Mapping class group actions and their applications to 3D gravity

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To my late grandfather

Zusammenfassung

In dieser Doktorarbeit untersuchen wir Wirkungen der Abbildungsklassengruppe der drei-dimensionalen Reshetikhin-Turaev topologischen Quantenfeldtheorie, motiviert durch Fragen in der drei-dimensionalen Quantengravitation, wo Durchschnitte der Abbildungsklassengruppe als Kandidaten für Gravitationszustandsummen gelten. Ein Hauptergebniss ist eine holographische Korrespondenz zwischen Durchschnitten der Abbildungsklassengruppe und einer konformen Feldtheorie, deren chirale Darstellungen der Abbildungsklassengruppe irreduzibel sind und eine Endlichkeiteigenschaft besitzen. Als wesentliches Beispiel finden wir heraus, dass modulare Fusionskategorien von Ising-Typ und ihre Reshetikhin-Turaev topologische Quantenfeldtheorien diese Eigenschaften erfüllen. Abschließend zeigen wir für eine modulare Fusionskategorie $\mathcal{C}$ dass, wenn die Darstellungen der Abbildungsklassengruppe jeder Fläche ohne markierte Punkte irreduzibel ist, es eine eindeutige unzerlegbare $\mathcal{C}$-Modulkategorie mit Modulspur, nämlich $\mathcal{C}$, gibt. Solche Modulkategorien beschreiben Oberflächendefekte in drei-dimensionalen Reshetikhin-Turaev topologische Quantenfeldtheorien. Dies verlinkt die Irreduzibilität der Abbildungsklassengruppendarstellungen und die Abwesenheit von nicht-trivialen Oberflächendefekten.

Abstract

In this thesis we study mapping class group actions of the three-dimensional Reshetikhin-Turaev topological quantum field theory motivated by questions in three-dimensional quantum gravity where mapping class group averages appear as candidates for gravity partition functions. One of the main results is a bulk-boundary correspondence between mapping class group averages and a rational conformal field theory whose chiral mapping class group representations are irreducible and obey a finiteness property. As primary examples we find that Ising-type modular fusion categories and their Reshetikhin-Turaev topological quantum field theories are characterised by these properties. Finally, for a given modular fusion category $\mathcal{C}$ we show that if the mapping class group representation on every surface without marked points is irreducible then there is a unique indecomposable $\mathcal{C}$-module category with module trace, namely $\mathcal{C}$ itself. Such module categories describe surface defects in three-dimensional Reshetikhin-Turaev topological quantum field theories. This links irreducibility of mapping class group representations and absence of non-trivial surface defects.

Relevant publications

This thesis is based on the following preprint:

- I. Romaidis and I. Runkel, *Mapping class group representations and Morita classes of algebras*, [2106.01454 [math.QA]].

and the following as of yet unpublished work:

- I. Romaidis and I. Runkel, in preparation.

The idea of linking irreducibility of mapping class group representations and Morita classes of algebras (later Theorem 7.1) was developed together with Ingo Runkel which resulted in the joint preprint mentioned above. These results are contained in Section 7 of this thesis. The details of the proof were worked out by me, which concerns in particular Lemmas 7.8 to 7.21.

The rest of the original work presented in this thesis was developed in the unpublished work in collaboration with Ingo Runkel mentioned above, expected to be published after the submission of this dissertation. This includes the correspondence between mapping class group averages and RCFT correlators in Section 5 and irreducibility and property F of Ising categories in Section 6. The idea of this correspondence was the original motivation for this dissertation as proposed by my supervisor Ingo Runkel. The explicit proofs of Theorems 5.4, 6.8 and 6.10 were worked out by me. The proof of Theorem 5.4 relies in combining some known results presented in Sections 3 and 5.1. The proofs of Theorems 6.8 and 6.10, which take up all of the Section 6.2, make use of explicit computations specific to Ising categories with the first one emulating the proof given in [JLLSW] for irreducibility with respect to surfaces without marked points and restricted to the Ising CFT. Finally, I have obtained the proof of Proposition 7.22 in Section 7.4 by mainly combining results of [ENOM].

Both works are the result of a collaboration with my supervisor Ingo Runkel, whose contributions and ideas I fully acknowledge.

Eidesstattliche Versicherung

Hiermit versichere ich an Eides statt, dass ich die vorliegende Dissertation selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

I hereby declare upon oath that I have written the present dissertation independently and have not used further resources and aids than those stated in the dissertation.

Hamburg, 21.08.2022

Iordanis Romaidis

Contents

1 Introduction

Topological quantum field theory (TQFT) combines geometry with various algebraic and higher categorical structures which makes it a powerful computational tool to use in the study of quantum field theory as well as representation theory. An essential feature of a TQFT is its underlying *modular functor* which encodes a family of mapping class group representations. Mapping class group averages appear in 3D quantum gravity while looking for a computable approximation of the path integral. This surprising aspect of mapping class group actions offers a natural gateway between a candidate theory for 3D quantum gravity in the bulk and a 2D conformal field theory (CFT) on the boundary in the spirit of the AdS/CFT correspondence. More precisely, the goal of this work is to establish a correspondence between mapping class group averages and rational CFT (or RCFT) using the 3D Reshetikhin-Turaev (RT) TQFT.

An n-dimensional TQFT in the Atiyah-Segal formulation is a symmetric monoidal functor from a bordism category into the category of vector spaces. The bordism category captures the geometries which may include various structures such as Wilson lines, defects and boundaries.

Let $\mathcal{C}$ be a modular fusion category (MFC), that is, a finitely semisimple ribbon category with simple tensor unit whose braiding is non-degenerate. Famously, such a category gives rise to a three-dimensional topological quantum field theory [RT, Tu]

$$\mathcal{Z}^{\mathcal{C}} : \widehat{\mathrm{Bord}}_3^{\mathcal{C}} \to \mathrm{Vect} \tag{1.1}$$

called the Reshetikhin-Turaev TQFT of $\mathcal{C}$. The bordism category here consists of 3-dimensional bordisms with embedded $\mathcal{C}$-coloured ribbon graphs which are morphisms between *decorated surfaces* or just *d-surfaces* (surfaces with framed points and $\mathcal{C}$-labels). Due to gluing anomalies the bordism category includes some additional structures, denoted by the hat.

The *mapping class group* $\mathrm{Mod}^{\mathcal{C}}(\Sigma)$ of a d-surface Σ consists of isotopy classes of diffeomorphisms which preserve the relevant structure [FM, Tu]. The RT state space $V^{\mathcal{C}}(\Sigma)$ on a d-surface is a (projective) representation of $\mathrm{Mod}^{\mathcal{C}}(\Sigma)$. For our discussions it is often sufficient to restrict to the pure mapping class group $\mathrm{PMod}_{g,n}$ of a connected d-surface, i.e. the subgroup consisting of mapping classes which fix every marked point. Here the integers g and n denote the genus and number of framed points of Σ. The generators of the pure mapping class group are Dehn twists along the curves depicted in Figure 1.1 [FM].

In more detail, let I denote a choice of representatives of the isomorphism classes of simple objects in $\mathcal{C}$ and write $L = \bigoplus_{i\in I} i \otimes i^*$. Let $X_1, \dots, X_n$ be the point labels of a genus g d-surface. Then $\mathrm{PMod}_{g,n}$ acts projectively on the Hom-space

$$V_{g,n}^{\mathcal{C}} := \mathcal{C}(\mathbb{1}, X_1 \otimes \cdots \otimes X_n \otimes L^{\otimes g}) , \tag{1.2}$$

and we recall this action in Section 3. Having an explicit description of this action will prove to be useful in the case of Ising categories.

Defects

Defects provide conceptually a richer geometry, where the top-dimensional manifold (worldvolume) may include embedded manifolds of lower-dimension. The theory is allowed to behave differently on these lower-dimensional manifolds, called *strata*, thus justifying the term *defect*. This allows one to study the interplay between theories separated by codimension-one defects. Defects can be used to gain insight about symmetries in the theory and as we shall see defects in RT TQFT offer a topological approach to RCFT.

Similarly, *physical* or *free* boundaries are codimension-one boundaries of the worldvolume where the theory abruptly ends. They are distinguished from gluing boundaries (which are present in a functorial field theory) and they can be thought as defects between the theory and a trivial theory.

An n-dimensional TQFT with defects has been axiomatised in [CMS] for $n = 3$ and then in [CRS1] for $n \geq 3$ as symmetric monoidal functors from a defect bordism category $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ to Vect. Bordisms are now *stratified* and each stratum carries a defect label from the defect datum $\mathbb{D}$. Defects in RT TQFT have been studied in [KaS, FSV] and a construction of a functorial defect RT TQFT has been given in [CRS2]. The construction works with a single governing bulk theory associated to an MFC $\mathcal{C}$. Surface defects are labelled by symmetric Δ-separable Frobenius algebras and line defects are certain multi-modules over algebras, which label incident surface defects. This has been then extended to multiple bulk theories in [KMRS]. An equivalent interpretation of RT defects is obtained by the language of module categories. A surface defect internal in $\mathcal{C}$, i.e. separating two regions labelled by $\mathcal{C}$, corresponds to a left $\mathcal{C}$-module category with module trace. Module categories with module trace have been introduced in [Sch] and are in direct correspondence with Morita classes of symmetric special Frobenius algebras.

Boundary conditions are treated in a similar fashion. We relate bordisms with free boundary to stratified bordisms by taking the double, defined by gluing two copies of the manifold along the free boundary which then defines a stratification on the resulting manifold. This is summarised by a symmetric monoidal functor

$$\widehat{(-)} : \mathrm{Bord}_n^{\mathrm{bnd}} \to \mathrm{Bord}_n^{\mathrm{str}} \tag{1.3}$$

from the category of bordisms with boundary to the category of stratified bordisms. This gives a way to construct from a defect TQFT $\mathcal{Z}$ a TQFT with boundary conditions $\hat{\mathcal{Z}}$ by applying the defect TQFT on the double. For an MFC $\mathcal{C}$ the Drinfeld centre $\mathcal{Z}(\mathcal{C})$ is equivalent to $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$. Conceptually, the TQFT of $\mathcal{Z}(\mathcal{C})$ with boundary can be seen as the

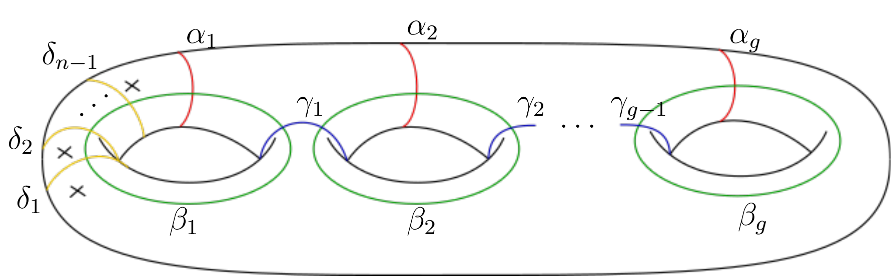


Figure 1.1: Generators of the unframed pure mapping class group.

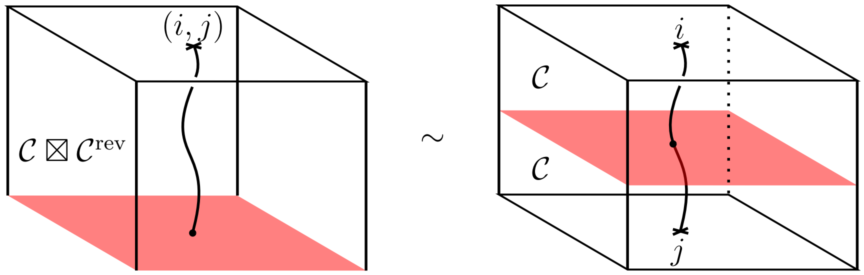


Figure 1.2: The construction of correlators via boundary in $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ or defects in $\mathcal{C}$.

double $\hat{\mathcal{Z}}^{\mathcal{C},\text{def}}$ of the defect RT TQFT of type $\mathcal{C}$. This provides two pictures for the TQFT approach to RCFT.

Rational conformal field theory

Two-dimensional CFT is a mathematically well-understood QFT which goes beyond TQFTs and is employed even in the study of higher-dimensional QFTs and string theory. In particular, rational CFT is a well-behaved theory characterised by the nice property of semisimplicity. In this text we work with the approach of [FRS, FjFRS1, FjFRS2] to RCFT using the defect RT TQFT of $\mathcal{C}$, which corresponds to the representation category of the associated chiral algebra [H]. In this case, chiral conformal blocks on a surface Σ correspond to states in $V^{\mathcal{C}}(\Sigma)$ and a correlator is an element in the double, i.e.

$$\mathrm{Cor}(\Sigma) \in V^{\mathcal{C}}(\hat{\Sigma}) \ . \tag{1.4}$$

Correlators are subject to *modular invariance* (invariant under the action of $\mathrm{Mod}^{\mathcal{C}}(\Sigma)$) and the *factorisation property* (compatibility with cuttings on the surface). We mostly focus on modular invariance and we will later see that under certain conditions on the modular functor the factorisation property is automatic.

The construction of [FjFRS1] is essentially using a surface defect labelled by some symmetric special Frobenius algebra A to obtain a consistent system of correlators $\{\mathrm{Cor}^{\mathcal{C}}_A(\Sigma)\}_\Sigma$. A d-surface Σ here carries $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ labels as it includes a holomorphic and an antiholomorphic part. The correlator $\mathrm{Cor}^{\mathcal{C}}_A(\Sigma)$ is obtained by taking the cylinder $\Sigma \times [0, 1]$ and interpreting $\Sigma \times \{1/2\}$ as a surface defect labelled by A. The bulk fields connect to the surface defect via Wilson lines. Equivalently, it is obtained by the TQFT of $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ on the cylinder but with one boundary interpreted as a free boundary. These two pictures are illustrated in Figure 1.2.

Mapping class group averages in quantum gravity

As already mentioned, mapping class group averages appear in three-dimensional quantum gravity. A special feature of gravity in three dimensions is the lack of local degrees of freedom. This implies that classical solutions to the equations of motion of Einstein gravity

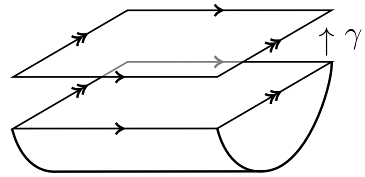
$M_\gamma =$ 

Figure 1.3: The manifold obtained by twisting the torus boundary of the solid torus by γ.

have constant curvature. Classical solutions are obtained by a quotient of a model geometry by a group of isometries. For a negative cosmological constant $\Lambda < 0$ the corresponding model geometry (for Euclidean signature) is that of the hyperbolic space $\mathbb{H}^3$ (or Euclidean Anti-de-Sitter space AdS_3).

A particular solution is *thermal* AdS_3 which is obtained from the Anti-de-Sitter spacetime by imposing an additional periodicity condition (see Section 8). The result is topologically a solid torus and a torus with conformal parameter τ as its asymptotic boundary. This solution is considered as the vacuum geometry of Einstein gravity.

Famously, Brown and Henneaux [BH] showed that the algebra of asymptotic symmetries of gravity on AdS_3 of radius l is isomorphic to two copies of the Virasoro algebra Vir_c with central charge

$$c = \frac{3l}{2G} \tag{1.5}$$

where G is Newton's constant and assumed to be much smaller than l. This result was one of the first hints for the conjecture of an AdS/CFT correspondence by Maldacena [Ma].

The path integral approach requires studying the gravity partition function

$$Z = \sum_{M \text{ topologies}} \int \mathcal{D}g \, e^{iS[g]} \tag{1.6}$$

with a fixed conformal boundary Σ for the topologies M. However, this is a highly ill-defined object. In the semi-classical limit $c \gg 1$ the classical solutions have dominating contributions to the path integral [MW], which makes a semi-classical approximation to the partition function a meaningful object to consider. A family of classical solutions M_γ with a conformal torus as the asymptotic boundary is obtained by taking thermal AdS_3 and twisting its boundary by a mapping class of the torus $\gamma \in Sl(2, \mathbb{Z})$ (see Figure 1.3). The contribution of thermal AdS_3 is $Z_{\text{vac}} = |\chi_0(\tau)|^2$ where $\chi_0(\tau)$ is the vacuum character. The contribution of M_γ is obtained by the action of γ on Z_{vac}. Therefore, if it exists, the semi-classical torus partition function is a mapping class group average of the vacuum contribution Z_{vac}. This is the motivating idea for studying mapping class group averages as candidates of gravity.

Mapping class group averages and RCFT

Our boundary theory is an RCFT with an associated MFC $\mathcal{C}$. In order to obtain a well-defined notion of a mapping class group average we impose a finiteness condition (see also

Definition 5.2):

Definition 1.1. Let Σ be a d-surface such that the representation image $G := V^{\mathcal{C}}(\mathrm{Mod}^{\mathcal{C}}(\Sigma))$ is finite in $\mathrm{End}(V^{\mathcal{C}}(\Sigma))$. Then, define the linear map

$$\langle - \rangle_\Sigma : V^{\mathcal{C}}(\hat{\Sigma}) \to V^{\mathcal{C}}(\hat{\Sigma})^{\mathrm{Mod}(\Sigma)}, \ x \mapsto \langle x \rangle_\Sigma := \frac{1}{|G|} \sum_{g \in G} g.x \tag{1.7}$$

where $\mathrm{Mod}(\Sigma)$ acts diagonally on $V^{\mathcal{C}}(\hat{\Sigma})$.

Let $\mathrm{d}_\Sigma : V^{\mathcal{C}}(\hat{\Sigma}) \to \Bbbk$ be the linear map obtained from the TQFT pairing (3.16). Imposing an additional irreducibility condition we obtain the following correspondence theorem (see also Theorem 5.4) [RR2]:

Theorem 1.2. Let Σ be a d-surface such that the projective representation $V^{\mathcal{C}}(\Sigma)$ is irreducible and $V^{\mathcal{C}}(\mathrm{Mod}^{\mathcal{C}}(\Sigma)) \subset \mathrm{End}(V^{\mathcal{C}}(\Sigma))$ is finite and let $x \in V^{\mathcal{C}}(\hat{\Sigma})$ be an element such that $\mathrm{d}_\Sigma(x) \neq 0$. In addition, suppose that A is a symmetric special Frobenius algebra in $\mathcal{C}$. Then, there exists $\lambda_\Sigma \in \Bbbk$ such that

$$\mathrm{Cor}^{\mathcal{C}}_A(\Sigma) = \lambda_\Sigma \, \langle x \rangle_\Sigma \ . \tag{1.8}$$

The element x is the *seed* and $\mathrm{d}_\Sigma(x) \neq 0$ is a non-degeneracy condition explained in Section 3. For example, for the torus $\Sigma = T^2$ the seed x can be chosen to be the contribution of the solid torus or in other words the vacuum seed.

The above result is in the spirit of AdS/CFT and it gives a class of examples where the bulk theory is an RT TQFT which is under particularly good mathematical control and an RCFT with chiral MFC $\mathcal{C}$ when $\mathcal{C}$ obeys these additional finiteness and irreducibility properties. To be precise, the bulk theory is modelled by the RT TQFT of $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ with partition functions given by mapping class group averages. It is thus sensible to study the finiteness and irreducibility properties and find examples. An example theory and guiding example of this work is the Ising CFT. In this setting, where $\mathcal{C}$ is the MFC associated to Ising, the statement of Theorem 1.2 was a known result of [CGHMV] for $\Sigma = T^2$ later extended to higher genus surfaces without marked points in [JLLSW]. The reason is precisely because the associated MFC satisfies the finiteness and irreducibility properties for all such surfaces. The first property is also referred to as property F, terminology borrowed from [NR] which used this with respect to braid group representations.

There are 16 MFCs of Ising-type fusion [DGNO] for which we prove the following result (see Theorems 6.8 and 6.10) [RR2]:

Theorem 1.3. Let $\mathcal{C}$ be an Ising-type MFC and let $\Sigma_{g,n}$ be an extended surface whose marked points are labelled by simple objects. Then, $V^{\mathcal{C}}(\Sigma_{g,n})$ is an irreducible projective representation of the pure mapping class group and the representation image $V^{\mathcal{C}}(\mathrm{PMod}_{g,n})$ is finite.

This extends the results of [JLLSW] to surfaces with framed points and to all 16 Ising categories.

Another known example of property F with respect to mapping class groups is for $\mathcal{C} = \mathrm{Rep}(D^\omega G)$ [G], i.e. twisted Dijkgraaf-Witten theories of a finite group G. There are further results of property F with respect to braid group representations [GrN, GRR, NR, RW], which are related to genus 0 d-surfaces. It is expected and conjectured for this case that property F is equivalent to the weak integral property. A fusion category $\mathcal{A}$ is *weakly integral* if its Frobenius-Perron dimension $\mathrm{FPdim}(\mathcal{A})$ is an integer.

Examples of $\mathcal{C}$ where all $V_g^{\mathcal{C}}$ are irreducible are when $\mathcal{C}$ is of Ising-type [JLLSW, RR2], and when it is given by $\mathcal{C}(sl(2), k)$ – the modular fusion category for the affine Lie algebra $\widehat{sl}(2)$ at level k – when $k+2$ is prime [Ro]. We will review examples for both properties in Section 6.1.

Absence of defects

In [RR1] we studied the irreducibility property in more detail and we found that such theories admit no non-trivial surface defects in the following way.

An algebra $A \in \mathcal{C}$ is called *non-degenerate* if its trace pairing is non-degenerate. Non-degenerate algebras carry a symmetric Frobenius structure. An algebra is called *simple* if it is simple as a bimodule over itself. Two algebras A, B are *Morita-equivalent* if there are bimodules ${}_AX_B$ and ${}_BY_A$ in $\mathcal{C}$ such that $X \otimes_B Y \cong A$ and $Y \otimes_A X \cong B$ as bimodules.

The main result of [RR1] is (see Theorem 7.1):

Theorem 1.4. Let $\mathcal{C}$ be a modular fusion category over an algebraically closed field of characteristic zero. If the projective mapping class group representations $V_g^{\mathcal{C}}$ are irreducible for all $g \geq 0$, then every simple non-degenerate algebra in $\mathcal{C}$ is Morita-equivalent to the tensor unit.

Suppose now that the modular fusion category $\mathcal{C}$ is defined over $\mathbb{C}$. In this case, $\mathcal{C}$ is called *pseudo-unitary* if all simple objects have positive quantum dimension, see [ENO1, Sec. 8] for details. Combining Theorem 1.4 with results on the existence of module traces in [Sch], it turns out we can drop the non-degeneracy condition (see Corollary 7.3):

Corollary 1.5. Suppose that in addition to the hypotheses in Theorem 1.4, $\mathcal{C}$ is defined over $\mathbb{C}$ and is pseudo-unitary. Then all simple algebras in $\mathcal{C}$ are Morita-equivalent to the tensor unit.

Recalling the brief introduction to defects, the statement of Theorem 1.4 can be interpreted as an absence of surface defects.

Absence of defects is maybe not surprising in the context of an AdS/CFT correspondence as it is conjectured [HO] that quantum gravity has no global symmetries, also known as absence of global symmetries. Since our model bulk theory is a RT TQFT of $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ (or Turaev-Viro TQFT of type $\mathcal{C}$) its global symmetries are *invertible* defects. Invertibility is defined with respect to the fusion of topological defects. In terms of module categories, these correspond to invertible $\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}$-module categories or equivalently invertible bimodule categories over $\mathcal{C}$ (see Section 2.3).

The relation of absence of defects for $\mathcal{C}$ and absence of invertible defects for $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ is not entirely clear. However, we prove the following Proposition (see Proposition 7.22):

Proposition 1.6. Let $\mathcal{C}$ be a non-degenerate braided fusion category which has no non-trivial indecomposable left module categories. Then, its invertible bimodule categories (up to equivalence) are in one-to-one correspondence with isomorphism classes of tensor autoequivalences on $\mathcal{C}$.

Indeed, for all examples, with the same hypothesis we are aware of, the group of tensor autoequivalences is trivial [EM].

In summary, the examples considered in this thesis are particularly tractable models for a bulk-boundary correspondence of 3D gravity and 2D CFT and they capture some key features like a well-defined average over classical solutions and absence of global symmetries.

Thesis structure

The aim of this thesis is to study mathematical questions motivated by problems in quantum gravity. Excluding the introduction, Sections 2 to 7 form the mathematical part of this thesis using the formal language of tensor category theory, TQFTs and RCFTs.

Section 8 plays the role of a translator and gives gravitational context to this thesis. On the one hand, it is meant to be read by a mathematician to gain some insight into the topics of quantum gravity in three dimensions. On the other hand, a physicist can safely skip the previous sections and may read the last section as a physics introduction to this work.

The thesis is organized as follows:

- In Section 2.1 we introduce and review basic notions in tensor category theory mostly following [BK, EGNO, TV2] such as modular fusion categories. Algebras, modules and their graphical calculus are reviewed in Section 2.2 following the conventions in [FRS]. Finally, Section 2.3 offers a brief exposition in module categories.

- Section 3.1 introduces the RT TQFT [Tu] by first defining the relevant bordism category of decorated surfaces and the associated mapping class groupoid. We make a note on gluing anomalies and their treatment, but most of the emphasis is put on mapping class groups [FM]. In Section 3.2 we give an explicit description of the mapping class group representations for an MFC $\mathcal{C}$.

- Section 4 discusses defects and boundaries in TQFTs. More precisely, Section 4.1 gives the definitions of stratified manifolds and TQFTs with defects as in [CRS1]. In addition, we summarise the construction of the defect RT TQFT based on [CRS2]. Section 4.2 considers TQFTs with boundaries for which we review manifolds with corners [SP] and the bordism category with free boundaries. Moreover, we give the construction of the double enabling to pass from bordisms with boundaries to bordisms with defects.

- Section 5.1 summarises the TQFT approach to RCFT of [FRS, FjFRS1]. We define mapping class group averages in Section 5.2 which we use to establish our correspondence Theorem in Section 5.3.

- In Section 6 we study the properties of irreducibility and finiteness. A review of known examples is given in Section 6.1. One of the primary results of this thesis is irreducibility and finiteness property of Ising categories proven in Sections 6.2.

- Section 7 is primarily a presentation of our results in [RR1]. We review the notion of the universal grading group in Section 7.1 and give the explicit construction of mapping class group invariants in Section 7.2 which are used for the proof which is completed in Section 7.3. Section 7.4 addresses the question of global symmetries or invertible defects.

- Section 8 reviews some of the physics background and motivation for this thesis. We give context to our results in the spirit of a holographic duality between quantum gravity and CFT. Furthermore, we address and discuss some questions and interesting directions.

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2 Prerequisites

2.1 Tensor categories

We will briefly introduce some notions of tensor category theory and fix conventions and the graphical calculus used in this thesis. We will follow mainly the definitions of [EGNO, TV2] and the graphical calculus of [BK].

Throughout this thesis, the field $\Bbbk$ will be an algebraically closed field of characteristic zero, unless specified otherwise. The category of finite dimensional $\Bbbk$-vector spaces will be denoted by Vect as the field $\Bbbk$ will be clear from context.

A *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, associativity isomorphisms $\alpha : \otimes \circ (\otimes \times \mathrm{id}) \cong \otimes \circ (\mathrm{id} \times \otimes)$, a unit object $\mathbb{1} \in \mathcal{C}$ and unitality isomorphisms $l : \otimes \circ (\mathbb{1} \times \mathrm{id}) \cong \mathrm{id}$ and $r : \otimes \circ (\mathrm{id} \times \mathbb{1}) \cong \mathrm{id}$. The associators α and unitators l, r are subject to the pentagon and triangle axioms. Given a monoidal category $\mathcal{C}$, we write $\mathcal{C}^{\mathrm{mop}}$ for the monoidal category with underlying category $\mathcal{C}$, but opposite monoidal product

$$A \otimes^{\mathrm{mop}} B := B \otimes A.$$

Its associators are given by $\alpha^{\mathrm{mop}}_{X,Y,Z} = \alpha^{-1}_{Z,Y,X}$ and its unitators by $l^{\mathrm{mop}}_X = r_X$ and $r^{\mathrm{mop}}_X = l_X$. The category $\mathcal{C}^{\mathrm{mop}}$ is not to be confused with the opposite category $\mathcal{C}^{\mathrm{op}}$ obtained from reversing all morphism directions. The category $\mathcal{C}^{\mathrm{op}}$ is also given a monoidal structure, when $\mathcal{C}$ is monoidal, equipped with the same bifunctor $\otimes$ and with associators and unitators given by the respective inverses.

A functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories $\mathcal{C}, \mathcal{D}$ is called *monoidal* if it is equipped with structure isomorphisms $F_2 : F \circ \otimes_{\mathcal{C}} \cong \otimes_{\mathcal{D}} \circ (F \times F)$ and $F_0 : F(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{1}_{\mathcal{D}}$ subject to compatibility conditions. By a natural transformation between two monoidal functors, we will always mean a natural transformation which is compatible with the monoidal structures.

A braiding on $\mathcal{C}$ is a family of natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ which satisfy the hexagon axioms, i.e. they are compatible with associators and unitators. Without loss of generality, we only consider strict monoidal categories, for which the associators and unitators are trivial. A monoidal category $\mathcal{C}$ equipped with a braiding c is called *braided* monoidal. An object T in a braided monoidal category $\mathcal{C}$ is called *transparent* if its double braiding with any other object in $\mathcal{C}$ is trivial, i.e.

$$c_{X,T} \circ c_{T,X} = \mathrm{id}_{T \otimes X} \tag{2.1}$$

for any $X \in \mathcal{C}$. A trivial example of a transparent object in any braided monoidal category is its unit object $\mathbb{1}$. By $\mathcal{C}^{\mathrm{rev}}$ we denote the braided category with the same underlying category equipped with the reverse braiding $c^{\mathrm{rev}}_{X,Y} = c^{-1}_{Y,X}$ (see Figure 2.1). A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between two braided monoidal categories $\mathcal{C}, \mathcal{D}$ is called braided, if it is compatible with the braidings.

Given a monoidal category $\mathcal{C}$, there is a construction of a braided monoidal category $\mathcal{Z}(\mathcal{C})$, called the *Drinfeld centre* of $\mathcal{C}$, cf. [EGNO, Def. 7.13.1]. Objects in $\mathcal{Z}(\mathcal{C})$ are pairs (V, γ) consisting of an object V in $\mathcal{C}$ and a natural isomorphism $\gamma : \mathrm{id}_{\mathcal{C}} \otimes V \to V \otimes \mathrm{id}_{\mathcal{C}}$

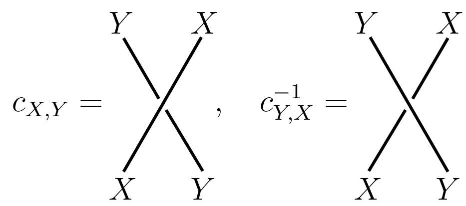


Figure 2.1: Graphical notation of a braiding and its inverse.

called a *half-braiding* and is subject to a hexagon axiom much like the braiding in a braided monoidal category. If $\mathcal{C}$ is braided monoidal then there is a natural braided monoidal functor

$$\mathcal{C} \to \mathcal{Z}(\mathcal{C}) \tag{2.2}$$

which takes an object X in $\mathcal{C}$ to the pair $(X, c_{-,X})$ where the braiding in $\mathcal{C}$ is used to define a half-braiding. Similarly, there is a braided monoidal functor

$$\mathcal{C}^{\text{rev}} \to \mathcal{Z}(\mathcal{C}) \tag{2.3}$$

which takes an object X to the pair $(X, c^{-1}_{X,-})$.

Let X be an object in a monoidal category $\mathcal{C}$. We say that ${}^*X$ is a *left dual* of X if there is a (left) evaluation map

$$ev^L_X : {}^*X \otimes X \to \mathbb{1} \tag{2.4}$$

and a (left) coevaluation map

$$coev^L_X : \mathbb{1} \to X \otimes {}^*X \tag{2.5}$$

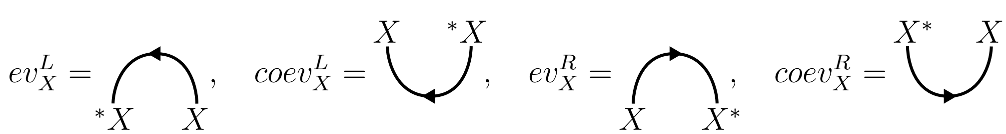
which satisfy the rigidity, or zig-zag, axioms. Similarly, we say that X^* is a *right dual* of X if there are (right) evaluation and coevaluation maps:

$$ev^R_X : X \otimes X^* \to \mathbb{1} \tag{2.6}$$

and

$$coev^R_X : \mathbb{1} \to X^* \otimes X \tag{2.7}$$

Graphically, these maps are represented as follows:



(2.8)

A *rigid category* is a monoidal category for which every object X has a left dual ${}^*X$ and a right dual X^*. Taking left duals and right duals defines monoidal functors

$${}^*(-), (-)^* : \mathcal{C} \to (\mathcal{C}^{\text{mop}})^{\text{op}} \ . \tag{2.9}$$

On morphisms, duals are obtained by using the evaluation and coevaluation maps. For instance, the right dual of a morphism $f : X \to Y$ is defined as:

$$f^* := (\text{id}_{X^*} \otimes ev^R_Y) \circ (\text{id}_{X^*} \otimes f \otimes \text{id}_{Y^*}) \circ (coev^R_X \otimes \text{id}_{Y^*})$$

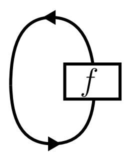
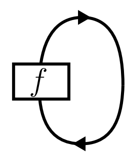
Similarly, for left duals. In a rigid category $\mathcal{C}$ one can canonically identify

$$^*(V^*) \cong V, \tag{2.10}$$

that is we have a canonical equivalence of monoidal functors $^*((-)^*) \cong \mathrm{id}_\mathcal{C}$.

Let $\mathcal{C}$ be rigid monoidal category. A *pivotal structure* on $\mathcal{C}$ is a monoidal isomorphism $\mathrm{id}_\mathcal{C} \xrightarrow{\simeq} (-)^{**}$ between the identity functor $\mathrm{id}_\mathcal{C}$ and the monoidal endofunctor obtained from taking double duals. A rigid category together with a pivotal structure is called a *pivotal category*. Functors between pivotal categories will be called *pivotal*, if they satisfy some additional compatibility conditions involving the pivotal structures.

The pivotal structure allows one to define the notion of left and right traces, as left duals can and will be canonically identified with right duals via $^*V \cong {}^*(V^{**}) \cong V^*$ where the pivotal structure gives the first isomorphism and the second is provided by the canonical isomorphism from rigidity (2.10). Let V be an object in a pivotal category $\mathcal{C}$ and $f : V \to V$ be an endomorphism of V. The *left trace* $\mathrm{tr}_L(f)$ respectively the *right trace* $\mathrm{tr}_R(f)$ are defined via

$\mathrm{tr}_L(f) =$  , $\quad \mathrm{tr}_R(f) =$  (2.11)

as endomorphisms of $\mathbb{1}$, where the canonical isomorphism $^*V \cong V^*$ is used but not included in the string diagram. In particular, this gives the notion of left and right dimensions by taking the respective traces of the identity morphism, i.e.

$$\dim_L(V) := \mathrm{tr}_L(\mathrm{id}_V) \quad \text{and} \quad \dim_R(V) := \mathrm{tr}_R(\mathrm{id}_V)\ . \tag{2.12}$$

A *spherical category* is a pivotal category where left traces coincide with right traces. In particular, in a spherical category we write $\mathrm{tr}(f) := \mathrm{tr}_L(f) = \mathrm{tr}_R(f)$ for the uniquely defined trace of f and $\dim(V) := \dim_L(V) = \dim_R(V)$ for the dimension of V.

Let $\mathcal{C}$ be a braided rigid monoidal category. A *ribbon structure* on $\mathcal{C}$ consists of a natural automorphism $\theta : \mathrm{id}_\mathcal{C} \xrightarrow{\simeq} \mathrm{id}_\mathcal{C}$ of the identity functor such that for any objects $V, W \in \mathcal{C}$

$$\theta_{V\otimes W} = c_{W,V} \circ c_{V,W} \circ (\theta_V \otimes \theta_W) \tag{2.13}$$

and $(\theta_V)^* = \theta_{V^*}$. A braided rigid monoidal category equipped with a ribbon structure is called *ribbon category* and the isomorphism θ is called the *twist*. A ribbon category inherits naturally the structure of a spherical category. From now on, we will not distinguish between left and right duals in pivotal categories as they are canonically identified, and we will use the right dual notation. For a ribbon category $\mathcal{C}$ with twist θ, the reversed ribbon category $\mathcal{C}^{\mathrm{rev}}$ is obtained by reversing both the braiding and the twist, i.e. $\theta^{\mathrm{rev}}_X = \theta^{-1}_X$. Any braided pivotal functor $F : \mathcal{C} \to \mathcal{D}$ between ribbon categories automatically preserves the ribbon structure in that $F(\theta) = \theta$, where θ denotes the respective twists. We may also refer to them as *ribbon* functors.

A *multitensor category* $\mathcal{C}$ is a locally finite[1] $\Bbbk$-linear abelian rigid monoidal category such that the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear. If in addition the unit $\mathbb{1}$ is simple, i.e. $\mathcal{C}(\mathbb{1}, \mathbb{1}) \cong \Bbbk$, then it is a *tensor category*. Moreover, a *multifusion category* is a finite semisimple multitensor category and a *fusion category* is a finite semisimple tensor category. The finiteness condition requires in addition to locally finiteness that there are only finitely many simple objects up to isomorphism and every simple object has a projective cover. For the details on abelian, locally finite and finite categories see [EGNO].

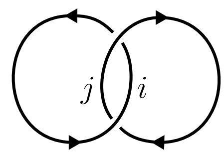
Deligne's tensor product[2] $\mathcal{C} \boxtimes \mathcal{D}$ of locally finite $\Bbbk$-linear abelian categories $\mathcal{C}, \mathcal{D}$ inherits a (braided) (multi)tensor or (multi)fusion category structure if $\mathcal{C}$ and $\mathcal{D}$ are also equipped with this structure [EGNO, Prop. 4.6.1]. Suppose $\mathcal{C}$ is a braided multitensor category. The functors (2.2) and (2.3) form a braided tensor functor

$$\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \to \mathcal{Z}(\mathcal{C}) \ . \tag{2.14}$$

Given a fusion category $\mathcal{C}$, we write I for the set of representatives of isomorphism classes of simple objects in $\mathcal{C}$ and choose $\mathbb{1}$ as the representative of $[\mathbb{1}]$. The duality in $\mathcal{C}$ induces an involution map $\overline{(\ )} : I \to I$, since for any $i \in I$ there exists $\bar{i}$ such that $\bar{i} \cong i^*$. We also define the object

$$L = \bigoplus_{i \in I} i \otimes i^* \ . \tag{2.15}$$

Suppose now $\mathcal{C}$ is a ribbon fusion category. Define an $|I| \times |I|$-matrix s with matrix elements

$$s_{i,j} = \mathrm{tr}(c_{j,i} \circ c_{i,j}) =$$  (2.16)

given by taking the trace of the double braiding. An object of great significance in this thesis is the modular fusion category which is defined in the following Proposition:

Proposition 2.1. A *modular fusion category* (MFC) is a ribbon fusion category $\mathcal{C}$ satisfying one of the following equivalent statements:

1. The s-matrix (2.16) is a non-degenerate matrix.

2. The Müger centre $(\mathcal{C})'$, which consists of all transparent objects in $\mathcal{C}$, is trivial, i.e. $(\mathcal{C})' \simeq \mathrm{Vect}$.

3. The (ribbon) tensor functor (2.14) is an equivalence, i.e. $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{C})$.

[1] The condition of locally finiteness requires that the category is enriched over finite dimensional vector spaces in that morphism spaces are finite dimensional and every object has finite length, see [EGNO, Def. 1.8.1].

[2] The Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ is equipped with a bilinear right biexact functor $\boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ and characterised by the universal property with respect to any other such abelian category $\mathcal{A}$ and bilinear right biexact functor $\mathcal{C} \times \mathcal{D} \to \mathcal{A}$.

The condition (ii) is equivalent to requiring that every transparent object is a direct sum of $\mathbb{1}$'s. We will often refer to the equivalence obtained in (iii) without specifying the exact functor of which it is obtained.

A trivial example of a (modular) fusion category is of course the category of finite dimensional vector spaces Vect. A more rich family of examples comes from the following observation: The Drinfeld centre $\mathcal{Z}(\mathcal{S})$ of a spherical fusion category $\mathcal{S}$ is a modular fusion category.

Let $\mathcal{C}$ be a fusion category, X an object and $i \in I$. We write N_X^i for the dimension of the morphism space $\mathcal{C}(i, X)$. That is the multiplicity of the simple i in X. An i-partition of X [TV2, Chapter 4] consists of a basis $\{p_\alpha^{(i)}\}_{\alpha=1,\dots,N_X^i}$ of the vector space $\mathcal{C}(i, X)$ and a basis $\{q_\alpha^{(i)}\}_{\alpha=1,\dots,N_X^i}$ of the vector space $\mathcal{C}(X, i)$ which are dual in the sense that

$$q_\alpha^{(i)} \circ p_\beta^{(i)} = \delta_{\alpha,\beta} \ . \tag{2.17}$$

By semisimplicity, taking the union of i-partitions for each $i \in I$, we get the property

$$\sum_{i \in I} \sum_\alpha p_\alpha^{(i)} \circ q_\alpha^{(i)} = \mathrm{id}_X. \tag{2.18}$$

We will use the following notation for a fixed i-partition, $p_\alpha^{(i)} = \alpha$ and $q_\alpha^{(i)} = \overline{\alpha}$.

