Ricci curvature on Courant algebroids

by

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Summary

This thesis is concerned with the Ricci curvature in generalized geometry. The central objects which are being considered are Courant algebroids. These are vector bundles equipped with certain structures that satisfy some compatibility conditions. We begin this thesis by introducing them and explaining some basic properties. After that, we address important objects on Courant algebroids. For instance, one defines generalized metrics as special symmetric bilinear forms on the vector bundle. Then, as in classical differential geometry, one can define generalized Levi-Civita connections as generalized connections which are torsion-free and compatible with the generalized metric. In contrast to classical differential geometry, those are not uniquely determined by their defining properties. Therefore, in order to define the Ricci curvature analogously to the classical theory, one needs to introduce so-called divergence operators, which can be associated to a generalized connection. It turns out that one can define the generalized Ricci curvature of a pair consisting of a generalized metric and a divergence operator on a Courant algebroid as follows. One chooses a generalized Levi-Civita connection which has this divergence operator. Then one defines the generalized Ricci curvature of the pair as the generalized Ricci curvature of this generalized connection. Next, we focus on the simplest class of Courant algebroids, the exact Courant algebroids. We determine the generalized Ricci curvature for an exact Courant algebroid, with arbitrary generalized metric and divergence operator. In the subsequent part we consider the case when there exists a frame on a arbitrary Courant algebroid, which is orthonormal with respect to a generalized metric on it. We construct a generalized Levi-Civita connection from the orthonormal frame and the datum of the Courant algebroid. Then we compute its divergence and Ricci curvature and determine how this depends on the choice of the orthonormal frame. We give an explicit formula for the difference of the generalized Ricci curvatures of two generalized Levi-Civita connections in terms of the difference of their divergence operators. In the last part of the thesis, we develop the theory of left-invariant generalized metrics on Lie groups. We study the generalized Einstein equation as the vanishing of the generalized Ricci curvature. For that, we compute the generalized Ricci curvature in terms of tensors on the sum of the Lie algebra and its dual, which encode the structure of the Courant algebroid, the generalized metric and the divergence operator. The resulting expression is a homogeneous polynomial of degree two in the coefficients of the Courant algebroid structure and the divergence operator with respect to a left-invariant orthonormal basis of the generalized metric. We conclude the thesis by determining all left-invariant generalized Einstein metrics on three-dimensional Lie groups.

Zusammenfassung

Diese Arbeit handelt von der Ricci-Krümmung in der verallgemeinerten Geometrie. Die zentralen Objekte, die betrachtet werden, sind Courant-Algebroide. Dies sind Vektorbündel ausgestattet mit gewissen Strukturen, die bestimmte Kompatibilitätsbedingungen erfüllen. Wir beginnen die Arbeit, indem wir diese einführen und grundlegende Eigenschaften erklären. Danach widmen wir uns wichtigen Objekten auf Courant-Algebroiden. So definiert man verallgemeinerte Metriken als spezielle symmetrische Bilinearformen auf dem Vektorbündel. Dann kann man, wie in der klassischen Differentialgeometrie, verallgemeinerte Levi-Civita-Zusammenhänge als verallgemeinerte Zusammenhänge definieren, die torsionsfrei und kompatibel mit der verallgemeinerten Metrik sind. Diese sind allerdings im Gegensatz zur klassischen Differentialgeometrie nicht durch ihre definierenden Eigenschaften eindeutig bestimmt. Um also analog zur klassischen Theorie die Ricci-Krümmung zu definieren, benötigt man sogenannte Divergenzoperatoren, die man verallgemeinerten Zusammenhängen zuordnen kann. Es stellt sich heraus, dass man die verallgemeinerte Ricci-Krümmung eines Paars bestehend aus einer verallgemeinerten Metrik und einem Divergenzoperator auf einem Courant-Algebroid wie folgt definieren kann. Man wählt einen verallgemeinerten Levi-Civita-Zusammenhang, der diese Divergenz hat. Dann definiert man die verallgemeinerte Ricci-Krümmung des Paars als verallgemeinerte Ricci-Krümmung dieses verallgemeinerten Zusammenhangs. Danach befassen wir uns mit der wohl einfachsten Klasse von Courant-Algebroiden, den exakten Courant-Algebroiden. Wir bestimmen die verallgemeinerte Ricci-Krümmung für exakte Courant-Algebroide mit beliebiger verallgemeinerter Metrik und Divergenz. Im nächsten Teil betrachten wir den Fall, dass ein Rahmen auf einem beliebigen Courant-Algebroid existiert, der orthonormal bezüglich einer verallgemeinerten Metrik darauf ist. Wir konstruieren einen verallgemeinerten Levi-Civita-Zusammenhang aus dem orthonormalen Rahmen und den Daten des Courant-Algebroids. Dann berechnen wir seine Divergenz und Ricci-Krümmung und bestimmen, wie dies von der Wahl des orthonormalen Rahmens abhängt. Wir geben eine explizite Formel für den Unterschied der verallgemeinerten Ricci-Krümmungen von zwei verallgemeinerten Levi-Civita-Zusammenhängen in Abhängigkeit der Differenz ihrer Divergenzen an. Im letzten Teil der Arbeit entwickeln wir die Theorie von links-invarianten verallgemeinerten Metriken auf Lie-Gruppen. Wir studieren die verallgemeinerte Einstein-Gleichung als das Verschwinden der verallgemeinerten Ricci-Krümmung. Dazu berechnen wir die verallgemeinerte Ricci-Krümmung in Abhängigkeit von Tensoren auf der Summe der Lie-Algebra und ihrem Dualraum, die die Courant-Algebroid Struktur, die verallgemeinerte Metrik und den Divergenzoperator ausdrücken. Der resultierende Ausdruck ist ein homogenes Polynom zweiten Grades in den Koeffizienten der Courant-Algebroid Struktur und des Divergenzoperators bezüglich einer links-invarianten orthonormalen Basis der verallgemeinerten Metrik. Wir beschließen die Arbeit, indem wir alle links-invarianten verallgemeinerten Einstein-Metriken auf dreidimensionalen Lie-Gruppen bestimmen.

