sequence of 1-morphisms a unique 2-isomorphism made up of combination of *associators* and *unitors* that connects the pair. As a consequence, given any choice of bracketings for the *source* and the *target* of a (string or pasting) diagram, there is a unique 2-morphism assigned to it. Moreover, for any two such choices, there is a unique isomorphism of 2-hom spaces connecting the corresponding pair of 2-morphisms. Therefore, a pasting diagram or a string diagram uniquely determines a *contractible groupoid*[1] in $\mathrm{Vect}_{\mathbb{k}}$ (as a subcategory thereof) whose vertices are the 2-hom spaces that correspond to different choices of bracketings, as well as a *coherent choice* of elements in each of the 2-hom spaces, which we call the *value* of the diagram.

Accordingly, every equation of string diagrams (or pasting diagrams) should be interpreted as an *infinite family* of equations in different 2-hom spaces, one for each simultaneous choice of bracketings for both sides of the equation. Alternatively, one invokes the *strictification theorem* for bicategories which states that every bicategory is canonically biequivalent to a canonical strict 2-category associated to it (see e.g. [Gur13, Chapter 2]), and treat any bicategory *as if they were strict.* We alternate between these two viewpoints in this thesis.

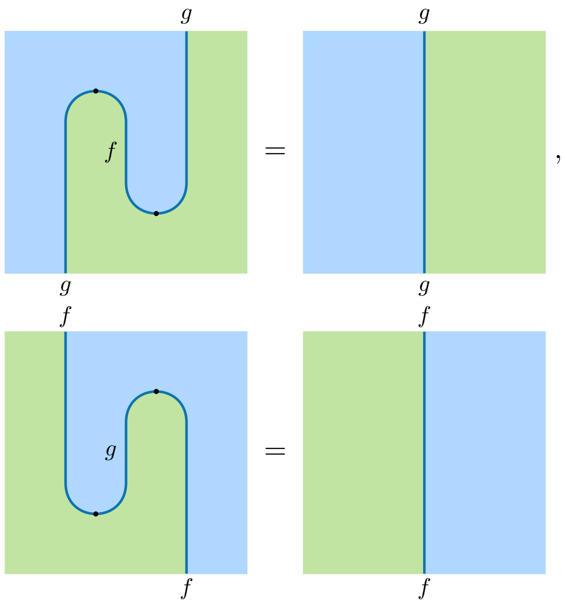
The rules of interpreting string diagrams as well as the axioms of a bicategory together guarantee that the value of a string diagram is unaffected by any isotopy of the diagram that fixes the orientation of the rectangular coupons while keeps the diagram *progressive* and the endpoints of its legs fixed. To allow for non-progressive string diagrams, i.e. those containing *"U-turns"*, one needs *dualities.*

[1] A groupoid $\mathcal{G}$ is called *contractible* if the unique functor $\mathcal{G} \to 1$ to the terminal category is an equivalence.

A *dual pair* (or an *adjoint pair*) in a bicategory $\mathcal{B}$ is a tuple $(f, g, \eta, \varepsilon)$ that consists of 1-morphisms $f \in \mathcal{B}(a, b)$, $g \in \mathcal{B}(b, a)$, a unit $\eta \colon \mathrm{id}_a \Rightarrow f \cdot g$, expressed by the string diagram 

, and a counit $\varepsilon \colon g \cdot f \Rightarrow \mathrm{id}_b$, drawn as 

, such that the following two *yanking equations* holds:



Note that we have made the identity 1-morphisms *invisible*. We call f the *left dual* (or the *left adjoint*) of g and write $f = {}^{\vee}g$. Conversely, g is called the *right dual* (or the *right adjoint*) of f and $g = f^{\vee}$. The usage of generalized "*the*" here is justified by the fact that left (resp. right) duals are defined up to unique isomorphisms.

A bicategory *with duals* is a bicategory $\mathcal{B}$ such that every 1-morphism therein has both left and right duals. By fixing for each 1-morphism a right dual, we obtain a pseudofunctor

$$(-)^{\vee} \colon \mathcal{B} \to \mathcal{B}^{\mathrm{coop}},$$

where $\mathcal{B}^{\mathrm{coop}}$ stands for the bicategory obtained from reversing both the 1- and the 2-morphisms in $\mathcal{B}$, and the pseudofunctor acts as identities on objects, sends every 1-morphism to its chosen right dual and every 2-morphism $\alpha \colon f \Rightarrow g$ to its *transpose* $\alpha^{\vee} \colon g^{\vee} \Rightarrow f^{\vee}$.

Definition 3.1.1. Let $\mathcal{B}$ be a bicategory with fixed left and right duals.

- A *pivotal structure* on $\mathcal{B}$ is an identity component pseudonatural transformation (i.e. every component 1-morphism is identity)

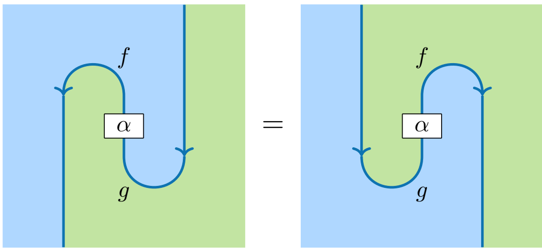
  $$\omega \colon \mathrm{id}_{\mathcal{B}} \Rightarrow (-)^{\vee\vee}.$$

  Equipped with a pivotal structure, $\mathcal{B}$ is called a *pivotal bicategory*.

- $\mathcal{B}$ is a *strictly pivotal bicategory* if the choice for duals satisfies

$$\mathrm{id}_{\mathcal{B}} = (-)^{\vee\vee}.$$

For a strictly pivotal bicategory $\mathcal{B}$, we have ${}^{\vee}f = ({}^{\vee}f)^{\vee\vee} = f^{\vee}$, therefore we can speak of the *dual* of a 1-morphism and denote it in string diagrams by a string with the same label but *opposite* direction. Consequently, non-progressive diagrams are meaningful for strictly pivotal bicategories. Moreover, for any 2-morphism $\alpha\colon f \Rightarrow g$ in a strictly pivotal bicategory $\mathcal{B}$,



This implies that we can now rotate any coupon in a string diagram by a full circle via isotopy without changing the value of the diagram. Note that both diagrams in (3.1.1) have their coupons *aligned with the frame*, meaning that the coupons' boundaries are parallel to the boundary of the standard square $I \times I$. Combining all of these, we arrive at the following conclusion:

Proposition 3.1.2. *Given a strictly pivotal bicategory $\mathcal{B}$, there is a well defined value for any not necessarily progressive string diagram whose coupons are not necessarily aligned with the frame, and the value is unchanged under any isotopy of the diagram that fixes the endpoints of its legs with rotations of the coupons allowed.*

Proof. Any string diagram Γ on the square $I \times I$ is isotopic (through an isotopy that fixes the endpoints of its legs) to a string diagram $\widetilde{\Gamma}$ whose coupons are aligned with the frame, and any other choice of $\widetilde{\Gamma}$ differs by an isotopy that fixes the endpoints and rotates the relevant coupons by full circles, which does not change the value according to (3.1.1). $\square$

In this thesis, we focus on pivotal bicategories of the following types.

Example 3.1.3. Given a pivotal tensor category $\mathcal{C}$, its *delooping* $\mathcal{BC}$, i.e. $\mathcal{C}$ viewed as a bicategory with a single object, is a pivotal bicategory with the same pivotal structure, now viewed as a pivotal structure for the bicategory $\mathcal{BC}$. Upon strictifying the pivotal structure of $\mathcal{C}$, which is always possible due to [NS07, Theorem 2.2], the bicategory $\mathcal{BC}$ becomes strictly pivotal.

Example 3.1.4. Given a pivotal tensor category $\mathcal{C}$, the bicategory $\mathcal{F}r(\mathcal{C})$ of simple special symmetric Frobenius algebras, bimodules and bimodule morphisms therein inherits a canonical pivotal structure from $\mathcal{C}$. If $\mathcal{C}$ is strictly pivotal, then so is $\mathcal{F}r(\mathcal{C})$.

3.2 Unframed graphical calculus for pivotal bicategories

In our previous discussion of string diagrams for bicategories, we used the *standard square* $I \times I$ as the canvas. The standard square comes with a canonical *2-framing*, i.e.

a section of its tangent bundle, where the value of the section at each point $p \in I \times I$ is $(0,1) \in \mathbb{R}^2 = T(I \times I)$, and asserting that a coupon is aligned with the frame is the same as saying the coupon is aligned with the canonical 2-framing of the standard square. Proposition 3.1.2 then says that the 2-framing in the *interior* of the square is irrelevant to the evaluation of any string diagram for a (strictly) pivotal bicategory: the value is unchanged under isotopies that do not necessarily preserve the alignment of the coupons with the 2-framing. However, so far we are still using the 2-framing at the boundary of the square: it tells us which part of the boundary is the *bottom*: this is the interval in $\partial(I \times I)$ where the values of the framing are pointing *inwards*; likewise it characterizes the top of the square. We use the top and the bottom of the square to separate the *output ports* from the *input ports* of the string diagram. As we will see in this section, the difference between input and output is immaterial to the graphical calculus for a pivotal bicategory, and the 2-framing of our canvas can be completely forgotten. This is a crucial step towards the formulation of *string-net models* – in a sense, they are generalizations of the graphical calculus where the canvas for a string diagram can be any compact *oriented* surface.

We fix in this section a strictly pivotal bicategory $\mathcal{B}$ (recall that we require each of its hom-categories to be $\Bbbk$-linear). Let us first establish the formal definition of a *partially $\mathcal{B}$-colored graph on an oriented surface* and the relevant notations. For the moment the relevant surface will be the *standard disk* only. However, more general types of surfaces will be considered in the following chapters.

Definition 3.2.1. A *partially $\mathcal{B}$-colored graph Γ* on a compact oriented surface Σ with possibly non-empty boundary is the following data:

- an underlying *directed finite* graph, i.e. a diagram of finite sets

$$\begin{tikzcd} E(\Gamma) \arrow[r, bend right, "\delta"'] & H(\Gamma) \arrow[l, two heads] \arrow[loop above, "i"] & I(\Gamma) \arrow[l, hook', "s"'] \arrow[r, "t"] & V(\Gamma) \end{tikzcd}$$

  where $V(\Gamma)$, $H(\Gamma)$ and $I(\Gamma)$ are the sets of *internal vertices*, *half-edges*, and half-edges that *touch an internal vertex*, respectively; the map t indicates the incidence of half-edges to the vertices, s is the canonical inclusion, i is a *fixed-point-free involution* that indicates the *juncture* of pairs of half-edges, whose set of *orbits* is $E(\Gamma)$, to be interpreted as the set of *edges*, and those edges that consist of pairs of internal half-edges are called *internal edges* whereas the rest are called *legs*; the map δ is a section of the canonical quotient map $H(\Gamma) \twoheadrightarrow E(\Gamma)$ and it picks out for each edge its *starting* half-edge, thereby *directs* the edges;

- an *embedding* of the *geometric realization*[2] $|\Gamma|$ of the underlying graph into the

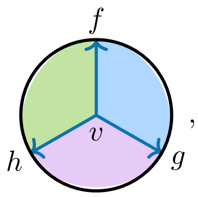
[2]We define $|\Gamma|$ to be the topological space $(\bigsqcup_{v \in V(\Gamma)} \{v\}) \sqcup (\bigsqcup_{e \in H(\Gamma)} [0,1]_e)/\sim$, where the equivalence relation is given by:

1. $\forall e \in H(\Gamma)$: $[0,1]_e \ni 1 \sim 1 \in [0,1]_{i(e)}$;
2. $\forall e \in I(\Gamma)$: $[0,1]_e \ni 0 \sim \{t(e)\}$.

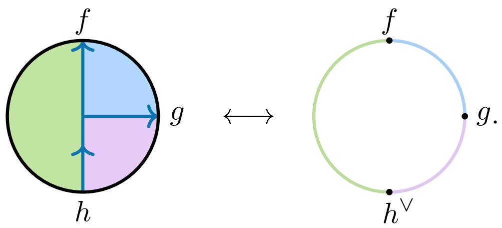
surface, such that the intersection of the boundary $\partial\Sigma$ with the image of $|\Gamma|$ is exactly the image of the *endpoints* of the legs, i.e. the image of $\{0 \in [0,1]_l \hookrightarrow |\Gamma|\}_{l\in H(\Gamma)\setminus I(\Gamma)}$;

- a coloring of the *patches*, i.e. the connected components of $\Sigma \setminus |\Gamma|$, with the objects of $\mathcal{B}$, as well as a coloring of the directed edges with the 1-morphisms of $\mathcal{B}$ in a way that is analogous to how we label the line defects in a worldsheet. Note that the vertices are *not* colored.

Let $D \subset \mathbb{C}$ be the *closed unit disk* centered at $0 \in \mathbb{C}$ with radius 1 (to be referred to as the *standard disk*), it has the standard circle $S^1 \subset \mathbb{C}$ as its boundary. By a *partially $\mathcal{B}$-colored corolla on the standard disk* we mean a partially $\mathcal{B}$-colored graph on D that has an underlying contractible directed finite graph with a single vertex, called the *center* of the corolla, whose image under the embedding is $0 \in D \subset \mathbb{C}$, and the image of each of its edges (the number of which is allowed to be zero) connects the center with a point on the boundary $\partial D = S^1$ by a *straight line* that is oriented either towards or away from the center. For instance, we have the following partially $\mathcal{B}$-colored corolla on D:

$K =$  , (3.2.1)

where $a, b, c \in \operatorname{obj}\mathcal{B}$ and correspond to the colors green, blue, and purple, respectively, and $f \in \mathcal{B}(a,b)$, $g \in \mathcal{B}(b,c)$, $h \in \mathcal{B}(c,a)$ are 1-morphisms. Here we have implicitly equipped the standard disk D with the orientation that goes counterclockwise. Note that due to the lack of coloring for the single vertex v, there is a canonical bijection between the set of partially $\mathcal{B}$-colored corollas and the set of $\mathcal{B}$-boundary data on S^1 (defined by replacing the specific pivotal bicategory $\mathcal{F}r(\mathcal{C})$ with $\mathcal{B}$ in Definition 2.5.1):

 (3.2.2)

We would like to associate to the center v of a partially $\mathcal{B}$-colored corolla K a vector space $H_v^{\mathcal{B}}$ that will be the *space of colors* for the vertex v. Note that the orientation of D naturally induces a cyclic order on the set $H(v)$ of half-edges incident to the vertex v: if we draw D as oriented counterclockwise (as we always do), then the cyclic order on $H(v)$ is *clockwise*. Recall that in (2.3.10), we defined the space of defect conditions for a point defect with the help of a *linear order* on the set of half-edges incident to it that is compatible with the cyclic order. What we are going to do in the following is essentially the same, but more formal.

A linear order on $H(v)$ that is compatible with the induced cyclic order can be uniquely determined by the choice of a *starting half-edge*, i.e. a *root*, $e \in H(v)$. Let's say, for the example of K in (3.2.1), that we choose e to be the half-edge that is labeled by the 1-morphism $h \in \mathcal{B}(c,a)$, or by abusing notation, $e = h$. We then associate to v with the choice of root $e = h$ a 2-hom space $h_v^{\mathcal{B}}(e)$ in $\mathcal{B}$ in an obvious way:

$$h_v^{\mathcal{B}}(e = h) \coloneqq \mathrm{End}_{\mathcal{B}}(c)(\mathrm{id}_c, h \cdot f \cdot g).$$

Likewise, for the choices $e = f$ and $e = g$, the associated 2-hom spaces are

$$h_v^{\mathcal{B}}(e = f) \coloneqq \mathrm{End}_{\mathcal{B}}(a)(\mathrm{id}_a, f \cdot g \cdot h)$$

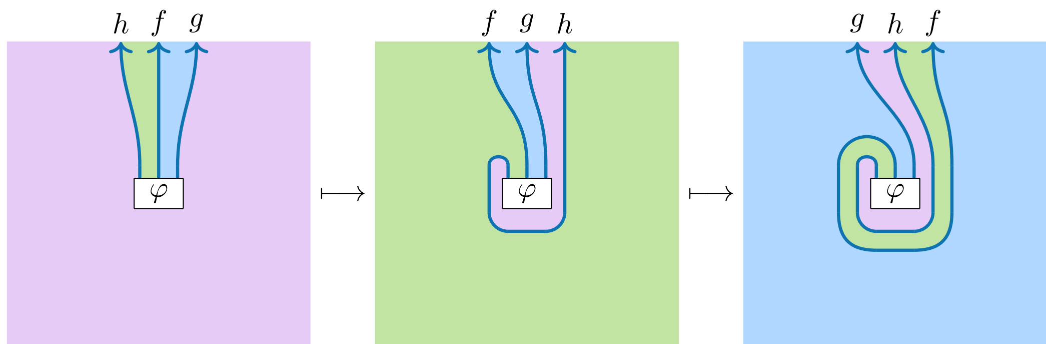
and

$$h_v^{\mathcal{B}}(e = g) \coloneqq \mathrm{End}_{\mathcal{B}}(b)(\mathrm{id}_b, g \cdot h \cdot f).$$

By using the units and counits of the dual pairs to turn the edges around, we canonically induce from a change of root an isomorphism between the assigned 2-hom spaces. For instance, the change of root from h to f then to g gives rise to the following chain of isomorphisms of vector spaces:

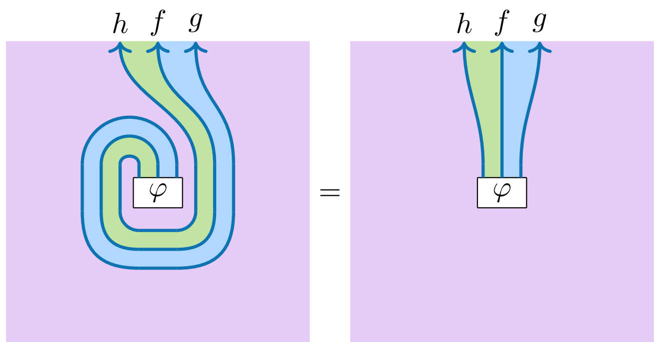
$$h_v^{\mathcal{B}}(h) \xrightarrow{\cong} h_v^{\mathcal{B}}(f) \xrightarrow{\cong} h_v^{\mathcal{B}}(g),$$

whose action on an arbitrary element $\varphi \in h_v^{\mathcal{B}}(h) = \mathrm{End}_{\mathcal{B}}(c)(\mathrm{id}_c, h \cdot f \cdot g)$ is given by



.

A crucial observation is that, by dragging the left most leg of the coupon counterclockwise to the right one more time, we get back the original element $\varphi \in h_v^{\mathcal{B}}(h)$, i.e.



,

which is due to the strict pivotality of $\mathcal{B}$, or more directly the yanking equations for duals together with (3.1.1). All in all, we obtained in this way a groupoid in $\mathrm{Vect}_{\Bbbk}$ that is generated by the following diagram:

$$\begin{tikzcd}
 & h_v^{\mathcal{B}}(f) \arrow[dl, leftrightarrow, "\cong"'] \arrow[dr, leftrightarrow, "\cong"] & \\
h_v^{\mathcal{B}}(h) \arrow[rr, leftrightarrow, "\cong"'] & & h_v^{\mathcal{B}}(g)
\end{tikzcd}$$

This groupoid is contractible, which means that for any ordered pair of objects therein, there is only one morphism in the groupoid between them, hence any composites of morphisms in this groupoid sharing the same source and target are the same. The upshot is that, by choosing an element in *any* of the three 2-hom spaces, we are *simultaneously* choosing an element in *each* of the spaces since the groupoid gives an *coherent* way to identify them. Inspired by this, we formulate as follows:

Let K be a partially $\mathcal{B}$-colored corolla on D with center v and $n = |H(v)| > 0$. Define:

1. a *contractible* groupoid $\mathcal{G}_v^{\mathcal{B}}$ with the set of objects being the set $H(v) = \{e_1, e_2, \cdots, e_n\}$ of half-edges incident to v, which is conveniently indexed according to an arbitrary choice of compatible linear order. The morphisms are generated by the diagram

$$\begin{tikzcd}
e_1 \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & e_2 \arrow[r, leftrightarrow] & e_3 \arrow[d, leftrightarrow] \\
e_n \arrow[r, leftrightarrow] & \cdots \arrow[r, leftrightarrow] & e_4
\end{tikzcd}$$

   with relations uniquely determined by asserting that the groupoid is contractible, i.e. that each hom-set $\mathcal{G}_v^{\mathcal{B}}(e_i, e_j)$ is the singleton set;

2. a functor

$$h_v^{\mathcal{B}} \colon \mathcal{G}_v^{\mathcal{B}} \to \mathrm{Vect}_{\Bbbk}$$

   that acts on objects by

$$e_i \mapsto h_v^{\mathcal{B}}(e_i) := \mathrm{End}_{\mathcal{B}}(a_i)(\mathrm{id}_{a_i}, f_i^{\epsilon_i} \cdot f_{i+1}^{\epsilon_{i+1}} \cdot \cdots \cdot f_{i+n-1}^{\epsilon_{i+n-1}}),$$

   where the indices are counted mod n and f_j is the color of the half-edge e_j with $f_j^{\epsilon_j} \in \mathcal{B}(a_j, a_{j+1})$, $\epsilon_j = +$ if e_j is directed away from v and $\epsilon_j = -$ otherwise; The action on the generating morphisms are given by "dragging the leg around".

We now define the desired vector space of colors $H_v^{\mathcal{B}}$ as a limit

$$H_v^{\mathcal{B}} := \lim h_v^{\mathcal{B}} \in \mathrm{Vect}_{\Bbbk} \tag{3.2.3}$$

for a partially colored corolla with at least one leg. For a corolla K_a with no leg and its single patch $D \setminus v_a$ colored with an object $a \in \mathcal{B}$, we define the space of color as

$$H_{v_a}^{\mathcal{B}} := \mathrm{End}_{\mathcal{B}}(a)(\mathrm{id}_a, \mathrm{id}_a). \tag{3.2.4}$$

Being the limit of a contractible groupoid, $H_v^{\mathcal{B}}$ is determined by an isomorphism

$$p_v^e \colon H_v^{\mathcal{B}} \xrightarrow{\cong} h_v^{\mathcal{B}}(e)$$

for any choice of $e \in H(v)$, which uniquely extends to a limit cone over $h_v^{\mathcal{B}}$ with every leg being an isomorphism of vector spaces. Therefore, by choosing a *color* for the vertex v, i.e. an element $c \in H_v^{\mathcal{B}}$, we produce a coherent family of 2-morphisms in $\mathcal{B}$

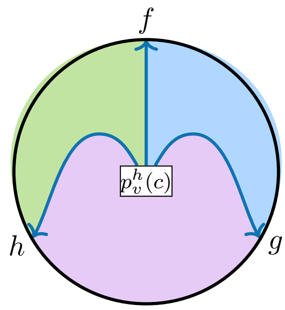
$$\{p_v^e(c) \in h_v^{\mathcal{B}}(e)\}_{e \in H(v)}$$

in the sense that any pair $(p_v^{e_i}(c), p_v^{e_j}(c))$ therein is connected by the unique isomorphism

$$h_v^{\mathcal{B}}(e_i) \xrightarrow{\cong} h_v^{\mathcal{B}}(e_j)$$

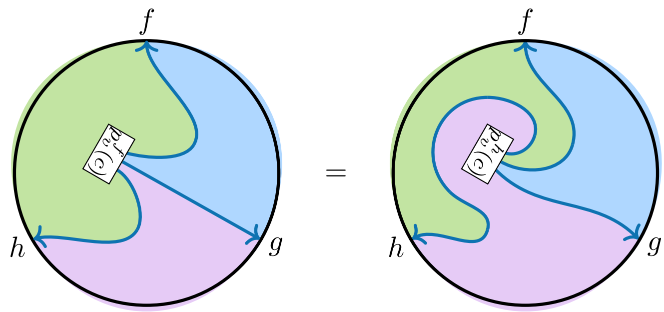
obtain by evaluating the functor $h_v^{\mathcal{B}}$ on the unique morphism $e_i \xrightarrow{\cong} e_j$ in the groupoid $\mathcal{G}_v^{\mathcal{B}}$.

Remark 3.2.2. The relation between the elements determined by the same color $c \in H_v^{\mathcal{B}}$ can be further clarified as follows: by choosing a root $e \in H(v)$, we can produce, up to isotopies that fix the boundary $\partial D = S^1$, a string diagram with a rectangular coupon on D in an obvious way, which is *isotopic* (rel. ∂D) to the string diagram produced by choosing *any other* root. We demonstrate this by the example in (3.2.1), if we choose $e = h$, we obtain the string diagram



; (3.2.5)

If we have chosen $e = f$, then the diagram will be



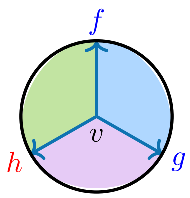
, (3.2.6)

which is isotopic to (3.2.5). It is crucial that the 2-hom spaces are labeled by the *half-edges* instead of their colors, because we need to know which of them has been chosen as the *root* when producing a string diagram on the standard disk. Therefore, one should really avoid conflating half-edges with their labels as *what we are doing here*, especially when all the half-edges are colored with the same 1-morphism.

Remark 3.2.3. So far we have only considered the type of 2-morphisms that have all non-trivial 1-morphisms in their *outputs*, which is quite restrictive for applications. To remedy this, we introduce the following notion: given a partially $\mathcal{B}$-colored corolla K, a *polarization* on its vertex v is a partition

$$H(v) = H^{\mathrm{in}}(v) \sqcup H^{\mathrm{out}}(v)$$

of the cyclically ordered set of half-edges into two linearly ordered sets of input and output half-edges, such that any two half-edges of the same type (either in- or output) are *consecutive* with respect to the cyclic order on $H(v)$ if they are consecutive with respect to the linear order on $H^{\mathrm{in}}(v)$ or $H^{\mathrm{out}}(v)$, which by definition is induced by the cyclic order on $H(v)$. For instance, we denote by



(3.2.7)

the polarization that has $H^{\mathrm{in}}(v) = \{h\}$ and $H^{\mathrm{out}}(v) = \{f, g\}$. Note that in the special cases where $H^{\mathrm{in}}(v) = \emptyset$, a polarization is reduced to a compatible linear order on $H(v)$. Therefore we have a canonical extension of the groupoid $\mathcal{G}_v^{\mathcal{B}}$:

$$\mathcal{G}_v^{\mathcal{B}} \overset{\simeq}{\hookrightarrow} \widehat{\mathcal{G}_v^{\mathcal{B}}} \xrightarrow{\simeq} 1$$

where the groupoid $\widehat{\mathcal{G}_v^{\mathcal{B}}}$ is contractible and generated by the set of all polarizations on v. Moreover, we have the following extension of the functor $h_v^{\mathcal{B}}$:

$$\begin{tikzcd}
\widehat{\mathcal{G}_v^{\mathcal{B}}} \arrow[r, dashed, "\widehat{h_v^{\mathcal{B}}}"] & \mathrm{Vect}_{\Bbbk} \\
\mathcal{G}_v^{\mathcal{B}} \arrow[u, hook, "\wr"] \arrow[ur, "h_v^{\mathcal{B}}"'] &
\end{tikzcd}$$

by defining for an arbitrary polarization k given by $H^{\mathrm{out}}(v) = \{e_i, e_{i+1}, \ldots, e_j\}$ and $H^{\mathrm{in}}(v) = \{e_{j+1}, e_{j+2}, \ldots, e_{i+n-1}\}$ with indices counted mod n,

$$\widehat{h_v^{\mathcal{B}}}(k) \coloneqq \mathrm{Hom}_{\mathcal{B}(a_i, a_{j+1})}(f_{i+n-1}^{-\epsilon_{i+n-1}} \cdot \cdots \cdot f_{j+2}^{-\epsilon_{j+2}} \cdot f_{j+1}^{-\epsilon_{j+1}}, f_i^{\epsilon_i} \cdot f_{i+1}^{\epsilon_{i+1}} \cdot \cdots \cdot f_j^{\epsilon_j}).$$

As an example, for K as in (3.2.1) and the polarization k as in (3.2.7), we have

$$\widehat{h_v^{\mathcal{B}}}(k) = \mathrm{Hom}_{\mathcal{B}(a,c)}(h^{\vee}, f \cdot g).$$

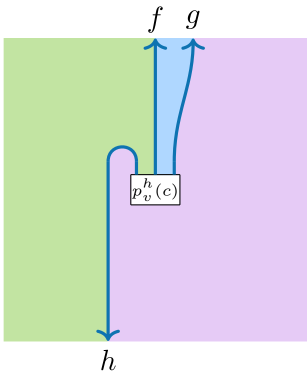
Consequently, $H_v^{\mathcal{B}}$ is also equipped with a unique limit cone over $\widehat{h_v^{\mathcal{B}}} \colon \widehat{\mathcal{G}_v^{\mathcal{B}}} \to \mathrm{Vect}_{\Bbbk}$ that restricts to the limit cone over $h_v^{\mathcal{B}}$, with the legs (which are isomorphisms) denoted by

$$\{p_v^k \colon H_v^{\mathcal{B}} \xrightarrow{\cong} \widehat{h_v^{\mathcal{B}}}(k)\}_{k \in \mathrm{obj}\, \widehat{\mathcal{G}_v^{\mathcal{B}}}}. \tag{3.2.8}$$

Now, given a color $c \in H_v^{\mathcal{B}}$, we obtain from it an entire family of 2-morphisms

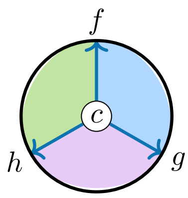
$$\{p_v^k(c) \in \widehat{h_v^{\mathcal{B}}}(k)\}_{k \in \mathrm{obj}\, \widehat{\mathcal{G}_v^{\mathcal{B}}}}$$

that are *mates* to each other. For instance, with k as in (3.2.7), the 2-morphism $p_v^k(c) \in \mathrm{Hom}_{\mathcal{B}(a,c)}(h^\vee, f \cdot g)$ can be expressed in terms of $p_v^h(c) \in \mathrm{End}_{\mathcal{B}}(c)(\mathrm{id}_c, h \cdot f \cdot g)$ as

$p_v^k(c) =$ .

Together with a choice of polarization, a color $c \in H_v^{\mathcal{B}}$ again determines an isotopy class of string diagrams with a rectangular coupon on D, and the isotopy class is independent of the choice of polarization.

From now on, we will place a circular coupon labeled by a color $c \in H_v^{\mathcal{B}}$ at the vertex v to represent the isotopy class of string diagrams the color produces, without choosing a specific polarization. For instance,



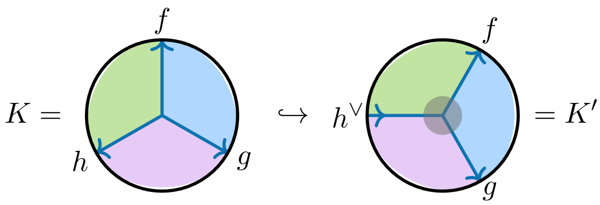
(3.2.9)

represents the isotopy class of string diagrams that contains both (3.2.5) and (3.2.6). Conversely, a diagram like (3.2.9) can be represented by a string diagram on D that corresponds to any choice of polarization and a 2-morphism in the associated 2-hom space.

Let us assign to a partially $\mathcal{B}$-colored corolla K the space of color for its center v and write

$$\mathsf{Cal}_{\mathcal{B}}(K) := H_v^{\mathcal{B}}. \tag{3.2.10}$$

This assignment is functorial with respect to any orientation preserving *embedding* of the standard disk to itself that induces a *local isomorphism* of the partially colored embedded corollas, where the colors of the patches are required to match while a half-edge is allowed to be mapped to either a half-edge with the same orientation and the same color or a half-edge with the opposite orientation and the dual color. For instance, we can have an embedding depicted by



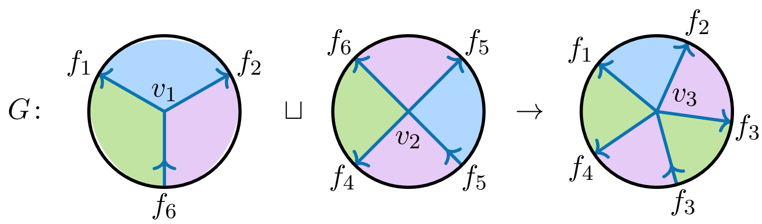
where the image of the corolla on the left is indicated by the shaded area, and the half-edge colored with the 1-morphism h is mapped to the one that is colored with the dual $h^\vee$. The action of the assignment (3.2.10) in this case is given by

$$\mathsf{Cal}_{\mathcal{B}}(K) = H_v^{\mathcal{B}} \xrightarrow{\cong} h_v^{\mathcal{B}}(e_h) = \mathrm{End}_{\mathcal{B}}(c)(\mathrm{id}_c, h \cdot f \cdot g) = h_{v'}^{\mathcal{B}}(e_{h^\vee}) \xrightarrow{\cong} H_{v'}^{\mathcal{B}} = \mathsf{Cal}_{\mathcal{B}}(K').$$

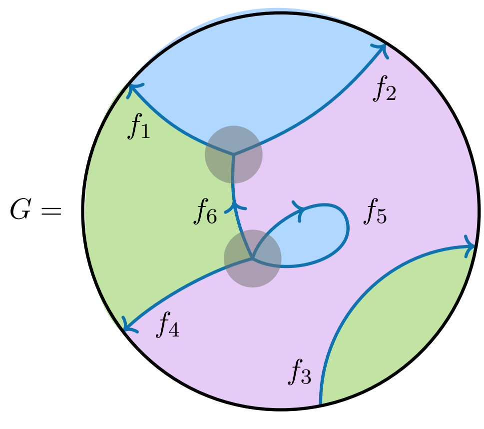
Note that due to this functoriality, for a *monochromatic* corolla K_{mnc}, i.e. a corolla whose half-edges are all colored with the same 1-morphism and oriented in the same way, the vector space $\mathsf{Cal}_{\mathcal{B}}(K_{\mathrm{mnc}})$ carries an action of the cyclic group of appropriate order.

We now extend the assignment $\mathsf{Cal}_{\mathcal{B}}$ to a symmetric monoidal functor from a category of partially $\mathcal{B}$-colored corollas and graphs to the category of vector spaces and linear maps, in order to capture the *unframed graphical calculus* for pivotal bicategories.

Let us first introduce the source category $\mathsf{Corollas}_{\mathcal{B}}$: it is a symmetric monoidal category with objects given by finite disjoint unions of partially $\mathcal{B}$-colored corollas. A morphism of the type $K_1 \sqcup \cdots \sqcup K_n \to K_{n+1}$ is given by a partially $\mathcal{B}$-colored graph G on the standard disk whose boundary datum coincides with that of K_{n+1}, together with an orientation preserving embedding $D_1 \sqcup \cdots \sqcup D_n \hookrightarrow D_{n+1}$ of the underlying disks of the source (the number of which is allowed to be zero) to the underlying disk of the target that induces a local isomorphism of the graphs, where the colorings of the patches are respected, while for a half-edge only the combination of its orientation and color is required to match, and each internal vertex of G is covered by the image of exactly one disk. For instance, we have

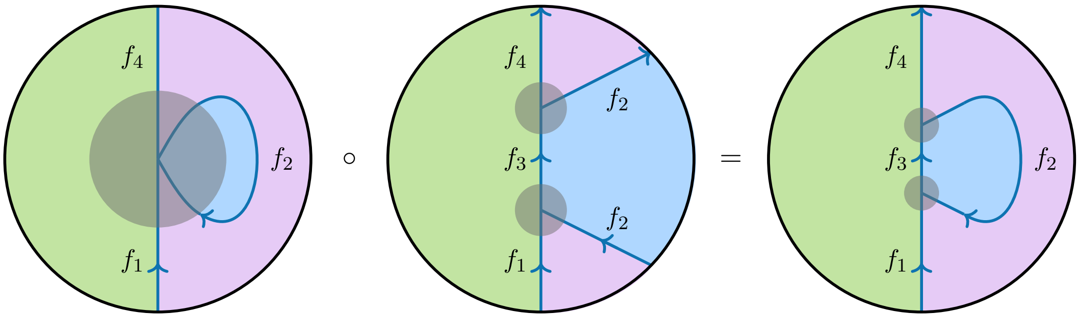


where



(3.2.11)

with the images of the disks indicated by the shaded area. General morphisms in $\mathsf{Corollas}_{\mathcal{B}}$ are obtained by taking disjoint unions. The composition $G_2 \circ G_1$ is given by *blowing up* the internal vertices of G_2 by G_1 using the embeddings of disks, for instance we have

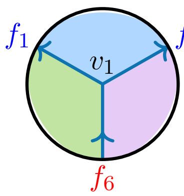
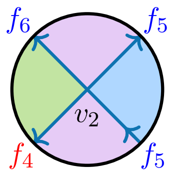


The monoidal product on $\mathsf{Corollas}_{\mathcal{B}}$ is given by disjoint union, and the symmetric braiding is the obvious one.

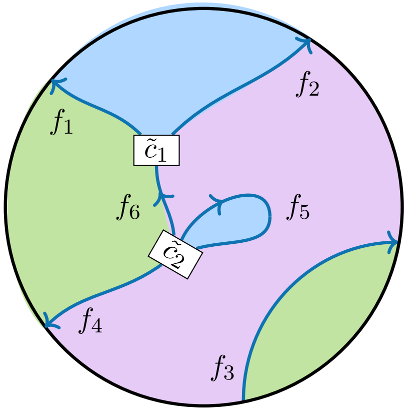
We now define the functor $\mathsf{Cal}_{\mathcal{B}} \colon \mathsf{Corollas}_{\mathcal{B}} \to \mathrm{Vect}_{\Bbbk}$. Its action on the objects is given by

$$\mathsf{Cal}_{\mathcal{B}}(K_1 \sqcup \cdots \sqcup K_n) \coloneqq \mathsf{Cal}_{\mathcal{B}}(K_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathsf{Cal}_{\mathcal{B}}(K_n) = H^{\mathcal{B}}_{v_1} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} H^{\mathcal{B}}_{v_n}, \tag{3.2.12}$$

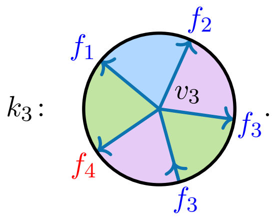
where it is implied that for the monoidal unit $\emptyset$ of $\mathsf{Corollas}_{\mathcal{B}}$, the value is defined to be $\mathsf{Cal}_{\mathcal{B}}(\emptyset) \coloneqq \Bbbk$. Its action on the morphisms is given by the following procedure, demonstrated on the morphism $G \colon K_1 \sqcup K_2 \to K_3$ given by (3.2.11). Take an arbitrary $c_1 \otimes_{\Bbbk} c_2 \in \mathsf{Cal}_{\mathcal{B}}(K_1) \otimes_{\Bbbk} \mathsf{Cal}_{\mathcal{B}}(K_2) = H^{\mathcal{B}}_{v_1} \otimes_{\Bbbk} H^{\mathcal{B}}_{v_2}$, we would like to assign to it an element $c_3 \in \mathsf{Cal}_{\mathcal{B}}(K_3) = H^{\mathcal{B}}_{v_3}$ and then obtain a linear map by linear extension. To this end, we first choose a polarization for each of the vertices $v_1 \in K_1$ and $v_2 \in K_2$, say,

$k_1 \colon$  and $k_2 \colon$  ,

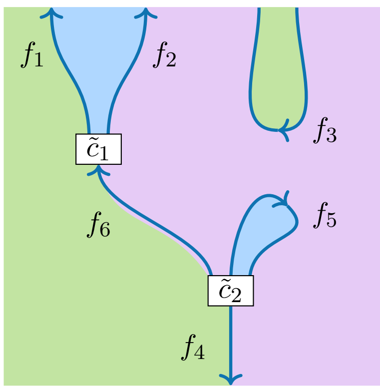
where the labels of the input half-edges are colored red and the labels of the outputs are colored green. Let $\widetilde{c}_i = p^{k_i}_{v_i}(c_i) \in \widehat{h^{\mathcal{B}}_{v_i}}(k_i)$ for $i = 1, 2$ (see (3.2.8) for the definition of p^k_v). According to Remark 3.2.3, we obtain (up to isotopies relative to the boundary of D) two string diagrams with rectangular coupons on D corresponding to the choice of colors and polarizations. These string diagrams are then pushed forward along the embedding $D_1 \sqcup D_2 \hookrightarrow D_3$, replacing the images of the corollas K_1 and K_2. In this way, up to isotopies fixing the boundary, we obtain a string diagram with rectangular coupons on D, which shares the same boundary datum with K_3:



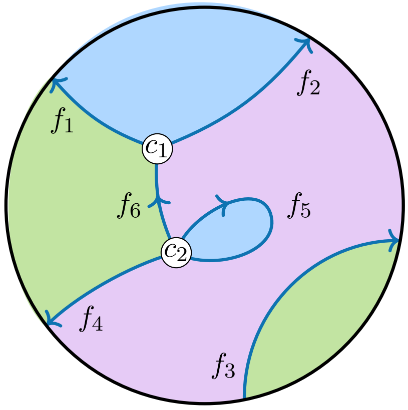
String diagrams with rectangular coupons (or, more generally, coupons with chosen *polarizations*) are called *polarized*. We now choose a polarization for the vertex $v_3 \in K_3$, e.g.