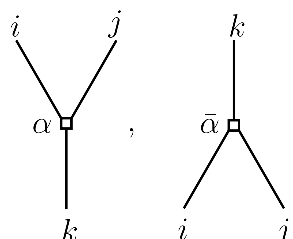
The *fusion coefficients* are defined by

$$N_{ij}^k := \dim \mathcal{C}(k, i \otimes j) \tag{2.19}$$

and satisfy

$$N_{ij}^k = N_{ji}^k = N_{i\overline{k}}^{\overline{j}} = N_{\overline{ij}}^{\overline{k}} \ . \tag{2.20}$$

For $i, j, k \in I$, a k-partition of $X = i \otimes j$ describes the fusion and split of i and j. Graphically, fusion basis elements are



(2.21)

When considering the fusion of four labels i, j, k, l in I, we have two natural decompositions, namely

$$\mathcal{C}(l, i \otimes j \otimes k) \cong \bigoplus_{m \in I} \mathcal{C}(l, m \otimes k) \otimes_{\Bbbk} \mathcal{C}(m, i \otimes j) \tag{2.22}$$

$$\cong \bigoplus_{n \in I} \mathcal{C}(l, i \otimes n) \otimes_{\Bbbk} \mathcal{C}(n, j \otimes k) \tag{2.23}$$

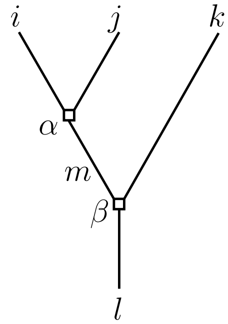
which gives two bases respectively. The transition matrix between these two bases is called an F-matrix and is defined by the relation (cf. [FRS, Eq. (2.39)])

$$(\alpha \otimes \mathrm{id}_k) \circ \beta = \sum_{n,\gamma,\delta} F_{\gamma n \delta,\, \alpha m \beta}^{(ijk)l} \, (\mathrm{id}_i \otimes \delta) \circ \gamma \ . \tag{2.24}$$

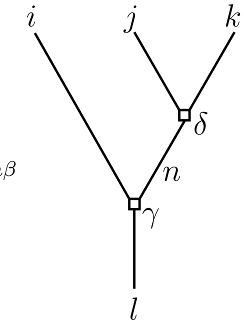
The matrix elements of the transformation inverse to F will be denoted by G:

$$(\mathrm{id}_i \otimes \delta) \circ \gamma = \sum_{m,\alpha,\beta} G^{(ijk)l}_{\alpha m\beta,\,\gamma n\delta}\,(\alpha \otimes \mathrm{id}_k) \circ \beta \ . \tag{2.25}$$

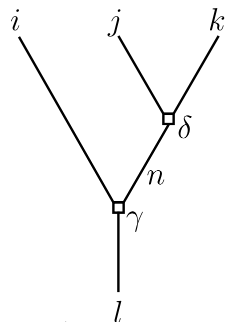
The string diagrams for equations (2.24) and (2.25) are



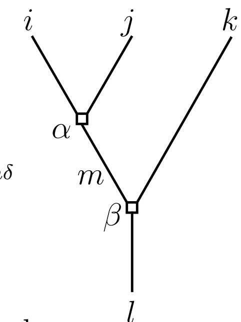
$$= \sum_{n,\gamma,\delta} F^{(ijk)l}_{\gamma n\delta,\,\alpha m\beta}$$



$$, \tag{2.26}$$



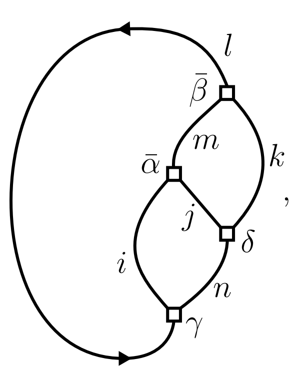
$$= \sum_{m,\alpha,\beta} G^{(ijk)l}_{\alpha m\beta,\,\gamma n\delta}$$



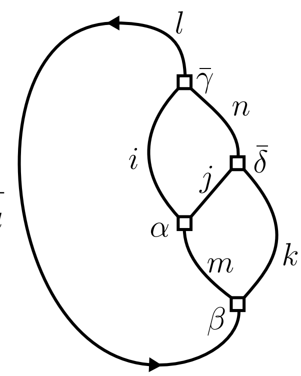
$$. \tag{2.27}$$

In a spherical fusion category we can compute them by:

$$F^{(ijk)l}_{\gamma n\delta,\,\alpha m\beta} = \frac{1}{d_l}$$



$$, \quad G^{(ijk)l}_{\alpha m\beta,\,\gamma n\delta} = \frac{1}{d_l}$$

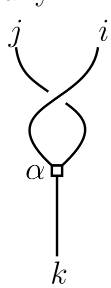


$$\tag{2.28}$$

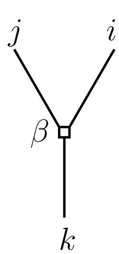
Let $\mathcal{C}$ be a braided fusion category. The R-matrix describes how the fusion basis changes under the braiding of $\mathcal{C}$. Namely, it is defined as

$$c_{i,j} \circ \alpha = \sum_{\beta} R^{(ij)k}_{\beta\alpha}\,\beta \quad , \quad c^{-1}_{j,i} \circ \alpha = \sum_{\beta} R^{-(ij)k}_{\beta\alpha}\,\beta \tag{2.29}$$

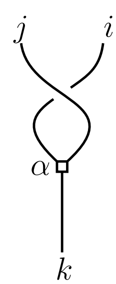
and graphically:



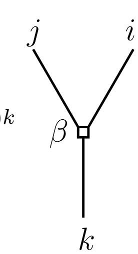
$$= \sum_{\beta} R^{(ij)k}_{\beta\alpha}$$



$$,$$



$$= \sum_{\beta} R^{-(ij)k}_{\beta\alpha}$$

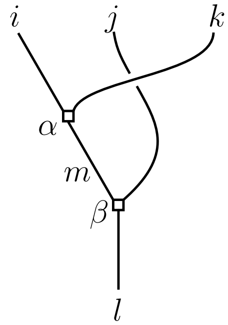


$$\tag{2.30}$$

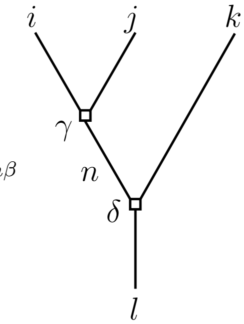
Another useful transformation we will use is the B-matrix, which is defined by

$$(\mathrm{id}_i \otimes c_{k,j}) \circ (\alpha \otimes \mathrm{id}_j) \circ \beta = \sum_{n,\gamma,\delta} B^{(ijk)l}_{\gamma n\delta,\, \alpha m\beta}\, (\gamma \otimes \mathrm{id}_k) \circ \delta \ . \tag{2.31}$$

Graphically, the B-matrix is defined as follows:



$$= \sum_{n,\gamma,\delta} B^{(ijk)l}_{\gamma n\delta,\, \alpha m\beta} \tag{2.32}$$



Using the F- and R-matrices, one can write for the expression on the left:

$$\begin{aligned}(\mathrm{id}_i \otimes c_{k,j}) \circ (\alpha \otimes \mathrm{id}_j) \circ \beta &\overset{(2.24)}{=} \sum_{p,\mu,\nu} F^{(ikj)l}_{\mu p\nu,\alpha m\beta} (\mathrm{id}_i \otimes c_{k,j}) \circ (\mathrm{id}_i \otimes \nu) \circ \mu \\ &\overset{(2.29)}{=} \sum_{p,\mu,\nu,\lambda} F^{(ikj)l}_{\mu p\nu,\alpha m\beta} R^{(kj)p}_{\lambda\nu} (\mathrm{id}_i \otimes \lambda) \circ \mu \\ &\overset{(2.25)}{=} \sum_{n,\gamma,\delta} \sum_{p,\mu,\nu,\lambda} F^{(ikj)l}_{\mu p\nu,\alpha m\beta} R^{(kj)p}_{\lambda\nu} G^{(ijk)l}_{\gamma n\delta,\mu p\lambda} (\gamma \otimes \mathrm{id}_k) \circ \delta \end{aligned} \tag{2.33}$$

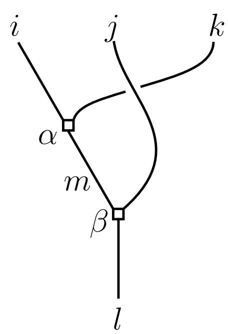
Inserting this into (2.31) and comparing both sides we obtain an expression for the B-matrix in terms of F- and R-matrices

$$B^{(ijk)l}_{\gamma n\delta,\alpha m\beta} = \sum_{p,\mu,\nu,\lambda} F^{(ikj)l}_{\mu p\nu,\alpha m\beta} R^{(kj)p}_{\lambda\nu} G^{(ijk)l}_{\gamma n\delta,\mu p\lambda} \ . \tag{2.34}$$

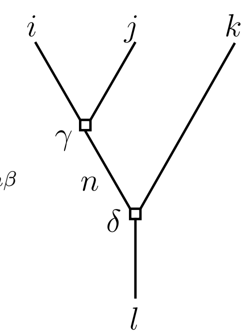
The reverse B-matrix is defined by

$$(\mathrm{id}_i \otimes c^{-1}_{j,k}) \circ (\alpha \otimes \mathrm{id}_j) \circ \beta = \sum_{n,\gamma,\delta} B^{-(ijk)l}_{\gamma n\delta,\, \alpha m\beta}\, (\gamma \otimes \mathrm{id}_k) \circ \delta \tag{2.35}$$

and pictorially:



$$= \sum_{n,\gamma,\delta} B^{-(ijk)l}_{\gamma n\delta,\, \alpha m\beta} \tag{2.36}$$



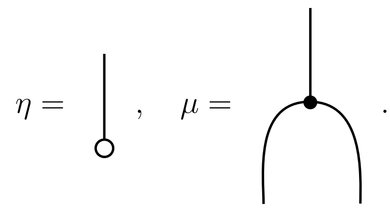
Its computation resembles that of the B-matrix according to (2.33) but using in the second step the reversed R-matrix from (2.29) because we work with the reverse braiding. One can easily verify that

$$B^{-(ijk)l}_{\gamma n\delta,\alpha m\beta} = \sum_{p,\mu,\nu,\lambda} F^{(ikj)l}_{\mu p\nu,\alpha m\beta} R^{-(kj)p}_{\lambda\nu} G^{(ijk)l}_{\gamma n\delta,\mu p\lambda} \ . \tag{2.37}$$

2.2 Algebras

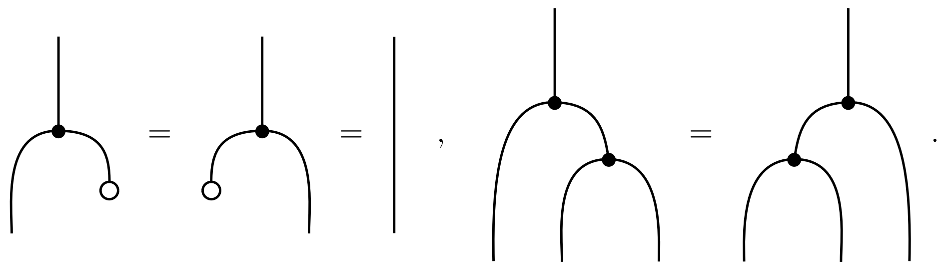
We now recall some algebraic notions following the conventions in [FRS].

Definition 2.2. Let $\mathcal{C}$ be a monoidal category. An *algebra* in $\mathcal{C}$ is an object $A \in \mathcal{C}$ equipped with morphisms $\eta : \mathbb{1} \to A$ (unit) and $\mu : A \otimes A \to A$ (product or multiplication) represented graphically[3]:



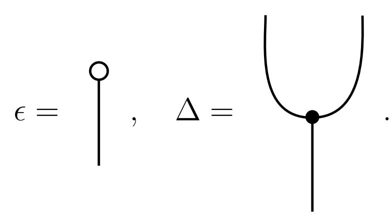
(2.38)

These morphisms are subject to the unitality and associativity conditions:



(2.39)

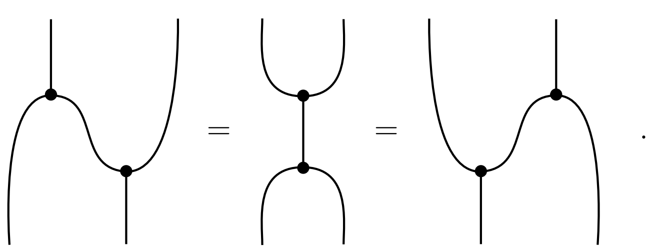
Dually, a *coalgebra* C in $\mathcal{C}$ is an algebra object in the opposite category $\mathcal{C}^{\mathrm{op}}$. It consists of morphisms $\epsilon : C \to \mathbb{1}$ (counit) and $\Delta : C \to C \otimes C$ (coproduct or comultiplication) represented by



(2.40)

that are subject to the counitality and coassociativity conditions (obtained from the reflection of (2.39) along the horizontal axis).

Definition 2.3. Let $\mathcal{C}$ be a monoidal category. A *Frobenius* algebra is an object A with an algebra structure (A, η, μ) and a coalgebra structure (A, ϵ, Δ) which are compatible in the following way:



(2.41)

[3] We often omit labels for string diagrams if labels are clear from context.

Let $\mathcal{C}$ be a braided category and (A, η, μ) an algebra in $\mathcal{C}$. Then, we define the *opposite* algebra $A^{\text{op}} = (A, \eta, \mu^{\text{op}})$ with opposite product

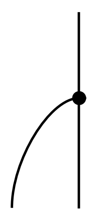
$$\mu^{\text{op}} := \mu \circ c_{A,A} \ . \tag{2.42}$$

The algebra A is called *commutative* if $\mu^{\text{op}} = \mu$. Similarly, for a coalgebra (C, ϵ, Δ) the opposite coalgebra $(C, \epsilon, \Delta^{\text{op}})$ is defined by the coproduct

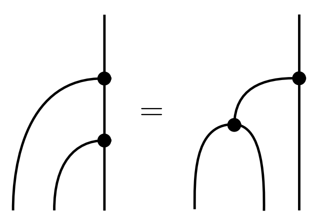
$$\Delta^{\text{op}} = c_{C,C}^{-1} \circ \Delta \tag{2.43}$$

and C is called *cocommutative* if $\Delta^{\text{op}} = \Delta$. Notice that composing the coproduct with the ordinary braiding $c_{C,C}$ instead of the reverse braiding in (2.43) would also give rise to a coalgebra structure. However, to be consistent with Frobenius algebras we fix the opposite coproduct as above. For instance, suppose that A is a Frobenius algebra with product μ and coproduct Δ. The opposite algebra A^{op} with product μ^{op} as in (2.42) and coproduct Δ^{op} as in (2.43) forms a Frobenius algebra.

Definition 2.4. 1. Let $\mathcal{C}$ be a monoidal category and A an algebra in $\mathcal{C}$. A *left A-module* is a pair $(M, \rho) \equiv {}_AM$ where M is a object in $\mathcal{C}$ and ρ is a left action of A on M, i.e. a morphism

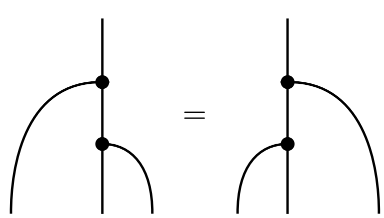
$\rho =$  $: A \otimes M \to M$ (2.44)

such that

 . (2.45)

A *right A-module* is a left A-module in $\mathcal{C}^{\text{mop}}$

2. Suppose A and B are two algberas. An A-B-bimodule M carries a left A-action $A \otimes M \to M$ and a right B-action $M \otimes B \to M$ which are compatible

 . (2.46)

We will write ${}_AM_B$ to indicate that M is an A-B-bimodule.

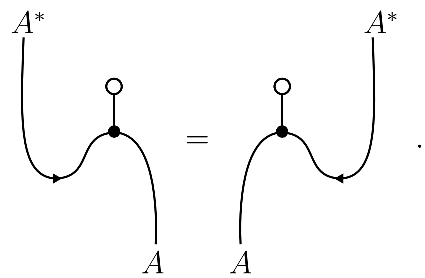
Remark 2.5. An algebra A has a canonical A-A-bimodule structure with actions given by its product. In this sense, the Frobenius condition (2.41) is equivalent to the coproduct Δ being an A-A-bimodule map.

Two algebras A and B are called *Morita equivalent*, if there exist an A-B-bimodule X and a B-A-bimodule Y such that $X \otimes_B Y \cong A$ and $Y \otimes_A X \cong B$ as bimodules.

Definition 2.6.

1. A Frobenius algebra A in $\mathcal{C}$ is Δ-*separable* if $\mu \circ \Delta = \mathrm{id}$, and for $\mathcal{C}$ fusion it is *special* if $\varepsilon \circ \eta \neq 0$ and $\mu \circ \Delta = \zeta\,\mathrm{id}$ for some $\zeta \in \Bbbk^\times$. We call A *normalised-special* if $\zeta = 1$, or, equivalently, if it is Δ-separable and special.

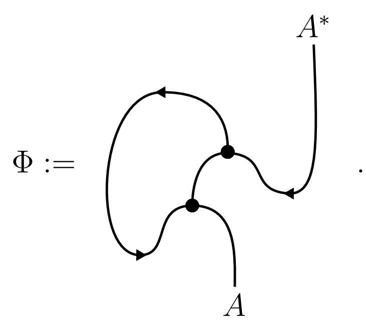
2. A Frobenius algebra A in a pivotal category $\mathcal{C}$ is called *symmetric* if



(2.47)

An algebra A in a tensor category $\mathcal{C}$ is called *simple* if it is simple as a bimodule over itself. It is called *haploid* if $\mathcal{C}(\mathbb{1}, A) = \Bbbk\,\eta$. A haploid algebra is automatically simple [FSc1, Lem. 4.5].

Given an algebra A in a pivotal category $\mathcal{C}$, define the morphism $\Phi : A \to A^*$ as



(2.48)

An algebra A is called *non-degenerate* if Φ is an isomorphism.

Lemma 2.7. Let A be a non-degenerate algebra in a pivotal tensor category $\mathcal{C}$. Then:

1. There is a unique coproduct and counit on A such that A becomes a symmetric Frobenius algebra, and such that the isomorphism $A \to A^*$ in (2.47) agrees with Φ in (2.48).

2. The Frobenius algebra in part 1 is Δ-separable, and it is normalised-special iff $\dim_{\mathcal{C}}(A) \neq 0$.

3. If $\mathcal{C}$ is spherical and A is simple, then $\dim_{\mathcal{C}}(A) \neq 0$.

Parts 1 and 2 of this lemma are proved in [KR1, Lem. 2.3]. Part 3 is proven in Lemma 2.9 using the additional constraint of sphericality.

Remark 2.8. The reason why we work with non-degenerate algebras instead of directly with Frobenius algebras is that being non-degenerate is a *property* of an algebra. Being Frobenius is, first of all, *more data* (coproduct and counit). It becomes a property when one adds the conditions of symmetry and Δ-separability. Namely, an algebra is non-degenerate iff it is Δ-separable symmetric Frobenius, cf. [KR1, Lem. 2.3]

The following Lemma proves part 3 of Lemma 2.7 and can be found in [RR1, Lem. A.1].

Lemma 2.9. Let $\mathcal{F}$ be a spherical multifusion category over an algebraically closed field $\Bbbk$ (of any characteristic) and let $A \in \mathcal{F}$ be a simple Δ-separable symmetric Frobenius algebra. Then $\dim_{\mathcal{F}}(A) \neq 0$.

Proof. To avoid cumbersome notation, in this proof we assume $\mathcal{F}$ to be strict. Let $A_{\mathrm{top}} := \mathcal{F}(\mathbb{1}, A)$ denote the *topological algebra* of A, cf. [FRS, Sec. 3.4]. It is an algebra over $\Bbbk$ via the product and unit

$$\mu_{\mathrm{top}}(x, y) := \mu \circ (x \otimes y) \quad , \quad 1_{\mathrm{top}} := \eta \ . \tag{2.49}$$

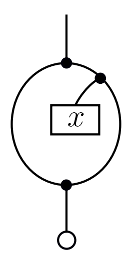
Define a pairing on A_{top} by

$$\langle x, y \rangle := \varepsilon \circ \mu_{\mathrm{top}}(x, y) \ . \tag{2.50}$$

Let $x \neq 0$ be an element in A_{top}. The non-degeneracy of A (cf. Remark 2.8) implies that $\Phi \circ x \neq 0$. Since $\mathcal{F}$ is semisimple, there is a $\varphi : A^* \to \mathbb{1}$ such that $\varphi \circ \Phi \circ x \neq 0$. Using the expression in (2.47) for Φ, it follows for $y = \big[\mathbb{1} \xrightarrow{\mathrm{coev}_A} A \otimes A^* \xrightarrow{\mathrm{id}_A \otimes \varphi} A\big]$ that

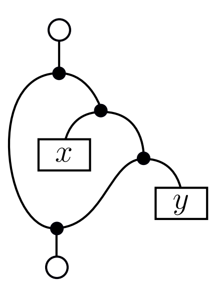
$$\langle x, y \rangle = \varphi \circ \Phi \circ x \neq 0 \ . \tag{2.51}$$

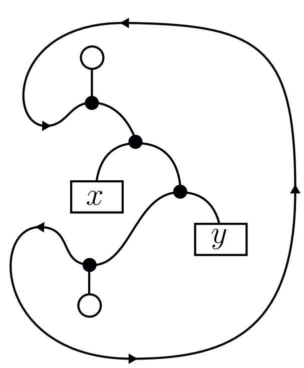
Therefore, the pairing on A_{top} is non-degenerate. Consider now the linear map $p : A_{\mathrm{top}} \to A_{\mathrm{top}}$ defined by

$p(x) =$ 

. (2.52)

Since A is Δ-separable, it follows that $p^2 = p$ and $p(\eta) = \eta$. The pairing satisfies the following invariance property:

$\langle p(x), y \rangle \overset{(1)}{=}$ 

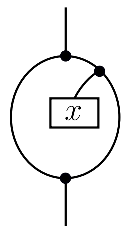
$\overset{(2)}{=}$ 

$\overset{(3)}{=} \langle x, p(y) \rangle$, (2.53) where (1) uses the associativity of A, (2) is immediate from sphericality of $\mathcal{F}$ and (3) can be verified using that A is a symmetric Frobenius algebra.

Write ${}_A\mathcal{F}_A(A, A)$ for the subspace of A-A-bimodule morphisms in $\mathcal{F}(A, A)$. Consider the linear map $\psi : {}_A\mathcal{F}_A(A, A) \to A_{\text{top}}$ given by $\psi(f) := f \circ \eta$. It satisfies

$$\psi(f) = f \circ \eta = f \circ p(\eta) = p(f \circ \eta) = p(\psi(f)) \ , \tag{2.54}$$

where we used that f is a bimodule morphism to exchange p with f. Hence, we have $\text{Im}(\psi) \subset \text{Im}(p)$. Conversely, let $x \in A_{\text{top}}$ and define $f_x \in \mathcal{F}(A, A)$ by

$f_x =$  . (2.55)

One checks that $f_x \in {}_A\mathcal{F}_A(A, A)$ and $\psi(f_x) = p(x)$, so that $\text{Im}(p) \subset \text{Im}(\psi)$, i.e. altogether $\text{Im}(p) = \text{Im}(\psi)$.

Since the algebra A is simple, ${}_A\mathcal{F}_A(A, A) = \Bbbk\,\text{id}$ (this uses that $\Bbbk$ is algebraically closed). Therefore, $\dim \text{Im}(p) \le 1$ and so in fact we have $\text{Im}(p) = \Bbbk\eta$. By non-degeneracy of the pairing in (2.50) we can find some y such that $\langle \eta, y \rangle \neq 0$. As η is a basis for $\text{Im}(p)$ there is $\lambda \in \Bbbk$ with $p(y) = \lambda\,\eta$. Using this, we compute

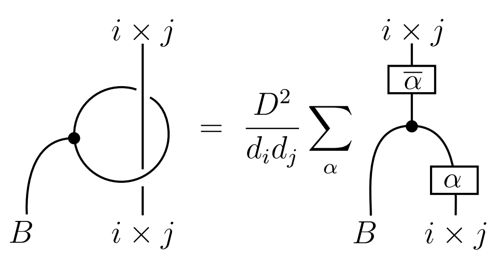
$$0 \neq \langle \eta, y \rangle = \langle p(\eta), y \rangle = \langle \eta, p(y) \rangle = \lambda \langle \eta, \eta \rangle \ . \tag{2.56}$$

Therefore $\langle \eta, \eta \rangle \neq 0$. Finally, $\dim_{\mathcal{F}}(A) = \varepsilon \circ \eta = \langle \eta, \eta \rangle \neq 0$. □

The condition that A is symmetric cannot be dropped from Lemma 2.9. For example, the two-dimensional Clifford algebra with one odd generator in the category of super vector spaces SVect is simple Δ-separable Frobenius (but not symmetric) and has dimension zero.

Suppose $\mathcal{C}$ is an MFC. For $B \in \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ let $\{\alpha\}$ be a basis of $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}(i \times j, B)$ and let $\{\overline{\alpha}\}$ be the dual basis of $\mathcal{C}\boxtimes\mathcal{C}^{\text{rev}}(B, i \times j)$ in the sense that $\overline{\alpha} \circ \beta = \delta_{\alpha,\beta}\,\text{id}_{i\times j}$, i.e. they define an $i \times j$-partition of B. A key ingredient in our proof of Theorem 7.1 will be the notion of a modular invariant algebra from [KR2, Def. 3.1] (using the alternative formulation in [KR2, Lem. 3.2]).

Definition 2.10. An algebra B in $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ is called *modular invariant* if $\theta_B = \text{id}_B$ and if the product is *S-invariant*, i.e.



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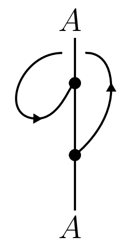
The full centre

We now recall the definition of the full centre of an algebra, as well as a result from [KR1] that will be used later for the proof of Theorem 7.2. We assume $\mathcal{C}$ to be a ribbon fusion category throughout this section.

Definition 2.11. The *left centre* $C_l(A)$ of a non-degenerate algebra A is the image of the idempotent $P_l : A \to A$,

$P_l =$  .

More details on the definition of left (and right) centres and their properties can be found e.g. in [FrFRS, Sec. 2.4]. The left centre inherits a natural structure of a non-degenerate algebra from the original algebra and it is also commutative [FrFRS, Prop. 2.37]. If A in Definition 2.11 is commutative, then the idempotent is trivial $P_l = \mathrm{id}_A$ and thus $C_l(A) = A$.

The tensor functor $T : \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \to \mathcal{C}$, $X \times Y \mapsto X \otimes Y$ admits a two-sided adjoint. Explicitly, the adjoint is given by $R : \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$, $X \mapsto \bigoplus_{i \in I} (X \otimes i^*) \times i$, see [KR2, Sec. 2.4].

Definition 2.12. Let $A \in \mathcal{C}$ be a non-degenerate algebra. The *full centre of* A is $Z(A) = C_l(R(A)) \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$.

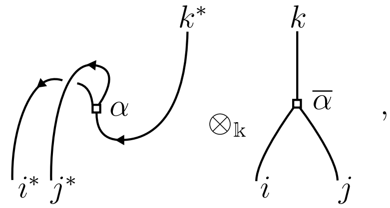
Remark 2.13. The full centre was first introduced in [FjFRS2, Def. 4.9]. Actually, one can assign to an algebra A in a monoidal category $\mathcal{M}$ a commutative algebra in the Drinfeld centre $\mathcal{Z}(\mathcal{M})$ which is characterised by a universal property [Da1]. The notion in Definition 2.12 is a special case of this more general characterisation.

The full centre is important in our construction because it produces modular invariant algebras. The following theorem is the first key input in our construction. It is shown in [KR1, Prop. 2.7] and [KR2, Thm. 3.18].

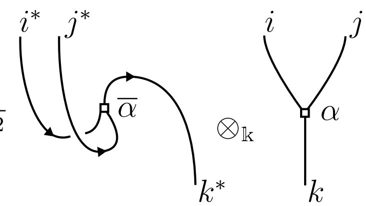
Theorem 2.14. Let $A \in \mathcal{C}$ be a simple non-degenerate algebra and let $\mathcal{C}$ be modular. Then the full centre $Z(A) \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ is a haploid commutative non-degenerate modular invariant algebra with $\dim_{\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}} Z(A) = D^2$.

Example 2.15. The fundamental example is to choose $A = \mathbb{1} \in \mathcal{C}$. We describe the Frobenius algebra structure of $Z(\mathbb{1})$ explicitly, as we will need it later. The expressions below are taken from [KR2, Eq. (2.58)], which gives $R(A)$, together with the observation that for $A = \mathbb{1}$ it is already commutative, and so equal to $Z(\mathbb{1})$. The underlying object of $Z(\mathbb{1})$ is $\bigoplus_{i \in I} i^* \times i$. The unit is given by the natural embedding of $\mathbb{1} \times \mathbb{1}$, while the counit is given by the projection to $\mathbb{1} \times \mathbb{1}$ times D^2. Let $\{\alpha\}$ be a basis of $\mathcal{C}(k, i \otimes j)$ and $\{\overline{\alpha}\}$ the dual basis in $\mathcal{C}(i \otimes j, k)$ in the sense that $\overline{\alpha} \circ \beta = \delta_{\alpha,\beta}$, i.e. a fusion basis as in (2.21). The product and coproduct are given by

$$\mu_{Z(\mathbb{1})} = \bigoplus_{i,j,k} \sum_{\alpha=1}^{N_{ij}^{\ k}}$$



$$\Delta_{Z(\mathbb{1})} = \bigoplus_{i,j,k} \sum_{\alpha=1}^{N_{ij}^{\ k}} \frac{d_i d_j}{d_k D^2}$$



(2.57)

The next theorem is the second key input for our construction, as it relates Morita equivalence to isomorphisms of full centres.

Theorem 2.16 ([KR1, Thm. 1.1])**.** Let A and B be simple non-degenerate algebras. Then the following are equivalent:

1. A and B are Morita equivalent.
2. $Z(A)$ and $Z(B)$ are isomorphic as algebras.

2.3 Module categories

The definitions in this section can be found in [ENOM] and [Sch].

Let $\mathcal{C}$ be a fusion category.

Definition 2.17. A left $\mathcal{C}$-module category is a finite semisimple $\mathbb{k}$-linear abelian category $\mathcal{M}$ together with a bilinear functor $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, a natural isomorphism

$$\lambda_{V,W,M} : (V \otimes W) \triangleright M \xrightarrow{\sim} V \triangleright (W \triangleright M) \tag{2.58}$$

and $l_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M$ for each object $V, W \in \mathcal{C}$ and $M \in \mathcal{M}$. The natural isomorphisms λ and l are subject to compatibility conditions, see [Os, Def. 2.6].

A (left) *$\mathcal{C}$-module functor* $F : \mathcal{M} \to \mathcal{N}$ between two left $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ is an abelian $\mathbb{k}$-linear functor equipped with a structure isomorphism

$$F(- \triangleright -) \cong - \triangleright F(-)$$

subject to compatibility conditions [Os, Def. 2.7]. Module natural transformations are natural transformations between such module functors, which respect their structure isomorphisms. Left $\mathcal{C}$-module categories, module functors and module natural transformations form the bicategory $\mathcal{C}$-Mod of $\mathcal{C}$-modules.

Two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are *Morita equivalent* if there is an equivalence $\mathcal{C}\text{-Mod} \simeq \mathcal{D}\text{-Mod}$ of bicategories. An equivalent condition is given by the following theorem due to [ENO2, Thm. 3.1] and [Mü, Rem. 3.18]:

Theorem 2.18. Two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if and only if the respective Drinfeld centres $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}(\mathcal{D})$ are braided equivalent.

Given a left $\mathcal{C}$-module category $\mathcal{M}$, the category of $\mathcal{C}$-module endofunctors on $\mathcal{M}$ forms a multi-fusion category

$$\mathcal{C}^*_{\mathcal{M}} := \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \tag{2.59}$$

which is also referred to as the *dual* of $\mathcal{C}$ with respect to $\mathcal{M}$. The tensor product is given by the composition of endofunctors. If $\mathcal{M}$ is an indecomposable module category, i.e. it is not equivalent to any product of non-trivial $\mathcal{C}$-module categories, the identity functor id_M is simple and therefore $\mathcal{C}^*_{\mathcal{M}}$ becomes a fusion category.

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor between fusion categories $\mathcal{C}$ and $\mathcal{D}$. There is a natural way to pull-back a $\mathcal{D}$-module structure to a $\mathcal{C}$-structure along F. Suppose $\mathcal{M}$ is a $\mathcal{D}$-module category. Then, $\mathcal{M}$ inherits the (pull-back) $\mathcal{C}$-module structure, with the action of some object X in $\mathcal{C}$ on an object M in $\mathcal{M}$ defined by

$$X \triangleright M := F(X) \triangleright M \ .$$

Example 2.19. The fusion category $\mathcal{C}$ can be seen as a left $\mathcal{C}$-module category by using the tensor product as the action, i.e. $\mathcal{M} = \mathcal{C}$ and $\triangleright = \otimes$. The coherence morphisms (2.58) correspond to the associators (which are trivial due to the strictness). This is the (left) *regular* module category over $\mathcal{C}$ and is denoted by ${}_{\mathcal{C}}\mathcal{C}$. Let A be an algebra in $\mathcal{C}$. Then, the category of right A-modules in $\mathcal{C}$, denoted by $\mathcal{C}_A$, forms a left $\mathcal{C}$-module category. The action $\mathcal{C}$ is given by tensoring on the left. More precisely, for a given right A-module (M, ρ) we define $V \triangleright (M, \rho) := (V \otimes M, \mathrm{id}_V \otimes \rho)$.

A *right* $\mathcal{C}$-module category is the same as a left $\mathcal{C}$-module category, but with an action from the right, i.e. a functor $\triangleleft : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ with corresponding coherence isomorphisms. Equivalently, a right $\mathcal{C}$-module is a left $\mathcal{C}^{\mathrm{mop}}$-module.

Let $\mathcal{M}$ be a left $\mathcal{C}$-module category. The left $\mathcal{C}$-action on $\mathcal{M}$ is equivalent to the left action of the opposite category $\mathcal{C}^{\mathrm{op}}$ on $\mathcal{M}^{\mathrm{op}}$. By pulling back this action via the dual functor $(-)^* : \mathcal{C}^{\mathrm{mop}} \to \mathcal{C}^{\mathrm{op}}$ we obtain a $\mathcal{C}^{\mathrm{mop}}$-module structure on $\mathcal{M}^{\mathrm{op}}$. In other words, $\mathcal{M}^{\mathrm{op}}$ is a right $\mathcal{C}$-module category with action

$$M \triangleleft X := X^* \triangleright M \tag{2.60}$$

for X in $\mathcal{C}$ and M in $\mathcal{M}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two fusion categories. A $\mathcal{C}$-$\mathcal{D}$-*bimodule category* $\mathcal{M}$ has both a left $\mathcal{C}$-module category structure and a right $\mathcal{D}$-module category structure with additional coherence isomorphisms

$$(X \triangleright M) \triangleleft Y \xrightarrow{\sim} X \triangleright (M \triangleleft Y) \tag{2.61}$$

where $X \in \mathcal{C}$, $M \in \mathcal{M}$ and $Y \in \mathcal{D}$. These natural isomorphisms are compatible with the left and right coherence isomorphisms [EGNO, Def. 7.1.7]. A $\mathcal{C}$-$\mathcal{D}$-bicategory is simply a left $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{mop}}$-category. We write ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ to denote the $\mathcal{C}$-$\mathcal{D}$-bimodule structure on $\mathcal{M}$. Once again, a natural example of a bimodule category is the fusion category $\mathcal{C}$ itself, seen as a $\mathcal{C}$-$\mathcal{C}$-bimodule category with action given by the tensor product from both sides.

Given a right module M and a left module N over an algebra A, one can take their balanced tensor product $M \otimes_A N$ which equalises the right action on M with the left action on N. There exists the following analogous construction for tensoring right module categories with left module categories over a fusion category $\mathcal{C}$.

Let $\mathcal{M}$ be a right $\mathcal{C}$-module category and $\mathcal{N}$ be a left $\mathcal{C}$-module category. Their *balanced tensor product* is a finite semisimple $\Bbbk$-linear abelian category $\mathcal{M} \boxtimes_{\mathcal{C}} N$, equipped with a $\Bbbk$-linear functor

$$F : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}.$$

The functor F is balanced in the sense that it is equipped with natural isomorphisms

$$F(- \triangleleft - \boxtimes -) \cong F(- \boxtimes - \triangleright -) \tag{2.62}$$

(compatible with the coherence isomorphisms). Moreover, it satisfies the universal property that for any other semisimple $\mathcal{A}$ the functor F induces an equivalence of categories:

$$\mathrm{Fun}_{bal}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{A}) \simeq \mathrm{Fun}(\mathcal{M} \boxtimes_{\mathcal{C}} N, \mathcal{A})$$

where the subscript on the left points at balanced functors in the sense of (2.62). Hence, the *balanced tensor product* is uniquely determined up to equivalence[4]. Its existence is proven in [ENOM, Prop. 3.5] presenting it as a category of module functors, i.e.

$$\mathcal{M} \boxtimes_{\mathcal{C}} N \cong \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}^{\mathrm{op}}, \mathcal{N}) \ . \tag{2.63}$$

Suppose ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ and ${}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$ are bimodule categories of the corresponding fusion categories indicated in the the subscripts. Their balanced tensor product $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ obtains a natural $\mathcal{C}$-$\mathcal{D}$-bimodule structure [ENOM, Rem. 3.6]. This allows one to define the tricategory **Fusion** [ENOM, Sec. 3.2] consisting of:

- Objects: Fusion categories.
- 1-morphisms: A 1-morphism $\mathcal{C} \to \mathcal{D}$ is a $\mathcal{C}$-$\mathcal{D}$-bimodule category.
- 2-morphisms: Bimodule functors.
- 3-morphisms: Bimodule natural isomorphisms.

The 1-category FUSION is obtained by truncation of **Fusion** and consists of equivalence classes of bimodule categories as morphisms. The composition of 1-morphisms in **Fusion** is given precisely by the balanced tensor product. The identity 1-morphism on a fusion category $\mathcal{C}$ according to this composition is the regular bimodule ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$. The bicategory of endomorphisms **Fusion**$(\mathcal{C}, \mathcal{C})$ is the monoidal bicategory consisting of $\mathcal{C}$-$\mathcal{C}$-bimodules with the balanced tensor product and it is denoted by $\mathcal{C}$-**Bimod**. Its truncation is a monoidal category consisting of bimodule categories and isomorphism classes of bimodule functors, and is denoted by ${}_{\mathcal{C}}$BIMOD.

[4] For any other $M \boxtimes N \to M \tilde{\boxtimes}_{\mathcal{C}} N$ satisfying the same universal property, their universal properties guarantees the existence of an equivalence $\Phi : M \boxtimes_{\mathcal{C}} N \xrightarrow{\sim} M \tilde{\boxtimes}_{\mathcal{C}} N$ such that $\Phi \circ \boxtimes_{\mathcal{C}} \cong \tilde{\boxtimes}_{\mathcal{C}}$. Such an equivalence Φ is unique up to natural isomorphism by the universal property of $\boxtimes_{\mathcal{C}}$

We are also interested in the invertibility of morphisms in **Fusion**, in particular the 1-morphisms (invertible 2- and 3-morphisms correspond to bimodule equivalences and bimodule natural isomorphisms). It is clear that a $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{M}$ is *invertible* if there exists some $\mathcal{D}$-$\mathcal{C}$-bimodule $\mathcal{N}$ and

$$\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \simeq \mathcal{C} \tag{2.64}$$

as $\mathcal{C}$-$\mathcal{C}$-bimodules and

$$\mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{D} \tag{2.65}$$

as $\mathcal{D}$-$\mathcal{D}$-bimodules. The existence of such $\mathcal{N}$ along with the above bimodule equivalences determines it uniquely up to bimodule equivalence. It is in fact equivalent to the definition from [ENOM, Def. 4.1]:

Definition 2.20. A $\mathcal{C}$-$\mathcal{D}$-bimodule category $\mathcal{M}$ is *invertible* if there exist bimodule equivalences

$$\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\mathrm{op}} \simeq \mathcal{C} \tag{2.66}$$

and

$$\mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{D} \ . \tag{2.67}$$

The maximal subgroupoid or *core* of the tricategory **Fusion** consists of fusion categories, invertible bimodule categories, bimodule equivalences and bimodule natural isomorphisms. It is called the *Brauer-Picard* 3-groupoid and it is denoted by

$$\mathbf{BrPic} = \mathit{core}(\mathbf{Fusion}) \subset \mathbf{Fusion} \ . \tag{2.68}$$

Its truncation is denoted by $\textsc{BrPic}$ and the group of its automorphisms on a fusion category $\mathcal{C}$ is the *Brauer-Picard group* $\textsc{BrPic}(\mathcal{C})$.

Module categories with traces

We review the notion of module categories with trace as introduced in [Sch].

Definition 2.21. Let $\mathcal{C}$ be a pivotal fusion category and let $\mathcal{M}$ be a left $\mathcal{C}$-module category. A *trace* Θ on $\mathcal{M}$ is a family of linear maps

$$\Theta_M : \mathcal{M}(M, M) \to \Bbbk \tag{2.69}$$

for each $M \in \mathcal{M}$ such that:

1. For any morphisms $f \in \mathcal{M}(M, N)$ and $g \in \mathcal{M}(N, M)$,

$$\Theta_M(g \circ f) \ = \ \Theta_N(f \circ g). \tag{2.70}$$

2. The pairing

$$\mathcal{M}(M, N) \times \mathcal{M}(N, M) \to \Bbbk, \ (f, g) \to \Theta_M(g \circ f) \tag{2.71}$$

is non-degenerate.