Contents

1 Introduction

The main objects of generalized geometry are the so-called Courant algebroids. Courant [C] and Dorfman [D] defined a natural bracket on the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$ in their study of Dirac structures. This was formalized by Liu, Weinstein and Xu in [LWX] with the introduction of Courant algebroids. These are defined as vector bundles E equipped with a vector bundle homomorphism $\pi : E \to TM$ (modeled on the natural projection $\mathbb{T}M \to TM$), a scalar product on E (modeled on the natural pairing of neutral signature on $\mathbb{T}M$) and a bracket (modeled on the bracket introduced by Courant and Dorfman[1]), which satisfy certain compatibility conditions. Among others, these concepts were later used by Hitchin [H] as a framework unifying complex and symplectic structures. The two latter can be viewed as particular instances of the notion of a generalized complex structure. The theory of generalized complex structures was developed in [Gu1, Gu2] including a geometrization of Barannikov's and Kontsevich's extended deformation theory [BK]. As we will explain below, also many other geometric structures were transferred to this generalized setting.

An important class of Courant algebroids are exact Courant algebroids. By definition those fit into the exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0,$$

where π^* is the dual map of π while identifying E with its dual E^* via the scalar product. In fact, they are always isomorphic to the generalized tangent bundle $\mathbb{T}M$ and were classified by Ševera in [S] in terms of the third de Rham cohomology $H^3_{dR}(M)$ of the underlying manifold M. Most of the results of the present work will be in the context of exact Courant algebroids.

The counterpart of pseudo-Riemannian metrics in generalized geometry as so-called generalized (pseudo-Riemannian) metrics. They were introduced by Gualtieri in [Gu1, Gu3] and further investigated in for example [CD, G]. Furthermore, they can be used, for instance, to unify and geometrize the structures involved in type II supergravity [CSW]. They also lead to many interesting concepts such as generalized Kähler structures [Gu1] and notions of curvature on Courant algebroids [G].

The fundamental theorem of Riemannian geometry states that there always exists a unique torsion-free, metric connection (i.e. a Levi-Civita connection) on a manifold. The ideas of connections and their torsion have been transferred to generalized geometry in [AX] and developed further in for example [Gu3]. After that a lot of work has been done on the theory of generalized connections. For instance, a characterization for the integrability of various generalized structures in terms of torsion-free generalized connections was presented in [CD]. However, as pointed out in [G], being torsion-free and compatible with a generalized metric does not determine the generalized connection uniquely. There is no immediate analogue of the fundamental theorem of Riemannian geometry in the generalized setting. This means in particular that it is not possible to define the curvature of a Courant algebroid without further specifications.

[1] In fact, the Courant bracket is the skew-symmetrization of the Dorfman bracket. The Dorfman bracket, while not being skew-symmetric, satisfies the Jacobi identity, as opposed to the Courant bracket.

However, there have been various works on generalizing the Ricci curvature in this setting (e.g. [CSW, G, SV1, SV2, St]). We will focus on the approach of [G]. In order to fix the ambivalence in the choice of metric, torsion-free generalized connection (also called generalized Levi-Civita connection), one introduces so-called divergence operators. Divergence operators can be associated to a generalized connection by taking a trace. If one defines the Ricci curvature of a generalized connection in a similar way as in the classical setting, one can show that it does not depend on the choice of generalized Levi-Civita connection, if one fixes its divergence.

Hence a generalized pseudo-Riemannian metric together with a divergence operator is not only sufficient to define a notion of generalized Ricci curvature but also to pose the generalized Einstein equation as the vanishing of the generalized Ricci curvature [GSt]. The generalized Einstein equations are related to equations that arise from physics in supergravity theories [CSW] for instance as solutions of the Hull-Strominger system [G]. They also appear as critical points of the so-called generalized Einstein-Hilbert action [GSt] and in complex geometry [IP]. Furthermore, a generalized geometry formulation of minimal six-dimensional supergravity has been given in [GS] with a particular case of the generalized Einstein equation as the main bosonic equation of motion. Examples of invariant Ricci-flat Bismut connections on compact homogeneous Riemannian manifolds have been constructed in [GSt, PR1, PR2]. They include non Bismut-flat examples [PR1, PR2] and give rise to invariant positive definite solutions of the generalized Einstein equation with the so-called Riemannian divergence operator.

We start this thesis by reviewing basic concepts in generalized geometry. First, we define Courant algebroids. We give the standard example $\mathbb{T}M$, and prove that a closed three-form defines a Courant algebroid structure on $\mathbb{T}M$. After that, we introduce several objects, such as generalized connections, their divergences and their torsions. Furthermore, we define generalized pseudo-Riemannian metrics on Courant algebroids. We describe various spaces of generalized connections, depending on their divergence, their torsion and their compatibility with a given generalized metric. We also introduce the curvature of a generalized connection and define the Ricci tensors of a pair consisting of a generalized metric and a divergence operator. This section follows [G] closely.

Section 3 treats exact Courant algebroids. We show that any exact Courant algebroid is isomorphic to the generalized tangent bundle $\mathbb{T}M$, equipped with the bracket induced by some closed three-form. We explain the classification of exact Courant algebroids by Ševera [S]. Building on [G], the remaining part of Section 3 is devoted to the curvature of exact Courant algebroids. We compute the Ricci tensors for an arbitrary generalized metric on an exact Courant algebroid for any divergence. We obtain a formula in Proposition 3.10 that differs from the formula in [G, Example 4.9] by a relative factor.