According to this choice, we then produce a string diagram on the standard square $I \times I$, which is unique up to isotopies fixing the top and the bottom of the square setwise:



This *rectified* string diagram uniquely determines a 2-morphism $\widetilde{c}_3 \in \widehat{h^{\mathcal{B}}_{v_3}}(k_3)$, which in turn determines a unique element $c_3 = (p^{k_3}_{v_3})^{-1}(\widetilde{c}_3) \in H^{\mathcal{B}}_{v_3} = \mathsf{Cal}_{\mathcal{B}}(K_3)$ that only depends on, besides the colors c_1 and c_2, the *isotopy class* (rel. boundary) of G and in particular does not depend on the auxiliary choice of polarizations. We write

$$c_3 = \Big\langle$$  $$\Big\rangle \tag{3.2.13}$$

and refer to c_3 as the *value* of the string diagram with *circular coupons* on the right hand side. String diagrams with circular coupons are called *unpolarized.* In this way, we obtain a linear map

$$\mathsf{Cal}_{\mathcal{B}}(G)\colon \mathsf{Cal}_{\mathcal{B}}(K_1 \sqcup K_2) = \mathsf{Cal}_{\mathcal{B}}(K_1) \otimes_{\Bbbk} \mathsf{Cal}_{\mathcal{B}}(K_2) \to \mathsf{Cal}_{\mathcal{B}}(K_3),$$

which is set to be the value of the morphism $G\colon K_1 \sqcup K_2 \to K_3$ under the functor. By declaring $\mathsf{Cal}_{\mathcal{B}}(G_1 \sqcup G_2) = \mathsf{Cal}_{\mathcal{B}}(G_1) \otimes_{\Bbbk} \mathsf{Cal}_{\mathcal{B}}(G_2)$, we finish the definition of the functor

$$\mathsf{Cal}_{\mathcal{B}}\colon \mathsf{Corollas}_{\mathcal{B}} \to \mathrm{Vect}_{\Bbbk}.$$

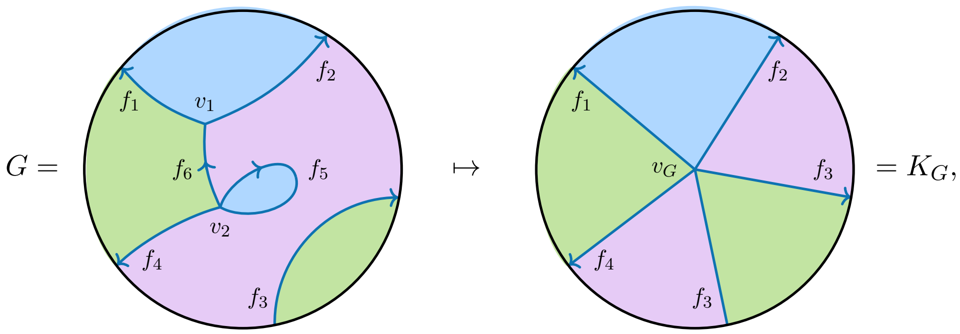
Remark 3.2.4. So far we have used corollas on standard disks as local models for internal vertices of embedded graphs. However, the notion of a space of color $H_v^{\mathcal{B}}$ and that of a polarization can be directly defined for an arbitrary internal vertex of a partially $\mathcal{B}$-colored embedded graph on any oriented surface and we can make sense of the evaluation of an *unpolarized* string diagram on D as in (3.2.13) without referring to embedded disks. Formally, given a partially $\mathcal{B}$-colored graph G on the standard disk with its set of internal vertices denoted by $V(G)$, by choosing for each internal vertex $v \in V(G)$ an embedding of some partially colored corolla K_v (which is also called a *parametrization by a corolla*), we can turn G into a morphism $\widehat{G}\colon \bigsqcup_{v\in V(G)} K_v \to K_G$ in the category $\mathsf{Corollas}_{\mathcal{B}}$, where K_G is the unique corolla on the standard disk that share the same boundary datum with G. Using the canonical isomorphism $\bigotimes_{v\in V(G)} H_v^{\mathcal{B}} \xrightarrow{\cong} \bigotimes_{v\in V(G)} \mathsf{Cal}_{\mathcal{B}}(K_v)$ induced by the embeddings of corollas, we obtain a linear map

$$\widehat{\mathsf{Cal}}_{\mathcal{B}}(G)\colon \bigotimes_{v\in V(G)} H_v^{\mathcal{B}} \xrightarrow{\cong} \bigotimes_{v\in V(G)} \mathsf{Cal}_{\mathcal{B}}(K_v) \xrightarrow{\mathsf{Cal}_{\mathcal{B}}(\widehat{G})} \mathsf{Cal}_{\mathcal{B}}(K_G) = H_{v_G}^{\mathcal{B}}. \tag{3.2.14}$$

This map is independent of the choice of embeddings of corollas due the the coherent isomorphisms between the vector spaces $\bigotimes_{v\in V(G)} \mathsf{Cal}_{\mathcal{B}}(K_v)$ obtained from different such choices and is moreover unaffected by any isotopy of G that fixes its boundary. We then define the value $\langle G_c \rangle \in H^{\mathcal{B}}_{v_G}$ of the *fully colored* graph G_c with the coloring of its internal vertices given by $c \in \bigotimes_{v \in V(G)} H^{\mathcal{B}}_v$ as

$$\langle G_c \rangle := \widehat{\mathsf{Cal}}_{\mathcal{B}}(G)(c) \in H^{\mathcal{B}}_{v_G} \tag{3.2.15}$$

and think of it as the color for the vertex $v_G \in K_G$ obtained by replacing G with the corolla K_G. For example, for



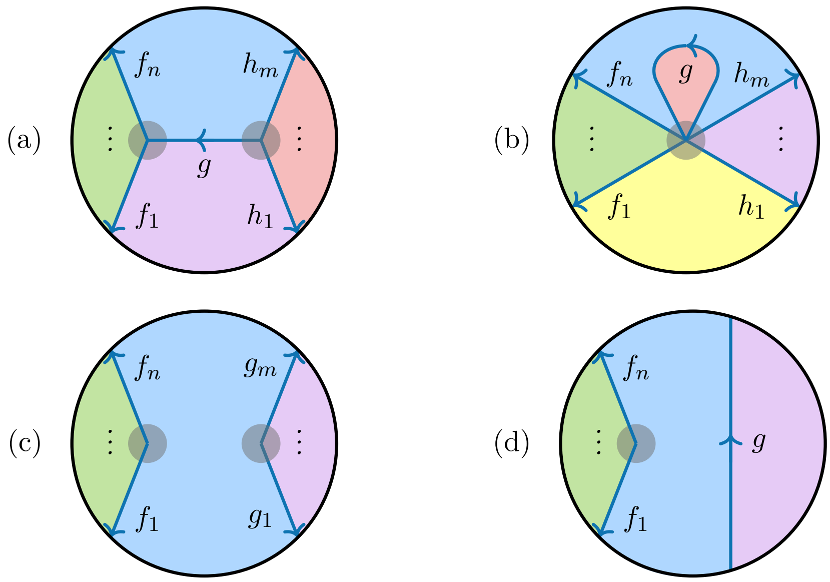
we have

$$\begin{aligned} \widehat{\mathsf{Cal}}_{\mathcal{B}}(G) \colon H^{\mathcal{B}}_{v_1} \otimes_{\Bbbk} H^{\mathcal{B}}_{v_2} &\to H^{\mathcal{B}}_{v_G} \\ c = c_1 \otimes_{\Bbbk} c_2 &\mapsto \langle G_c \rangle, \end{aligned}$$

and the value $\langle G_c \rangle \in H^{\mathcal{B}}_{v_G} = \mathsf{Cal}_{\mathcal{B}}(K_G)$ is the same as the one in (3.2.13), but with the color $c = c_1 \otimes_{\Bbbk} c_2$ understood as living in the vector space $H^{\mathcal{B}}_{v_1} \otimes_{\Bbbk} H^{\mathcal{B}}_{v_2}$ directly associated to the internal vertices of G.

We can think of the functor $\mathsf{Cal}_{\mathcal{B}} \colon \mathsf{Corollas}_{\mathcal{B}} \to \mathrm{Vect}_{\Bbbk}$ as a rule to assign a space of morphisms to each partially colored corolla and a *composition map* to each partially colored graph. An important observation is that the composition map configured by any partially colored graph can be decomposed into a sequence of maps that consist of *operadic compositions*, *partial trace maps*, *horizontal products*, and *whiskerings*, all of which will be introduced in Proposition 3.2.5. We call an internal edge that connects a pair of distinct internal vertices *regular*, and otherwise a *loop*. A partially colored embedded graph in the standard disk is called *trivial* if it is isotopic to a partially colored corolla or it does not contain any internal vertices. A morphism in $\mathsf{Corollas}_{\mathcal{B}}$ is called trivial if its underlying partially colored embedded graph is trivial. Compositions of the type $G_2 \diamond G_1 := (K_1 \sqcup K_2 \xrightarrow{G_1 \sqcup 1} K_3 \sqcup K_2 \xrightarrow{G_2} K_4)$ are called *partial compositions*.

Proposition 3.2.5. *Any non-trivial morphism $G \colon K_1 \sqcup \cdots \sqcup K_n \to K_{n+1}$ in $\mathsf{Corollas}_{\mathcal{B}}$ that has one partially colored embedded graph on the standard disk as its underlying graph can be decomposed into a finite partial composition of morphisms of the following types:*



Proof. 1) Assuming that the morphism G has a non-zero number of *regular* edges (otherwise we jump to the next step), we can pick a regular edge of G and embed the standard disk to a small disk shaped neighborhood containing it. The embedding pulls back a partially colored graph on the standard disk which gives rise to a morphism G_1 of type (a) after we parameterize each internal vertex by a corolla, and we have $G = G_1' \diamond G_1$, where G_1' is obtained by replacing the part of G that is contained in image of the aforementioned embedding with a corolla. Repeat this until $G = G_n' \diamond G_n \diamond \cdots \diamond G_1$, where all the G_i's are of type (a) and G_n' does not contain any regular edges. This can be reached within finite steps because each step reduces the number of internal vertices by 1 and we require all our graphs to be *finite*. If G_n' is isotopic to a corolla, then the composite $G_n' \diamond G_n$ is of type (a) and we are done. Otherwise, we proceed to the next step.

2) We now look for the *loops* in G_n'. Without loss of generality, we assume that G_n' contains a loop (otherwise we jump to the next step), we can then choose one of the loops and embed a standard disk to a small neighborhood that contains it. This gives us $G = G_1'' \diamond G_1^{(1)} \diamond G_n \diamond \cdots \diamond G_1$, where $G_1^{(1)}$ is of type (b). Repeat this step until $G = G_m'' \diamond G_m^{(1)} \diamond \cdots \diamond G_1^{(1)} \diamond G_n \diamond \cdots \diamond G_1$, where all the $G_i^{(1)}$'s are of type (b) and G_m'' does not contain any loops. If G_m'' is isotopic to a corolla, then the composite $G_m'' \diamond G_m^{(1)}$ is of type (b) and we are done. Otherwise, G_m'' must be a union of corollas and edges that do not contain internal vertices and we proceed to the next step.

3) Let's assume that G_m'' contains at least two internal vertices, if not we jump to the next step. We can then pick a pair of internal vertices of G_m'' and embed a standard disk to a small neighborhood containing them. This gives us the decomposition $G = G_1''' \diamond G_1^{(2)} \diamond G_m^{(1)} \diamond \cdots \diamond G_1^{(1)} \diamond G_n \diamond \cdots \diamond G_1$, where $G_1^{(2)}$ is of type (c). Repeating this process until $G = G_l''' \diamond G_l^{(2)} \diamond \cdots \diamond G_1^{(2)} \diamond G_m^{(1)} \diamond \cdots \diamond G_1^{(1)} \diamond G_n \diamond \cdots \diamond G_1$, where all the $G_i^{(2)}$'s are of type (c) and G_l''' contains only one internal vertex and no regular edges or loops. If G_l''' is isotopic to a corolla, then the composite $G_l''' \diamond G_l^{(2)}$ is of type (c) and

we are done. Otherwise, G_l''' must consist of only one corolla and multiple edges that are without internal vertices. If this is the case, then let's move on to the final step.

4) After repeatedly searching for disk shaped neighborhoods that are of type (d) and replacing them with corollas, we arrive at the desired decomposition

$$G = G_k^{(3)} \diamond \cdots \diamond G_1^{(3)} \diamond G_l^{(2)} \diamond \cdots \diamond G_1^{(2)} \diamond G_m^{(1)} \diamond \cdots \diamond G_1^{(1)} \diamond G_n \diamond \cdots \diamond G_1,$$

where all the $G_i^{(3)}$'s are of type (d). $\square$

We call the linear maps obtained from evaluating the functor $\mathsf{Cal}_{\mathcal{B}} \colon \mathsf{Corollas}_{\mathcal{B}} \to \mathrm{Vect}_{\Bbbk}$ at morphisms of the types (a), (b), (c), and (d) *operadic compositions*, *partial trace maps*, *horizontal products*, and *whiskerings*, respectively, for evident reasons. The implication of Proposition 3.2.5 is that a linear map obtained from evaluating the functor $\mathsf{Cal}_{\mathcal{B}}$ at any *non-trivial* morphism can be decomposed into either a partial composite of such elementary maps or a tensor product of such partial composites.

Let us denote by $\mathsf{Corollas}_{\mathcal{B}}^{\mathrm{conn}}$ the subcategory of $\mathsf{Corollas}_{\mathcal{B}}$ that consists of all the objects and only the morphisms whose underlying graphs are all connected in the corresponding disks they are embedded in[3]. We have the following corollary:

Corollary 3.2.6. *The subcategory* $\mathsf{Corollas}_{\mathcal{B}}^{\mathrm{conn}}$ *is spanned by the trivial morphisms with connected embedded graphs[4] and the morphisms of the types* (a) *and* (b) *under partial composition and monoidal product (which is given by disjoint union).*

Denote by $\mathsf{Cal}_{\mathcal{B}}^{\mathrm{conn}} \colon \mathsf{Corollas}_{\mathcal{B}}^{\mathrm{conn}} \hookrightarrow \mathsf{Corollas}_{\mathcal{B}} \xrightarrow{\mathsf{Cal}_{\mathcal{B}}} \mathrm{Vect}_{\Bbbk}$ the restriction of the functor $\mathsf{Cal}_{\mathcal{B}}$ to the subcategory $\mathsf{Corollas}_{\mathcal{B}}^{\mathrm{conn}}$. As a consequence of Corollary 3.2.6, evaluating $\mathsf{Cal}_{\mathcal{B}}^{\mathrm{conn}}$ at any non-trivial morphism in its domain produces a partial composite of operadic compositions and partial trace maps, or a tensor product of such partial composites.

Remark 3.2.7. The constructions we introduced in this section are inspired by [Cos04], where the author described various types of operads as symmetric monoidal functors defined on different categories of graphs. Indeed, when $\mathcal{B}$ contains a single object, i.e. when $\mathcal{B}$ is the delooping of some (strictly) pivotal tensor category, the functor $\mathsf{Cal}_{\mathcal{B}}$ gives us its underlying non-symmetric colored modular operad with horizontal products for operations (since we allow non-connected graphs to be embedded in the standard disk when defining the category $\mathsf{Corollas}_{\mathcal{B}}$), and the restricted functor $\mathsf{Cal}_{\mathcal{B}}^{\mathrm{conn}}$ gives us its underlying non-symmetric colored modular operad.

3.3 What rigid separable Frobenius functors preserve

Let $\mathcal{B}$ and $\mathcal{B}'$ be two strictly pivotal bicategories. Recall that a *lax functor* is a triple $(F, F^{(2)}, F^{(0)})$, where by F we denote both a *map of objects* $F \colon \mathrm{obj}\,\mathcal{B} \to \mathrm{obj}\,\mathcal{B}'$ and a *local functor* (i.e. a functor on the respective hom-category) $F \colon \mathcal{B}(a, b) \to \mathcal{B}'(Fa, Fb)$ for

[3] For instance, a disjoint union of several morphisms of type (a) belongs to $\mathsf{Corollas}_{\mathcal{B}}^{\mathrm{conn}}$ while a morphism of type (c) does not.

[4] Here we are excluding the morphisms whose underlying graphs contain several distinct edges on the same disk.

each pair of objects $a, b \in \mathcal{B}$, while $F^{(2)}$ and $F^{(0)}$ are families of natural transformations given by

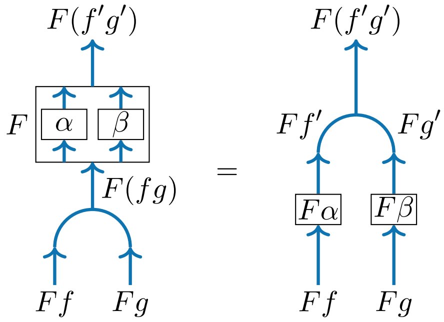
$$\begin{tikzcd}
\mathcal{B}(a,b) \times \mathcal{B}(b,c) \arrow[rr, "c"] \arrow[d, "F\times F"'] & & \mathcal{B}(a,c) \arrow[d, "F"] \\
\mathcal{B}'(Fa,Fb) \times \mathcal{B}'(Fb,Fc) \arrow[rr, "c'"'] \arrow[urr, phantom, "\Nearrow F^{(2)}"] & & \mathcal{B}'(Fa,Fc)
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{id}_a"] \arrow[dr, bend right, "\mathrm{id}_{Fa}"'] \arrow[dr, phantom, "\Nearrow F^{(0)}"] & \mathcal{B}(a,a) \arrow[d, "F"] \\
& \mathcal{B}'(Fa,Fa)
\end{tikzcd}$$

for every $a, b, c \in \mathcal{B}$, where c and c' are the horizontal composition functors of $\mathcal{B}$ and $\mathcal{B}'$, respectively. $F^{(2)}$ and $F^{(0)}$ are called the *lax functoriality constraint* and the *lax unity constraint* and they are required to be natural and satisfy a set of conditions that are akin to the axioms of an algebra (i.e. a monoid object) in a monoidal category. The *naturality* and the *lax associativity* of the lax functoriality constraint[5] $F^{(2)}$ can be expressed conveniently in terms of string diagrams:

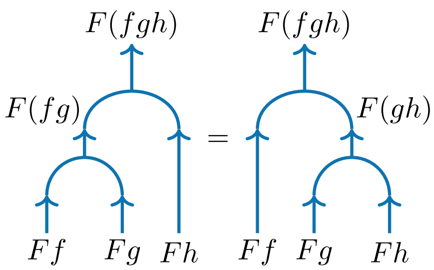
Naturality:



(3.3.1)

for every triple of objects $a, b, c \in \mathcal{B}$, 1-morphisms $f, f' \in \mathcal{B}(a,b)$, $g, g' \in \mathcal{B}(b,c)$ and 2-morphisms $\alpha\colon f \Rightarrow f'$, $\beta\colon g \Rightarrow g'$, where we used F followed by a window containing a string diagram to express the evaluation of the functor on the 2-morphism expressed by the string diagram, and the unnamed trivalent vertices stand for the corresponding components of the lax functoriality constraint $F^{(2)}$. Note that we have suppressed the colors that stand for different objects.

Lax associativity:

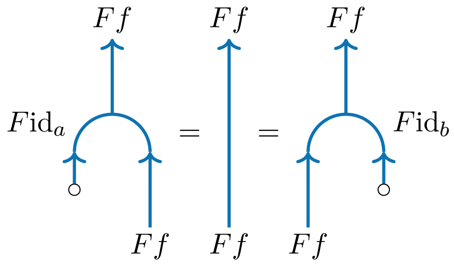


(3.3.2)

[5]Note that the naturality of the lax unity constraint is redundant, because the terminal category 1 contains only the identity morphism.

for every composable triple of 1-morphisms f, g, h in $\mathcal{B}$.

Lax left and right unity:



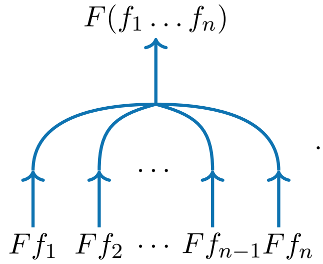
(3.3.3)

for every pair of objects $a, b \in \mathcal{B}$ and 1-morphism $f \in \mathcal{B}(a, b)$, where the unnamed nodes stand for the corresponding components of the lax unity constraint.

Due to the lax associativity, for every composable string of 1-morphisms $a_1 \xrightarrow{f_1} a_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_{n+1}$ in $\mathcal{B}$, we have a unique 2-morphism

$$F^{(n)}_{f_1,\ldots,f_n} : Ff_1 \cdots Ff_n \Rightarrow F(f_1 \cdots f_n)$$

obtained by composing suitable components of the lax functoriality constraint in arbitrary order, which we denote by the string diagram



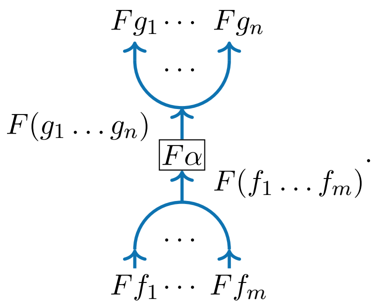
.

Dually, we have the notion of an *oplax functor* between two bicategories $\mathcal{B}$ and $\mathcal{B}'$ given by the triple $(G, G_{(2)}, G_{(0)})$, where $G_{(2)}$ and $G_{(0)}$ are called *oplax functoriality constraint* and *oplax unity constraint*. We refer to [JY21, Section 4.1] for the details.

Let $F : \mathcal{B} \to \mathcal{B}'$ be a functor with *both* lax and oplax structures, i.e. we have a tuple $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$. Then we can speak of the F*-conjugate* of a 2-morphism $\alpha : f_1 \ldots f_m \Rightarrow g_1 \ldots g_n$ defined by composing with the lax and oplax constraints. More precisely, if all the 1-morphisms in the domain and codomain are *non-trivial*, we define the F-conjugate to be

$$\alpha^F : Ff_1 \ldots Ff_m \xRightarrow{F^{(m)}_{f_1,\ldots,f_m}} F(f_1 \ldots f_m) \xRightarrow{F\alpha} F(g_1 \ldots g_n) \xRightarrow{F_{(n)\, g_1,\ldots,g_n}} Fg_1 \ldots Fg_n,$$

which is depicted by the string diagram



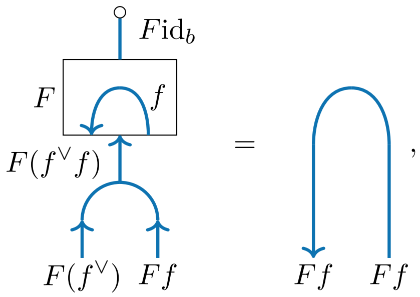
. (3.3.4)

If any of the 1-morphisms are identities, we need to insert the lax and/or oplax unity constraints accordingly.

In general, given a functor between bicategories with both lax and oplax structures, there is no reason for the conjugation to preserve compositions or partial traces, nor should it respect the coherent isomorphisms between the 2-hom spaces related by dualities. For this, we need to impose *properties* on the lax and oplax structures. The following notions are generalizations of their *monoidal* counterparts, see e.g. [MS10].

Definition 3.3.1. Let $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) \colon \mathcal{B} \to \mathcal{B}'$ be a functor between two strictly pivotal bicategories that is equipped with lax and oplax structures. It is

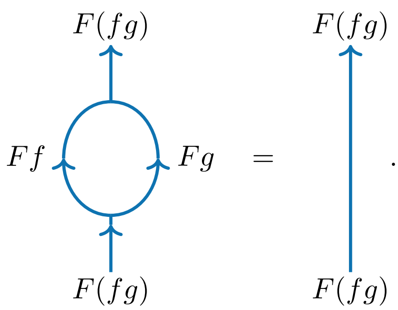
- *rigid*, if F-conjugation preserves the units and counits of the duals, i.e. for all $a, b \in \mathcal{B}$ and $f \in \mathcal{B}(a, b)$, we have $F(f^\vee) = (Ff)^\vee$ and

  (3.3.5)

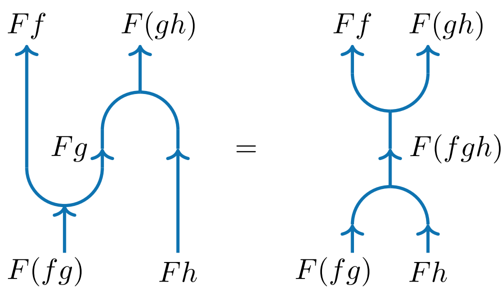
  etc.

- *separable*, if for every composable pair of 1-morphisms $a \xrightarrow{f} b \xrightarrow{g} c$, we have

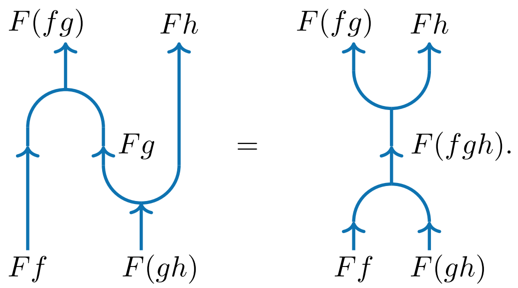
  (3.3.6)

- *Frobenius*, if for all composable triple of 1-morphisms $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$, we have

  (3.3.7)

and



(3.3.8)

It turns out that a *rigid separable Frobenius* functor *almost* preserves the unframed graphical calculus, as we shall see in the following theorem. Let K be a partially $\mathcal{B}$-colored corolla on the standard disk and $F \equiv (F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) \colon \mathcal{B} \to \mathcal{B}'$ a *rigid separable Frobenius* functor. The map of objects and the local functors entailed by F give rise to a symmetric monoidal functor $F_* \colon \mathsf{Corollas}_{\mathcal{B}} \to \mathsf{Corollas}_{\mathcal{B}'}$ by changing the colors of the patches and edges of the partially colored graphs. Moreover, the functor F_* can be restricted to $F_* \colon \mathsf{Corollas}^{\mathrm{conn}}_{\mathcal{B}} \to \mathsf{Corollas}^{\mathrm{conn}}_{\mathcal{B}'}$ because the change of colors does not affect the connectedness of the embedded graphs.

Theorem 3.3.2. *Let $F \colon \mathcal{B} \to \mathcal{B}'$ be a rigid Frobenius functor between two strictly pivotal bicategories. The F-conjugation canonically induces a monoidal natural transformation*

$$\begin{tikzcd}
\mathsf{Corollas}^{\mathrm{conn}}_{\mathcal{B}} \arrow[dd, "F_*"'] \arrow[dr, "\mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}}"] & \\
 \arrow[r, phantom, "\swarrow (-)^F"] & \mathrm{Vect}_{\Bbbk} \\
\mathsf{Corollas}^{\mathrm{conn}}_{\mathcal{B}'} \arrow[ur, "\mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}'}"'] &
\end{tikzcd}$$

which we also call the F-conjugation, whose component at a corolla $K \in \mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}}$ is given by

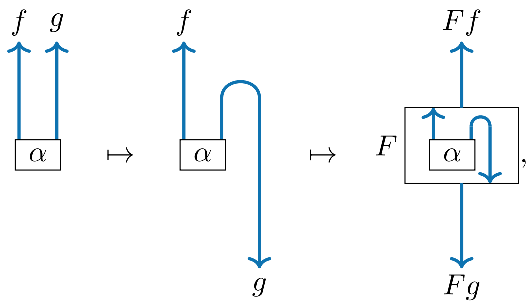
$$(-)^F_K \colon \mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}}(K) \xrightarrow{p^k_v} \widehat{h^{\mathcal{B}}_v}(k) \xrightarrow{(-)^F} \widehat{h^{\mathcal{B}'}_{v'}}(k) \xrightarrow{(p^k_{v'})^{-1}} \mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}}(F_* K) \tag{3.3.9}$$

for any choice of polarization k on the vertex $v \in K$, where all labels of the edges are treated as non-trivial 1-morphisms, whereas $(-)^F_{K_1 \sqcup K_2} \coloneqq (-)^F_{K_1} \otimes_{\Bbbk} (-)^F_{K_2}$.

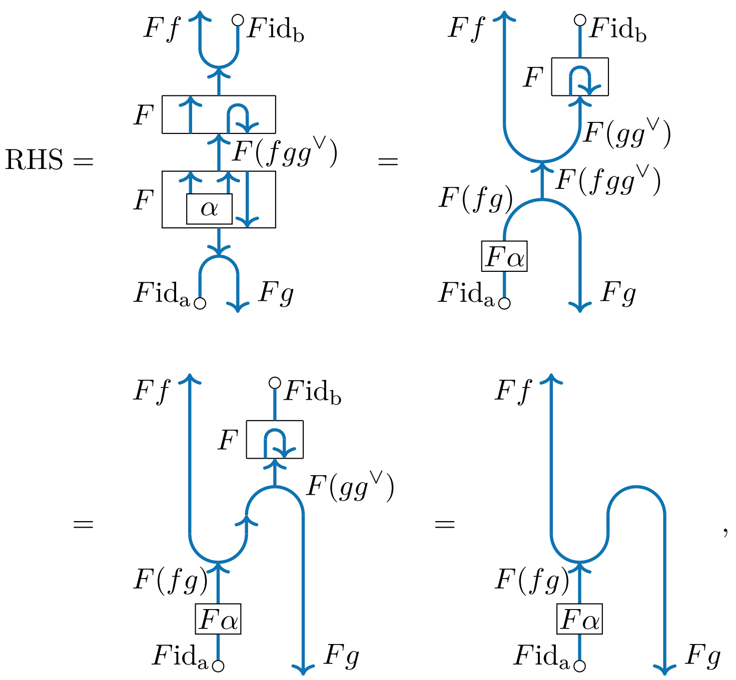
Proof. First we need to show that (3.3.9) is well defined, i.e. it is independent of the choice of polarization. This amounts to showing that the F-conjugation commutes with all canonical isomorphisms between 2-hom spaces that are related by duality. For simplicity, we consider a specific case, the argument for which can be generalized straightforwardly. Let us show that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}(a,a)}(\mathrm{id}_a, f \cdot g) \arrow[r, "\cong"] \arrow[d, "(-)^F"'] & \mathrm{Hom}_{\mathcal{B}(a,b)}(g^\vee, f) \arrow[d, "(-)^F"] \\
\mathrm{Hom}_{\mathcal{B}'(Fa,Fa)}(\mathrm{id}_{Fa}, Ff \cdot Fg) \arrow[r, "\cong"'] & \mathrm{Hom}_{\mathcal{B}'(Fa,Fb)}((Fg)^\vee, Ff)
\end{tikzcd}$$

commutes, where $a, b \in \mathcal{B}$ are arbitrary objects and $f \in \mathcal{B}(a, b)$ and $g \in \mathcal{B}(b, a)$ are *arbitrary* 1-morphisms (note that $F(g^\vee) = (Fg)^\vee$ due to the rigidity). To this end, we pick an arbitrary 2-morphism $\alpha \in \mathrm{Hom}_{\mathcal{B}(a,a)}(\mathrm{id}_a, fg)$ and chase the diagram. Note that since the 1-morphisms f and g are supposed to come from the coloring of edges, they are considered *non-trivial* and will be treated as such by the F-conjugation. First we go right then down which gives us



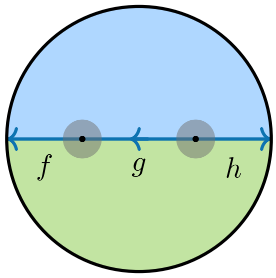
with



which is equal to tracing the diagram first down then right.

Now we need to show the naturality. The naturality squares for the trivial morphisms in $\mathsf{Corollas}_{\mathcal{B}}^{\mathrm{conn}}$ commute trivially. According to Corollary 3.2.6, we only need to verify the naturality for morphisms of the types (a) and (b), i.e. we need to verify that the F-conjugation commutes with the *operadic compositions* and the *partial trace maps*.

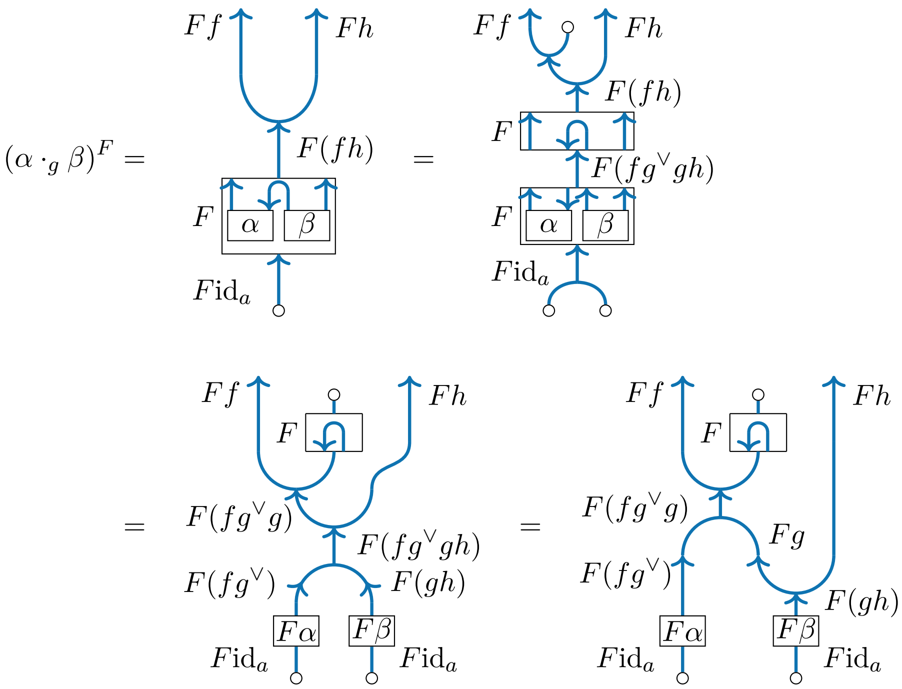
We first consider the operadic compositions. To keep things simple, let us pick a specific morphism of type (a), say $G \colon K_1 \sqcup K_2 \to K_3$ given by

$G =$ 

with $f, g, h \in \mathcal{B}(a, b)$. We need to show that the naturality square

$$\begin{tikzcd}
\mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}}(K_1) \otimes_{\Bbbk} \mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}}(K_2) \arrow[r, "\mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}}(G)"] \arrow[d, "(-)^F_{K_1} \otimes_{\Bbbk} (-)^F_{K_2}"'] & \mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}}(K_3) \arrow[d, "(-)^F_{K_3}"] \\
\mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}'}(F_* K_1) \otimes_{\Bbbk} \mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}'}(F_* K_2) \arrow[r, "\mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}'}(F_* G)"'] & \mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}'}(F_* K_3)
\end{tikzcd}$$

commutes. To this end, we choose a polarization for each of the three corollas as well as an element in each of the corresponding 2-hom spaces for K_1 and K_2. Let's say we have $\alpha \in \widehat{h^{\mathcal{B}}_{v_1}}(k_1) = \mathrm{Hom}_{\mathcal{B}(a,a)}(\mathrm{id}_a, fg^\vee)$ and $\beta \in \widehat{h^{\mathcal{B}}_{v_2}}(k_2) = \mathrm{Hom}_{\mathcal{B}(a,a)}(\mathrm{id}_a, gh)$. We need to show that composing α with β before taking the F-conjugate produces the same element in $\widehat{h^{\mathcal{B}}_{v_3}}(k_3) = \mathrm{Hom}_{\mathcal{B}(a,a)}(\mathrm{id}_a, fh)$ as taking the F-conjugates first then composing them. Indeed, we have



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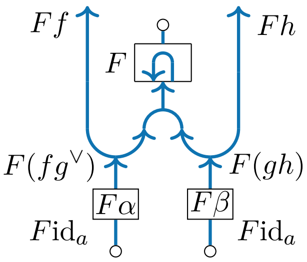

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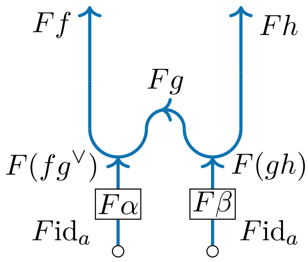
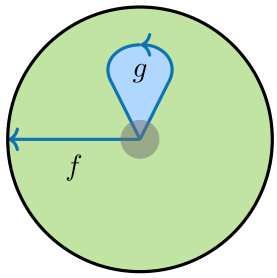


$= \alpha^F \cdot_{Fg} \beta^F.$

We now consider the partial trace maps. Let $G\colon K_1 \to K_2$ be the type (b) morphism in $\mathsf{Corollas}_{\mathcal{B}}^{\mathrm{conn}}$ given by

$G =$

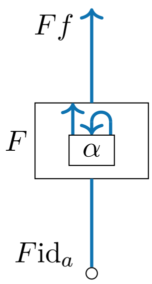


with $f \in \mathcal{B}(a,a)$ and $g \in \mathcal{B}(b,a)$. To show the commutativity of the naturality square

$$\begin{tikzcd}
\mathsf{Cal}_{\mathcal{B}}^{\mathrm{conn}}(K_1) \arrow[r, "\mathsf{Cal}_{\mathcal{B}}^{\mathrm{conn}}(G)"] \arrow[d, "(-)^F_{K_1}"'] & \mathsf{Cal}_{\mathcal{B}}^{\mathrm{conn}}(K_2) \arrow[d, "(-)^F_{K_2}"] \\
\mathsf{Cal}_{\mathcal{B}'}^{\mathrm{conn}}(F_*K_1) \arrow[r, "\mathsf{Cal}_{\mathcal{B}'}^{\mathrm{conn}}(F_*G)"'] & \mathsf{Cal}_{\mathcal{B}'}^{\mathrm{conn}}(F_*K_2)
\end{tikzcd}$$

we make a choice of the polarizations k_1 and k_2 for the corollas as well as a 2-morphism in $\widehat{h^{\mathcal{B}}_{v_1}}(k_1)$. Let's say we have $\alpha \in \widehat{h^{\mathcal{B}}_{v_1}}(k_1) = \mathrm{Hom}_{\mathcal{B}(a,a)}(\mathrm{id}_a, fg^\vee g)$ and $\widehat{h^{\mathcal{B}}_{v_2}}(k_2) = \mathrm{Hom}_{\mathcal{B}(a,a)}(\mathrm{id}_a, f)$. We then need to show that taking partial trace commute with the F-conjugation. Indeed,

$(\mathrm{tr}_g\, \alpha)^F =$



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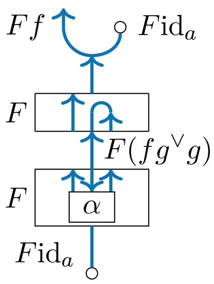

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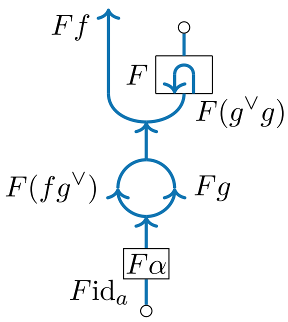

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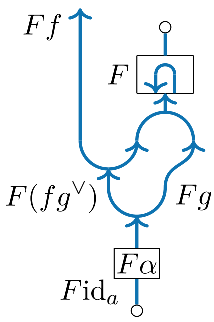
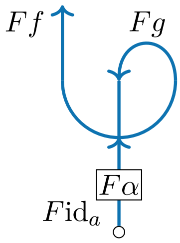
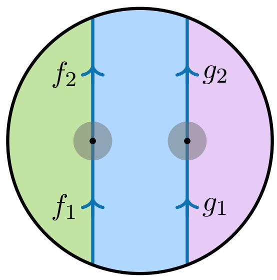


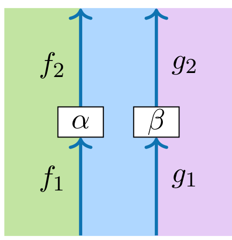
$=$  $= \operatorname{tr}_{Fg} \alpha^F.$

Therefore we did construct a natural transformation $(-)^F \colon \mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}} \Rightarrow \mathsf{Cal}^{\mathrm{conn}}_{\mathcal{B}'} \circ F_*$, which is evidently monoidal. $\square$

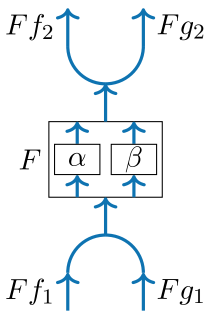
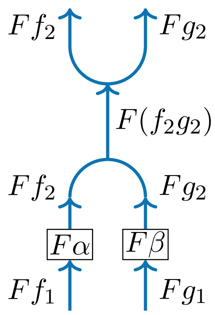
To understand why rigid separable Frobenius functors in general do not preserve the *complete* unframed graphical calculus, let us consider the following morphism $G \colon K_1 \sqcup K_2 \to K_3$ of type (c) given by

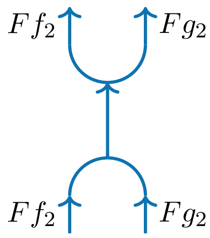
$G =$ 

with $f_1, f_2 \in \mathcal{B}(a, b)$ and $g_1, g_2 \in \mathcal{B}(b, c)$, which leads to the horizontal product



upon choosing appropriate polarizations and 2-morphisms. Let us compute the F-conjugate of the horizontal product:

$(\alpha \cdot \beta)^F =$  $=$  , which deviates from $\alpha^F \cdot \beta^F = F\alpha \cdot F\beta$ by the idempotent



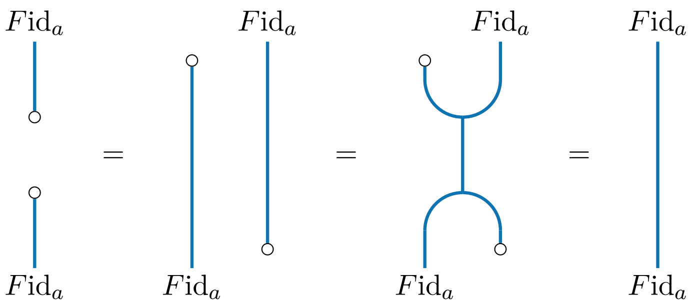
. (3.3.10)

Let us call a functor $F\colon \mathcal{B} \to \mathcal{B}'$ equipped with both lax and oplax structures whose lax functoriality constraint is the *inverse* of its oplax functoriality constraint *strongly separable*. It immediately follows that a rigid *strongly separable* Frobenius functor preserves the *horizontal products* as well as the *whiskerings*, hence the complete unframed graphical calculus of its domain pivotal bicategory. However, the notion of a strongly separable Frobenius functor is in fact redundant. Recall that a pseudofunctor can be regarded as a functor with lax and oplax structures, with the two being inverse to each other.

Proposition 3.3.3. *Let $F\colon \mathcal{B} \to \mathcal{B}'$ be a functor between two bicategories that is equipped with lax and oplax structures.*

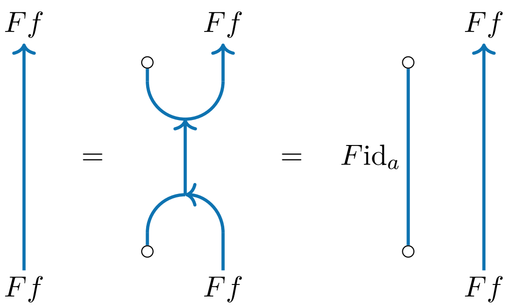
- *F is strongly separable if and only if F is a pseudofunctor.*
- *F is Frobenius if F is a pseudofunctor.*

Proof. 1) We first show that being strongly separable is equivalent to being a pseudofunctor. It is trivial that a pseudofunctor is necessarily strongly separable. To show that F being strongly separable implies it being a pseudofunctor, we need to show that the lax and oplax unity constraints are inverse to each other. Let $a \in \mathcal{B}$ be an arbitrary object, we have



,

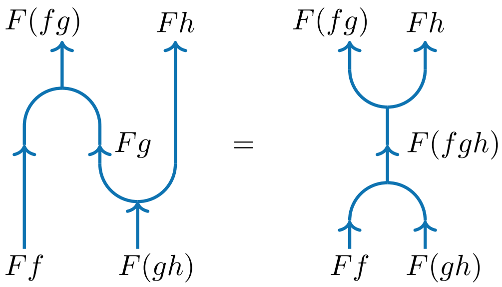
where we have used the premise that F is strongly separable for the second equality. On the other hand, for a 1-morphism $f \in \mathcal{B}(a, b)$ whose space of endomorphisms is non-zero (if there does not exist such an f for any $b \in \mathcal{B}$, then the statement we are proving holds trivially true), we have



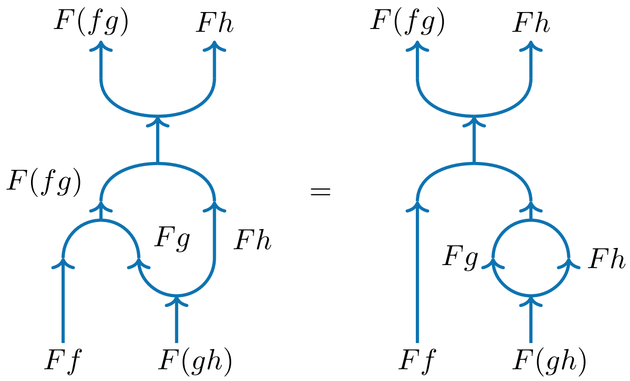
,

therefore $F_{(0)\,a} \circ F_a^{(0)} = \mathrm{id}_{Fa}$.