The trace Θ is called a *module trace*, if in addition, for all $V \in \mathcal{C}, M \in \mathcal{M}$,

$$\Theta_{V \triangleright M} = \Theta_M \circ \mathrm{tr}_L^{\mathcal{C}} \tag{2.72}$$

where the linear map $\mathrm{tr}_L^{\mathcal{C}} : \mathcal{M}(V \triangleright M, V \triangleright M) \to \mathcal{M}(M, M)$ is obtained by tracing on the left the V-argument using the pivotal structure of $\mathcal{C}$.

Remark 2.22. Given a pivotal fusion category $\mathcal{C}$, there is a natural module trace on the regular module category ${}_{\mathcal{C}}\mathcal{C}$ which is given by the left trace, i.e. $\Theta = \mathrm{tr}_L$. The properties (2.70) and (2.72) are easy to check using the definition of the left trace. That the left (or right) trace of a pivotal fusion category is non-degenerate in the sense of (2.71) is more involved and relies on the fusion structure [Sch, Rem. 3.8], [Tu, Lem. II.4.2.3].

Let $\mathcal{M}$ be a $\mathcal{C}$-module category with (module)-trace Θ. Then, any non-zero scalar $\lambda \in \Bbbk^\times$ gives rise to a further (module) trace $\lambda\Theta$. In fact, any module trace on the left regular module category ${}_{\mathcal{C}}\mathcal{C}$ is obtained by a multiple of the left trace.

Finally, we state a correspondence theorem of module categories with trace and symmetric special haploid Frobenius algebras due to Schaumann.

Theorem 2.23 ([Sch])**.** Let $\mathcal{C}$ be a pivotal fusion category. There is a one-to-one correspondence between:

1. Indecomposable left $\mathcal{C}$-module categories with module trace (up to equivalence).

2. Morita classes of symmetric special haploid Frobenius algebras in $\mathcal{C}$.

3 Reshetikhin-Turaev TQFT and mapping class group representations

In this section we introduce Reshetikhin-Turaev TQFT and its underlying modular functor. We review the notions of mapping class groups and their explicit (projective) action on the Reshetikhin-Turaev state spaces.

3.1 Reshetikhin-Turaev topological quantum field theories

The Reshetikhin-Tureav TQFT is a construction of a 3-dimensional TQFT associated to an MFC $\mathcal{C}$. In this section, we recall the geometric input of this TQFT and the underlying modular functor, namely the decorated bordism category $\mathrm{Bord}^{\mathcal{C}}$ and the mapping class groupoid $\mathrm{Mod}^{\mathcal{C}}$ following [Tu, Chapter IV].

A *connected decorated surface*, or just d-surface, is a (smooth, compact, oriented, closed)[5] surface Σ together with a finite set of framed points. A framed point is a point $p \in \Sigma$ equipped with a tangent vector v_p, a label $X \in \mathcal{C}$ and a sign $\epsilon_p \in \{\pm\}$. More generally, a d-surface is a coproduct, or disjoint union, of connected d-surfaces where one includes the empty coproduct, meaning the empty set $\emptyset$. We will refer to framed points also as punctures or just marked points, where the framing is always implied unless specified otherwise. The *negation* of a d-surface Σ is the d-surface obtained by orientation reversal $-\Sigma$ and replacing the tangent vectors by $-v_p$ and signs by $-\epsilon_p$. For simplicity we often leave signs unspecified for which we assume every sign to be positive. The negation of such labels will result to the corresponding dual objects which will become more clear later. Given a d-surface Σ, its *double* is the d-surface

$$\hat{\Sigma} = \Sigma \amalg -\Sigma \ . \tag{3.1}$$

A diffeomorphism of d-surfaces is a diffeomorphism between the underlying surfaces, which in addition preserves the framed points. Two such diffeomorphisms are *isotopic*, if they are isotopic through diffeomophisms of decorated surfaces.

A *decorated 3-manifold* is a smooth, oriented, compact 3-dimensional manifold with an embedded $\mathcal{C}$-coloured ribbon graph such that all ribbon ends lie on the boundary ∂M. The colouring and framing of the ribbon graph give the boundary ∂M the structure of a decorated surface. A *decorated bordism* $\Sigma \to \Sigma'$ between two decorated surfaces is a decorated 3-manifold M together with a boundary parametrisation, i.e. a d-diffeomorphism $\partial M \xrightarrow{\simeq} -\Sigma \amalg \Sigma'$. Two d-bordisms are *equivalent* if there is a diffeomorphism compatible with the ribbon graphs and boundary parametrisations.

The d-bordism category $\mathrm{Bord}^{\mathcal{C}}$ is the symmetric monoidal category formed by d-surfaces as objects and classes of d-bordisms as morphisms. The monoidal product is given by the disjoint union and the unit is the empty set $\emptyset$.

[5]For the purpose of this text, we may always assume that a surface is smooth, compact, oriented and closed unless specified otherwise. Smoothness is not a strictly stronger condition here, as for $n \leq 3$ topological and smooth n-manifolds are equivalent.

The construction of Reshetikhin-Turaev TQFT (RT-TQFT) assigns to any d-surface Σ a finite dimensional vector space $V^{\mathcal{C}}(\Sigma)$, called the state space, and to a d-bordism $(M;\Sigma,\Sigma',\varphi)$ it assigns a linear map

$$\mathcal{Z}^{\mathcal{C}}(M;\Sigma,\Sigma',\varphi) : V^{\mathcal{C}}(\Sigma) \to V^{\mathcal{C}}(\Sigma') \ , \tag{3.2}$$

called operator invariant, between the respective state spaces. In the tuple $(M;\Sigma,\Sigma',\varphi)$ the element φ denotes the boundary parametrisation $\varphi : \partial M \to -\Sigma \amalg \Sigma'$ which admits Σ as the *ingoing* and Σ' as the *outgoing* boundary. The RT-TQFT turns out to have gluing anomalies, which means that the operator invariants preserve the gluing of bordisms only up to scalars, i.e. functoriality holds only projectively. In order to fix that and produce an honest symmetric monoidal functor, one introduces the extended version of the bordism category.

The extended bordism category $\widehat{\mathrm{Bord}}^{\mathcal{C}}$ has objects of the form (Σ,λ) where $\Sigma \in \mathrm{Bord}^{\mathcal{C}}$ and $\lambda \subset H_1(\Sigma;\mathbb{R})$ is a Lagrangian subspace. Its morphisms are given by pairs (M,n) where M is a morphism in $\mathrm{Bord}^{\mathcal{C}}$ and n is an integer, also referred to as the weight. This gives an anomaly-free TQFT as a symmetric monoidal functor

$$\mathcal{Z}^{\mathcal{C}} : \widehat{\mathrm{Bord}}^{\mathcal{C}} \to \mathrm{Vect} \ . \tag{3.3}$$

Even though, the complete RT TQFT will be present in the thesis, we will mostly focus on one particular feature, that of mapping class group representations. These representations and how they behave are encoded by the modular functor.

The *mapping class groupoid* $\mathrm{Mod}^{\mathcal{C}}$ is the symmetric monoidal groupoid consisting of decorated surfaces as objects and isotopy classes of diffeomorphisms of decorated surfaces as morphisms. The mapping class groupoid can be considered as the subgroupoid of the bordism category via the symmetric monoidal functor

$$M : \mathrm{Mod}^{\mathcal{C}} \hookrightarrow \mathrm{Bord}^{\mathcal{C}} \tag{3.4}$$

which acts as the identity on objects and to a mapping class f it assigns the bordism

$$M(f) := (\Sigma \times [0,1];\Sigma,\Sigma, f \amalg \mathrm{id}) \tag{3.5}$$

where f appears in the boundary parametrisation by "twisting" the ingoing boundary. The bordism $M(f)$ is called the *mapping cylinder* of f as it consists of the underlying cylinder over Σ and the mapping class f, which appears in the boundary parametrisation.

Restricting the RT TQFT on the mapping class groupoid via the functor in (3.4) yields the modular functor. However, as with the TQFT itself functoriality holds only projectively due to the gluing anomalies. Similarly, one introduces the extended mapping class groupoid $\widehat{\mathrm{Mod}}^{\mathcal{C}}$ with the same objects as $\widehat{\mathrm{Bord}}^{\mathcal{C}}$ and pairs (f,n) as morphisms, where f is a mapping class and n is an integer. The outcome is now a symmetric monoidal functor

$$V^{\mathcal{C}} : \widehat{\mathrm{Mod}}^{\mathcal{C}} \to \mathrm{Vect} \tag{3.6}$$

called the modular functor. We will often omit writing $\mathcal{C}$ in the superscript when it is clear from the context.

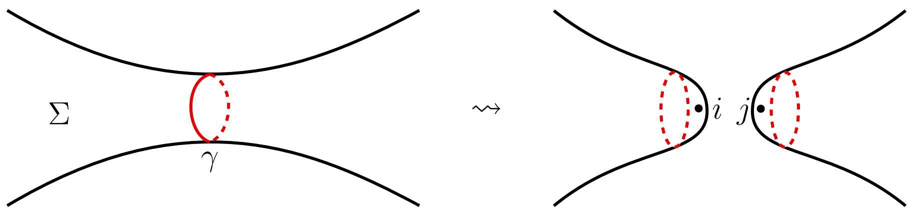


Figure 3.1: Cutting a d-surface Σ along a simple closed curve γ and the resulting d-surface obtained by inserting labels i and j in the filled holes.

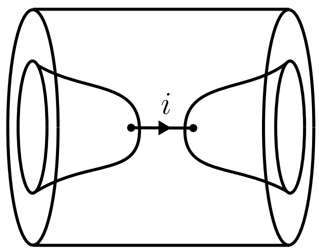


Figure 3.2: A local picture of N_i^γ around the cut along γ.

An important ingredient of the modular functor is formed by the gluing isomorphisms which describe how the state spaces behave under cutting and gluing of surfaces. Let γ be a simple closed curve on a d-surface Σ. By cutting along γ we obtain a surface $\Sigma\backslash\gamma$ which has two boundary circles $\gamma^{(1)}$ and $\gamma^{(2)}$ obtained from cutting. Denote by $\Sigma^\gamma(i,j)$ the d-surface obtained from filling the holes in $\Sigma\backslash\gamma$ with disks and inserting in the middle of each disk a framed point labelled by an object i for the $\gamma^{(1)}$ component and an object j for the $\gamma^{(2)}$ component, see Figure 3.1.

Then, there are gluing isomorphisms

$$g_{\gamma,\Sigma} = \bigoplus_{i\in I} g_i : \bigoplus_{i\in I} V(\Sigma^\gamma(i^*,i)) \to V(\Sigma) \tag{3.7}$$

which are natural in Σ and compatible with the symmetric monoidal structure. These isomorphisms are provided by the RT TQFT as follows: Define the 3-manifold

$$N_i^\gamma := (\Sigma^\gamma(i^*,i) \times [0,1])/\sim \tag{3.8}$$

where $\sim$ identifies on $\Sigma^\gamma(i^*,i) \times \{1\}$ the glued-in disks, therefore resulting in Σ. The manifold N_i^γ has an incoming boundary at 0 which is $\Sigma^\gamma(i^*,i)$ and an outgoing boundary at 1 being Σ. The two framed points on the ingoing boundary are connected via an i-labelled ribbon in the interior of N_i^γ, see Figure 3.2 and [FjFRS1, Sec. 2.6] for more details. Therefore, it defines a bordism

$$N_i^\gamma : \Sigma^\gamma(i^*,i) \to \Sigma \tag{3.9}$$

in $\mathrm{Bord}^{\mathcal{C}}$. The TQFT evaluation on this bordism defines the i'th summand g_i in (3.7), i.e. $g_i = \mathcal{Z}^{\mathcal{C}}(N_i^\gamma)$.

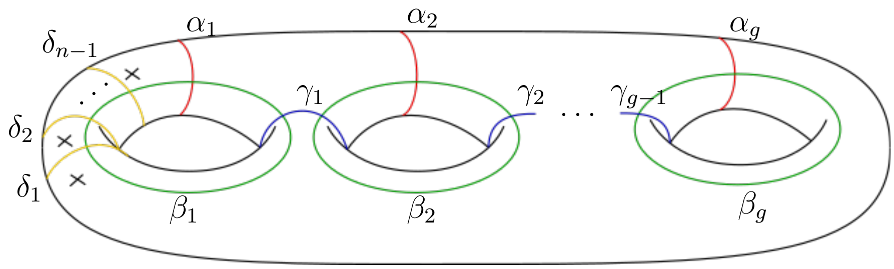


Figure 3.3: Generator curves of the (unframed) pure mapping class group.

For a d-surface Σ, the group of automorphisms $\mathrm{Mod}^{\mathcal{C}}(\Sigma) := \mathrm{Mod}^{\mathcal{C}}(\Sigma, \Sigma)$ is called the *mapping class group of* Σ. This is closely related to the ordinary geometric definition of mapping class groups as in [FM, Chapter 2]. For instance, let $(X, \epsilon) \in ob(\mathcal{C}) \times \{\pm\}$ be the label of every framed point on a d-surface Σ. Then, $\mathrm{Mod}^{\mathcal{C}}(\Sigma)$ corresponds to the framed mapping class group $\mathrm{Mod}(\Sigma)$ as in [FM, Chapter 2]. Here, we include a superscript of $\mathcal{C}$ to indicate that framed points carry labels. In general, $\mathrm{Mod}^{\mathcal{C}}(\Sigma)$ will be a subgroup of $\mathrm{Mod}(\Sigma)$ as its elements are required to also preserve labels.

Furthermore, consider the subgroup of diffeomorphisms (up to isotopy), which fix each framed point pointwise. This subgroup is called the (framed) *pure mapping class group* and we denote this by $\mathrm{PMod}(\Sigma)$, where we omit the superscript of $\mathcal{C}$ as such diffeomorphisms preserve by definition the label of each framed point. The mapping class group $\mathrm{Mod}(\Sigma_{g,n})$ of a surface $\Sigma_{g,n}$ with genus g and n framed points will be also denoted by $\mathrm{Mod}_{g,n}$ for short and the associated pure mapping class group by $\mathrm{PMod}_{g,n}$. In fact, there is a short exact sequence

$$1 \to \mathrm{PMod}_{g,n} \to \mathrm{Mod}_{g,n} \to S_n \to 1 \tag{3.10}$$

where the first arrow is the inclusion of $\mathrm{PMod}_{g,n}$ in $\mathrm{Mod}_{g,n}$ and the second arrow is determined the restriction of mapping classes onto the set of framed points. The *unframed* version is obtained by simply omitting the data of tangent vectors in the description above. The unframed pure mapping class group is finitely generated by Dehn twists around the simple closed curves shown in figure 3.3 [FM, Section 4.4.4]. A generating set for the framed pure mapping class group $\mathrm{PMod}_{g,n}$ is then obtained by adding Dehn twists around each point to the generating set.

Example 3.1. Genus 0 mapping class groups are directly related to braid groups. Let $\mathbb{D}^2_n$ denote the unit disk with n framed points on the x-axis ordered from left to right and framings directed to the right. For labels $X_1, \dots, X_n$ in $\mathcal{C}$ the braid group with such labels can be defined as the mapping class group of the disk

$$\mathrm{B}^{\mathcal{C}}_n(X_1, \dots, X_n) := \mathrm{Mod}(\mathbb{D}^2_n(X_1, \dots, X_n), \partial) \tag{3.11}$$

where the definition of the mapping class group is modified for surface with boundary, by requiring that diffeomorphisms fix pointwise the boundary. As before, we omit writing the labels $X_1, \dots, X_n$ when it is clear from context. Forgetting the labels or equivalently taking $X = X_1 = \cdots = X_n$ recovers the ordinary notion of the framed n-braid group denoted by B_n. Moreover, if one forgets the framing, then one obtains the unframed version.

The pure braid group PB_n is the subgroup in B_n consisting of diffeomorphisms which fix every framed point and the analogue of (3.10) holds.

By capping the boundary of $\mathbb{D}^2_n$ with a disk one obtains the sphere with the same framed points (and possible labels). Pick any point p on the glued disk (and any framing) such that the sphere has $n+1$ framed points. There is an isomorphism [FM, Sec. 4.2.5]

$$B_n \xrightarrow{\sim} \mathrm{Mod}(\mathbb{S}^2_{n+1})_p \tag{3.12}$$

where $\mathrm{Mod}(\mathbb{S}^2_{n+1})_p$ is the stabiliser of the "new" framed point p. This is by extending a mapping class to the glued disk via the identity. If we include $\mathcal{C}$-labels $X_1, \dots, X_n$, we may write $B_n^{\mathcal{C}}(X_1, \dots, X_n) \cong \mathrm{Mod}_{\mathbb{S}^2_{n+1}(X_1,\dots,X_n,Y)}$ where Y is the label of p and we assume $Y \notin \{X_1, \dots, X_n\}$.

The modular functor obtained from the RT TQFT as described above gives rise to representations over the mapping class groups. Due to the gluing anomaly there are two ways of describing these representations: as honest representations over extended mapping class groups $\widehat{\mathrm{Mod}}^{\mathcal{C}}(\Sigma)$ or as projective representations over the (non-extended) mapping class groups $\mathrm{Mod}^{\mathcal{C}}(\Sigma)$.

Let (Σ, λ) be an extended d-surface, i.e. in $\widehat{\mathrm{Mod}}^{\mathcal{C}}$ and (f, n) an element in the extended mapping class group $\widehat{\mathrm{Mod}}^{\mathcal{C}}(\Sigma)$. Then, the (extended) modular functor (3.6) encodes by $V(f, n)$ the action of (f, n) on the state space $V^{\mathcal{C}}(\Sigma, \lambda)$. This functorial assignment makes $V^{\mathcal{C}}(\Sigma, \lambda)$ into a representation over $\widehat{\mathrm{Mod}}^{\mathcal{C}}(\Sigma)$.

Forgetting the extended structure and working with the gluing anomalies, we get projective representations $V^{\mathcal{C}}(\Sigma)$ over the mapping class group $\mathrm{Mod}^{\mathcal{C}}(\Sigma)$. From now on, for simplicity we will only work with the non-extended mapping class groups as the projectivity of the associated representations will not pose a problem in our discussions. Notice that the extended mapping class group relates to the ordinary via the short exact sequence

$$0 \to \mathbb{Z} \to \widehat{\mathrm{Mod}}_{g,n} \to \mathrm{Mod}_{g,n} \to 1 \tag{3.13}$$

where the first arrow maps $n \in \mathbb{Z}$ to the weighted identity mapping class (id, n) and the second arrow is the projection onto the first factor $(f, n) \mapsto f$.

Remark 3.2. For an MFC $\mathcal{C}$ define the scalars

$$p_\pm = \sum_{i \in I} \theta_i^{\pm 1} d_i^2. \tag{3.14}$$

The projective factors appearing from the gluing anomalies are integer powers of the so-called *anomaly factor* p_+/p_-. If $p_+/p_- = 1$ then $\mathcal{C}$ is called anomaly-free and its RT TQFT resp. its modular functor descend to honest symmetric monoidal functors on the non-extended bordism category resp. non-extended mapping class groupoid.

The Drinfeld centre $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ of any MFC is automatically anomaly-free. Its RT TQFT of Turaev-Viro type in the sense that $\mathcal{Z}^{\mathcal{Z}(\mathcal{C})} \cong \mathcal{Z}^{\mathrm{TV},\mathcal{C}}$ with the latter being the Turaev-Viro TQFT of $\mathcal{C}$ [TV].

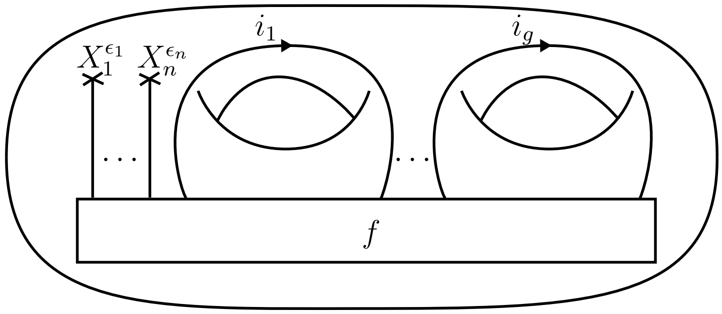


Figure 3.4: A handlebody with embedded ribbon graph and boundary $\Sigma_{g,n}$. The coupon is labelled by a morphism $f \in \mathcal{C}(\mathbb{1}, X_1^{\epsilon_1} \otimes \cdots \otimes X_n^{\epsilon_n} \otimes L^{\otimes g})$.

Let Σ be a d-surface and fix an ordering of its labels $(X_1, \epsilon_1), \ldots, (X_n, \epsilon_n)$. The state space $V(\Sigma_{g,n}) \equiv V_{g,n}$ is isomorphic to the morphism space

$$\mathcal{C}(\mathbb{1}, X_1^{\epsilon_1} \otimes \cdots \otimes X_n^{\epsilon_n} \otimes L^{\otimes g}), \tag{3.15}$$

where $X^+ \equiv X$, $X^- \equiv X^*$ and $L = \bigoplus_{i \in I} i \otimes i^*$ from (2.15). For a vector $f \in V_{g,n}$, consider the handlebody with an embedded ribbon graph in Figure 3.4, where the i'th ribbon strand is directed upwards respectively downwards if $\epsilon_i = +$ respectively $\epsilon_i = -$ and the coupon is labelled by the morphism f. In this figure, the framing is the one where all tangent vectors point to the right. This represents a d-bordism $\emptyset \to \Sigma_{g,n}$. Then, f is obtained by evaluating the RT TQFT on this bordism, which gives a geometric picture of the vectors in $V_{g,n}$.

An important property of the modular functor associated to the RT-TQFT is that there exists a non-degenerate pairing

$$\mathrm{d}_\Sigma : V^{\mathcal{C}}(\Sigma) \otimes V^{\mathcal{C}}(-\Sigma) \to \Bbbk \tag{3.16}$$

(see [Tu, Eq. III.(1.2.4)] and [Tu, Thm. III.2.1.1]). Here, natural means that the pairing is compatible with d-diffeomorphisms. In particular, it is invariant under the action of the mapping class group, i.e. for any $(x, y) \in V^{\mathcal{C}}(\Sigma) \times V^{\mathcal{C}}(-\Sigma)$ and any mapping class $f \in \mathrm{Mod}(\Sigma)$

$$\mathrm{d}_\Sigma(f.x, f.y) \;=\; \mathrm{d}_\Sigma(x, y)\ . \tag{3.17}$$

In fact, d_Σ is the result of applying the RT TQFT on the cylinder $\Sigma \times [0, 1]$ seen as a bordism $\Sigma \amalg -\Sigma \to \emptyset$. For a mapping class f on Σ the diffeomorphism $f \times \mathrm{id}_{[0,1]}$ does not change the equivalence class of $\Sigma \times [0, 1]$, which implies invariance of the pairing under mapping class group action.

Example 3.3. Let Σ be a d-surface without marked points and let H_Σ be the handlebody bounding Σ. Evaluating the TQFT on H_Σ seen as a bordism $\emptyset \to \Sigma$ gives a vector

$$x := \mathcal{Z}^{\mathcal{C}}(H_\Sigma) \in V^{\mathcal{C}}(\Sigma) \tag{3.18}$$

Similarly, evaluating the handlebody with reversed orientation $-H_\Sigma : \emptyset \to -\Sigma$ gives a vector

$$\hat{x} := \mathcal{Z}^{\mathcal{C}}(-H_\Sigma) \in V^{\mathcal{C}}(-\Sigma)\ . \tag{3.19}$$

One can show that x and $\hat{x}$ are dual with respect to the pairing d_Σ in the sense that

$$\mathrm{d}_\Sigma(x, \hat{x}) \neq 0 \ . \tag{3.20}$$

There are two different ways of showing (3.20): the first one uses a topological argument subject to general 3D TQFTs and the second one involves computing the pairing explicitly for RT TQFTs. We will describe the first one and refer to specific equations in [Tu] for the second one.

By definition (and functoriality of TQFTs) $\mathrm{d}_\Sigma(x, \hat{x})$ is obtained by gluing H_Σ and $-H_\Sigma$ onto the (bend down) cylinder $\Sigma \times [0, 1]$ and then applying the TQFT on this closed manifold. Notice that gluing only $-H_\Sigma$ onto the cylinder amounts to the bordism $H_\Sigma : \Sigma \to \emptyset$ and thus the resulting closed manifold M is obtained by gluing the handlebody H_Σ to itself via the identity. Topologically, M is the connected sum of g copies of $\mathbb{S}^1 \times \mathbb{S}^2$ where g is the genus of H_Σ, i.e. $M = \#^g(\mathbb{S}^1 \times \mathbb{S}^2)$. For (normalised) TQFTs where $\mathcal{Z}(\mathbb{S}^3) \neq 0$ and $V(\mathbb{S}^2) \cong \Bbbk$ we have

$$\mathrm{d}_\Sigma(x, \hat{x}) = \mathcal{Z}(\#^g(\mathcal{S}^1 \times \mathcal{S}^2)) = D^g \mathcal{Z}(\mathbb{S}^1 \times \mathbb{S}^2)^g = D^g \neq 0 \tag{3.21}$$

where $D := \mathcal{Z}(\mathbb{S}^3)^{-1}$ and we apply multiple times the multiplicative property $\mathcal{Z}(M_1 \# M_2) = \mathcal{Z}(\mathbb{S}^3)^{-1} \mathcal{Z}(M_1)\mathcal{Z}(M_2)$ of such normalised TQFTs [CR, Prop. 2.10]. For RT TQFTs D is the usual fixed square root of the global dimension, i.e. $D^2 = \sum_{i \in I} d_i^2$.

For the second approach, which uses the explicit formulas, use the fact that the cylinder is presented by the special ribbon graph shown in [Tu, Fig. IV.2.4] and use formula [Tu, Eq. (IV.2.3.a)].

3.2 Mapping class group action on RT state spaces

We will only need explicit expressions for the action of the pure mapping class group. Fix a surface $\Sigma_{g,n}$ of genus g and n framed points labelled by simple objects $l_1, \dots, l_n \in I$. Recall that the pure mapping class group $\mathrm{PMod}_{g,n}$ is the subgroup consisting of mapping classes that fix the framed points pointwise. For a simple closed curve γ on $\Sigma_{g,n}$ one can define the Dehn twist T_γ along γ as a mapping class in $\mathrm{PMod}_{g,n}$. It is obtained by cutting out an annular neighbourhood of γ, performing a full twist and gluing it back to the surface. A set of generators of the pure mapping class group is given as the set of Dehn twists along the curves in Figure 3.5 and also (due to the framings) Dehn twists T_{λ_k} around each framed point, see [FM, Corollary 4.15]. We further define the so-called S-transformations by $S_k := T_{\alpha_k} \circ T_{\beta_k} \circ T_{\alpha_k}$ and we replace in our generating set the Dehn twists T_{β_k} by the corresponding S-transformations S_k.

Following [Tu, Ch. IV] we can now describe explicitly how these generators act on vectors of the state space $V^{\mathcal{C}}(\Sigma_{g,n})$ up to projectivity. Recall from (3.15) that this vector space is identified with the morphism space

$$\mathcal{C}(\mathbb{1}, l_1 \otimes \dots \otimes l_n \otimes L^{\otimes g}) \tag{3.22}$$

which in turn, by definition of L decomposes into a direct sum

$$V^{\mathcal{C}}(\Sigma_{g,n}) = \bigoplus_{i_1,\dots,i_g \in I} \mathcal{C}(\mathbb{1}, l_1 \otimes \dots \otimes l_n \otimes i_1 \otimes i_1^* \otimes \dots \otimes i_g \otimes i_g^*) \ . \tag{3.23}$$

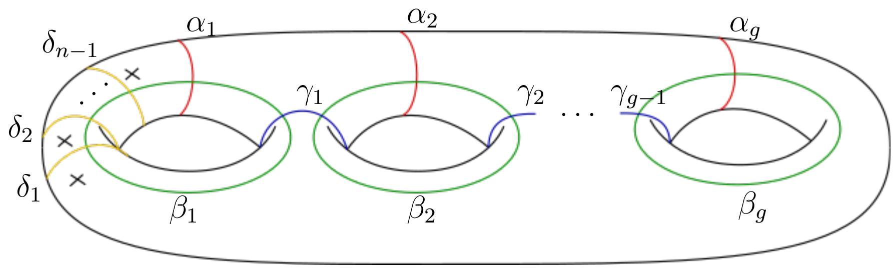


Figure 3.5: Generators of the unframed pure mapping class group.

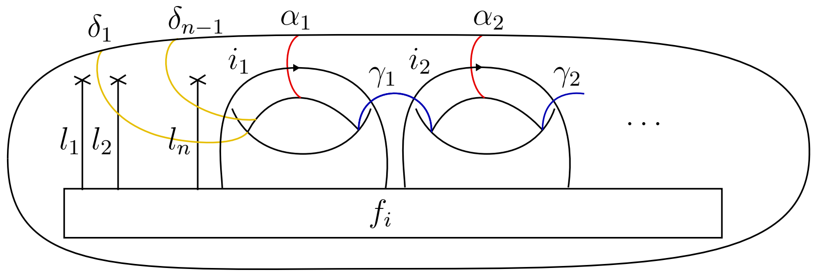
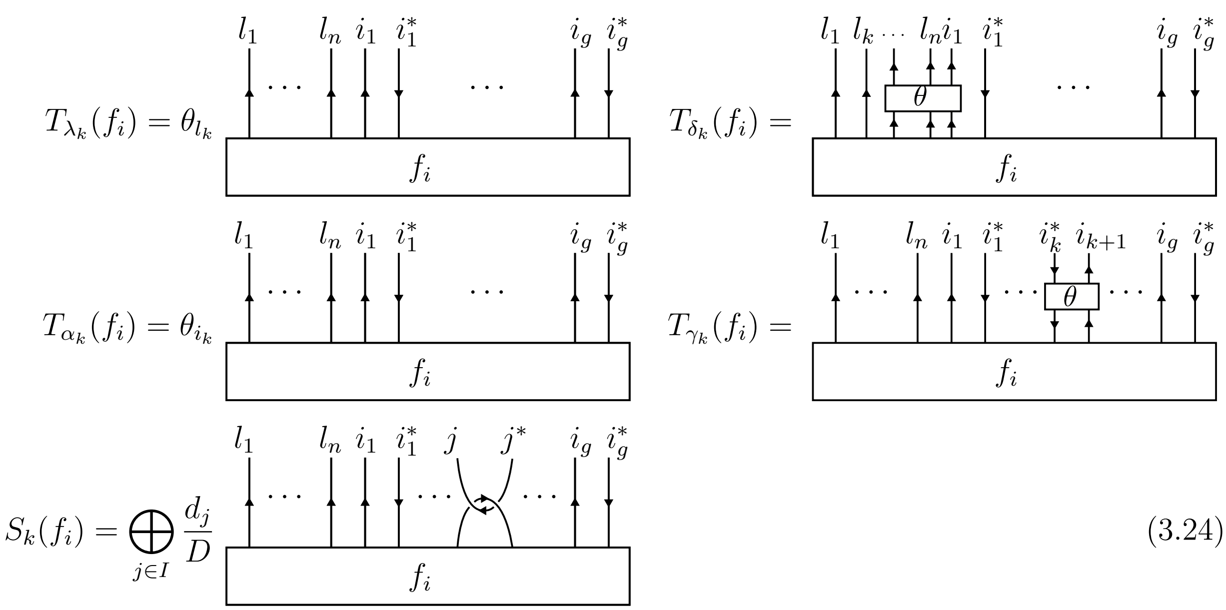


Figure 3.6: Arrangement of coupon and ribbons in the handlebody used to derive the mapping class group action on the Hom-space (3.22).

For $i = (i_1, \dots, i_g) \in I^g$ we denote by V_i the corresponding summand in (3.23). For a vector $f_i \in V_i$, the mapping class group generators act as follows (up to a projective factor):



(3.24)

Here, we used θ to denote the twist morphism in $\mathcal{C}$ and $\theta_i \in \Bbbk^\times$ to denote the twist eigenvalue corresponding to a simple object i. Note that all generators except for S_k map the direct summand V_i back to itself and have eigenvalues in the set of twist eigenvalues $\{\theta_i\}_{i \in I}$.

To arrive at these expressions, consider a handlebody with a coupon labelled by f_i as in Figure 3.6. The pure mapping class group acts on the state space assigned to the boundary surface via mapping cylinders as in (3.5). Thus the action of a mapping class h on f_i is obtained by gluing the mapping cylinder $M(h)$ over $\Sigma_{g,n}$ to the handlebody.

For a simple closed curve γ on $\Sigma_{g,n}$, which is contractible with respect to the handlebody, the Dehn twist T_γ acts by twisting the ribbons passing through the disc in the handlebody bounded by the curve. This is the origin of the twists θ in the first four equations in (3.24).

The generator S_k does not correspond to a Dehn twist along a curve, but rather to a composition of those. Details on the derivation of the last equation in (3.24) can be found e.g. in [BK, Def. 3.1.15].

4 Surface defects in TQFTs

In this section we will review defects and boundary conditions in 3D TQFT. The algebraic objects that were defined in Section 2 such as symmetric special Frobenius algebras and module categories with trace will become relevant as they model topological defects and boundary conditions for TQFTs of Reshetikhin-Turaev type.

Defects provide a model of introducing embedded manifolds of lower dimension, called *strata*, whose nature describes the interplay of theories meeting there. For instance, a defect of codimension one may divide two theories which live in the bulk (the top-dimensional regions). Lower dimensional defects encode information about the adjacent higher dimensional defects. Boundaries are codimension one boundaries of the worldvolume where the theory ends. These are not to be confused with gluing boundaries, which encode the functorial nature of the field theory. A familiar example of such boundaries are in the 2D case of open-closed TQFTs.

4.1 Strata and TQFTs with defects

In the case of three-dimensional TQFTs, in which we are interested, codimension one defects correspond to surface defects or *domain walls* and one may include lower-dimensional defects such as line defects and point defects. Three-dimensional and later n-dimensional TQFTs with defects have been axiomatised in [CMS] and [CRS1] as in the Atiyah-Segal formulation as a symmetric monoidal functor now on the source category $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ of bordisms with defects. This is achieved by enlarging the ordinary bordism category Bord_n (consisting of closed oriented $(n-1)$-manifolds and n-bordisms) into $\mathrm{Bord}_n^{\mathrm{str}}$ of *stratified* bordisms.

A closed *stratified* n-manifold means a smooth, oriented, compact n-manifold M without boundary and with a filtration $M = F_n \supset \cdots \supset F_0 \supset F_{-1} = \emptyset$ such that:

- $M_k = F_k \backslash F_{k-1}$ is a k-dimensional open smooth manifold with choice of orientation such that the orientation of M_n agrees with that of M. The connected components M_k^α are the k-strata of M.
- If $M_k^\alpha \cap \overline{M}_j^\beta$ is non-empty, then M_k^α is contained in $\overline{M}_j^\beta$.
- There are finitely many strata.

Generalising to *stratified* n-manifolds M with boundary ∂M, one requires that:

- The filtration of M induces a stratification in the interior of M.
- $M_k = F_k \backslash F_{k-1}$ is a properly embedded submanifold and therefore $\partial M_k \subset \partial M$, such that all strata meet transversally the boundary of M.
- The restriction of the filtration on the boundary induces a stratification on ∂M, with k-strata in ∂M corresponding to $(k+1)$-strata in M inheriting their associated orientation.

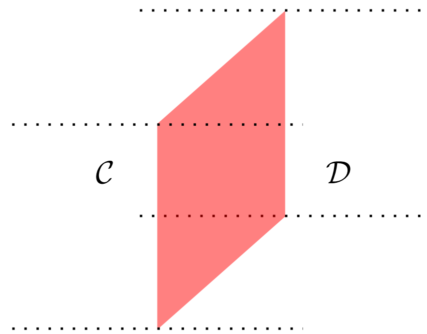


Figure 4.1: A surface defect between two theories labelled by $\mathcal{C}$ and $\mathcal{D}$.

A *morphism* between stratified manifolds $f : M \to M'$ is a map that is compatible with the stratificiations. It sends k-strata in M to k-strata in M', for details see [CRS1].

Based on these definitions one can define stratified n-bordisms in the obvious way and obtain the definition of the symmetric monoidal category $\mathrm{Bord}_n^{\mathrm{str}}$ of stratified bordisms. A defect datum $\mathbb{D}$ provides defect labels for strata of any dimension. In particular, it consists of label sets D_k for each $0 \leq k \leq n$ and additional adjacency maps which describe the relation between labels on k-strata and their adjacent higher-dimensional strata. A defect of dimension k is then a k-stratum carrying a label of D_k. Decorating in this fashion stratified bordisms defines the defect bordism category $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ (see [CRS1] for details).

Definition 4.1. An n-dimensional TQFT with defects $\mathbb{D}$ is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect} \ . \tag{4.1}$$

Defects are naturally described by higher categorical structures, as there is an associated tricategory of defects (or n-category for n-dimensional TQFTs with defects). Objects are n-dimensional defects and k-morphisms are $(n-k)$-dimensional defects with a natural composition provided by their topological nature.

Defects in TQFTs of RT type have been first studied in [KaS, FSV] and an explicit construction of a defect RT TQFT appeared in [CRS2]. The construction in [CRS2] assumed that a single RT TQFT lives in the bulk with its associated modular fusion category $\mathcal{C}$ serving as the single label in $D_3 = *$. Defect TQFTs with different bulk theories of RT type have been constructed in [KMRS]. We will also use the associated MFC to refer to the type of the bulk theory living in some 3-stratum.

Suppose that a surface defect separates two 3-dimensional regions where theories of type $\mathcal{C}$ and $\mathcal{D}$ live. According to [FSV] such a surface defect exists if there is a braided equivalence

$$\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{A}) \tag{4.2}$$

where $\mathcal{Z}(\mathcal{A})$ is the Drinfeld centre of some pivotal fusion category $\mathcal{A}$. In other words, domain walls between RT TQFTs of type $\mathcal{C}$ and $\mathcal{D}$ exist if $\mathcal{C}$ and $\mathcal{D}$ in the same Witt class.

The fusion category $\mathcal{A}$ is interpreted as the category of Wilson lines on the surface, with the fusion structure originating from the topological manipulation of such Wilson lines. Bulk Wilson lines approaching from the bulk labelled by $\mathcal{C}$ can come arbitrarily close to the surface defect and be seen as Wilson lines on the surface. However, since they are not bound to the surface they can "pass through" other surface Wilson lines. All in all, this produces a braided functor

$$\mathcal{C} \to \mathcal{Z}(\mathcal{A}) \tag{4.3}$$

where the half-braiding in $\mathcal{Z}(\mathcal{A})$ corresponds to the "passing through" operation. Similarly, bulk Wilson lines approaching from the other side give a braided functor

$$\mathcal{D}^{\mathrm{rev}} \to \mathcal{Z}(\mathcal{A}) \ . \tag{4.4}$$

The functors (4.3) and (4.4) form the braided equivalence (4.2).

Introducing the fusion category $\mathcal{A}$ allowed one to view Wilson lines on a fixed surface defect. Suppose now there is a line defect separating the fixed surface defect with some other surface defect. A similar manipulation of surface Wilson lines close to this line defect leads one to the structure of an $\mathcal{A}$-module category. Further analysis of *sphere* defects leads to the existence of a module trace on the $\mathcal{A}$-module categories, cf. [KMRS, Prop. 3.9]. For that reason, it is argued that the relevant bicategory of surface defects from $\mathcal{C}$ to $\mathcal{D}$ is the bicategory $\mathcal{A}\text{-Mod}^{tr}$ of $\mathcal{A}$-module categories with module trace. This description is independent of the choice of $\mathcal{A}$ as fixing any other such pivotal fusion category $\mathcal{A}'$ corresponding to another surface defect one can show $\mathcal{Z}(\mathcal{A}) \simeq \mathcal{Z}(\mathcal{A}')$, or equivalently $\mathcal{A}\text{-Mod} \simeq \mathcal{A}'\text{-Mod}$ (Theorem 2.18).

Remark 4.2. One may consider defects internal to the theory $\mathcal{C}$, i.e. when $\mathcal{C}$ lives in each 3-dimensional stratum ($D_3 = *$). In this case, we obtain a natural choice for the fusion category $\mathcal{A}$ in (4.2); $\mathcal{C}$ itself as it is modular fusion and we have the braided equivalence from Proposition 2.1

$$\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{C}) \ . \tag{4.5}$$

Therefore, module categories (with module trace) over $\mathcal{C}$ appear naturally in the study of surface defects in $\mathcal{C}$.

The construction of RT TQFT with defects can be made more explicit using the notion of symmetric Δ-separable Frobenius algebras instead of module categories with traces following [CRS2]. We briefly recall how this construction of the RT TQFT with defects associated to a single MFC $\mathcal{C}$ works. The defect labelling sets are:

- $D_3^{\mathcal{C}} = \{\mathcal{C}\}$,
- $D_2^{\mathcal{C}} = \{\Delta\text{-separable symmetric Frobenius algebras in } \mathcal{C}\}$,
- $D_1^{\mathcal{C}} = \{\text{Cyclic multi-modules}\}$.

Multi-modules are modules over a tensor product $A_1 \otimes \cdots \otimes A_n$ of algebras A_i. Regarding labels of line defects, the algebras A_i correspond to the algebras labelling the adjacent surface defects. We do not recall the full definition of cyclic multi-modules for which we refer to Section 2 in [CRS2].