In the subsequent section we perform several computations in orthonormal frames. We start with an arbitrary Courant algebroid and a generalized pseudo-Riemannian metric on it. Furthermore, we assume that there is an orthonormal frame with respect to the generalized metric. One goal is to derive formulas for certain objects in terms of this orthonormal frame. In Section 4.1 we present equations for the Dorfman bracket, such as symmetry properties, the Leibniz rule and the Jacobi identity, with respect to the orthonormal frame. In Section 4.2 we construct a particular generalized Levi-Civita connection only from the orthonormal frame and the Dorfman bracket, by defining in Equation (9) the generalized connection to be a multiple of the Dorfman bracket on the frame. Furthermore, Lemma 4.7 determines the divergence of this generalized Levi-Civita connection. In Section 4.3 we compute the generalized curvature tensors and the generalized Ricci tensors of this connection. In particular this yields a concrete formula for the Ricci tensor of the pair given by the generalized metric and the divergence operator, which is the divergence of this generalized Levi-Civita connection. This raises the question, how the Ricci tensors depend on the choice we made in the orthonormal frame. This question is answered in Section 4.4. We give an explicit formula in Proposition 4.17 for how the Ricci tensors depend on the difference of the divergence operators of two generalized connections that are constructed form two different orthonormal frames.

In the remaining part, which contains our main results, we focus on left-invariant generalized pseudo-Riemannian metrics on Lie groups G. We develop the theory on arbitrary Lie groups in Section 5 and, based on that theory, provide a complete classification of left-invariant solutions of the generalized Einstein equation on three-dimensional Lie groups in Section 6. This is a joint work with Vicente Cortés [CK] and parts of it can be seen as an application of the results of Section 4.

First we show in Proposition 5.5 that, up to an isomorphism, the generalized metric $\mathcal{G}$ and the Courant algebroid structure are encoded in a pair (g, H) consisting of a left-invariant pseudo-Riemannian metric g and a left-invariant closed three-form H on G. Then we describe the space of left-invariant torsion-free and metric generalized connections D on $(G, \mathcal{G}_g, H)$ as a finite-dimensional affine space modeled on the generalized first prolongation of $\mathfrak{so}(\mathfrak{g} \oplus \mathfrak{g}^*)$ in Proposition 5.10, where $\mathcal{G}_g$ denotes the generalized metric determined by g and $\mathfrak{g}$ the Lie algebra of the Lie group G. Such generalized connections D are called left-invariant Levi-Civita generalized connections. As part of the proof, we construct a canonical left-invariant Levi-Civita generalized connection D^0, which can serve as an origin in the above affine space.

A left-invariant divergence operator on $\Gamma(\mathbb{T}G)$ can be identified with an element $\delta \in E^*$, where $E = \mathfrak{g} \oplus \mathfrak{g}^*$. We say that a left-invariant generalized connection D has divergence operator δ if $\delta_D = \delta$, where $\delta_D(v) := \mathrm{tr}(Dv)$, $v \in E$. Here D is identified with an element of $E^* \otimes \mathfrak{so}(E)$, $E \ni u \mapsto D_u \in \mathfrak{so}(E)$. For instance, we have $\delta_{D^0} = 0$ for the canonical left-invariant Levi-Civita generalized connection D^0, compare Proposition 5.17. In Proposition 5.19 we specify for every $\delta \in E^*$ a left-invariant Levi-Civita generalized connection D such that $\delta_D = \delta$. We end Section 5.4 by observing that $\delta = 0$ is not the only canonical choice of left-invariant divergence operator on a Lie group. A more general choice is to take δ as a fixed multiple of the trace-form τ of $\mathfrak{g}$. The choice $\delta^{\mathcal{G}} = -\tau \circ \pi \in E^*$, where $\pi : E \to \mathfrak{g}$ is the canonical projection, corresponds precisely to the divergence operator associated with the generalized connection trivially extending the Levi-Civita connection of any left-invariant pseudo-Riemannian metric, as shown in Proposition 5.20. The latter choice does therefore coincide with what is called the Riemannian divergence operator [GSt].

In Section 5.5 we define the Ricci curvature of any pseudo-Riemannian generalized Lie group $(G, \mathcal{G}_g, H, \delta)$ with prescribed divergence operator $\delta \in E^*$ as a certain element in $E^* \otimes E^*$ (see Definition 5.21). Then we express it in terms of the algebraic data on the Lie algebra $\mathfrak{g}$. The starting point is the computation of the tensorial part of the curvature of the canonical Levi-Civita generalized connection D^0 in Proposition 5.22 as a homogeneous quadratic polynomial expression in the Dorfman bracket $\mathcal{B} = [\cdot, \cdot]_H$. The Ricci curvature of any pseudo-Riemannian generalized Lie group $(G, \mathcal{G}_g, H, \delta = 0)$ with zero divergence operator is then obtained as a Corollary 5.23. These results are then generalized to arbitrary δ by considering $D = D^0 + S$, where S is an arbitrary element of the first generalized prolongation of $\mathfrak{so}(E)$, leading to Lemma 5.26, Proposition 5.27 and Theorem 5.28.

For illustration we give here the explicit expression for the Ricci curvature

$$Ric_\delta \in E_-^* \otimes E_+^* \oplus E_+^* \otimes E_-^*$$

of a pseudo-Riemannian generalized Lie group $(G, \mathcal{G}_g, H, \delta)$, where $E_\pm$ stands for the eigenspaces of the generalized metric. For $u_\pm \in E_\pm$ and using the projections $\mathrm{pr}_{E_\pm} : E \to E_\pm$ we consider the linear maps

$$\Gamma_{u_\pm} := \mathrm{pr}_{E_\pm} \circ \mathcal{B}(u_\pm, \cdot)|_{E_\mp} : E_\mp \to E_\pm.$$

Theorem 1.1. Let $(G, \mathcal{G}_g, H, \delta)$ be any pseudo-Riemannian generalized Lie group. Then its Ricci curvature is given by

$$\begin{aligned} Ric_\delta(u_-, u_+) &= -\operatorname{tr}\left(\Gamma_{u_-} \circ \Gamma_{u_+}\right) + \delta(\mathrm{pr}_{E_+}\mathcal{B}(u_-, u_+)), \\ Ric_\delta(u_+, u_-) &= -\operatorname{tr}\left(\Gamma_{u_-} \circ \Gamma_{u_+}\right) + \delta(\mathrm{pr}_{E_-}\mathcal{B}(u_+, u_-)). \end{aligned}$$

This implies that the tensor Ric_δ is polynomial of degree two and homogeneous in the pair $(\mathcal{B}, \delta)$. Note that it depends on the generalized metric and thus on g through the projections $\mathrm{pr}_{E_\pm}$. An equivalent convenient component expression in an adapted basis is given in Theorem 5.28, where also symmetry properties of Ric_δ are discussed.