2) We now prove that "F is a pseudofunctor" $\Rightarrow$ "F is Frobenius". We show that



and leave the analogous proof of (3.3.7) to the reader:

LHS = 

= RHS.

□

Therefore, the notion of a strongly separable Frobenius functor is equivalent to that of a pseudofunctor.

Corollary 3.3.4. *Let $F\colon \mathcal{B} \to \mathcal{B}'$ be a rigid pseudofunctor between two strictly pivotal bicategories. The F-conjugation defined in Theorem 3.3.2 canonically extends to a monoidal natural transformation*

$$\begin{tikzcd}
\mathsf{Corollas}_{\mathcal{B}} \arrow[d, "F_*"'] \arrow[dr, "\mathsf{Cal}_{\mathcal{B}}"] \arrow[d, phantom, shift left=6ex, "\Swarrow (-)^F"] & \\
\mathsf{Corollas}_{\mathcal{B}'} \arrow[r, "\mathsf{Cal}_{\mathcal{B}'}"'] & \mathrm{Vect}_{\Bbbk}
\end{tikzcd}$$

We end this chapter by an example of a rigid separable Frobenius functor that will be important for our construction of RCFT correlators.

Example 3.3.5. Let $\mathcal{C}$ be a strictly pivotal fusion category, and $\mathcal{F}r(\mathcal{C})$ the strictly pivotal bicategory of simple special symmetric Frobenius algebras in $\mathcal{C}$. Recall that $\mathcal{C}$ can be viewed as a strictly pivotal bicategory $\mathcal{BC}$ (i.e. its *delooping*, see Example 3.1.3) with a single object $*$. Consider the functor

$$\mathcal{U}\colon \mathcal{F}r(\mathcal{C}) \to \mathcal{BC}$$

that is defined by

$$\begin{tikzcd} A \arrow[d, bend right=50, "X"'] \arrow[d, bend left=50, "Y"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] \\ B \end{tikzcd} \qquad \overset{\mathcal{U}}{\longmapsto} \qquad \begin{tikzcd} * \arrow[d, bend right=50, "\dot{X}"'] \arrow[d, bend left=50, "\dot{Y}"] \arrow[d, phantom, "\overset{\dot{\alpha}}{\Longrightarrow}"] \\ * \end{tikzcd}$$

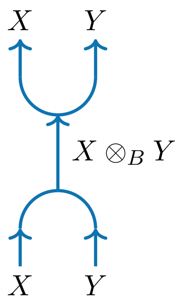
for arbitrary Frobenius algebras $A, B \in \mathcal{F}r(\mathcal{C})$, A-B-bimodules X, Y and bimodule morphism $\alpha\colon X \Rightarrow Y$, i.e. $\mathcal{U}$ sends all objects in $\mathcal{F}r(\mathcal{C})$ to the sole object $*$ in $\mathcal{BC}$ and every bimodule (resp. bimodule morphism) to its underlying object (resp. morphism) in $\mathcal{C}$. In the following, we suppress the dot and use the same symbol for a bimodule (resp. bimodule morphism) and its underlying object (resp. morphism) when the context is clear.

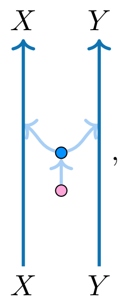
$\mathcal{U}$ is canonically a *rigid separable Frobenius* functor, with the components of the lax and oplax functoriality constraints

$$\mathcal{U}^{(2)}_{X,Y}\colon X \otimes Y \to X \otimes_B Y \quad \text{and} \quad \mathcal{U}_{(2)\,X,Y}\colon X \otimes_B Y \to X \otimes Y$$

at $(X, Y) \in \mathcal{F}r(\mathcal{C})(A, B) \times \mathcal{F}r(\mathcal{C})(B, C)$ given by the splitting of idempotent:

$$\mathcal{U}^{(2)}_{X,Y} \circ \mathcal{U}_{(2)\,X,Y} = \mathrm{id}_{X \otimes_B Y}, \tag{3.3.11}$$

$$\mathcal{U}_{(2)\,X,Y} \circ \mathcal{U}^{(2)}_{X,Y} =$$


$$=$$


$$, \tag{3.3.12}$$

and the lax and oplax unity constraints given by the units and counits of the Frobenius algebras. Note that when verifying the axioms, one needs to insert the associators and unitors accordingly.

4 String-net models for pivotal bicategories

In this chapter, whenever we mention an unspecified strictly pivotal bicategory, we assume it to be small and locally small, i.e. its objects form a set and all its hom-categories are small.

4.1 The definition of string-net spaces

First recall the notion of an $\mathcal{F}r(\mathcal{C})$-boundary datum (see Definition 2.5.1) on a compact oriented 1-manifold, and note that it can be easily generalized by replacing $\mathcal{F}r(\mathcal{C})$ with any bicategory. Let Σ be a compact oriented surface and b a $\mathcal{B}$-boundary datum on $\partial\Sigma$, where $\mathcal{B}$ is a strictly pivotal bicategory. A (fully) $\mathcal{B}$-colored graph Γ on Σ with $\mathcal{B}$-boundary datum b on $\partial\Sigma$ is a partially $\mathcal{B}$-colored graph $\mathring{\Gamma}$ on Σ together with a coloring of its internal vertices, i.e. a choice of an element in the vector space $H_v^{\mathcal{B}}$ for each internal vertex $v \in V(\mathring{\Gamma})$, such that the canonical embedding $\partial\Sigma \hookrightarrow \Sigma$ viewed as an *outgoing* parametrization of the boundary itself *pulls back* a $\mathcal{B}$-boundary datum on $\partial\Sigma$ that is b. For instance, the complemented worldsheet $\widetilde{\mathcal{S}}$ in Example 2.4.2 can be viewed as an $\mathcal{F}r(\mathcal{C})$-colored graph on the ambient surface $\Sigma_{\mathcal{S}}$ with the $\mathcal{F}r(\mathcal{C})$-boundary datum given schematically by

(4.1.1)

where the coloring of internal vertices (interpreted as point defects in the RCFT context) is furnished by the choice of linear orders and the corresponding 2-morphisms.

Denote by $\mathrm{G}(\Sigma, \mathsf{b})$ the set of all $\mathcal{B}$-colored graphs on Σ with $\mathcal{B}$-boundary datum b and $\Bbbk\mathrm{G}(\Sigma, \mathsf{b})$ the $\Bbbk$-vector space generated by it. We now define the *string-net space* for the pair (Σ, b) as follows:

Definition 4.1.1. Let $\mathcal{B}$ be a strictly pivotal bicategory, Σ a compact oriented surface, and b a $\mathcal{B}$-boundary datum on $\partial\Sigma$. The *string-net space* $\mathrm{SN}^\circ_{\mathcal{B}}(\Sigma, \mathsf{b})$ is the quotient

$$\mathrm{SN}^\circ_{\mathcal{B}}(\Sigma, \mathsf{b}) \coloneqq \Bbbk\mathrm{G}(\Sigma, \mathsf{b})/\mathrm{N}(\Sigma, \mathsf{b}), \tag{4.1.2}$$

where $\mathrm{N}(\Sigma, \mathsf{b})$ is the subspace of $\Bbbk\mathrm{G}(\Sigma, \mathsf{b})$ generated by *null graphs*: a null graph is an element $\sum_i \lambda_i \Gamma_i$ of $\Bbbk\mathrm{G}(\Sigma, \mathsf{b})$ such that there exists an embedding of the standard disk

$D \hookrightarrow \operatorname{int} \Sigma$ to the *interior* of the surface, whose image of the boundary of disk intersects with all the Γ_i's *transversally* at the *edges* and does not touch any vertices, such that the Γ_i's coincide outside of (the image of) D , as well as $\sum_i \lambda_i \langle \Gamma_i \cap D \rangle_D = 0$, i.e. the values of the graphs pulled back by the embedding sum up to zero.

We call a vector in the quotient space $\mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma, \mathsf{b})$ that is the image of an element of the generating set $\mathrm{G}(\Sigma, \mathsf{b})$ of $\Bbbk\mathrm{G}(\Sigma, \mathsf{b})$ a *bare string-net*, or just a *string-net*. The reason for the qualification "bare" and for the choice of the notation $\mathrm{SN}^{\circ}_{\mathcal{B}}$ is that later, we will construct a *Karoubified* version of string-net spaces, where the boundary data are replaced by certain *idempotents*. A *string-net* that has a $\mathcal{B}$-colored graph Γ as representative is denoted by $[\Gamma]$; by abuse of language, the term *string-net* is also used for individual ($\mathcal{B}$-colored) graphs that represent an element $[\Gamma] \in \mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma, \mathsf{b})$. The string-net space is linear in the color of each vertex of a graph and additive with respect to taking direct sums of objects labeling the edges. Due to the nature of the (unframed) graphical calculus, isotopic graphs represent the same string-net. Furthermore, all identities valid in the graphical calculus for $\mathcal{B}$ also hold inside any disk embedded in Σ. Thus for string-nets the graphical calculus for $\mathcal{B}$ applies locally on Σ.

Homeomorphisms of the surface Σ act naturally on embedded graphs. Since in the string-net space isotopic graphs are identified, this action descend to an action of the mapping class group $\mathrm{Map}(\Sigma)$ of the surface on $\mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma, \mathsf{b})$. String-nets with *matching boundary data* can be *concatenated*, in a manner similar to the sewing of worldsheets.

Remark 4.1.2. Recall that a color $c \in H^{\mathcal{B}}_v$ for an internal vertex v of a partially $\mathcal{B}$-colored graph $\mathring{\Gamma}$ is completely determined by a choice of 2-morphism $c_k \in \widehat{h^{\mathcal{B}}_v}(k)$ for any choice of polarization on v, and the 2-morphisms for different choices of polarization are related by coherent isomorphisms. Since the coherent isomorphisms are devised in such a way that the string diagrams produced according to different choices of polarizations share the same value when restricted to embedded disks, it is equally good, if not more convenient for calculation, to represent string-nets with *polarized* string diagrams, i.e. string diagrams with *rectangular coupons*.

Example 4.1.3. Let b be an $\mathcal{B}$-boundary datum on $S^1 = \partial D$. Recall that there is a unique partially colored corolla K^{b} associated to it according to (3.2.2). We have a canonical isomorphism

$$\mathrm{SN}^{\circ}_{\mathcal{B}}(D, \mathsf{b}) \xrightarrow{\cong} \mathsf{Cal}_{\mathcal{B}}(K^{\mathsf{b}})$$

given by the evaluation

$$[\Gamma] \mapsto \Gamma \mapsto c_\Gamma \in \bigotimes_{v \in V(\mathring{\Gamma})} H^{\mathcal{B}}_v \xmapsto{\widehat{\mathsf{Cal}}_{\mathcal{B}}(\mathring{\Gamma})} \langle \Gamma \rangle,$$

where c_Γ stands for the coloring of the internal vertex of the underlying partially colored graph $\mathring{\Gamma}$ given by Γ. The linear map is well defined because the graphical calculus is *local* in nature; it is injective since graphs with the same value under the evaluation on D must have the same value when evaluated on a slightly smaller disk embedded in D (after being replaced by isotopic graphs when necessary) hence they represent the same string-net; it is also surjective because given any element $c \in \mathsf{Cal}_{\mathcal{B}}(K^{\mathsf{b}})$, we can color the center of K^{b} with it and produce a fully colored corolla K^{b}_c whose value is c itself.

Remark 4.1.4. When the input bicategory is the delooping of a spherical fusion category $\mathcal{C}$, Definition 4.1.1 is reduced to the definition of string-net spaces for $\mathcal{C}$ given in [KJ11] and we write

$$\mathrm{SN}^{\circ}_{\mathcal{C}}(\Sigma, \mathsf{b}) \equiv \mathrm{SN}^{\circ}_{\mathcal{BC}}(\Sigma, \mathsf{b}).$$

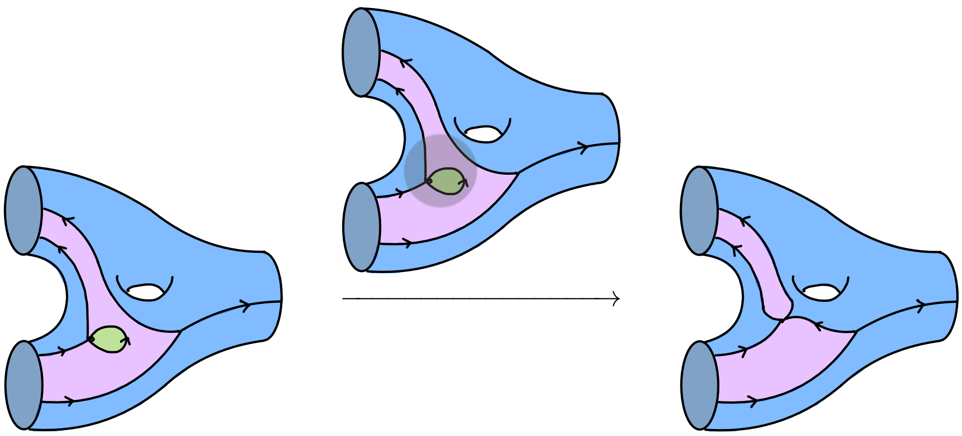
We hereby emphasize that our definition does not require *any* homological properties such as semisimplicity or finiteness.

4.2 String-net spaces as colimits

In the last section, we defined the string-net spaces as quotients of vector spaces. We now show that they are actually colimits in a very nice way.

Definition 4.2.1. Let $\mathcal{B}$ be a strictly pivotal bicategory, Σ a compact oriented surface, and b a $\mathcal{B}$-boundary datum on $\partial\Sigma$.

1. We define $\mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma, \mathsf{b})$ to be the category with:
   - objects: *partially* $\mathcal{B}$-colored graphs on Σ that have b as their boundary data;
   - morphisms: they are freely generated under composition (upon adjoining identities) by the morphisms exemplified by the following example (where the labels of the edges are suppressed):

   (4.2.1)

   here the morphism is given by an embedding of the standard disk D into the interior of the surface, such that the image of ∂D intersects with the edges of the partially colored graph $\mathring{\Gamma}_1$ in the domain transversally and does not touch any vertices, and the codomain is the graph $\mathring{\Gamma}_2$ obtained by replacing $\mathring{\Gamma}_1 \cap D$ with the image of the unique partially colored corolla on D associated with the boundary datum on $S^1 = \partial D$ pulled back by the embedding.

2. We define

$$\mathcal{E}^{\Sigma,\mathsf{b}}_{\mathcal{B}} \colon \mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma, \mathsf{b}) \to \mathrm{Vect}_{\Bbbk} \tag{4.2.2}$$

   to be the *evaluation functor* that

- sends an object (i.e. a partially $\mathcal{B}$-colored graph on Σ) $\mathring{\Gamma}$ to the vector space $\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(\mathring{\Gamma}) := \bigotimes_{v \in V(\mathring{\Gamma})} H_v^{\mathcal{B}}$;
- sends a generating morphism $\gamma \colon \mathring{\Gamma}_1 \to \mathring{\Gamma}_2$ to the linear map

$$\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(\gamma) \colon \bigotimes_{v \in V(\mathring{\Gamma}_1)} H_v^{\mathcal{B}} \xrightarrow{\mathrm{id} \otimes_{\Bbbk} \widehat{\mathsf{Cal}}_{\mathcal{B}}(\mathring{\Gamma}_1 \cap D_\gamma)} \bigotimes_{v' \in V(\mathring{\Gamma}_2)} H_{v'}^{\mathcal{B}}$$

obtained by applying the unframed graphical calculus to the partially colored graph on D pulled back by the embedding.

Theorem 4.2.2. *Let $\mathcal{B}$ be a strictly pivotal bicategory, Σ a compact oriented surface, and b a $\mathcal{B}$-boundary datum on $\partial\Sigma$. We have*

$$\mathrm{SN}_{\mathcal{B}}^{\circ}(\Sigma, \mathsf{b}) = \operatorname{colim} \mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}},$$

where the legs of the cocone are given by

$$\begin{aligned} \mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(\mathring{\Gamma}) = \bigotimes_{v \in V(\mathring{\Gamma})} H_v^{\mathcal{B}} &\to \mathrm{SN}_{\mathcal{B}}^{\circ}(\Sigma, \mathsf{b}) \\ c = \bigotimes_{v \in V(\mathring{\Gamma})} c_v &\mapsto [\mathring{\Gamma}_c] \end{aligned} \tag{4.2.3}$$

for every $\mathring{\Gamma} \in \mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma, \mathsf{b})$, here $\mathring{\Gamma}_c$ is the fully colored graph obtained by coloring $\mathring{\Gamma}$ with $c \in \bigotimes_{v \in V(\mathring{\Gamma})} H_v^{\mathcal{B}}$.

Proof. It is evident that (4.2.3) gives rise to a cocone. To show that the cocone is initial, consider an arbitrary cocone given by

$$\{f_{\mathring{\Gamma}} \colon \mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(\mathring{\Gamma}) \to V\}_{\mathring{\Gamma} \in \mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma,\mathsf{b})}, \tag{4.2.4}$$

where V is an arbitrary $\Bbbk$-vector space. We need to show that there is a unique linear map

$$f \colon \mathrm{SN}_{\mathcal{B}}^{\circ}(\Sigma, \mathsf{b}) \to V$$

making the diagram

$$\begin{tikzcd} \mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(\mathring{\Gamma}) \arrow[d] \arrow[dr, "f_{\mathring{\Gamma}}"] & \\ \mathrm{SN}_{\mathcal{B}}^{\circ}(\Sigma, \mathsf{b}) \arrow[r, dashed, "f"'] & V \end{tikzcd}$$

for every object $\mathring{\Gamma} \in \mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma, \mathsf{b})$ commute. Indeed, the desired linear map is given by

$$f \colon [\Gamma] \mapsto \Gamma \mapsto c_\Gamma \in \mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(\mathring{\Gamma}) \mapsto f_{\mathring{\Gamma}}(c_\Gamma).$$

To show that the linear map is well-defined, assume that the string-net $[\Gamma]$ is represented by two different colored graphs Γ and Γ'. By the definition of the string-net space, the underlying partially colored graphs $\mathring{\Gamma}$ and $\mathring{\Gamma}'$ are necessarily connected by a *zigzag*

$$\begin{tikzcd}
 & & \dots \arrow[r] & \mathring{\Gamma}' \\
 & \vdots \arrow[d] & & \\
\mathring{\Gamma}^{(1)} \arrow[r] \arrow[d] & \mathring{\Gamma}^{(2)} & & \\
\mathring{\Gamma} & & &
\end{tikzcd}$$

in the category $\mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma, \mathsf{b})$, and the corresponding vertex colors c_Γ and $c_{\Gamma'}$ are related by the zigzag in $\mathrm{Vect}_{\Bbbk}$ obtained by applying the functor $\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}} \colon \mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma, \mathsf{b}) \to \mathrm{Vect}_{\Bbbk}$, therefore are mapped to the same element in V by the legs of the cocone (4.2.4). By construction, f makes the relevant diagrams commute and is unique. $\square$

The recognition of the string-net space $\mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma, \mathsf{b})$ as a colimit provides a new perspective of the canonical map class group action: let $\xi \in \mathrm{Map}(\Sigma)$ be a mapping class group element and x a representing homeomorphism thereof. We have a canonical natural isomorphism

$$\begin{tikzcd}
\mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma, \mathsf{b}) \arrow[dd, "x_*"'] \arrow[dr, "\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}"] & \\
 \arrow[r, phantom, "\Swarrow\cong"] & \mathrm{Vect}_{\Bbbk} \\
\mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma, \mathsf{b}) \arrow[ur, "\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}"'] &
\end{tikzcd}$$

whose component at an object $\mathring{\Gamma} \in \mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\Sigma, \mathsf{b})$ is given by the canonical identification

$$\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(\mathring{\Gamma}) = \bigotimes_{v \in V(\mathring{\Gamma})} H_v^{\mathcal{B}} \xrightarrow{\cong} \bigotimes_{v' \in V(x_*\mathring{\Gamma})} H_{v'}^{\mathcal{B}} = \mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(x_*\mathring{\Gamma}),$$

where x_* is the endofunctor obtained by pushing forward the partially colored graphs. Now consider

$$\begin{tikzcd}
\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(\mathring{\Gamma}) = \bigotimes_{v \in V(\mathring{\Gamma})} H_v^{\mathcal{B}} \arrow[r] \arrow[d, "\cong"'] & \mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma, \mathsf{b}) \arrow[d, dashed, "\mathrm{SN}^{\circ}_{\mathcal{B}}(\xi,\mathsf{b})"] \\
\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(x_*\mathring{\Gamma}) = \bigotimes_{v' \in V(x_*\mathring{\Gamma})} H_{v'}^{\mathcal{B}} \arrow[r] & \mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma, \mathsf{b})
\end{tikzcd}$$

since the composite indicated by the color violet is a component of a natural transformation followed by a leg of a cocone, it is the leg of a cocone under $\mathcal{E}_{\mathcal{B}}^{\varSigma,\mathsf{b}}$. As a consequence, there is a unique endomorphism

$$\mathrm{SN}^{\circ}_{\mathcal{B}}(\xi,\mathsf{b})\colon \mathrm{SN}^{\circ}_{\mathcal{B}}(\varSigma,\mathsf{b}) \to \mathrm{SN}^{\circ}_{\mathcal{B}}(\varSigma,\mathsf{b})$$

of the string-net space $\mathrm{SN}^{\circ}_{\mathcal{B}}(\varSigma,\mathsf{b})$ filling the dashed vertical arrow making the square commute. By a straightforward diagram-chase we see that this endomorphism coincides with the action of $\xi = [x] \in \mathrm{Map}(\varSigma)$ on $\mathrm{SN}^{\circ}_{\mathcal{B}}(\varSigma,\mathsf{b})$.

4.3 Functoriality under rigid pseudofunctors

As we have seen in Section 3.3, *rigid pseudofunctors* preserve the unframed graphical calculus for strictly pivotal bicategories. Since the string-net spaces are built from such graphical calculus, it is correct to expect that a rigid pseudofunctor induces canonical linear maps between string-net spaces by *changing the colors*.

Theorem 4.3.1. *Let $\varSigma$ be a compact oriented surface, $\mathcal{B}$ and $\mathcal{B}'$ two strictly pivotal bicategories, $F\colon \mathcal{B} \to \mathcal{B}'$ a rigid pseudofunctor, and b a $\mathcal{B}$-boundary datum on $\partial\varSigma$. There is a canonical* $\mathrm{Map}(\varSigma)$*-intertwiner*

$$\mathrm{SN}^{\circ}_{F}(\varSigma,\mathsf{b})\colon \mathrm{SN}^{\circ}_{\mathcal{B}}(\varSigma,\mathsf{b}) \to \mathrm{SN}^{\circ}_{\mathcal{B}'}(\varSigma,F_*\mathsf{b}),$$

where $F_*\mathsf{b}$ is the $\mathcal{B}'$-boundary datum on $\partial\varSigma$ obtained by changing the coloring according to the map of objects and the local functors of F. The linear map is defined by sending each representing $\mathcal{B}$-colored graph to the $\mathcal{B}'$-colored graph obtained by the F-conjugation (3.3.4)*. Moreover, the collection of such intertwiners corresponding to different surfaces and boundary data is compatible with the concatenation of string-nets.*

Proof. Consider the natural transformation

$$\begin{tikzcd}
\mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\varSigma,\mathsf{b}) \arrow[dd, "F_*"'] \arrow[dr, "\mathcal{E}_{\mathcal{B}}^{\varSigma,\mathsf{b}}"] & \\
 \arrow[r, phantom, "\Swarrow (-)^F"] & \mathrm{Vect}_{\Bbbk} \\
\mathcal{G}\mathrm{raphs}_{\mathcal{B}'}(\varSigma,F_*\mathsf{b}) \arrow[ur, "\mathcal{E}_{\mathcal{B}'}^{\varSigma,F_*\mathsf{b}}"'] &
\end{tikzcd}$$

whose component at $\mathring{\varGamma} \in \mathcal{G}\mathrm{raphs}_{\mathcal{B}}(\varSigma,\mathsf{b})$ is given by the F-conjugation

$$(-)^F_{\mathring{\varGamma}}\colon \mathcal{E}_{\mathcal{B}}^{\varSigma,\mathsf{b}}(\mathring{\varGamma}) = \bigotimes_{v\in V(\mathring{\varGamma})} H_v^{\mathcal{B}} \to \bigotimes_{v'\in V(F_*\mathring{\varGamma})} H_{v'}^{\mathcal{B}'} = \mathcal{E}_{\mathcal{B}'}^{\varSigma,F_*\mathsf{b}}(\mathring{\varGamma}).$$

The naturality square for a generating morphism $\gamma\colon \mathring{\varGamma}_1 \to \mathring{\varGamma}_2$ reads

$$\begin{tikzcd}
\bigotimes_{v_1\in V(\mathring{\varGamma}_1)} H_{v_1}^{\mathcal{B}} \arrow[r, "\mathrm{id}\otimes_{\Bbbk}\widehat{\mathsf{Cal}}_{\mathcal{B}}(\mathring{\varGamma}_1\cap D_\gamma)"] \arrow[d, "(-)^F_{\mathring{\varGamma}_1}"'] & \bigotimes_{v_2\in V(\mathring{\varGamma}_2)} H_{v_2}^{\mathcal{B}} \arrow[d, "(-)^F_{\mathring{\varGamma}_2}"] \\
\bigotimes_{v_1'\in V(F_*\mathring{\varGamma}_1)} H_{v_1'}^{\mathcal{B}} \arrow[r, "\mathrm{id}\otimes_{\Bbbk}\widehat{\mathsf{Cal}}_{\mathcal{B}'}(F_*\mathring{\varGamma}_1\cap D_{F_*\gamma})"'] & \bigotimes_{v_2'\in V(F_*\mathring{\varGamma}_2)} H_{v_2'}^{\mathcal{B}}
\end{tikzcd}$$

whose commutativity follows from Corollary 3.3.4. Now consider

$$\begin{tikzcd}
\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}(\mathring{\Gamma}) = \bigotimes\limits_{v\in V(\mathring{\Gamma})} H_v^{\mathcal{B}} \arrow[r] \arrow[d, "(-)^F_{\mathring{\Gamma}}"'] & \mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma,\mathsf{b}) \arrow[d, dashed, "\mathrm{SN}^{\circ}_{F}(\Sigma,\mathsf{b})"] \\
\mathcal{E}_{\mathcal{B}'}^{\Sigma,F_*\mathsf{b}}(F_*\mathring{\Gamma}) = \bigotimes\limits_{v'\in V(F_*\mathring{\Gamma})} H_{v'}^{\mathcal{B}} \arrow[r] & \mathrm{SN}^{\circ}_{\mathcal{B}'}(\Sigma,F_*\mathsf{b})
\end{tikzcd}$$

since the composite indicated by the color violet is a component of a natural transformation followed by a leg of a cocone, it is the leg of a cocone under $\mathcal{E}_{\mathcal{B}}^{\Sigma,\mathsf{b}}$. Therefore, there is a unique linear map

$$\mathrm{SN}^{\circ}_{F}(\Sigma,\mathsf{b})\colon \mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma,\mathsf{b}) \to \mathrm{SN}^{\circ}_{\mathcal{B}'}(\Sigma,F_*\mathsf{b})$$

making the square commute. By an easy diagram-chase, we see that this linear map is as described in the theorem. The equivariance and the compatibility with concatenation is evident. $\square$

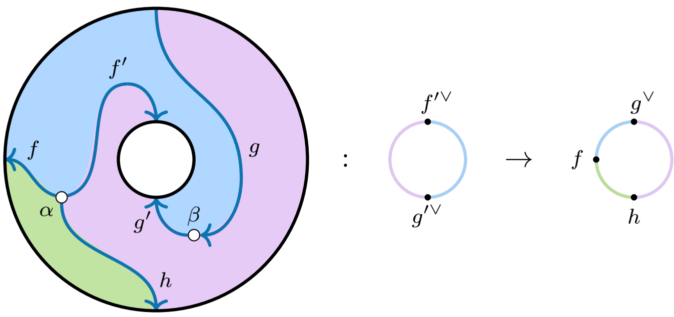
Note that since in general a *rigid separable Frobenius functor* only preserve horizontal products and whiskerings up to idempotents of the type (3.3.10), the corresponding change of colors provided by conjugation *does not* descend to linear maps between string-net spaces.

4.4 Cylinder categories over circles

The notion of string-net spaces for a strictly pivotal bicategory $\mathcal{B}$ allows us to promote the set of $\mathcal{B}$-boundary data on a *closed oriented* 1-manifold to a (small) $\Bbbk$-linear category.

Definition 4.4.1. Let $\mathcal{B}$ be a strictly pivotal bicategory and ℓ a closed oriented 1-manifold. If ℓ is non-empty, the *cylinder category* for $\mathcal{B}$ over ℓ is the category $\mathrm{Cyl}^{\circ}(\mathcal{B},\ell)$ with

- objects: given by all $\mathcal{B}$-boundary data on ℓ;
- morphisms: a morphisms between two boundary data is given by a string-net on the cylinder $\ell \times I$ matching the boundary data, for instance:



(4.4.1)

is a morphism in $\mathrm{Cyl}^{\circ}(\mathcal{B},S^1)$, with an appropriate $\mathcal{B}$-coloring. Note that the boundary component $\ell \times \{0\}$ is considered *in-going* and supports the domain boundary datum, hence the opposite convention for the edge labels. The composition is given by the concatenation of string-nets.

If ℓ is the empty 1-manifold $\emptyset$, we define $\mathrm{Cyl}^{\circ}(\mathcal{B}, \emptyset) \coloneqq \mathrm{Vect}_{\Bbbk}$.

Note that for each compact oriented surface Σ, the string-net construction provides a functor

$$\mathrm{SN}^{\circ}_{\mathcal{B}}(\Sigma, -) \colon \mathrm{Cyl}^{\circ}(\mathcal{B}, \partial\Sigma) \to \mathrm{Vect}_{\Bbbk},$$

where the morphisms of the cylinder category act by concatenation.

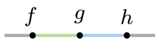
Remark 4.4.2. Since the cylinder over a 1-manifold comes with a canonical 2-framing, one can define the cylinder categories for a bicategory without a pivotal structure by requiring the use of rectangular coupons that are *aligned with the frame.*

4.5 Pointed pivotal bicategories and cylinder categories over intervals

We have defined the cylinder category over a *closed* oriented 1-manifold in the last section. One might wonder: what about the 1-manifolds with boundaries? It turns out to be the most appropriate to define the cylinder categories over them for a *pointed* strictly pivotal bicategory $(\mathcal{B}, *_{\mathcal{B}})$, i.e. a strictly pivotal bicategory with a distinguished object $*_{\mathcal{B}} \in \mathcal{B}$.

Definition 4.5.1. Let $(\mathcal{B}, *_{\mathcal{B}})$ be a pointed strictly pivotal bicategory and ℓ a compact oriented 1-manifold with possibly non-empty boundary. The cylinder category for $(\mathcal{B}, *_{\mathcal{B}})$ over ℓ is the category $\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \ell)$ with

- objects: given by the $\mathcal{B}$-boundary data whose 1-cells adjacent to the boundary components of ℓ are all colored with the distinguished object $*_{\mathcal{B}}$, which are called the $(\mathcal{B}, *_{\mathcal{B}})$-*boundary data* on ℓ. For instance,

  $\mathsf{b} =$ 

  is an object in $\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, I)$, where we used the color gray to denote the distinguished object $*_{\mathcal{B}}$.

- morphisms: given by string-nets on the cylinder $\ell \times I$ over ℓ. The composition is given by concatenating string-nets.

Note that for a *closed* oriented 1-manifold ℓ, we have $\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \ell) \equiv \mathrm{Cyl}^{\circ}(\mathcal{B}, \ell)$.

Example 4.5.2. The delooping $\mathcal{BC}$ of a strictly pivotal tensor category $\mathcal{C}$ is automatically pointed due to having only a single object, therefore the restriction on the objects in Definition 4.5.1 is vacuous in this case. We therefore write

$$\mathrm{Cyl}^{\circ}(\mathcal{C}, \ell) \equiv \mathrm{Cyl}^{\circ}(\mathcal{BC}, *, \ell)$$

and call an object therein a $\mathcal{C}$-*boundary value* on ℓ.

Example 4.5.3. The strictly pivotal bicategory $\mathcal{F}r(\mathcal{C})$ of simple special symmetric Frobenius algebras in a strictly pivotal tensor category $\mathcal{C}$ is canonically pointed with

$$*_{\mathcal{F}r(\mathcal{C})} \coloneqq \mathbb{1} \in \mathcal{F}r(\mathcal{C}),$$

i.e. the tensor unit of $\mathcal{C}$ canonically viewed as a Frobenius algebra. The $\mathcal{F}r(\mathcal{C})$-boundary data on the standard interval I obtained by parametrizations of the boundaries of the complemented worldsheets are all objects in the cylinder category $\mathrm{Cyl}^{\circ}(\mathcal{F}r(\mathcal{C}), \mathbb{1}, I)$.

4.6 Functoriality under embeddings

We claimed that the *appropriate* categorical input for the definition of a cylinder category over a compact oriented 1-manifold with boundary is a *pointed* strictly pivotal bicategory. In this section we provide the reason behind this claim.

Recall that, for two arbitrary 1-manifolds ℓ_1 and ℓ_2, any continuous map $f\colon \ell_1 \to \ell_2$ extends canonically to a map $f \times I\colon \ell_1 \times I \to \ell_2 \times I$ between the cylinders by the formula $f \times I\colon (p,t) \mapsto (f(p),t)$. Therefore, an orientation *preserving* automorphism of the circle $x\colon S^1 \to S^1$ induces a functor

$$\mathrm{Cyl}^\circ(\mathcal{B},x)\colon \mathrm{Cyl}^\circ(\mathcal{B},S^1) \to \mathrm{Cyl}^\circ(\mathcal{B},S^1)$$

by pushing forward the objects via x and the morphisms via $x \times I$. Notably, the cylinder category $\mathrm{Cyl}^\circ(\mathcal{B},S^1)$ carries an action of the circle group $\mathrm{U}(1) \subset \mathrm{Aut}(S^1)$.

One soon realizes that when the input bicategory $\mathcal{B}$ has more than one object, there is no functoriality under the embeddings of general oriented 1-manifolds if we assign to each 1-manifold the *naive* version of cylinder category over it. The remedy is to fix a distinguished object.

First consider an orientation preserving embedding $f\colon \bigsqcup_{i=1}^{n} I \hookrightarrow I$ of n copies of the standard interval into itself. Due to the restriction on the $\mathcal{B}$-coloring of the 1-cells, now it is possible to define a functor

$$\mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},f)\colon \mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},\bigsqcup_{i=1}^{n} I) = \prod_{i=1}^{n} \mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},I) \to \mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},I) \tag{4.6.1}$$

by pushing forward the objects and morphisms via f and $f \times I$, respectively, then color the complement $I \setminus \mathrm{im}\, f$ (resp. $I^2 \setminus \mathrm{im}(f \times I)$) with the distinguished object $*_\mathcal{B}$. As a special case, the embedding of the empty 1-manifold $\emptyset \hookrightarrow I$ induces a functor of the type $\mathrm{Cyl}^\circ(\mathcal{B},\emptyset) = \mathrm{Vect}_\Bbbk \to \mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},I)$ by sending $\Bbbk$ to the $\mathcal{B}$-boundary datum b_I^* on I that has the interval completely colored with $*_\mathcal{B}$. (This in fact works with the embedding $\emptyset \hookrightarrow \ell$ for any oriented 1-manifold ℓ. In this sense, *all* cylinder categories are canonically pointed by pointing the input bicategory.) By fixing a binary embedding $I \sqcup I \to I$, (4.6.1) endows the category $\mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},I)$ with a monoidal structure, where the tensor unit is given by b_I^*. Moreover, by sending each object in $\mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},I)$ to the corresponding horizontal composite of the coloring 1-morphisms, we establish an equivalence of tensor categories $\mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},I) \simeq \mathrm{End}_\mathcal{B}(*_\mathcal{B})$, where the monoidal product for the endomorphism category is given by the horizontal composition.

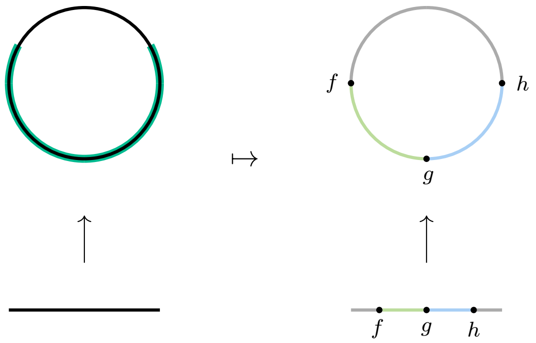
Proposition 4.6.1. *Let $(\mathcal{B},*_\mathcal{B})$ be a pointed strictly pivotal bicategory. There is a canonical monoidal equivalence*

$$\mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},I) \simeq \mathrm{End}_\mathcal{B}(*_\mathcal{B}).$$

Now consider an embedding $f\colon I \hookrightarrow S^1$. Again by pushing forward the objects (resp. morphisms) via the embedding (resp. $f \times I$) and color the complement of the image with the distinguished color, we obtain a functor

$$\mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},f)\colon \mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},I) \to \mathrm{Cyl}^\circ(\mathcal{B},S^1) \equiv \mathrm{Cyl}^\circ(\mathcal{B},,*_\mathcal{B},S^1).$$

This can be demonstrated by the following figure:



Combining all of these, we have the following statement.

Proposition 4.6.2. *The assignment of cylinder categories for a pointed pivotal bicategory $(\mathcal{B}, *_{\mathcal{B}})$ to compact oriented 1-manifolds canonically extends to a symmetric monoidal functor*[1]

$$\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, -) \colon \mathrm{Emb}_1^{\mathrm{or}} \to \mathrm{Cat}_{\Bbbk},$$

where the domain category $\mathrm{Emb}_1^{\mathrm{or}}$ is the symmetric monoidal category of compact oriented 1-manifolds and orientation preserving embeddings whose monoidal product is given by disjoint union, whereas the codomain is the symmetric monoidal category of small $\Bbbk$-linear categories and $\Bbbk$-linear functors whose monoidal product is given by the Cartesian product.[2]

Lastly, let ℓ be a compact oriented 1-manifold and $\bar{\ell}$ the same underlying 1-manifold but with the opposite orientation. Due to the strict pivotality of $\mathcal{B}$, we have a canonical isomorphism

$$\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \ell)^{\mathrm{op}} \xrightarrow{\cong} \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \bar{\ell})$$

which sends an object in $\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \ell)^{\mathrm{op}}$ to the boundary datum on $\bar{\ell}$ obtained by taking the duals of the coloring 1-morphisms. With the help of this identification, an orientation *reversing* embedding $f \colon \ell_1 \to \ell_2$ induces a functor

$$\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, f) \colon \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \ell_1)^{\mathrm{op}} \to \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \ell_2).$$

Remark 4.6.3. Let us choose the 2-framings for the cylinders $I \times I$ and $S^1 \times I$ to be the ones that uniformly point from $(p, 0)$ towards $(p, 1)$ for p in the interval I and the circle S^1, respectively. This allows us to define the cylinder categories $\mathrm{Cyl}^{\circ}(\mathcal{B}, a, I)$ and $\mathrm{Cyl}^{\circ}(\mathcal{B}, S^1)$ for a bicategory $\mathcal{B}$ without a pivotal structure and every object $a \in \mathcal{B}$, by restricting to *progressive* string diagrams. Upon fixing an embedding $I \hookrightarrow S^1$, we obtain a functor

$$\bigsqcup_{a \in \mathcal{B}} \mathcal{B}(a, a) \simeq \bigsqcup_{a \in \mathcal{B}} \mathrm{Cyl}^{\circ}(\mathcal{B}, a, I) \xrightarrow{\bigsqcup_{a \in \mathcal{B}} \mathrm{Cyl}^{\circ}(\mathcal{B}, a, f)} \mathrm{Cyl}^{\circ}(\mathcal{B}, S^1). \tag{4.6.2}$$

[1] In fact, the functor $\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, -)$ factors through the forgetful functor $\mathrm{Cat}_{\Bbbk}^{\mathrm{pointed}} \to \mathrm{Cat}_{\Bbbk}$, because as already mentioned, every cylinder category is pointed by the embedding of the empty manifold $\emptyset$.

[2] Even better, we actually obtain a symmetric monoidal 2-functor $\mathcal{C}\mathrm{yl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, -) \colon \mathcal{E}\mathrm{mb}_1^{\mathrm{or}} \to \mathcal{C}\mathrm{at}_{\Bbbk}$, where $\mathcal{E}\mathrm{mb}_1^{\mathrm{or}}$ is the symmetric monoidal (2,1)-category of compact oriented 1-manifolds, orientation preserving embeddings, and isotopy classes of isotopies between embeddings.

This functor admits the structure of a *categorified trace* on $\mathcal{B}$: a categorified trace on a bicategory $\mathcal{B}$ with values in a category $\mathcal{A}$ is a functor

$$[\![-]\!]\colon \coprod_{a\in\mathcal{B}} \mathcal{B}(a,a) \to \mathcal{A}$$

from the hom-categories of $\mathcal{B}$ to the category $\mathcal{A}$ equipped with a natural isomorphism

$$\theta\colon [\![fg]\!] \xrightarrow{\cong} [\![gf]\!]$$

where $f\colon a \to b$ and $g\colon b \to a$ are cyclically composable 1-morphisms in $\mathcal{B}$, such that two hexagon and two triangle diagrams commute. When the natural isomorphism θ is an *involution*, the categorified trace is called *symmetric*, also known as a *shadow* [Pon10, PS13]. An important example of a symmetric categorified trace/shadow is provided by the *topological Hochschild homology (THH) of spectral categories* [CP19, Theorem 2.17]. Recently [Ber22, HR21], the notion of topological Hochschild homology was extended to that of a *bicategory*, and it was shown [HR21, Theorem 3.19] that the THH of a bicategory $\mathcal{B}$, which is a category $\mathrm{THH}(\mathcal{B})$ given by a pseudocolimit of a certain 2-truncated cyclic bar construction, is canonically endowed with the structure of a *universal shadow* on $\mathcal{B}$ in the sense that we have a canonical equivalence of categories

$$\mathrm{Fun}(\mathrm{THH}(\mathcal{B}),\mathcal{D}) \xrightarrow{\simeq} \mathcal{S}\mathrm{ha}(\mathcal{B},\mathcal{D})$$

for every category $\mathcal{D}$, where $\mathcal{S}\mathrm{ha}(\mathcal{B},\mathcal{D})$ is the category of shadows on $\mathcal{B}$ with values in $\mathcal{D}$. We expect that the cylinder category $\mathrm{Cyl}^{\circ}(\mathcal{B}, S^1)$ provides a *non-symmetric* analogue of $\mathrm{THH}(\mathcal{B})$ – the *topological cyclic homology* of the bicategory $\mathcal{B}$. More precisely: we expect that the non-symmetric categorified traces they give rise to are universal, and that the cylinder categories over the circle are pseudocolimits of a variant of the cyclic bar construction where the simplicial category Δ is replaced by the *cyclic category* ΔC introduced in [Con83].