Let M be a stratified decorated bordism in $\mathrm{Bord}_3(\mathbb{D}^{\mathcal{C}})$. One assigns to M an associated d-bordism $\tilde{M}$ in $\mathrm{Bord}_3^{\mathcal{C}}$, i.e. a bordism with an embedded ribbon $\mathcal{C}$-coloured graph, in the following way:

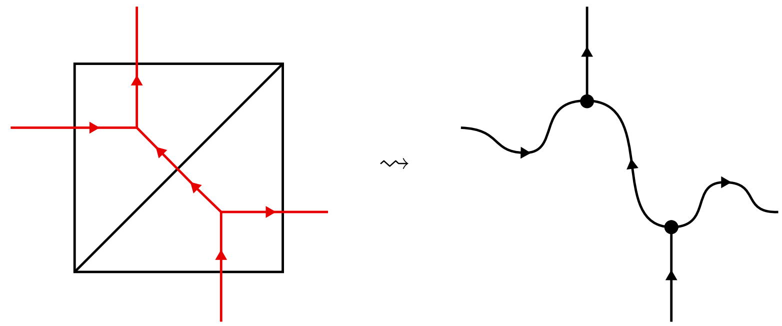


Figure 4.2: The triangulation on the left gives rise to a ribbon graph labelled by A by its Poincaré dual.

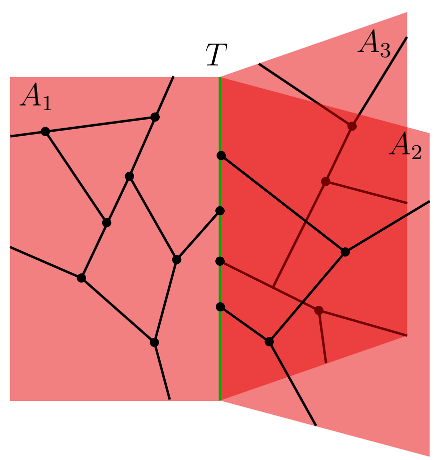


Figure 4.3: A line defect labelled by a cyclic multi-module T carrying a module structure over the adjacent surface defect algebras A_1, A_2, A_3. Surface defects are triangulated, replaced by ribbon graphs labelled by the corresponding algebra and finally attached to the ribbon labelled by M using the module structure.

- Pick a triangulation for every 2-stratum in M labelled by A. Its Poincaré dual is thickened into a ribbon network with coupons. Each ribbon is coloured by A and each coupon is labelled by the product μ or the coproduct Δ of A depending on the orientation.
- Every 1-stratum in M is also thickened into a ribbon coloured by the corresponding multi-module and attached to it one finds the algebras of the incident 2-strata via the corresponding module maps.

The first step is illustrated in Figure 4.2 whereas the construction involving line defects is sketched in Figure 4.3. Evaluating the RT TQFT on the d-bordism $\tilde{M}$ is independent of the choice of triangulation in the interior. A further limit construction gets rid of the dependence on the boundary and completes the construction of the defect TQFT, cf. Construction 5.5 in [CRS2].

Theorem 4.3. The above construction defines a 3-dimensional TQFT with defects, i.e. a symmetric monoidal functor

$$\mathcal{Z}^{\mathcal{C},\mathrm{def}} : \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) \to \mathrm{Vect} \ . \tag{4.6}$$

Remark 4.4. One can compute using the above steps that the contribution of a sphere defect labelled by a symmetric Δ-separable Frobenius algebra A is given by its dimension $\dim_{\mathcal{C}}(A)$. As it is pointed out in [CRS2, Rem. 1.2.(ii)] this is not invariant under Morita equivalence. To obtain a Morita invariant defect theory one has to introduce so-called Euler defects. These form an invertible[6] defect TQFT. Each defect N contributes by $\psi^{\chi_{\mathrm{sym}}(N)}$, where ψ is a non-zero scalar[7] and

$$\chi_{\mathrm{sym}}(N) := 2\chi(N) - \chi(\partial N)$$

is the *symmetric Euler characteristic* of N. One can tensor $\mathcal{Z}^{\mathcal{C},\mathrm{def}}$ with such an invertible Euler defect TQFT to obtain a Morita invariant theory.

The above construction has been then enhanced to include domain walls between distinct MFCs $\mathcal{C}$ and $\mathcal{D}$ [KMRS], which are according to (4.2) Witt equivalent. For Witt equivalent MFCs $\mathcal{C} \sim \mathcal{D}$ one can find commutative Δ-separable Frobenius algebras A, B in an auxiliary MFC $\mathcal{E}$ such that their categories of local modules are equivalent to $\mathcal{C}$ and $\mathcal{D}$, namely $\mathcal{E}_A^{\mathrm{loc}} \simeq \mathcal{C}$ and $\mathcal{E}_B^{\mathrm{loc}} \simeq \mathcal{D}$ [KMRS, Rem. 2.15]. Then, surface defects correspond to Δ-separable symmetric Frobenius algebras over (A, B) in $\mathcal{E}$ which means that in addition to their Δ-separable symmetric Frobenius algebra structure they carry an A-B-bimodule structure subject to some additional compatibility conditions. For details of such defects and the explicit construction we refer to [KMRS, Sec. 2.3].

4.2 Boundaries and TQFTs with boundary conditions

Boundaries in TQFTs can be treated similarly to defects as they can be thought as domain walls between the TQFT of interest and the *trivial*[8] TQFT. We will give a definition of a general n-dimensional TQFT with boundaries and construct such a TQFT for an associated TQFT with defects using the doubling procedure.

We recall manifolds with corners while adjusting the definitions of [SP, Chap. 3] for corners of maximal codimension 2.

Let M be a topological n-manifold with boundary. A *chart* at $m \in M$ is given by a continuous map

$$\phi : U \to \mathbb{R}^{n-2} \times \mathbb{R}^2_{\geq 0} \tag{4.7}$$

for U an open neighbourhood of m and ϕ homeomorphic onto its image. The *index* of m is the number of zero coordinates of $\phi(m)$ in $\mathbb{R}^2_{\geq 0}$ and denoted by $\mathrm{index}(m)$. Two charts $\phi_i : U_i \to \mathbb{R}^{n-2} \times \mathbb{R}^2_{\geq 0}$ for $i = 1, 2$ are *compatible* if their transition map

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}^2_{\geq 0} \tag{4.8}$$

[6] Invertible in the monoidal category of defect TQFTs, see [CRS1, Sec. 2.3.2].

[7] The non-zero scalar may depend on the stratum dimension and therefore we really have non-zero scalars ψ_k for each dimension k.

[8] It factors through the (symmetric monoidal) category with one object and one morphism, which is a zero-object in the 2-category of symmetric monoidal categories.

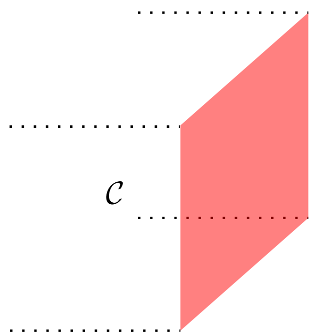


Figure 4.4: Boundary condition for a theory labelled by $\mathcal{C}$.

is diffeomorphic[9] to its image.

Definition 4.5. A *manifold with corners* is a topological, compact, oriented n-manifold with a maximal atlas consisting of compatible charts at every $m \in M$.

Due to compatibility of the charts, the index is independent of the chart and it can be interpreted as the distinction of bulk, boundary and corners. For instance, the set $\{m \in M \,|\, \text{index}(m) = 0\}$ coincides with the interior M°. The constraint of smoothness of the transition maps (4.8) gives M° the structure of a smooth n-manifold.

A *connected face* in a manifold M with corners is the closure of a connected component of $\{m \in M \,|\, \text{index}(m) = 1\}$. A *manifold with faces* is a manifold with corners such that every point m of index 2 (a corner point) belongs to exactly two distinct connected faces. A *face* in such a manifold is the disjoint union of connected faces and is itself an $(n-1)$-manifold with boundary.

Definition 4.6. A $\langle 2 \rangle$*-manifold* is a manifold with faces M and with distinguished faces $\partial_0 M$ and $\partial_1 M$ such that $\partial M = \partial_0 M \cup \partial_1 M$. In addition, if $m \in \partial_0 M \cap \partial_1 M$, then m has $\text{index}(m) = 2$.

In other words, the faces $\partial_0 M$ and $\partial_1 M$ meet exactly on the corners and every corner point belongs to both $\partial_0 M$ and $\partial_1 M$ (as faces are *disjoint* unions of connected faces). We can now define an n-bordism (with corners) between $(n-1)$-manifolds with free boundary as follows: Let Σ and Σ' be smooth, compact, oriented $(n-1)$-dimensional manifolds with boundary. A bordism from Σ to Σ' is a $\langle 2 \rangle$-manifold with a diffeomorphism

$$\partial_0 M \cong -\Sigma \amalg \Sigma' \, . \tag{4.9}$$

Notice that there is no requirement of a boundary parametrisation for $\partial_1 M$, since $\partial_1 M$ will not be treated as a gluing boundary but as the free boundary. For that reason, we will write $\partial_1 \equiv \partial^f$ to distinguish from the bordisms used to define the extended bordism category in [SP].

Two bordisms $M, M' : \Sigma \to \Sigma'$ are *equivalent* if there is a diffeomorphism $f : M \to M'$ compatible with the boundary parametrisations of $\partial_0 M$ and $\partial_0 M'$.

[9]The smooth structure on $\mathbb{R}^{n-2} \times \mathbb{R}^2_{\geq 0}$ is the one induced from its inclusion in $\mathbb{R}^n$.

Definition 4.7. The bordism category $\mathrm{Bord}_n^{\mathrm{bnd}}$ consists of:

- Objects are smooth, oriented, compact $(n-1)$-manifolds (without corners) possibly with boundary.
- Morphisms $\Sigma \to \Sigma'$ are equivalence classes of bordisms with corners from $\Sigma \to \Sigma'$.

$\mathrm{Bord}_n^{\mathrm{bnd}}$ is a symmetric monoidal category with monoidal product given by the disjoint union. Composition is given by the gluing of two bordisms with corners. The smooth structure on the gluing is a priori ambiguous. Suppose, for instance, that M is a bordism from Σ to Σ' and N is a bordism from Σ' to Σ''. One has to specify collars of Σ' in M and in N, which induce a smooth structure on the gluing $M \cup_{\Sigma'} N$. However, different choices of collars lead to different smooth structures. The resulting smooth structures are related by a non-canonical diffeomorphism. In our definition of $\mathrm{Bord}_n^{\mathrm{bnd}}$ this problem is eliminated by two facts:

1. Any $\langle 2\rangle$-manifold admits collars for each of its faces [La, Lem. 2.1.6].
2. Bordisms are taken up to a diffeomorphism, which is compatible with the boundary parametrisations. The non-canonical diffeomorphism relating two different smooth structures on $M \cup_{\Sigma'} N$ is equal to the identity outside a neighbourhood of Σ' in $M \cup_{\Sigma'} N$ [SP, Thm. 3.3]. In particular, it gives an equivalence of bordisms.

There is a obvious (symmetric monoidal) functor

$$\mathrm{Bord}_n \to \mathrm{Bord}_n^{\mathrm{bnd}} \tag{4.10}$$

which is the inclusion of surfaces without boundary and ordinary bordisms, albeit not full.

We can now define the double of objects and morphisms in $\mathrm{Bord}_n^{\mathrm{bnd}}$ to obtain stratified manifolds.

Let $\Sigma \in \mathrm{Bord}_n^{\mathrm{bnd}}$. Its double is the manifold

$$\hat{\Sigma} := \Sigma \amalg -\Sigma / \sim \quad \text{with} \quad (x,+) \sim (x,-) \quad \forall x \in \partial\Sigma \ . \tag{4.11}$$

This is a closed stratified $(n-1)$-manifold with filtration $\hat{\Sigma} = F_{n-1} \supset F_{n-2} = \partial\Sigma \supset F_{n-3} = \emptyset$ where $\partial\Sigma \hookrightarrow \hat{\Sigma}$ is naturally embedded via the identification in (4.11). Thus, $\pi_0(\partial\Sigma)$ is the collection of all $(n-2)$-strata and there are no lower-dimensional ones. Keep in mind that we secretly make a choice of a collar for $\partial\Sigma$ to extend to a smooth structure on $\hat{\Sigma}$. We will only recall this choice when it becomes relevant[10].

Similarly, we define for a $\langle 2\rangle$-manifold M its double by

$$\hat{M} := M \amalg -M / \sim \quad \text{with} \quad (m,+) \sim (m,-) \quad \forall m \in \partial^f M \ . \tag{4.12}$$

where we only identify boundary points in the free boundary $\partial^f M$. This is a stratified n-manifold with filtration $M = F_n \supset F_{n-1} = \partial^f M \supset F_{n-2} = \emptyset$ with stratified boundary $\widehat{\partial_0 M}$. Once again, we make a choice of a collar for the face $\partial^f M$ to define the smooth structure on $\hat{M}$.

We summarise the above discussion in the following Proposition:

[10] Different choices of collars will lead to naturally isomorphic functors $\mathrm{Bord}_n^{\mathrm{bnd}} \to \mathrm{Bord}_n^{\mathrm{str}}$ in the sense of Proposition 4.8.

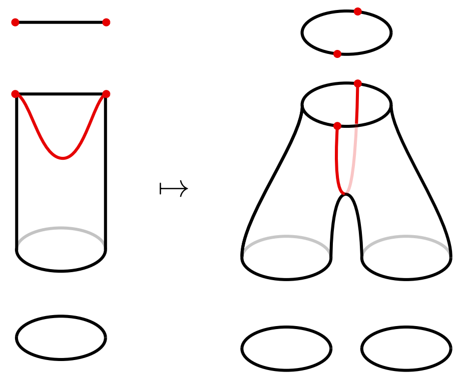


Figure 4.5: The double of a bordism in $\mathrm{Bord}_2^{\mathrm{bnd}}$.

Proposition 4.8. The double defines a symmetric monoidal functor

$$\widehat{(-)} : \mathrm{Bord}_n^{\mathrm{bnd}} \to \mathrm{Bord}_n^{\mathrm{str}} \,. \tag{4.13}$$

Example 4.9. In $\mathrm{Bord}_2^{\mathrm{bnd}}$, in addition to circles, intervals $[0,1]$ are allowed as objects and they each have two boundary components. The double of the interval is clearly a circle with two 0-strata.

For a non-trivial example of a double of a bordism, consider the bordism $\mathbb{S}^1 \to [0,1]$ on the left side of Figure 4.5, where the outgoing boundary component of $\partial_0 M$ meets on its respective boundary points the free boundary $\partial^f M$, which also happens to be an interval. The double of this bordism on the right is a pair of pants, stratified by a single 1-stratum with end points on its outgoing boundary.

Based on the bordism category $\mathrm{Bord}_n^{\mathrm{bnd}}$, one can consider the notion of n-dimensional TQFTs with boundary conditions. For that we include a set $\mathbb{B}$ of boundary conditions, which will be the labelling set of free boundaries.

For a set $\mathbb{B}$ define the bordism category $\mathrm{Bord}_n^{\mathrm{bnd}}(\mathbb{B})$ by labelling free boundaries as follows:

- Objects are smooth, oriented, compact $(n-1)$-manifolds with a map $b_\Sigma : \pi_0(\partial\Sigma) \to \mathbb{B}$, which labels each connected boundary component by an element in $\mathbb{B}$.
- A bordism M with corners is now equipped with a map $b_M : \pi_0(\partial^f M) \to \mathbb{B}$ labelling each free boundary component with a boundary condition. This induces a $\mathbb{B}$-boundary labelling of $\partial_0 M$. We further ask that the boundary parametrisation respects this labelling. Two bordisms are equivalent, if there is a diffeomorphism, which is compatible with the free boundary labels and the boundary parametrisations. Equivalence classes of such bordisms form the morphisms in $\mathrm{Bord}_n^{\mathrm{bnd}}(\mathbb{B})$.

In the above definition, we have no labels for the bulk regions. In terms of TQFT, we therefore consider a single TQFT with boundary conditions.

We will now define a TQFT with boundary as a symmetric monoidal functor on the bordism category $\mathrm{Bord}_3^{\mathrm{bnd}}$ of bordisms with free boundaries and relate it to a given TQFT with defects.

Definition 4.10. An n-dimensional TQFT with boundary conditions given by $\mathbb{B}$ is a symmetric monoidal functor

$$\mathcal{Z} : \mathrm{Bord}_n^{\mathrm{bnd}}(\mathbb{B}) \to \mathrm{Vect} \ . \tag{4.14}$$

Let $\mathbb{D}$ be a defect datum for n-dimensional bordisms as in [CRS1] with a single bulk label, i.e. $D_n = *$. Choose as a candidate for boundary conditions the set $\hat{\mathbb{D}} := D_{n-1}$. Then, the functor from Proposition 4.8 extends to a functor

$$\widehat{(-)} : \mathrm{Bord}_n^{\mathrm{bnd}}(\hat{\mathbb{D}}) \to \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}). \tag{4.15}$$

by labelling each stratum in $\hat{M}$ by the boundary label of the corresponded connected component of $\partial^f M$.

Let $\mathcal{Z}$ be an n-dimensional defect TQFT with defect datum $\mathbb{D}$, such that $\mathbb{D}_n = *$. In other words, all defects are internal to a single governing bulk theory. Then, composing $\mathcal{Z}$ with the symmetric monoidal functor (4.15) yields a symmetric monoidal functor

$$\hat{\mathcal{Z}} : \mathrm{Bord}_n^{\mathrm{bnd}}(\hat{\mathbb{D}}) \to \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \xrightarrow{\mathcal{Z}} \mathrm{Vect} \tag{4.16}$$

which will be called the *double theory* of $\mathcal{Z}$.

Boundaries in RT TQFT of type $\mathcal{C}$ have been analysed similarly to defects. A boundary condition exists according to [FSV] if there is *central functor*

$$F : \mathcal{C} \to \mathcal{A} \tag{4.17}$$

to some fusion category $\mathcal{A}$. A central functor F is endowed with a lift $\tilde{F} : \mathcal{C} \to \mathcal{Z}(\mathcal{A})$ with respect to the forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$. For the existence of a boundary condition, $\tilde{F}$ is further required to be an equivalence and therefore a Witt trivialisation of $\mathcal{C}$, i.e. $\mathcal{C}$ is in the Witt class of Vect. This relates to the previous discussion of surface defects as a boundary for $\mathcal{C}$ may be viewed as a surface defect separating $\mathcal{C}$ from the trivial theory Vect and the central functor (4.17) corresponds to the functor (4.2) for $\mathcal{D} = \mathrm{Vect}$. Hence, RT TQFTs which admit boundaries are of Turaev-Viro type.

For example, the RT TQFT associated to the double $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ has a natural boundary condition of $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{C})$. Its boundary conditions are therefore classified by $\mathcal{C}$-module categories. Let $\mathcal{S}$ be a spherical fusion category. The construction of Turaev-Viro TQFT $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ associated to $\mathcal{S}$ has been extended to a TQFT with boundary conditions [KK].

Suppose $\mathcal{C}$ is an MFC. Let $\mathcal{Z}^{\mathcal{C},\mathrm{def}} : \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) \to \mathrm{Vect}$ be the RT TQFT with defects from Theorem 4.3 and $\mathcal{Z}^{\mathrm{TV},\mathcal{C}} : \mathrm{Bord}_3^{\mathrm{bnd}}(\hat{\mathbb{D}}^{\mathcal{C}}) \to \mathrm{Vect}$ be the Turaev-Viro TQFT with boundary associated to the (underlying) spherical fusion category $\mathcal{C}$.

Proposition 4.11. There is a natural isomorphism of TQFTs $\hat{\mathcal{Z}}^{\mathcal{C}} \cong \mathcal{Z}^{\mathrm{TV},\mathcal{C}}$, as symmetric monoidal functors on the ordinary bordism category.

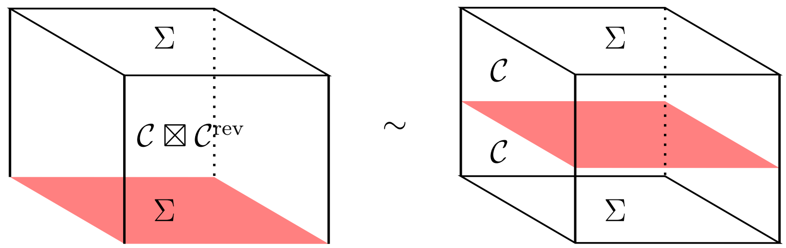


Figure 4.6: The cylinder over Σ on the left viewed as a bordism in $\emptyset \to \Sigma$ in $\mathrm{Bord}_3^{\mathrm{bnd}}$ and its double on the right viewed as a defect bordism $\emptyset \to \hat{\Sigma}$ with a surface defect $\Sigma \times \{1/2\}$ in $\mathrm{Bord}_3^{\mathrm{str}}$.

Proof. Let Σ be a closed surface, i.e. $\partial\Sigma = \emptyset$. Then, we have the following isomorphisms:

$$\begin{aligned} \hat{\mathcal{Z}}^{\mathcal{C}} := \mathcal{Z}^{\mathcal{C}}(\hat{\Sigma}) &= \mathcal{Z}^{\mathcal{C}}(\Sigma \amalg -\Sigma) \\ &\cong \mathcal{Z}^{\mathcal{C}}(\Sigma) \otimes \mathcal{Z}^{\mathcal{C}}(-\Sigma) \\ &\cong \mathcal{Z}^{\mathcal{C}}(\Sigma) \otimes \mathcal{Z}^{\mathcal{C}^{\mathrm{rev}}}(\Sigma) \\ &\cong \mathcal{Z}^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(\Sigma) \\ &\cong \mathcal{Z}^{\mathcal{Z}(\mathcal{C})}(\Sigma) \cong \mathcal{Z}^{\mathrm{TV},\mathcal{C}}(\Sigma)\ . \end{aligned} \tag{4.18}$$

The last isomorphism is the result of comparing Turaev-Viro TQFTs with RT TQFTs from [TV]. All these isomorphisms are natural and therefore we have a natural isomorphism of

$$\hat{\mathcal{Z}}^{\mathcal{C}} \cong \mathcal{Z}^{\mathrm{TV},\mathcal{C}} : \mathrm{Bord}_n \to \mathrm{Vect}\,.$$

□

Remark 4.12. It would be interesting to compare the TQFT with boundary $\widehat{\mathcal{Z}^{\mathcal{C},\mathrm{def}}}$ obtained from the double of the RT TQFT of $\mathcal{C}$ with the Turaev-Viro TQFT of $\mathcal{C}$ with boundary as treated in [KK]. It is natural to expect that these are isomorphic enhancing Proposition 4.11. In other words, there should be an isomorphism of TQFTs with boundary $\widehat{\mathcal{Z}^{\mathcal{C},\mathrm{def}}} \cong \mathcal{Z}^{\mathrm{TV},\mathcal{C}}$, i.e. the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) & \xrightarrow{\mathcal{Z}^{\mathcal{C},\mathrm{def}}} & \mathrm{Vect} \\ \widehat{(-)}\big\uparrow & \nearrow_{\mathcal{Z}^{\mathrm{TV},\mathcal{C}}} & \\ \mathrm{Bord}_3^{\mathrm{bnd}}(\hat{\mathbb{D}}^{\mathcal{C}}) & & \end{array} \tag{4.19}$$

Example 4.13. Let Σ be a surface without boundary. The cylinder $M_\Sigma = \Sigma \times I$ over Σ typically represents the identity bordism in Bord_3. However, we view here M as a bordism $\emptyset \to \Sigma$ in $\mathrm{Bord}_3^{\mathrm{bnd}}$ by renaming the incoming boundary a free boundary, i.e. $\partial_f M = \Sigma$ and $\partial_0 M = \Sigma$. Its double $\hat{M}_\Sigma$ is a stratified manifold, which may be viewed as the cylinder $\Sigma \times I$ with a single surface defect, namely the surface $\Sigma \times \{1/2\}$.

Fix a boundary condition for Σ in M_Σ which corresponds to fixing a defect label for $\Sigma \times \{1/2\}$ in $\hat{M}_\Sigma$. Proposition 4.11 implies that under the isomorphism of TQFTs $\Psi : \widehat{\mathcal{Z}^{\mathcal{C},\mathrm{def}}} \cong \mathcal{Z}^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}$ we have an equality:

$$\mathcal{Z}^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(M_\Sigma) = \Psi_\Sigma(\mathcal{Z}^{\mathcal{C},\mathrm{def}}(\hat{M}_\Sigma)) \tag{4.20}$$

This equality is represented by Figure 4.6. The bulk region on the left is labelled by $\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}$ whereas the bulk regions on the right are labelled by $\mathcal{C}$.

5 Correlators as mapping class group averages

The goal of this section is to establish a correspondence between RCFT correlators and MCG averages. This is surprisingly motivated by physics considerations in 3D quantum gravity, which will be explained in the later Section 8. In 5.1 we briefly recall the basics of rational CFT and its RT TQFT description. In 5.2 we define mapping class group averages under a finiteness condition and in 5.3 we relate for certain theories RCFT correlators with MCG averages.

5.1 RCFT correlators

A 2-dimensional rational CFT is characterised by its chiral algebra $\mathcal{V}$, which can be modelled as a rational vertex operator algebra (VOA), and its correlation functions that live on Riemann surfaces with field insertions.

Suppose C is a Riemann surface with n framed points labelled by representations $V_1, \ldots, V_n$ over the chiral algebra $\mathcal{V}$. The CFT assigns a vector space $\mathcal{B}(C)$ which is a subspace of linear functionals on $V_1 \otimes \cdots \otimes V_n$. This is the space of (chiral) *conformal blocks* which are the building blocks from which the correlation functions are made. Let us fix the genus to be g and the number of framed points to be n with corresponding labels $V_1, \ldots, V_n$. The set of isomorphism classes of Riemann surfaces of this type forms the so-called *moduli space* $\mathcal{M}_{g,n}$. By using the assignment

$$C \mapsto \mathcal{B}(C), \tag{5.1}$$

one can consider the vector bundle of conformal blocks over the moduli space

$$\mathcal{B}_{g,n} \to \mathcal{M}_{g,n} \tag{5.2}$$

where $\mathcal{B}_{g,n}$ consists of pairs $([C], \mathcal{B}(C))$ with an isomorphism class $[C] \in \mathcal{M}_{g,n}$ of a Riemann surface C of genus g and n punctures and the corresponding space of conformal blocks $\mathcal{B}(C)$[11]. This bundle is equipped with the projectively flat Knizhnik-Zamolodchikov connection ∇_{KZ} with respect to which horizontal sections correspond to conformal blocks of our CFT. The above results in what is called a *chiral* CFT.

Due to the non-trivial monodromies, conformal blocks are multi-valued. To determine the *full* CFT one has to determine a consistent system of correlators which are subject to the following conditions: i) They solve the chiral Ward identities. ii) They are single-valued. iii) They solve *sewing (or gluing) constraints.*

In a rational CFT the representation category $\mathcal{C} = \mathrm{Rep}(\mathcal{V})$ of the associated chiral algebra is shown to be an MFC [H]. The TQFT approach to RCFT in [FRS] utilises the RT TQFT associated to $\mathcal{C}$ to provide a solution to finding a consistent system of correlators. The RT TQFT of $\mathcal{C}$ or $\mathcal{C}$ itself will be referred to as the *chiral* theory.

The topological nature of the problem can be described roughly as follows: Let C be complex curve with framed points and fixed labels. Let $U(C)$ denote the underlying

[11] This is well-defined as for any two isomorphic Riemann surfaces $C \simeq C'$ the corresponding vector spaces $\mathcal{B}(C) = \mathcal{B}(C')$ are identical.

topological surface by forgetting the complex structure, but keeping track of the framed points with their insertions. In particular, $U(C)$ is a d-surface. The conformal block space on the complex curve C is isomorphic to the state space on $U(C)$, i.e. there exists an isomorphism

$$\Psi_C : \mathcal{B}(C) \to V^{\mathcal{C}}(U(C)) \ . \tag{5.3}$$

This isomorphism is natural in C in the following sense: Let γ be a curve in the moduli space $\mathcal{M}_{g,n}$ starting at $[C]$ and ending at $[C]'$. On the one hand, the connection on (5.2) provides an isomorphism

$$\Gamma_\gamma : \mathcal{B}(C) \xrightarrow{\sim} \mathcal{B}(C') \tag{5.4}$$

via parallel transport along γ. On the other hand, the curve γ induces a isomorphism of the underlying surfaces $\tilde{\gamma} \in \mathrm{Mod}^{\mathcal{C}}(U(C), U(C)')$. Naturality of the isomorphism in (5.3) means

$$\Psi_{C'} \circ \Gamma_\gamma \propto V^{\mathcal{C}}(\tilde{\gamma}) \circ \Psi_C \ . \tag{5.5}$$

We will therefore refer to the state space $V^{\mathcal{C}}(\Sigma)$ for a surface Σ also as the space of chiral conformal blocks, even though the complex-analytic structure is forgotten. To be precise, it is the space of projectively flat sections over the moduli space on Σ, which recovers the complex-analytic description. Similarly, correlators are viewed as elements in the state space of the double $\hat{\Sigma}$.

We summarise the topological description of correlators:

- The correlator $\mathrm{Cor}(\Sigma)$ on a surface Σ is an element of the state space $V^{\mathcal{C}}(\hat{\Sigma})$ of the double surface Σ, i.e.

  $$\mathrm{Cor}(\Sigma) \in V^{\mathcal{C}}(\hat{\Sigma}) \ . \tag{5.6}$$

  This is known as *holomorphic factorisation* in that correlators solve the chiral Ward identities.

- The correlator $\mathrm{Cor}(\Sigma)$ is invariant under the diagonal action of the mapping class group $\mathrm{Mod}(\Sigma)$, i.e.

  $$\mathrm{Cor}(\Sigma) \in V^{\mathcal{C}}(\hat{\Sigma})^{\mathrm{Mod}(\Sigma)} \ . \tag{5.7}$$

  This invariance, also called modular invariance, implies that correlation functions are single-valued.

- Correlators solve the sewing constraints, i.e. they behave nicely under cutting and gluing of a surface along some boundary circle. This property is captured in terms of the modular functor by the gluing isomorphism g from (3.7). Let Σ be a d-surface and γ be a simple closed curve on Σ. Then

  $$\mathrm{Cor}(\Sigma) = g_{\gamma,\hat{\Sigma}} \left(\sum_{i,j,\lambda} \mathrm{Cor}(\Sigma_\gamma(i,j)) \right) \tag{5.8}$$

  where $\Sigma_\gamma(i,j)$ is the cut surface with label $(i,j) \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ resp. its dual and λ runs over possible multiplicities. The sum is over intermediate states which means over all simple labels $i, j \in I$. Finally, the $g_{\gamma,\hat{\Sigma}}$ is the gluing isomorphism (3.7) when

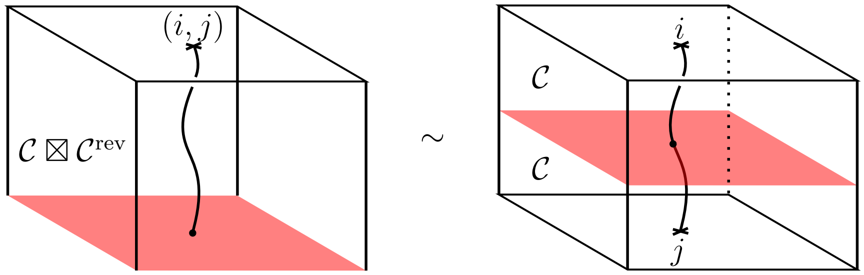


Figure 5.1: The construction of correlators via boundary in $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ or defects in $\mathcal{C}$.

gluing along γ in both copies in $\hat{\Sigma}$. This is also called the *factorisation property* of correlators and for the sphere with four marked points this is the realisation of an operator product expansion (OPE).

The explicit construction of RCFT correlators for a given chiral theory $\mathcal{C}$ relies further on a symmetric special Frobenius algebra A in $\mathcal{C}$. The corresponding CFT correlators depending on A are:

$$\mathrm{Cor}_A^{\mathcal{C}}(\Sigma) \in V^{\mathcal{C}}(\hat{\Sigma}) \tag{5.9}$$

Recall that symmetric special Frobenius algebras give rise to surface defects in $\mathcal{C}$. We can exploit this to construct the correlators given in [FjFRS1] in terms of the RT TQFT with defects from Theorem 4.3.

The doubling procedure introduced in Section 4 can be extended to include Wilson lines. Let Σ be a d-surface with framed points labelled by pairs $(i, j) \in \mathcal{C} \times \mathcal{C}$ which correspond to bulk field insertions. Consider the defect cylinder

$$\hat{M}_\Sigma : \emptyset \to \hat{\Sigma}$$

from Example 4.13 with the surface defect $\Sigma \times \{1/2\}$ labelled by the symmetric special Frobenius algebra A. The candidate for the RCFT correlator on Σ is then defined as

$$\mathrm{Cor}_A^{\mathcal{C}}(\Sigma) = \mathcal{Z}^{\mathcal{C},\mathrm{def}}(\hat{M}_\Sigma) \tag{5.10}$$

by applying the defect RT TQFT on the right.

Applying Proposition 4.11 and Remark 4.12 we may also write

$$\mathrm{Cor}_A^{\mathcal{C}}(\Sigma) = \mathcal{Z}^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}},\mathrm{bnd}}(M_\Sigma) \in V^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{def}}}(\Sigma) \tag{5.11}$$

where the TQFT of $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ is applied to the cylinder M_Σ from example 4.13. The two approaches (5.10) and (5.11) are illustrated by Figure 5.1.

The following theorem is a result from [FjFRS1, Thm. 2.2, Sec. 5]:

Theorem 5.1. Let $\mathcal{C}$ be an MFC and A a symmetric special Frobenius algebra in $\mathcal{C}$. The correlators $\{\mathrm{Cor}_A^{\mathcal{C}}\}_\Sigma$ defined in (5.10) form a consistent system of correlators, i.e. they satisfy (5.6), (5.7) and (5.8).

The result is even stronger in that they prove in [FjFRS2] that Morita equivalent symmetric special Frobenius algebras in $\mathcal{C}$ lead to the same correlators and thus equivalent CFTs. This forms in fact a one-to-one correspondence between 2D RCFTs and pairs $(\mathcal{C}, [A])$ of an MFC $\mathcal{C}$ and a Morita class of a symmetric special Frobenius algebra A in $\mathcal{C}$.

5.2 Mapping class group averages

We now define the mapping class group averages, which should be thought as candidates for quantum gravity correlators (discussed in Section 8).

For a given MFC $\mathcal{C}$ we ask for a finiteness condition on its mapping class group representations. If the representation image is finite, and in particular has finite orbits we give the following definition of a mapping class group average.

Definition 5.2. Let Σ be a d-surface such that the representation image $G := V^{\mathcal{C}}(\mathrm{Mod}(\Sigma))$ is finite in $\mathrm{End}(V^{\mathcal{C}}(\Sigma))$. Then, define the linear map

$$\langle - \rangle_\Sigma : V^{\mathcal{C}}(\hat{\Sigma}) \to V^{\mathcal{C}}(\hat{\Sigma})^{\mathrm{Mod}(\Sigma)},\ x \mapsto \langle x \rangle_\Sigma := \frac{1}{|G|} \sum_{g \in G} g.x \tag{5.12}$$

where $\mathrm{Mod}(\Sigma)$ acts diagonally on $V^{\mathcal{C}}(\hat{\Sigma})$.

Remark 5.3.

1. One can express the mapping class group average as an orbit sum

$$\langle x \rangle_\Sigma := \frac{1}{|G|} \sum_{g \in G} g.x = \frac{1}{|\mathcal{O}_x|} \sum_{y \in \mathcal{O}_x} y \tag{5.13}$$

   where $\mathcal{O}_x$ is the mapping class group orbit of x. In fact to define $\langle x \rangle_\Sigma$ it is sufficient to require finiteness of its orbit and not of the representation image. To go beyond the finiteness requirement itself would require a good notion of a measure. This fails for the mapping class group itself as it is non-amenable as pointed out in [JLLSW]. However, we are not aware if there is still hope for its representation image.

2. The definition above depends on the finiteness condition as for an infinite orbit, there is no natural regularisation procedure which allows the definition of an average.

   However, for the torus $\Sigma = T^2$ the finiteness condition is always satisfied. This follows from a result of [NS] showing that the kernel of the associated mapping class group representation is a congruence group and thus a finite index group.

3. The semisimplicity of the MFC $\mathcal{C}$ proves to play an important role in the finiteness condition. For non-semisimple modular tensor categories[12] [Ly] constructed projective mapping class group representations generalising the semisimple case. In [DGGPR, DGGPR2] they extended these representations to a 3-dimensional TQFT.

[12] We used the term modular fusion category instead of the traditional modular tensor category, which also assumed an underlying semisimple tensor category, to allow for a more general definition of modular tensor categories. A *modular tensor category* is then a finite ribbon tensor category with trivial Müger centre as condition 2 in Proposition 2.1.

A standard example of non-semisimple modular tensor category is the representation category of the quantum group $U_q(sl_2)$. The finiteness condition fails for such theories after the following observation: Let γ be an essential closed curve on a surface Σ without framed points. The Dehn twist along γ has infinite order in the mapping class group representations [DGGPR2, Prop. 5.1].

Even though going beyond semisimplicity is of great interest as it allows one to go from rational to logarithmic CFTs, we will only work with the semisimple case without needing to fix the above problem.

5.3 Main correspondence

The state space of the double $\hat{\Sigma}$ can be decomposed using the tensoriality isomorphisms of the modular functor, i.e.

$$V^{\mathcal{C}}(\hat{\Sigma}) \cong V^{\mathcal{C}}(\Sigma) \otimes V^{\mathcal{C}}(-\Sigma). \tag{5.14}$$

Naturality of these isomorphisms implies in particular that they are $\mathrm{Mod}(\Sigma)$-intertwiners.

By abuse of notation let $\mathrm{d}_\Sigma : V^{\mathcal{C}}(\hat{\Sigma}) \to \Bbbk$ denote the pullback of the pairing (3.16). If in addition to the finiteness property $\mathcal{C}$ obeys an irreducibility property, we get a relation between conformal correlators and mapping class group averages as stated in the following theorem.

Theorem 5.4. Let Σ be a d-surface such that the projective representation $V^{\mathcal{C}}(\Sigma)$ is irreducible and $V^{\mathcal{C}}(\mathrm{Mod}(\Sigma)) \subset \mathrm{End}(V^{\mathcal{C}}(\Sigma))$ is finite and let $x \in V^{\mathcal{C}}(\hat{\Sigma})$ be an element such that $\mathrm{d}_\Sigma(x) \neq 0$. In addition, suppose that A is a symmetric special Frobenius algebra in $\mathcal{C}$. Then, there exists $\lambda_\Sigma \in \Bbbk$ such that

$$\mathrm{Cor}_A^{\mathcal{C}}(\Sigma) = \lambda_\Sigma \left\langle x \right\rangle_\Sigma \ . \tag{5.15}$$

The proof will follow after the following Lemma:

Lemma 5.5. Let Σ be a d-surface such that the projective representation $V^{\mathcal{C}}(\Sigma)$ is irreducible. The space of mapping class group invariants on the double $\hat{\Sigma}$ is one-dimensional, i.e.

$$V^{\mathcal{C}}(\hat{\Sigma})^{\mathrm{Mod}(\Sigma)} \cong \Bbbk \ . \tag{5.16}$$

Proof. The non-degeneracy of the pairing $\mathrm{d}_\Sigma : V^{\mathcal{C}}(\Sigma) \otimes V^{\mathcal{C}}(-\Sigma) \to \Bbbk$ gives an isomorphism $\psi : V^{\mathcal{C}}(-\Sigma) \xrightarrow{\sim} V^{\mathcal{C}}(\Sigma)^*$, $y \mapsto \mathrm{d}_\Sigma(-, y)$. This is a $\mathrm{Mod}(\Sigma)$-intertwiner (with respect to the dual action on $V^{\mathcal{C}}(\Sigma)^*$). Indeed, for any $y \in V^{\mathcal{C}}(-\Sigma)$ and $f \in \mathrm{Mod}(\Sigma)$:

$$\psi(f.y) := \mathrm{d}_\Sigma(-, f.y) \overset{(3.17)}{=} \mathrm{d}_\Sigma(f^{-1}.-, y) = f.\psi(y)$$

Furthermore, there is an obvious $\mathrm{Mod}(\Sigma)$-isomorphism $V^{\mathcal{C}}(\Sigma) \otimes V^{\mathcal{C}}(\Sigma)^* \cong \mathrm{End}(V^{\mathcal{C}}(\Sigma))$, which intertwines the diagonal action on the left with the conjugation action on the right. Combining this with (5.14) we obtain a $\mathrm{Mod}(\Sigma)$-isomorphism

$$V^{\mathcal{C}}(\Sigma) \cong \mathrm{End}(V^{\mathcal{C}}(\Sigma)) \ .$$

In particular, they have isomorphic spaces of $\mathrm{Mod}(\Sigma)$-invariants. The space of $\mathrm{Mod}(\Sigma)$-invariants in $\mathrm{End}(V^{\mathcal{C}}(\Sigma))$ coincides with the subspace $\mathrm{End}_{\mathrm{Mod}(\Sigma)}(V^{\mathcal{C}}(\Sigma))$ of $\mathrm{Mod}(\Sigma)$-intertwiners. By Schur's Lemma, irreducibility of $V^{\mathcal{C}}(\Sigma)$ implies $\mathrm{End}_{\mathrm{Mod}(\Sigma)}(V^{\mathcal{C}}(\Sigma)) \cong \Bbbk$ which concludes the proof. □

Now we can prove Theorem 5.4:

Proof of Theorem 5.4. Irreducibility of $V^{\mathcal{C}}(\Sigma)$ implies by Lemma 5.5 that the space of modular invariants $V^{\mathcal{C}}(\hat{\Sigma})^{\mathrm{Mod}(\Sigma)}$ is 1-dimensional. The correlators $\mathrm{Cor}^{\mathcal{C}}_A(\Sigma)$ are mapping class group invariant (5.7) and so is the mapping class group average $\langle x \rangle_\Sigma$ by definition and, therefore, they both lie in the same 1-dimensional space. For the statement to hold, we only need to show that $\langle x \rangle_\Sigma$ is non-zero. To show that this is the case, it is sufficient to check that $\mathrm{d}_\Sigma(\langle x \rangle_\Sigma) \neq 0$. This follows from the invariance property (3.17) and linearity of d_Σ, namely

$$\mathrm{d}_\Sigma(\langle x \rangle_\Sigma) = \frac{1}{|G|} \sum_{g \in G} \mathrm{d}_\Sigma(g.x) = \frac{1}{|G|} \sum_{g \in G} \mathrm{d}_\Sigma(x) = \mathrm{d}_\Sigma(x) \neq 0 \ . \tag{5.17}$$

□

The element x in Theorem 5.4 is referred to as the *seed*. Since correlators are mapping class group invariant, they are invariant under averaging, i.e.

$$\langle \mathrm{Cor}^{\mathcal{C}}_A(\Sigma) \rangle_\Sigma \ = \ \mathrm{Cor}^{\mathcal{C}}_A(\Sigma) \ . \tag{5.18}$$

However, they are not interesting choices of seed elements.