To derive an explicit expression for Ric_δ in terms of the data $(\mathfrak{g}, g, H)$ rather than $(\mathfrak{g}, g, \mathcal{B})$ it suffices to express the Dorfman bracket $\mathcal{B}$ in terms of the Lie bracket and the three-form H, see Proposition 5.29. In Proposition 5.30 we show that the underlying metric g of an Einstein generalized pseudo-Riemannian Lie group (i.e. a left-invariant solution of $Ric_\delta = 0$) can be freely rescaled without changing the Einstein property, provided that the three-form and the divergence are appropriately rescaled. In Proposition 5.32 we relate the Ricci curvature Ric_δ of the pseudo-Riemannian generalized Lie group to the Ricci curvature of the left-invariant pseudo-Riemannian metric g. We show that $(G, \mathcal{G}_g, H = 0, \delta = 0)$ is generalized Einstein if and only if g satisfies a certain gradient Ricci soliton equation (31) involving the trace-form τ of $\mathfrak{g}$. In particular, in the special case when $\mathfrak{g}$ is unimodular, the generalized Einstein equation reduces to the Einstein (vacuum) equation for g.

Next we describe how, building on the general results of Section 5, in Section 6 we determine all left-invariant solutions $(H, \mathcal{G}, \delta)$ to the Einstein equation on three-dimensional Lie groups G, up to isomorphism. Here H stands for the three-form which, together with the Lie bracket, determines the exact Courant algebroid structure, $\mathcal{G}$ stands for the generalized pseudo-Riemannian metric and δ for the divergence required to define the Ricci curvature uniquely. The data $(G, H, \mathcal{G}, \delta)$ can be simply referred to as a generalized Einstein Lie group (three-dimensional in our case).

Up to isomorphism, we can assume from the start that $\mathcal{G} = \mathcal{G}_g$ is associated with a left-invariant pseudo-Riemannian metric g on G, compare Proposition 5.5. In the remaining part of the introduction we will therefore simply speak of solutions (H, g, δ) on $\mathfrak{g}$, or more precisely as generalized Einstein structures on $\mathfrak{g}$. In particular, we identify the left-invariant structures (H, g, δ) with tensors

$$H \in \bigwedge^3 \mathfrak{g}^*, \quad g \in \mathrm{Sym}^2\mathfrak{g}^* \quad \text{and} \quad \delta \in E^* = (\mathfrak{g} \oplus \mathfrak{g}^*)^*.$$

As a preliminary, we explain in Section 6.1 how, using the metric g, the Lie bracket of $\mathfrak{g}$ can be encoded in an endomorphism $L \in \mathrm{End}\,\mathfrak{g}$. Irrespective of the signature of g, the endomorphism L happens to be g-symmetric if and only if the Lie algebra is unimodular. This allows for the choice of an orthonormal basis of $(\mathfrak{g}, g)$ in which L takes one of five parameter-dependent normal forms, provided that $\mathfrak{g}$ is unimodular, see Proposition 6.2. Moreover, the Jacobi identity does not impose any constraint on the normal form.

After these preliminaries, we give in Section 6.2 the classification of solutions with zero divergence, that is solutions of the type $(H, g, \delta = 0)$, beginning with the class of unimodular Lie algebras. The final results can be roughly summarized as follows, see Theorem 6.4, Theorem 6.8 and Remark 6.6.

Theorem 1.2. Any **divergence-free** generalized Einstein structure on a three-dimensional **unimodular** Lie algebra is isomorphic to one in the following classes (described explicitly in Theorem 6.4).

1. $\mathfrak{g}$ is abelian and $H = 0$. The metric g is flat of any signature.
2. $\mathfrak{g}$ is simple, $H \neq 0$ and the metric g is of non-zero constant curvature. It is definite if and only if $\mathfrak{g} = \mathfrak{so}(3)$ and indefinite if and only if $\mathfrak{g} = \mathfrak{so}(2,1)$.
3. $H = 0$, g is flat and $\mathfrak{g}$ is one of the following metabelian Lie algebras: $\mathfrak{g} = \mathfrak{e}(2)$ or $\mathfrak{g} = \mathfrak{e}(1,1)$, where $\mathfrak{e}(p,q)$ denotes the Lie algebra of the isometry group of $\mathbb{R}^{p,q}$ (the affine pseudo-orthogonal Lie algebra). The metric is definite on $[\mathfrak{g}, \mathfrak{g}]$ if and only if $\mathfrak{g} = \mathfrak{e}(2)$.
4. $\mathfrak{g} = \mathfrak{heis}$ is the Heisenberg algebra, $H = 0$ and g is flat and indefinite.

We note that the above list of Lie algebras,

$$\mathbb{R}^3, \mathfrak{so}(3), \mathfrak{so}(2,1), \mathfrak{e}(2), \mathfrak{e}(1,1), \mathfrak{heis},$$

is precisely the list of all unimodular three-dimensional Lie algebras.

Theorem 1.3. Any **divergence-free** generalized Einstein structure on a three-dimensional **non-unimodular** Lie algebra is of the type $(H = 0, g)$, where g is indefinite, non-degenerate on the unimodular kernel $\mathfrak{u} = \ker \tau$, $\tau = \operatorname{tr} \circ \operatorname{ad}$, and belongs to a certain one-parameter family of metrics on the metabelian Lie algebra

$$\mathbb{R} \ltimes_A \mathbb{R}^2, \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The family of metrics (described in Theorem 6.8) consists of Ricci solitons which are not of constant curvature.

The classification in the case of non-zero divergence is the content of Section 6.3. The unimodular case is covered in Section 6.3.1, the non-unimodular case in Section 6.3.2. To keep the introduction succinct we do only summarize the isomorphism types of the Lie algebras resulting from our classification without listing the detailed solutions, which can be found in Theorem 6.12, Proposition 6.14 and Proposition 6.15.