4.7 Idempotent completion and the string-net models for spherical fusion categories

Recall that given a category $\mathcal{A}$, the *Karoubi envelope* $\mathrm{Kar}(\mathcal{A})$ is the idempotent completion of $\mathcal{A}$ whose objects are idempotents in $\mathcal{A}$, and a morphism $f \in \mathrm{Kar}(\mathcal{A})(p_1, p_2)$ between idempotents $p_i \in \mathrm{End}_{\mathcal{A}}(a_i)$ has an underlying morphism $f\colon a_1 \to a_2$ in $\mathcal{A}$, such that $f \circ p_1 = f = p_2 \circ f$. Note that the identity of an idempotent $p \in \mathrm{End}_{\mathcal{A}}(a)$ viewed as an object in $\mathrm{Kar}(\mathcal{A})$ is given by p itself, while id_a in general does not belong in $\mathrm{End}_{\mathrm{Kar}(\mathcal{A})}(p)$, unless $p = \mathrm{id}_a$. The Karoubi envelope $\mathrm{Kar}(\mathcal{A})$ comes with a canonical fully faithful functor

$$\mathcal{A} \to \mathrm{Kar}(\mathcal{A}) \tag{4.7.1}$$

that sends an object $a \in \mathcal{A}$ to the idempotent id_a, which is universal among the functors from $\mathcal{A}$ that have a chosen *splitting* for every idempotent in their codomains. Moreover, the functor (4.7.1) is *cofinal*[3] in the sense that we can restrict functors from $\mathrm{Kar}(\mathcal{A})$

[3]Note that we are using the terminology used in [Bor94] and [Lur09], which differs from the one used in [ML98] and [Joh02] where they are called *final functors* instead.

to functors from $\mathcal{A}$ along (4.7.1) without changing their *colimits* (see [Lur09, Lemma 5.1.4.6] for the $(\infty,1)$-version of the statement).

Let ℓ be a compact oriented and not necessarily closed 1-manifold and $(\mathcal{B}, *_{\mathcal{B}})$ a pointed strictly pivotal bicategory. We define the *Karoubified* cylinder category for $(\mathcal{B}, *_{\mathcal{B}})$ over ℓ to be the Karoubi envelope

$$\mathrm{Cyl}(\mathcal{B}, *_{\mathcal{B}}, \ell) \coloneqq \mathrm{Kar}(\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \ell)).$$

An object $\mathsf{B} \in \mathrm{Cyl}(\mathcal{B}, *_{\mathcal{B}}, \ell)$ is given by an idempotent $\mathsf{B}\colon \mathsf{B}^{\circ} \to \mathsf{B}^{\circ}$ in the ordinary cylinder category $\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \ell)$ and is called a *thickened* $(\mathcal{B}, *_{\mathcal{B}})$-boundary datum on ℓ. Subsequently, we define the *Karoubified* string-net space $\mathrm{SN}_{\mathcal{B}}(\varSigma, \mathsf{B})$ for a compact oriented surface $\varSigma$ and a *thickened* $(\mathcal{B}, *_{\mathcal{B}})$-boundary datum $\mathsf{B} \in \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \partial\varSigma) \equiv \mathrm{Cyl}^{\circ}(\mathcal{B}, \partial\varSigma)$ to be the subspace

$$\mathrm{SN}_{\mathcal{B}}(\varSigma, \mathsf{B}) \coloneqq \mathrm{SN}^{\circ}_{\mathcal{B}}(\varSigma, \mathsf{B}^{\circ})^{\mathsf{B}} \subset \mathrm{SN}^{\circ}_{\mathcal{B}}(\varSigma, \mathsf{B}^{\circ}) \tag{4.7.2}$$

consisting of string-nets that are *invariant* under the concatenation with B. In this way, we obtain a functor

$$\mathrm{SN}_{\mathcal{B}}(\varSigma, -)\colon \mathrm{Cyl}(\mathcal{B}, \partial\varSigma) \to \mathrm{Vect}_{\Bbbk}$$

for every compact oriented surface $\varSigma$. Note that the assignment $\mathrm{Cyl}(\mathcal{B}, *_{\mathcal{B}}, -)$ is also functorial with respect to the embeddings of 1-manifolds.

The reason for introducing the notions of Karoubified cylinder categories and string-net spaces is the following. Let $\mathcal{C}$ be a spherical fusion category (with tacitly strictified pivotal structure) and write

$$\mathrm{Cyl}(\mathcal{C}, \ell) \equiv \mathrm{Cyl}(\mathcal{B}\mathcal{C}, *, \ell)$$

and

$$\mathrm{SN}_{\mathcal{C}}(\varSigma, \mathsf{B}) \equiv \mathrm{SN}_{\mathcal{B}\mathcal{C}}(\varSigma, \mathsf{B}).$$

First of all, since, according to Proposition 4.6.1, $\mathrm{Cyl}^{\circ}(\mathcal{C}, I) \simeq \mathrm{End}_{\mathcal{B}\mathcal{C}}(*) = \mathcal{C}$ and $\mathcal{C}$ is idempotent complete, there is a unique monoidal equivalence

$$\Phi_I\colon \mathrm{Cyl}(\mathcal{C}, I) \simeq \mathcal{C} \tag{4.7.3}$$

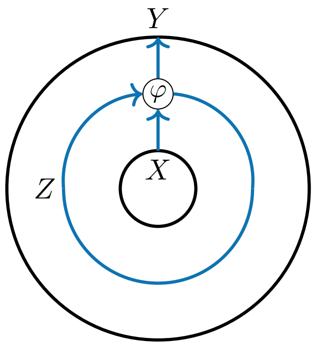
up to a choice of a spitting for each idempotent in $\mathcal{C}$. Moreover, by regarding $-1 \in S^1$ as the distinguished point in S^1, we have a canonical fully faithful functor ([KJ11, Theorem 6.4])

$$\Phi^{\circ}_{S^1}\colon \mathrm{Cyl}^{\circ}(\mathcal{C}, S^1) \to \mathcal{Z}(\mathcal{C})$$

which sends a $\mathcal{C}$-boundary value whose marked points (staring from $-1 \in S^1$ going clockwise for one cycle) are colored with $X_1, \dots, X_n \in \mathcal{C}$ to $L(X_1 \otimes \cdots \otimes X_n) \in \mathcal{Z}(\mathcal{C})$, where

$$\begin{aligned} L\colon \mathcal{C} &\to \mathcal{Z}(\mathcal{C}) \\ X &\mapsto \Big(\int^{C \in \mathcal{C}} C^{\vee} \otimes X \otimes C = \bigoplus_{i \in \mathcal{I}(\mathcal{C})} i^{\vee} \otimes X \otimes i, \gamma_{LX}\Big) \end{aligned}$$

is the two-sided adjoint (the adjoint is two-sided because $\mathcal{C}$ is *unimodular*, see [Shi17]) of the forgetful functor $U\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, and by γ_{LX} we denote the canonical half-braiding obtained from the universal action of the central monad of $\mathcal{C}$. See [KJ11, Theorem 8.2] or for more general unimodular finite tensor categories [Shi17, Theorem 4.10]. To describe the action of $\Phi^{\circ}_{S^1}$ on morphisms, it suffices to consider $\varphi \in \mathcal{C}(X, Z^{\vee} \otimes Y \otimes Z)$ and set

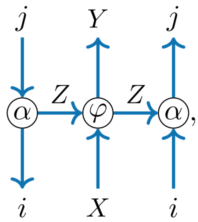
$\widetilde{\varphi} :=$  $\in \mathrm{Hom}_{\mathrm{Cyl}^{\circ}(\mathcal{C},S^1)}(\mathsf{b}_X, \mathsf{b}_Y).$

The morphism $\Phi^{\circ}_{S^1}(\widetilde{\varphi})\colon L(X) \to L(Y)$ in $\mathcal{Z}(\mathcal{C})$ is then defined by the dinatural family

$$C_0^{\vee} \otimes X \otimes C_0 \xrightarrow{\mathrm{id}\otimes\varphi\otimes\mathrm{id}} C_0^{\vee} \otimes Z^{\vee} \otimes Y \otimes Z \otimes C_0 \xrightarrow{\imath_{Y;Z\otimes C_0}} L(Y),$$

where $\imath_{Y;-}$ is the dinatural structure morphism of the coend $L(Y)$. Using that $\mathcal{C}$ is finitely semisimple, we have

$$\Phi^{\circ}_{S^1}(\widetilde{\varphi}) = \sum_{i,j\in\mathcal{I}(\mathcal{C})} d_j \quad \text{(4.7.4)}$$



where $d_j \equiv \dim(j)$ is the quantum dimension of $j \in \mathcal{C}$ and we have used the convention that the appearance of a pair of circular coupons labeled by the same Greek letter indicates the summation over a dual pair of bases of the relevant hom-spaces with respect to the non-degenerate pairing given by

$$\mathcal{C}(U^{\vee}, V^{\vee}) \otimes_{\Bbbk} \mathcal{C}(U, V) \xrightarrow{(\mathrm{ev}^{\mathrm{r}}_V)_* \circ (\mathrm{coev}^{\mathrm{l}}_U)^*} \mathcal{C}(\mathbb{1}, \mathbb{1}) = \Bbbk.$$

Since $\mathcal{Z}(\mathcal{C})$ is idempotent complete, after choosing a splitting for each idempotent in $\mathcal{Z}(\mathcal{C})$ there is a unique extension of $\Phi^{\circ}_{S^1}$ to a functor

$$\Phi_{S^1}\colon \mathrm{Cyl}(\mathcal{C}, S^1) = \mathrm{Kar}(\mathrm{Cyl}^{\circ}(\mathcal{C}, S^1)) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C}) \tag{4.7.5}$$

which is moreover an equivalence[4] ([KJ11, Theorem 6.4]). The functor (4.7.5) sends a thickened $\mathcal{C}$-boundary value $\mathsf{B} \in \mathrm{Cyl}(\mathcal{C}, S^1)$ to the image $\mathrm{im}(\Phi^{\circ}_{S^1}(\mathsf{B})) \in \mathcal{Z}(\mathcal{C})$ given by the chosen splitting with $\Phi^{\circ}_{S^1}(\mathsf{B}) \in \mathrm{End}_{\mathcal{Z}(\mathcal{C})}(\Phi^{\circ}_{S^1}(\mathsf{B}^{\circ}))$. In fact, the equivalence (4.7.5)

[4] Note that for this result, the semisimplicity of $\mathcal{C}$ is important. If $\mathcal{C}$ is a non-semisimple spherical *finite* category, the functor Φ_{S^1} is still fully faithful but not essentially surjective.

can be extended for an arbitrary closed oriented 1-manifold $\dot{\ell}$ with a marked point in each connected component thereof to a canonical equivalence

$$\Phi_{\dot{\ell}} \colon \mathrm{Cyl}(\mathcal{C}, \dot{\ell}) \xrightarrow{\simeq} \prod_{i \in \pi_0(\dot{\ell})} \mathcal{Z}(\mathcal{C}).$$

Now consider an orientation preserving embedding $f \colon I \hookrightarrow S^1$ whose image does not contain the distinguished point $-1 \in S^1$. In view of the definition of the equivalences (4.7.3) and (4.7.5), we have the following commutative diagram in $\mathrm{Cat}_{\Bbbk}$:

$$\begin{tikzcd} \mathrm{Cyl}(\mathcal{C}, I) \arrow[r, "\mathrm{Cyl}(\mathcal{C}, f)"] \arrow[d, "\Phi_I"'] & \mathrm{Cyl}(\mathcal{C}, S^1) \arrow[d, "\Phi_{S^1}"] \\ \mathcal{C} \arrow[r, "L"'] & \mathcal{Z}(\mathcal{C}) \end{tikzcd} \tag{4.7.6}$$

This can be better understood given the result from [KJT21] which states that the assignment of the Karoubified cylinder categories for a spherical fusion category to compact oriented 1-manifolds satisfies *excision* hence provides 1-dimensional *factorization homologies*. Calculated via excision, the functor $\mathrm{Cyl}(\mathcal{C}, f)$ is transported via the canonical equivalences to

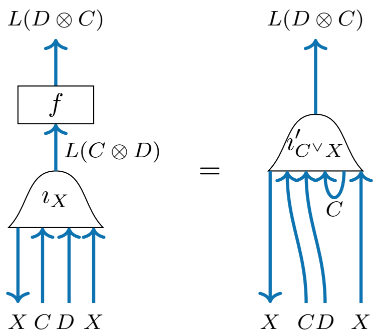
$$\mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C} \boxtimes_{\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}} \mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C}), \tag{4.7.7}$$

where the right most equivalence uses the pivotal structure of $\mathcal{C}$. Note that morally speaking, $\mathrm{Cyl}(\mathcal{C}, S^1)$ gives us the *cocenter* or equivalently, the *twisted* center $\mathcal{Z}^{\mathrm{D}}(\mathcal{C})$, i.e. the Drinfeld center twisted by the double dual functor, which does not admit a monoidal structure *a priori* and can be realized by the relative Deligne product $\mathcal{C} \boxtimes_{\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}} \mathcal{C}$ according to [FSS17, Proposition 2.18 (i)]. Here the composite $\mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C} \boxtimes_{\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}} \mathcal{C}$ is the left adjoint L^{D} of the forgetful functor $U^{\mathrm{D}} \colon \mathcal{Z}^{\mathrm{D}}(\mathcal{C}) = \mathcal{C} \boxtimes_{\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}} \mathcal{C} \to \mathcal{C}$, which leads to $L \colon \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ by postcomposing with the equivalence $\mathcal{C} \boxtimes_{\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}} \mathcal{C} \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C})$. In the same spirit, embedding two copies of the interval I into S^1 gives rise to the functor $L \circ \otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. Geometrically, one sees that

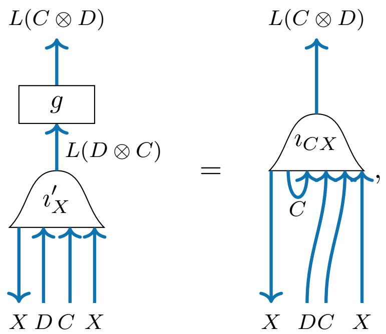
$$L(C \otimes D) \cong L(D \otimes C) \tag{4.7.8}$$

for any pair of objects $C, D \in \mathcal{C}$, where a *non-canonical* isomorphism is provided by an isotopy between the two corresponding embedding of the type $I \sqcup I \hookrightarrow S^1$. One can also construct an isomorphism explicitly as follows:

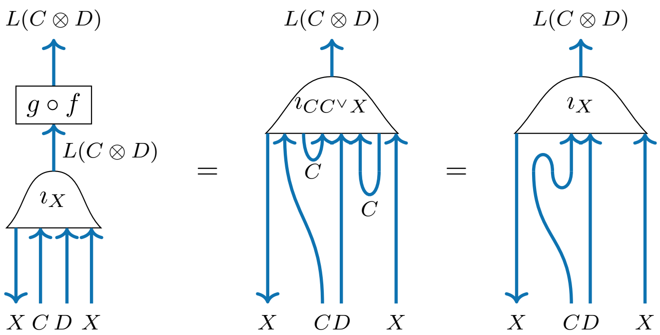
Consider the morphisms $f \colon L(C \otimes D) \rightleftharpoons L(D \otimes C) \colon g$ given by



and



where the jellyfishes stand for the dinatural structure morphisms of the coends. The composite $g \circ f$ satisfies



where the second equality follows from dinaturality. Together with (one of) the yanking equation, this shows that $g \circ f = \mathrm{id}_{L(C\otimes D)}$. Similarly one sees that g is also the right inverse of f. Moreover, it is readily checked that all morphisms involved are compatible with the half-braidings, so that they are actually morphisms in $\mathcal{Z}(\mathcal{C})$.

We conclude this section by recalling the following important theorem obtained in [KJ11] and [Goo18]:

Theorem 4.7.1. *Let $\mathcal{C}$ be a spherical fusion category.*

1. *The Karoubified string-net construction* $\mathrm{SN}_\mathcal{C}$ *canonically extends to a 3-2-1 topological field theory that is canonically equivalent to* $\mathrm{TV}_\mathcal{C}$*, i.e. the once-extended Turaev-Viro state sum TFT for* $\mathcal{C}$*.*

2. *Let Σ be a compact oriented surface whose boundary $\partial\Sigma$ has a marked point in each of its connected components*[5] *and* $\mathsf{B} \in \mathrm{Cyl}(\mathcal{C}, \partial\Sigma)$ *a thickened boundary value. Denote by*
$$\Phi_{\partial\Sigma} \colon \mathrm{Cyl}(\mathcal{C}, \partial\Sigma) \xrightarrow{\simeq} \prod_{i\in\pi_0(\partial\Sigma)} \mathcal{Z}(\mathcal{C})$$
the canonical equivalence and $\prod_{i\in\pi_0(\partial\Sigma)} Y_i = \Phi_{\partial\Sigma}(\mathsf{B}) \in \prod_{i\in\pi_0(\partial\Sigma)} \mathcal{Z}(\mathcal{C})$*. Then there is an isomorphism*
$$\mathrm{SN}_\mathcal{C}(\Sigma, \mathsf{B}) \cong \mathcal{Z}(\mathcal{C})(\mathbb{1}, \bigotimes_{i\in\pi_0(\partial\Sigma)} Y_i \otimes K^{\otimes g}) \tag{4.7.9}$$

[5] For instance, a marking on Σ (in the sense of [BKJ00]) provides such marked points.

of $\mathrm{Map}(\Sigma)$*-representations, where* $K := \int^{Z \in \mathcal{Z}(\mathcal{C})} Z^\vee \otimes Z \in \mathcal{Z}(\mathcal{C})$ *is the canonical coend for* $\mathcal{Z}(\mathcal{C})$ *and the* $\mathrm{Map}(\Sigma)$*-action on the RHS is given by Lyubashenko in [Lyu96], which is fixed upon a choice of a marking on* Σ *that connects the marked points on* $\partial\Sigma$*.*

4.8 Factorization

Consider a bordism $\Sigma\colon \alpha \nrightarrow \beta$ given by an underlying compact oriented surface Σ and in-going parametrization $\phi_-\colon \bar{\alpha} \hookrightarrow \partial\Sigma$ as well as out-going parametrization $\phi_+\colon \beta \hookrightarrow \partial\Sigma$, where $\alpha, \beta \in \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ are disjoint unions of intervals and circles. By the functoriality of $\mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, -)$ and $\mathrm{Cyl}(\mathcal{B}, *_{\mathcal{B}}, -)$ under the embeddings of 1-manifolds, we obtain a $\Bbbk$-linear profunctor

$$\mathcal{S}\mathrm{N}^\circ_{\mathcal{B}}(\Sigma; -, \sim)\colon \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \alpha) \nrightarrow \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \beta)$$

as the composite

$$\mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \alpha)^{\mathrm{op}} \times \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \beta) \xrightarrow{(\phi_-)_* \sqcup (\phi_+)_*} \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \partial\Sigma) \xrightarrow{\mathrm{SN}^\circ_{\mathcal{B}}(\Sigma, -)} \mathrm{Vect}_\Bbbk$$

where we have abbreviated $\phi_* \equiv \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \phi)$, and likewise its Karoubified version

$$\mathcal{S}\mathrm{N}_{\mathcal{B}}(\Sigma; -, \sim)\colon \mathrm{Cyl}(\mathcal{B}, *_{\mathcal{B}}, \alpha) \nrightarrow \mathrm{Cyl}(\mathcal{B}, *_{\mathcal{B}}, \beta).$$

We are now ready to state:

Theorem 4.8.1. *Let* $\Sigma\colon \alpha \sqcup \beta \nrightarrow \beta \sqcup \gamma$ *be a bordism, where* $\alpha, \beta, \gamma \in \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ *are disjoint unions of intervals and circles. Then the dinatural family*

$$\{s^{\Sigma}_{-,\mathsf{b}_0,\sim}\colon \mathcal{S}\mathrm{N}^\circ_{\mathcal{B}}(\Sigma; -, \mathsf{b}_0, \mathsf{b}_0, \sim) \Rightarrow \mathcal{S}\mathrm{N}^\circ_{\mathcal{B}}(\cup_\beta \Sigma; -, \sim)\}_{\mathsf{b}_0 \in \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \beta)} \tag{4.8.1}$$

whose components are given by the sewing of string-nets, exhibits $\mathcal{S}\mathrm{N}^\circ_{\mathcal{B}}(\cup_\beta \Sigma; -, \sim)$ *as the coend*

$$\int^{\mathsf{b} \in \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \beta)} \mathcal{S}\mathrm{N}^\circ_{\mathcal{B}}(\Sigma; -, \mathsf{b}, \mathsf{b}, \sim)\colon \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \alpha) \nrightarrow \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \gamma).$$

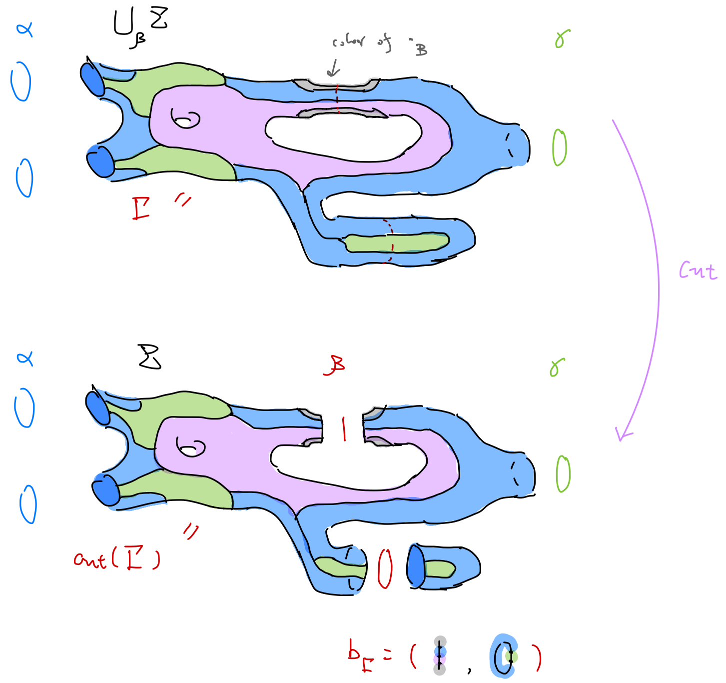
Proof. We need to show that the dinatural family (4.8.1) is universal. To this end, consider a dinatural family

$$\{g_{\mathsf{b}_0}\colon \mathcal{S}\mathrm{N}^\circ_{\mathcal{B}}(\Sigma; \mathsf{a}_0, \mathsf{b}_0, \mathsf{b}_0, \mathsf{c}_0) \to V\}_{\mathsf{b}_0 \in \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \beta)}$$

for arbitrary boundary data $\mathsf{a}_0 \in \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \alpha)$ and $\mathsf{c}_0 \in \mathrm{Cyl}^\circ(\mathcal{B}, *_{\mathcal{B}}, \gamma)$ and $\Bbbk$-vector space $V \in \mathrm{Vect}_\Bbbk$. Define a linear map

$$\begin{aligned} g\colon \mathcal{S}\mathrm{N}^\circ_{\mathcal{B}}(\cup_\beta \Sigma; \mathsf{a}_0, \mathsf{c}_0) &\to V \\ [\Gamma] &\mapsto [\mathrm{cut}(\Gamma)] \mapsto g_{\mathsf{b}_\Gamma}([\mathrm{cut}(\Gamma)]), \end{aligned} \tag{4.8.2}$$

where $\mathrm{cut}(\Gamma)$ is the fully colored graph on Σ obtained by cutting the representative graph Γ along β and $\mathsf{b}_\Gamma \in \mathrm{Cyl}^\circ(\mathcal{B}, *_\mathcal{B}, \beta)$ is the boundary datum obtained from the cut. The scenario can be presented schematically by:



The linear map (4.8.2) is well-defined because all isotopies and the generating local relations provided by the unframed graphical calculus are contained within embedded discs hence a different choice of representative for the string-net $[\Gamma] \in \mathcal{S}\mathrm{N}^\circ_\mathcal{B}(\cup_\beta \Sigma; \mathsf{a}_0, \mathsf{c}_0)$ only differs by an action of some morphism in the cylinder category $\mathrm{Cyl}^\circ(\mathcal{B}, *_\mathcal{B}, \beta)$, as well as the premise that the family $\{g_{\mathsf{b}_0}\}_{\mathsf{b}_0 \in \mathrm{Cyl}^\circ(\mathcal{B}, *_\mathcal{B}, \beta)}$ is dinatural. The uniqueness of the map is by design, hence the dinatural family at each component does exhibit a coend at the level of vector spaces and linear maps. It is standard that the componentwisely defined coends are automatically enhanced to a coend at the level of linear functors and linear natural transformations. □

There is an analogous statement for the Karoubified string-net. In order to show that, we make the following observation: Let $\mathcal{A}$ be a category, recall the notion of the *twisted arrow category* $\mathrm{Tw}(\mathcal{A})$: it is the category with objects given by the morphisms in $\mathcal{A}$, and a morphism in $\mathrm{Tw}(\mathcal{A})$ from $f\colon a \to b$ to $f'\colon a' \to b'$ is given by a pair $(g, h) \in \mathcal{A}^{\mathrm{op}}(a, a') \times \mathcal{A}(b, b')$ such that the square

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[d, "h"] \\ a' \arrow[u, "g"] \arrow[r, "f'"'] & b' \end{tikzcd}$$

commutes. The twisted arrow category $\mathrm{Tw}(\mathcal{A})$ comes with a canonical projection functor

$$\pi_\mathcal{A} \colon \mathrm{Tw}(\mathcal{A}) \to \mathcal{A}^{\mathrm{op}} \times \mathcal{A}$$

that sends an object $f\colon a \to b$ to $(a,b) \in \mathcal{A}^{\mathrm{op}} \times \mathcal{A}$ and keeps the morphisms as they are. The relevance of this construction to us is that given a functor

$$F\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Vect}_{\Bbbk},$$

we can identify its coend with a *colimit* over the twisted arrow category:

$$\int^{a\in\mathcal{A}} F(a,a) = \operatorname{colim}(\mathrm{Tw}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}}}} \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \xrightarrow{F} \mathrm{Vect}_{\Bbbk}), \tag{4.8.3}$$

see e.g. [Lor21, Section 1.2].

Lemma 4.8.2. *Let $\mathcal{A}$ be a category and $F\colon \mathrm{Kar}(\mathcal{A})^{\mathrm{op}} \times \mathrm{Kar}(\mathcal{A}) \to \mathrm{Vect}_{\Bbbk}$ a functor whose coend exists. Then*

$$\int^{a\in\mathcal{A}} F|_{\mathcal{A}}(a,a) = \int^{A\in\mathrm{Kar}(\mathcal{A})} F(A,A),$$

where $F|_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Kar}(\mathcal{A})^{\mathrm{op}} \times \mathrm{Kar}(\mathcal{A}) \xrightarrow{F} \mathrm{Vect}_{\Bbbk}$ is the restriction of F along the canonical embedding.

Proof. Due to the commutativity of the square

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \arrow[r, "G"] \arrow[d, "\pi^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}}}"'] & \mathrm{Tw}(\mathrm{Kar}(\mathcal{A})^{\mathrm{op}})^{\mathrm{op}} \arrow[d, "\pi^{\mathrm{op}}_{\mathrm{Kar}(\mathcal{A})^{\mathrm{op}}}"] \\
\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \arrow[r] & \mathrm{Kar}(\mathcal{A})^{\mathrm{op}} \times \mathrm{Kar}(\mathcal{A})
\end{tikzcd}$$

it suffices to show that the functor

$$G\colon \mathrm{Tw}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Tw}(\mathrm{Kar}(\mathcal{A})^{\mathrm{op}})^{\mathrm{op}}$$
$$(a \xleftarrow{f} b) \mapsto (\mathrm{id}_a \xleftarrow{f} \mathrm{id}_b)$$

is cofinal, because then we have

$$\begin{aligned}
\int^{a\in\mathcal{A}} F|_{\mathcal{A}}(a,a) &= \operatorname{colim}(\mathrm{Tw}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}}}} \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Kar}(\mathcal{A})^{\mathrm{op}} \times \mathrm{Kar}(\mathcal{A}) \xrightarrow{F} \mathrm{Vect}_{\Bbbk}) \\
&= \operatorname{colim}(\mathrm{Tw}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{G} \mathrm{Tw}(\mathrm{Kar}(\mathcal{A})^{\mathrm{op}})^{\mathrm{op}} \\
&\qquad \xrightarrow{\pi^{\mathrm{op}}_{\mathrm{Kar}(\mathcal{A})^{\mathrm{op}}}} \mathrm{Kar}(\mathcal{A})^{\mathrm{op}} \times \mathrm{Kar}(\mathcal{A}) \xrightarrow{F} \mathrm{Vect}_{\Bbbk}) \\
&= \operatorname{colim}(\mathrm{Tw}(\mathrm{Kar}(\mathcal{A})^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}_{\mathrm{Kar}(\mathcal{A})^{\mathrm{op}}}} \mathrm{Kar}(\mathcal{A})^{\mathrm{op}} \times \mathrm{Kar}(\mathcal{A}) \xrightarrow{F} \mathrm{Vect}_{\Bbbk}) \\
&= \int^{A\in\mathrm{Kar}(\mathcal{A})} F(A,A).
\end{aligned}$$

Recall that $G\colon \mathrm{Tw}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Tw}(\mathrm{Kar}(\mathcal{A})^{\mathrm{op}})^{\mathrm{op}}$ being cofinal is equivalent to that for every object $(p \xleftarrow{g} q) \in \mathrm{Tw}(\mathrm{Kar}(\mathcal{A})^{\mathrm{op}})^{\mathrm{op}}$, where $p \in \mathrm{End}_{\mathcal{A}}(a)$ and $q \in \mathrm{End}_{\mathcal{A}}(b)$ are idempotents, the comma category $g \downarrow G$ is connected, i.e. it is non-empty and for each pair of objects therein, there exists a zigzag connecting them. First consider the square

$$\begin{tikzcd}
p & q \arrow[l, "g"'] \arrow[d, "q"] \\
\mathrm{id}_a \arrow[u, "p"] & \mathrm{id}_b \arrow[l, "g"]
\end{tikzcd}$$

it is commutative due to the defining condition for g to be a morphism of the type $q \to p$. This commutative square provides us an object $(g \to Gg) \in g \downarrow G$, therefore $g \downarrow G$ is non-empty. Then assume that we have a pair of objects $g \to Gf$ and $g \to Gh$ in the comma category $g \downarrow G$ given by the commutative squares

$$\begin{tikzcd}
p & q \arrow[l, "g"'] \arrow[d, "s"] \\
\mathrm{id}_c \arrow[u, "r"] & \mathrm{id}_d \arrow[l, "f"]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
p & q \arrow[l, "g"'] \arrow[d, "s'"] \\
\mathrm{id}_{c'} \arrow[u, "r'"] & \mathrm{id}_{d'} \arrow[l, "h"]
\end{tikzcd}$$

respectively. Observe that the following diagram

$$\begin{tikzcd}
& \mathrm{id}_c \arrow[ddl, "r"'] \arrow[dr, "r"] & & \mathrm{id}_d \arrow[ll, "f"'] & \\
& & \mathrm{id}_a \arrow[dll, "p"'] & & \mathrm{id}_b \arrow[ul, "s"'] \arrow[ll, "g"'] \arrow[ddl, "s'"] \\
p & & q \arrow[ll, "g"] \arrow[uur, "s"] \arrow[urr, "q"] \arrow[dr, "s'"] & & \\
& \mathrm{id}_{c'} \arrow[ul, "r'"] \arrow[uur, "r'"] & & \mathrm{id}_{d'} \arrow[ll, "h"] &
\end{tikzcd}$$

commutes and hence the two squares colored with violet give rise to a span

$$(g \to Gf) \longleftarrow (g \to Gg) \longrightarrow (g \to Gh)$$

in the comma category $g \downarrow G$. Therefore, $g \downarrow G$ is connected for every $g \in \mathrm{Tw}(\mathrm{Kar}(\mathcal{A})^{\mathrm{op}})^{\mathrm{op}}$ hence $G \colon \mathrm{Tw}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Tw}(\mathrm{Kar}(\mathcal{A})^{\mathrm{op}})^{\mathrm{op}}$ is cofinal. $\square$

Corollary 4.8.3. *Let $\Sigma \colon \alpha \sqcup \beta \nrightarrow \beta \sqcup \gamma$ be a bordism, where $\alpha, \beta, \gamma \in \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ are disjoint unions of intervals and circles. Then the dinatural family*

$$\{\widehat{s}^{\Sigma}_{-,\mathsf{b}_0,\sim} \colon \mathcal{S}\mathrm{N}_{\mathcal{B}}(\Sigma; -, \mathsf{B}_0, \mathsf{B}_0, \sim) \Rightarrow \mathcal{S}\mathrm{N}_{\mathcal{B}}(\cup_\beta \Sigma; -, \sim)\}_{\mathsf{B}_0 \in \mathrm{Cyl}(\mathcal{B}, *_{\mathcal{B}}, \beta)} \tag{4.8.4}$$

whose components are given by the sewing of string-nets, exhibits $\mathcal{S}\mathrm{N}_{\mathcal{B}}(\cup_\beta \Sigma; -, \sim)$ as the coend

$$\int^{\mathsf{B} \in \mathrm{Cyl}(\mathcal{B}, *_{\mathcal{B}}, \beta)} \mathcal{S}\mathrm{N}_{\mathcal{B}}(\Sigma; -, \mathsf{B}, \mathsf{B}, \sim) \colon \mathrm{Cyl}(\mathcal{B}, *_{\mathcal{B}}, \alpha) \nrightarrow \mathrm{Cyl}(\mathcal{B}, *_{\mathcal{B}}, \gamma).$$

Proof. Applying Theorem 4.8.1 and Lemma 4.8.2, we have

$$\mathcal{S}\mathrm{N}_{\mathcal{B}}(\cup_\beta \Sigma; , \mathsf{A}_0, \mathsf{C}_0) = \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\cup_\beta \Sigma; , \mathsf{A}^{\circ}_0, \mathsf{C}^{\circ}_0)^{(\mathsf{A}_0, \mathsf{C}_0)}$$

$$= \int^{\mathsf{b}\in \mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},\beta)} \mathcal{S}\mathrm{N}^\circ_\mathcal{B}(\Sigma; \mathsf{A}^\circ_0, \mathsf{b}, \mathsf{b}, \mathsf{C}^\circ_0)^{(\mathsf{A}_0,\mathsf{C}_0)}$$
$$= \int^{\mathsf{b}\in \mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},\beta)} \mathcal{S}\mathrm{N}_\mathcal{B}(\Sigma; \mathsf{A}_0, \mathrm{id}_\mathsf{b}, \mathrm{id}_\mathsf{b}, \mathsf{C}_0)$$
$$= \int^{\mathsf{B}\in \mathrm{Cyl}(\mathcal{B},*_\mathcal{B},\beta)} \mathcal{S}\mathrm{N}_\mathcal{B}(\Sigma; \mathsf{A}_0, \mathsf{B}, \mathsf{B}, \mathsf{C}_0)$$

for every $\mathsf{A}_0 \in \mathrm{Cyl}(\mathcal{B}, *_\mathcal{B}, \alpha)$ and $\mathsf{C}_0 \in \mathrm{Cyl}(\mathcal{B}, *_\mathcal{B}, \gamma)$. □

4.9 Open-closed modular functors from string-net models

An implication of Theorem 4.8.1 is that for any composable pair of bordisms $\Sigma\colon \alpha \nrightarrow \beta$ and $\Sigma'\colon \beta \nrightarrow \gamma$, the dinatural family of sewing maps exhibits a coend

$$\mathcal{S}\mathrm{N}^\circ_\mathcal{B}(\Sigma \cup_\beta \Sigma'; -, \sim) = \int^{\mathsf{b}\in \mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},\beta)} \mathcal{S}\mathrm{N}^\circ_\mathcal{B}(\Sigma \sqcup \Sigma'; -, \mathsf{b}, \mathsf{b}, \sim)$$
$$= \int^{\mathsf{b}\in \mathrm{Cyl}^\circ(\mathcal{B},*_\mathcal{B},\beta)} \mathcal{S}\mathrm{N}^\circ_\mathcal{B}(\Sigma; -, \mathsf{b}) \otimes_\Bbbk \mathcal{S}\mathrm{N}^\circ_\mathcal{B}(\Sigma'; \mathsf{b}, \sim).$$

Therefore, the assignment

$$\alpha \mapsto \mathrm{Cyl}^\circ(\mathcal{B}, *_\mathcal{B}, \alpha)$$

and

$$\Sigma \mapsto \mathcal{S}\mathrm{N}^\circ_\mathcal{B}(\Sigma; -, \sim)$$

extends to a symmetric monoidal pseudofunctor

$$\mathcal{S}\mathrm{N}^\circ_\mathcal{B}\colon \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \to \mathcal{P}\mathrm{rof}_\Bbbk,$$

i.e. an open-closed modular functor. Likewise, Corollary 4.8.3 implies that the Karoubified string-net spaces give rise to an open-closed modular functor

$$\mathcal{S}\mathrm{N}_\mathcal{B}\colon \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \to \mathcal{P}\mathrm{rof}_\Bbbk$$

as well.

Now, let $\mathcal{C}$ be the modular fusion category of chiral data for a rational conformal theory. The results listed in Section 4.7 assert that $\mathcal{S}\mathrm{N}_\mathcal{C}$ satisfy all the conditions from Problem 2.6.2 hence provides an open-closed modular functor that models the conformal blocks. In other words, by defining

$$\mathrm{Bl}_\mathcal{C}(\alpha) \coloneqq \mathrm{Cyl}(\mathcal{C}, \alpha)$$

for every $\alpha \in \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$, and

$$\mathrm{Bl}_\mathcal{C}(\Sigma; -, \sim) \coloneqq \mathcal{S}\mathrm{N}_\mathcal{C}(\Sigma; -, \sim)$$

for every bordism Σ, we obtain a solution to Problem 2.6.2.

Remark 4.9.1. As mentioned in Section 4.7, we have $\mathrm{Bl}_\mathcal{C}(I) = \mathrm{Cyl}(\mathcal{C}, I) \simeq \mathcal{C}$ and $\mathrm{Bl}_\mathcal{C}(S^1) = \mathrm{Cyl}(\mathcal{C}, S^1) \simeq \mathcal{Z}(\mathcal{C})$ which are finitely semisimple. Therefore for any bordism $\Sigma\colon \alpha \nrightarrow \beta$, $\mathrm{Bl}_\mathcal{C}(\Sigma; -, \sim) = \mathcal{S}\mathrm{N}_\mathcal{C}(\Sigma; -, \sim)$ is exact in each variable and can be replaced by an exact functor $\widehat{\mathrm{Bl}}_\mathcal{C}(\Sigma; - \boxtimes \sim)\colon \mathrm{Bl}_\mathcal{C}(\alpha)^{\mathrm{op}} \boxtimes \mathrm{Bl}_\mathcal{C}(\beta) \to \mathrm{Vect}_\Bbbk$. Consequently, for a modular fusion category $\mathcal{C}$, the open-closed modular functor $\mathrm{Bl}_{\mathcal{C}}$ factors through the forgetful functor

$$\mathcal{P}\mathrm{rof}_{\mathbb{k}}^{\mathcal{L}\mathrm{ex}} \to \mathcal{P}\mathrm{rof}_{\mathbb{k}},$$

where $\mathcal{P}\mathrm{rof}_{\mathbb{k}}^{\mathcal{L}\mathrm{ex}}$ is the symmetric monoidal bicategory of finite $\mathbb{k}$-linear abelian categories, left exact profunctors and natural transformations, the horizontal composition of which is given by *left exact* coends (in the sense of [FS17a]) and the monoidal product given by the Deligne product.

Finally, note that by taking $(\mathcal{F}r(\mathcal{C}), \mathbb{1})$ as the decorating pointed pivotal bicategory, we obtain an open-closed modular functor

$$\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{F}r(\mathcal{C})} \colon \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \to \mathcal{P}\mathrm{rof}_{\mathbb{k}}$$

that turns out to be also relevant to the RCFT (see Chapters 8 and 9).

5 String-net construction of correlators

In this chapter, we use the string-nets to construct the field maps and a consistent system of correlators requested in Problem 2.6.3 and Problem 2.6.7. The construction involves calculating in the *spherical fusion* category $\mathcal{C}$ of chiral data (recall that a modular fusion category is in particular spherical) via string diagrams, so let us first introduce the relevant notations and results.

Recall that a pivotal fusion category $\mathcal{C}$ is *spherical* if for every object $X \in \mathcal{C}$, the left and right dimensions are equal:

$$\dim_{\mathrm{l}}(X) = \dim_{\mathrm{r}}(X) = \dim(X).$$

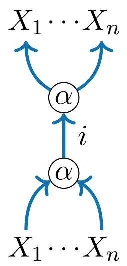
The *global dimension* of a spherical fusion category $\mathcal{C}$ is the number

$$D_{\mathcal{C}}^2 \coloneqq \sum_{i \in \mathcal{I}(\mathcal{C})} d_i^2$$

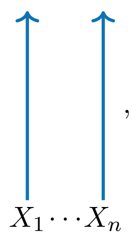
(no choice of square root implied); this number is non-zero [ENO05, Theorem 2.3].

Let $X_1, \ldots, X_n \in \mathcal{C}$ be objects in a spherical fusion category. We have the *completeness relation*

$$\bigoplus_{i \in \mathcal{I}(\mathcal{C})} d_i$$



$=$



, (5.0.1)

where we have used the summation convention mentioned after (4.7.4).

There is a virtual object of particular interest that is called the *canonical color* (or *Kirby color*, or *surgery color*). It is denoted by an undirected purple line and defined as the morphism

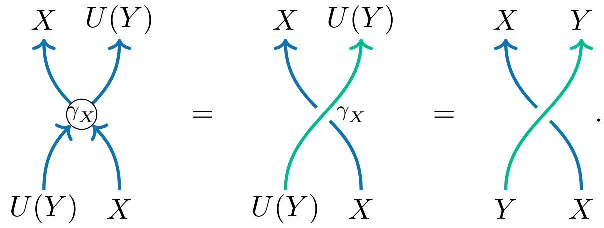
$$\coloneqq \sum_{i \in \mathcal{I}(\mathcal{C})} \frac{d_i}{D_{\mathcal{C}}^2}$$

$$\in \mathrm{End}_{\mathcal{C}}\Big(\bigoplus_{i \in \mathcal{I}(\mathcal{C})} i\Big). \tag{5.0.2}$$

Recall that an object in the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is given by a pair $Y = (U(Y), \gamma)$, where $U(Y) \in \mathcal{C}$ is its underlying object in $\mathcal{C}$ and the natural isomorphism

$$\gamma \colon U(Y) \otimes - \Rightarrow - \otimes U(Y)$$

is the half-braiding of Y which has to obey a hexagon diagram. We draw the component $\gamma_X : U(Y) \otimes X \to X \otimes U(Y)$ of the half-braiding γ at $X \in \mathcal{C}$ as:



(5.0.3)

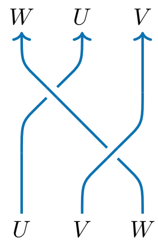
Here in the last picture we slightly abused the notation by using the label $Y \in \mathcal{Z}(\mathcal{C})$ in the description of a morphism in $\mathcal{C}$, the reason behind which is that by doing so, the morphism indicated by the over-crossing of the *green* line cannot be mixed up with a braiding in $\mathcal{C}$ (if the spherical fusion category $\mathcal{C}$ is equipped with one, e.g. when $\mathcal{C}$ is the modular fusion category of the chiral data) and thus be omitted. For clarity, in addition we draw the strands labeled by objects of $\mathcal{C}$ and strands labeled by objects of $\mathcal{Z}(\mathcal{C})$ in two different color, i.e. in dark blue and green, respectively.