Consider a handlebody H_Σ with boundary Σ without framed points, seen as a bordism $H_\Sigma : \emptyset \to \Sigma$. The RT TQFT evaluated on its double offers a natural choice of a seed

$$x_{\mathrm{vac}} := \mathcal{Z}^{\mathcal{C}}(\hat{H}_\Sigma) \tag{5.19}$$

called *vacuum seed*. As we will see in Section 8 x_{vac} will be the vacuum contribution to the gravity path integral. For the torus $\Sigma = T^2$ the element $\mathcal{Z}^{\mathcal{C}}(H_{T^2})$ corresponds to the vacuum character $\chi_0(\tau)$ [FRS, Eq. (5.15),(5.16)] (τ is the conformal parameter) and x_{vac} to $|\chi_0(\tau)|^2$. Finally, notice that x_{vac} satisfies $\mathrm{d}_\Sigma(x_{\mathrm{vac}}) \neq 0$ which follows directly from Example 3.3 (where $x_{\mathrm{vac}} = x \otimes \hat{x}$).

Remark 5.6. 1. Theorem 5.4 gives a correspondence between conformal correlators and mapping class group averages. In particular, for theories (MFCs) $\mathcal{C}$ such that the hypothesis holds for any d-surface, one can obtain any conformal correlator via a mapping class group average (up to a non-zero scalar).

In fact, for the irreducibility property we have found in [RR1] (see later Theorem 7.1) the following result:

> Let $\mathcal{C}$ be an MFC such that the mapping class group representations $V^{\mathcal{C}}(\Sigma)$ are irreducible for any surface Σ (with no insertions). Then, there exists a unique Morita class of simple symmetric special Frobenius algebras, namely that of the trivial algebra $\mathbb{1}$.

The above statement implies in particular that there is a unique CFT with MFC $\mathcal{C}$. In such a setting we may write $\mathrm{Cor}^{\mathcal{C}}(\Sigma)$ for the unique CFT correlator.

2. The finiteness condition in Theorem 5.4 will also be referred to as *property F*, or prop F for short. This terminology was originally used when studying a finiteness property for braided fusion categories [NR]. Namely, we will say that an MFC $\mathcal{C}$ has property F, if for every d-surface Σ the group $V^{\mathcal{C}}(\mathrm{Mod}(\Sigma))$ is finite. We may also specify to only a family of surfaces, for example with respect to surfaces without framed points.

The correspondence (5.15) has been established for the vacuum seed x_{vac} and the Ising CFT on the torus in [CGHMV] and then extended to partition functions on higher genus surfaces in [JLLSW]. In the following section we will extend this to surfaces with framed points for all Ising-type MFCs.

6 Irreducibility and property F

In this section we discuss theories with the irreducibility and/or finiteness property used in the hypothesis of the main correspondence Theorem 5.4. We review in 6.1 some known examples and in 6.2 we extend in the case of Ising categories some existing results.

6.1 Irreducibility and property F examples

A trivial example which has irreducible representations and property F with respect to any d-surface Σ is Vect. There are further trivial examples of theories with irreducible representations when restricting to specific surfaces. For instance, any MFC $\mathcal{C}$ has an irreducible representation on the sphere, as $V^{\mathcal{C}}(\mathbb{S}^2) \cong \Bbbk$. However, we are interested in examples with an apparent irreducibility property, present on a large family of surfaces.

The only non-trivial examples with irreducible representations $V^{\mathcal{C}}(\Sigma)$'s for all surfaces Σ we are aware of are Ising-type categories and the MFC $\mathcal{C}(sl(2),k)$ associated to the affine Lie algebra $\widehat{sl}(2)$ at certain levels $k \in \mathbb{Z}_{>0}$. Let us list these examples, as well as some non-examples.

1. It is shown in [Ro] that for $\mathcal{C} = \mathcal{C}(sl(2),k)$ and $r = k+2$ prime, all projective representations $V^{\mathcal{C}}(\Sigma_{g,0})$, $g \geq 0$ are irreducible.

   Most of the remaining cases can be excluded already by looking at $g=1$. Namely by [GQ, App. A] and [CIZ, Prop. 1], invariants in the representation $V_{g=1}^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}$ are obtained from divisors d of r, with divisors d and r/d describing the same invariant subspace, and where d with $d^2 = r$ is excluded. Thus, when $r \geq 3$ is not a prime or a square of a prime, the space of invariants satisfies $\dim(V_1^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}})^{\mathrm{Mod}_1} > 1$, and so by Lemma 5.5, $V_1^{\mathcal{C}}$ is not irreducible.

   On the other hand, for $k=2$ ($r=4$), one obtains a category of Ising-type, for which all $V_g^{\mathcal{C}}$ are irreducible, see point 2. Some results on the irreducibility of $V_{g\geq 2}^{\mathcal{C}}$ for the remaining cases of $r = p^2$ with $p > 2$ prime can be found in [Kor] but this remains an open problem and we hope to return to this in the future.

2. The Ising model without marked points is studied in [CGHMV, JLLSW]. Irreducibility of all $V_g^{\mathcal{C}}$, $g \geq 0$ is shown in [JLLSW, Sec. 4.3]. In the next section we will extend this result to all 16 Ising-type MFCs and to surfaces with marked points.

3. Let $\mathcal{C} = \mathcal{C}(sl(N),k)$ be the MFC for the affine Lie algebra $\widehat{sl}(N)$ at any level $k \in \mathbb{Z}_{>0}$, for $N \geq 3$. It is shown in [AF2, Thm. 3.6] that the $V_g^{\mathcal{C}}$ are reducible for each $g \geq 1$.

4. For $\mathcal{C}(sl(2),k)$, irreducibility has also been studied for the mapping class group of surfaces with marked points in [KoS] where it is shown with respect to surfaces which contain at least one marked point labelled by the fundamental label 1. In [KM] they studied irreducibility for surfaces with boundary. For Ising-type MFCs, irreducibility in the presence of marked points will be shown in the next section.

Definition 6.1. We say that an MFC $\mathcal{C}$ has *property F* with respect to a d-surface Σ if the associated mapping class group image $V^{\mathcal{C}}(\mathrm{Mod}^{\mathcal{C}}(\Sigma))$ is a finite group.

Remark 6.2. 1. It does not matter if we define property F with respect to the whole mapping class group $\mathrm{Mod}^{\mathcal{C}}$ or just the pure mapping class group PMod. This is because the following conditions are equivalent:

a) $V^{\mathcal{C}}(\mathrm{Mod}^{\mathcal{C}}(\Sigma))$ is finite.

b) $V^{\mathcal{C}}(\mathrm{PMod}(\Sigma))$ is finite.

a) $\Rightarrow$ b): The pure mapping class group is a subgroup and therefore $V^{\mathcal{C}}(\mathrm{PMod}(\Sigma)) \subset V^{\mathcal{C}}(\mathrm{Mod}^{\mathcal{C}}(\Sigma))$.

b) $\Rightarrow$ a): The mapping class group $\mathrm{Mod}^{\mathcal{C}}(\Sigma)$ is an extension of $\mathrm{PMod}(\Sigma)$ by a subgroup H of the symmetric group S_n, cf. (3.10), where n is the number of framed points and H is determined by allowed permutations of framed points. In particular H is a finite group and $V^{\mathcal{C}}(\mathrm{Mod}^{\mathcal{C}}(\Sigma))$ is finite as an extension of a finite group by a finite group.

Therefore, we will not distinguish between the two conditions and work with the pure mapping class group instead.

2. Let $T_1, \ldots, T_n$ denote the Dehn twists around each framed point in $\mathrm{PMod}_{g,n}$. Such Dehn twists act on the state space by scalar multiplication of θ_l where l is the label of the corresponding point, see (3.24). Since MFCs have twist eigenvalues of finite order, the image of the subgroup $\langle T_1, \ldots, T_n \rangle$ will always be finite. Hence, it is sufficient to check property F with respect to the remaining generators which are Dehn twists around the curves in Figure 3.5, corresponding to the unframed mapping class group $\mathrm{PMod}^{\mathrm{un}}_{g,n}$.

Similar to irreducibility, there are trivial examples where property F is present. For example, according to the above Remark, if $\mathrm{PMod}^{\mathrm{un}}_{g,n}$ is trivial (for $(g, n) = (0, n)$ with $n \leq 3$) then $\mathrm{PMod}_{g,n}$ is fully generated by Dehn twist around its marked points and its representation image is finite.

Property F on higher genus surfaces

A non-trivial result of [NS], briefly mentioned in Remark 5.6, guarantees the presence of property F on the torus $\Sigma = T^2$ for any MFC $\mathcal{C}$. However, we are also interested in examples on a larger family of surfaces, particularly of higher genus. Such examples include:

1. The MFC $\mathcal{C} = \mathrm{Rep}(D^{\omega}G)$ of representations of the twisted Drinfeld double of a finite group G is shown in [G] to have property F with respect to any d-surface (any genus and number of framed points).

2. In [JLLSW] it is shown that the MFC associated to the Ising CFT has property F with respect to surfaces without marked points. In the next section we extend this to all 16 Ising MFCs and all d-surfaces.

Property F on genus 0 surfaces

As mentioned before, the term property F is borrowed from [NR] where such a finiteness property is studied with respect to braid group representations.

Let $\mathcal{B}$ be a braided fusion category. For an object $X \in \mathcal{B}$, the endomorphism space $\mathrm{End}(X^{\otimes n})$ carries a natural action of the braid group by mapping a braid generator in B_n to the corresponding braiding of the strands labelled by X. The braided fusion category is said to have property $\mathrm{F}_{\mathrm{braid}}$ with respect to X if the braid group image is finite for all n. If this is the case for every X and any $n \in \mathbb{N}$, then $\mathcal{B}$ is said to have property $\mathrm{F}_{\mathrm{braid}}$ as a braided fusion category.

Remark 6.3. Since braid groups can be seen as mapping class groups as in Example 3.1, we may compare the above mentioned notion of finiteness with that of the mapping class group.

Let $\mathcal{C}$ be a MFC and X an object in $\mathcal{C}$. Let $\mathbb{S}^2(X, \dots, X, i)$ be the sphere with the first n points labelled by X and the $(n+1)$'th point labelled by $i \in I$. By semisimplicity property $\mathrm{F}_{\mathrm{braid}}$ with respect to X (as a braided fusion category) is equivalent to property F with respect to $\mathbb{S}^2(X, \dots, X, i)$ for all $i \in I$. I am not aware of a counterexample MFC which satisfies property $\mathrm{F}_{\mathrm{braid}}$ but not property F.

Before discussing results on property $\mathrm{F}_{\mathrm{braid}}$, we recall the following notions.

A fusion category $\mathcal{A}$ is *weakly integral* if its Frobenius-Perron dimension is an integer, i.e. $\mathrm{FPdim}(\mathcal{A}) \in \mathbb{N}$. This is equivalent to the condition that $\mathrm{FPdim}(X)^2 \in \mathbb{N}$ for all simple objects X [ENO1, Prop. 8.27]. If $\mathrm{FPdim}(X) \in \mathbb{N}$ for all simple objects X, then $\mathcal{A}$ is *integral*. An example of a weakly integral (but non-integral) fusion category is an Ising-type category, which has $\mathrm{FPdim}(\sigma) = \sqrt{2}$ for the spin object $\sigma \in I$. Ising categories will be introduced in the next section in detail.

A fusion category is *group-theoretical* if it is Morita equivalent to a pointed fusion category. Every pointed fusion category is equivalent to Vect_G^ω for some group G. Equivalently, a group-theoretical fusion category $\mathcal{A}$ has Drinfeld centre $\mathcal{Z}(\mathcal{A}) \simeq \mathrm{Rep}(D^\omega G)$. Every group-theoretical fusion category is integral, but the converse does not hold [Ni].

There is a further notion of weakly group-theoretical, which we do not recall in detail. The defining property of weakly group-theoretical fusion categories is that they are Morita equivalent to nilpotent[13] fusion categories. Weakly group-theoretical fusion categories are weakly integral and the converse is conjectured to be true.

Regarding property $\mathrm{F}_{\mathrm{braid}}$, it is conjectured by the authors in [NR, Sec. 1] that:

> A braided fusion category $\mathcal{B}$ has property $\mathrm{F}_{\mathrm{braid}}$ if and only if it is weakly integral.

This conjecture has been verified in one direction for weakly group-theoretical braided fusion categories [GrN]. This extends an earlier result that group-theoretical braided fusion categories have property $\mathrm{F}_{\mathrm{braid}}$ by [ERW] as well as some weakly group-theoretical examples

[13] A fusion category $\mathcal{N}$ is *nilpotent* if the sequence $\mathcal{N} \supset \mathcal{N}_{\mathrm{ad}} \supset (\mathcal{N}_{\mathrm{ad}})_{\mathrm{ad}} \supset \dots$, constructed by taking adjoint subcategories, converges to Vect. Adjoint subcategories or rather adjoint subrings will be recalled briefly in Section 7.1.

[RW, GRR]. The proof also relies on a result by [Na] showing that the core of any weakly group-theoretical fusion category is equivalent to either a pointed fusion category or the Deligne product of a pointed fusion category with an Ising category.

Remark 6.4. 1. One motivation for studying braid group representations and particularly their finiteness or lack thereof is due to topological quantum computing. To have universal quantum computation one is interested in dense (thus infinite) braid group images. A famous example is that of the Fibonacci category Fib. Moreover, for a family of non weakly integral examples coming from $\mathcal{C}(\mathfrak{g}, k)$ there are infinite images [NR]. Since for $\mathfrak{g} = sl_2$ the associated category is weakly integral only if $k \in \{2, 3, 4, 6\}$.

2. From part 1 one can observe that there are many levels which admit irreducible mapping class group representations but not $\mathrm{F}_{\mathrm{braid}}$, e.g. for all $k \geq 5$ with $k+2$ prime. Conversely, property F does not imply irreducibility (e.g. levels $k = 4, 6$).

6.2 Irreducibility and property F of Ising

In this section, we will review Ising-type MFCs and compute their associated mapping class group representations explicitly. Thereafter, we will present a proof that such categories satisfy property F and the irreducibility property, thus extending the result of [JLLSW] to surfaces with marked points and all 16 Ising-type MFCs.

6.2.1 Ising categories and their mapping class group representations

We introduce Ising categories following [DGNO, App. B] and compute their mapping class group action explicitly following the previous section. Up to equivalence there are 16 Ising modular fusion categories. This family is parametrized by an 8'th root of -1, which will be denoted by ζ, and a sign $\nu \in \{\pm\}$. The first determines the braided structure of the category, whereas the second determines the spherical structure. We will also need

$$\lambda := \zeta^2 + \zeta^{-2} \tag{6.1}$$

which satisfies $\lambda^2 = 2$.

The Ising category $\mathcal{C}(\zeta, \nu)$ has three simple objects $\mathbb{1}, \varepsilon, \sigma$ and the fusion rules are given in the following table:

$\otimes$	$\mathbb{1}$	ε	σ
$\mathbb{1}$	$\mathbb{1}$	ε	σ
ε	ε	$\mathbb{1}$	σ
σ	σ	σ	$\mathbb{1} \oplus \varepsilon$

In particular, there are no multiplicities in the fusion basis $N_{ij}^k \in \{0, 1\}$. This simplifies the notation of R,F,B-matrices and the associated graphs. The dimensions are $d_{\mathbb{1}} = d_\varepsilon = 1$, $d_\sigma = \nu\lambda$, and the twist values are $\theta_{\mathbb{1}} = 1$, $\theta_\varepsilon = -1$ and $\theta_\sigma = \nu\zeta^{-1}$.

The braidings in terms of R-matrices are given by:

$$R^{(\varepsilon\varepsilon)\mathbb{1}} = -1, \quad R^{(\varepsilon\sigma)\sigma} = R^{(\sigma\varepsilon)\sigma} = \zeta^4, \quad R^{(\sigma\sigma)\mathbb{1}} = \zeta, \quad R^{(\sigma\sigma)\varepsilon} = \zeta^{-3}.$$

The (non-trivial) F-matrices are given as:

$$F^{(\varepsilon\sigma\varepsilon)\sigma} = F^{(\sigma\varepsilon\sigma)\varepsilon} = -1$$
$$F^{(\sigma\sigma\sigma)\sigma} = \begin{pmatrix} F^{(\sigma\sigma\sigma)\sigma}_{\mathbb{1}\mathbb{1}} & F^{(\sigma\sigma\sigma)\sigma}_{\mathbb{1}\varepsilon} \\ F^{(\sigma\sigma\sigma)\sigma}_{\varepsilon\mathbb{1}} & F^{(\sigma\sigma\sigma)\sigma}_{\sigma\sigma} \end{pmatrix} = \frac{1}{\lambda}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The S-matrix is

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & \nu\lambda \\ 1 & 1 & -\nu\lambda \\ \nu\lambda & -\nu\lambda & 0 \end{pmatrix} \tag{6.2}$$

where the order of the basis is $\{\mathbb{1}, \varepsilon, \sigma\}$.

Notice that the F-matrices are self-inverse, i.e. $F^{(ijk)l} = G^{(ijk)l}$. It will be convenient to use the following notation: $\{\varepsilon_a\}_{a\in\mathbb{Z}_2}$, where $\varepsilon_0 = \mathbb{1}$ and $\varepsilon_1 = \varepsilon$. With this notation we have for example: $R^{(\sigma\sigma)\varepsilon_a} = \zeta^{1-4a}$.

Using the above data one can also compute the B-matrices according to (2.34) and (2.37). When $\mathrm{Hom}(l, i\otimes j\otimes k)$ is 1-dimensional (in other words when at least one of the labels is not σ) we have the following non-trivial B-matrices:

$$\begin{aligned} B^{\pm(\varepsilon_a\varepsilon\varepsilon)\varepsilon_a} &= (-1) \\ B^{\pm(\varepsilon_a\sigma\sigma)\varepsilon_b} &= \zeta^{\pm(1-4[a+b])}\ . \end{aligned} \tag{6.3}$$

These follow easily from R- and F-matrices. Note also that we write $[a+b]$ in the end to emphasize that we take $a+b \mod 2$ (a and b live in $\mathbb{Z}_2$). Otherwise, the expression would be wrong. When all labels i, j, k and l are σ-labels the corresponding Hom-space is 2-dimensional. By inserting R- and F-matrices into (2.34) and (2.37) one obtains the following 2×2-matrices:

$$B^{\pm(\sigma\sigma\sigma)\sigma} = \zeta^{\pm}\begin{pmatrix} \zeta^{\mp 2} & \zeta^{\pm 2} \\ \zeta^{\pm 2} & \zeta^{\mp 2} \end{pmatrix}\ . \tag{6.4}$$

Remark 6.5. Half of the 16 Ising categories are unitary and the other half are non-unitary dictated by the positivity respectively negativity of d_σ.

The Ising category associated to the Ising CFT as well as the $\mathcal{C}(sl_2, k)$ at level $k = 2$ are among the unitary Ising categories.

6.2.2 Modular Action of Ising Categories

In this section, we will explicitly describe how the pure mapping group acts on the associated state spaces in the case of Ising categories. Therefore, we fix an Ising category $\mathcal{C} = \mathcal{C}(\zeta, \nu)$ as introduced in section 6.2.1.

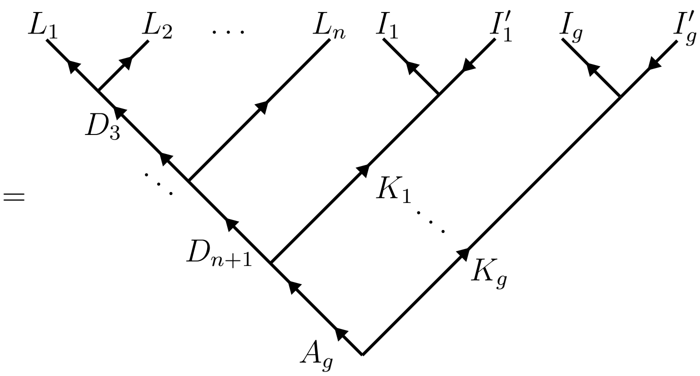
Let Σ be a surface of genus g and n framed points labelled by simple objects $l_1, \ldots, l_n$, i.e. the underlying vector space of the state space is

$$V_{g,n} = \mathcal{C}(\mathbb{1}, l_1 \otimes \ldots l_n \otimes L^{\otimes g})\ . \tag{6.5}$$

We now describe a basis of this space that we will use to give the action of the generators as listed in (3.24).

Consider the oriented graph

$\Gamma :=$



, (6.6)

where $L_1, \dots, L_n, D_3, \dots, D_{n+1}, K_1, \dots, K_g, A_2, \dots, A_g, I_1, \dots, I_g, I'_1, \dots, I'_g$ are the edges of the graph. This choice of notation will become convenient later.

A coloring is a map $\chi : E(\Gamma) \to I$, which assigns to every edge of the graph a label in I. We say that the coloring is *admissible* if at any trivalent vertex of the graph the associated fusion is non-zero. For instance, the coloring χ is admissible at the first vertex in (6.6) if $N_{\chi(L_1)\chi(L_2)}^{\chi(D_3)} \neq 0$. Let $\mathrm{col}(\Gamma)$ denote the set of admissible colorings of the graph Γ. Moreover, we consider the subset $\mathrm{col}(\Gamma)^\circ \subset \mathrm{col}(\Gamma)$ consisting of all colorings χ such that

$$\chi(L_1) = l_1 \ , \ \dots \ , \ \chi(L_n) = l_n \quad \text{and} \quad \chi(I_k) = \chi(I'_k) \ , \ \ k = 1, \dots, g \ . \tag{6.7}$$

Notice that for $\chi \in \mathrm{col}(\Gamma)^\circ$ we have $\chi(K_m) \in \{\mathbb{1}, \varepsilon\}$ for all $m = 1, \dots, g$ as $\chi(K_m) = \sigma$ is not admissible due to the condition of $\chi(I_m) = \chi(I'_m)$ and the Ising fusion rules.

In terms of this notation, for $\chi \in \mathrm{col}(\Gamma)^\circ$ the graph Γ coloured by χ represents a vector $\hat{\chi} \in V_{g,n}$, with $V_{g,n}$ as in (6.5), and where the embedding $\chi(I_k) \otimes \chi(I_k)^* \to L$ is implicit. Altogether, the set

$$\{\hat{\chi}\}_{\chi \in \mathrm{col}(\Gamma)^\circ} \tag{6.8}$$

forms a basis of $V_{g,n}$.

We proceed by computing the mapping class group action on $V(\Sigma_{g,n})$ with respect to this basis using the equations in (3.24). It is clear that

$$T_{\lambda_m} \hat{\chi} = \theta_{l_m} \hat{\chi} \tag{6.9}$$

and

$$T_{\alpha_m} \hat{\chi} = \theta_{\chi(I_m)} \hat{\chi} \ . \tag{6.10}$$

To give the expression for the action of S_m on a basis vector $\hat{\chi}$ we make the following distinction:

For $\chi(I_m) = \varepsilon_p$, which by admissibility also implies $\chi(K_m) = \mathbb{1}$, we compute

$$S_m \hat{\chi} = \frac{1}{2} \big(\hat{\chi}|_{I_m = \mathbb{1}} + \hat{\chi}|_{I_m = \varepsilon} + (-1)^p \nu \lambda \, \hat{\chi}|_{I_m = \sigma} \big) \tag{6.11}$$

where the slash notation indicates that for example $\hat{\chi}|_{I_m=\sigma}$ represents the vector with the same colouring χ everywhere up to the edge I_m, which is now labelled by σ. The factor $\frac{1}{2}$ appears as $D = 2$ for Ising categories.

For $\chi(I_m) = \sigma$ and $\chi(K_m) = \mathbb{1}$ we find

$$S_m\hat{\chi} = \frac{\lambda}{2}\left(\hat{\chi}|_{I_m=\mathbb{1}} - \hat{\chi}|_{I_m=\varepsilon}\right) \ . \tag{6.12}$$

Equations (6.11) and (6.12) can be formed into one:

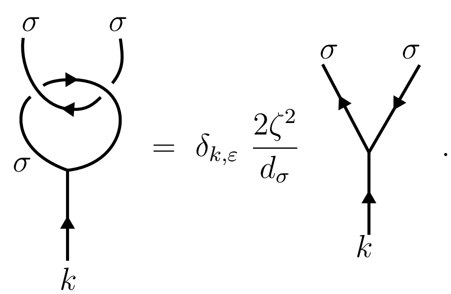
$$S_m\hat{\chi} = \sum_{j\in I} S_{\chi(I_m),j}\hat{\chi}|_{I_m=j} \tag{6.13}$$

using the S-matrix S from (6.2) where χ such that $\chi(K_m) = \mathbb{1}$.

The last case is when $\chi(K_m) = \varepsilon$ and hence $\chi(I_m) = \sigma$. The result is

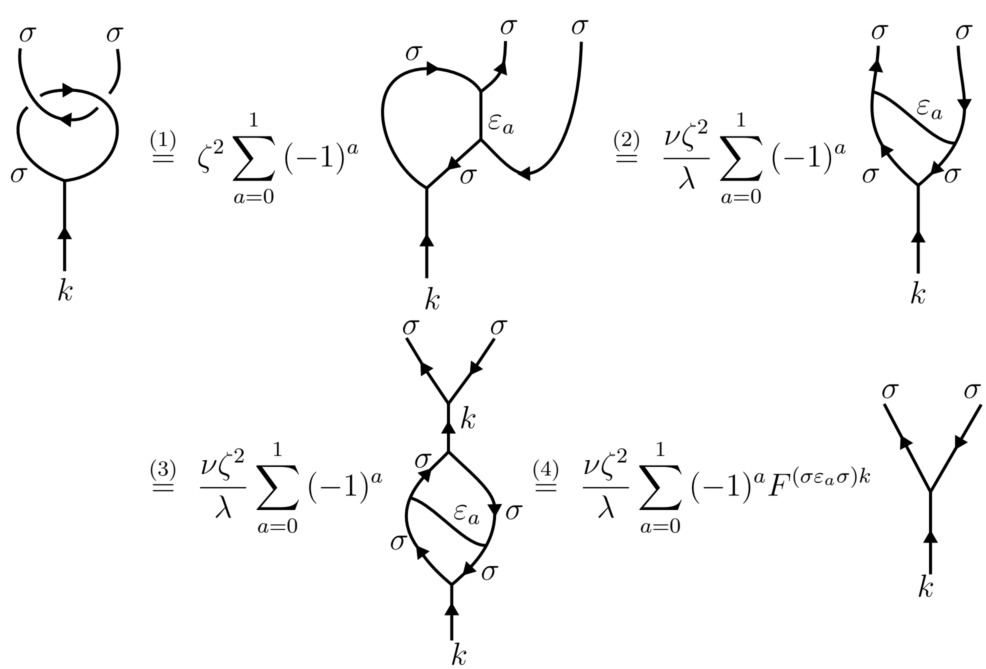
$$S_m\hat{\chi} = \zeta^2\hat{\chi} \tag{6.14}$$

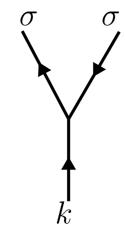
which follows from the computation



(6.15)

In fact, let me demonstrate how to perform the above computation to familiarise the reader with string diagrammatic computations involving Ising categories:



$$\overset{(5)}{=} \ \delta_{k,\varepsilon}\, \frac{2\zeta^2}{d_\sigma}$$  . (6.16)

In (1) the left diagram is isotoped to the right and the double braiding of σ with itself decomposes into summands of $\mathbb{1}$ and ε. The phase $\zeta^2(-1)^a$ is the result of the double R-matrix applied twice. Step (2) might seem enigmatic but it is obtained by changing the orientation of the two fusion vertices by bending the σ-labelled strings. The factor ν/λ is the result of this change of basis (we refer to [TV2, Chap. 4.7] for a general graphical calculus with fusion bases). Step (3) is fusing the two σ strings on the top and step (4) follows from the definition of F-matrices (here a one dimensional matrix). Finally, (5) is the result of:

$$\frac{\nu\zeta^2}{\lambda}\sum_{a=0}^{1}(-1)^a F^{(\sigma\varepsilon_a\sigma)k} = \frac{\nu\zeta^2}{\lambda}\sum_{a=0}^{1}(-1)^{a(1+p(k))} = \delta_{k,\varepsilon}\frac{2\nu\zeta^2}{\lambda} = \delta_{k,\varepsilon}\frac{2\zeta^2}{d_\sigma}\ . \tag{6.17}$$

This completes the proof of (6.15).

The action of T_{γ_m} is described by twisting with θ the tensor product of two neighbouring strands as seen in equation (3.24). If the fusion is again a simple object, the action produces just the twist eigenvalue of this simple object. That is, let $\chi \in \mathrm{col}(\Gamma)^\circ$ such that $\chi(I_m)\otimes\chi(I_{m+1})$ is simple. Then, we have

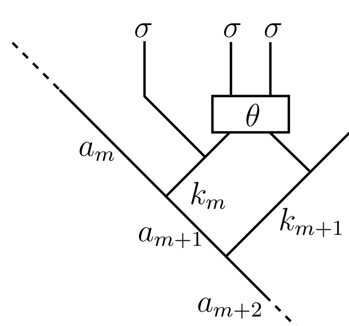
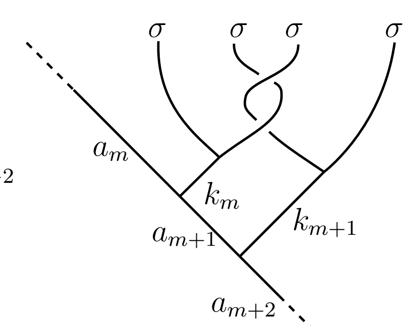
$$T_{\gamma_m}\hat{\chi} = \theta_{\chi(I_m)\otimes\chi(I_{m+1})}\,\hat{\chi}\ . \tag{6.18}$$

If $\chi(I_m)\otimes\chi(I_{m+1})$ is not simple, then we have $\chi(I_m)=\chi(I_{m+1})=\sigma$ according to the Ising fusion rules. In this case, we claim

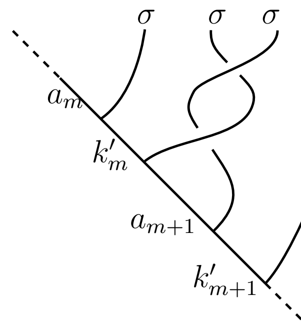
$$T_{\gamma_m}\hat{\chi} = \hat{\chi}|_{\{K_m,K_{m+1},A_{m+1}\}\otimes\varepsilon} \tag{6.19}$$

where $\{K_m,K_{m+1},A_{m+1}\}\otimes\varepsilon$ indicates that we change the colouring of the edges K_m, K_{m+1} and A_{m+1} to $\chi(K_m)\otimes\varepsilon, \chi(K_{m+1})\otimes\varepsilon$ and $\chi(A_{m+1})\otimes\varepsilon$ respectively.

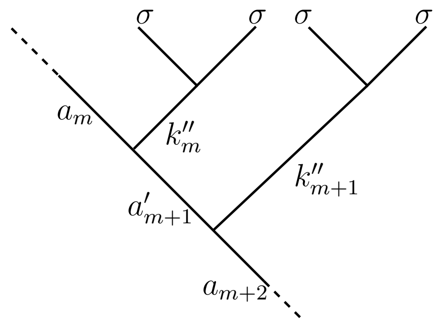
Let us expand on how (6.19) is obtained. Fix $\hat{\chi}$ with labels $\chi(I_m)=\chi(I_{m+1})$ and to simplify notation write $k_i \equiv \chi(K_m), a_i \equiv \chi(A_i)$ for its labels:

$T_{\gamma_m}\hat{\chi} =$  $= \zeta^{-2}$ 

$$= \zeta^{-2} \sum_{k'_m, k'_{m+1}} F^{(a_m \sigma\sigma) a_{m+1}}_{k_m k'_m} F^{(a_{m+1} \sigma\sigma) a_{m+2}}_{k_{m+1} k'_{m+1}}$$



$$= \zeta^{-2} \sum_{\substack{k'_m, k''_m, \\ k'_{m+1}, k''_{m+1}, \\ a'_{m+1}}} M_{k'_m k''_m k'_{m+1} k''_{m+1} a'_{m+1}} \qquad (6.20)$$



where $M_{k'_m k''_m k'_{m+1} k''_{m+1} a'_{m+1}}$ in the last expression is defined as:

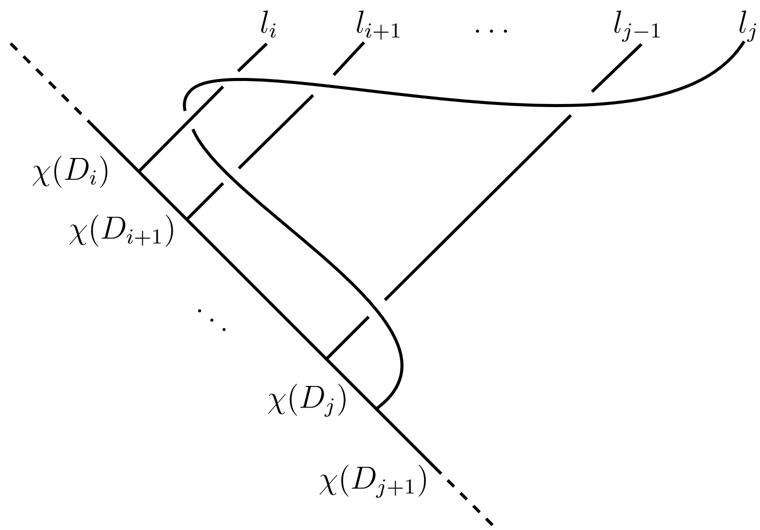
$$F^{(a_m \sigma\sigma) a_{m+1}}_{k_m k'_m} F^{(a_{m+1} \sigma\sigma) a_{m+2}}_{k_{m+1} k'_{m+1}} (B^{(k'_m \sigma\sigma) k'_{m+1}})^2_{a_{m+1} a'_{m+1}} F^{(a_m \sigma\sigma) a'_{m+1}}_{k'_m k''_m} F^{(a'_{m+1} \sigma\sigma) a_{m+2}}_{k'_{m+1} k''_{m+1}} \qquad (6.21)$$

The last term in (6.20) might seem complicated at first, but it is only matrix multiplication. Using the explicit formulas we found in Section 6.2.1 we obtain (6.19).

To compute the action of T_{δ_m} we will use the tensoriality property of the twist as we did essentially for T_{γ_m}. For instance, the action of T_{δ_m} on $\hat{\chi}$ is the composition of pure braids in strands labelled by $\chi(L_{m+1}), \ldots, \chi(L_n), \chi(I_1)$ and products of the twist eigenvalues of these labels. Therefore, we study the action of pure braids.

We start by considering pure braids in $l_1, \ldots, l_n$. Let

$$T_{ij}\ \hat{\chi}\ = \qquad (6.22)$$



be the pure braid between the l_i- and l_j-coloured strands.

If l_i and l_j have a unique fusion, i.e. if $l_i \otimes l_j$ is a simple object, then T_{ij} acts by a phase $\theta_{l_i\otimes l_j}\theta_{l_i}^{-1}\theta_{l_j}^{-1}$, which is obtained by expressing the double braid using the twist and its inverses.

Otherwise, for $l_i = l_j = \sigma$, we prove the following lemma.

Lemma 6.6. If $l_i = l_j = \sigma$, then

$$T_{ij}\hat{\chi} = \zeta^2(-1)^{p(\chi(D_{j+1}))}\zeta^{4m}\hat{\chi}|_{\{D_{i+1},\dots,D_j\}\otimes\varepsilon}$$

where $p(\sigma) \equiv 0$ and some $m \in \mathbb{Z}_4$ which may depend on $\chi(D_i),\dots,\chi(D_j)$ but it does not depend on $\chi(D_{j+1})$.

Proof. We prove this by induction on $|i-j|$. We start with $j = i+1$. This is the case of applying the B-matrix twice on the strands l_i and l_{i+1}. We compute this for the different admissible labels of $D_{i+1} = D_j$.

- For $\chi(D_{i+1}) = \sigma$, let $\chi(D_i) = \varepsilon_a$ and $\chi(D_{i+2}) = \varepsilon_b$ be the admissible labels. Then, we find:
$$T_{ii+1}\hat{\chi} = (B^{(\varepsilon_a\sigma\sigma)\varepsilon_b})^2\hat{\chi} \overset{(6.3)}{=} \zeta^2(-1)^{a+b}\hat{\chi}$$
- For $\chi(D_{i+1}) = \varepsilon_a$, we get admissible labels $\chi(D_i) = \sigma$ and $\chi(D_{i+2}) = \sigma$. The result is:
$$T_{ii+1}\hat{\chi} \overset{(6.4)}{=} \zeta^2(-1)^a\hat{\chi}|_{D_{i+1}\otimes\varepsilon}\ .$$

Assuming that the statement is true for fixed i and j, we make the induction step and prove it for i and $j+1$:

$$T_{ij+1}\hat{\chi} = \zeta^2\zeta^{4m}\sum_{s_{j+1},d'_{j+1}}(-1)^{p(s_{j+1})}B^{-(d_jl_{j+1}l_j)d_{j+2}}_{d_{j+1}s_{j+1}}B^{(\varepsilon\otimes d_jl_jl_{j+1})d_{j+2}}_{s_{j+1}d'_{j+1}}$$
$$\hat{\chi}|_{\{D_{i+1},\dots,D_j\}\otimes\varepsilon,D_{j+1}=d'_{j+1}}.$$

where m depends only on $d_i \equiv \chi(D_i),\dots,\chi(D_j) \equiv d_j$ allowing us to take the factor ζ^{4m} out of the sum[14]. The above is obtained by using the inverse B-matrix, the induction assumption and the B-matrix again[15]. After computing this for all labels, one verifies the statement. We give the phases for each case:

- For $l_j = \varepsilon$: $\zeta^2\zeta^{4m}(-1)^{p(d_{j+2})+1}$.
- For $l_j = \sigma$:
  1. for $d_{j+2} = \varepsilon_a$: $\zeta^2\zeta^{4m+2p(d_j)}(-1)^a$
  2. for $d_{j+2} = \sigma$: $\zeta^2\zeta^{4m}\zeta^4(-1)^{p(d_{j+1})}$

[14] This is the whole reason why we assume this special dependence of m on the D-labels; for the induction to work.

[15] There is a pure braid relation $T_{ij+1} = (\mathrm{id}\otimes c^{-1}_{j-1,j})\circ T_{ij}\circ(\mathrm{id}\otimes c_{j-1,j})$ where T_{ij} denotes the pure braid of strands i and j as before and c denotes the braid generator with c^{-1} the inverse braid.

□

Similar to this, we can prove the following lemma, which includes braids with i_1.

Lemma 6.7. If $l_i = i_1 = \sigma$, then

$$T_{l_i i_1}\ \hat{\chi}\ =\ \zeta^2\ \zeta^{4m}\ \hat{\chi}|_{\{D_{i+1},\dots,D_{n+1},K_1\}\otimes\varepsilon}$$

where $p(\sigma) \equiv 0$ and $m \in \mathbb{Z}_4$ which may depend on $\chi(D_i), \dots, \chi(D_{n+1})$ but not on $\chi(A_2)$.

6.2.3 Irreducibility property of Ising categories

In this section we prove our second main result:

Theorem 6.8. Let $\mathcal{C} = \mathcal{C}(\zeta, \nu)$ be an Ising-type MFC and let $\Sigma_{g,n}$ be an extended surface whose marked points are labelled by simple objects. Then, $V^{\mathcal{C}}(\Sigma_{g,n})$ is an irreducible projective representation of the pure mapping class group.

Remark 6.9.

1. The case without marked points, i.e. $n = 0$, was already shown in [JLLSW, Sec. 4.3] (for one of the 16 Ising-type categories). It turns out that the same method works in all 16 cases and that it can be easily adapted to the case with marked points, and so our proof follows closely that of [JLLSW].
2. Irreducibility with respect to the pure mapping class group $\mathrm{PMod}_{g,n}$ implies irreducibility with respect to the mapping class group $\mathrm{Mod}^{\mathcal{C}}(\Sigma_{g,n})$, as $\mathrm{PMod}_{g,n}$ is a subgroup.