Theorem 1.4. Any three-dimensional **unimodular** Lie algebra $\mathfrak{g}$ admits a generalized Einstein structure with **non-zero divergence** as well as a divergence-free solution. (See Theorem 6.12).

Theorem 1.5. Let (H, g, δ) be a generalized Einstein structure with **non-zero divergence** on a three-dimensional **non-unimodular** Lie algebra $\mathfrak{g}$. Then either

1. the unimodular kernel of $\mathfrak{g}$ is non-degenerate (with respect to g) and $\mathfrak{g} = \mathbb{R} \ltimes_A \mathbb{R}^2$ where either

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } H = 0 \quad \text{or} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \ \lambda \in (-1, 1], \text{ and } H \neq 0,$$

   see Proposition 6.14 for a complete description of (H, g, δ), or

2. its unimodular kernel is degenerate, $H = 0$ and $\mathfrak{g} = \mathbb{R} \ltimes_A \mathbb{R}^2$, where $A \in \mathfrak{gl}(2, \mathbb{R})$ is arbitrary with only real eigenvalues and such that $\operatorname{tr} A \neq 0$. (See Proposition 6.15.)

In Proposition 6.16 we indicate for which of the left-invariant generalized Einstein structures the divergence δ coincides with the Riemannian divergence. We find that this is not only the case for all divergence-free solutions on unimodular Lie algebras but also for some of the non-unimodular cases with non-zero divergence. In the latter case, the unimodular kernel can be both degenerate or non-degenerate with respect to the metric g.

For better overview the results of our classification are summarized in the tables of Section 7.

2 Courant algebroids

We begin by recalling the basic definitions and properties of Courant algebroids. After that we will review generalized connections as well as their torsion and divergence. Finally, we introduce generalized metrics and generalized Levi-Civita connections and define some curvature quantities related to them. Further details can for example be found in [CD] and [G]. Another reference, which we use, is [Co].

Definition 2.1. A Courant algebroid over a smooth manifold M is a vector bundle $E \to M$ equipped with the so-called scalar product, a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle \in \Gamma(S^2(E^*))$, the Dorfman bracket, which is a bilinear bracket $[\cdot, \cdot]$ on the smooth sections $\Gamma(E)$, and a homomorphism of vector bundles $\pi : E \to TM$, called the anchor, which satisfy the following conditions for all sections $u, v, w \in \Gamma(E)$ and functions $f \in C^\infty(M)$

(C1) $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$,

(C2) $\pi([u, v]) = [\pi(u), \pi(v)]$, where the second bracket denotes the Lie bracket of vector fields,

(C3) $[u, fv] = \pi(u)(f)v + f[u, v]$,

(C4) $\pi(u)\langle v, w\rangle = \langle [u, v], w\rangle + \langle v, [u, w]\rangle$,

(C5) $\langle [u, v] + [v, u], w\rangle = \pi(w)\langle u, v\rangle$.

In what follows, we denote by $\pi^* : T^*M \to E$ the map which is the composition of the dual map of π followed by the isomorphism between E^* and E induced by the scalar product $\langle \cdot, \cdot \rangle$. Explicitly, it is given by $\langle \pi^*(\xi), v\rangle = \xi(\pi(v))$ for all $\xi \in \Omega^1(M)$ and $v \in \Gamma(E)$.

Lemma 2.2. The axioms (C2) and (C3) can be obtained from the other axioms (compare [CD, U]). Furthermore axiom (C5) is equivalent to

$$2[u, u] = \pi^* d\langle u, u\rangle$$

for all $u \in \Gamma(E)$.

Proof. Using (C1) and (C4) we compute for arbitrary sections $e, u, v, w \in \Gamma(E)$

$$\begin{aligned}
[\pi(u), \pi(v)]\langle w, e\rangle &= \pi(u)\pi(v)\langle w, e\rangle - \pi(v)\pi(u)\langle w, e\rangle \\
&= \pi(u)(\langle [v, w]e, \rangle + \langle w, [v, e]\rangle) - \pi(v)(\langle [u, w], e\rangle + \langle w, [u, e]\rangle) \\
&= \langle [u, [v, w]], e\rangle + \langle [v, w], [u, e]\rangle + \langle [u, w], [v, e]\rangle + \langle w, [u, [v, e]]\rangle \\
&\quad - \langle [v, [u, w]], e\rangle - \langle [u, w], [v, e]\rangle - \langle [v, w], [u, e]\rangle - \langle w, [v, [u, e]]\rangle \\
&= \langle [u, [v, w]], e\rangle - \langle [v, [u, w]], e\rangle + \langle w, [u, [v, e]]\rangle - \langle w, [v, [u, e]]\rangle \\
&= \langle [[u, v], w], e\rangle + \langle w, [[u, v], e]\rangle \\
&= \pi([u, v])\langle w, e\rangle.
\end{aligned}$$

This shows axiom (C2). Axiom (C3) can be shown from (C4) as follows

$$\begin{aligned}\langle [u, fv], w\rangle &= \pi(u)\langle fv, w\rangle - \langle fv, [u,w]\rangle \\ &= \pi(u)(f)\langle v, w\rangle + f\pi(u)\langle v, w\rangle - \langle fv, [u,w]\rangle \\ &= \pi(u)(f)\langle v, w\rangle + f\langle [u,v], w\rangle + f\langle v, [u,w]\rangle - \langle fv, [u,w]\rangle \\ &= \langle \pi(u)(f)v + f[u,v], w\rangle,\end{aligned}$$

where $f \in C^\infty(M)$. Finally note that by (C5) we have

$$\langle 2[u,u], w\rangle = \pi(w)\langle u, u\rangle = (d\langle u,u\rangle)(\pi(w)) = \langle \pi^* d\langle u,u\rangle, w\rangle.$$

The other direction is obtained by polarization of the above calculation. □

In fact the equivalent form of axiom (C5) is sometimes used in the definition of Courant algebroids. The standard example of a Courant algebroid and the basic motivation for the theory is given by the following example.