Lastly, recall that for a *modular fusion category* $\mathcal{C}$ (and more generally for a *non-degenerate braided finite category*, see [Shi19]), the functor

$$\begin{aligned} \Xi_{\mathcal{C}} : \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} &\to \mathcal{Z}(\mathcal{C}) \\ U \boxtimes V &\mapsto (U \otimes V, \gamma_{\Xi_{\mathcal{C}}(U \boxtimes V)}) \end{aligned} \tag{5.0.4}$$

is a braided equivalence, where the component of the half-braiding $\gamma_{\Xi_{\mathcal{C}}(U\boxtimes V)}$ at an object $W \in \mathcal{C}$ is given by

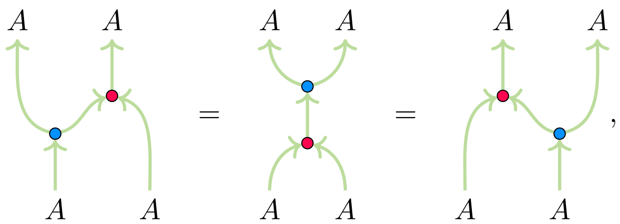
$\gamma_{\Xi_{\mathcal{C}}(U\boxtimes V);W} :=$



. (5.0.5)

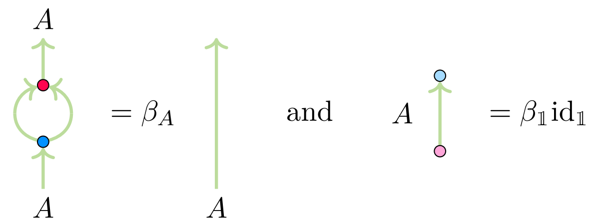
5.1 Frobenius algebras and their modules

The notion of a Frobenius algebra is of great important to 2-dimensional topological and conformal field theories. Let $\mathcal{C}$ be a monoidal category, an object $A \in \mathcal{C}$ equipped with the structure of a unital algebra and that of a counital coalgebra is a *Frobenius algebra* if the following relation holds:



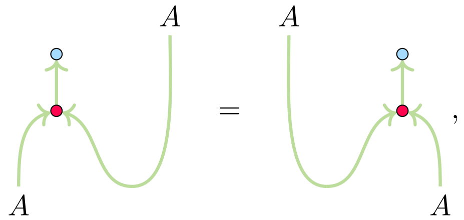
(5.1.1)

where the trivalent red and blue coupons stand for the multiplication $\mu\colon A \otimes A \to A$ and comultiplication $\Delta\colon A \to A \otimes A$, respectively. A Frobenius algebra $A \in \mathcal{C}$ is *special* if



(5.1.2)

for non-zero $\beta_A, \beta_{\mathbb{1}} \in \Bbbk$, where the pink and the light blue coupons stand for the unit $\eta\colon \mathbb{1} \to A$ and the counit $\varepsilon\colon A \to \mathbb{1}$ of A, respectively. If $\mathcal{C}$ has, in addition, a pivotal structure, then a Frobenius algebra $A \in \mathcal{C}$ is called *symmetric* if we have



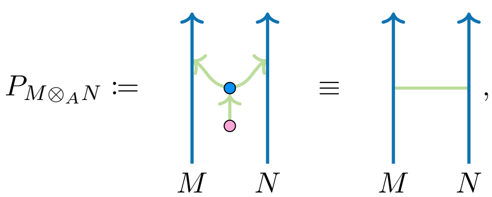
(5.1.3)

where the pivotal structure is left implicit. Note that for a special symmetric Frobenius algebra $A \in \mathcal{C}$, the constants β_A and $\beta_{\mathbb{1}}$ obey

$$\beta_A \beta_{\mathbb{1}} = \dim_{\mathrm{l}}(A) = \dim_{\mathrm{r}}(A) \neq 0.$$

Note that $\beta_A, \beta_{\mathbb{1}} \in \Bbbk$ are required to be non-zero, therefore their product is non-zero. In this thesis, we will always assume (without loss of generality) that the comultiplication of a special symmetric Frobenius algebra $A \in \mathcal{C}$ is normalized such that $\beta_A = 1$ and $\beta_{\mathbb{1}} = \dim_{\mathrm{l}}(A) = \dim_{\mathrm{r}}(A)$.

Let M, N be a right module and a left module over a special symmetric Frobenius algebra $A \in \mathcal{C}$. The relative tensor product $M \otimes_A N$ can be realized as the image of the idempotent



(5.1.4)

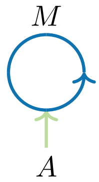
where we have omitted the orientation of the A-line and the occurrence of the unit on the right hand side, using that the dual $A^\vee = {}^\vee A$ is canonically identified with A itself via the unique isomorphism (5.1.3). Note that for any object $X \in \mathcal{C}$, the hom-space $\mathcal{C}(X, M \otimes_A N)$ is canonically isomorphic to the subspace (which is in fact a retract) $\mathcal{C}^{(A)}(X, M \otimes N) \subset \mathcal{C}(X, M \otimes N)$ that consists of those morphisms which are invariant under postcomposing with the idempotent $P_{M \otimes_A N}$. We therefore tacitly identify

$$\mathcal{C}(X, M \otimes_A N) = \mathcal{C}^{(A)}(X, M \otimes N). \tag{5.1.5}$$

Analogously, we identify the space $\mathcal{C}(M \otimes_A N, X)$ with the subspace $\mathcal{C}^{(A)}(M \otimes N, X)$ of $\mathcal{C}(M \otimes N, X)$ that consists of morphisms that are invariant under precomposing with the idempotent $P_{M\otimes_A N}$, and similarly for the analogous morphism spaces involving three or more bimodules.

We will need the following lemmas.

Lemma 5.1.1. *For a simple special symmetric Frobenius algebra A in a spherical fusion category $\mathcal{C}$ and M a right A-module the equality*

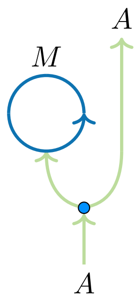


$$= \quad \frac{\dim(M)}{\dim(A)} \tag{5.1.6}$$

holds.

Proof. Consider the endomorphism

$$f_M :=$$



of A, obtained by combining the left hand side of (5.1.6) with the coproduct of A. Since A is symmetric Frobenius, f_M is not just a morphism in $\mathcal{C}$, but also a morphism of A-bimodules with respect to the regular A-bimodule structure on A. Since A is simple, this implies that f_M is a multiple of the identity morphism, i.e. $f_M = \xi_M \mathrm{id}_A$, for some $\xi_M \in \Bbbk$. Postcomposing with the counit then shows that the morphism on the left hand side of (5.1.6) equals $\xi_M \varepsilon$. Further precomposing with the unit gives $\dim(M) = \xi_M \varepsilon \circ \eta$. Since A is special, this implies (5.1.6). □

Denote by d_m for a simple right A-module m the dimension of the object $\dot{m} \in \mathcal{C}$ that underlies $m \in \mathrm{mod}^{\mathcal{C}}\text{-}A$.

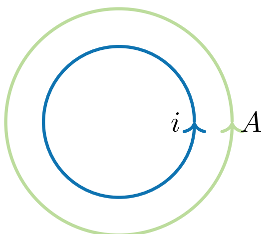
Lemma 5.1.2. *For a special symmetric Frobenius algebra A in a spherical fusion category $\mathcal{C}$ we have*

$$\sum_{m\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} d_m^2 = \dim(A) D_{\mathcal{C}}^2, \tag{5.1.7}$$

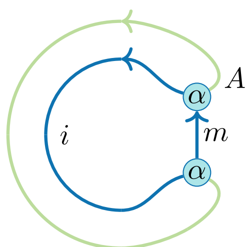
with $D_{\mathcal{C}}^2$ being the global dimension of $\mathcal{C}$.

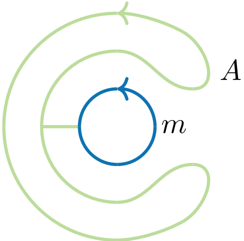
Proof. We have

$$\dim(A) D_{\mathcal{C}}^2 = \sum_{i\in\mathcal{I}(\mathcal{C})} d_i$$

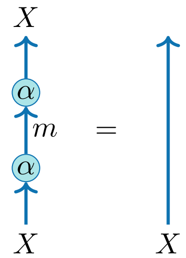


$$= \sum_{\substack{i\in\mathcal{I}(\mathcal{C}) \\ m\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)}} d_i d_m$$

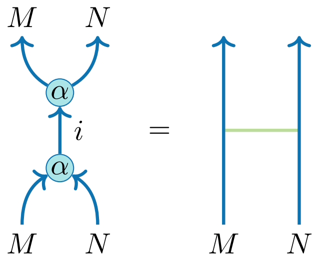


$$= \sum_{m \in \mathcal{I}(\mathcal{C})} d_m$$  $$= \sum_{m \in \mathcal{I}(\mathcal{C})} d_m^2. \qquad (5.1.8)$$

Here in the second equality we used the identity

$$\bigoplus_{m \in \mathcal{I}(A\text{-mod}^{\mathcal{C}}\text{-}B)} d_m$$  (5.1.9)

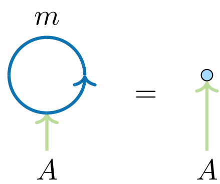
valid for any $A, B \in \mathcal{F}r(\mathcal{C})$ and $X \in A\text{-mod}^{\mathcal{C}}\text{-}B$, where the α-summation is over a basis of $A\text{-mod}^{\mathcal{C}}\text{-}B(m, X)$; this is a variant of the completeness relation (5.0.1). The third equality of (5.1.8) follows from the identity [FFRS06a, Lemma 4.3]

$$\bigoplus_{i \in \mathcal{I}(\mathcal{C})} d_i$$  (5.1.10)

which holds for any $M \in \text{mod}^{\mathcal{C}}\text{-}A$ and $N \in A\text{-mod}^{\mathcal{C}}$, with the α-summation being over a basis of $\mathcal{C}^{(A)}(i, M \otimes N) \cong \mathcal{C}(X, M \otimes_A N)$, where $\mathcal{C}^{(A)}(i, M \otimes N)$ stands for the subspace of $\mathcal{C}(i, M \otimes N)$ that is invariant under postcomposing with the idempotent (5.1.4). □

Combining Lemma 5.1.1 with Lemma 5.1.2 we obtain:

Corollary 5.1.3. *For a simple special symmetric Frobenius algebra A in a spherical fusion category $\mathcal{C}$ the equality*

$$\sum_{m \in \mathcal{I}(\text{mod}^{\mathcal{C}}\text{-}A)} \frac{d_m}{D_{\mathcal{C}}^2}$$  (5.1.11)

holds.

5.2 Field functors

We are now ready to construct the field maps (see Problem 2.6.3). In fact, we are going to construct a *field functor*

$$\mathbb{F}_\ell \colon \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \mathbb{1}, \ell) \to \mathrm{Bl}_{\mathcal{C}}(\ell) = \mathrm{Cyl}(\mathcal{C}, \ell)$$

for every compact oriented 1-manifold ℓ with possibly non-empty boundary, such that for every worldsheet $\mathcal{S}$ with ambient bordism $\Sigma_{\mathcal{S}}\colon \ell_{\text{in}} \nrightarrow \ell_{\text{out}}$, (2.6.6) and (2.6.7) are satisfied, i.e.

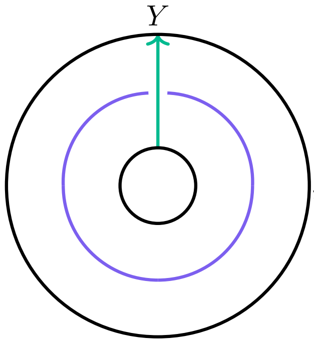
$$\mathbb{F}_{\ell_{\text{in}}}(\mathsf{b}_{\ell_{\text{in}}}) \xmapsto{\Phi_{\ell_{\text{in}}}} \mathbb{F}^{\mathcal{S}}_{\text{in}} \in \prod_{i=1}^{p_{\text{in}}} \mathcal{C} \times \prod_{j=1}^{q_{\text{in}}} \mathcal{Z}(\mathcal{C}) \tag{5.2.1}$$

and

$$\mathbb{F}_{\ell_{\text{out}}}(\mathsf{b}_{\ell_{\text{out}}}) \xmapsto{\Phi_{\ell_{\text{out}}}} \mathbb{F}^{\mathcal{S}}_{\text{out}} \in \prod_{i=1}^{p_{\text{out}}} \mathcal{C} \times \prod_{j=1}^{q_{\text{out}}} \mathcal{Z}(\mathcal{C}) \tag{5.2.2}$$

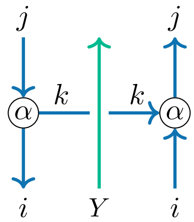
where $\mathbb{F}^{\mathcal{S}}_{\text{in}}$ and $\mathbb{F}^{\mathcal{S}}_{\text{out}}$ are the combinations of field contents associated to the in-going and out-going sewing boundaries. We start with giving two examples of thickened $\mathcal{C}$-boundary values on S^1, i.e. objects in the Karoubified cylinder category $\mathrm{Cyl}(\mathcal{C}, S^1)$:

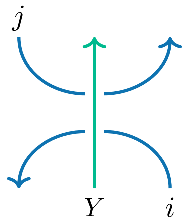
Example 5.2.1. For $Y = (U(Y), \gamma) \in \mathcal{Z}(\mathcal{C})$ consider the string-net

$p^{\text{can}}_Y :=$ 

,

where the unlabeled edge in purple that runs along the non-contractible cycle of the cylinder, stands for the canonical color (5.0.2), and the half baring γ is indicated by an over crossing, as explained below (5.0.3). Using the completeness relation (5.0.1) it is readily seen that p^{can}_Y is an idempotent (see e.g. [SY21, Remark 2.6]), therefore supplies an object $p^{\text{can}}_Y \in \mathrm{Cyl}(\mathcal{C}, S^1)$ in the Karoubified cylinder category for $\mathcal{C}$ over S^1. We have

$\Phi^{\circ}_{S^1}(p^{\text{can}}_Y) = \sum_{i,j,k\in\mathcal{I}(\mathcal{C})} \frac{d_j d_k}{D^2_{\mathcal{C}}}$ 

$= \sum_{i,j\in\mathcal{I}(\mathcal{C})} \frac{d_j}{D^2_{\mathcal{C}}}$ 

by the definition of the fully faithful functor $\Phi^{\circ}_{S^1}\colon \mathrm{Cyl}^{\circ}(\mathcal{C}, S^1) \to \mathcal{Z}(\mathcal{C})$, see (4.7.4). It is then easily seen [KJ11, Lemma 8.3] that the image of the morphism $\Phi^{\circ}_{S^1}(p^{\text{can}}_Y) \in \mathcal{Z}(\mathcal{C})(LU(Y), LU(Y))$ is (up to a canonical isomorphism) the object $Y \in \mathcal{Z}(\mathcal{C})$. We write[1]

$$\mathsf{B}^{\text{can}}_Y := p^{\text{can}}_Y \in \mathrm{Cyl}(\mathcal{C}, S^1) \tag{5.2.3}$$

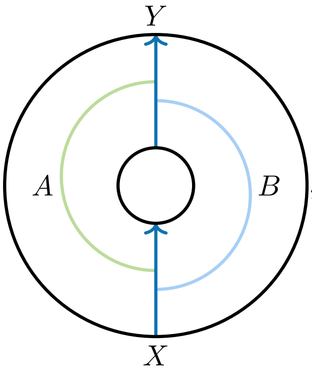
for every $Y \in \mathcal{Z}(\mathcal{C})$. We thus have

$$\Phi_{S^1}(\mathsf{B}^{\text{can}}_Y) = \mathrm{im}(\Phi^{\circ}_{S^1}(p^{\text{can}}_Y)) = Y \in \mathcal{Z}(\mathcal{C})$$

for every $Y \in \mathcal{Z}(\mathcal{C})$, where $\Phi_{S^1}\colon \mathrm{Cyl}(\mathcal{C}, S^1) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C})$ is the canonical equivalence (4.7.5).

[1] Strictly speaking, one needs to choose a marked point on S^1 before defining these objects.

Example 5.2.2. For any simple special symmetric Frobenius algebras A and B in $\mathcal{C}$ and any two A-B-bimodules X and Y consider the string-net

$$p_{X,Y} :=$$ 

,

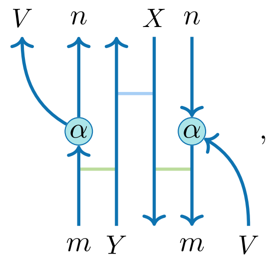
where we used the convention in (5.1.4), i.e. we omitted the orientations and the units for the strands labeled by the Frobenius algebras. Using that A and B are special Frobenius algebras, one sees again directly that $p_{X,Y}$ is an idempotent and thus provides an object $p_{X,Y} \in \mathrm{Cyl}(\mathcal{C}, S^1)$ in the Karoubified cylinder category for $\mathcal{C}$ over S^1. Furthermore we find

$$\mathrm{im}(\Phi^{\circ}_{S^1}(p_{X,Y})) = \underline{\mathrm{Nat}}(G^X, G^Y) = \mathbb{D}^{X,Y} \in \mathcal{Z}(\mathcal{C}).$$

Let us explain this isomorphism in detail, using manifestly that $\mathcal{C}$ is finitely semisimple (the statement is, however, in fact valid beyond semisimplicity). Recall from (2.3.5) that by inserting the explicit form (2.3.3) of the functors G^X and G^Y, we have

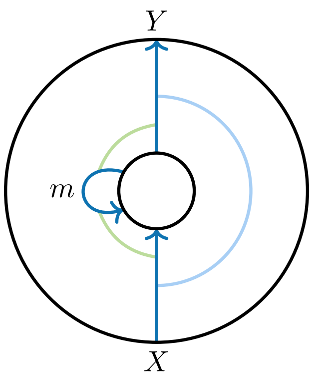
$$\mathbb{D}^{X,Y} = \Big(\bigoplus_{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} m \otimes_A Y \otimes_B X^{\vee} \otimes_A m^{\vee}, \gamma_{\mathbb{D}^{X,Y}} \Big) \in \mathcal{Z}(\mathcal{C}),$$

where the component of the half-braiding $\gamma_{\mathbb{D}^{X,Y}}$ at an object $V \in \mathcal{C}$ is given by

$$\gamma_{\mathbb{D}^{X,Y};V} = \bigoplus_{m,n \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} d_m$$ 

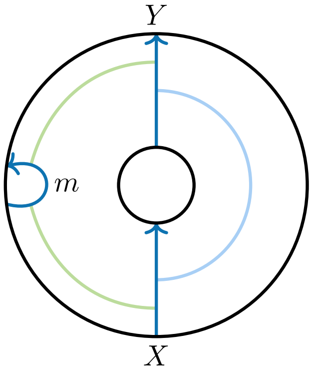
,

where the α-summation is over module morphisms. Now consider the two string-nets

$$e_{X,Y} := \sum_{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} \frac{d_m}{D^2_{\mathcal{C}}}$$ 

(5.2.4)

and

$$r_{X,Y} \coloneqq \sum_{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} \frac{d_m}{D_{\mathcal{C}}^2}$$  $$(5.2.5)$$

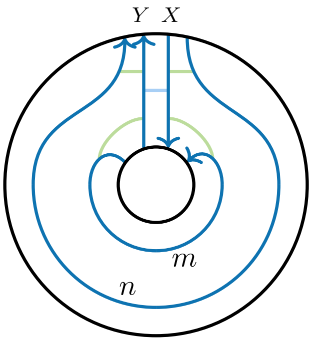
Using that A and B are simple special Frobenius and invoking Corollary 5.1.3 we see that

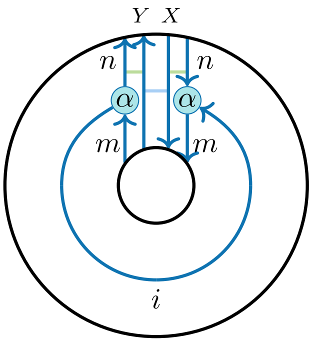
$$e_{X,Y} \circ r_{X,Y} = p_{X,Y}. \tag{5.2.6}$$

For the composition in the opposite order we get

$$r_{X,Y} \circ e_{X,Y} = p^{\mathrm{can}}_{\mathbb{D}X,Y}\,, \tag{5.2.7}$$

shown by the following identities of string-nets:

$$r_{X,Y} \circ e_{X,Y} = \sum_{m,n \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} \frac{d_m}{D_{\mathcal{C}}^4}$$ 

$$= \sum_{\substack{i \in \mathcal{I}(\mathcal{C}) \\ m,n \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)}} \frac{d_i d_m}{D_{\mathcal{C}}^2}$$  $$= p^{\mathrm{can}}_{\mathbb{D}X,Y}\,,$$

where the cylinders are drawn in a slightly different manner – deformed by a homeomorphism – than in (5.2.4) and (5.2.5), and the middle equality uses (5.1.10). Write

$$\mathsf{B}_{X,Y} \coloneqq p_{X,Y} \in \mathrm{Cyl}(\mathcal{C}, S^1)$$

and according to (5.2.6) and (5.2.7), we have the isomorphism

$$e_{X,Y} \colon \mathsf{B}^{\mathrm{can}}_{\mathbb{D}X,Y} \rightleftharpoons \mathsf{B}_{X,Y} \colon r_{X,Y} \tag{5.2.8}$$

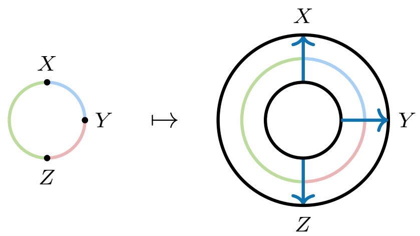
in the Karoubified cylinder category $\mathrm{Cyl}(\mathcal{C}, S^1)$, which furnishes an identification

$$\mathrm{im}(\Phi^{\circ}_{S^1}(p_{X,Y})) = \Phi_{S^1}(\mathsf{B}_{X,Y}) = \Phi_{S^1}(\mathsf{B}^{\mathrm{can}}_{\mathbb{D}^{X,Y}}) = \mathbb{D}^{X,Y}.$$

Example 5.2.2 inspires us to define the action of the field functor

$$\mathbb{F}_{S^1} \colon \mathrm{Cyl}^{\circ}(\mathcal{F}r(\mathcal{C}), S^1) \to \mathrm{Bl}_{\mathcal{C}}(S^1) = \mathrm{Cyl}(\mathcal{C}, S^1)$$

for the standard circle S^1 to be (demonstrated using a generic object in the domain category):

$\mathsf{b} =$ 

$=: \mathbb{F}_{S^1}(\mathsf{b}).$ (5.2.9)

That is, we take the identity morphism id_{b} of an $\mathcal{F}r(\mathcal{C})$-boundary datum $\mathsf{b} \in \mathrm{Cyl}^{\circ}(\mathcal{F}r(\mathcal{C}), S^1)$ and remove the colors of the patches, replacing each of the colors with a single strand colored with the corresponding Frobenius algebra; this produces an idempotent $\mathbb{F}_{S^1}(\mathsf{b}) = p_{\mathsf{b}}$ in $\mathrm{Cyl}^{\circ}(\mathcal{C}, S^1)$ (i.e. an object in $\mathrm{Cyl}(\mathcal{C}, S^1)$) that contains all the information about the $\mathcal{F}r(\mathcal{C})$-boundary datum b, and is sent to the corresponding field content $\mathbb{D}_{\mathsf{b}} \in \mathcal{Z}(\mathcal{C})$ by the equivalence $\Phi_{S^1} \colon \mathrm{Cyl}(\mathcal{C}, S^1) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C})$ according to an argument similar to that of Example 5.2.2.

Before prescribing the action of the field functor $\mathbb{F}_{S^1}$ on the morphisms of $\mathrm{Cyl}^{\circ}(\mathcal{F}r(\mathcal{C}), S^1)$, let us introduce the notion of a *Frobenius graph.*

Definition 5.2.3. Let Γ be a fully $\mathcal{F}r(\mathcal{C})$-colored graph on a compact oriented surface Σ, ϑ the 2-cell underlying a patch, i.e. a connected component of the complement $\Sigma \setminus \Gamma$, and $A \in \mathcal{F}r(\mathcal{C})$ the color of the patch. A *Frobenius graph on* ϑ is a fully $\mathcal{C}$-colored graph Γ_ϑ (this in particular means that the edges are *oriented*) on ϑ having the following features:

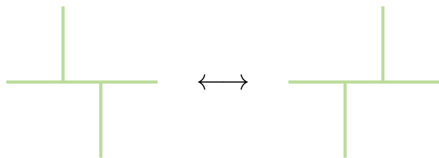
- The vertices of Γ_ϑ are either univalent or trivalent. Each of the vertices of Γ_ϑ that lies on the boundary $\partial\vartheta$ is univalent and lies in the interior of an edge of the $\mathcal{F}r(\mathcal{C})$-colored graph Γ.
- Each edge of Γ_ϑ is labeled by the underlying object of the Frobenius algebra A.
- Each trivalent vertex is labeled either by the multiplication μ or comultiplication Δ of A, and each univalent vertex in the interior of ϑ either by the unit η or the counit ε; each univalent vertex on the boundary $\partial\vartheta$ is labeled by the structure morphism for the left or right A-module that labels the corresponding edge of Γ.

A *full Frobenius graph on* ϑ is a Frobenius graph Γ_ϑ on ϑ such that in addition the following condition is satisfied:

- Γ_ϑ is *full* in the sense that adding any further edges and vertices in a way compatible with the previous requirements results in a graph that can be reduced to Γ_ϑ by a finite sequence of moves provided by the axioms of a special symmetric Frobenius algebra and its modules.

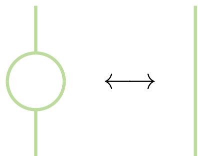
When drawing a Frobenius graph, it is convenient to use the *simplified* graphical notation for morphisms involving symmetric Frobenius algebras in which, as explained below (5.1.4), orientations as well as univalent vertices are omitted. We refer to the thus obtained simplified version of a (full) Frobenius graph as a *simplified* (full) Frobenius graph. Upon such simplification, the moves provided by the axioms of a special symmetric Frobenius algebra and its modules can be gathered into the following list of *elementary Frobenius moves*:

- The a-move:



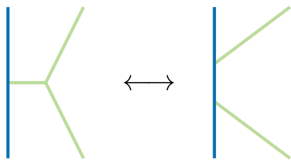
(5.2.10)

- The b-move:



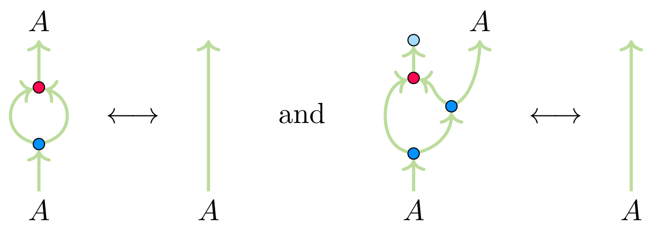
(5.2.11)

- The r-move:

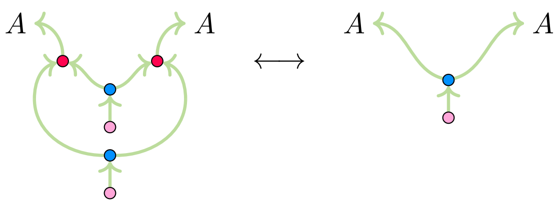


(5.2.12)

The blue line in (5.2.12) stands for a bimodule, and we used one side of the action to demonstrate the r-move. The composite of a finite sequence of elementary Frobenius moves is called a *Frobenius move*. Note that each of the elementary Frobenius moves between simplified Frobenius graphs stands for a whole family of moves between non-simplified graphs. For instance, the moves

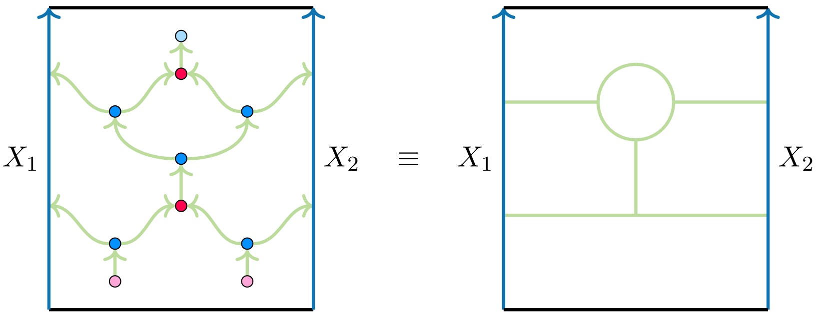


as well as



constitute three possible realizations of the b-move as a move between non-simplified Frobenius graphs.

Example 5.2.4. The following pictures show an example of a full Frobenius graph on a 2-cell (left) and its simplified version (right):



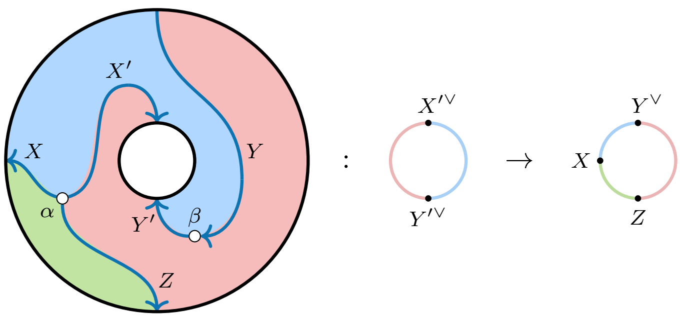
Note that this Frobenius graph can be reduced to the one of the form $P_{X_1 \otimes_A X_2}$, see (5.1.4), by a suitable Frobenius move.

Remark 5.2.5. (i) A convenient way to construct a (simplified) Frobenius graph Γ_ϑ on a given 2-cell ϑ is to remove sufficiently many disks D_i from ϑ and take Γ_ϑ to be the graph that is obtained from $\vartheta \setminus \cup_i D_i$ as a retract. This procedure of "punching holes" is similar to the way in which the presence of a triangulation of the worldsheet in the TFT construction of correlators is explained in [KS11] and [FSV13, Section 6].
(ii) If the 2-cell ϑ is labeled by the monoidal unit $\mathbb{1} \in \mathcal{C}$, like e.g. any of the additional transparent 2-cells in a complemented worldsheet $\widetilde{S}$ (see Section 2.4), then each edge of a full $\mathbb{1}$-graph on ϑ is labeled by $\mathbb{1}$ and each vertex by an identity morphism. As a consequence the graph is *transparent* and may be suppressed completely.

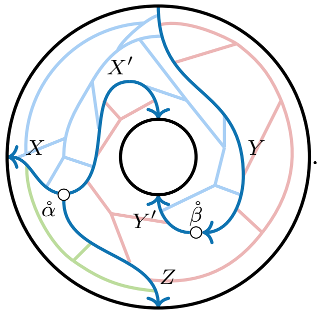
Lemma 5.2.6. *Any two full Frobenius graphs on a 2-cell ϑ are related by a Frobenius move up to isotopy.*

Proof. The assertion follows by combining the following two statements, each of which is easy to verify. First, any two realizations of a *simplified* (not necessarily full) Frobenius graph as a non-simplified Frobenius graph are related by a (not necessarily unique) Frobenius move up to isotopy. Second, any two simplified full Frobenius graphs on a 2-cell ϑ can be related by a finite sequence of the elementary Frobenius moves that are shown in (5.2.10), (5.2.11), and (5.2.12). □

We now demonstrate the action of $\mathbb{F}_{S^1}$ on morphisms by an explicit example. Let $\Phi\colon \mathsf{b}_1 \to \mathsf{b}_2$ be the morphism in the cylinder category $\mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), S^1)$ given by



for Frobenius algebras $A, B, C \in \mathcal{F}r(\mathcal{C})$, bimodules $X \in A\text{-mod}^{\mathcal{C}}\text{-}B$, $X' \in B\text{-mod}^{\mathcal{C}}\text{-}C$, $Z \in C\text{-mod}^{\mathcal{C}}\text{-}A$, $Y, Y' \in C\text{-mod}^{\mathcal{C}}\text{-}B$, and vertex colors α, β given by a choice of polarizations and bimodule morphisms. First we remove the colors of the patches and replace the labels of the edges by their underlying objects in $\mathcal{C}$ and the labels of the vertices by their $\mathcal{U}$-conjugation (see Example 3.3.5 for the definition of the rigid separable Frobenius functor $\mathcal{U}\colon \mathcal{F}r(\mathcal{C}) \to \mathcal{BC}$), with the latter denoted by for instance $\mathring{\alpha} \equiv \alpha^{\mathcal{U}}$. Note that the $\mathcal{U}$-conjugates necessarily intertwine with the actions of the Frobenius algebras. We then put a *full* Frobenius graph Γ_ϑ on each of the 2-cells ϑ and obtain a fully $\mathcal{C}$-colored graph Γ_Φ on the cylinder $S^1 \times I$, for instance

$\Gamma_\Phi =$  . (5.2.13)

Due to Lemma 5.2.6 and the fact that the Frobenius graphs Γ_ϑ are full, Γ_Φ gives rise to a unique string-net upon taking the equivalence class $[\Gamma_\Phi]$ which is independent of the choice of full Frobenius graphs. In fact, the string-net $[\Gamma_\Phi]$ depends only on the $\mathcal{F}r(\mathcal{C})$-colored string-net underlying the morphism Φ which is an equivalence class of $\mathcal{F}r(\mathcal{C})$-colored graphs on the cylinder $S^1 \times I$: this is a corollary of Theorem 8.1.1. Moreover, the string-net $[\Gamma_\Phi]$ is invariant under precomposing with the idempotent $\mathbb{F}_{S^1}(\mathsf{b}_1)$ and postcomposing with the idempotent $\mathbb{F}_{S^1}(\mathsf{b}_2)$ and thus we define

$$\mathbb{F}_{S^1}(\Phi) := [\Gamma_\Phi] \in \mathrm{Hom}_{\mathrm{Cyl}(\mathcal{C},S^1)}(\mathbb{F}_{S^1}(\mathsf{b}_1), \mathbb{F}_{S^1}(\mathsf{b}_2)). \tag{5.2.14}$$

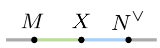
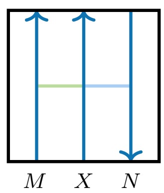
Again, because of Lemma 5.2.6, the assignments (5.2.9) and (5.2.14) are compatible with the concatenation of string-nets (see the proof of Theorem 5.3.3), therefore define a functor

$$\mathbb{F}_{S^1}\colon \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), S^1) \to \mathrm{Bl}_{\mathcal{C}}(S^1) = \mathrm{Cyl}(\mathcal{C}, S^1). \tag{5.2.15}$$

Similarly, one defines the field functor for the standard interval I

$$\mathbb{F}_I\colon \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \mathbb{1}, I) \to \mathrm{Bl}_{\mathcal{C}}(I) = \mathrm{Cyl}(\mathcal{C}, I) \tag{5.2.16}$$

by, for example:

$\mathsf{b} =$  $\mapsto$  $=: \mathbb{F}_I(\mathsf{b})$ (5.2.17)

where $M \in \mathrm{mod}^{\mathcal{C}}\text{-}A$, $N \in \mathrm{mod}^{\mathcal{C}}\text{-}B$ and $X \in A\text{-}\mathrm{mod}^{\mathcal{C}}\text{-}B$, and the action on morphisms is analogous to the one given by (5.2.13). Under the equivalence

$$\Phi_I\colon \mathrm{Cyl}(\mathcal{C}, I) \xrightarrow{\simeq} \mathcal{C},$$

the right hand side of (5.2.17) is sent to $\mathbb{B}_{\mathsf{b}} = \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}B}(N, M \otimes_A X) = M \otimes_A X \otimes_B N^\vee \in \mathcal{C}$, therefore $\mathbb{F}_I$ also satisfy the requirement listed in Problem 2.6.3. Extending (5.2.15) and (5.2.16) by taking the Cartesian product, we thus obtain a field functor $\mathbb{F}_\ell \colon \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \mathbb{1}, \ell) \to \mathrm{Bl}_{\mathcal{C}}(\ell)$ for every object $\ell \in \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$.

It is also straight forward to generalize the field functors and define a functor

$$\mathbb{F}_{\partial\Sigma} \colon \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \partial\Sigma) \to \mathrm{Cyl}(\mathcal{C}, \partial\Sigma) \tag{5.2.18}$$

for a compact oriented surface Σ (note that $\partial\Sigma$ is necessarily a closed 1-manifold).

5.3 Correlators

Let $\mathcal{S}$ be a worldsheet and $\Sigma_{\mathcal{S}} \colon \ell_{\mathrm{in}} \nrightarrow \ell_{\mathrm{out}}$ its ambient bordism. As defined in (2.6.9), the space of conformal blocks for the worldsheet $\mathcal{S}$ is given by

$$\begin{aligned} \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) &= \mathrm{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathbb{F}_{\ell_{\mathrm{in}}}(\mathsf{b}_{\ell_{\mathrm{in}}}), \mathbb{F}_{\ell_{\mathrm{out}}}(\mathsf{b}_{\ell_{\mathrm{out}}})) \\ &= \mathcal{S}\mathrm{N}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathbb{F}_{\ell_{\mathrm{in}}}(\mathsf{b}_{\ell_{\mathrm{in}}}), \mathbb{F}_{\ell_{\mathrm{out}}}(\mathsf{b}_{\ell_{\mathrm{out}}})), \end{aligned}$$

where $\mathsf{b}_{\ell_{\mathrm{in}}} \in \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \mathbb{1}, \ell_{\mathrm{in}})$ and $\mathsf{b}_{\ell_{\mathrm{out}}} \in \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \mathbb{1}, \ell_{\mathrm{out}})$ are the $\mathcal{F}r(\mathcal{C})$-boundary data pulled back by the parametrization maps of the ambient bordism $\Sigma_{\mathcal{S}}$ and the profunctor $\mathcal{S}\mathrm{N}_{\mathcal{C}}$ is given by the composite

$$\mathcal{S}\mathrm{N}_{\mathcal{C}} \colon \mathrm{Cyl}(\mathcal{C}, \ell_{\mathrm{in}})^{\mathrm{op}} \times \mathrm{Cyl}(\mathcal{C}, \ell_{\mathrm{out}}) \to \mathrm{Cyl}(\mathcal{C}, \partial\Sigma_{\mathcal{S}}) \xrightarrow{\mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, -)} \mathrm{Vect}_{\Bbbk},$$

where we left the functor induced by the parametrization maps unnamed.

We have a commutative diagram

$$\begin{tikzcd} \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \mathbb{1}, \ell_{\mathrm{in}})^{\mathrm{op}} \times \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \mathbb{1}, \ell_{\mathrm{out}}) \arrow[r] \arrow[d, "\mathbb{F}^{\mathrm{op}}_{\ell_{\mathrm{in}}} \times \mathbb{F}_{\ell_{\mathrm{in}}}"'] & \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \mathbb{1}, \partial\Sigma_{\mathcal{S}}) \arrow[d, "\mathbb{F}_{\partial\Sigma_{\mathcal{S}}}"] \\ \mathrm{Cyl}(\mathcal{C}, \ell_{\mathrm{in}})^{\mathrm{op}} \times \mathrm{Cyl}(\mathcal{C}, \ell_{\mathrm{out}}) \arrow[r] & \mathrm{Cyl}(\mathcal{C}, \partial\Sigma_{\mathcal{S}}) \end{tikzcd}$$

composed of the field functors for ℓ_{in} and ℓ_{out}, the functor $\mathbb{F}_{\partial\Sigma}$, and the functors induced by the parametrization maps of $\Sigma_{\mathcal{S}}$ (they are the unnamed horizontal arrows). Note that the functor denoted by the upper unnamed horizontal arrow sends $(\mathsf{b}_{\ell_{\mathrm{in}}}, \mathsf{b}_{\ell_{\mathrm{out}}}) \in \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \mathbb{1}, \ell_{\mathrm{in}})^{\mathrm{op}} \times \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \mathbb{1}, \ell_{\mathrm{out}})$ to the $\mathcal{F}r(\mathcal{C})$-boundary datum

$$\mathsf{b}_{\mathcal{S}} \in \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \partial\Sigma_{\mathcal{S}})$$

that is the boundary datum of the complemented worldsheet $\widetilde{\mathcal{S}}$ viewed as a fully $\mathcal{F}r(\mathcal{C})$-colored graph on $\Sigma_{\mathcal{S}}$, therefore *independent* of the choice of in- and outgoing parametrizations of the ambient bordism $\Sigma_{\mathcal{S}}$. For example, for the worldsheet (2.3.9) whose complemented worldsheet is given by (2.4.2), the boundary datum $\mathsf{b}_{\mathcal{S}}$ is given by (4.1.1). Consequently, we have

$$\mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) = \mathcal{S}\mathrm{N}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathbb{F}_{\ell_{\mathrm{in}}}(\mathsf{b}_{\ell_{\mathrm{in}}}), \mathbb{F}_{\ell_{\mathrm{out}}}(\mathsf{b}_{\ell_{\mathrm{out}}}))$$

$$= \mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, \mathbb{F}_{\partial\Sigma}(\mathsf{b}_{\mathcal{S}})), \tag{5.3.1}$$

which means that the space of conformal blocks for a worldsheet $\mathcal{S}$, realized as a space of string-nets, is in fact *parametrization independent.*

We are now ready to prescribe the correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ for a worldsheet $\mathcal{S}$, which is a vector in the space of conformal blocks $\mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) = \mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, \mathbb{F}_{\partial\Sigma}(\mathsf{b}_{\mathcal{S}}))$.

Definition 5.3.1. Let $\mathcal{S}$ be a worldsheet with ambient surface $\Sigma_{\mathcal{S}}$. We define the correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \in \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) = \mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, \mathbb{F}_{\partial\Sigma}(\mathsf{b}_{\mathcal{S}}))$ to be the vector obtained by the following two-step procedure:

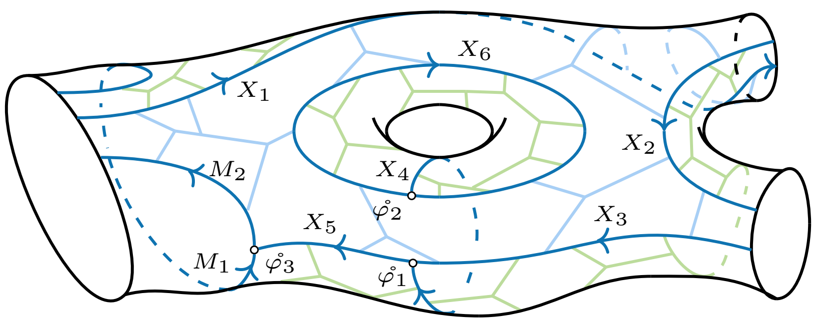
1. Take the complemented worldsheet $\widetilde{\mathcal{S}}$ viewed as a fully $\mathcal{F}r(\mathcal{C})$-colored graph on the surface $\Sigma_{\mathcal{S}}$ and remove the colors of the patches, replace the edge colors (i.e. the line-defect conditions) with their underlying objects in $\mathcal{C}$ and the vertex colors with their $\mathcal{U}$-conjugates, thereby producing a fully $\mathcal{C}$-colored graph $\Gamma_{\mathcal{S}}$ on $\Sigma_{\mathcal{S}}$, which we refer to as the *partial defect network for* $\mathcal{S}$.