Proof of Theorem 6.8. Let $\Sigma_{g,n}$ be a d-surface with simple point labels and set $V_{g,n} := V^{\mathcal{C}}(\Sigma_{g,n})$. To prove irreducibility, we will show that the inclusion $\Bbbk[V(\mathrm{PMod}_{g,n})] \subset \mathrm{End}(V_{g,n})$ is actually an equality. In terms of the basis $\{\hat{\chi}\}$ of $V_{g,n}$ from (6.8), denote by $E(\hat{\chi}, \hat{\chi}')$ the elementary matrix in $\mathrm{End}(V_{g,n})$, which maps $\hat{\chi}$ to $\hat{\chi}'$ and which maps all other basis elements to zero. We have completed the proof once we have shown that

$$\forall \chi, \chi' \in \mathrm{col}(\Gamma)^\circ\ :\ E(\hat{\chi}, \hat{\chi}') \in \Bbbk[V(\mathrm{PMod}_{g,n})]\ . \tag{6.23}$$

As a first step towards this goal, define the operators:

$$P_{\mathbb{1}}(\gamma) = \frac{1}{16}\sum_{k=1}^{16} V(T_\gamma^k)\ ,\quad P_\sigma(\gamma) = \frac{(\mathbb{1} - V(T_\gamma^2))}{1-\zeta^{-2}}\ ,\quad P_\varepsilon(\gamma) = \mathbb{1} - P_{\mathbb{1}}(\gamma) - P_\sigma(\gamma)\ , \tag{6.24}$$

where γ is a simple closed curve and T_γ is the Dehn twist around γ. One checks[16] that the $P_j(\gamma)$, $j \in \{\mathbb{1}, \sigma, \varepsilon\}$, for γ a simple closed from Figure 6.1, are pairwise orthogonal idempotents. Note that for all simple closed curves γ, by construction

$$P_{\mathbb{1}}(\gamma)\,,\ P_\sigma(\gamma)\,,\ P_\varepsilon(\gamma)\ \in\ \Bbbk[V(\mathrm{PMod}_{g,n})]\ . \tag{6.25}$$

[16] Notice that $\frac{1}{16}\sum_{k=1}^{16}\theta_i^k = \delta_{\mathbb{1},i}$ and $\frac{1-\theta_i^2}{1-\zeta^{-2}} = \delta_{\sigma,i}$.

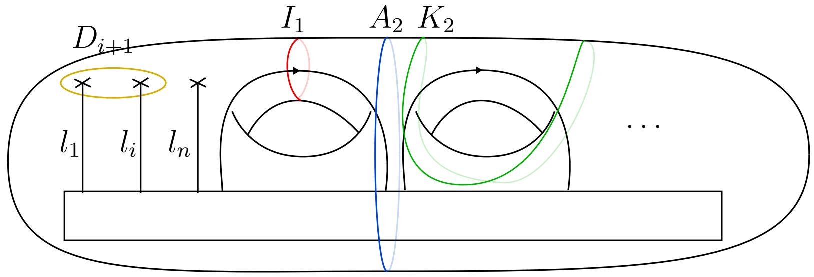


Figure 6.1: Curves of type D, I, A, K on the surface based on the handlebody with an embedded graph and a coupon.

Let a be one of the edge labels D_m, I_m, A_m, or K_m as used in (6.6) and denote by γ_a the corresponding simple closed curve as shown in Figure 6.1. The Dehn twist along γ_a acts diagonally on each basis vector $\hat{\chi}$, giving twist eigenvalue of the edge colour $\chi(a)$. From this it is easy to check that the idempotents $P_j(\gamma_a)$ for $j \in I$ project onto those basis elements where the edge a is labelled by j:

$$P_j(\gamma_a)\ \hat{\chi} \ = \ \delta_{j,\chi(a)}\ \hat{\chi}\ . \tag{6.26}$$

This implies that diagonal maps of the form $E(\hat{\chi},\hat{\chi}) \equiv E(\hat{\chi})$ are in $\Bbbk[V(\mathrm{Mod}_{g,n})]$ as they are realized as a product of P_j maps, namely

$$E(\hat{\chi}) = \prod_{E \text{ edge}} P_{\chi(E)}(E)\ . \tag{6.27}$$

More precisely, E runs over the edges $D_3,\dots,D_{n+1},K_1,\dots,K_g,A_2,\dots,A_g,I_1,\dots,I_g$ and by abuse of notation they also denote the corresponding curves in Figure 6.1. We will now describe how to move through different basis elements by changing the respective labels, i.e. how to construct the rest of the $E(\hat{\chi},\hat{\chi}')$ maps using mapping classes. We will use the computations of section 6.2.2.

Changing only I_m labels: Let $\chi \in \mathrm{col}(\Gamma)^\circ$ be a colouring such that $\chi(K_m) = \mathbb{1}$. Then, the colourings $\{\chi|_{I_m=j}\}_{j\in I}$ are all admissible. To jump between the corresponding basis vectors, we use equations (6.11) and (6.12) to show

$$E(\hat{\chi}|_{I_m=\varepsilon_p};\hat{\chi}|_{I_m=\varepsilon_{p+1}}) = 2P_{\varepsilon_{p+1}}(I_m)S_m E(\hat{\chi}|_{I_m=\varepsilon_p}) \tag{6.28}$$

and

$$E(\hat{\chi}|_{I_m=\varepsilon_p};\hat{\chi}|_{I_m=\sigma}) = (-1)^p \lambda P_\sigma(I_m)S_m E(\hat{\chi}|_{I_m=\varepsilon_p}) \tag{6.29}$$

and

$$E(\hat{\chi}|_{I_m=\sigma};\hat{\chi}|_{I_m=\varepsilon_p}) = (-1)^p \lambda P_{\varepsilon_p}(I_m)S_m E(\hat{\chi}|_{I_m=\sigma}). \tag{6.30}$$

Note that $\chi(K_m) = \varepsilon$ only allows σ as an I_m-label.

Changing K_m-labels: We will now describe how to change K_m-labels while keeping the labels of $D_3,\dots,D_{n+1}$ unchanged. Therefore, we will fix labels $d_3 \equiv \chi(D_3),\dots,d_{n+1} \equiv \chi(D_{n+1})$ and only consider basis elements with such labels. The labels of $A_2,\dots,A_g$ are in fact uniquely determined by the labels for $D_{n+1},K_1,\dots,K_g$ and they are all in the label set $\{\mathbb{1},\varepsilon\}$.

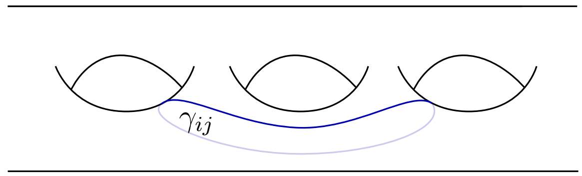


Figure 6.2: The curve γ_{ij} connects the i'th with the j'th genus.

Consider first the case where $d_{n+1} = \mathbb{1}$. In this case, $\chi(K_1) = \cdots = \chi(K_g) = \mathbb{1}$ and $\chi(I_1) = \cdots = \chi(I_g) = \mathbb{1}$ are admissible and let χ_0 denote this colouring and the already fixed labels for $D_3, \ldots, D_{n+1}$. For any other basis vector $\hat{\chi}$ (with the same D-labels $d_{n+1} = \mathbb{1}$ as mentioned) we will construct the operators $E(\hat{\chi}_0, \hat{\chi})$. Let $\{K_{m_l}\}_{l=1,\ldots,L}$ be the maximal subset of K-edges such that $\chi(K_{m_l}) = \varepsilon$ and $m_1 < \cdots < m_L$. In particular, we have $\chi(I_{m_l}) = \sigma$ for all l. The fusion rules imply that the number of non-trivial K-labels L is even as we have fixed $d_{n+1} = \mathbb{1}$. Then

$$E(\hat{\chi}_0, \hat{\chi}) \;=\; E(\hat{\chi'}, \hat{\chi})\, T_{\gamma_{m_{L-1},m_L}} \cdots T_{\gamma_{m_3,m_4}}\, T_{\gamma_{m_1,m_2}}\, E(\hat{\chi}_0, \hat{\chi}_0|_{I_{m_l}=\sigma}) \tag{6.31}$$

where the curves $\gamma_{i,j}$ connect the i'th with the j'th hole as shown in Figure 6.2. The action of $T_{\gamma_{m_i m_{i+1}}}$ is obtained similar to (6.19) and changes the labels of K_{m_i} and $K_{m_{i+1}}$ to ε. The operator $E(\hat{\chi}_0, \hat{\chi}_0|_{\{I_{m_1},\ldots,I_{m_L}\}=\sigma})$ changes the I_{m_l} labels to $\chi(I_{m_l}) = \sigma$ and is a product of the operators in (6.29). The vector $\hat{\chi}'$ is obtained from the colouring χ, but with $\chi(I_m) = \mathbb{1}$ for all $m \notin \{m_1, \ldots, m_L\}$. Similarly, the operator $E(\hat{\chi'}, \hat{\chi})$ is obtained as a product of operators in (6.28), (6.29).

In the opposite direction, one gets

$$E(\hat{\chi}, \hat{\chi}_0) \;=\; E(\hat{\chi}', \hat{\chi}_0)\, T_{\gamma_{m_{L-1},m_L}} \cdots T_{\gamma_{m_1,m_2}}\, E(\hat{\chi}) \tag{6.32}$$

where $\hat{\chi}'$ now is obtained by χ but with all K_m-labels set to $\mathbb{1}$.

Consider now the case where $d_{n+1} = \varepsilon$. In this case, fix the colouring χ_0 with labels $\chi_0(K_1) = \varepsilon, \chi_0(I_1) = \sigma$ and $\chi_0(K_2) = \cdots = \chi_0(K_g) = \mathbb{1}$ and $\chi(I_2) = \cdots = \chi(I_g) = \mathbb{1}$. As before, let $\hat{\chi}$ be any basis element (now with $d_{n+1} = \varepsilon$) and consider the maximal subsequence $\{K_{m_l}\}_{l=1,\ldots,L}$ with $\chi(K_{m_l}) = \varepsilon$, which is in this case has an odd number of elements L. Then, similar to the previous considerations we get

$$E(\hat{\chi}_0, \chi) \;=\; E(\hat{\chi}', \chi)\, T_{\gamma_{m_{L-1},m_L}} \cdots T_{\gamma_{m_2,m_3}}\, T_{\gamma_{1,m_1}}\, E(\hat{\chi}_0, \hat{\chi}_0|_{I_{m_l}=\sigma}) \tag{6.33}$$

where we omit the γ_{1,m_1} Dehn twist if $m_1 = 1$. Similarly,

$$E(\hat{\chi}, \hat{\chi}_0) \;=\; E(\hat{\chi}', \hat{\chi}_0)\, T_{\gamma_{m_{L-1},m_L}} \cdots T_{\gamma_{m_2,m_3}}\, T_{\gamma_{1,m_1}}\, E(\hat{\chi}, \hat{\chi}|_{I_1=\sigma}) \;. \tag{6.34}$$

To conclude[17], we have constructed for any two basis elements $\hat{\chi}, \hat{\chi}'$ with fixed labels $d_3, \ldots, d_{n+1}$ the operators $E(\hat{\chi}, \hat{\chi}')$, namely by passing through the distinguished basis element χ_0, i.e.

$$E(\hat{\chi}, \hat{\chi}') \;=\; E(\hat{\chi}_0, \hat{\chi}')\, E(\hat{\chi}, \hat{\chi}_0) \;. \tag{6.35}$$

[17]The label σ is not admissible for d_{n+1} due to the Ising fusion rules, $\mathcal{C}(\sigma, L^g) = 0$ for Ising categories.

Changing D_m-labels: Changing the labels D_m-labels which previously were left unchanged, is the final step to complete the irreducibility proof. For D_m to have two admissible labels, i.e. such that both $\mathbb{1}$ and ε are admissible, we have either $\#(\sigma-$labelled marked points$) \geq 4$ or $\#(\sigma -$ labelled marked points$) \geq 2$ and $g \geq 1$. Therefore, consider the case where $\mathbb{1}$ and ε are both admissible for D_m. There exists some $m_- < m$ such that $l_{m_-} = \sigma$ and let m_- be the maximal such index, which directly implies that the only admissible label for D_{m_-} is σ.

If there exists some $m_+ \geq m$ such that $l_{m_+} = \sigma$, let m_+ be the minimal such index. Then, consider the curve δ_{m_-,m_+} that encircles the m_-'th and m_+'th marked point and all the marked points in between. Then, using the result of Lemma 6.6

$$E(\hat{\chi}, \hat{\chi}|_{D_m \otimes \varepsilon}) = \alpha T_{\delta_{m_-,m_+}} E(\hat{\chi}) \tag{6.36}$$

where α is a phase. To see how Lemma 6.6 is applied, note that the twist of the product of the strands from m_- to m_+ will lead to a pure braid thereof multiplied by their respective twist eigenvalues. The twist eigenvalues are absorbed in the phase α and the pure braid is a product of the pure braid generators whose action we described in Lemma 6.6.

If however there does not exist such $m_+ \geq m$, then $g \geq 1$ and therefore consider the curve δ_{m-1} in Figure 3.6. Then,

$$E(\hat{\chi}, \hat{\chi}|_{D_m \otimes \varepsilon}) = \alpha T_{\delta_{m-1}} E(\hat{\chi}) \tag{6.37}$$

where the phase α appears according to Lemma 6.7. □

6.2.4 Property F of Ising Categories

In this section we prove that Ising categories have property F with respect to any extended surface.

Theorem 6.10. Let $\mathcal{C} = \mathcal{C}(\zeta, \nu)$ be any Ising-type modular fusion category. Then, $\mathcal{C}$ has property F with respect to d-surfaces.

To prove that Ising categories have property F, we will give for any surface $\Sigma_{g,n}$ with simple point labels $l_1, \ldots, l_n$ a certain set $X \equiv X_{g,n}(l_1, \ldots, l_n) \subset V_{g,n}$ such that:

1. X is finite
2. X spans $V(\Sigma_{g,n})$
3. X is $\mathrm{PMod}(\Sigma_{g,n})$-invariant

This is sufficient to show property F. The representation image of $\mathrm{PMod}(\Sigma_{g,n})$ is contained in the subgroup of transformations that preserve X by the invariance condition on X, i.e. $V(\mathrm{PMod}(\Sigma_{g,n})) \subset \{T \in \mathrm{GL}(V_{g,n}) \mid T(X) = X\}$. The latter group is isomorphic to the group of bijections on X, $\mathrm{Aut}(X)$, which follows from the fact that X spans $V_{g,n}$. Since X is a finite set and the group of bijections consists of $|X|!$ elements, the image $V(\mathrm{PMod}(\Sigma_{g,n}))$ has finitely many elements.

Remark 6.11. As already mentioned in Remark 6.2 property F with respect to the pure mapping class group PMod is equivalent to property F with respect to the non-pure mapping class group $\mathrm{Mod}^{\mathcal{C}}$. In fact, consider a surface $\Sigma_{g,n}$ carrying the same point labels $l_1 = \cdots = l_n$ and a finite $\mathrm{PMod}_{g,n}$-invariant set $X_{g,n}$. Pick a section $s : S_n \to \mathrm{Mod}_{g,n}$ of the short exact sequence (3.10) and define the set

$$X^s_{g,n} := \{s(t).x \mid x \in X_{g,n}, t \in S_n\} \subset V_{g,n} \ . \tag{6.38}$$

This set is also finite as both $X_{g,n}$ and S_n are finite. Let $f \in \mathrm{Mod}_{g,n}$ be a mapping class and $t \in S_n$ any permutation. Then, $s(\pi(f) \circ t^{-1}) \circ f \circ s(t) \in \ker(\pi) = \mathrm{PMod}_{g,n}$, i.e. there exists $g \in \mathrm{PMod}_{g,n}$ such that $f \circ s(t) = s(\pi(f) \circ t) \circ g$. Then, for some $x \in X_{g,n}$ we have

$$f \circ s(t).x = s(\pi(f) \circ t) \circ g.x = s(\pi(f) \circ t).x' \tag{6.39}$$

for some $x' \in X_{g,n}$ which is provided by the $\mathrm{PMod}_{g,n}$-invariance.

The same can be applied for distinct point labels, when $\mathrm{Mod}_{g,n}$ is the extension of $\mathrm{PMod}_{g,n}$ by some subgroup of the permutation group S_n.

An analogous argument works for property F for the extended mapping class group $\widehat{\mathrm{Mod}}_{g,n}$. Taking the weighted mapping class (id, n) which is the identity mapping class with weight n we have by definition of the TQFT

$$V(\mathrm{id}, n) = \left(\frac{p_+}{p_-}\right)^{-n/2} \mathrm{id}, \tag{6.40}$$

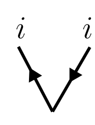
where $p_+/p_- = \theta_\sigma^2$ is the anomaly factor of the Ising MFC which follows from a quick calculation using (3.14). Since θ_σ has a finite order $\theta_\sigma^{16} = 1$, the set X can be extended by multiplying all elements with powers of θ_σ such that it is $\widehat{\mathrm{Mod}}_{g,n}$-invariant while remaining finite.

Before proving Theorem 6.10 in its generality, we will start with the torus case and the sphere with four marked points to give an idea of how the elements of $X_{g,n}$ will look like.

Torus Case

The mapping class group of the torus $\mathrm{PMod}_{1,0} = \mathrm{Mod}_{1,0}$ is isomorphic to the group $\mathrm{SL}(2, \mathbb{Z})$ with generators T and S. The two generators consist of the Dehn twist T_α along the single meridian α, see Figure 6.3, and the S-transformation.

The state space $V_{1,0}$ has basis elements

$e_i \ =$ 

(6.41)

where $i \in I = \{\mathbb{1}, \varepsilon, \sigma\}$ and the action of the generators as described in section 6.2.2 is given explicitly as:

$$T_\alpha(e_i) = \theta_i e_i$$

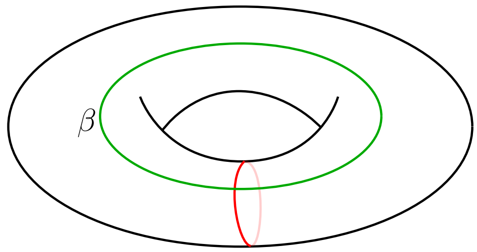


Figure 6.3: Dehn twists which generate the mapping class group of the torus

$$S(e_{\mathbb{1}}) = \frac{1}{2}(e_{\mathbb{1}} + e_\varepsilon + \lambda e_\sigma)$$
$$S(e_\varepsilon) = \frac{1}{2}(e_{\mathbb{1}} + e_\varepsilon - \lambda e_\sigma)$$
$$S(e_\sigma) = \frac{\lambda}{2}(e_{\mathbb{1}} - e_\varepsilon) \tag{6.42}$$

Now, for the elements $e_\pm := e_{\mathbb{1}} \pm e_\varepsilon$ one can easily check using the above equations that

$$T_\alpha e_\pm = e_\mp, \quad T_\alpha \lambda e_\sigma = \nu\zeta^{-1}\lambda e_\sigma \tag{6.43}$$

and

$$S(e_+) = e_+, \quad S(e_-) = \lambda e_\sigma, \quad S(\lambda e_\sigma) = e_- \tag{6.44}$$

where the last equality also holds trivially from the fact that $S^2 = \mathrm{id}$. Therefore, the set $X_{1,0} := \{\theta_\sigma^m e_\pm, \theta_\sigma^m \lambda e_\sigma \,|\, m \in \mathbb{Z}_{16}\}$ is invariant under the generators T_α and S.

Before we conclude invariance under the whole mapping class group, we have to consider that the representation as presented here is projective. The projective factors of the MCG representation, not only for the torus, are integer powers of the anomaly factor $\sqrt{p_+/p_-}$. In the case of Ising categories, these are integer powers of θ_σ. In the torus case, we notice that $X_{1,0}$ is already invariant under multiplication of such factors and is therefore $\mathrm{PMod}_{1,0}$. Not to mention that it is obviously finite and spans the state space, it satisfies the desired conditions.

Sphere with four framed points

We continue by considering the sphere with four framed points with labels $l_1, \ldots, l_4$. This is the smallest number of points when the state space V has $\dim(V) > 1$ when all framed points are labelled by σ.

The pure mapping class group of the sphere is generated by Dehn twists T_{τ_1} and T_{τ_2} along the curves τ_1 and τ_2 shown in Figure 6.4 as well as the Dehn twists around each framed point.

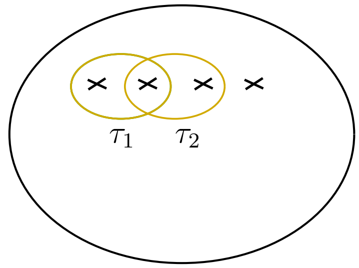
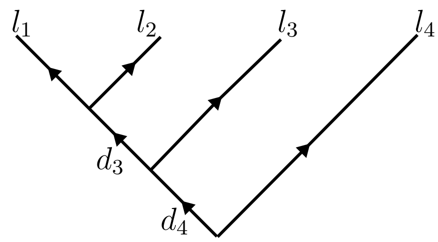


Figure 6.4: Generators of the (pure) mapping class group of the sphere with four marked points.

The basis elements of $V_{0,4}$ are

e^{d_3,d_4} = 

(6.45)

where d_3, d_4 run over admissible labels. In particular, $d_4 = l_4$ is the only admissible label and thus we have only dependence on d_3. As already mentioned, d_3 has more than one admissible choices only when $l_1 = \cdots = l_4 = \sigma$.

The action on the basis is given explicitly as

$$T_{\tau_1} e^{d_3} = \theta_{d_3} e^{d_3} \tag{6.46}$$

and

$$T_{\tau_2} e^{d_3} = \sum_{d_3', q} \theta_q F^{(l_1 l_2 l_3) l_4}_{d_3 q} G^{(l_1 l_2 l_3) l_4}_{q d_3'} e^{d_3'} \ . \tag{6.47}$$

For the action of T_{τ_2}, we notice that if any of the point labels is distinct from σ, then e^{d_3} is an eigenvector (as the state space is one-dimensional) and the eigenvalue is expressed as an integer power of θ_σ. However, if $l_1 = \cdots = l_4 = \sigma$ then $\mathbb{1}$ and ε are both admissible labels for d_3 and we compute by using twists and F-matrices[18]

$$T_{\tau_2} e^{\varepsilon_a} = \frac{1}{2}(1 + (-1)^{a+1}) e^{\mathbb{1}} + \frac{1}{2}(1 + (-1)^{a}) e^{\varepsilon} = e^{\varepsilon_{a+1}} \ . \tag{6.48}$$

The above calculations imply that

$$X_{0,4}(l_1, l_2, l_{3,4}) = \{\theta_\sigma^k e^{d_3} \,|\, k \in \mathbb{Z}_16, d_3\} \tag{6.49}$$

is $\mathrm{PMod}_{0,4}$-invariant. Notice however that it is not $\mathrm{Mod}_{0,4}$-invariant.

[18] Alternatively, notice that $\theta_\sigma^2 (B^{(\sigma\sigma\sigma)\sigma})^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ using (6.4).

Closed Surfaces

We choose to prove property F with respect to closed surfaces as the proof can be then easily be extended to the case with framed points. Recall the basis elements obtained from colourings of the graph $\Gamma_{g,0}$ in (6.6). For any colouring $\chi \in \mathrm{col}(\Gamma)^\circ$ we consider an alternate colouring $\tilde{\chi}$, in which we change every I_m labelled by $\mathbb{1}$ to the plus label $+$ and every I_m labelled by ε to the minus label $-$. This gives a new colouring set $\widetilde{\mathrm{col}}(\Gamma)^\circ$ consisting of these alternate colourings. Let $\tilde{\chi}$ be a colouring in $\widetilde{\mathrm{col}}(\Gamma)^\circ$ with $I_{m_1}, \dots, I_{m_L}$ the maximal subsequence of edges labelled by signs $\pm$ with the rest of I-edges labelled by σ. We define the associated vector in $V_{g,0}$ by

$$[\tilde{\chi}] = \sum_{p_1,\dots,p_L=0,1} \tilde{\chi}(I_{m_1})^{p_1} \cdots \tilde{\chi}(I_{m_L})^{p_L} \hat{\chi}|_{I_{m_1}=\varepsilon_{p_1},\dots,I_{m_L}=\varepsilon_{p_L}} \tag{6.50}$$

where $\hat{\chi}|_{I_{m_1}=\varepsilon_{p_1},\dots,I_{m_L}=\varepsilon_{p_L}}$ is the basis element corresponding to the colouring in $\mathrm{col}(\Gamma)^\circ$ obtained from $\tilde{\chi}$ by changing the indicated $\pm$-labelled I edges accordingly. This notation is the higher genus analogue of $e_\pm$ for the torus case.

Proposition 6.12. The finite set

$$X_{g,0} = \{\theta_\sigma^k \lambda^{m_{\tilde{\chi}}}[\tilde{\chi}] \,|\, k \in \mathbb{Z}_{16}, \tilde{\chi} \in \widetilde{\mathrm{col}}(\Gamma)^\circ\},$$

where $m_{\tilde{\chi}} = |\{k \,|\, \tilde{\chi}(I_k) = \sigma\}|$, spans the state space and is $\mathrm{Mod}_{g,0}-$invariant.

Proof. 1. The fact that this set spans the state space comes from the definition of the $\pm$-notation. Let $\hat{\chi}$ be some basis element of the state space and let $I_{m_1}, \dots, I_{m_L}$ be the maximal subsequence of I-edges such that $\chi(I_{m_1}) = \varepsilon_{p_1}, \dots, \chi(I_{m_L}) = \varepsilon_{p_L}$ (all non-sigma labels). Then, by inverting (6.50):

$$\hat{\chi} = \frac{1}{2^L} \sum_{\nu_1,\dots,\nu_L=\pm} \nu_1^{p_1} \cdots \nu_L^{p_L} [\tilde{\chi}|_{I_{m_1}=\nu_1,\dots,I_{m_L}=\nu_L}]$$

where $\tilde{\chi}|_{I_{m_1}=\nu_1,\dots,I_{m_L}=\nu_L}$ is the alternate colouring obtained from χ by changing non σ-labels of I-edges into $\pm$-labels.

2. We now prove $\mathrm{Mod}_{g,0}$-invariance. It is invariant under T_{α_m}'s as on the basis, one has from (6.10):

$$T_{\alpha_m}\hat{\chi} = \theta_{\chi(I_m)}\hat{\chi}$$

and from the observation in (6.43) it follows for an element of $X_{g,0}$:

$$T_{\alpha_m}\theta_\sigma^k \lambda^{m_{\tilde{\chi}}}[\tilde{\chi}] = \begin{cases} \theta_\sigma^{k+1}\lambda^{m_{\tilde{\chi}}}[\tilde{\chi}] & ,\tilde{\chi}(I_m) = \sigma \\ \theta_\sigma^k \lambda^{m_{\tilde{\chi}}}[\tilde{\chi}|_{I_m=-\tilde{\chi}(I_m)}] & ,\tilde{\chi}(I_m) = \pm \end{cases}$$

Recall from (6.11), (6.12) and (6.14) how S_m acts on our fixed basis. Then, one can easily check that the action on the set elements is given by

$$S_m\theta_\sigma^k \lambda^{m_{\tilde{\chi}}}[\tilde{\chi}] = \begin{cases} \theta_\sigma^k \lambda^{m_{\tilde{\chi}}}[\tilde{\chi}] & \tilde{\chi}(I_m) = + \\ \theta_\sigma^k \lambda^{m_{\tilde{\chi}}+1}[\tilde{\chi}|_{I_m=\sigma}] & \tilde{\chi}(I_m) = - \\ \theta_\sigma^k \lambda^{m_{\tilde{\chi}}-1}[\tilde{\chi}|_{I_m=-}] & \tilde{\chi}(I_m) = \sigma, \tilde{\chi}(K_m) = \mathbb{1} \\ \zeta^2\theta_\sigma^k \lambda^{m_{\tilde{\chi}}}[\tilde{\chi}|_{I_m=\sigma}] & \tilde{\chi}(I_m) = \sigma, \tilde{\chi}(K_m) = \varepsilon \end{cases}$$

which means that the set $X_{g,0}$ is S_m-invariant.

Recall from (6.18) and (6.19) how T_{γ_m} acts on the fixed basis. This results in

$$T_{\gamma_m}\theta_\sigma^k\lambda^{m_{\tilde\chi}}[\tilde\chi] = \begin{cases} \theta_\sigma^k\lambda^{m_{\tilde\chi}}[\tilde\chi|_{I_m=-\tilde\chi(I_m),I_{m+1}=-\tilde\chi(I_{m+1})}] & \tilde\chi(I_m),\tilde\chi(I_{m+1})=\pm \\ \theta_\sigma^{k+1}\lambda^{m_{\tilde\chi}}[\tilde\chi] & \tilde\chi(I_m)=\pm,\tilde\chi(I_{m+1})=\sigma \\ \theta_\sigma^k\lambda^{m_{\tilde\chi}}[\tilde\chi|_{\{K_m,K_{m+1},A_{k+1}\}\otimes\varepsilon}] & \tilde\chi(I_m)=\tilde\chi(I_{m+1})=\sigma \end{cases}$$

This concludes $T_{\gamma_m}-$invariance.

Invariance under the generators implies invariance under $\mathrm{Mod}_{g,0}$, even though the representation is only projective as discussed in section 6.2.4 for the torus case. This is because the projective factors can be expressed as integer powers of θ_σ.

□

General Case

In this section, we will complete the proof of Proposition 6.10 by proving the following proposition. Here, $\widetilde{\mathrm{col}}(\Gamma)^\circ$ is obtained again by $\mathrm{col}(\Gamma)^\circ$ as previously by changing I-labels to allow $\pm$-labels.

Proposition 6.13. The finite set

$$X_{g,n} = \{\theta_\sigma^k\lambda^{m_{\tilde\chi}}[\tilde\chi]\,|\,k\in\mathbb{Z}_{16},\tilde\chi\in\widetilde{\mathrm{col}}(\Gamma)^\circ\},$$

where $m_{\tilde\chi}=|\{k\,|\,\tilde\chi(I_k)=\sigma\}|$, spans the state space and is $\mathrm{PMod}_{g,n}-$invariant.

Proof. The invariance under the generators T_{α_k}, S_k and T_{γ_k} follows in exactly the same way as in Proposition 6.12 as the action of these generators does not depend on or affect the labels of $L_1,\dots,L_n,D_3,\dots,D_{n+1}$. It only remains to find out what happens when we act with the T_δ generators.

To show invariance under T_{δ_k}'s, it is sufficient to prove that the set is invariant under pure braids in $l_1,\dots,l_n,i_1$ strands as discussed in section 6.2.2. The proof follows from Lemmata 6.6 and 6.7. □

7 Irreducible mapping class group representations and absence of surface defects

Theories with irreducible representations on surfaces without marked points have an interesting feature which is an absence of surface defects. This is implied by the following Theorem we prove in [RR1]:

Theorem 7.1. Let $\mathcal{C}$ be a MFC such that the projective mapping class group representations $V_g^{\mathcal{C}}$ are irreducible for all $g \geq 0$. Then $\mathcal{C}$ has a unique Morita class of simple non-degenerate algebras, namely the Morita class of the tensor unit $\mathbb{1}$.

The proof is contained in Sections 7.2 and 7.3.

In [AF1, Thm. 1] the following closely related statement is shown:

> Let $A \in \mathcal{C}$ be a simple non-degenerate algebra such that $Z(A)$ is not isomorphic to $Z(\mathbb{1})$ as an object in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$. Then all projective mapping class group representations $V_g^{\mathcal{C}}$, $g \geq 1$ are reducible.

In contrapositive form this reads: Suppose there is a $g \geq 1$ such that $V_g^{\mathcal{C}}$ is irreducible. Then for every simple non-degenerate algebra A one has that $Z(A)$ is isomorphic to $Z(\mathbb{1})$ as an object in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$.

From this point of view, on the one hand, Theorem 7.1 needs the stronger assumption that $V_g^{\mathcal{C}}$ is irreducible for all $g \geq 0$ (however, see Remark 7.2 (1) below). On the other hand, under these assumptions it gives a stronger result, namely together with Theorem 2.16 it follows that $Z(A) \cong Z(\mathbb{1})$ as algebras in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$, and not just as objects. This confirms an expectation formulated in [AF1, Rem. 1], at least under our stronger assumptions. The method to prove Theorem 7.1 is different from that used in [AF1], and thus may be of independent interest.

Note that it is not at all obvious that there are examples where $Z(A) \cong Z(\mathbb{1})$ as objects but not as algebras. Such examples were first provided in [Da2].[19] In fact, that paper provides examples of Lagrangian algebras, but each such algebra can be realised as a full centre by [KR2, Thm. 3.22] (see also [DMNO, Prop. 4.8] for a more general statement).

Remark 7.2.

1. In the proof of Theorem 7.1 we actually need irreducibility of the representations $V_g^{\mathcal{C}}$ only for $1 \leq g \leq 3N+2$, where N is a bound introduced in Section 7.1 in terms of the adjoint subring. The place in the proof where this maximal g occurs is pointed out in Remark 7.20. The constant N in turn is trivially bounded by the number of isomorphism classes of simple objects, $N \leq |I|$. In other words, one can relax the hypothesis of Theorem 7.1 to assume irreducibility only for $V_g^{\mathcal{C}}$ with $1 \leq g \leq 3N+2$.

[19] In these examples it is not required that all simple non-degenerate algebras have $Z(A) \cong Z(\mathbb{1})$ as objects. Thus these examples do not yet imply that the conclusion of Theorem 7.1 is indeed stronger than that of [AF1, Thm. 1].

2. The condition of irreducibility on surfaces with marked points is not necessary. Nonetheless, let us for the moment consider the surface $\Sigma_{0,3}$, i.e. the sphere with three punctures, and assume that the punctures are labelled by simple objects, say $i, j, k \in I$. The (framed, pure) mapping class group $\mathrm{Mod}(\Sigma_{0,3})$ acts on $V^{\mathcal{C}}(\Sigma_{0,3})$ by rotation of the framing at the marked points, and so by a scalar given by the corresponding twist eigenvalue. If $V^{\mathcal{C}}(\Sigma_{0,3})$ is non-zero, for $\mathrm{Mod}(\Sigma_{0,3})$ to act irreducibly we must hence have $\dim V^{\mathcal{C}}(\Sigma_{0,3}) = 1$.

   On the other hand, $V^{\mathcal{C}}(\Sigma_{0,3}) = \mathcal{C}(\mathbb{1}, i \otimes j \otimes k)$. Thus, requiring irreducibility of the mapping class group action also on surfaces with marked points implies in particular that the fusion coefficients of $\mathcal{C}$ must satisfy $N_{ij}^{\;\bar{k}} \in \{0, 1\}$. Considering only surfaces without punctures, as we do, does not a priori impose this requirement, but we do not know any example where the $V_g^{\mathcal{C}}$, $g \geq 0$ are irreducible but $N_{ij}^{\;\bar{k}} > 1$ can occur.

3. Due to the defect analysis in Section 4 one can interpret the statement of the Theorem as an absence of surface defects, in the sense that $\mathcal{C}$ (or rather its RT TQFT) has no non-trivial surface defects. More precisely any defect is an invertible Euler defect (see Remark 4.4).

Examples which satisfy the criterion of Theorem 7.1 have been reviewed in Section 6.1. In all these examples it was already known that there is a unique Morita class of simple non-degererate algebras.

Theorem 7.1 can be reformulated using module categories with module trace from Section 2.3. From [Sch, Thm. 6.6, Prop. 6.8] we get the following reformulation of Theorem 7.1:

Theorem 7.1 (v2)**.** Let $\mathcal{C}$ be a MFC such that the projective mapping class group representations $V_g^{\mathcal{C}}$ are irreducible for all $g \geq 0$. Then there is up to equivalence a unique semisimple indecomposable $\mathcal{C}$-module category with module trace, namely $\mathcal{C}$ itself.

As an application of this point of view, let us explain how under certain conditions the non-degeneracy of a simple algebra is implied. The MFC $\mathcal{C}$ is called *pseudo-unitary* if $\Bbbk = \mathbb{C}$ and if the quantum dimensions of all simple objects are positive. By [Sch, Prop. 5.8], for pseudo-unitary $\mathcal{C}$, a semisimple $\mathcal{C}$-module category can be equipped with a module trace. Hence in this situation we can drop the existence of a module trace from Theorem 7.1 (v2). We obtain the following corollary to Theorem 7.1 (see also [Sch, Cor. 6.11]):

Corollary 7.3. Suppose that in addition to the hypotheses in Theorem 7.1, $\mathcal{C}$ is pseudo-unitary. Then all simple algebras in $\mathcal{C}$ are Morita-equivalent to the tensor unit.

Before going into the details, let us briefly sketch the proof of Theorem 7.1. By Theorem 2.16 it suffices to show that for any simple non-degenerate algebra A we have $Z(A) \cong Z(\mathbb{1})$ as algebras. To obtain this isomorphism we proceed in several steps:

1. In Section 7.1 we will review the notion of the adjoint subring and universal grading group as well as the bound N mentioned in Remark 7.2.

2. In Section 7.2, we will use irreducibility on the torus and obtain multiplication constants $(\lambda_{ij}^k)_\beta^\alpha$ relating the structure morphisms of $Z(A)$ to those of $Z(\mathbb{1})$. Furthermore, we use irreducibility for genus 2 to obtain constants λ_{ij}^k independent of the multiplicity labels α, β. We then use irreducibility for $g > 2$ to obtain constraints on the λ_{ij}^k.

3. In Section 7.3 we construct a sequence of algebra isomorphisms using the results of the previous step and the universal grading group to arrive to an algebra isomorphism $Z(A) \cong Z(\mathbb{1})$.

7.1 The adjoint subring and the universal grading group

We briefly recall from [GeN] the notion of the universal grading group and of the adjoint subring (see also [EGNO, Ch. 3]).

Let $\mathcal{F}$ be a fusion category and I a set of representatives of isomorphism classes of simple objects in $\mathcal{F}$. The duality on $\mathcal{F}$ defines an involution $\overline{(\ )} : I \to I$ by requiring that $\bar{i} \cong i^*$. Then, the Grothendieck ring $\mathrm{Gr}(\mathcal{F}) \equiv R$ is a unital based ring with basis $\{b_i\}_{i\in I}$. The ring R is *transitive* in the sense that for any $i, j \in I$ there exists $k \in I$ such that $N_{ik}^{j} \neq 0$.

Definition 7.4. The *adjoint subring* $R_{ad} \subset R$ is generated by all basis elements contained in $b_i b_{\bar{i}}$ for some $i \in I$. We denote by $I_{ad} \subset I$ the index set of the basis $\{b_i\}_{i\in I_{ad}}$ of R_{ad}.

It is shown in [GeN, Thm. 3.5] that the ring R decomposes into a direct sum of indecomposable based R_{ad}-bimodules $R = \bigoplus_{g\in G} R_g$, and that the product of R induces a group structure on G with $R_e = R_{ad}$. In particular, R is a faithful G-graded ring. The set $I_g \subset I$ will denote the index set of the basis $\{b_i\}_{i\in I_g}$ of R_g. Transitivity of R now implies that R_{ad} acts transitively on R_g for each $g \in G$: for all $x, y \in I_g$ there is $i \in I_{ad}$ such that $N_{xi}^{\ y} \neq 0$.

Definition 7.5. The group G is called the *universal grading group of R*.

We define a filtration on R_{ad} as follows. For $i \in I_{ad}$ let $n(i)$ be the minimal integer such that b_i is contained in $b_{m_1} b_{\overline{m}_1} \dots b_{m_{n(i)}} b_{\overline{m}_{n(i)}}$ for some $m_1, \dots, m_{n(i)} \in I$. Such labels exist by the definition of the adjoint subring. For the unit we set $n(\mathbb{1}) = 0$. Setting

$$R_{ad}^{(n)} \;=\; \langle\, b_i \in R_{ad} \mid n(i) \leq n \,\rangle \;. \tag{7.1}$$

we get the filtration

$$\mathbb{Z}\, b_{\mathbb{1}} = R_{ad}^{(0)} \subset R_{ad}^{(1)} \subset R_{ad}^{(2)} \subset \cdots \tag{7.2}$$

of R_{ad}. Let N denote the minimal number such that $R_{ad}^{(N)} = R_{ad}$. Since the filtration is strictly increasing until degree N, and since $I_{ad} \subset I$, we trivially have that $N \leq |I|$.

7.2 Mapping class group invariants and structure constants

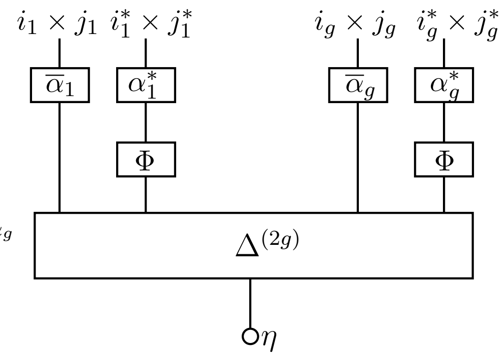
We give an explicit construction to mapping class group invariants closely related to RCFT partition functions on surfaces without marked points and exploit irreducibility to relate the algebraic structure of the full centre.

Let $\mathcal{C}$ be an MFC and B be a symmetric Frobenius algebra in $\mathcal{C}$. For an integer $n \geq 2$ we write $\Delta^{(n)} : B \to B^{\otimes n}$ for the iterated coproduct, so that $\Delta = \Delta^{(2)}$. Let $\mathbb{L}$ denote the object defined in (2.15), but now in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$. For $g \geq 1$ define the elements

$$C(B)_g \;\in\; V^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(\Sigma_g) = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}(\mathbb{1} \times \mathbb{1}, \mathbb{L}^{\otimes g}) \tag{7.3}$$

by setting

$$C(B)_g \;:=\; \bigoplus_{i_1,j_1,\dots,i_g,j_g} \sum_{\alpha_1,\dots,\alpha_g}$$



$. \qquad (7.4)$

For $g = 0$ we have $\Sigma_g = S^2$ and we set

$$C(B)_0 \;:=\; \varepsilon \circ \eta \;\in\; V^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(S^2) = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}(\mathbb{1} \times \mathbb{1}, \mathbb{1} \times \mathbb{1}) = \Bbbk\, \mathrm{id}_{\mathbb{1}\times\mathbb{1}} \;. \tag{7.5}$$

The next proposition follows from [FRS, KR2, KLR], but it can also be shown directly and we give a short proof here for the convenience of the reader.