Example 2.3. On the so-called generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, we have a natural scalar product $\langle X+\xi, Y+\eta\rangle := \frac{1}{2}(\xi(Y)+\eta(X))$, where X and Y are vector fields and ξ and η are one-forms. Projection on the first factor defines a natural anchor map $\pi : \mathbb{T}M \to TM$, and we define the Dorfman bracket on sections $X+\xi, Y+\eta \in \Gamma(\mathbb{T}M)$ by

$$[X+\xi, Y+\eta] := [X,Y] + L_X\eta - L_Y\xi + d(\xi(Y)).$$

The bracket was first given in [D] and (in a skew-symmetric form) in [C].

The following proposition shows that this indeed defines a Courant algebroid.

Proposition 2.4. Let $H \in \Gamma(\Lambda^3 T^*M)$ be a three-form on M. Then the data $\pi(X+\xi) := X$ as the anchor, $\langle X+\xi, Y+\eta\rangle := \frac{1}{2}(\xi(Y)+\eta(X))$ as the scalar product, and

$$[X+\xi, Y+\eta]_H := [X,Y] + L_X\eta - L_Y\xi + d(\xi(Y)) + H(X,Y,\cdot)$$

as the Dorfman bracket defines a structure of a Courant algebroid on $\mathbb{T}M$ if and only if the three-form H is closed. In that case the Courant algebroid is also called the (H-twisted) generalized tangent bundle.

Proof. By Lemma 2.2, it is enough to check axioms (C1), (C4) and (C5) from Definition 2.1. For that, let $u = X+\xi, v = Y+\eta, w = Z+\zeta \in \mathbb{T}M$. First we compute

$$\begin{aligned}2\pi(u)\langle v, w\rangle &= X\left(\zeta(Y)+\eta(Z)\right) = \left(L_X\eta\right)(Z) + \eta([X,Z]) + \left(L_X\zeta\right)(Y) + \zeta([X,Y]), \\ 2\langle [u,v]_H, w\rangle &= 2\langle [X,Y] + L_X\eta - L_Y\xi + d(\xi(Y)) + H(X,Y,\cdot), Z+\zeta\rangle \\ &= \left(L_X\eta\right)(Z) - \left(L_Y\xi\right)(Z) + Z(\xi(Y)) + H(X,Y,Z) + \zeta([X,Y]), \\ 2\langle v, [u,w]_H\rangle &= 2\langle Y+\eta, [X,Z] + L_X\zeta - L_Z\xi + d(\xi(Z)) + H(X,Z,\cdot)\rangle \\ &= \left(L_X\zeta\right)(Y) - \left(L_Z\xi\right)(Y) + Y(\xi(Z)) + H(X,Z,Y) + \eta([X,Z])\end{aligned}$$

Axiom (C4) now follows from

$$-\left(L_Y\xi\right)(Z)+Y(\xi(Z))-\left(L_Z\xi\right)(Y)+Z(\xi(Y))=\xi([Y,Z])+\xi([Z,Y])=0.$$

Furthermore we see from the expression for $2\langle[u,v]_H,w\rangle$ that

$$2\langle[u,v]_H+[v,u]_H,w\rangle=Z\left(\xi(Y)\right)+Z\left(\eta(X)\right)=2w\langle u,v\rangle,$$

which is axiom (C5).

Now we want to show that axiom (C1) holds if and only if the three-form H is closed. For that we show first that (C1) holds for the bracket $[\cdot,\cdot]_0$ that is if $H=0$, as in Example 2.3. Note that (C1) is equivalent to $[u,[v,w]_0]_0-[[u,v]_0,w]_0-[v,[u,w]_0]_0=0$, where

$$\begin{aligned}
[u,[v,w]_0]_0 &= [u,[Y,Z]+L_Y\zeta-L_Z\eta+d\left(\eta\left(Z\right)\right)]_0\\
&= [X,[Y,Z]]+L_X\left(L_Y\zeta-L_Z\eta+d\left(\eta\left(Z\right)\right)\right)-L_{[Y,Z]}\xi+d\left(\xi([Y,Z])\right),\\
[[u,v]_0,w]_0 &= [[X,Y]+L_X\eta-L_Y\xi+d(\xi(Y)),w]_0\\
&= [[X,Y],Z]+L_{[X,Y]}\zeta-L_Z(L_X\eta-L_Y\xi+d(\xi(Y)))\\
&\qquad\qquad +d\left(\left(L_X\eta-L_Y\xi+d(\xi(Y))\right)(Z)\right),\\
[v,[u,w]_0]_0 &= [Y,[X,Z]]+L_Y\left(L_X\zeta-L_Z\xi+d(\xi(Z))\right)-L_{[X,Z]}\eta+d\left(\eta([X,Z])\right).
\end{aligned}$$

The vector part cancels due to the Jacobi identity for the Lie bracket of vector fields and the terms only involving Lie derivatives of one forms cancel because of the compatibility of the Lie derivative on forms and the Lie bracket on vector fields. Therefore

$$\begin{aligned}
&[u,[v,w]_0]_0-[[u,v]_0,w]_0-[v,[u,w]_0]_0\\
=&L_X\left(d\left(\eta\left(Z\right)\right)\right)+d\left(\xi([Y,Z])\right)+L_Z(d(\xi(Y)))-d\left(\left(L_X\eta\right)(Z)\right)+d\left(\left(L_Y\xi\right)(Z)\right)\\
&\quad -d\big(d(\xi(Y))(Z)\big)-L_Y\left(d(\xi(Z))\right)-d\left(\eta([X,Z])\right)\\
=&d\left(\xi([Y,Z])\right)+d(L_Z(\xi(Y)))+d\left(\left(L_Y\xi\right)(Z)\right)-d\big(d(\xi(Y))(Z)\big)-d\left(L_Y(\xi(Z))\right)\\
=&d\left(\left(L_Z\xi\right)(Y)\right)+d\left(\left(L_Y\xi\right)(Z)\right)-d\big(d(\xi(Y))(Z)\big)-d\left(L_Y(\xi(Z))\right)\\
=&d\left(\left(L_Z\xi\right)(Y)\right)+d\left(\left(L_Y\xi\right)(Z)\right)-d\left(\left(L_Z\xi\right)(Y)\right)-d\left(\xi([Z,Y])\right)-d\left(L_Y(\xi(Z))\right)\\
=&d\left(\left(L_Y\xi\right)(Z)+\xi\left(L_YZ\right)-\left(L_Y(\xi(Z))\right)\right)\\
=&0,
\end{aligned}$$