2. Modify the graph $\Gamma_{\mathcal{S}}$ by putting a full Frobenius graph Γ_{ϑ} on each of the 2-cells of $\widetilde{\mathcal{S}}$. The correlator for the worldsheet $\mathcal{S}$ is defined to be the string-net it represents:

$$\mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \coloneqq [\Gamma_{\mathcal{S}} \cup \bigcup_{\vartheta\in\widetilde{\mathcal{S}}} \Gamma_{\vartheta}] \in \mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, \mathbb{F}_{\partial\Sigma}(\mathsf{b}_{\mathcal{S}})). \tag{5.3.2}$$

Note that, analogous to how we defined the actions of the field functors on morphisms, the definition of $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ involves a choice of full Frobenius graph for each 2-cell of $\widetilde{\mathcal{S}}$. However, by Lemma 5.2.6, any two such choices are related by a Frobenius move. Now each Frobenius move corresponds to an equality of string diagrams in $\mathcal{C}$, and thus also to an equality of string-nets. Therefore, the correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ does not depend on these choices and is well defined.

Example 5.3.2. Let $\mathcal{S}$ be the worldsheet given by (2.3.9). Its correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ is the string-net represented by the $\mathcal{C}$-colored graph



We now show that Definition 5.3.1 indeed provides a consistent system of correlators:

Theorem 5.3.3. *The assignment of the vector* $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \in \mathrm{Bl}_{\mathcal{C}}(\mathcal{S}) = \mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathcal{S}}, \mathbb{F}_{\partial\Sigma}(\mathsf{b}_{\mathcal{S}}))$ *to every worldsheet* $\mathcal{S}$ *given by Definition 5.3.1 provides a consistent system of correlators in the sense of Problem 2.6.7.*

Proof. (i) Invariance under the mapping class group $\mathrm{Map}(\mathcal{S})$: By definition, any mapping class group element in $\mathrm{Map}(\mathcal{S})$ maps the partial defect network $\Gamma_{\mathcal{S}}$ to itself. Thus when studying the correlator (5.3.2) we only have to deal with the full Frobenius graphs for the 2-cells. Let Γ_ϑ be such a graph and $\xi \in \mathrm{Map}(\mathcal{S})$ a mapping class group element. Then the graph $\xi(\Gamma_\vartheta)$ is clearly again a Frobenius graph; the following consideration shows that it is even a *full* Frobenius graph: Let $\xi(\Gamma_\vartheta)^+$ be a Frobenius graph obtained by adding an edge labeled by the corresponding Frobenius algebra. Then the graph $\xi^{-1}(\xi(\Gamma_\vartheta)^+)$ can be obtained from Γ_ϑ by adding a Frobenius line. Since Γ_ϑ is full, it is related to $\xi^{-1}(\xi(\Gamma_\vartheta)^+)$ by some Frobenius move. Upon applying ξ, that move transports to a Frobenius move that relates $\xi(\Gamma_\vartheta)$ to $\xi(\Gamma_\vartheta)^+$. Hence just like Γ_ϑ, $\xi(\Gamma_\vartheta)$ is also a full Frobenius graph; by Lemma 5.2.6, they are thus related by a Frobenius move. Since, as already pointed out, a Frobenius move corresponds to an equality of string-nets, and this implies that the two graphs on $\Sigma_{\mathcal{S}}$ that are related by replacing Γ_ϑ with $\xi(\Gamma_\vartheta)$ represent one and the same string-net. Altogether we then have

$$\xi(\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})) = [\xi(\Gamma_{\mathcal{S}} \cup \bigcup_{\vartheta\in\widetilde{\mathcal{S}}} \Gamma_\vartheta)] = [\Gamma_{\mathcal{S}} \cup \bigcup_{\vartheta\in\widetilde{\mathcal{S}}} \xi(\Gamma_\vartheta)] = \mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$$

for every $\xi \in \mathrm{Map}(\mathcal{S})$.
(ii) Compatibility with sewing: Without loss of generality we can restrict our attention to what happens when two 2-cells ϑ' and ϑ'', both labeled by the same Frobenius algebra, are sewn to a single new 2-cell $\vartheta = \vartheta' \cup_\alpha \vartheta''$, whereby the full Frobenius graphs $\Gamma_{\vartheta'}$ and $\Gamma_{\vartheta''}$ combine to a graph $\Gamma := \Gamma_{\vartheta'} \cup_\alpha \Gamma_{\vartheta''}$. The graph Γ is clearly a Frobenius graph, and we can show that it is even a *full* Frobenius graph. To this end, we add an edge labeled by the corresponding Frobenius algebra to Γ, resulting in a new Frobenius graph Γ^+, and show that Γ^+ is related to Γ by a Frobenius move. If the new edge of Γ^+ lies entirely in either ϑ' or ϑ'' (regarded as embedded in ϑ), then with obvious notation we have either $\Gamma^+ = \Gamma^+_{\vartheta'} \cup_\alpha \Gamma_{\vartheta''}$ or $\Gamma^+ = \Gamma_{\vartheta'} \cup_\alpha \Gamma^+_{\vartheta''}$, so that the statement follows immediately, just because $\Gamma_{\vartheta'}$ and $\Gamma_{\vartheta''}$ are both full. Otherwise, i.e. if the new edge lies partly in ϑ' and partly in ϑ'', we can perform a suitable Frobenius move together with isotopy to transform the graph Γ in such a way that we deal again with the previous situation. □

Remark 5.3.4. As we will see in Theorem 8.1.1, the construction of correlators gives rise to a $\mathrm{Map}(\Sigma)$-intertwiner

$$\begin{aligned} \mathrm{UCor}_{\mathcal{C}}(\Sigma, \mathsf{b})\colon \mathrm{SN}^\circ_{\mathcal{F}r(\mathcal{C})}(\Sigma, \mathsf{b}) &\to \mathrm{SN}_{\mathcal{C}}(\Sigma, \mathbb{F}_{\partial\Sigma}(\mathsf{b})) \\ [\widetilde{\mathcal{S}}] &\to \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \end{aligned}$$

for each compact oriented surface Σ and $\mathcal{F}r(\mathcal{C})$-boundary datum $\mathsf{b} \in \mathrm{Cyl}^\circ(\mathcal{F}r(\mathcal{C}), \partial\Sigma)$, by viewing each $\mathcal{F}r(\mathcal{C})$-colored graph as a complemented worldsheet. When specialized to the cylinders, these linear maps reproduce the actions of the field functors on the morphisms in the cylinder categories. In this sense, the field functors describe the correlators for the neighborhoods around the sewing boundaries.

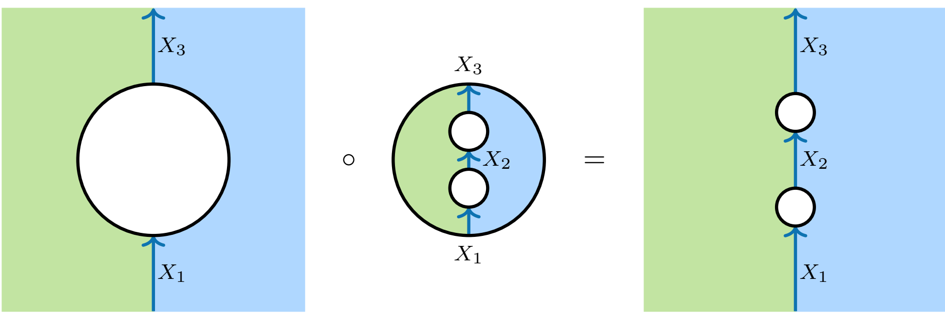
6 Correlators of particular interest

The prescription of correlators presented in the previous chapter applies to all worldsheets of the theory. However, a few specific correlators are of particular interest: concretely, partition functions on the one hand, and correlators which determine operator products, i.e. composition morphisms on the field contents, on the other hand. The present chapter provides detailed information about such correlators. Similar to the case of point defects (point defects on a surface can be composed in two directions – in general, defects of codimension k can be composed in k directions), there are two ways of forming a product of defect fields: either along a defect line or accompanying the fusion of two defect lines. The former is analyzed in Section 6.1, and two variants of the latter, which are related by a braiding, are exhibited in Section 6.2. In Section 6.3 we specialize from defect fields to bulk fields – the most basic type of field insertions in the bulk, and in Section 6.4 we discuss the torus partition function. The final Sections 6.5 and 6.6 of the chapter are devoted to the operator product of boundary fields and the bulk-boundary operator products, respectively.

6.1 Vertical operator product

A crucial feature of a full conformal field theory are the the *operator products* among the various types of fields. In the case of defect fields, there are two ways of forming a product, either along a given defect line, or such that a fusion of two defect lines is involved. In terms of the description of defect field contents as internal natural transformations, these correspond to the vertical and horizontal compositions [FS21a]. The vertical and horizontal products coincide if the relevant line defects are all trivial, i.e. labeled by identity 1-morphisms in $\mathcal{F}r(\mathcal{C})$: in such cases, we deal with bulk fields. The purpose of the present section is to exhibit how the algebraic notion of vertical product of defect fields can be related to the string-net construction of correlators.

Thinking of defect fields in terms of two-pronged defect junctions, as already visualized in (2.3.6), the situation on the worldsheet that is relevant for the vertical composition amounts to a sewing operation, according to



(6.1.1)

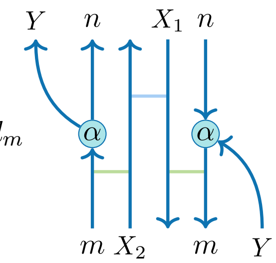
It is worth stressing that a string-net correlator as constructed in Section 5.3 is directly assigned to the worldsheet $\mathcal{S}$ using the geometric information encoded in its complemented form $\widetilde{\mathcal{S}}$, without making use of further auxiliary data such as, say, a fine marking of the surface $\Sigma_{\mathcal{S}}$ (the latter is, for example, needed in the Lego-Teichmüller based approach of [FS17b]). In contrast, relating algebraic structures – like the compositions of internal natural transformations that formalize operator products – to correlators does require such auxiliary data. Via the string-net construction of correlators one thus achieves a more invariant description of operator products than what can be seen based on the underlying purely algebraic structures alone.

Let $A, B \in \mathcal{F}r(\mathcal{C})$ be simple special symmetric Frobenius algebras and $X_1, X_2 \in A\text{-mod}^{\mathcal{C}}\text{-}B$ bimodules. First recall the explicit form of the defect field content $\mathbb{D}^{X_1,X_2}$:

$$\mathbb{D}^{X_1,X_2} = \underline{\mathrm{Nat}}(G^{X_1}, G^{X_2}) = \Big(\bigoplus_{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} m \otimes_A X_2 \otimes_B X_1^\vee \otimes_A m^\vee, \gamma_{\mathbb{D}^{X_1,X_2}} \Big) \in \mathcal{Z}(\mathcal{C}),$$

where the half-braiding $\gamma_{\mathbb{D}^{X_1,X_2}}$ comes from the universal coaction of the central comonad of $\mathcal{C}$ and can be explicitly expressed in components as

$$\gamma_{\mathbb{D}^{X_1,X_2};Y} := \bigoplus_{m,n \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} d_m \tag{6.1.2}$$



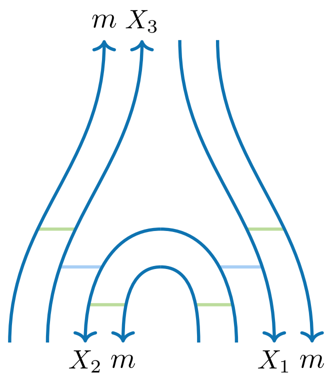
for every $Y \in \mathcal{C}$.

The vertical composition

$$\underline{\mu}_{\mathrm{ver}} \equiv \underline{\mu}_{\mathrm{ver}}(X_1, X_2, X_3) \in \mathcal{Z}(\mathcal{C})(\mathbb{D}^{X_2,X_3} \otimes \mathbb{D}^{X_1,X_2}, \mathbb{D}^{X_1,X_3})$$

of internal natural transformations is nothing but a particular instance of the canonical composition morphism of internal homs [FS21b, Definition 23]. Just like for ordinary natural transformations it amounts to the composition of components (and due to semisimplicity, it is enough to restrict to components labeled by simple objects), according to

$$\underline{\mu}_{\mathrm{ver}} := \bigoplus_{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)}$$



$$\in \mathcal{Z}(\mathcal{C})(\mathbb{D}^{X_2,X_3} \otimes \mathbb{D}^{X_1,X_2}, \mathbb{D}^{X_1,X_3}). \tag{6.1.3}$$

To relate this morphism to a string-net correlator, take the surface to be *a pair of pants* $\Sigma_{\text{p.o.p.}}$ – a sphere with three boundary circles, two of them regarded as incoming and one as outgoing – and fix a marking w without cuts (in the sense of [FS17b, Definition 2.3]) on $\Sigma_{\text{p.o.p.}}$. Given these data, we will specify a worldsheet $\mathcal{S}_w^{\text{ver}} \equiv \mathcal{S}_w^{\text{ver}}(X_1, X_2, X_3)$ with ambient surface $\Sigma_{\text{p.o.p.}}$ as well as a linear isomorphism

$$\varphi_w \colon \mathcal{Z}(\mathcal{C})(\mathbb{D}^{X_2,X_3} \otimes \mathbb{D}^{X_1,X_2}, \mathbb{D}^{X_1,X_3}) \xrightarrow{\cong} \mathrm{SN}_{\mathcal{C}}(\Sigma_{\text{p.o.p.}}, \mathbb{F}_{\partial\Sigma_{\text{p.o.p.}}}(\mathsf{b}_{\mathcal{S}_w^{\text{ver}}})), \tag{6.1.4}$$

in such a way that φ_w maps the vertical composition (6.1.3) to the correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S}_w^{\text{ver}})$ that the string-net construction yields for the worldsheet $\mathcal{S}_w^{\text{ver}}$, i.e.

$$\varphi_w(\underline{\mu}_{\text{ver}}) = \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}_w^{\text{ver}}). \tag{6.1.5}$$

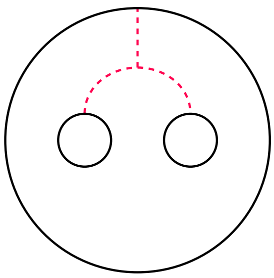
We first present the isomorphism (6.1.4) for a particular marking w. The prescription for any other marking without cuts is then obtained via an action of the mapping class group, as will be described in more detail below. Moreover, for describing the isomorphism (6.1.4) it is convenient to split it up into a composition of two simpler isomorphisms: In a first step, we consider an isomorphism

$$\varphi_w^{\text{can}} \colon \mathcal{Z}(\mathcal{C})(\mathbb{D}^{X_2,X_3} \otimes \mathbb{D}^{X_1,X_2}, \mathbb{D}^{X_1,X_3}) \xrightarrow{\cong} \mathrm{SN}_{\mathcal{C}}(\Sigma_{\text{p.o.p.}}, \mathsf{B}_w^{\text{can}})$$

to the string-net space for a pair of paint with a different boundary value $\mathsf{B}_w^{\text{can}}$, namely one that involves the boundary value $\mathsf{B}_Y^{\text{can}}$, as defined in Example 5.2.1, for $Y \in \mathcal{Z}(\mathcal{C})$ being the object $\mathbb{D}^{X_1,X_2}$, $\mathbb{D}^{X_2,X_3}$ and $\mathbb{D}^{X_1,X_3}$, respectively. The second step consists of implementing the isomorphism (5.2.8) between $\mathsf{B}_{\mathbb{D}^{X,Y}}^{\text{can}}$ and $\mathsf{B}_{X,Y}$ as objects in the Karoubified cylinder category $\mathrm{Cyl}(\mathcal{C}, S^1)$.

For a standard pair of pants $\Sigma_{\text{p.o.p.}}$, which we draw as a disk with two holes, the marking of our choice is

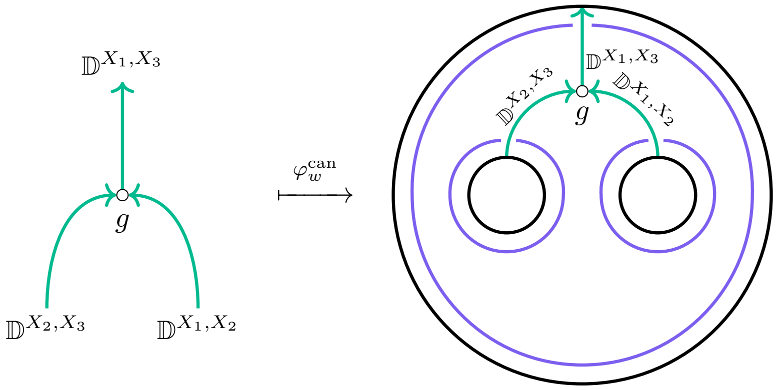
$$w := \tag{6.1.6}$$



The corresponding boundary value $\mathsf{B}_w^{\text{can}}$ is

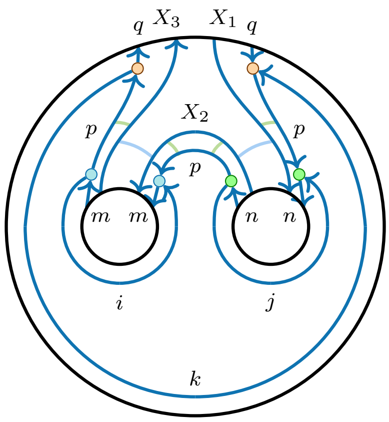
$$\mathsf{B}_w^{\text{can}} = (\overline{\mathsf{B}}_{\mathbb{D}^{X_2,X_3}}^{\text{can}}, \overline{\mathsf{B}}_{\mathbb{D}^{X_2,X_3}}^{\text{can}}, \mathsf{B}_{\mathbb{D}^{X_1,X_3}}^{\text{can}}) \in \mathrm{Cyl}(\mathcal{C}, S^1)^{\times 3} = \mathrm{Cyl}(\mathcal{C}, \partial\Sigma_{\text{p.o.p.}}), \tag{6.1.7}$$

where we denote by $\overline{\mathsf{B}}_Y^{\text{can}}$ the thickened boundary value obtained from $\mathsf{B}_Y^{\text{can}}$ by reversing the orientation of the strand that is labeled by $Y \in \mathcal{Z}(\mathcal{C})$. The isomorphism φ_w^{can} is then defined by

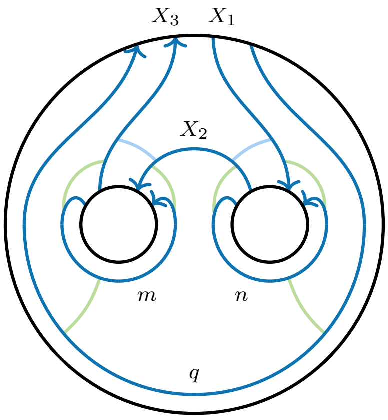


Hereby the vertical composition $\underline{\mu}_{\mathrm{ver}}$ is mapped to the string-net

$$\varphi_w^{\mathrm{can}}(\underline{\mu}_{\mathrm{ver}}) = \bigoplus_{\substack{i,j,k\in\mathcal{I}(\mathcal{C}) \\ m,n,p,q\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)}} \frac{d_i d_j d_k d_m d_n d_p}{D_{\mathcal{C}}^6}$$



$$= \bigoplus_{m,n,q\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} \frac{d_m d_n}{D_{\mathcal{C}}^4} \tag{6.1.8}$$



in $\mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathrm{p.o.p.}}, \mathsf{B}_w^{\mathrm{can}})$. Here the second equality holds as a consequence of the identity (5.1.10) and Corollary 5.1.3. (Also, in the first picture – and likewise in several other pictures later on – we omit, for lack of space, the summation labels of the module morphisms; the appropriate parings are instead indicated by matching colors.) The second step in the construction of φ_w then amounts to setting

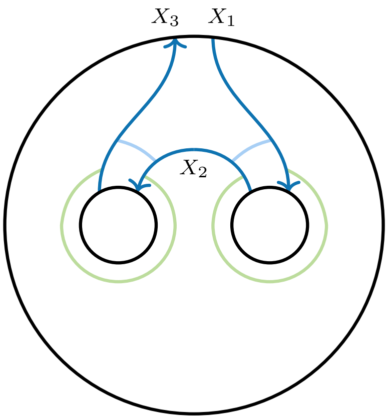
$$\varphi_w(-) \coloneqq e_{X_1,X_3} \circ \varphi_w^{\mathrm{can}} \circ (r_{X_2,X_3} \times r_{X_1,X_2}), \tag{6.1.9}$$

with the morphisms $e_{-,-}$ and $r_{-,-}$ as defined in (5.2.4) and (5.2.5), respectively. Accordingly, the boundary value B_w is given by

$$\mathsf{B}_w = (\overline{B}_{X_2,X_3}, \overline{B}_{X_1,X_2}, B_{X_1,X_3}) \in \mathrm{Cyl}(\mathcal{C}, S^1)^{\times 3} = \mathrm{Cyl}(\mathcal{C}, \partial\Sigma_{\text{p.o.p.}}). \tag{6.1.10}$$

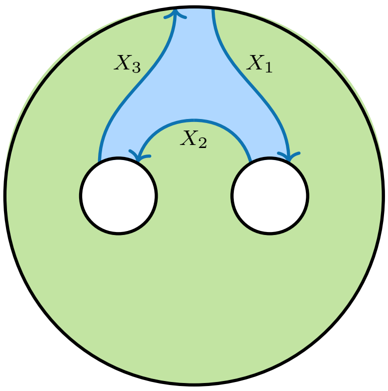
Via (6.1.9), the string-net (6.1.8) gets mapped to

$$\varphi_w(\underline{\mu}_{\text{ver}}) = \qquad \in \mathrm{SN}_{\mathcal{C}}(\Sigma_{\text{p.o.p.}}, \mathsf{B}_w). \tag{6.1.11}$$



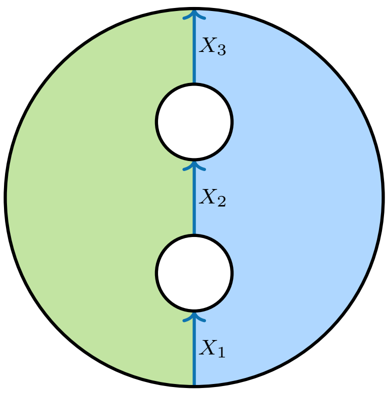
We can now read off that we have indeed achieved to express $\varphi_w(\underline{\mu}_{\text{ver}})$ as a string-net correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S}_w^{\text{ver}})$, namely as the one for the worldsheet

$$\mathcal{S}_w^{\text{ver}} := \tag{6.1.12}$$



that is, a pair of pants with three defect lines labeled by X_1, X_2 and X_3, respectively, which pairwise connect the boundary circles.

Remark 6.1.1. By a homeomorphism, the worldsheet (6.1.12) can be redrawn as follows:



(6.1.13)

This way of representing $\mathcal{S}_w^{\text{ver}}$ slightly obscures its relevance for the algebraic notion of vertical operator product. On the other hand, it clarifies the relation to other approaches: It makes it obvious that one deals with an "operator product along a defect line", and it is precisely what is commonly called the *fundamental worldsheet* for three defect fields on the sphere, see e.g. [FRS05, Section 4.5].

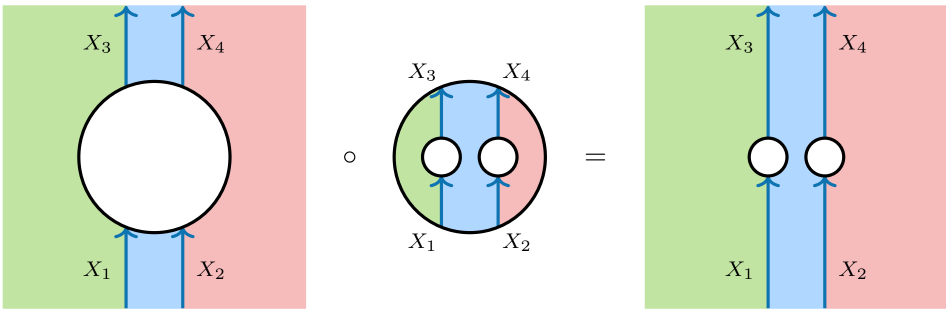
We finally comment on the dependence of the construction on the marking of the surface. A different choice w' of a marking without cuts on the pair of pants leads to a string-net which, in general, differs from (6.1.11). The two worldsheets are related by the unique element $\xi_{w,w'}$ of the mapping class group of Σ that corresponds to (see [FS17b, Section 3.1]) to a move of markings mapping w to w'. In particular, two markings w and w' without cut and with the same end points on the boundary circles give the same worldsheet, and thus the same correlator, if and only if they are isotopic.

As a particular case, any two markings without cuts give the same correlator when each of the three defect fields is actually a bulk field, i.e. when $X_1 = X_2 = X_3 = A = B$.

6.2 Horizontal operator products

We now analyze the horizontal composition of defect fields, in analogy with the study of the vertical composition in Section 6.1. Horizontal composition happens in combination with the *fusion of line defects*, in which two parallel segments of defect lines get replaced by a single one. That such a fusion process is possible is an ingredient of the description of line defects in other approaches to CFT. In the present setting, fusion is algebraically realized as the tensor product over the relevant Frobenius algebra, which is the horizontal composition of the 1-morphisms in $\mathcal{F}r(\mathcal{C})$.

In the same way as done in (6.1.1) for the vertical operator product, the horizontal composition can be expressed as a specific sewing operation, according to



(6.2.1)

When attempting to translate this picture into an expression for composition

$$\mathbb{D}^{X_2,X_4} \otimes \mathbb{D}^{X_1,X_3} \to \mathbb{D}^{X_1 \otimes_B X_2, X_3 \otimes_B X_4}$$

of internal natural transformations, some care is needed. To explain the subtleties involved, it is convenient to recall first how the horizontal composition of *ordinary* natural transformations is described in components: The horizontal products of two natural transformations d^{G_1,G_3} between functors $G_1, G_3 \colon \mathcal{M} \to \mathcal{M}'$ and d^{G_2,G_4} between functors $G_2, G_4 \colon \mathcal{M}' \to \mathcal{M}''$ – a natural transformation from $G_2 \circ G_1$ to $G_4 \circ G_3$, which are functors from $\mathcal{M}$ to $\mathcal{M}''$ – amounts to a suitable composition of their components, i.e. of morphisms $d_M^{G_1,G_3} \in \mathcal{M}'(G_1(M), G_3(M))$ and $d_{M'}^{G_2,G_4} \in \mathcal{M}''(G_2(M'), G_4(M'))$, respectively. In more detail, the composition can be expressed both as

$$G_2 \circ G_1(M) \xrightarrow{G_2(d_M^{G_1,G_3})} G_2 \circ G_3(M) \xrightarrow{d_{G_3(M)}^{G_2,G_4}} G_4 \circ G_3(M), \tag{6.2.2}$$

and as

$$G_2 \circ G_1(M) \xrightarrow{d_{G_1(M)}^{G_2,G_4}} G_2 \circ G_3(M) \xrightarrow{G_4(d_M^{G_1,G_3})} G_4 \circ G_3(M). \tag{6.2.3}$$

Equality of the two composite (6.2.2) and (6.2.3) for all $M \in \mathcal{M}$ holds by naturality of d^{G_2,G_4}. Now, in the case of *internal* natural transformations $\mathbb{D}^{X_1,X_3}$ and $\mathbb{D}^{X_2,X_4}$, the components are internal homs, and accordingly it is appropriate to interpret the dinatural structure morphisms

$$\jmath_M \equiv \jmath_M^{F,F'} : \underline{\mathrm{Nat}}(F, F') \to \underline{\mathrm{Hom}}_{\mathcal{N}}(F(M), F'(M))$$

of the end $\underline{\mathrm{Nat}}(F, F') = \int_{M \in \mathcal{M}} \underline{\mathrm{Hom}}_{\mathcal{N}}(F(M), F'(M))$ as the "projection to components". The analogues of the morphisms (6.2.2) and (6.2.3) are then the composites

$$\begin{aligned}
&\underline{\mathrm{Nat}}(G_2, G_4) \otimes \underline{\mathrm{Nat}}(G_1, G_3) \\
&\quad\xrightarrow{\jmath_{G_3(M)}^{G_2,G_4} \otimes \jmath_M^{G_1,G_3}} \underline{\mathrm{Hom}}_{\mathcal{M}''}(G_2 \circ G_3(M), G_4 \circ G_3(M)) \otimes \underline{\mathrm{Hom}}_{\mathcal{M}'}(G_1(M), G_3(M)) \\
&\quad\xrightarrow{\mathrm{id} \otimes \underline{G_2}} \underline{\mathrm{Hom}}_{\mathcal{M}''}(G_2 \circ G_3(M), G_4 \circ G_3(M)) \otimes \underline{\mathrm{Hom}}_{\mathcal{M}''}(G_2 \circ G_1(M), G_2 \circ G_3(M)) \\
&\quad\xrightarrow{\underline{\mu}} \underline{\mathrm{Hom}}_{\mathcal{M}''}(G_2 \circ G_1(M), G_4 \circ G_3(M))
\end{aligned} \tag{6.2.4}$$

and

$$\begin{aligned}
&\underline{\mathrm{Nat}}(G_1, G_3) \otimes \underline{\mathrm{Nat}}(G_2, G_4) \\
&\quad\xrightarrow{\jmath_M^{G_1,G_3} \otimes \jmath_{G_1(M)}^{G_2,G_4}} \underline{\mathrm{Hom}}_{\mathcal{M}'}(G_1(M), G_3(M)) \otimes \underline{\mathrm{Hom}}_{\mathcal{M}''}(G_2 \circ G_1(M), G_4 \circ G_1(M)) \\
&\quad\xrightarrow{\underline{G_4} \otimes \mathrm{id}} \underline{\mathrm{Hom}}_{\mathcal{M}''}(G_4 \circ G_1(M), G_4 \circ G_3(M)) \otimes \underline{\mathrm{Hom}}_{\mathcal{M}''}(G_2 \circ G_1(M), G_4 \circ G_1(M)) \\
&\quad\xrightarrow{\underline{\mu}} \underline{\mathrm{Hom}}_{\mathcal{M}''}(G_2 \circ G_1(M), G_4 \circ G_3(M))
\end{aligned} \tag{6.2.5}$$

for $M \in \mathcal{M}$, respectively. Here $\underline{\mu}$ is the standard composition of internal homs, while the morphisms of the type

$$\underline{G}\colon \underline{\mathrm{Hom}}_{\mathcal{M}}(M, M') \to \underline{\mathrm{Hom}}_{\mathcal{N}}(G(M), G(M'))$$

is defined via applying the Yoneda lemma to the composite

$$\begin{aligned}
&\mathcal{C}(C, \underline{\mathrm{Hom}}_{\mathcal{M}}(M, M')) \xrightarrow{\cong} \mathcal{M}(C \triangleright M, M') \\
&\qquad\xrightarrow{G} \mathcal{M}(G(C \triangleright M), G(M')) \xrightarrow{\cong} \mathcal{M}(C \triangleright G(M), G(M')) \\
&\qquad\qquad\xrightarrow{\cong} \mathcal{C}(C, \underline{\mathrm{Hom}}_{\mathcal{N}}(G(M), G(M')))
\end{aligned}$$

for every $C \in \mathcal{C}$.

It is straightforward to see that both of the families (6.2.4) and (6.2.5) are dinatural in $M \in \mathcal{M}$. Owing to the universal property of $\underline{\mathrm{Nat}}$ as an end, they thus factorize to unique morphisms

$$\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}} \colon \underline{\mathrm{Nat}}(G_2, G_4) \otimes \underline{\mathrm{Nat}}(G_1, G_3) \to \underline{\mathrm{Nat}}(G_2 \circ G_1, G_4 \circ G_3) \tag{6.2.6}$$

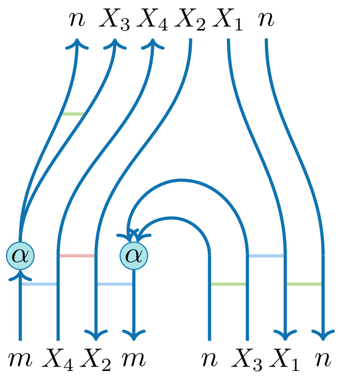
and

$$\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}} \colon \underline{\mathrm{Nat}}(G_1, G_3) \otimes \underline{\mathrm{Nat}}(G_2, G_4) \to \underline{\mathrm{Nat}}(G_2 \circ G_1, G_4 \circ G_3) \tag{6.2.7}$$

respectively.

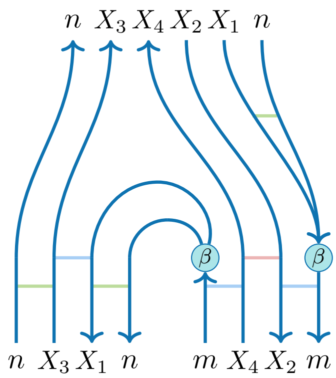
We call the morphisms $\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}$ and $\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}}$ the *left* and *right horizontal operator product*. The reason for this choice of terminology is the description of (6.2.6) and (6.2.7) in terms of string diagrams, in which the basis morphisms that are summed over are located in the left and right half of the graph, respectively: we have

$$\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}} = \bigoplus_{\substack{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}B) \\ n \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)}} d_m \tag{6.2.8}$$



and

$$\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}} = \bigoplus_{\substack{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}B) \\ n \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)}} d_m \tag{6.2.9}$$



Let us now express these two compositions through string-nets. Recall from Section 6.1 that this requires the specification of auxiliary data beyond the structure of a worldsheet (and its ambient surface), concretely the choice of a marking without cuts. Note that these auxiliary data do not appear in the purely algebraic treatment of horizontal composition that is given in [FS21b]. (Also, in our discussion of the horizontal composition we partly deviate from the exposition in [FS21b].)

In the same vein as done for the vertical composition $\underline{\mu}_{\mathrm{ver}}$, we will now determine the image of the morphism $\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}} \in \mathcal{Z}(\mathcal{C})(\mathbb{D}^{X_2,X_4} \otimes \mathbb{D}^{X_1,X_3}, \mathbb{D}^{X_1 \otimes_B X_2, X_3 \otimes_B X_4})$ under the composite map

$$\varphi_w \colon \mathcal{Z}(\mathcal{C})(\mathbb{D}^{X_2,X_4} \otimes \mathbb{D}^{X_1,X_3}, \mathbb{D}^{X_1 \otimes_B X_2, X_3 \otimes_B X_4}) \xrightarrow{\varphi_w^{\mathrm{can}}} \mathrm{SN}_{\mathcal{C}}(\Sigma_{\mathrm{p.o.p.}}, \mathsf{B}_w^{\mathrm{can}})$$

$$\xrightarrow{e_{X_1\otimes_B X_2, X_3\otimes_B X_4}\circ(-)\circ(r_{X_2,X_4}\times r_{X_1,X_3})} \mathrm{SN}_{\mathcal{C}}(\Sigma_{\text{p.o.p.}}, \mathsf{B}_w), \tag{6.2.10}$$

where w is the marking (6.1.6) on the pair of pants $\Sigma_{\text{p.o.p.}}$, the isomorphism φ_w^{can} is defined analogously as in (6.1.9), and the boundary values are

$$\mathsf{B}_w^{\text{can}} = \mathsf{B}_w^{\text{can}} = (\overline{\mathsf{B}}^{\text{can}}_{\mathbb{D}^{X_2,X_4}}, \overline{\mathsf{B}}^{\text{can}}_{\mathbb{D}^{X_1,X_3}}, \mathsf{B}^{\text{can}}_{\mathbb{D}^{X_1\otimes_B X_2, X_3\otimes_B X_4}})$$

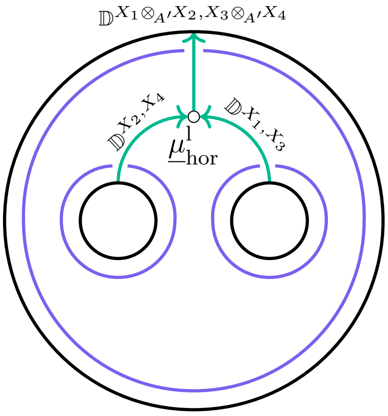
and

$$\mathsf{B}_w = (\overline{B}_{X_2,X_4}, \overline{B}_{X_1,X_3}, B_{X_1\otimes_B X_2, X_3\otimes_B X_4})$$

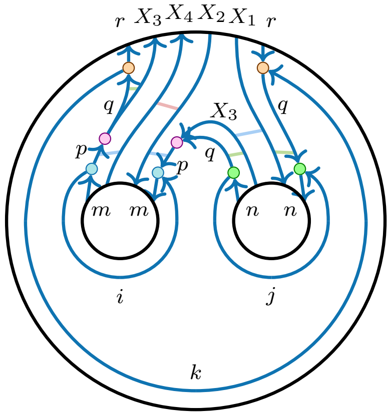
analogously as in (6.1.7) and (6.1.10).

By direct calculation we obtain

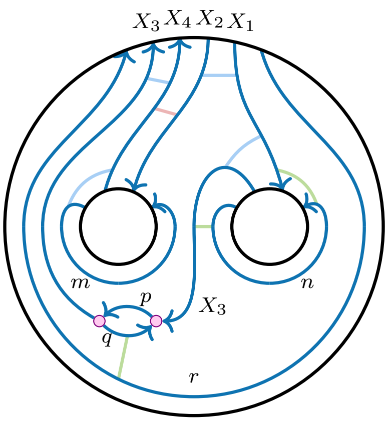
$$\varphi_w^{\text{can}}(\underline{\mu}^{1}_{\text{hor}}) =$$



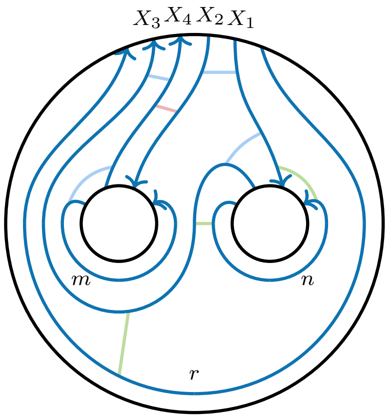
$$= \sum_{\substack{i,j,k\in\mathcal{I}(\mathcal{C})\\ m,p\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}B)\\ n,q,r\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}C)}} \frac{d_i d_j d_k d_m d_n d_p d_q}{D_{\mathcal{C}}^6}$$



$$= \sum_{\substack{m,p\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}B)\\ n,q,r\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}C)}} \frac{d_m d_n d_p d_q}{D_{\mathcal{C}}^6}$$



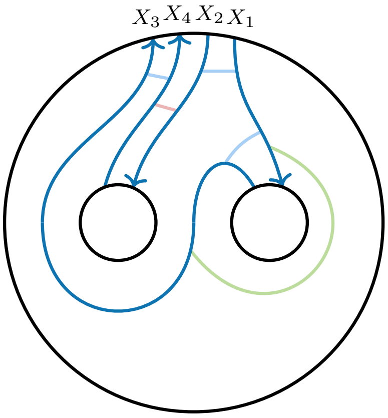
$$= \sum_{\substack{m\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}B)\\ n,r\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}C)}} \frac{d_m d_n}{D_{\mathcal{C}}^4}$$



This implies that $\varphi_w(\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}})$ gives the worldsheet for the left horizontal composition as follows:

$$\varphi_w(\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}) = e_{X_1\otimes_B X_2, X_3\otimes_B X_4} \circ \varphi_w^{\mathrm{can}}(\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}) \circ (r_{X_2,X_4} \times r_{X_1,X_3})$$

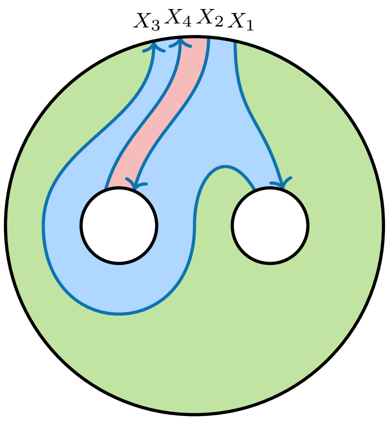
$$=$$



$$= \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}^{\mathrm{l}}_{\mathrm{hor}}) \tag{6.2.11}$$

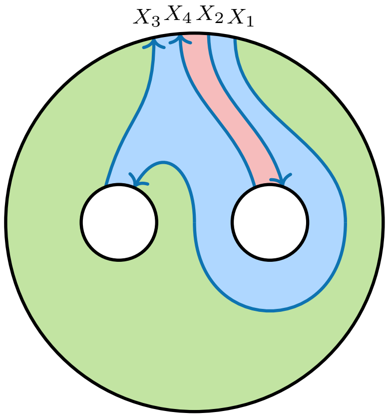
with

$$\mathcal{S}^{\mathrm{l}}_{\mathrm{hor}} = \tag{6.2.12}$$



Likewise, the image of $\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}}$ under φ_w gives the worldsheet for the right horizontal composition:

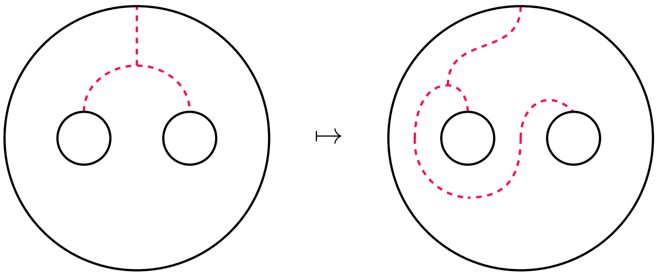
$$\varphi_w(\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}}) = \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}^{\mathrm{r}}_{\mathrm{hor}}) \qquad \text{with} \qquad \mathcal{S}^{\mathrm{r}}_{\mathrm{hor}} = \tag{6.2.13}$$



Remark 6.2.1. (i) Up to isotopy we have

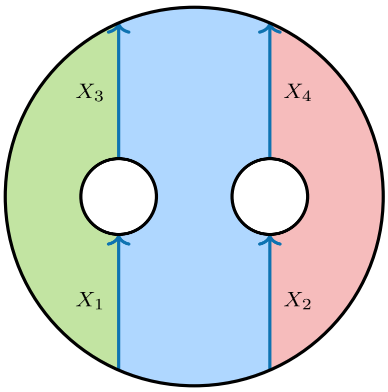
$$\mathcal{S}^{\mathrm{l}}_{\mathrm{hor}} = \beta(\mathcal{S}^{\mathrm{r}}_{\mathrm{hor}}), \tag{6.2.14}$$

where β is the action of the *braid move*, which (when expressed in terms of markings) acts as



(6.2.15)

As we will observe in Remark 7.3.2 below, this implies that the left and right horizontal compositions $\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}$ and $\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}}$ merely differ by a half-braiding.
(ii) By applying suitable homeomorphism, both worldsheets $\mathcal{S}^{\mathrm{l}}_{\mathrm{hor}}$ and $\mathcal{S}^{\mathrm{r}}_{\mathrm{hor}}$ may be redrawn as

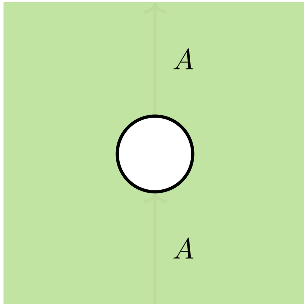


(6.2.16)

which is the form familiar from the the sewing operation shown in picture (6.2.1).