Proposition 7.6. Let $B \in \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ be a non-degenerate modular invariant algebra. Then for each $g \geq 0$ the vector $C(B)_g \in V^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(\Sigma_g)$ is Mod_g-invariant.

Proof. For $g = 0$ there is nothing to show. Let thus $g \geq 1$. We need to check that the generators in (3.24) (for the MFC $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$) leave $C(B)_g$ invariant.

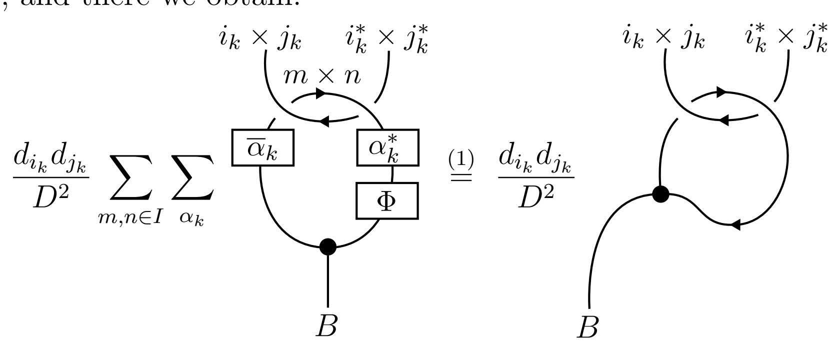
T_{α_k}: Invariance is immediate from the fact that B has a trivial twist, i.e. $\theta_B = \mathrm{id}_B$.

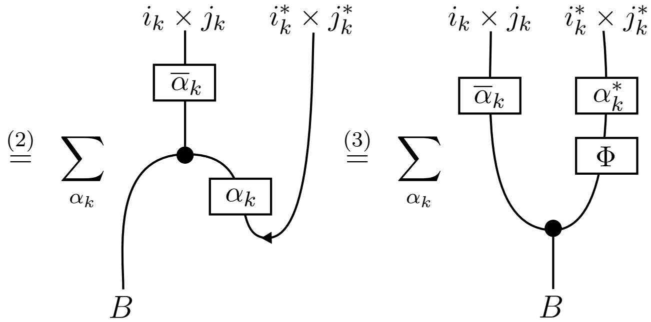
T_{γ_k}: Choose the iterated coproduct $\Delta^{(2g)}$ such that the $2k$'th and $(2k+1)$'th strand form the output of one coproduct, i.e. write

$$\Delta^{(2g)} = \left(\mathrm{id}_{B^{\otimes(2k-1)}} \otimes \Delta \otimes \mathrm{id}_{B^{\otimes(2g-2k-1)}} \right) \circ \Delta^{(2g-1)} \;.$$

Invariance under T_{γ_k} now boils down to the observation that $\theta_{B\otimes B} \circ \Delta = \Delta \circ \theta_B = \Delta$.

S_k: Choose the iterated coproduct $\Delta^{(2g)}$ such that the $(2k-1)$'th and $2k$'th strand form the output of one coproduct. Applying S_k to $C(B)_g$ only affects the $(2k-1)$'th and $2k$'th strand, and there we obtain:





For the first expression in this computation, recall that we have to evaluate the formula for S_k in (3.24) for $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$. We take $j \rightsquigarrow i_k \times j_k$ and $i_k \rightsquigarrow m \times n$ in (3.24), so that the prefactor there becomes $d_{i_k} d_{j_k}/D^2$. In step (1) we carry out the sum over m, n and α_k which gives the identity on B, and we use that B is Δ-separable and symmetric to remove Φ. Step (2) is precisely S-invariance of B as in Definition 2.10. Step (3) is easier to see backwards, and again uses that B is Δ-separable and symmetric.

This shows that $S_k \circ C(B)_g = C(B)_g$.

□

Remark 7.7.

1. The construction of mapping class group invariants as in (7.4) first appeared in the study of consistent systems of correlators for rational 2d conformal field theories via 3d topological quantum field theories [FRS, FjFRS1]. Recall from Section 5.1 that $V^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(\Sigma)$ describes the space of holomorphic times antiholomorphic conformal blocks, and a vector $\mathrm{Cor}(\Sigma) \in V^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(\Sigma)$ describes a bulk correlation function on Σ. To be consistent, the collection $\{\mathrm{Cor}(\Sigma)\}$ has to satisfy modular invariance and factorisation conditions. Here, we only make use of the former.

2. The categorical form of the modular invariance condition for algebras first appeared in [Kon, Sec. 6.1] in the context of vertex operator algebras and has been investigated in detail in [KR2]. The notion of a Cardy algebra from [Kon] was used in [KLR] to classify solutions to the open/closed factorisation and modular invariance conditions. In this context, the algebra B in Proposition 7.6 corresponds to the closed part of a Cardy algebra, and (7.4) is the correlator for a closed genus-g surface.

3. The classification of solutions to the consistency conditions in [KLR] relied on semisimplicty of $\mathcal{C}$ and $\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}$. A more general approach applicable to non-semisimple modular tensor categories has been developed in [FSSt, FSc2]. See in particular [FSSt, Eqn. (5.3)] for the generalisation of (7.4) and [FSc2, Def. 4.9] for the definition of modular invariant algebras in this non-semisimple setting. These ingredients will be important when trying to generalise the present results to non-semisimple modular tensor categories.

Let $A \in \mathcal{C}$ be a simple non-degenerate algebra. By Theorem 2.14 and Lemma 2.7 the full centre $Z(A)$ is a haploid normalised-special commutative symmetric modular invariant Frobenius algebra.

Lemma 7.8. The full centre $Z(A)$ has the same underlying object as $Z(\mathbb{1})$, i.e. $Z(A) \cong \bigoplus_{i\in I} i^* \times i$ as objects in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$.

Proof. By [KR2, Eq. (3.7)], the matrix $Z(A)_{ij} = \dim \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}(i \times j, Z(A))$ commutes with the S-generator, and it commutes with the T-generator since $Z(A)$ has trivial twist. By irreducibility of $V^{\mathcal{C}}_{g=1}$, the space of invariants in $V^{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}_{g=1}$ is one-dimensional (Lemma 5.5). Hence there exists a constant $\lambda \in \Bbbk$ such that

$$Z(A)_{ij} = \lambda\, Z(\mathbb{1})_{ij} = \lambda\, \delta_{\bar{i}j} \ . \tag{7.6}$$

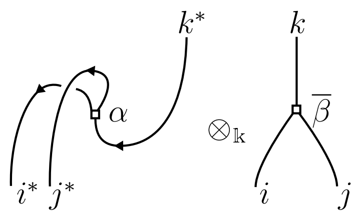
By haploidity, $Z(A)_{\mathbb{1}\mathbb{1}} = 1$ and therefore $\lambda = 1$. Altogether, $\dim \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}(i \times j, Z(A)) = \delta_{\bar{i},j}$ i.e. the underlying object of $Z(A)$ is $\bigoplus_{i\in I} i^* \times i$. □

Denote by $e_i : i^* \times i \to Z(A)$ and $r_i : Z(A) \to i^* \times i$ the embedding and projection of $i^* \times i$ as a subobject of $Z(A)$, i.e. $r_i \circ e_i = \mathrm{id}$. Given the underlying object of $Z(A)$ as in Lemma 7.8, we now make a general ansatz for the Frobenius algebra structure on $Z(A)$. Namely, in terms of constants η_0, ε_0, $(\lambda^k_{ij})^\beta_\alpha$, $(\lambda^{ij}_k)^\alpha_\beta \in \Bbbk$ we set

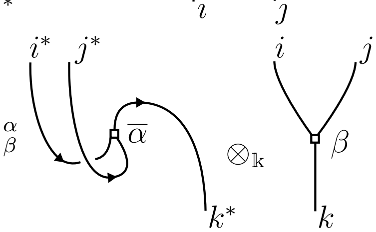
$$\eta_{Z(A)} = \eta_0\, e_{\mathbb{1}}$$

$$\varepsilon_{Z(A)} = \varepsilon_0\, D^2\, r_{\mathbb{1}}$$

$$\mu_{Z(A)} = \bigoplus_{i,j,k} \sum_{\alpha,\beta=1}^{N_{ij}{}^k} (\lambda^k_{ij})^\beta_\alpha$$



$$\Delta_{Z(A)} = \bigoplus_{i,j,k} \sum_{\alpha,\beta=1}^{N_{ij}{}^k} \frac{d_i d_j}{d_k D^2} (\lambda^{ij}_k)^\alpha_\beta$$



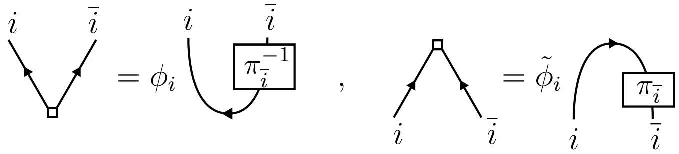
. (7.7)

On the right hand side of $\mu_{Z(A)}$ we did not spell out the embedding and projection morphisms $r_k \circ (\cdots) \circ (e_i \otimes e_j)$, and dito for $\Delta_{Z(A)}$.

Lemma 7.9. The elements $C(Z(A))_g$ from (7.4) are non-zero for every $g \geq 0$.

Proof. The element $C(Z(A))_0$ is non-zero since $\varepsilon_{Z(A)} \circ \eta_{Z(A)} = \dim_{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(Z(A)) = D^2$, where the first equality follows from the symmetric normalised-special Frobenius algebra structure. Now, let $g \geq 1$ and consider in (7.4) the summand of $C(Z(A))_g$ where $i_m = j_m = \mathbb{1}$ for $m = 1, \ldots, g$. Since $\mathbb{1} \times \mathbb{1}$ appears in $Z(A)$ with multiplicity one, there is no sum over multiplicities. Up to factors of $\varepsilon_0 D^2 \neq 0$, the result is the same as composing all out-going $Z(A)$-factors with the counit $\varepsilon_{Z(A)}$. The overall expression then reduces to $\varepsilon_{Z(A)} \circ \eta_{Z(A)} = D^2$. □

As in [FRS, Sec. 2.2], for every $i \in I$ fix an isomorphism $\pi_i : i \to \bar{i}^*$, which exists by definition of the involution $i \mapsto \bar{i}$. We use these isomorphisms to express the fusion basis in $\mathcal{C}(i \otimes \bar{i}, \mathbb{1})$ and its dual in terms of dualities in $\mathcal{C}$. Namely, there exist $\phi_i, \tilde{\phi}_i \in \Bbbk^\times$ such that



(7.8)

as the respective morphism spaces are one-dimensional. By the normalisation chosen for fusion bases, one obtains

$$\phi_i \, \tilde{\phi}_i = \frac{1}{d_i} \ . \tag{7.9}$$

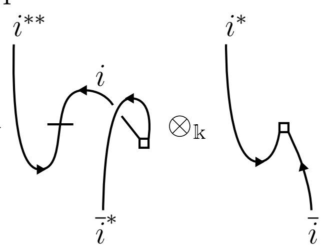
In the following lemma we will give the isomorphism Φ for $Z(A)$ in a basis, which will later be used to express the modular invariants $C(Z(A))_g$.

Lemma 7.10. For any $i \in I$, we have

$$e_i^* \circ \Phi_{Z(A)} \circ e_{\bar{i}} \;=\; \frac{D^2 \theta_i}{d_i} \, \varepsilon_0 \, \lambda_{i\bar{i}}^{\mathbb{1}} \, (\pi_{\bar{i}}^{-1})^* \otimes_{\Bbbk} \pi_{\bar{i}} \quad : \quad \bar{i}^* \times \bar{i} \longrightarrow i^{**} \times i^* \ .$$

Proof. By using (7.7) in the expression for Φ on the left hand side of (2.47), one obtains:

$$e_i^* \circ \Phi_{Z(A)} \circ e_{\bar{i}} = D^2 \varepsilon_0 \lambda_{i\bar{i}}^{\mathbb{1}}$$



$$= \frac{D^2 \, \theta_i}{d_i} \, \varepsilon_0 \, \lambda_{i\bar{i}}^{\mathbb{1}} \, (\pi_{\bar{i}}^{-1})^* \otimes_{\Bbbk} \pi_{\bar{i}} \ ,$$

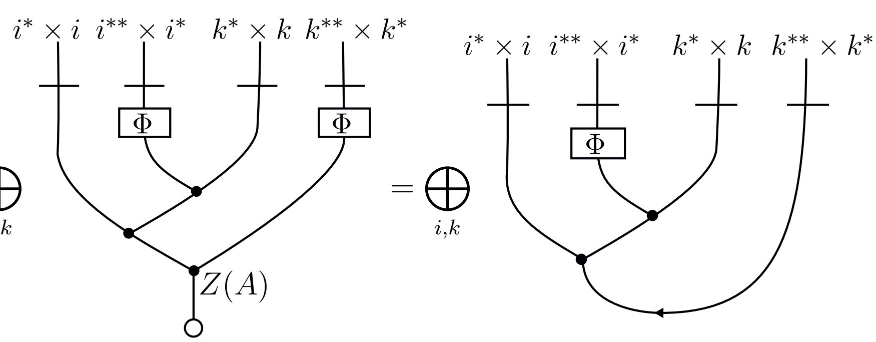
where the horizontal line denotes the identity id_{i^*}. The last equality follows from (7.8) and (7.9). □

The allowed structure constants are non-zero, diagonal and independent of the multiplicity index:

Lemma 7.11. For $i, j, k \in I$ with $N_{ij}^{\ k} \neq 0$ we have $(\lambda_{ij}^k)_\alpha^\beta = \delta_{\alpha,\beta} \, \lambda_{ij}^k$ and $(\lambda_k^{ij})_\alpha^\beta = \delta_{\alpha,\beta} \, \lambda_k^{ij}$ with $\lambda_{ij}^k, \lambda_k^{ij} \neq 0$.

Proof. Consider the modular invariant vector $C(Z(A))_{g=2}$ as defined in (7.4), which is given as[20]

$$C(Z(A))_2 = \bigoplus_{i,k}$$

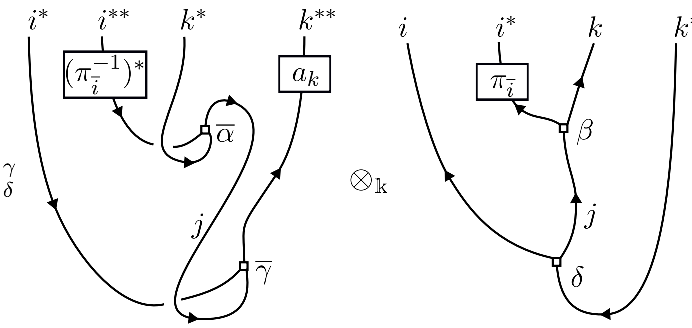


(7.10)

[20]To be precise, the element $C(Z(A))_2$ is defined in the isomorphic morphism space $\bigoplus_{i,k} \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}(\mathbb{1} \times \mathbb{1}, (\bar{i} \times i) \otimes (\bar{i}^* \times i^*) \otimes (\bar{k} \times k) \otimes (\bar{k}^* \times k^*))$, but we find it convenient to use the form given here, rather than to include the isomorphisms $\pi_{\bar{i}}^{-1} \times \mathrm{id}_i$ and $\pi_{\bar{i}}^* \times \mathrm{id}_i$, etc.

where the horizontal lines denote the embeddings and projections, and where we used that $Z(A)$ is symmetric Frobenius to remove one of the Φ's. Using Lemma 7.10, we can write $C(Z(A))_2$ explicitly:

$$\bigoplus_{i,k} \sum_j \sum_{\alpha,\beta,\gamma,\delta} \frac{d_i\theta_i}{D^2}\, \varepsilon_0\, \lambda^{\mathbb{1}}_{i\bar{i}}\, (\lambda^{\bar{i}k}_j)^\alpha_\beta\, (\lambda^{ij}_k)^\gamma_\delta$$



where $a_k : k \to k^{**}$ denotes the pivotal structure isomorphism.

Since $C(Z(A))_2$ and $C(Z(\mathbb{1}))_2$ are both modular invariant vectors and the space of invariants is one-dimensional, and since $C(Z(A))_2$ and $C(Z(\mathbb{1}))_2$ are non-zero by Lemma 7.9 there exists $\lambda_2 \in \mathbb{k}^\times$ such that

$$C(Z(A))_2 \;=\; \lambda_2\, C(Z(\mathbb{1}))_2\;. \tag{7.11}$$

Let $i, j, k \in I$ such that $N_{ij}^{\;k} \neq 0$, then by comparing both sides of (7.11) and using Example 2.15 we obtain

$$\varepsilon_0\, \lambda^{\mathbb{1}}_{i\bar{i}}\, (\lambda^{\bar{i}k}_j)^\alpha_\beta\, (\lambda^{ij}_k)^\gamma_\delta \;=\; \delta_{\alpha,\beta}\, \delta_{\gamma,\delta}\, \lambda_2\;. \tag{7.12}$$

For $\alpha = \beta$ and $\gamma = \delta$ we get

$$\varepsilon_0\, \lambda^{\mathbb{1}}_{i\bar{i}}\, (\lambda^{\bar{i}k}_j)^\alpha_\alpha\, (\lambda^{ij}_k)^\gamma_\gamma \;=\; \lambda_2 \neq 0\;. \tag{7.13}$$

This shows that $(\lambda^{\bar{i}k}_j)^\alpha_\alpha$ and $(\lambda^{ij}_k)^\gamma_\gamma$ are non-zero and independent of α, γ. I.e. for $N_{ij}^{\;k} \neq 0$ there exists $\lambda^{ij}_k \in \mathbb{k}^\times$ such that $(\lambda^{ij}_k)^\gamma_\gamma = \lambda^{ij}_k$ for $\gamma = 1, \dots, N_{ij}^{\;k}$. Taking $\alpha = \beta$ but $\gamma \neq \delta$ in (7.12) gives the desired form for the comultiplication structure constants

$$(\lambda^{ij}_k)^\gamma_\delta \;=\; \delta_{\gamma,\delta}\, \lambda^{ij}_k\;. \tag{7.14}$$

To get also the expression for the structure constants of the product as claimed in the lemma, insert (7.7) into $\mu = ((\varepsilon \circ \mu) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)$. This gives

$$(\lambda^k_{ij})^\beta_\alpha \;=\; \delta_{\alpha,\beta}\, \varepsilon_0\, \lambda^{\mathbb{1}}_{i\bar{i}}\, \lambda^{\bar{i}k}_j \;=\; \delta_{\alpha,\beta}\, \lambda^k_{ij}\;, \tag{7.15}$$

with $\lambda^k_{ij} = \varepsilon_0\, \lambda^{\mathbb{1}}_{i\bar{i}}\, \lambda^{\bar{i}k}_j \neq 0$. □

Lemma 7.12. The structure constants obey the following properties:

1. (Unitality and counitality) $\lambda^i_{\mathbb{1}i} = \lambda^i_{i\mathbb{1}} = \eta_0^{-1}$ and $\lambda^{\mathbb{1}i}_i = \lambda^{i\mathbb{1}}_i = \varepsilon_0^{-1}$
2. (Commutativity) $\lambda^k_{ij} = \lambda^k_{ji}$
3. (Index lowering and raising) $\lambda^k_{ij} = \varepsilon_0\, \lambda^{\mathbb{1}}_{i\bar{i}}\, \lambda^{\bar{i}k}_j$ and $\lambda^{ij}_k = \eta_0\, \lambda^{i\bar{i}}_{\mathbb{1}}\, \lambda^j_{\bar{i}k}$

Proof. Part 1 follows directly by evaluating the unitality and the counitality conditions for the morphisms in (7.7).

For part 2, let $i, j, k \in I$ with $N_{ij}{}^k \neq 0$. From Lemma 7.11 we get

$$r_k \circ \mu_{Z(A)} \circ (e_i \otimes e_j) = \lambda^k_{ij}\, r_k \circ \mu_{Z(\mathbb{1})} \circ (e_i \otimes e_j)\ . \tag{7.16}$$

Composing both sides of this equation with the braiding $c_{j,i} : j \otimes i \to i \otimes j$ and using naturality, we get

$$\begin{aligned} r_k \circ \mu_{Z(A)} \circ c_{Z(A),Z(A)} \circ (e_j \otimes e_i) &= \lambda^k_{ij}\, r_k \circ \mu_{Z(\mathbb{1})} \circ c_{Z(\mathbb{1}),Z(\mathbb{1})} \circ (e_j \otimes e_i) \\ &= \lambda^k_{ij}\, r_k \circ \mu_{Z(\mathbb{1})} \circ (e_j \otimes e_i)\ . \end{aligned} \tag{7.17}$$

In the last step we used the commutativity of $Z(\mathbb{1})$. Making use of commutativity of $Z(A)$, i.e. $\mu_{Z(A)} \circ c_{Z(A),Z(A)} = \mu_{Z(A)}$, finally implies part 2.

In part 3, the first equality was already given at the end of the previous proof, and the second follows analogously by inserting (7.7) into $\Delta = (\mathrm{id} \otimes \mu) \circ ((\Delta \circ \eta) \otimes \mathrm{id})$. □

7.3 Sequence of isomorphisms

Given an object automorphism $f \in \mathrm{Aut}(Z(A))$, one can give an isomorphic (haploid commutative symmetric normalised-special modular invariant) Frobenius algebra $\tilde{Z} \equiv f_*(Z(A))$. Its underlying object is again $Z(A)$ but its structure morphisms are

$$\begin{aligned} \tilde{\mu} &= f \circ \mu_{Z(A)} \circ (f^{-1} \otimes f^{-1})\ , & \tilde{\eta} &= f \circ \eta_{Z(A)}\ , \\ \tilde{\Delta} &= (f \otimes f) \circ \Delta_{Z(A)} \circ f^{-1}\ , & \tilde{\varepsilon} &= \varepsilon_{Z(A)} \circ f^{-1}\ . \end{aligned} \tag{7.18}$$

This is the unique Frobenius algebra structure such that $f : Z(A) \to \tilde{Z}$ is an isomorphism of Frobenius algebras.

The isomorphism f is determined by invertible scalars $\{f_i\}$ as the underlying object of $Z(A)$ is $\bigoplus_{i\in I} i^* \times i$:

$$f = \sum_{i\in I} f_i\, e_i \circ r_i\ . \tag{7.19}$$

The new structure constants are then given by

$$\tilde{\lambda}^k_{ij} = \frac{f_k}{f_i f_j}\lambda^k_{ij}\ , \quad \tilde{\lambda}^{ij}_k = \frac{f_i f_j}{f_k}\lambda^{ij}_k\ , \quad \tilde{\eta}_0 = f_{\mathbb{1}}\, \eta_0\ , \quad \tilde{\varepsilon}_0 = f_{\mathbb{1}}^{-1}\, \varepsilon_0\ . \tag{7.20}$$

The new constants defined as above still obey the equations of Lemmata 7.12 and 7.14.

We will find a sequence of such Frobenius algebra isomorphisms that take $Z(A)$ into $Z(\mathbb{1})$. In other words, we need to find (a sequence of) transformations f_i such that $\tilde{\eta}_0 = \tilde{\varepsilon}_0 = 1$ and $\tilde{\lambda}^k_{ij} = \tilde{\lambda}^{ij}_k = 1$ whenever $N_{ij}{}^k \neq 0$.

First Step

Our first step will be to normalise the constants $\lambda_{i\mathbb{1}}^{i}, \lambda_{\mathbb{1}i}^{i}$ and $\lambda_{i\bar{i}}^{\mathbb{1}}$. We do this by fixing f_i such that $f_{\bar{i}} f_i = \lambda_{\mathbb{1}\mathbb{1}}^{\mathbb{1}} \lambda_{i\bar{i}}^{\mathbb{1}}$ (for instance pick any square root $f_i = f_{\bar{i}} = \sqrt{\lambda_{\mathbb{1}\mathbb{1}}^{\mathbb{1}} \lambda_{i\bar{i}}^{\mathbb{1}}}$) and fix $f_{\mathbb{1}} = \lambda_{\mathbb{1}\mathbb{1}}^{\mathbb{1}}$. For example,

$$\tilde{\lambda}_{i\bar{i}}^{\mathbb{1}} \;=\; \frac{f_{\mathbb{1}}}{f_i\, f_{\bar{i}}}\, \lambda_{i\bar{i}}^{\mathbb{1}} \;=\; \frac{\lambda_{\mathbb{1}\mathbb{1}}^{\mathbb{1}}}{\lambda_{\mathbb{1}\mathbb{1}}^{\mathbb{1}} \lambda_{i\bar{i}}^{\mathbb{1}}}\, \lambda_{i\bar{i}}^{\mathbb{1}} \;=\; 1\ . \tag{7.21}$$

Assuming we applied this isomorphism, we may now start with constants such that $\lambda_{i\mathbb{1}}^{i} = \lambda_{\mathbb{1}i}^{i} = 1 = \lambda_{i\bar{i}}^{\mathbb{1}}$ for all $i \in I$. By Lemma 7.12, this implies $\eta_0 = \varepsilon_0 = 1$, as well as

$$\lambda_k^{ij} = \lambda_{\bar{i}k}^{j}\ . \tag{7.22}$$

i.e. we can raise or lower indices by conjugating the respective label. To avoid confusion, we will denote this algebra by Z, which is isomorphic to $Z(A)$ as a Frobenius algebra.

The above conditions on λ_{ij}^{k}, λ_k^{ij}, η_0, ε_0 are preserved by isomorphisms f that satisfy

$$f_{\mathbb{1}} = 1 \quad \text{and} \quad f_i f_{\bar{i}} = 1 \;\; \text{for all } i \in I\ . \tag{7.23}$$

Second Step

To exploit the irreducibility of the $V_g^{\mathcal{C}}$ for higher genus, it is convenient to introduce the notion of an I-fusion tree.

Definition 7.13.

- A *3-valent tree* is a tree graph, where each vertex is 3-valent with one incoming edge and two outgoing edges.
- An *I-fusion tree* is a 3-valent tree such that each edge is labelled by an element in I, and such that at each vertex v the following condition is satisfied: if the incoming edge at v is labelled k and the two outgoing edges at v are labelled i, j, then ${N_{ij}}^{k} \neq 0$.

  The outgoing edges of an I-fusion tree are ordered (we will label them $1, \dots, m$).
- Let $i, j_1, \dots, j_m \in I$. An $(i; j_1, \dots, j_m)$-fusion tree is an I-fusion tree such that the incoming edge is labelled by i and the outgoing edges are labelled by $j_1, \dots, j_m$.

We stress that an I-fusion tree Ω is *not* a string diagram in $\mathcal{C}$. Namely, Ω only records labels in I and does not include a specific morphism at each vertex.

Let Z be a Frobenius algebra isomorphic to $Z(A)$ as a Frobenius algebra, and with structure constants λ_k^{ij}, etc., normalised as in the first step. To a vertex v of an I-fusion tree with incoming label k and outgoing labels i, j we assign the number $\lambda(v) := \lambda_k^{ij}$. To the whole I-fusion tree we assign the product of the structure constants at each vertex,

$$\lambda : \{I\text{-fusion trees}\} \longrightarrow \Bbbk^{\times} \ \ , \quad \Omega \;\longmapsto\; \lambda(\Omega) = \prod_{v \text{ vertex}} \lambda(v)\ . \tag{7.24}$$

Lemma 7.14. Let $i_1, \ldots, i_g \in I$ and Ω be a $(\mathbb{1}; i_1, \bar{i}_1, \ldots, i_g, \bar{i}_g)$-fusion tree. Then

$$\lambda(\Omega) = 1 \ , \tag{7.25}$$

independent of the choice of $i_1, \ldots, i_g$ and Ω.

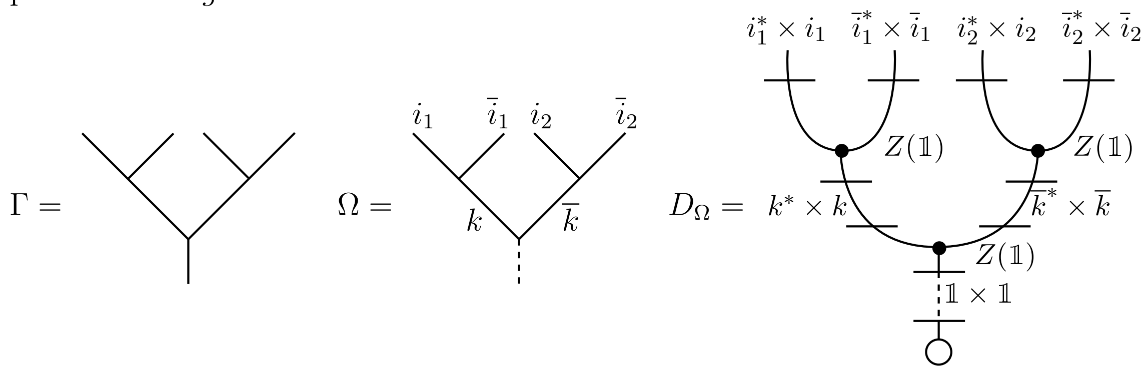
Proof. By irreducibility of the $V_g^{\mathcal{C}}$ and by Lemma 7.9 there is a $\lambda_g \in \Bbbk^\times$ such that

$$C(Z)_g = \lambda_g \, C(Z(\mathbb{1}))_g \ . \tag{7.26}$$

Fix a 3-valent tree Γ with $2g$ leaves. By decorating each vertex with the coproduct, each such tree gives a realisation of the iterated coproduct $\Delta^{(2g)} : Z \to Z^{\otimes 2g}$. Using labellings of Γ by I, we get a direct sum decomposition

$$\Delta_Z^{(2g)} \circ \eta_Z = \bigoplus_{\Omega} \lambda(\Omega) \, D_\Omega \ . \tag{7.27}$$

Here, the direct sum runs over I-fusion trees Ω with underlying unlabelled tree Γ, where the unique incoming edge is labelled by $\mathbb{1}$. The factor $\lambda(\Omega)$ is the product of structure constants as defined in (7.24). D_Ω is the summand of $\Delta_Z^{(2g)} \circ \eta_Z$ where for an edge labelled k the corresponding tensor factor Z is projected to $k^* \times k$. The following example illustrates the procedure for $g = 2$:



(7.28)

Here, the coproduct is that of $Z(\mathbb{1})$ as given in (2.57), for which all structure constants are 1.

The important point to realise is that the D_Ω are linearly independent for the different choices of Ω (but for fixed Γ). This can be seen for example by composing with the corresponding dual graph with in- and outgoing edges exchanged, which provides a non-degenerate pairing.

We can thus evaluate (7.26) summand by summand. For Z we get a factor $\lambda(\Omega)$ as in (7.24), while for $Z(\mathbb{1})$ the structure constants are all 1. Altogether we obtain, for all I-fusion trees Ω with underlying 3-valent tree Γ,

$$\lambda(\Omega) \, D_\Omega = \lambda_g \, D_\Omega \ . \tag{7.29}$$

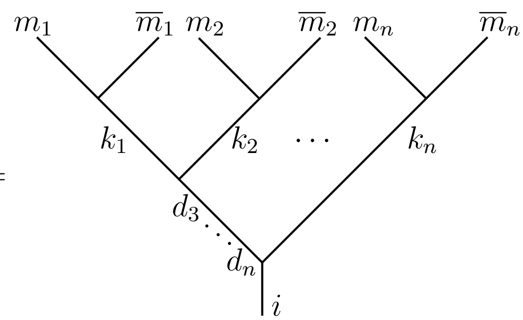
Finally, to compute λ_g, take the I-fusion tree Ω where all edges are labelled by $\mathbb{1}$. Since $\lambda_{\mathbb{1}}^{\mathbb{1}\mathbb{1}} = 1$, this results in $\lambda(\Omega) = 1$. □

The next lemma is the first place where the universal grading group becomes important. Namely, the elements in the neutral component R_{ad} are precisely those that "can be created by handles" (cf. Figure 3.4). This property can be used to set the corresponding structure constants to 1:

Lemma 7.15. There exist f_i satisfying (7.23) such that $\tilde{\lambda}_k^{ij} = 1$ for all $i, j, k \in I_{ad}$ with ${N_{ij}}^k \neq 0$.

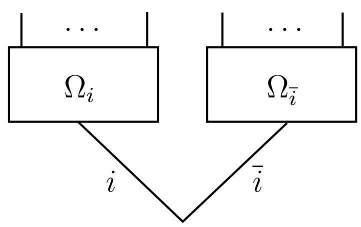
Proof. Recall the filtration of the adjoint subring given in (7.2). For $b_i \in R_{ad}^{(n)}$, $i \neq \mathbb{1}$ there are $m_1, \dots, m_n$ such that b_i is contained $b_{m_1} b_{\overline{m}_1} \dots b_{m_n} b_{\overline{m}_n}$. In other words, there exists a $(i; m_1, \overline{m}_1, \dots, m_n, \overline{m}_n)$-fusion tree

$\Omega_i \;=$



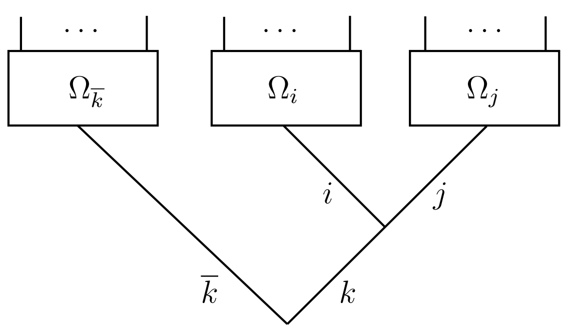
. (7.30)

We set $f_i = \lambda(\Omega_i)$. To check that the condition $f_i f_{\bar{i}} = 1$ in (7.23) is satisfied, apply Lemma 7.14 to the fusion tree



. (7.31)

For $i, j, k \in I_{ad}$ and their associated fusion trees Ω_i, Ω_j and $\Omega_{\overline{k}}$, consider the graph



. (7.32)

Lemma 7.14 gives the condition

$$\lambda(\Omega_{\overline{k}})\, \lambda(\Omega_i)\, \lambda(\Omega_j)\, \lambda_k^{ij} \;=\; 1 \;. \tag{7.33}$$

Substituting $f_i = \lambda(\Omega_i)$ and recalling that $f_k f_{\overline{k}} = 1$ finally gives $\lambda_k^{ij} = \frac{f_k}{f_i f_j}$, i.e. $\tilde{\lambda}_k^{ij} = 1$. □

We will need to know how the $\lambda(\Omega_i)$ change in the new normalisation given by the f_i. We have

$$\tilde{\lambda}(\Omega_i) \;=\; \frac{f_{m_1} f_{\overline{m}_1} \cdots f_{m_n} f_{\overline{m}_n}}{f_i} \lambda(\Omega_i) \;=\; 1 \;, \tag{7.34}$$

since $f_m f_{\overline{m}} = 1$ and $f_i = \lambda(\Omega_i)$.

Note that in the proof of Lemma 7.15 we have only used the irreducibility of $V_g^{\mathcal{C}}$ up to $g = 3N$, where N was defined in Section 7.1 to be the maximal degree in the filtration of R_{ad}. Below we will need to go up to $g = 3N + 2$, see Remark 7.20.

Let $i, j, k \in I$ be such that $N_{ij}^{\;\;k} \neq 0$. At this point we have achieved $\lambda_k^{ij} = 1$ whenever at least one of i, j, k is given by $\mathbb{1}$ (Step 1), and $\lambda_k^{ij} = 1$ for $i, j, k \in I_{ad}$ (Step 2). We are still free to choose all f_i with $i \notin I_{ad}$, subject to (7.23). Recall that in the proof of Lemma 7.15 we fixed a fusion graph Ω_i for each $i \in I_{ad}$, and that by (7.34) we have in the new normalisation:

$$\lambda(\Omega_i) = 1 \quad \text{for} \;\; i \in I_{ad} \;. \tag{7.35}$$

Third Step

The following lemma is an extension of Lemma 7.14 to allow any reordering of the outgoing labels.

Lemma 7.16. Let $i_1, \dots, i_g \in I$, $\sigma \in S_{2g}$ and Ω be a $(\mathbb{1}; (i_1, \bar{i}_1 \dots, i_g, \bar{i}_g).\sigma)$-fusion tree where the permutation σ acts by changing the order of the $2g$ outgoing labels accordingly. Then $\lambda(\Omega) = 1$.

Proof. Let β_{2g} be any $2g$-braid, whose underlying permutation is $\sigma \in S_{2g}$. Since Z and $Z(\mathbb{1})$ are cocommutative, we have $\beta_{2g} \circ \Delta^{(2g)} = \Delta^{(2g)}$ for both of them.

We proceed as in the proof of Lemma 7.14 by expressing $\Delta_Z \circ \eta_Z$ as a direct sum where such fusion trees appear. Using cocommutativity of Z and $Z(\mathbb{1})$, we get a direct sum decomposition

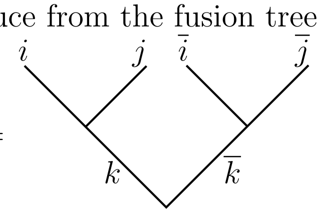
$$\Delta_Z^{(2g)} \circ \eta_Z = \beta_{2g} \circ \Delta_Z^{(2g)} \circ \eta_Z = \bigoplus_{\Omega} \lambda(\Omega)\, \beta_{2g} \circ D_\Omega \quad , \quad \Delta_{Z(\mathbb{1})}^{(2g)} \circ \eta_{Z(\mathbb{1})} = \bigoplus_{\Omega} \beta_{2g} \circ D_\Omega \;. \tag{7.36}$$

where the direct sum is over I-fusion trees Ω. Next, insert this into the definition of $C(Z)_g$ and use $C(Z)_g = C(Z(\mathbb{1}))$ as in the proof of Lemma 7.14. Comparing linearly independent terms gives

$$\lambda(\Omega) = 1 \tag{7.37}$$

for any $(\mathbb{1}; (i_1, \bar{i}_1 \dots, i_g, \bar{i}_g).\sigma)$-fusion tree Ω. □

Using Lemma 7.16, one can deduce from the fusion tree

$\Omega =$ 

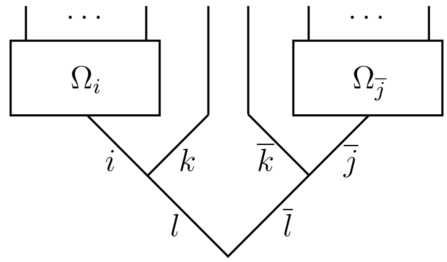
(7.38)

the equality

$$\lambda_{\bar{k}}^{\bar{i}\bar{j}} = (\lambda_k^{ij})^{-1} \;. \tag{7.39}$$

Lemma 7.17. Let $k, l \in I_g$ and $i, j \in I_{ad}$ such that $N_{ik}^l, N_{jk}^l \neq 0$. Then, $\lambda_l^{ik} = \lambda_l^{jk}$.

Proof. Recall the fusion trees Ω_i we picked for each $i \in I_{ad}$ in Step 2. Consider the fusion tree



(7.40)

By Lemma 7.14 this gives the identity $\lambda(\Omega_i)\,\lambda(\Omega_{\bar{j}})\,\lambda_l^{ik}\,\lambda_{\bar{l}}^{\bar{k}\bar{j}} = 1$. Together with (7.35) and (7.39), we conclude $\lambda_l^{ik} = \lambda_l^{jk}$. □

Lemma 7.18. There exist f_i satisfying (7.23) for all $i \in I$ and $f_i = 1$ for $i \in I_{ad}$, such that $\tilde{\lambda}_{k'}^{ik} = 1$ if $i \in I_{ad}$ and $N_{ik}^{k'} \neq 0$.

Proof. For each $g \in G$, fix an element $k_g \in I_g$, such that $k_e = \mathbb{1}$. By transitivity there exists some $i_g \in I_{ad}$ such that $N_{i_g \bar{k}_g}^{k_{g^{-1}}} \neq 0$. Then, find and fix f_{k_g} for every g such that

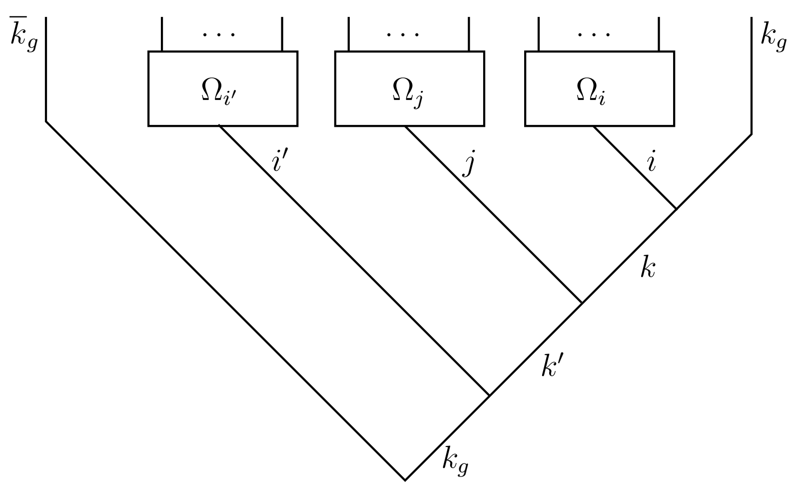
$$f_{k_g} f_{k_{g^{-1}}} = \lambda_{k_{g^{-1}}}^{i_g \bar{k}_g} \tag{7.41}$$

and such that $f_{\mathbb{1}} = 1$. For instance $f_{k_g} = f_{k_{g^{-1}}} = \left(\lambda_{k_{g^{-1}}}^{i_g \bar{k}_g}\right)^{\frac{1}{2}}$ is a consistent choice, since $\lambda_{k_g}^{i_g \bar{k}_{g^{-1}}} = \lambda_{k_{g^{-1}}}^{i_g \bar{k}_g}$ by raising and lowering indices (Lemma 7.12). Note that this choice satisfies $f_{\mathbb{1}} = 1$. Let $k \in I_g$ and $i \in I_{ad}$ be such that $N_{ik_g}^k \neq 0$ and define

$$f_k = \lambda_k^{ik_g} f_{k_g} \; . \tag{7.42}$$

This is independent of the choice of i by Lemma 7.17 and is consistent with $k = k_g$ as $\lambda_{k_g}^{ik_g} = 1$ (choose $i = \mathbb{1}$).