which proves (C1) in the case $H=0$. Let now H be an arbitrary three-form. Then (C1) is equivalent to $[u,[v,w]_H]_H-[[u,v]_H,w]_H-[v,[u,w]_H]_H=0$, where

$$\begin{aligned}
[u,[v,w]_H]_H &= [u,[v,w]_0+H(X,Y,\cdot)]_H\\
&= [u,[v,w]_0]_0+H(X,[Y,Z],\cdot)+L_X(H(Y,Z,\cdot))\\
[[u,v]_H,w]_H &= [[u,v]_0+H(X,Y,\cdot),w]_H\\
&= [[u,v]_0,w]_0+H([X,Y],Z,\cdot)-L_Z(H(X,Y,\cdot))+d(H(X,Y,Z))\\
[v,[u,w]_H]_H &= [v,[u,w]_0]_0+H(Y,[X,Z],\cdot)+L_Y(H(X,Z,\cdot)).
\end{aligned}$$

Using that the Jacobi identity holds for $[\cdot,\cdot]_0$, we see for $e = W + \varphi \in \mathbb{T}M$

$$\begin{aligned}
&\langle [u,[v,w]_H]_H - [[u,v]_H,w]_H - [v,[u,w]_H]_H, e\rangle \\
=&H(X,[Y,Z],W) + L_X(H(Y,Z,\cdot))(W) - H([X,Y],Z,W) + L_Z(H(X,Y,\cdot))(W) \\
&\qquad - d(H(X,Y,Z))(W) - H(Y,[X,Z],W) - L_Y(H(X,Z,\cdot))(W) \\
=&H(X,[Y,Z],W) - H([X,Y],Z,W) - H(Y,[X,Z],W) - WH(X,Y,Z) \\
&\qquad + XH(Y,Z,W) - H(Y,Z,[X,W]) + ZH(X,Y,W) - H(X,Y,[Z,W]) \\
&\qquad\qquad - YH(X,Z,W) + H(X,Z,[Y,W]) \\
=&(dH)(X,Y,Z,W),
\end{aligned}$$

where we have used the invariant formula for the exterior derivative of a three-form in the last line. This proves that (C1) holds if and only if $dH = 0$. □

We now give some properties of the map $\pi^* : T^*M \to E$.

Lemma 2.5. Let E be a Courant algebroid with anchor $\pi : E \to TM$. Then

(i) $\pi^* \circ \pi = 0$,

(ii) $\pi^*(T^*M)$ is an isotropic subbundle, that is $\langle \pi^*(\xi), \pi^*(\eta)\rangle = 0$ for all $\xi, \eta \in \Gamma(T^*M)$,

(iii) $[\pi^*(\xi), \pi^*(\eta)] = 0$ for all $\xi, \eta \in \Gamma(T^*M)$.

Proof. Note first that $T_p^*M = \{d_p f \mid f = \langle v, v\rangle, v \in \Gamma(E)\}$ for any $p \in M$. Therefore it is sufficient to consider a one-form of the form $\xi = df$ for $f = \langle v, v\rangle$ with $v \in \Gamma(E)$. By the equivalent form of axiom (C5) from Lemma 2.2 we have $\pi^*(\xi) = 2[v, v]$. To show that the vector field $\pi(\pi^*(\xi))$ is zero, we evaluate an arbitrary one form $\eta \in \Gamma(T^*M)$ on it and obtain

$$\eta\left(\pi\left(\pi^*(\xi)\right)\right) = \langle \pi^*(\eta), \pi^*(\xi)\rangle = 2\left\langle \pi^*(\eta), [v,v]\right\rangle = 2\eta(\pi([v,v])) = 2\eta([\pi(v),\pi(v)]) = 0.$$

This shows parts (i) and (ii) of the lemma. To prove part (iii) we compute for arbitrary $e \in \Gamma(E)$

$$\begin{aligned}
\left\langle [\pi^*(\xi), \pi^*(\eta)], e\right\rangle =& \pi\left(\pi^*(\xi)\right)\left\langle \pi^*(\eta), e\right\rangle - \left\langle \pi^*(\eta), [\pi^*(\xi), e]\right\rangle \\
=& -\eta\left(\pi\left[\pi^*(\xi), e\right]\right) \\
=& -\eta\left([\pi(\pi^*(\xi)), \pi(e)]\right) \\
=& \ 0.
\end{aligned}$$
□

2.1 Generalized connections

In this and the following sections we will introduce the concept of a generalized connection. Following [G], we will also describe the spaces of connections that are torsion-free, have a fixed divergence, or are compatible with a given generalized metric.

Definition 2.6. [AX] Let E be a Courant algebroid. A generalized connection on E is a linear map

$$D : \Gamma(E) \to \Gamma(E^* \otimes E), \quad v \mapsto Dv = (u \mapsto D_u v),$$

such that

(i) $D_u(fv) = \pi(u)(f)v + fD_u v$ (anchored Leibniz rule), and

(ii) $\pi(u)\langle v, w\rangle = \langle D_u v, w\rangle + \langle v, D_u w\rangle$

for all $u, v, w \in \Gamma(E)$ and $f \in C^\infty(M)$.

Example 2.7. [CD] Let $\nabla^E : \Gamma(E) \to \Gamma(T^*M \otimes E)$ be an orthogonal connection on $(E, \langle\cdot,\cdot\rangle)$, that is $\langle\nabla^E_X u, v\rangle + \langle u, \nabla^E_X v\rangle = X\langle u, v\rangle$ for all $X \in \Gamma(TM)$ and $u, v \in \Gamma(E)$. Then $D_u v \coloneqq \nabla^E_{\pi(u)} v$ defines a generalized connection on E.