6.3 Bulk algebras

The most basic type of field insertion in the bulk is the one obtained for sewing circle $c = c^A_{\mathrm{bulk}}$ where $A \in \mathcal{F}r(\mathcal{C})$ is a simple special symmetric Frobenius algebra in $\mathcal{C}$, whose local neighborhood in the worldsheet is depicted by:



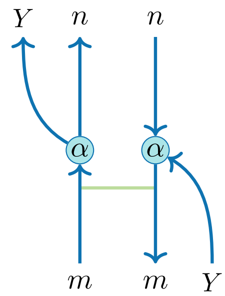
Recall that the associated field content is given by the *full center* of A (a canonically isomorphic form is given in [FFRS06b] whereas an axiomatic definition of the concept is given in [Dav10]):

$$
\begin{aligned}
Z(A) = \mathbb{D}^{A,A} &= \underline{\mathrm{Nat}}(\mathrm{id}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}, \mathrm{id}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}) \\
&= \int_{M \in \mathrm{mod}^{\mathcal{C}}\text{-}A} \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M, M) \in \mathcal{Z}(\mathcal{C}).
\end{aligned} \tag{6.3.1}
$$

In the semisimple case of our interest, we thus have

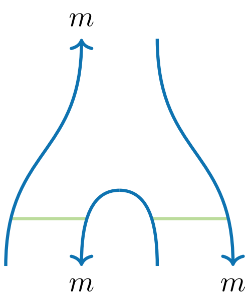
$$
Z(A) = \mathbb{D}^{A,A} = \Big(\bigoplus_{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} m \otimes_A m^\vee, \gamma_{Z(A)} \Big) \in \mathcal{Z}(\mathcal{C}). \tag{6.3.2}
$$

The half-braiding $\gamma_{Z(A)}$ is the one already described in (6.1.2), which now specializes to

$$
\gamma_{Z(A);Y} = \bigoplus_{m,n \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} d_m
$$


$$
\text{for } Y \in \mathcal{C}. \tag{6.3.3}
$$

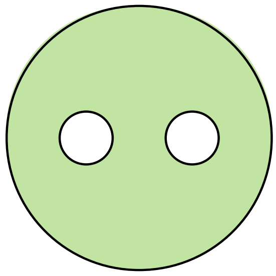
As already mentioned, the special case (6.3.1) of defect field contents is known as the field content of *bulk fields* of a CFT. For bulk fields, the vertical product $\underline{\mu}_{\mathrm{ver}}(A, A, A)$ as defined in (6.1.3) and the left and right horizontal products $\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}(A, A, A)$ and $\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}}(A, A, A)$ as defined in (6.2.8) and in (6.2.9), respectively, all coincide (up to transformation by a canonical isomorphism) with the multiplication

$$
\mu_{Z(A)} = \bigoplus_{m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} \tag{6.3.4}
$$


By the same strategy that we already pursued in the more general case of defect fields we can, after fixing a marking w without cuts on the pair of pants $\Sigma_{\text{p.o.p.}}$, map the product (6.3.4) to the string-net correlator on the worldsheet $\mathcal{S}$: as a special case of (6.1.5) we get

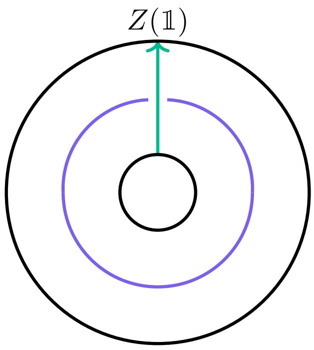
$$\varphi_w(\mu_{Z(A)}) = \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \in \mathrm{SN}_{\mathcal{C}}(\Sigma_{\text{p.o.p.}}, \mathsf{B}_w^A)$$

with

$\mathcal{S} \equiv \mathcal{S}_w :=$ 

Note that the so obtained worldsheet $\mathcal{S}$ is nothing but any of the worldsheets $\mathcal{S}_w^{\text{ver}}$, $\mathcal{S}_{\text{hor}}^{\text{l}}$ and $\mathcal{S}_{\text{hor}}^{\text{r}}$ as displayed in (6.1.12), (6.2.12) and (6.2.13), respectively, each specialized to the case that the 2-cells are both labeled by the same Frobenius algebra and the three defect lines are all transparently labeled. Owing to their transparency, the defect lines can be omitted from the worldsheet, and the boundary value B_w^A – a choice of points on $S^1 \sqcup S^1 \sqcup S^1$ at which the transparent defect lines start or end – is immaterial. For the same reason, the choice of marking w is actually irrelevant as well. It follows that $\mu_{Z(A)}$ endows the object $Z(A) \in \mathcal{Z}(\mathcal{C})$ with the structure of a commutative special symmetric Frobenius algebra, and that the correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ is invariant under the total mapping class group $\mathrm{Map}(\Sigma_{\text{p.o.p.}})$ of the pair of pants. Further, invoking [KR09, Theorem 3.4], it follows that $Z(A)$ is even a *modular* Frobenius algebra in the modular fusion category $\mathcal{Z}(\mathcal{C})$ in the sense of [FS17b].

Remark 6.3.1. The Frobenius algebra $Z(\mathbb{1})$ that is obtained when A is the monoidal unit $\mathbb{1}$ of $\mathcal{C}$ is the *Cardy bulk algebra* considered in [SY21]. It should be noted that in [SY21] a less natural boundary value

$\mathsf{B}_{Z(\mathbb{1})}^{\text{can}} :=$ 

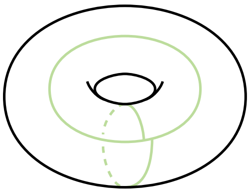
is considered. This choice results in more complicated graphs than the ones appearing above, having additional edges around the boundary circles. However, the boundary values B_w^A and $\mathsf{B}_{Z(\mathbb{1})}^{\text{can}}$ are in fact isomorphic, and as a consequence the present description and the one used in [SY21] indeed yield the same correlator for the pair of paints that is colored with the trivial algebra $\mathbb{1} \in \mathcal{C}$.

6.4 Torus partition function

Next we consider the correlator $\mathrm{Cor}_{\mathcal{C}}(T)$ for a torus $T \equiv T_A$ without sewing boundaries and without non-transparent defect lines and colored with a simple special symmetric Frobenius algebra $A \in \mathcal{F}r(\mathcal{C})$. This correlator, commonly called the *torus partition function*, is of much interest. For instance, in rational CFT it allows one to directly read off the decomposition of the bulk field object $Z(A) = \mathbb{D}^{A,A}$ into simple objects of $\mathcal{Z}(\mathcal{C})$. Also, modular invariance of $\mathrm{Cor}_{\mathcal{C}}(T)$ is an important constraint on the consistency of a full conformal field theory, to the extent that often it has even been assumed, erroneously, to be even a sufficient condition for consistency.

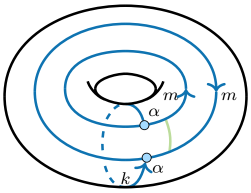
The string-net form of the torus partition follows immediately from Definition 5.3.1: we have

$$\mathrm{Cor}_{\mathcal{C}}(T) = \qquad (6.4.1)$$

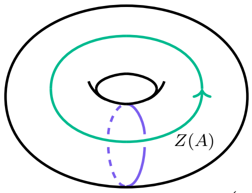


with the green lines labeled with the Frobenius algebra $A \in \mathcal{F}r(\mathcal{C})$. Using the expression (6.3.2) for the bulk field content $Z(A) = \mathbb{D}^{A,A}$ together with Corollary 5.1.3 and the identity (5.1.10), this can be rewritten as

$$\mathrm{Cor}_{\mathcal{C}}(T) = \bigoplus_{\substack{k \in \mathcal{I}(\mathcal{C}) \\ m \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)}} \frac{d_k d_m}{D_{\mathcal{C}}^2}$$



$$=$$



(6.4.2)

Moreover, since $\mathcal{C}$ is semisimple the object $Z(A) \in \mathcal{Z}(\mathcal{C}) \simeq \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$ can be decomposed into a direct sum of simple objects of $\mathcal{Z}(\mathcal{C})$ as

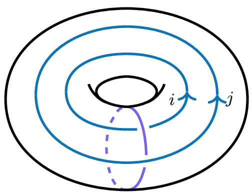
$$Z(A) = \bigoplus_{i,j \in \mathcal{I}(\mathcal{C})} Z(A)_{i,j} \otimes_{\Bbbk} \Xi_{\mathcal{C}}(i \boxtimes j).$$

Recall that $\Xi_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ is the canonical braided equivalence, while $Z(A)_{i,j}$ are the vector spaces

$$\begin{aligned} Z(A)_{i,j} &:= \mathcal{Z}(\mathcal{C})(\Xi_{\mathcal{C}}(i \boxtimes j), \underline{\mathrm{Nat}}(G^A, G^A)) \\ &\cong [\mathrm{mod}^{\mathcal{C}}\text{-}A, \mathrm{mod}^{\mathcal{C}}\text{-}B](z \triangleright G^A, G^A) \\ &\cong A\text{-}\mathrm{mod}^{\mathcal{C}}\text{-}A(i \otimes^- A \otimes^+ j, A). \end{aligned}$$

This in particular reproduces the decomposition rule for bulk field contents. For the torus partition function we thus get

$$\mathrm{Cor}_{\mathcal{C}}(T) = \sum_{i,j \in \mathcal{I}(\mathcal{C})} z(A)_{i,j} \qquad (6.4.3)$$

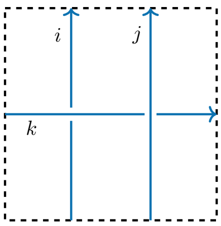


with $z(A)_{i,j} := \dim_{\Bbbk}(Z(A)_{i,j})$.

Remark 6.4.1. The full Frobenius graph in the string-net correlator (6.4.1) is the same as the one appearing in the TFT construction of $\mathrm{Cor}_{\mathcal{C}}(T)$. In that case, the torus in which the graph is embedded is the subset $\{0\} \times T$ of the 3-manifold $[-1, 1] \times T$ (see [FRS02, Equation (5.24)]). An analogous relationship between the two constructions holds for all other partition functions, i.e. for the correlator of any worldsheet that neither has field insertions nor contains any physical boundaries or non-trivial defect lines.

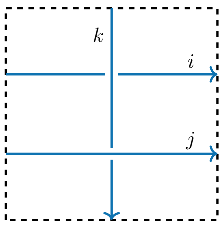
The expression (6.4.1) for $\mathrm{Cor}_{\mathcal{C}}(T)$ shows manifestly that the torus partition function is invariant under the geometric action of the modular group $\mathrm{Map}(T) \cong \mathrm{PSL}(2, \mathbb{Z})$ of the torus. The result (6.4.3) allows us to translate this geometric invariance to the algebraic modular invariance of the $|\mathcal{I}(\mathcal{C})| \times |\mathcal{I}(\mathcal{C})|$-matrix $z(A) = (z(A)_{i,j})$. To see this, first note that the set $\{G_{i,j} | i, j \in \mathcal{I}(\mathcal{C})\}$ with

$$G_{i,j} := \sum_{k \in \mathcal{I}(\mathcal{C})} \frac{d_k}{D_{\mathcal{C}}^2} \tag{6.4.4}$$



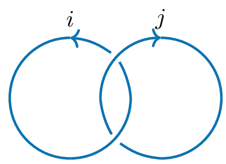
is a basis of the string-net space $\mathrm{SN}_{\mathcal{C}}(T)$ [Run20, Proposition 4.8]. It is therefore sufficient to show that the image

$$S(G_{i,j}) = \sum_{k \in \mathcal{I}(\mathcal{C})} \frac{d_k}{D_{\mathcal{C}}^2}$$



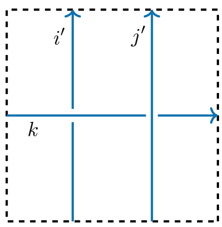
of $G_{i,j}$ under the modular S-transformation satisfies

$$S(G_{i,j}) = \sum_{i',j' \in \mathcal{I}(\mathcal{C})} \frac{s_{i^\vee,i'} s_{j,j'}}{D_{\mathcal{C}}^2} G_{i',j'} \qquad \text{with} \qquad s_{i,j} := \tag{6.4.5}$$

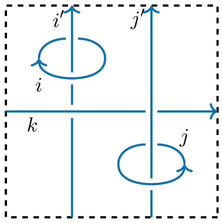


The validity of (6.4.5) is established by the following chain of equalities:

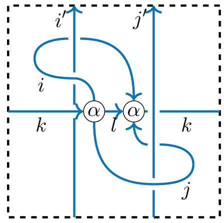
$$\sum_{i',j' \in \mathcal{I}(\mathcal{C})} \frac{s_{i^\vee,i'} s_{j,j'}}{D_{\mathcal{C}}^2} G_{i',j'} = \sum_{i',j',k \in \mathcal{I}(\mathcal{C})} \frac{s_{i^\vee,i'} s_{j,j'} d_k}{D_{\mathcal{C}}^4}$$



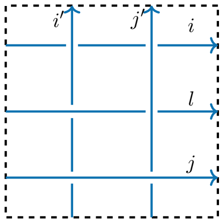
$$= \sum_{i',j',k\in\mathcal{I}(\mathcal{C})} \frac{d_{i'}d_{j'}d_k}{D_{\mathcal{C}}^4}$$



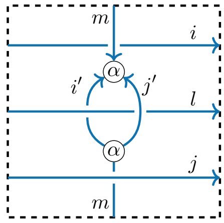
$$= \sum_{i',j',k,l\in\mathcal{I}(\mathcal{C})} \frac{d_{i'}d_{j'}d_k d_l}{D_{\mathcal{C}}^4}$$



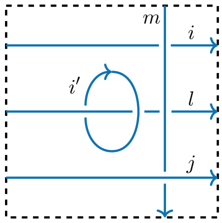
$$= \sum_{i',j',l\in\mathcal{I}(\mathcal{C})} \frac{d_{i'}d_{j'}d_l}{D_{\mathcal{C}}^4}$$



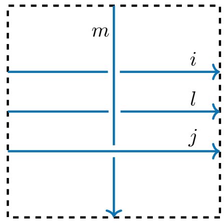
$$= \sum_{i',j',l,m\in\mathcal{I}(\mathcal{C})} \frac{d_{i'}d_{j'}d_l d_m}{D_{\mathcal{C}}^4}$$



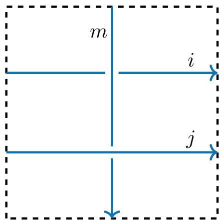
$$= \sum_{i',l,m\in\mathcal{I}(\mathcal{C})} \frac{d_{i'}d_l d_m}{D_{\mathcal{C}}^4}$$



$$= \sum_{l,m\in\mathcal{I}(\mathcal{C})} \frac{d_l d_m}{D_{\mathcal{C}}^2}\delta_{l,0}$$



$$= \sum_{m\in\mathcal{I}(\mathcal{C})} \frac{d_m}{D_{\mathcal{C}}^2}$$



$$= S(G_{ij}). \qquad (6.4.6)$$

Recalling the expansion (6.4.3), it follows that

$$\begin{aligned} S(\mathrm{SN}_{\mathcal{C}}(T)) &= \sum_{i,j\in\mathcal{I}(\mathcal{C})} z(A)_{i,j} S(G_{i,j}) \\ &= \frac{1}{D_{\mathcal{C}}^2} \sum_{i,i',j,j'\in\mathcal{I}(\mathcal{C})} s_{i',i^\vee} z(A)_{i,j} s_{j,j'} G_{i',j'} = \sum_{i',j'\in\mathcal{I}(\mathcal{C})} (S^{-1}z(A)S)_{i',j'} G_{i',j'} \end{aligned}$$

with $S_{i,j} := s_{i,j}/D_{\mathcal{C}}$. Hence indeed we have the equivalence of the two invariance (geometric v. algebraic):

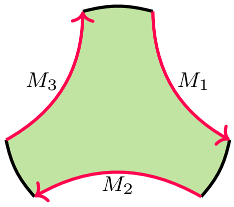
$$S(\mathrm{SN}_{\mathcal{C}}(T)) = \mathrm{SN}_{\mathcal{C}}(T) \quad \Longleftrightarrow \quad [S, z(A)] = 0. \qquad (6.4.7)$$

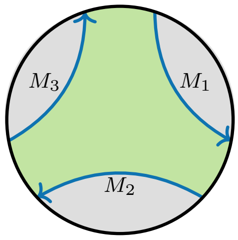
Similarly, invariance of the correlator $\mathrm{Cor}_{\mathcal{C}}(T)$ under the modular T-transformation is equivalent to $[T, z(A)] = 0$ which, in turn, is equivalent to the statement that the object $Z(A) \in \mathcal{Z}(\mathcal{C})$ has trivial twist.

Remark 6.4.2. A calculation similar to the one in (6.4.6) has been presented in [Har21]. The consideration therein are within the framework of the tube category of the modular fusion category $\mathcal{C}$ and of its category of representations, which correspond to the cylinder category $\mathrm{Cyl}^\circ(\mathcal{C}, S^1)$ and its idempotent completion $\mathrm{Cyl}(\mathcal{C}, S^1)$, respectively.

6.5 Boundary operator product

Among the correlators involving only boundary field contents, the most basic one is the correlator for a disk with three sewing intervals, which describes the operator product of boundary insertions. The worldsheet for this correlator is a disk D with a single 2-cell, labeled by a simple special symmetric Frobenius algebra A, and with three physical boundary segments labeled by right A-modules M_1, M_2 and M_3. We denote this worldsheet by D_{M_1,M_2,M_3}, and the corresponding complemented worldsheet by $\widetilde{D} \equiv \widetilde{D}_{M_1,M_2,M_3}$. In pictures,

$D_{M_1,M_2,M_3} =$ 

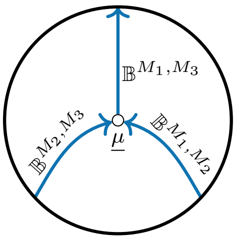
and $\widetilde{D}_{M_1,M_2,M_3} =$ 

(6.5.1)

Analogously as we did in the case of defect fields, we want to obtain the complemented worldsheet $\widetilde{D}$ from a suitable string-net on the disk D that encodes the algebraic information about the *boundary operator product*. We choose conventions such that two of the boundary fields, say $\mathbb{B}^{M_1,M_2}$ and $\mathbb{B}^{M_2,M_3}$ are incoming, while the third is outgoing and is thus given by $\mathbb{B}^{M_1,M_3}$. Recall that these field contents are internal homs, $\mathbb{B}^{M_i,M_j} = \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M_i, M_j)$. The natural candidate for their operator product is thus the canonical composition

$$\underline{\mu}(M_1, M_2, M_3) \in \mathcal{C}(\underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M_2, M_3) \otimes \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M_1, M_2), \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M_1, M_3))$$

of internal homs. Accordingly we consider the string-net represented by the graph

$\mathit{\Gamma}_{D;M_1,M_2,M_3}$: 

$\in \mathrm{SN}_{\mathcal{C}}(D, \mathsf{B}_{M_1,M_2,M_3})$. (6.5.2)

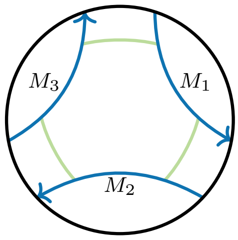
where the boundary value B_{M_1,M_2,M_3} is the circle $S^1 = \partial D$ with three points labeled by $(\mathbb{B}^{M_1,M_2})^\vee$, $(\mathbb{B}^{M_2,M_3})^\vee$ and $\mathbb{B}^{M_1,M_3}$. Note that (using Example 4.1.3)

$$\begin{aligned}\mathrm{SN}_{\mathcal{C}}(D, \mathsf{B}_{M_1,M_2,M_3}) &= \mathrm{SN}^{\circ}_{\mathcal{C}}(D, \mathsf{B}^{\circ}_{M_1,M_2,M_3}) \\ &\cong\in \mathcal{C}(\underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M_2, M_3) \otimes \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M_1, M_2), \underline{\mathrm{Hom}}_{\mathrm{mod}^{\mathcal{C}}\text{-}A}(M_1, M_3)),\end{aligned}$$

where we used the fact that the choice of in- and outgoing boundary field contents endows the corolla with a polarization.

Now since $\mathcal{C}$ is pivotal, we have $\mathbb{B}^{M_i,M_j} = M_j \otimes_A M_i$, and we then identify the boundary value B_{M_1,M_2,M_3} with an isomorphic one so as to rewrite the representing graph (6.5.2) as

$$\Gamma'_{D;M_1,M_2,M_3} = \qquad (6.5.3)$$



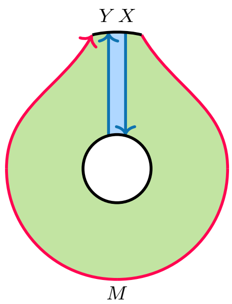
We can thus read off that we indeed have

$$[\Gamma'_{D;M_1,M_2,M_3}] = \mathrm{Cor}_{\mathcal{C}}(D_{M_1,M_2,M_3}).$$

6.6 Bulk-boundary operator product

Among the correlators involving both bulk and boundary field contents, the most basic one is the correlator for one bulk and one boundary insertion on a disk. This correlator encodes a connection between bulk and boundary field contents, which is called the *bulk-boundary operator product.* Since in our approach defect fields can be treated in much the same way as bulk fields, we consider here the more general situation of one defect and one boundary insertion on a disk. The corresponding worldsheet (with sewing boundaries), which we denote by $C_{X,Y;M}$, is the cylinder

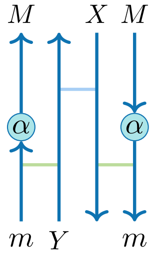
$$C_{X,Y;M} = \qquad (6.6.1)$$



where the two 2-cells in green and in blue are labeled by simple special symmetric Frobenius algebra $A, B \in \mathcal{F}r(\mathcal{C})$, respectively, the two line defects labeled by A-B-bimodules $X, Y \in A\text{-mod}^{\mathcal{C}}\text{-}B$ and the physical boundary (in red) labeled by a right A-module $M \in \mathrm{mod}^{\mathcal{C}}\text{-}A$.

The crucial algebraic datum describing the connection between bulk and boundary in this situation is the component

$$\jmath_M = \bigoplus_{m\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} d_m$$



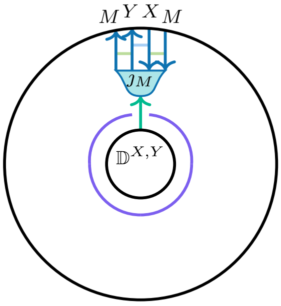
$$\in \mathcal{C}\big(\int_{M'\in\mathrm{mod}^{\mathcal{C}}\text{-}A} M' \otimes_A Y \otimes_B X^\vee \otimes_A M'^\vee, M \otimes_A Y \otimes_B X^\vee \otimes_A M^\vee\big) \qquad (6.6.2)$$

at the object $M \otimes_A Y \otimes_B X^\vee \otimes_A M^\vee = \mathbb{B}^{M\otimes_A X, M\otimes_A Y}$ of the structure morphism $\jmath$ of the end

$$\int_{M'\in \mathrm{mod}^{\mathcal{C}}\text{-}A} M' \otimes_A Y \otimes_B X^\vee \otimes_A M'^\vee = U(\mathbb{D}^{X,Y}) \in \mathcal{C}.$$

Accordingly we consider the string-net (represented by the graph)

$$\mathit{\Gamma}_{D;X,Y;M} :=$$



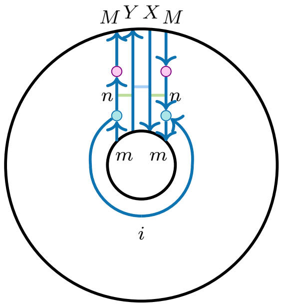
$$\in \mathrm{SN}_{\mathcal{C}}(S^1 \times I, \overline{\mathsf{B}}^{\mathrm{can}}_{\mathbb{D}^{X,Y}} \times \mathsf{B}_{X,Y;M}).$$

Here the thickened boundary value $\mathsf{B}^{\mathrm{can}}_{\mathbb{D}^{X,Y}}$ is the one defined according to (5.2.3), while $\mathsf{B}_{X,Y;M}$ in the first place consists of four points on S^1 labeled by $M, Y, X^\vee$ and $M^\vee$, respectively, together with an idempotent consisting of three relative tensor product idempotents (see (5.1.4)) combined, but is in an obvious manner isomorphic to the boundary value that consists of a single point on S^1 labeled by $\mathbb{B}^{M\otimes_A X, M\otimes_A Y}$ together with the identity cylinder. Using (4.7.6) and Theorem 4.7.1, we note that

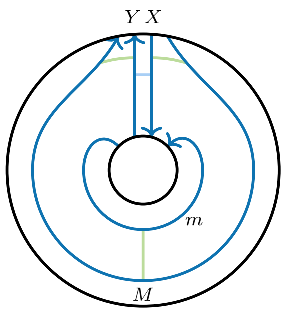
$$\begin{aligned}\mathrm{SN}_{\mathcal{C}}(S^1 \times I, \overline{\mathsf{B}}^{\mathrm{can}}_{\mathbb{D}^{X,Y}} \times \mathsf{B}_{X,Y;M}) &\cong \mathcal{Z}(\mathcal{C})(\mathbb{D}^{X,Y}, L(\mathbb{B}^{M\otimes_A X, M\otimes_A Y})) \\ &\cong \mathcal{C}(U(\mathbb{D}^{X,Y}), \mathbb{B}^{M\otimes_A X, M\otimes_A Y}),\end{aligned}$$

where we used that, $\mathcal{C}$ being finitely semisimple, L is also right adjoint to the forgetful functor $U\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. Inserting the explicit form (6.6.2) and the identities (5.1.9) and (5.1.10), we can write

$$[\mathit{\Gamma}_{D;X,Y;M}] = \sum_{\substack{i\in\mathcal{I}(\mathcal{C})\\ m,n\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)}} \frac{d_i d_m d_n}{D_{\mathcal{C}}^2}$$



$$= \sum_{m\in\mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} \frac{d_m}{D_{\mathcal{C}}^2}$$



Now recall from Example 5.2.2 the idempotent $r_{X,Y}$ defined in (5.2.5), which furnishes an isomorphism $\mathsf{B}_{X,Y} \xrightarrow{\cong} \mathsf{B}^{\mathrm{can}}_{\mathbb{D}^{X,Y}}$ in the Karoubified cylinder category $\mathrm{Cyl}(\mathcal{C}, S^1)$. Pre-composing with $r_{X,Y}$ yields the string-net

$$[\Gamma_{D;X,Y;M}] \circ r_{X,Y} = \mathrm{Cor}_{\mathcal{C}}(C_{X,Y;M}).$$

In short, the component (6.6.2) of the structure morphism of the end $U(\mathbb{D}^{X,Y})$ indeed reproduces the string-net correlator for the worldsheet $C_{X,Y;M}$. In particular, specializing to the case that $A = B \in \mathcal{F}r(\mathcal{C})$ and that $X = A = Y \in A\text{-mod}^{\mathcal{C}}\text{-}B$ as bimodules, we obtain the worldsheet describing the bulk-boundary operator product.

7 Internal Eckmann-Hilton relation

The exchange law for ordinary natural transformation turns out to have an analogue for internal natural transformations. To obtain this analogue, which we call the *internal Eckmann-Hilton relation*, we introduce three braided colored operads. The string-net correlator defined in Section 5.3 provides a morphism from the braided colored operad of worldsheets to the one of string-nets. We compose this morphism with a morphism from the string-net operad to a certain (braided) endomorphism operad. We can then derive the internal Eckmann-Hilton relation by invoking compatibility of the so obtained composite morphism with operadic composition. The result illustrates the utility of string-nets for understanding algebra in braided tensor categories.

7.1 An internalized Eckmann-Hilton argument

Recall the diagram (2.3.7) which gives a Poincaré-dual view of (two-pronged) defect fields. In this section we consider multiple operator products of defect fields for which the description in terms of (2.3.7) looks as follows (for better readability we suppress the labels of the defect fields):

$$\begin{tikzcd}[column sep=large]
\mathrm{mod}^{\mathcal{C}}\text{-}A \arrow[r, bend left=60, "X_5"] \arrow[r, "X_3"] \arrow[r, bend right=60, "X_1"'] \arrow[r, phantom, shift left=4, "\mathbb{D}\Uparrow"] \arrow[r, phantom, shift right=4, "\mathbb{D}\Uparrow"] & \mathrm{mod}^{\mathcal{C}}\text{-}B \arrow[r, bend left=60, "X_6"] \arrow[r, "X_4"] \arrow[r, bend right=60, "X_2"'] \arrow[r, phantom, shift left=4, "\mathbb{D}\Uparrow"] \arrow[r, phantom, shift right=4, "\mathbb{D}\Uparrow"] & \mathrm{mod}^{\mathcal{C}}\text{-}C
\end{tikzcd}$$

with $A, B, C \in \mathcal{F}r(\mathcal{C})$ simple special symmetric Frobenius algebras.

Now recall that for ordinary natural transformations (and, more generally, for 2-morphisms in a bicategory), the horizontal and vertical compositions satisfy the exchange law

$$\mu_{\mathrm{ver}} \circ (\mu_{\mathrm{hor}} \otimes \mu_{\mathrm{hor}}) = \mu_{\mathrm{hor}}(\mu_{\mathrm{ver}} \otimes \mu_{\mathrm{ver}}). \tag{7.1.1}$$

It should be appreciated that this is a statement about elements of sets, and it implicitly relies on the fact that the category Set of sets is a *symmetric* monoidal category.

In contrast, internal natural transformation are objects of the category $\mathcal{Z}(\mathcal{C})$ which is braided but (generically) not symmetric. However, we will show in Section 7.3 the following generalization to the internalized setting:

Theorem 7.1.1. *Let $\mathcal{C}$ be a spherical fusion category and $A, B, C \in \mathcal{F}r(\mathcal{C})$ simple special* *symmetric Frobenius algebras in $\mathcal{C}$. Then the diagram*

$$
\begin{tikzcd}
 & \mathbb{D}^{X_4,X_6} \otimes \mathbb{D}^{X_2,X_4} \otimes \mathbb{D}^{X_3,X_5} \otimes \mathbb{D}^{X_1,X_3} \arrow[dl, "\mathrm{id}\otimes\gamma_{\mathbb{D}^{X_2,X_4};\mathbb{D}^{X_3,X_5}}\otimes\mathrm{id}"'] \arrow[dd, "\underline{\mu}_{\mathrm{ver}}\otimes\underline{\mu}_{\mathrm{ver}}"] \\
\mathbb{D}^{X_4,X_6} \otimes \mathbb{D}^{X_3,X_5} \otimes \mathbb{D}^{X_2,X_4} \otimes \mathbb{D}^{X_1,X_3} \arrow[dd, "\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}\otimes\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}"'] & \\
 & \mathbb{D}^{X_2,X_6} \otimes \mathbb{D}^{X_1,X_5} \arrow[dd, "\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}"] \\
\mathbb{D}^{X_3\otimes_B X_4,X_5\otimes_B X_6} \otimes \mathbb{D}^{X_1\otimes_B X_2,X_3\otimes_B X_4} \arrow[dr, "\underline{\mu}_{\mathrm{ver}}"] & \\
 & \mathbb{D}^{X_1\otimes_B X_2,X_5\otimes_B X_6}
\end{tikzcd}
\tag{7.1.2}
$$

involving the horizontal and vertical compositions of internal natural transformations and the half-braiding of the defect field content $\mathbb{D}^{X_2,X_4} \in \mathcal{Z}(\mathcal{C})$ commutes for all $X_1, X_3, X_5 \in A\text{-mod}^{\mathcal{C}}\text{-}B$ and all $X_2, X_4, X_6 \in B\text{-mod}^{\mathcal{C}}\text{-}C$.

Since the equality (7.1.1) provides the prototype for the Eckmann-Hilton argument, we refer to the commutativity of (7.1.2) as the *internal Eckmann-Hilton relation.*

Remark 7.1.2. (i) The construction of string-net correlators introduced in Section 5.3, which will be the key component of our proof of Theorem 7.1.1, operates under the premise that $\mathcal{C}$ is a modular fusion category. However, it can be observed that the whole construction only requires $\mathcal{C}$ to be *spherical fusion.* Therefore, Theorem 7.1.1 is valid for any spherical fusion category.
(ii) Theorem 7.1.1 renders Proposition 14 in [FS21b] precise. In the latter, neither the relevant (left or right) horizontal composition nor the relevant half-braiding were specified.
(iii) There is, of course, also a variant in which the *right* rather than left horizontal composition appears: the diagram

$$
\begin{tikzcd}
 & \mathbb{D}^{X_3,X_5} \otimes \mathbb{D}^{X_1,X_3} \otimes \mathbb{D}^{X_4,X_6} \otimes \mathbb{D}^{X_2,X_4} \arrow[dl, "\mathrm{id}\otimes\gamma_{\mathbb{D}^{X_1,X_3};\mathbb{D}^{X_4,X_6}}\otimes\mathrm{id}"'] \arrow[dd, "\underline{\mu}_{\mathrm{ver}}\otimes\underline{\mu}_{\mathrm{ver}}"] \\
\mathbb{D}^{X_3,X_5} \otimes \mathbb{D}^{X_4,X_6} \otimes \mathbb{D}^{X_1,X_3} \otimes \mathbb{D}^{X_2,X_4} \arrow[dd, "\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}}\otimes\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}}"'] & \\
 & \mathbb{D}^{X_1,X_5} \otimes \mathbb{D}^{X_2,X_6} \arrow[dd, "\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}}"] \\
\mathbb{D}^{X_3\otimes_B X_4,X_5\otimes_B X_6} \otimes \mathbb{D}^{X_1\otimes_B X_2,X_3\otimes_B X_4} \arrow[dr, "\underline{\mu}_{\mathrm{ver}}"] & \\
 & \mathbb{D}^{X_1\otimes_B X_2,X_5\otimes_B X_6}
\end{tikzcd}
\tag{7.1.3}
$$

commutes as well. This follows directly by combining Theorem 7.1.1 with the identity (7.3.10) which will be established below.

7.2 Three braided colored operads

Preparing and giving the proof of Theorem 7.1.1 will occupy most of the rest of this section. A main ingredient are certain *braided colored operads* in Set [Yau21]. With the help of string-nets we will establish morphisms between these operads. The operads in question are:

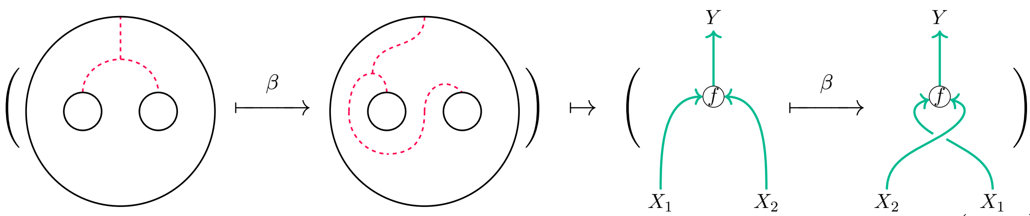
- The braided colored operad $\mathsf{WS}_{\mathcal{C}}$ of *genus-0 worldsheets* (with topological defects) in an RCFT whose chiral data is given by the modular fusion category $\mathcal{C}$.
  The set of colors is $\operatorname{obj}\mathrm{Cyl}^{\circ}(\mathcal{F}r(\mathcal{C}), S^1)$, i.e. the set of $\mathcal{F}r(\mathcal{C})$-boundary data on the circle S^1. The sets of operations of $\mathsf{WS}_{\mathcal{C}}$ consists of all genus-0 worldsheets (more precisely: the n-ary operation is given by a genus-0 worldsheet with $n+1$ boundary circles), with the boundary data on their boundary circles regarded as in- and outputs, and taken up to isotopy (rel. boundary). The operadic composition on $\mathsf{WS}_{\mathcal{C}}$ is obtained by identifying the braid group B_n as a subgroup of $\mathrm{Map}(\Sigma^0_{n+1})$, with Σ^0_{n+1} being a standard sphere with $n+1$ holes.

- The braided colored operad $\mathsf{SN}_{\mathcal{C}}$ of *$\mathcal{C}$-colored string-nets.*
  The set of colors of $\mathsf{SN}_{\mathcal{C}}$ is $\operatorname{obj}\mathrm{Cyl}(\mathcal{C}, S^1)$, i.e. the set of thickened $\mathcal{C}$-boundary values on the circle S^1. The sets of operations consists of the sets underlying the string-net spaces on genus-0 surfaces with appropriate boundary values, e.g.

$$\mathsf{SN}_{\mathcal{C}}\begin{pmatrix} & \mathsf{C} & \\ \mathsf{A} & & \mathsf{B}\end{pmatrix} = \mathrm{SN}_{\mathcal{C}}(\Sigma^0_3, \overline{\mathsf{A}} \times \overline{\mathsf{B}} \times \mathsf{C}).$$

  The operadic composition on $\mathsf{SN}_{\mathcal{C}}$ is the concatenation of string-nets, and the braid group action is again obtained by identifying B_n as a subgroup of $\mathrm{Map}(\Sigma^0_{n+1})$ which acts on the string-net space via pushforward.

- The braided colored *endomorphism operad* $\mathsf{Hom}_{\mathcal{Z}(\mathcal{C})}$.
  The colors of $\mathsf{Hom}_{\mathcal{Z}(\mathcal{C})}$ are the objects of the Drinfeld center $\mathcal{Z}(\mathcal{C})$. The sets of operations consists of the sets underlying the hom-spaces of $\mathcal{Z}(\mathcal{C})$ whose domains are given by the tensor products of the inputs. The operadic composition on $\mathsf{Hom}_{\mathcal{Z}(\mathcal{C})}$ is the composition of morphisms. We define the braid group action to be generated by



(7.2.1)

  for $X_1, X_2, Y \in \mathcal{Z}(\mathcal{C})$, with the over braiding standing for the half-braiding $\gamma_{X_2;X_1}$ (i.e. the diagram is interpreted in the braided tensor category $\mathcal{Z}(\mathcal{C})$).

Remark 7.2.1. For both $\mathsf{SN}_{\mathcal{C}}$ and $\mathsf{Hom}_{\mathcal{Z}(\mathcal{C})}$ there is an obvious linear version, in which the operations are given by the vector spaces $\mathrm{SN}_{\mathcal{C}}(-)$ and $\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(-)$, respectively, instead of by their underlying sets. We could indeed formulate the present considerations in a linear setting, by linearizing also the worldsheet operad via the $\mathcal{F}r(\mathcal{C})$-colored (bare) string-nets $\mathrm{SN}^{\circ}_{\mathcal{C}}(-)$, and applying the universal correlators that will be introduced in Chapter 8.

7.3 Proof of the relation

Now we define morphisms

$$\mathsf{Cor}_{\mathcal{C}} \colon \mathsf{WS}_{\mathcal{C}} \to \mathsf{SN}_{\mathcal{C}} \tag{7.3.1}$$

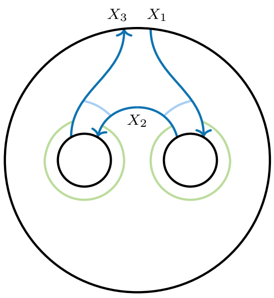
and

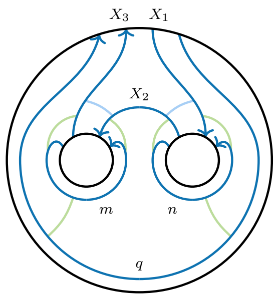
$$\varphi_{\mathsf{Hom}} \colon \mathsf{SN}_{\mathcal{C}} \to \mathsf{Hom}_{\mathcal{Z}(\mathcal{C})} \tag{7.3.2}$$

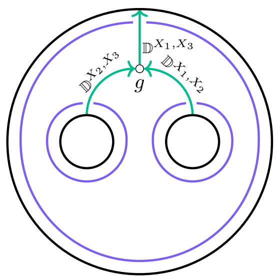
of braided colored operads in Set by the following prescriptions:

- $\mathsf{Cor}_{\mathcal{C}}$ acts on colors by sending a $\mathcal{F}r(\mathcal{C})$-boundary datum b to the corresponding thickened $\mathcal{C}$-boundary value $\mathbb{F}_{S^1}(\mathsf{b}) \in \mathrm{Cyl}(\mathcal{C}, S^1)$. $\mathsf{Cor}_{\mathcal{C}}$ acts on operations by sending a worldsheet $\mathcal{S}$ to its string-net correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$. Compatibility with the operadic compositions and braid group equivariance of these prescriptions are evident.

- The definition of φ_{Hom} is slightly more involved. It depends on two types of auxiliary data: for each genus-0 surface Σ a marking $w = w(\Sigma)$ without cuts, and for each object $\mathsf{B} \in \mathrm{Cyl}(\mathcal{C}, S^1)$ an isomorphism $\psi = \psi(\mathsf{B})$ from B to its "canonical form" $\mathsf{B}^{\mathrm{can}}_{\Phi(\mathsf{B})}$, i.e. the one that is analogous to (5.2.3), with $\Phi \equiv \Phi_{\partial\Sigma} \colon \mathrm{Cyl}(\mathcal{C}, S^1) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C})$ the canonical equivalence. With a fixed choice for these data, φ_{Hom} acts on colors as $\mathsf{B} \mapsto \Phi(\mathsf{B})$. Its action on operations is by the inverse of the isomorphisms $\mathcal{Z}(\mathcal{C})(\ldots,\ldots) \xrightarrow{\cong} \mathrm{SN}_{\mathcal{C}}(\Sigma, \mathsf{B})$ that are analogous to (6.1.9). To give an example, when choosing again the marking (6.1.6) on the pair of pants we have

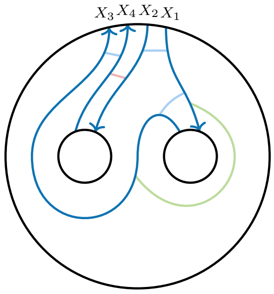
$\varphi_{\mathsf{Hom}} \colon$ 

$\xmapsto{\Psi} \displaystyle\sum_{m,n,q \in \mathcal{I}(\mathrm{mod}^{\mathcal{C}}\text{-}A)} \frac{d_m d_n}{D^4_{\mathcal{C}}}$ 

$=$ 

$$\xmapsto{\Psi} \quad \underline{\mu}_{\mathrm{ver}} \in \mathsf{Hom}_{\mathcal{Z}(\mathcal{C})} \begin{pmatrix} & \mathbb{D}^{X_1,X_3} & \\ \mathbb{D}^{X_2,X_3} & & \mathbb{D}^{X_1,X_2} \end{pmatrix},$$

where Ψ stands for pre- and post-composition with appropriate isomorphisms ψ and φ_w is given by (6.1.9). Similarly, with φ_w as in (6.2.10) we have

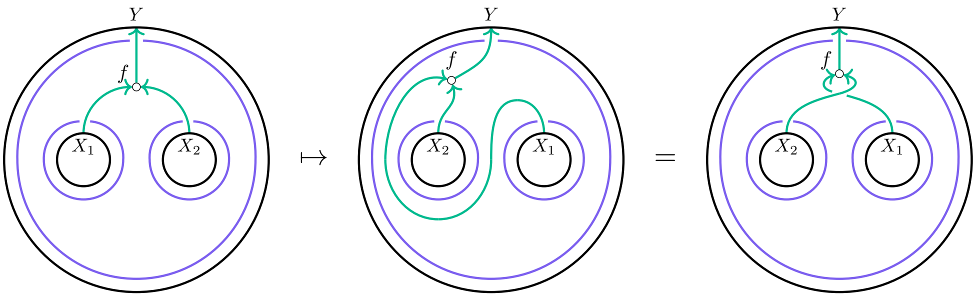


$$\xmapsto{\varphi_{\mathsf{Hom}}} \quad \underline{\mu}^{1}_{\mathrm{hor}} \in \mathsf{Hom}_{\mathcal{Z}(\mathcal{C})}\begin{pmatrix} \mathbb{D}^{X_1\otimes_B X_2, X_3\otimes_B X_4} \\ \mathbb{D}^{X_2,X_4} \quad \mathbb{D}^{X_1,X_3} \end{pmatrix}.$$

Operadic composition results in a marking with cuts with an internal edge e which connects the outgoing circle of the inserted surface to the new root of the marking. Compatibility with the operadic compositions is achieved by complementing the prescription for φ_{Hom} given by the requirement to replace this marking by the marking without cuts that is obtained by contracting the internal edge e. (This is analogous to the F-move on markings, compare Figure 5.7 of [BKJ00].)