Let $k, k' \in I_g$ and $i, i', j \in I_{ad}$ such that $N_{i'k'}^{k_g}, N_{jk}^{k'}, N_{ik_g}^{k} \neq 0$. The fusion tree



(7.43)

implies

$$\lambda_k^{ik_g}\,\lambda_{k'}^{jk}\,\lambda_{k_g}^{i'k'} = 1 \ . \tag{7.44}$$

From (7.39) and Lemma 7.12 we get $\lambda_{k_g}^{i'k'} = (\lambda_{k'}^{\overline{i'}k_g})^{-1}$. Inserting this in (7.44) and using (7.42) gives $\lambda_{k'}^{jk} = \frac{f_{k'}}{f_k}$. Furthermore,

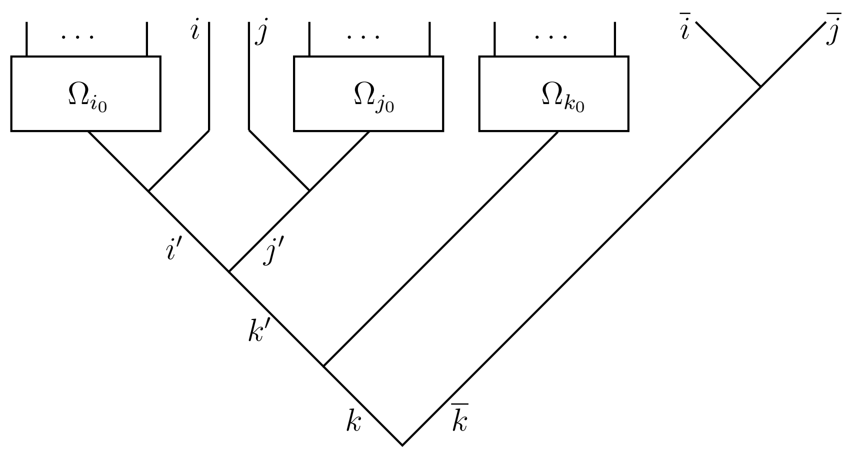
$$f_k f_{\overline{k}} \overset{(7.42)}{=} \lambda_k^{ik_g}\lambda_{\overline{k}}^{jk_{g^{-1}}} f_{k_g} f_{k_{g^{-1}}} \overset{(7.41)}{=} \lambda_k^{ik_g}\lambda_{\overline{k}}^{jk_{g^{-1}}}\lambda_{k_{g^{-1}}}^{i_g\overline{k}_g} = \lambda_k^{ik_g}\lambda_{\overline{k}_{g^{-1}}}^{jk}\lambda_{k_g}^{i_g\overline{k}_{g^{-1}}} \overset{(*)}{=} 1 \ , \tag{7.45}$$

where $(*)$ follows from setting $k' = \overline{k}_{g^{-1}}$ in (7.44). □

Therefore, in the new normalisation we have $\lambda_{k'}^{ik} = 1$ if $N_{ik}^{\ k'} \neq 0$ and $i \in I_{ad}$.

Lemma 7.19. The structure constants depend only on the universal grading group, i.e. for $i, i' \in I_g, j, j' \in I_h, k, k' \in I_{gh}$ with $N_{ij}^{\ k}, N_{i'j'}^{\ k'} \neq 0$ we have $\lambda_k^{ij} = \lambda_{k'}^{i'j'}$.

Proof. By transitivity of R, there exist $i_0, j_0, k_0 \in I_{ad}$ such that $N_{i_0 i}^{\ i'}, N_{jj_0}^{\ j'}$ and $N_{k'k_0}^{\ k}$ are non-zero. Consider the fusion tree



(7.46)

which implies $\lambda_{k'}^{i'j'}\lambda_{\overline{k}}^{\overline{i}\overline{j}} = 1$ and so by (7.38) also $\lambda_{k'}^{i'j'} = \lambda_k^{ij}$. □

Remark 7.20. The proof of Lemma 7.19 above is the place where the maximal genus g occurs for which we use irreducibility of $V_g^{\mathcal{C}}$, namely $g = 3N + 2$.

The conditions on λ_k^{ij} achieved up to this point are preserved by renormalisation constants f_i which satisfy (7.23) as well as

$$f_i = 1 \ \text{ for all } i \in I_{ad} \quad , \quad f_i = f_j \ \text{ whenever } i, j \in I_g \text{ for some } g \ . \tag{7.47}$$

Final Step

To conclude the proof, we will use group cohomology for the universal grading group G. Namely, we define a 2-cochain $\omega : G \times G \to \Bbbk^\times$ as follows. Given $g, h \in G$, pick $b_i \in R_g$, $b_j \in R_h$ as well as a $b_k \in R_{gh}$ that appears in the product $b_i b_j$. Then $N_{ij}^{\ k} \neq 0$ and we set

$$\omega(g, h) := \lambda_k^{ij} \ . \tag{7.48}$$

By Lemma 7.19, this is independent of the choice of i, j, k.

Lemma 7.21. The 2-cochain ω is a symmetric normalised 2-cocycle.

Proof. That ω is normalised, i.e. that $\omega(e,g) = 1 = \omega(g,e)$, is just the normalisation condition $\lambda_i^{\mathbb{1}i} = 1 = \lambda_i^{\mathbb{1}i}$ achieved in step 1. Symmetry of ω, that is $\omega(g,h) = \omega(h,g)$ follows from the commutativity property of λ_k^{ij} in Lemma 7.12.

To show the cocycle condition we will use coassociativity of the algebra Z. Given $f,g,h \in G$, pick $b_i \in R_f$, $b_j \in R_g$, and $b_k \in R_h$. Then choose $l \in I$ such that b_l is a summand in the product $b_i b_j b_k$. This implies that $b_l \in R_{fgh}$.

In terms of structure constants, one side of the coassociativity condition for Z can be rewritten as

$$
\begin{aligned}
&(r_i \otimes r_j \otimes r_k) \circ (\Delta_Z \otimes \mathrm{id}) \circ \Delta_Z \circ e_l \\
&\overset{(1)}{=} \sum_{p\in I} (r_i \otimes r_j \otimes r_k) \circ (\Delta_Z \otimes \mathrm{id}) \circ ((e_p \circ r_p) \otimes \mathrm{id}) \circ \Delta_Z \circ e_l \\
&\overset{(2)}{=} \sum_{p\in I} \lambda_p^{ij}\, \lambda_l^{pk}\, (r_i \otimes r_j \otimes r_k) \circ (\Delta_{Z(\mathbb{1})} \otimes \mathrm{id}) \circ ((e_p \circ r_p) \otimes \mathrm{id}) \circ \Delta_{Z(\mathbb{1})} \circ e_l \\
&\overset{(3)}{=} \omega(f,g)\,\omega(fg,h) \sum_{p\in I} (r_i \otimes r_j \otimes r_k) \circ (\Delta_{Z(\mathbb{1})} \otimes \mathrm{id}) \circ ((e_p \circ r_p) \otimes \mathrm{id}) \circ \Delta_{Z(\mathbb{1})} \circ e_l \\
&\overset{(4)}{=} \omega(f,g)\,\omega(fg,h)\ (r_i \otimes r_j \otimes r_k) \circ (\Delta_{Z(\mathbb{1})} \otimes \mathrm{id}) \circ \Delta_{Z(\mathbb{1})} \circ e_l
\end{aligned}
\tag{7.49}
$$

In step 1 we expanded id_Z into a direct sum over its component simple summands. This allows us in step 2 to insert the factors of λ which give the difference between Δ_Z and $\Delta_{Z(\mathbb{1})}$ in each simple summand. (In this expression we take λ's to be zero if their indices are not allowed by fusion.) The key step is equality 3. Here one uses that by the properties of the universal grading group, all $p \in I$ which give a nonzero contribution must have $b_p \in R_{fg}$, for else $N_{ij}^{\ p} = 0$. Thus, if we replace λ by ω via (7.48), the prefactor becomes independent of p and can be taking out of the sum. The sum over p can then be carried out giving the result of step 4.

An analogous computation for the other side of the coassociativity condition for Z gives

$$
\begin{aligned}
&(r_i \otimes r_j \otimes r_k) \circ (\mathrm{id} \otimes \Delta_Z) \circ \Delta_Z \circ e_l \\
&= \omega(g,h)\,\omega(f,gh)\ (r_i \otimes r_j \otimes r_k) \circ (\mathrm{id} \otimes \Delta_{Z(\mathbb{1})}) \circ \Delta_{Z(\mathbb{1})} \circ e_l\ .
\end{aligned}
\tag{7.50}
$$

Comparing the two expressions and using coassociativity of Z and $Z(\mathbb{1})$ results in

$$
\omega(f,g)\,\omega(fg,h) \;=\; \omega(g,h)\,\omega(f,gh)\ , \tag{7.51}
$$

which is the cocycle condition. □

In group cohomology there is a short exact sequence

$$
0 \to \mathrm{Ext}(G, \Bbbk^\times) \to H^2(G, \Bbbk^\times) \to \mathrm{Hom}(\Lambda^2 G, \Bbbk^\times) \to 0\ , \tag{7.52}
$$

see [Br, Exercise V.6.5]. However, $\mathrm{Ext}(G, \Bbbk^\times) = 0$ (as $\Bbbk$ is algebraically closed, $\Bbbk^\times$ is a divisible group, and so injective as an abelian group). The second map in (7.52) is given by

$$
\psi \;\longmapsto\; \Big(g \wedge h \mapsto \frac{\psi(g,h)}{\psi(h,g)} \Big)\ , \tag{7.53}
$$

and so any symmetric 2-cocycle is a coboundary.

In particular, by Lemma 7.21 ω is a coboundary, that is, there exist $\gamma_g \in \Bbbk^\times$ such that

$$\omega(g,h) \;=\; \frac{\gamma_g\,\gamma_h}{\gamma_{gh}}\;. \tag{7.54}$$

As ω is normalised, we have $\gamma_e = 1$. Now choose $f_i = \gamma_g^{-1}$ whenever $i \in I_g$. This choice satisfies the conditions in (7.47). To see that also (7.23) holds, note that $f_i f_{\bar{i}} = (\gamma_g \gamma_{g^{-1}})^{-1} = \omega(g, g^{-1})^{-1}$. But by (7.48) we have $\omega(g, g^{-1}) = \lambda_{\mathbb{1}}^{i\bar{i}} = 1$, by step 1. This finally gives

$$\lambda_k^{ij} \;=\; \frac{f_k}{f_i\, f_j}\;. \tag{7.55}$$

We have now completed the proof that $Z(A) \cong Z(\mathbb{1})$ as algebras and thereby the proof of Theorem 7.1.

7.4 Global symmetries

Previously, we have found that irreducibility of mapping class group representations of an MFC $\mathcal{C}$ is linked to the absence of surface defects in the associated RT TQFT of $\mathcal{C}$. In Section 8, Theorem 5.4 will receive some gravitational context in that the bulk gravity theory will be modelled after the RT TQFT of $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$. Since it is conjectured in [HO] that a 3D quantum gravity has no global symmetries, we study global symmetries of the RT TQFT of $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ under the assumption of an absence of surface defects of the (chiral) TQFT of $\mathcal{C}$.

Global symmetries of a 3D TQFT are in direct correspondence with invertible surface defects and thus we want to study invertible $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$-module categories or in other words invertible $\mathcal{C}$-bimodule categories.

We show the following statement:

Proposition 7.22. Let $\mathcal{C}$ be a braided fusion category which has no non-trivial indecomposable left module categories. Then, its invertible bimodule categories (up to equivalence) are in one-to-one bijection with isomorphism classes of tensor autoequivalences on $\mathcal{C}$.

The proof of Proposition 7.22 is given by Corollary 7.27.

Remark 7.23. To get an absence of global symmetries statement we would like to prove that all invertible bimodule categories are trivial or, equivalently by Proposition 7.22, that the group of tensor autoequivalences on $\mathcal{C}$ is trivial. As we shall see, there is evidence where this holds for the examples discussed in Example 7.29. We hope to revisit this problem in the future.

We introduce the definition of quasi-trivial bimodule categories as in [ENOM, Sec. 4.3]

Definition 7.24. An invertible $\mathcal{C}$-$\mathcal{C}$-bimodule category $\mathcal{M}$ is called *quasi-trivial* if as a left module category it is equivalent to the left regular module category $\mathcal{C}$.

Recall that *inner equivalence* in a fusion category $\mathcal{C}$ means conjugation by an invertible object. An *outer equivalence* is a tensor autoequivalence modulo inner equivalences. The group of isomorphism classes of outer equivalences of $\mathcal{C}$ will be denoted by $\mathrm{Out}(\mathcal{C})$. If $\mathcal{C}$ is braided, then inner equivalences are isomorphic to the identity and thus $\mathrm{Out}(\mathcal{C}) = \mathrm{Aut}_{\otimes}(\mathcal{C})$.

Given a tensor autoequivalence $\phi \in \mathrm{Aut}_{\otimes}(\mathcal{C})$, one can consider the bimodule category ${}_{\mathrm{id}}\mathcal{C}_{\phi}$ with the regular action from the left and the right action twisted by ϕ, i.e. $M \triangleleft X := M \otimes \phi(X)$. This bimodule is clearly quasi-trivial.

The statement of the following lemma is contained in [ENOM, Sec. 4.3] but without detailed proof:

Lemma 7.25. Let $\mathcal{C}$ be a fusion category. A quasi-trivial bimodule category $\mathcal{M}$ is determined uniquely up to bimodule equivalence by the isomorphism class of an outer equivalence $[\phi] \in \mathrm{Out}(\mathcal{C})$. Namely, there exists a unique $[\phi]$ such that ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{C}} \simeq {}_{\mathrm{id}}\mathcal{C}_{\phi}$ as bimodule categories.

Proof. Let $\mathcal{M}$ be a quasi-trivial bimodule category. Without loss of generality suppose $\mathcal{C}$ is the underlying left module category. The bimodule isomorphisms

$$b_{X,M,Y} : (X \triangleright M) \triangleleft Y = (X \otimes M) \triangleleft Y \xrightarrow{\simeq} X \otimes (M \triangleleft Y) = X \triangleright (M \triangleleft Y) \tag{7.56}$$

give in particular the natural isomorphism

$$b_{M,\mathbb{1},X} : M \triangleleft X = (M \otimes \mathbb{1}) \triangleleft X \xrightarrow{\simeq} M \otimes (\mathbb{1} \triangleleft X). \tag{7.57}$$

Define the endofunctor $\phi : \mathcal{C} \to \mathcal{C}$ by $\phi := (\mathbb{1} \triangleleft -)$. It follows from the bimodule category structure that ϕ is a tensor autoequivalence[21], i.e. $\phi \in \mathrm{Aut}_{\otimes}(\mathcal{C})$. The bimodule isomorphisms provide the equivalence $\mathcal{M} \simeq {}_{\mathrm{id}}\mathcal{C}_{\phi}$ as bimodules. This concludes the existence part of the statement.

We now show the uniqueness up to inner equivalence. Suppose $\phi, \psi \in \mathrm{Aut}_{\otimes}(\mathcal{C})$ such that there is a bimodule equivalence $F : {}_{\mathrm{id}}\mathcal{C}_{\phi} \simeq {}_{\mathrm{id}}\mathcal{C}_{\psi}$. This equivalence comes equipped with isomorphisms

$$F(X \otimes M \otimes \phi(Y)) \cong X \otimes F(M) \otimes \psi(Y) \tag{7.58}$$

and in particular, we have natural isomorphisms

$$\phi(X) \otimes F(\mathbb{1}) \cong F(\phi(X) \otimes \mathbb{1}) = F(\phi(X)) \cong F(\mathbb{1}) \otimes \psi(X)\ . \tag{7.59}$$

The object $P := F(\mathbb{1})$ is invertible as $\mathbb{1} \cong F(F^{-1}(\mathbb{1})) \cong F^{-1}(\mathbb{1}) \otimes F(\mathbb{1})$. Hence, as tensor functors we have $\psi \cong P^* \otimes \phi \otimes P$ (the choice of left or right dual here does not play a role due to [EGNO, Prop. 2.11.3]).

Conversely, if $\psi = P^* \otimes \phi \otimes P$ for some invertible object $P \in \mathcal{C}$, the functor $F = \mathrm{id} \otimes P$ is a bimodule equivalence.

Therefore, we have proved that any quasi-trivial bimodule category $\mathcal{M}$ is determined uniquely by a tensor auto-equivalence ϕ up to inner equivalence, i.e. by its outer equivalence class $[\phi]$. □

[21] For instance, one has isomorphisms $\phi(X \otimes Y) = \mathbb{1} \triangleleft (X \otimes Y) \cong (\mathbb{1} \triangleleft X) \triangleleft Y = \phi(X) \triangleleft Y = (\phi(X) \otimes \mathbb{1}) \triangleleft Y \cong \phi(X) \otimes \phi(Y)$.

Lemma 7.26. Let $\mathcal{C}$ be a fusion category which has no non-trivial indecomposable left module categories. Then, every invertible bimodule category is quasi-trivial.

Proof. By [ENOM, Cor. 4.4] any invertible bimodule category $\mathcal{M}$ is indecomposable as a left module category. This follows directly from the invertibility property

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{M}^{\mathrm{op}} \simeq \mathcal{C}$$

as bimodule categories. However, since $\mathcal{C}$ has only the left regular category $\mathcal{C}$ as an indecomposable left module category up to equivalence, $\mathcal{M}$ is equivalent to $\mathcal{C}$ as left module categories and thus quasi-trivial. □

Given a fusion category $\mathcal{C}$, the Brauer-Picard group consists of all invertible bimodule categories up to equivalence, where multiplication is given by the balanced tensor product $\boxtimes_{\mathcal{C}}$.

Corollary 7.27. Let $\mathcal{C}$ be a braided fusion category with no non-trivial indecomposable left module categories. Then, the Brauer-Picard group $\mathrm{BrPic}(\mathcal{C})$ is isomorphic to $\mathrm{Aut}_{\otimes}(\mathcal{C})$.

Proof. Since $\mathcal{C}$ is braided, inner equivalences are trivial and hence $\mathrm{Out}(\mathcal{C}) = \mathrm{Aut}_{\otimes}(\mathcal{C})$. By Lemma 7.25 and Lemma 7.26 it follows that the map

$$\mathrm{Out}(\mathcal{C}) \to \mathrm{BrPic}(\mathcal{C}),\ [\phi] \to {}_{\mathrm{id}}\mathcal{C}_{\phi} \tag{7.60}$$

is an isomorphism. □

Recall that for a braided fusion category $\mathcal{C}$, there is a natural way to consider left module categories as bimodule categories by using the braiding of $\mathcal{C}$. This allows to define invertibility for left module categories. The group such invertible module categories up to equivalence is called the Picard group $\mathrm{Pic}(\mathcal{C})$.

Lemma 7.28. Let $\mathcal{C}$ be a non-degenerate braided fusion category with no non-trivial left module categories. Then, the group of braided auto-equivalences is trivial, i.e. $\mathrm{Aut}^{\mathrm{br}}(\mathcal{C}) = 1$.

Proof. By [ENOM, Thm. 5.2] the Picard group $\mathrm{Pic}(\mathcal{C})$ is isomorphic to $\mathrm{Aut}^{\mathrm{br}}(\mathcal{C})$. However, by hypothesis $\mathrm{Pic}(\mathcal{C}) = 1$. □

Example 7.29. In [EM] the group $\mathrm{Aut}_{\otimes}(\mathcal{C})$ is determined for $\mathcal{C} = \mathcal{C}(sl_r, k)$. In particular, we can check that these groups are trivial for all the known examples of MFCs with irreducibility property. This implies that the bulk theory, we are interested in for mapping class group averages, has no invertible defects or equivalently no global symmetries as proposed in Remark 7.23.

8 Applications to 3D gravity

8.1 Geometry of classical solutions to euclidean 3D gravity

One fundamental property of gravity in three dimensions is that it has no local degrees of freedom. The curvature tensor is completely determined by the Ricci tensor and therefore solutions of Einstein gravity have constant curvature. In the study of gravity with a negative cosmological constant $\Lambda < 0$, which equates to a negative constant curvature R, such solutions are locally isometric to the three-dimensional Anti-de-Sitter spacetime AdS_3. The AdS_3 spacetime here is identified with the 3-dimensional hyperbolic space $\mathbb{H}^3$ since we consider only Euclidean signature. This space is a subspace of the four-dimensional Minkowski space $\mathbb{M}^4$:

$$\mathbb{H}^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{M}^4 | -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1/\Lambda,\ x_0 > 0\} \tag{8.1}$$

where $\Lambda < 0$ is the negative cosmological constant and its metric is induced by the Minkowski metric. By a change of coordinates[22], the AdS_3-metric can be expressed as

$$ds^2 = l^2(d\rho^2 + (\cosh\rho)^2 dt_E^2 + (\sinh\rho)^2 d\phi^2) \tag{8.2}$$

where l is called the AdS radius and coincides with the curvature R. It is related to the cosmological constant as we can guess from (8.1) by $l^2 = -1/\Lambda$. We also have parameters $\rho \in \mathbb{R}_{\geq 0}$, a spatial cycle parameter $\phi \in [0, 2\pi)$ and Euclidean time $t_E \in \mathbb{R}$. By introducing the complex coordinate $z = -t_E + i\phi$ with the obvious identification $z \sim z + i2\pi$, one finds that in the asymptotic boundary $\rho \to \infty$ the metric becomes flat with the associated $|dz|^2$. This asymptotic behaviour indicates that the conformal boundary of AdS_3 is precisely the Riemann sphere $\mathbb{CP}^1$.

The group of isometries of AdS_3 is $\mathrm{Isom}(\mathrm{AdS}_3) = Sl(2,\mathbb{C})/\mathbb{Z}_2$, which is isomorphic to the proper orthochronous Lorentz group $SO(3,1)^+$, i.e. the connected component of the Lorentz group containing the identity. The global geometry of solutions of Einstein gravity are obtained by gluing patches together which are locally isometric to AdS_3. Such a solution is globally isometric to a quotient U/Γ of a certain domain U in AdS_3 by a subgroup of isometries $\Gamma \subset Sl(2,\mathbb{C})/\mathbb{Z}_2$. The Anti-de-Sitter spacetime has a rather trivial topology as it is homeomorphic to the three-dimensional ball $\mathbb{D}^3$ with conformal boundary given by the Riemann sphere $\mathbb{CP}^1$. However, the gluing isometries of Γ in the quotient U/Γ are responsible for non-trivial topologies both for the 3-dimensional part and for the conformal boundary.

As already mentioned the group of isometries of (Euclidean) AdS_3 coincides with $SO(3,1)^+$ which happens to be the conformal group on the Euclidean plane. This can be viewed, at least on a preliminary level, as a motivation for an $\mathrm{AdS}_3\,/\,\mathrm{CFT}_2$ correspondence[23]. The story becomes even more compelling when studying boundary conditions and asymptotic symmetries. A fundamental solution of 3D gravity with $\Lambda < 0$ with such a boundary

[22]Set $x_0 = l\cosh\rho\sinh t_E, x_1 = l\cosh\rho\cosh t_E, x_2 = l\sinh\rho\cos\phi, x_3 = l\sinh\rho\sin\phi$.

[23]The idea of an AdS/CFT correspondence originates from Maldacena [Ma] and later Witten [W] matching type IIB supergravity on $\mathrm{AdS}_5 \times \mathbb{S}^5$ with 4D $\mathcal{N} = 4$ super Yang-Mills theory.

condition is thermal AdS_3 and is locally the ordinary Anti-de-Sitter spacetime of radius l with metric (8.2) but with an additional boundary condition given by

$$z \sim z + i2\pi\tau$$

where τ is a complex number. Recall that z was already subject to $z \sim z + i2\pi$. This gives thermal AdS_3 the topology of a solid torus and its conformal boundary is the torus T^2 with conformal (or modular) parameter τ.

The classical phase space of gravity can be thought as the space of solutions, asymptotically AdS, which obey boundary conditions (e.g. thermal AdS_3) modulo *small* diffeomorphisms which respect the boundary conditions. Small diffeomorphisms are diffeomorphisms homotopic to the identity. By computing the relevant Poisson structure in the limit $l \gg G$, Brown and Henneaux [BH] found that the algebra of asymptotic symmetries, which is generated by such diffeomorphisms, is isomorphic to two copies of the Virasoro algebra Vir_c of central charge

$$c = \frac{3l}{2G} \; . \tag{8.3}$$

8.2 Path integral formulation of quantum gravity

By fixing a conformal boundary, i.e. a Riemann surface Σ, the path integral approach to the pure Einstein gravity partition function suggests to study the path integral

$$Z = \sum_{M \text{ topologies}} \int \mathcal{D}g \, e^{iS[g]} \tag{8.4}$$

where the sum is over diffeomorphism classes of smooth 3-manifolds M with conformal boundary Σ and we perform a path integral over Riemannian metrics g with conformal boundary Σ. Here, $S[g]$ is the Einstein-Hilbert action. Of course, this expression is far from a well-defined object as not only the "geometries" M are not specified but also the integration of such seems far-fetched. However, one might try to compute sensible approximations. The semi-classical approach of [MW] suggests that the dominating contributions to (8.4) are precisely classical solutions. In this semi-classical limit, the central charge is large $c \gg 1$ which translates to a weak coupling. Fixing a genus 1 Riemman surface, i.e. a torus with conformal parameter τ, they consider the solutions of type U/Γ, where U is a certain domain of AdS_3 and Γ a group of isometries. When Γ is freely generated by one element, i.e. $\Gamma \cong \mathbb{Z}$, the solutions M_γ are topologically solid tori and they are parameterised by elements γ in the modular group

$$Sl(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

To be precise, some elements of $Sl(2,\mathbb{Z})$ lead to equivalent solutions. The solutions are uniquely determined by a pair (c,d) of coprime integers up to sign, i.e. $M_{c,d}$. The manifold $M_{0,1}$ has already appeared earlier as it represents precisely thermal AdS_3. The manifold $M_{1,0}$ is special, since it represents the Euclidean BTZ black hole, and is the result of [BTZ] showing the existence of black holes in 3D Einstein gravity. In the BTZ black hole, time

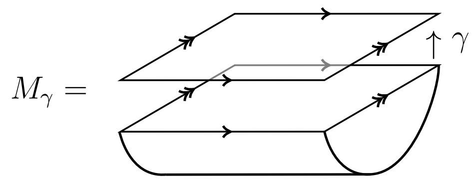


Figure 8.1: The manifold obtained by twisting the torus boundary of the solid torus by γ.

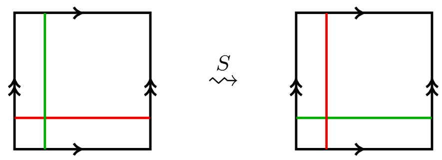


Figure 8.2: The S-transformation on the torus exchanges the meridian and the parallel.

becomes contractible and $\rho = 0$ represents the black hole horizon. This can be understood in the following topological consideration: The manifolds M_γ are obtained from the thermal AdS_3 manifold $M_{0,1}$ by gluing the torus boundary using $\gamma \in Sl(2, \mathbb{Z})$[24], see Figure 8.1. This corresponds to the transformation

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}. \tag{8.5}$$

In $M_{0,1}$ the *spatial* loop corresponds to the meridian and the *time* loop to the parallel of the boundary torus. The BTZ black hole $M_{1,0}$ can be obtained by the S-generator of $Sl(2, \mathbb{Z})$. Hence, the spatial loop becomes now the parallel and therefore is non-contractible, whereas the time loop becomes the meridian and therefore contractible, see Figure 8.2.

Other choices of the group Γ lead to geometries different than the solid tori solutions M_γ. For instance, if Γ consists of two free generators, then one obtains hyperbolic 3-manifolds with a cusp. If Γ contains a torsion element, the result is an orbifold. Evaluating the contribution of these more complex geometries to the path integral is not as clear as for the solid tori geometries and therefore are excluded from the semi-classical consideration of [MW][25]. Hence, equation (8.4) becomes in this semi-classical approximation

$$Z_{\text{grav}} = \sum_{\gamma \in Sl(2,\mathbb{Z})} Z(M_\gamma) \tag{8.6}$$

where $Z(M_\gamma)$ is the contribution of M_γ. Since some of these include the same contributions,

[24] Recall that $Sl(2, \mathbb{Z})$ coincides with the mapping class group of the torus.

[25] The contribution of cusp geometries is assumed to vanish in the semi-classical limit, whereas orbifolds which contain singularities might have a physical interpretation of containing massive particles [MW, Sec. 2.1].

one may regularise and redefine

$$Z_{\text{grav}} = \sum_{Z_\gamma \in \mathcal{O}_{\text{vac}}} Z_\gamma \tag{8.7}$$

where $Z_{\text{vac}} = Z(M_{0,1})$ is the "vacuum" contribution of thermal AdS_3 and $\mathcal{O}_{\text{vac}}$ is the orbit of Z_{vac} with respect to the mapping class group of the torus $Sl(2,\mathbb{Z})$. This sum is seemingly significantly more manageable. Indeed, for rational CFT considerations one arrives at a well-defined computable sum.

8.3 RCFT considerations

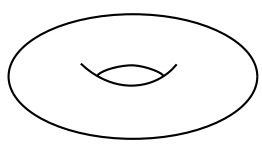
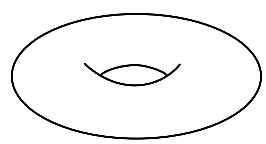
In [CGHMV] they used precisely this semi-classical approach to study gravity duals of unitary minimal models (a family of RCFTs). Even though for these models $c < 1$ and therefore is not subject to the limit $c \gg 1$, it is reasonable to study the semi-classical contribution as it should be included in the overall path integral and it produces the property of mapping class group or modular invariance, on which we will expand later. Nevertheless, assuming the vacuum contribution (or vacuum seed) to be

$$Z_{\text{vac}} = |\chi_0(\tau)|^2$$

where $\chi_0(\tau)$ is the holomorphic vacuum character, the contribution of M_γ is given by

$$Z_\gamma = |\chi_0(\gamma.\tau)|^2$$

where $\gamma.\tau$ is specified by the modular transformation (8.5). Topologically, the vacuum seed is obtained by the evaluation of the corresponding RT-TQFT on two copies of the solid torus:

$\mathcal{Z}^{\mathcal{C}}$ ( $\sqcup\ -$ ) (8.8)

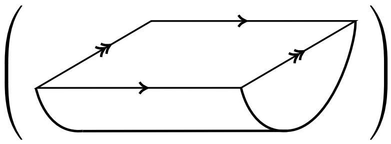
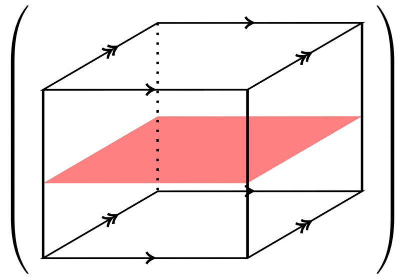
As mentioned in Remark 5.6, these minimal models and more general RCFTs satisfy a special property, which allows the sum to be finite, in particular well-defined. This is the discussed property F of mapping class group representations. Thus, given a RCFT on the boundary, we may write (8.5) in terms of the mapping class group averages as defined in Section 5:

$$Z_{\text{grav}} := \langle Z_{\text{vac}} \rangle_{T^2} \ . \tag{8.9}$$

The analysis of [CGHMV] showed that the expression in (8.9) is equal (up to a non-zero scalar) to a unitary minimal model CFT partition function, precisely for the Ising and the tricritical Ising model, i.e.

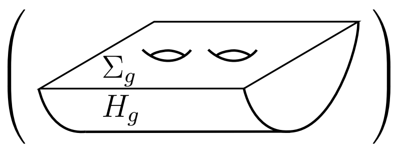
$$\langle Z^{\mathcal{C}}_{\text{vac}} \rangle_{T^2} \sim Z^{\mathcal{C}}_{\text{CFT}}(T^2) \tag{8.10}$$

for $\mathcal{C}$ the MFC corresponding to the Ising CFT, respectively the three-state Potts model. Topologically, (8.10) is

$$\langle \hat{\mathcal{Z}}^{\mathcal{C}}$$

$$\rangle_{T^2} = \mathcal{Z}^{\mathcal{C}}$$

(8.11)

For other central charges, it produces a sum which may include independent partition functions and even unphysical modular invariants.

So far, we have only discussed gravity with a conformal boundary of a torus, but one can consider boundary conditions which lead to higher genera Riemannian surfaces on the boundary. Choosing appropriate gluing isometries Γ for the quotient space U/Γ one can create geometries, which are topologically handlebodies of genus g, similar to the solid tori in the $g = 1$ case. We may assume the vacuum contribution to be given by the holomorphic factorised of the TQFT on the empty handlebody, i.e.

$$Z_{\text{vac}} = \hat{\mathcal{Z}}^{\mathcal{C}}$$


(8.12)

we might want to write Z_{grav} as the mapping class group average of Z_{vac}. In general, this might prove to be problematic as such the action of a mapping class group of a higher genus surface (even for RCFTs) might have infinite orbits (e.g. for the tricritical Ising CFT [JLLSW, Sec. 4.5]), with no clear way of regularisation of the sum. We bypass this problem by working with MFCs which obey property F and thus allow for a *working* definition of such mapping class group averages as finite sums. In this framework, we use Definition 5.2 and define the gravity partition function candidate:

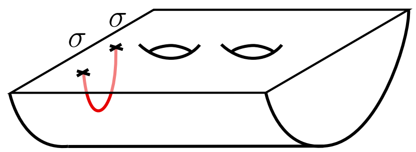
$$Z_{\text{grav}} := \langle Z_{\text{vac}} \rangle_{\Sigma} \tag{8.13}$$

It is a result by [JLLSW] that (8.10) extends to any genus g for the Ising CFT. Unfortunately, the tricritical Ising model fails to extend.

In Section 5, we took this one step further and give this correspondence for boundary surfaces, which may include field insertions. These insertions are to be connected via Wilson lines in the bulk theory. We found that an essential property which ensures this correspondence for general RCFTs is irreducibility of the (chiral) mapping class group representations, i.e. the mapping class group representations encoded by the modular functor $V^{\mathcal{C}}$ of the MFC $\mathcal{C}$. The result is

$$\langle x \rangle_{\Sigma} \sim \mathrm{Cor}_{\mathcal{C}}(\Sigma) \tag{8.14}$$

where x is an element in the full conformal block space and subject to the non-degeneracy conditions we described. The element x can be thought as a generalised vacuum seed, but we do not argue what is an appropriate physical choice. One might for instance choose in the Ising model for a surface with two insertions of the spin field σ the double of



(8.15)

which also satisfies the non-degeneracy condition similar to the empty handlebody from Example 3.3.

8.4 Summing over more Topologies

In the path integral (8.4), naively one should sum over topologies M and perform a path integral over geometries g on M, which solve the asymptotic boundary conditions. However, in the semi-classical approach the topologies which have been considered were that of the solid tori for genus 1, or handlebodies for higher genera boundary surfaces. It is a natural and an important question, if we can perform a sum over more topologies, but we will not address it directly here as there is no obvioius way to do this. This consideration is made and discussed in [BM] with interesting results in lower-dimensions. Let us instead make the following observation:

Let Σ be a closed oriented surface and $\mathcal{B}_\Sigma$ be the set of all 3-manifolds M that bound Σ (up to diffeomorphism which restricts to the identity on the boundary). The set $\mathcal{B}_\Sigma$ coincides directly with the morphism space of the bordism category from $\emptyset$ to Σ, i.e.

$$\mathcal{B}_\Sigma = \mathrm{Bord}_3(\emptyset, \Sigma). \tag{8.16}$$

This encodes the more complicated topologies we would like to include in our summation. The mapping class group $\mathrm{Mod}(\Sigma)$ has a natural action on $\mathcal{B}_\Sigma$. Therefore, we can decompose $\mathcal{B}_\Sigma$ into the disjoint union of mapping class group orbits

$$\mathcal{B}_\Sigma = \amalg_{i\in\mathcal{O}} \mathcal{B}_i \tag{8.17}$$

Let us consider the example of the torus. The (black hole) family

$$\{M_{c,d} \,|\, (c,d) \in \mathbb{Z}^2/\pm, \gcd(c,d) = 1\} \tag{8.18}$$

is the mapping class group orbit of thermal AdS_3, or topologically the solid torus. Similarly, for a genus g surface Σ we may write

$$\mathcal{B}_{H_g} = \{H_g^\gamma | \gamma \in \mathrm{Mod}(\Sigma)\} \tag{8.19}$$

for the orbit of the genus g handlebody H_g, which bounds Σ. The orbit elements H_g^γ are obtained by the twisting the boundary of H_g via the corresponding mapping class γ. Some H_g^γ's might represent the same bordism (for instance Dehn twists along meridians do not change the topology). These orbits are typically infinite. However, the image of such orbits under a 3D TQFT

$$\mathcal{Z}(\mathcal{B}_i) = \{\mathcal{Z}(M) | M \in \mathcal{B}_i\} \tag{8.20}$$

can become finite. Indeed, if a MFC $\mathcal{C}$ has property F with respect to a surface Σ, then the RT TQFT gives a finite image

$$\mathcal{Z}^{\mathcal{C}}(\mathcal{B}_i)$$

for each orbit $\mathcal{B}_i$.

If there is a sensible definition (regularisation) for the "sum over topologies"

$$\sum_{M\in\mathcal{B}_\Sigma} \mathcal{Z}(M),$$

then we can expand it as

$$\sum_{M\in\mathcal{B}_\Sigma} \mathcal{Z}(M) = \sum_{i\in\mathcal{O}} \sum_{v\in\mathcal{Z}(\mathcal{B}_i)} v \tag{8.21}$$

where the second sum $\sum_{v\in\mathcal{Z}(\mathcal{B}_i)} v$ is finite and mapping class group invariant by definition.

In particular, if the space of such mapping class group invariants is one-dimensional then computing only the finite mapping class group averages, which we studied in this thesis, will lead us to the correct ansatz for the more general sum over all topologies. As we have seen, irreducible mapping class group representations, say of the RT TQFT associated to $\mathcal{C}$, lead to such one-dimensional spaces of invariants in the double theory (evaluating on the double surface Σ). In the picture of the double theory, where one copy corresponds to holomorphic blocks and the other to antiholomorphic blocks, the topologies we have considered are two copies of the corresponding handlebodies, i.e."diagonal" topologies. However, one can consider topologies with multiple boundary components, i.e."wormholes".

8.5 Absence of global symmetries

In Section 7 we have studied global symmetries in the topological bulk theory of $\mathcal{Z}(\mathcal{C})$, which models the gravity dual candidate by providing mapping class group average constructions. It is a conjecture by [HO] that quantum gravitational theories do not admit any global symmetries. Hence, we would to prove that the RT TQFT of $\mathcal{Z}(\mathcal{C})$ for a MFC $\mathcal{C}$ subject to the hypothesis in Theorem 7.1 has no non-trivial global symmetries.

Global symmetries of TQFTs correspond to invertible module categories, see Section 4. Since the Drinfeld centre $\mathcal{Z}(\mathcal{C})$ obeys the factorisation property

$$\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$$

as $\mathcal{C}$ is non-degenerate (as a MFC), invertible module categories over $\mathcal{Z}(\mathcal{C})$ coincide with invertible bimodule categories over $\mathcal{C}$. The MFCs from Theorem 7.1 have the irreducibility property with respect to mapping class groups. However, because we want to study an algebraic structure such as module and bimodule categories, we prefer to start with the following hypothesis:

Let $\mathcal{C}$ have no non-trivial defects.

This is a hypothesis that holds for the MFCs of interest as it follows by Theorem 7.1, albeit strictly weaker than irreducibility (see Remark 7.2). Under this assumption, we have reduced the question of global symmetries to that of tensor autoequivalences on

$\mathcal{C}$ (see Proposition 7.22). Indeed, any examples we are aware of, which are subject to the irreducibility property and therefore examples with no non-trivial defects, possess only trivial tensor autoequivalences (see Example 7.29). Hence, they produce no global symmetries in the bulk theory of $\mathcal{Z}(\mathcal{C})$.

8.6 Beyond irreducibility

So far, irreducibility has been used as a tool to obtain a bulk-boundary correspondence, ensuring there is unique sensible boundary theory. This concerns only certain theories like the Ising CFT with this property. When there are more mapping class group invariants, mapping class group averages can produce a linear combination of such as for most of the minimal model CFTs [CGHMV] or [MMS]. There are some unknowns for these cases which go beyond irreducibility.

1. One possibility or strategy is that including more topologies (than the mapping class group average) will put more constraints on the mapping class group invariant and single out one partition function on the boundary.

2. The alternative idea, which experiences a recent emergence in physics literature [CM], is to consider an ensemble of CFTs on the conformal boundary rather than a single CFT. This is motivated already by the lower-dimensional $\mathrm{AdS}_2\,/\,\mathrm{CFT}_1$ correspondence [SSS] with Jackiw-Teitelboim (JT) gravity as the gravity dual of the Sachdev-Ye-Kitaev (SYK) model, which consists a random matrix ensemble.

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