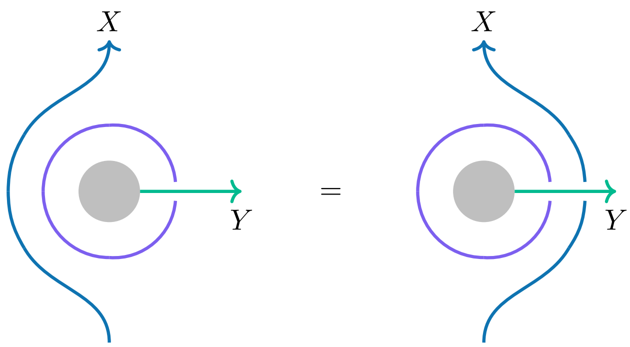
Remark 7.3.1. A different choice of the isomorphism between the boundary values and their canonical forms results in composing every element in a given morphisms space with one and the same isomorphism or its inverse. Similarly, a different choice of the markings results in composing them with braiding and twist isomorphisms. The choices we make are particularly convenient for revealing the internal Eckmann-Hilton relation, but other choices will lead to the same relation as well.

Let us verify that our prescription (7.2.1) leads to the correct braid group action on $\mathsf{Hom}_{\mathcal{Z}(\mathcal{C})}$: Under the move β that is shown on the left hand side of (7.2.1) we have



(7.3.3)

where the equality holds by the cloaking relation [SY21, Lemma 3.7]



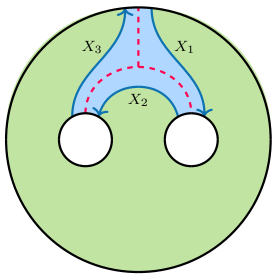
(7.3.4)

Note that the over braiding on the right hand side of (7.3.3) is the half-braiding $\gamma_{X_2;X_1}$. Thus we indeed obtain the braid group action on $\mathsf{Hom}_{\mathcal{Z}(\mathcal{C})}$ as defined in (7.2.1). (A priori, in (7.2.1) one might have considered instead the inverse half-braiding $\gamma^{-1}_{X_1;X_2}$; the present calculation shows that the choice made in (7.2.1) is the correct one.)

Proof of Theorem 7.1.1. Consider the composite

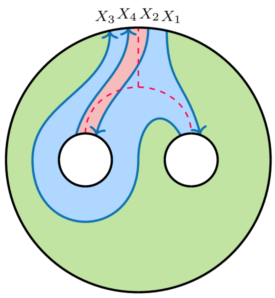
$$\widetilde{\mathrm{Cor}}_{\mathcal{C}} := \varphi_{\mathsf{Hom}} \circ \mathrm{Cor}_{\mathcal{C}} \colon \mathsf{WS}_{\mathcal{C}} \to \mathsf{Hom}_{\mathcal{Z}(\mathcal{C})}$$

of the morphism (7.3.1) and (7.3.2). We have

$\mathcal{S}^{w}_{\mathrm{ver}} \overset{(6.1.12)}{=\!=}$ 

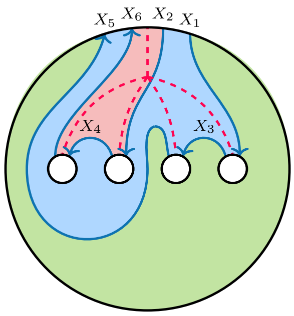
$\xmapsto{\widetilde{\mathrm{Cor}}_{\mathcal{C}}} \underline{\mu}_{\mathrm{ver}}$ (7.3.5)

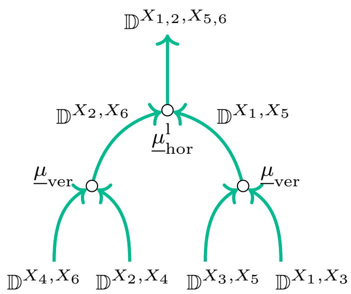
and

$\mathcal{S}^{\mathrm{l}}_{\mathrm{hor}} \overset{(6.2.12)}{=\!=}$ 

$\xmapsto{\widetilde{\mathrm{Cor}}_{\mathcal{C}}} \underline{\mu}^{\mathrm{l}}_{\mathrm{hor}}.$ (7.3.6)

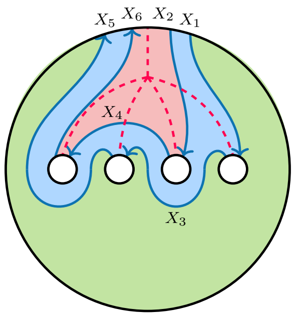
Invoking the compatibility of $\widetilde{\mathrm{Cor}}_{\mathcal{C}}$ with the operadic composition, we further get

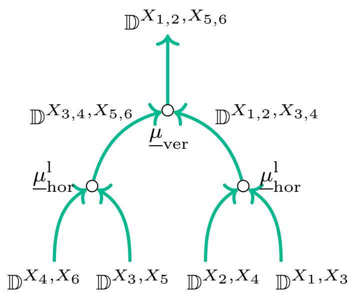
$\mathcal{S}_{\mathrm{h;v,v}} :=$ 

$\xmapsto{\widetilde{\mathrm{Cor}}_{\mathcal{C}}}$ 

(7.3.7)

as well as

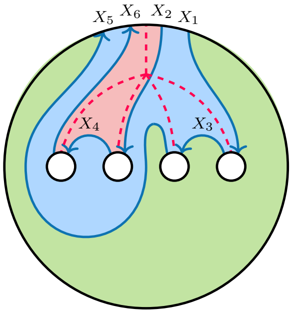
$\mathcal{S}_{\mathrm{v;h,h}} :=$ 

$\xmapsto{\widetilde{\mathrm{Cor}}_{\mathcal{C}}}$ 

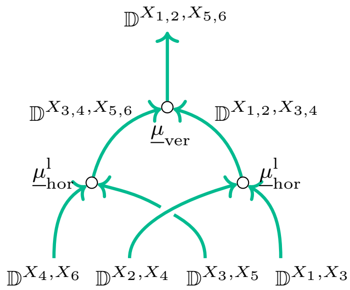
(7.3.8)

where we used the short-hand $X_{i,j} = X_i \otimes_B X_j$. Moreover, applying the braid group element $\beta_{2,3}$ to (7.3.8) gives

$$\beta_{2,3}(\mathcal{S}_{\mathrm{v;h,h}}) \coloneqq$$



$$\xmapsto{\widetilde{\mathrm{Cor}}_{\mathcal{C}}}$$



(7.3.9)

Notice that

$$\beta_{2,3}(\mathcal{S}_{\mathrm{v;h,h}}) = \mathcal{S}_{\mathrm{h;v,v}}.$$

By comparison of (7.3.7) and (7.3.9) we arrive at the desired equality that states the commutativity of the diagram (7.1.2). ☐

Remark 7.3.2. By similar arguments one sees that the equality $\mathcal{S}^{\mathrm{l}}_{\mathrm{hor}} = \beta(\mathcal{S}^{\mathrm{r}}_{\mathrm{hor}})$ obtained in (6.2.14) gets mapped under $\widetilde{\mathrm{Cor}}_{\mathcal{C}}$ to

$$\underline{\mu}^{\mathrm{l}}_{\mathrm{hor}} = \beta(\underline{\mu}^{\mathrm{r}}_{\mathrm{hor}}) = \underline{\mu}^{\mathrm{r}}_{\mathrm{hor}} \circ \gamma_{\mathbb{D}^{X_2,X_4};\mathbb{D}^{X_1,X_3}}, \tag{7.3.10}$$

showing that the left and right horizontal compositions are related by a half-braiding.

Remark 7.3.3. Consider the special case that $A = B = C$ and that all defect fields involved are actually bulk fields $\mathbb{D}^{A,A}$, for which the vertical and horizontal compositions coincide. The commutativity of the diagram (7.1.2) amounts to the statement that the bulk algebra $\mathbb{D}^{A,A} = Z(A)$ in $\mathcal{Z}(\mathcal{C})$ is braided commutative (compare [FS21b, Corollary 15]).

8 Universal correlators

8.1 Quantum worldsheets and the universal correlators

Using the (bare) string-net construction for the strictly pivotal bicategory $\mathcal{F}r(\mathcal{C})$, we can enrich our description of the correlators with further information and sharpen the concept of the mapping class group of a worldsheet. First, note that worldsheets related by local relations provided by the unframed graphical calculus for $\mathcal{F}r(\mathcal{C})$ have the same boundary data and thus have the same space of conformal blocks. Their correlators thus take value in the same vector space. We claim that a much stronger statement holds: such worldsheets that represent the same $\mathcal{F}r(\mathcal{C})$-colored string-net share the same correlator. Concretely, consider a worldsheet $\mathcal{S}$ with ambient surface Σ. Let $\mathsf{b} \in \mathrm{Cyl}^{\circ}(\mathcal{F}r(\mathcal{C}), \partial\Sigma)$ be the $\mathcal{F}r(\mathcal{C})$-boundary datum of its complemented worldsheet $\widetilde{\mathcal{S}}$, and

$$\begin{aligned} \delta_{\widetilde{\mathcal{S}}} \colon \mathbb{k} &\to \mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma, \mathsf{b}) \\ 1 &\mapsto [\widetilde{\mathcal{S}}] \end{aligned}$$

the linear map that picks out the string-net represented by the complemented worldsheet (viewed as a fully $\mathcal{F}r(\mathcal{C})$-colored graph on Σ). Likewise,

$$\begin{aligned} \delta_{\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})} \colon \mathbb{k} &\to \mathrm{SN}_{\mathcal{C}}(\Sigma, \mathbb{F}_{\partial\Sigma}(\mathsf{b})) \\ 1 &\mapsto \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}) \end{aligned}$$

is the map that picks out the correlator for $\mathcal{S}$. Note that by default, the $\mathbb{k}$-vector space $\mathbb{k}$ carries the trivial $\mathrm{Map}(\Sigma)$-action, and the string-net spaces (bare or Karoubified, colored with any strictly pivotal bicategory) are acted on by the mapping class groups via pushforward.

Theorem 8.1.1. *Let Σ be a compact oriented surface and $\mathsf{b} \in \mathrm{Cyl}^{\circ}(\mathcal{F}r(\mathcal{C}), \partial\Sigma)$ an $\mathcal{F}r(\mathcal{C})$-boundary datum on $\partial\Sigma$. There exists a unique $\mathrm{Map}(\Sigma)$-intertwiner*

$$\mathrm{UCor}_{\mathcal{C}}(\Sigma, \mathsf{b}) \colon \mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma, \mathsf{b}) \to \mathrm{SN}_{\mathcal{C}}(\Sigma, \mathbb{F}_{\partial\Sigma}(\mathsf{b})), \tag{8.1.1}$$

to be referred to as the universal correlator for the pair (Σ, b), such that for every worldsheet $\mathcal{S}$ that has Σ as its ambient bordism and b as the boundary datum of its complemented worldsheet $\widetilde{\mathcal{S}}$, the following diagram of $\mathrm{Map}(\Sigma)$-representations

$$\begin{tikzcd} \mathbb{k} \arrow[r, "\delta_{\widetilde{\mathcal{S}}}"] \arrow[dr, "\delta_{\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})}"'] & \mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma, \mathsf{b}) \arrow[d, "\mathrm{UCor}_{\mathcal{C}}(\Sigma,\mathsf{b})"] \\ & \mathrm{SN}_{\mathcal{C}}(\Sigma, \mathbb{F}_{\partial\Sigma}(\mathsf{b})) \end{tikzcd}$$

commutes. Moreover, the collection of universal correlators is compatible with sewing.

Proof. (i) Since every representative of a string-net in $\mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\varSigma, \mathsf{b})$ can be viewed as a complemented worldsheet, the commuting triangles force the universal correlator to be

$$\begin{aligned} \mathrm{UCor}_{\mathcal{C}}(\varSigma, \mathsf{b})\colon \mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\varSigma, \mathsf{b}) &\to \mathrm{SN}_{\mathcal{C}}(\varSigma, \mathbb{F}_{\partial\varSigma}(\mathsf{b})) \\ [\widetilde{\mathcal{S}}] &\to \mathrm{Cor}_{\mathcal{C}}(\mathcal{S}), \end{aligned} \tag{8.1.2}$$

we first need to show that this is indeed well defined as a linear map, which is equivalent to the statement that any two worldsheets related by the unframed graphical calculus for the pivotal bicategory $\mathcal{F}r(\mathcal{C})$ share the same correlator. To this end, we assume (without loss of generality) that $\widetilde{\mathcal{S}}_1$ and $\widetilde{\mathcal{S}}_2$ are two complemented worldsheets who are identical outside of an embedded disk $D \hookrightarrow \varSigma$ and share the same value on the disk, i.e. $\langle \widetilde{\mathcal{S}}_1 \cap D\rangle_{\mathcal{F}r(\mathcal{C})} = \langle \widetilde{\mathcal{S}}_2 \cap D\rangle_{\mathcal{F}r(\mathcal{C})}$. Now let $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S}_1) = [\varGamma_1]$ and $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S}_2) = [\varGamma_2]$ be the correlators for the corresponding worldsheets, with the representative graphs $\varGamma_1$ and $\varGamma_2$ chosen in a way such that they coincide outside of the embedded disk, and for each of them, there are no Frobenius lines within the disk except for each pair of distinct connected components (within the disk) of the partial defect network (see part 1 of Definition 5.3.1), a single Frobenius line connecting them: it is always possible to choose the representative graphs $\varGamma_1$ and $\varGamma_2$ as such because all $\mathcal{U}$-conjugates of bimodule morphisms commute with the action of Frobenius algebras and all the Frobenius graphs involved are *full.* We then only need to show that $\langle \varGamma_1 \cap D\rangle_{\mathcal{C}} = \langle \varGamma_2 \cap D\rangle_{\mathcal{C}}$. This is true because the functor

$$\mathcal{U}\colon \mathcal{F}r(\mathcal{C}) \to \mathcal{B}\mathcal{C}$$

introduced in Example 3.3.5 (with the canonical lax and oplax structures) is *rigid separable Frobenius* and therefore the $\mathcal{U}$-conjugation preserves *operadic compositions* and *partial trace maps* (Theorem 3.3.2), whereas the presence of the Frobenius lines connecting the connected components (on the embedded disk) of the partial defect networks compensates the fact that the $\mathcal{U}$-conjugation only preserves *horizontal products* and *whiskerings* up to idempotents of the type (3.3.12). It is readily clear that the linear map (8.1.2) intertwines with the mapping class group actions.

(ii) That the collection of universal correlators is compatible with sewing translates exactly to the statement that the prescription of string-net correlators is compatible with sewing. □

Theorem 8.1.1 implies that the correlator for every worldsheet $\mathcal{S}$ can be obtained by precomposing the universal correlator with the map $\delta_{\widetilde{\mathcal{S}}}$. Since precomposition is nothing but the pullback along a function, the situation may be regarded as an analogue – linearized and at one categorical level lower – of the description of a principal G-bundle via the universal bundle on the classifying space BG of a group G. This is the reason behind the choice of the name *"universal correlator"*.

It is a major insight that it is the equivalence class $[\widetilde{\mathcal{S}}]$ in the vector space $\mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\varSigma, \mathsf{b})$, rather than the worldsheet $\mathcal{S}$ itself, that is amenable to the "observation" of any aspects of correlators. It is thus appropriate to refer to the class $[\widetilde{\mathcal{S}}] \in \mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\varSigma, \mathsf{b})$ as the "observable worldsheet" that correspond to a given "classical" worldsheet $\mathcal{S}$, or also as the corresponding *quantum worldsheet.* Note that for being able to introduce this concept, allowing for the presence of defects is essential.

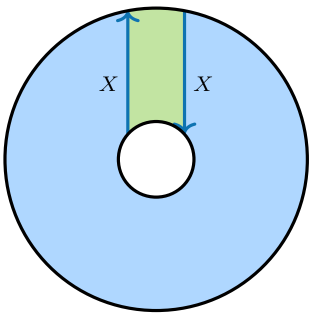
8.2 Mapping class group of a quantum worldsheet

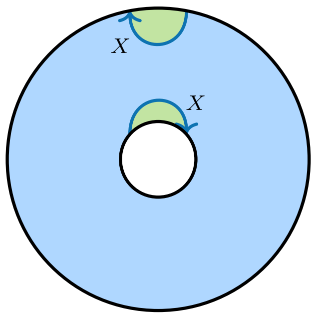
Another crucial insight is that, since the universal correlators (8.1.2) intertwine with the mapping class group actions on the string-net spaces, there is a more refined notion of the mapping class group that, morally speaking, should supersede the Definition 2.6.6 of $\mathrm{Map}(\mathcal{S})$, namely as the stabilizer of the quantum worldsheet $[\widetilde{\mathcal{S}}] \in \mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma, \mathsf{b})$ under the $\mathrm{Map}(\Sigma)$-action on $\mathrm{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma, \mathsf{b})$:

$$\widehat{\mathrm{Map}}(\mathcal{S}) := \mathrm{Stab}_{\mathrm{Map}(\Sigma)}([\widetilde{\mathcal{S}}]). \tag{8.2.1}$$

We think of $\widehat{\mathrm{Map}}(\mathcal{S})$ as the *mapping class group of the quantum worldsheet* $[\widetilde{\mathcal{S}}]$. By design, the correlator $\mathrm{Cor}_{\mathcal{C}}(\mathcal{S})$ for a worldsheet $\mathcal{S}$ is invariant under the action of the mapping class group $\widehat{\mathrm{Map}}(\mathcal{S})$ of the associated quantum worldsheet $[\widetilde{\mathcal{S}}]$.

As an illustration, let $X \in B\text{-mod}^{\mathcal{C}}\text{-}A$ be an invertible bimodule and consider the following worldsheets $\mathcal{S}_1$ and $\mathcal{S}_2$ which are identified by the local relations provided by the unframed graphical calculus for $\mathcal{F}r(\mathcal{C})$:

$\mathcal{S}_1 :=$ 

and $\mathcal{S}_2 := \dfrac{\dim(B)}{\dim(X)}$ 

represent the same quantum worldsheet, i.e.

$$[\widetilde{\mathcal{S}}_1] = [\widetilde{\mathcal{S}}_2].$$

In this case, the mapping class group $\mathrm{Map}(\mathcal{S}_1)$ of the worldsheet $\mathcal{S}_1$ does not include Dehn twists along the boundary circles, whereas $\mathrm{Map}(\mathcal{S}_2)$, hence also $\widehat{\mathrm{Map}}(\mathcal{S}_1) = \widehat{\mathrm{Map}}(\mathcal{S}_2) \supset \mathrm{Map}(\mathcal{S}_2)$, does.

9 A double categorical perspective

Introduced by Ehresmann [Ehr63], the notion of a double category is the less popular cousin of that of a bicategory. However, the flexibility that is intrinsic to double categories allows for their applications in both formal [Kou19, Kou22, Mye20] and applied [BC20, Cou20, Mye21] category theory. And it turns out, our string-net construction of CFT correlators can be better understood within the double categorical framework. Recall that the string-net construction gives rise to two sets of (apparently related) assignments: On one hand, one assigns linear categories to 1-manifolds and functors to embeddings, and on the other hand, in addition to the assignment of categories, one assigns profunctors to open-closed bordisms; our prescription of string-net correlators also involves two closely related assignments: for every 1-manifold, we have a field functor between cylinder categories, and for every bordism we have a natural transformation between string-net profunctors (with field functors inserted in the target profunctor) whose components are the universal correlators (see Theorem 9.3.1, which is an enhancement of Theorem 8.1.1 that states in a precise way that the collection of universal correlators is natural). The observation that will be presented in the last chapter of this thesis is that the open closed modular functors provided by the string-net construction can be naturally promoted to *symmetric monoidal double functors*, and that the field maps and universal correlators can be assembled into a *monoidal vertical transformation*. In a sense, the double categorical point of view incorporates *more locality* of the field theory.

We refer to [Cou20, Section A.2] for a nice exposition of the fundamentals of double category theory.

9.1 The double categories $\mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ and $\mathbb{P}\mathrm{rof}_{\Bbbk}$

A *(pseudo) double category* is a *pseudocategory object* internal to the 2-category (with finite limits) $\mathcal{C}\mathrm{at}$ of small categories, functors and natural transformations, whereas a *strict double category* is a *category object* internal to the category (with finite limits) Cat of small categories and functors. Invoking [GP99, Theorem 7.5] which states that every double category can be (functorially) strictified, we will treat every double category (and every double functor) as if they were strict.

Unraveling the concise definition, a double category $\mathbb{A}$ has:

- a collection of *objects*: $a, b, c, \ldots \in \mathbb{A}$;
- a collection of *vertical 1-morphisms*: $f: a \to b$, $g: b \to c, \ldots$ whose composition is *strictly* unital and associative;
- a collection of *horizontal 1-morphisms*: $P: a \nrightarrow b$, $Q: b \nrightarrow c, \ldots$ whose composition is *weakly* unital and associative; (Note that we use the marked arrow $\nrightarrow$ for *horizontal* 1-morphisms.)

- a collection of *2-morphisms*: $\begin{tikzcd} a \arrow[r, "P", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & b \arrow[d, "g"] \\ a' \arrow[r, "Q"', "|" marking] & b' \end{tikzcd}$, ... that can be composed both vertically and horizontally, e.g.

$$\begin{tikzcd} a \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & b \arrow[d] \\ a' \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha'"] & b' \arrow[d] \\ a'' \arrow[r, "|" marking] & b'' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & b \arrow[r, "|" marking] \arrow[d] \arrow[dr, phantom, "\beta"] & c \arrow[d] \\ a' \arrow[r, "|" marking] & b' \arrow[r, "|" marking] & c' \end{tikzcd}$$

  with the vertical composition being strictly unital and associative whereas the horizontal composition unital and associative only up to coherent isomorphisms; moreover the two types of compositions satisfy a *middle-four exchange law* that is akin to that for bicategories. Note that the squares *cannot* be interpreted as commutative squares since the composition of different types of 1-morphisms is not defined within the double category[1].

The framework of *internal categories* provides the notions of *(pseudo) double functors* and *vertical transformations* (or *double transformations*) [Cou20, Definition A.2.8 & A.2.9], which gives rise to the symmetric monoidal 2-category $\mathcal{D}$bl of double categories, double functors and vertical transformations. A *symmetric monoidal double category* is a *symmetric pseudomonoid* in $\mathcal{D}$bl, see [Cou20, Section A.2.1] for the unpacked description thereof.

There are plenty of naturally occurring examples of symmetric monoidal double categories: sets, functions, and (co)spans; sets, functions, and relations; rings, ring morphisms, and bimodules; manifolds, diffeomorphisms, and bordisms; categories, functors, and cofunctors, (here we have listed the objects, vertical 1-morphisms, horizontal 1-morphisms of the double categories and left the 2-morphisms implicit) and so on. In short: most of the commonly known symmetric monoidal *bicategories* are secretly the *horizontal bicategory* (i.e. the bicategory obtained from keeping only the horizontal 1-morphisms and *globular* 2-morphisms) of a symmetric monoidal double category that has enough *conjoints* and *companions* (see [Mye21] for a neat graphical description of both concepts), the rationale of which can be found in [WHS19].

The two symmetric monoidal double categories of our interest are:

Example 9.1.1. We have the following symmetric monoidal double categories:

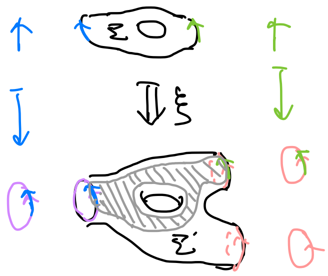
1. The symmetric monoidal double category $\mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ with:
    - objects: disjoint unions of the standard interval I and the standard circle S^1 endowed with the default orientations (see Definition 2.6.1), denoted by α, β, γ, etc.;

[1] There is a double category $\square\mathcal{A}$ whose vertical and horizontal 1-morphisms are both given by the morphisms in a category $\mathcal{A}$ and whose 2-morphisms are given by commutative squares in $\mathcal{A}$. However, the composition of a vertical morphism with a horizontal morphism, though possible due to the *contents* of $\square\mathcal{A}$, is not intrinsic to the framework of double categories.

- vertical 1-morphisms: orientation preserving embeddings, denoted by $f\colon \alpha \to \beta$, etc., with composition given by the composition of embeddings;
- horizontal 1-morphisms: open-closed bordisms, denoted by $\Sigma\colon \alpha \nrightarrow \beta$, etc, with composition given by sewing;
- 2-morphisms: isotopy classes of orientation preserving embeddings that restrict to the embeddings of the parameterizing 1-manifolds, an example for a 2-morphism of the type

$$\begin{tikzcd} \alpha \arrow[r, "\Sigma", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\xi"] & \beta \arrow[d, "g"] \\ \alpha' \arrow[r, "\Sigma'"', "\shortmid" marking] & \beta' \end{tikzcd}$$

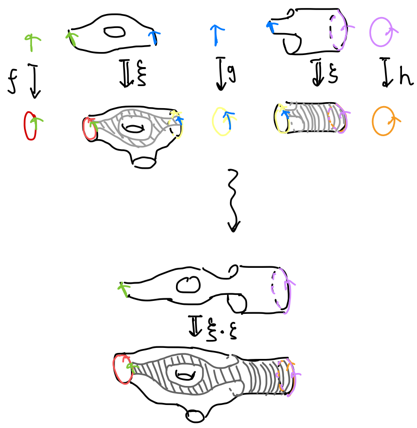
is the one that, for suitable Σ and Σ', is depicted as:



The vertical composition of 2-morphisms is induced by composition of embeddings, whereas the horizontal composition of 2-morphisms is induced by sewing. For instance, the horizontal composite

$$\begin{tikzcd} \alpha \arrow[r, "\Sigma_1", "\shortmid" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\xi"] & \beta \arrow[r, "\Sigma_2", "\shortmid" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\zeta"] & \gamma \arrow[d, "h"] \\ \alpha' \arrow[r, "\Sigma_1'"', "\shortmid" marking] & \beta' \arrow[r, "\Sigma_2'"', "\shortmid" marking] & \gamma' \end{tikzcd}$$

is depicted as:



The monoidal product is given by disjoint union.

2. The symmetric monoidal double category $\mathbb{P}\mathrm{rof}_{\Bbbk}$ with:
   - objects: small $\Bbbk$-linear categories, denoted by A, B, C, etc;
   - vertical 1-morphisms: $\Bbbk$-linear functors, denoted by $F\colon A \to B$, etc, with composition given by the composition of functors;
   - horizontal 1-morphisms: $\Bbbk$-linear profunctors, denoted by $P\colon A \nrightarrow B$, etc, with composition given by coends;
   - 2-morphisms: natural transformations (with functors inserted in the target profunctor), for instance:

$$\begin{tikzcd} A \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, "G"] \\ A' \arrow[r, "Q"', "\shortmid"{marking}] & B' \end{tikzcd}$$

     is given by a natural transformation

$$\varphi\colon P(-,\sim) \Rightarrow Q(F(-), G(\sim)).$$

     The vertical composition of 2-morphisms is given by the vertical composition of natural transformations (accompanied by the composition of functors) and the horizontal composition is induced by taking coends[2]. The monoidal product is given by the Cartesian product.

Remark 9.1.2. Both of the symmetric monoidal double categories $\mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ and $\mathbb{P}\mathrm{rof}_{\Bbbk}$ are so called *proarrow equipments* in the sense that every vertical 1-morphism can be "bent" into a horizontal 1-morphism in two ways, giving rise to its *conjoint* and *companion*. For instance, an embedding of a 1-manifold can be transformed into two bordisms by taking the mapping cylinders, whereas we can construct two profunctors from a given functor $F\colon A \to B$ by taking $B(F-,\sim)$ and $B(-,F\sim)$. Consequently [WHS19], upon taking horizontal bicategories, we obtain two *symmetric monoidal bicategories*. Indeed, the symmetric monoidal bicategory $\mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ is the full sub-(2,1)-category of the horizontal bicategory $\mathcal{H}(\mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}})$ (i.e. we keep all the objects and 1-morphisms of $\mathcal{H}(\mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}})$ but discard all those non-invertible 2-morphisms) and the symmetric monoidal bicategory $\mathcal{P}\mathrm{rof}_{\Bbbk}$ is the horizontal bicategory $\mathcal{H}(\mathbb{P}\mathrm{rof}_{\Bbbk})$.

[2] In detail: the component of the horizontal composite of the 2-morphism $\varphi\colon P(-_A,\sim_B) \Rightarrow P'(F-_A, G\sim_B)$ with $\psi\colon Q(-_B,\sim_C) \Rightarrow Q'(G-_B, H\sim_C)$ at $(a,c) \in A^{\mathrm{op}} \times C$ is given by the dinatural family

$$\begin{aligned} P(a,b_0) \otimes_{\Bbbk} Q(b_0,c) &\xrightarrow{\varphi_{a,b_0} \otimes_{\Bbbk} \psi_{b_0,c}} P'(Fa, Gb_0) \otimes_{\Bbbk} Q'(Gb_0, Hc) \\ &\longrightarrow \int^{b\in B} P'(Fa,Fb) \otimes_{\Bbbk} Q'(Gb,Gc) \end{aligned}$$

for every $b_0 \in B$.

9.2 String-net models as double functors

Given two double categories $\mathbb{A}$ and $\mathbb{B}$, we have the notion of a double functor $F\colon \mathbb{A} \to \mathbb{B}$. F transforms a 2-morphism in $\mathbb{A}$ to a 2-morphism in $\mathbb{B}$:

$$\begin{tikzcd} a \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & b \arrow[d, "g"] \\ a' \arrow[r, "Q"', "\shortmid"{marking}] & b' \end{tikzcd} \mapsto \begin{tikzcd} Fa \arrow[r, "FP", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & Fb \arrow[d, "Fg"] \\ Fa' \arrow[r, "FQ"', "\shortmid"{marking}] & Fb' \end{tikzcd}$$

such that (along with the preservation of units) both the vertical and the horizontal compositions are preserved:

$$\begin{tikzcd} Fa \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "F\alpha"] & Fb \arrow[d] \\ Fa' \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "F\alpha'"] & Fb' \arrow[d] \\ Fa'' \arrow[r, "\shortmid"{marking}] & Fb'' \end{tikzcd} = \begin{tikzcd} Fa \arrow[r, "\shortmid"{marking}] \arrow[d] & Fb \arrow[d] \\ Fa' \arrow[r, phantom, "F(\alpha'\circ\alpha)"] \arrow[d] & Fb' \arrow[d] \\ Fa'' \arrow[r, "\shortmid"{marking}] & Fb'' \end{tikzcd} \tag{9.2.1}$$

and

$$\begin{tikzcd} Fa \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "F\alpha"] & Fb \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "F\beta"] & Fc \arrow[d] \\ Fa' \arrow[r, "\shortmid"{marking}] & Fb' \arrow[r, "\shortmid"{marking}] & Fc' \end{tikzcd} = \begin{tikzcd} Fa \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[drr, phantom, "F(\alpha\cdot\beta)"] & Fb \arrow[r, "\shortmid"{marking}] & Fc \arrow[d] \\ Fa' \arrow[r, "\shortmid"{marking}] & Fb' \arrow[r, "\shortmid"{marking}] & Fc' \end{tikzcd} \tag{9.2.2}$$

for all composable pairs of 2-morphisms. Note that here we either interpret the equalities as up to simultaneous insertions of associators and unitors, or invoke the strictification theorem [GP99, Theorem 7.5] for double categories and double functors.

Let $(\mathcal{B}, *_{\mathcal{B}})$ be a pointed strictly pivotal bicategory. Recall that we have a symmetric monoidal pseudofunctor:

$$\begin{aligned} \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}\colon \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} &\to \mathcal{P}\mathrm{rof}_{\Bbbk} \\ \alpha &\mapsto \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\alpha) \equiv \mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, \alpha) \\ \Sigma &\mapsto \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\Sigma) \equiv \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\Sigma; -, \sim) \\ \xi &\mapsto \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\xi) \end{aligned}$$

which we call an open-closed modular functor, where the assignment of cylinder categories to 1-manifolds is functorial under the embeddings. Consequently, we can define a *double functor*:

$$\mathbb{SN}^{\circ}_{\mathcal{B}}\colon \mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \to \mathbb{P}\mathrm{rof}_{\Bbbk} \tag{9.2.3}$$

by

$$\begin{tikzcd} \alpha \arrow[r, "\Sigma", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\xi"] & \beta \arrow[d, "g"] \\ \alpha' \arrow[r, "\Sigma'"', "\shortmid"{marking}] & \beta' \end{tikzcd} \mapsto \begin{tikzcd} \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\alpha) \arrow[r, "\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\Sigma)", "\shortmid"{marking}] \arrow[d, "\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(f)"'] \arrow[dr, phantom, "\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\xi)"] & \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\beta) \arrow[d, "\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(g)"] \\ \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\alpha') \arrow[r, "\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\Sigma')"', "\shortmid"{marking}] & \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}(\beta') \end{tikzcd}$$

where we have extended the action of $\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}$ on mapping class group elements to that on *isotopy classes of embeddings*. The preservation of vertical composition (9.2.1) and of horizontal composition (9.2.2) is guaranteed by the functoriality of the pseudofunctor $\mathrm{SN}^{\circ}_{\mathcal{B}}$ and that of the functor $\mathrm{Cyl}^{\circ}(\mathcal{B}, *_{\mathcal{B}}, -)$. In fact, the double functor $\mathbb{SN}^{\circ}_{\mathcal{B}}$ is canonically symmetric monoidal (see [Cou20, Section A.2.2] for the definition). There is also the similar statement for the Karoubified string-nets. All together we have:

Theorem 9.2.1. *Let $(\mathcal{B}, *_{\mathcal{B}})$ be a pointed strictly pivotal bicategory. We have symmetric monoidal double functors*

$$\mathbb{SN}^{\circ}_{\mathcal{B}}, \mathbb{SN}_{\mathcal{B}} \colon \mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \to \mathbb{P}\mathrm{rof}_{\Bbbk}$$

that canonically extend the open-closed modular functors

$$\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{B}}, \mathcal{S}\mathrm{N}_{\mathcal{B}} \colon \mathcal{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}} \to \mathcal{P}\mathrm{rof}_{\Bbbk}.$$

9.3 Universal correlators as a vertical transformation

Now, Let $F, G \colon \mathbb{A} \to \mathbb{B}$ be two double functors. A *vertical transformation* $\theta \colon F \Rightarrow G$ is given by a family of *vertical 1-morphisms* in $\mathbb{B}$:

$$\{\theta_a \colon Fa \to Ga\}_{a \in \mathbb{A}}$$

and a family of 2-morphisms in $\mathbb{B}$ that is parameterized by the *horizontal 1-morphisms* in $\mathbb{A}$:

$$\begin{tikzcd} Fa \arrow[r, "FP", "\shortmid"{marking}] \arrow[d, "\theta_a"'] \arrow[dr, phantom, "\theta_P"] & Fb \arrow[d, "\theta_b"] \\ Ga \arrow[r, "GP"', "\shortmid"{marking}] & Gb \end{tikzcd}$$

for every $P \colon a \nrightarrow b$ in $\mathbb{A}$, such that the following two properties are satisfied:
Horizontal functoriality:

$$\begin{tikzcd} Fa \arrow[r, "FP", "\shortmid"{marking}] \arrow[d, "\theta_a"'] \arrow[dr, phantom, "\theta_P"] & Fb \arrow[r, "FQ", "\shortmid"{marking}] \arrow[d, "\theta_b"] \arrow[dr, phantom, "\theta_Q"] & Fc \arrow[d, "\theta_c"] \\ Ga \arrow[r, "GP"', "\shortmid"{marking}] & Gb \arrow[r, "GQ"', "\shortmid"{marking}] & Gc \end{tikzcd} = \begin{tikzcd} Fa \arrow[r, "F(P\cdot Q)", "\shortmid"{marking}] \arrow[d, "\theta_a"'] \arrow[dr, phantom, "\theta_{P\cdot Q}"] & Fc \arrow[d, "\theta_c"] \\ Ga \arrow[r, "G(P\cdot Q)"', "\shortmid"{marking}] & Gc \end{tikzcd}$$

Note that our presentation of the horizontal functoriality uses the strictness assumption for the double category $\mathbb{A}$ and the double functors F and G.
Vertical naturality:

$$\begin{tikzcd} Fa \arrow[r, "FP", "\shortmid"{marking}] \arrow[d, "Ff"'] \arrow[dr, phantom, "F\alpha"] & Fb \arrow[d, "Fg"] \\ Fa' \arrow[r, "FP'"', "\shortmid"{marking}] \arrow[d, "\theta_a"'] \arrow[dr, phantom, "\theta_{P'}"] & Fb' \arrow[d, "\theta_b"] \\ Ga' \arrow[r, "GP'"', "\shortmid"{marking}] & Gb' \end{tikzcd} = \begin{tikzcd} Fa \arrow[r, "FP", "\shortmid"{marking}] \arrow[d, "\theta_a"'] \arrow[dr, phantom, "\theta_P"] & Fb \arrow[d, "\theta_b"] \\ Ga \arrow[r, "GP"', "\shortmid"{marking}] \arrow[d, "Gf"'] \arrow[dr, phantom, "G\alpha"] & Gb \arrow[d, "Gg"] \\ Ga' \arrow[r, "GP'"', "\shortmid"{marking}] & Gb' \end{tikzcd}$$

It turns out the family of field maps as well as that of universal correlators allows for the definition of a vertical transformation

$$\mathbb{C}\mathrm{ft}_{\mathcal{C}} \colon \mathbb{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})} \Rightarrow \mathbb{SN}_{\mathcal{C}} \tag{9.3.1}$$

via setting the family of vertical 1-morphisms to be

$$\{\mathbb{F}_{\alpha} \colon \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\alpha) \to \mathcal{S}\mathrm{N}_{\mathcal{C}}(\alpha)\}_{\alpha \in \mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}}$$

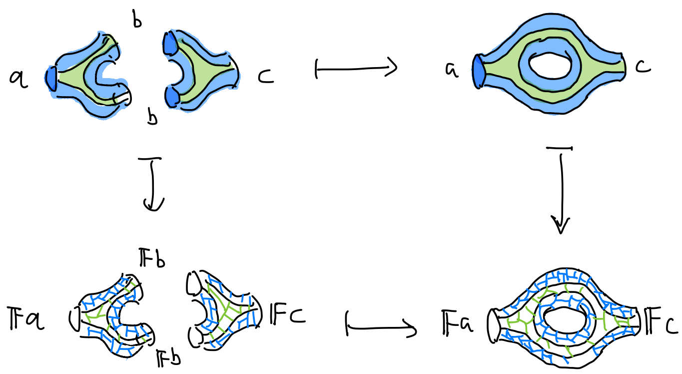
and the family of 2-morphisms to be given by

$$\begin{tikzcd}
\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\alpha) \arrow[r, "\mathcal{S}\mathrm{N}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma)", "\shortmid" marking] \arrow[d, "\mathbb{F}_{\alpha}"'] \arrow[dr, phantom, "\mathrm{UCor}_{\mathcal{C}}(\Sigma)"] & \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\beta) \arrow[d, "\mathbb{F}_{\beta}"] \\
\mathcal{S}\mathrm{N}_{\mathcal{C}}(\alpha) \arrow[r, "\mathcal{S}\mathrm{N}_{\mathcal{C}}(\Sigma)"', "\shortmid" marking] & \mathcal{S}\mathrm{N}_{\mathcal{C}}(\alpha)
\end{tikzcd}$$

where the components of the natural transformation

$$\mathrm{UCor}_{\mathcal{C}}(\Sigma) \colon \mathcal{S}\mathrm{N}^{\circ}_{\mathcal{F}r(\mathcal{C})}(\Sigma; -, \sim) \Rightarrow \mathcal{S}\mathrm{N}_{\mathcal{C}}(\Sigma, \mathbb{F}_{\alpha}-, \mathbb{F}_{\beta}\sim) \tag{9.3.2}$$

is given by the universal correlators. The horizontal functoriality then translates to the compatibility of $\mathrm{UCor}_{\mathcal{C}}$ with sewing:



and the vertical naturality correspond to the compatibility with embeddings (this is evident), which implies the invariance under the actions of the mapping class groups of *quantum worldsheets*. In fact, the so defined vertical transformation is monoidal ([Cou20, Definition A.2.15]): the verification is straightforward and uses the fact that both the field functors and the string-net modular functors are monoidal.

Theorem 9.3.1. *Let $\mathcal{C}$ be a modular fusion category. There is a canonical monoidal vertical transformation $\mathbb{C}\mathrm{ft}_{\mathcal{C}} \colon \mathbb{SN}^{\circ}_{\mathcal{F}r(\mathcal{C})} \Rightarrow \mathbb{SN}_{\mathcal{C}}$ whose components at objects are given by the field functors $\{\mathbb{F}_{\alpha}\}_{\alpha \in \mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}}$, and whose components at horizontal 1-morphisms are given by the universal correlators $\{\mathrm{UCor}_{\mathcal{C}}(\Sigma, -)\}$.*

Finally, we state the functoriality of the string-net construction as double functors under rigid pseudofunctors (Section 3.3), which follows directly from Corollary 3.3.4:

Theorem 9.3.2. *Let $\mathcal{B}, \mathcal{B}'$ be two strictly pivotal bicategories and $F \colon \mathcal{B} \to \mathcal{B}'$ a rigid pseudofunctors. There is a canonical vertical transformation*

$$\mathbb{SN}_{F} \colon \mathbb{SN}_{\mathcal{B}} \Rightarrow \mathbb{SN}_{\mathcal{B}'} \tag{9.3.3}$$

that is given by the change of colors via the F-conjugation.

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