Let $D, \tilde{D}$ be two generalized connections on E and $\eta \coloneqq D - \tilde{D}$. Then η is tensorial because of

$$\eta_u fv = D_u fv - \tilde{D}_u fv = \pi(u)(v)v + fD_u v - \pi(u)(f)v - f\tilde{D}_u v = f\eta_u v.$$

Furthermore

$$\begin{aligned}\langle\eta_u v, w\rangle &= \langle D_u v, w\rangle - \langle\tilde{D}_u v, w\rangle \\ &= \pi(u)\langle v, w\rangle - \langle v, D_u w\rangle - \pi(u)\langle v, w\rangle + \langle v, \tilde{D}_u w\rangle \\ &= -\langle v, \eta_u w\rangle\end{aligned}$$

and hence $\eta \in \Gamma(E^* \otimes \mathfrak{so}(E))$. Since for a generalized connection D and $\eta \in \Gamma(E^* \otimes \mathfrak{so}(E))$ the map $D + \eta$ is a generalized connection again, the space of generalized connections $\mathcal{D}$ is an affine space modeled on $\Gamma(E^* \otimes \mathfrak{so}(E))$.

Therefore we can, and will, assume that $rk(E) > 1$. Because otherwise there is only one generalized connection, since $rk(E \otimes \mathfrak{so}(E)) = 0$.

Lemma 2.8. If $rk(E) = 1$ and $e \in \Gamma(E)$ such that $\langle e, e\rangle = \pm 1$, then the unique generalized connection on E is given by

$$D_u fe = \pi(u)(f)e,$$

for any $u \in \Gamma(E)$ and $f \in C^\infty(M)$.

Proof. First note that due to the compatibility of D with the scalar product we have

$$0 = \pi(u)\langle e, e\rangle = \langle D_u e, e\rangle + \langle e, D_u e\rangle = 2\langle D_u e, e\rangle$$

and therefore $D_u e = 0$ for all $u \in \Gamma(E)$. Then for any $f \in C^\infty(M)$

$$D_u fe = \pi(u)(f)e + fD_u e = \pi(u)(f)e. \qquad \square$$

2.2 Torsion

Several equivalent formulations for the torsion have been given in the literature (see [AX, Gu3, G]). We give the formulation as in [CD].

Definition 2.9. The torsion $T_D \in \Gamma(\Lambda^2 E^* \otimes E)$ of a generalized connection D is defined to be

$$T_D(u,v) = D_u v - D_v u - [u,v] + (Du)^* v$$

for all $u, v \in \Gamma(E)$. Here $(Du)^*$ denotes the adjoint of Du with respect to $\langle \cdot, \cdot \rangle$, namely $\langle (Du)^* v, w \rangle = \langle D_w u, v \rangle$.

We will also denote

$$T_D(u,v,w) := \langle T_D(u,v), w \rangle = \langle D_u v - D_v u - [u,v], w \rangle + \langle D_w u, v \rangle.$$

Lemma 2.10. The torsion of a generalized connection defines a skew-symmetric three-form on E, that is $T_D \in \Gamma(\Lambda^3 E^*)$.

Proof. First we show skew-symmetry. Using (C5) and the compatibility of D with the scalar product, we see

$$\begin{aligned} T_D(u,v,w) &= \langle D_u v - D_v u - [u,v], w \rangle + \langle D_w u, v \rangle \\ &= \langle D_u v - D_v u, w \rangle + \langle [v,u], w \rangle - \pi(w)\langle u, v \rangle + \langle D_w u, v \rangle \\ &= \langle -D_v u + D_u v + [v,u], w \rangle - \langle D_w v, u \rangle \\ &= -T_D(v,u,w). \end{aligned}$$

With the help of axiom (C4) and again the compatibility of D with the scalar product, we compute

$$\begin{aligned} T_D(u,v,w) &= \langle D_u v - D_v u - [u,v], w \rangle + \langle D_w u, v \rangle \\ &= \langle D_u v - D_v u, w \rangle + \langle v, [u,w] \rangle - \pi(u)\langle v, w \rangle + \langle D_w u, v \rangle \\ &= -\langle v, D_u w \rangle - \langle D_v u, w \rangle + \langle v, [u,w] \rangle + \langle D_w u, v \rangle \\ &= \langle -D_u w + D_w u + [u,w], v \rangle - \langle D_v u, w \rangle \\ &= -T_D(u,w,v). \end{aligned}$$

Since this proves skew-symmetry, it is enough to check tensoriality only in the third component, which is immediate. ☐

Now, following [G, GSt], we want to describe the space of generalized connections with a given torsion.

Lemma 2.11. Let $T \in \Gamma(\Lambda^3 E^*)$ and consider the set $\mathcal{D}^T := \{D \in \mathcal{D} \mid T_D = T\} \subset \mathcal{D}$ of generalized connections with torsion T. Let $D \in \mathcal{D}^T$ and $\eta \in \Gamma(E^* \otimes \mathfrak{so}(E))$. Then $\tilde{D} = D + \eta$ is in $\mathcal{D}^T$ if and only if

$$\sum_{\mathfrak{S}} \langle \eta_u v, w \rangle = 0.$$

Here $\mathfrak{S}$ indicates the sum over the cyclic permutations of u, v and w.

Proof. We compute

$$\begin{aligned} T_{\tilde{D}}(u,v,w) &= \langle D_u v + \eta_u v - D_v u - \eta_v u - [u,v], w\rangle + \langle D_w u + \eta_w u, v\rangle \\ &= T_D(u,v,w) + \langle \eta_u v, w\rangle - \langle \eta_v u, w\rangle + \langle \eta_w u, v\rangle \\ &= T_D(u,v,w) + \langle \eta_u v, w\rangle + \langle \eta_v w, u\rangle + \langle \eta_w u, v\rangle. \end{aligned}$$

So $T_D = T_{\tilde{D}}$ if and only if

$$\sum_{\mathfrak{S}} \langle \eta_u v, w\rangle = 0,$$

as claimed. □

Hence we can regard $\mathcal{D}^T$ as an affine space modeled on Σ, where $\Sigma \subset \Gamma(E^* \otimes \mathfrak{so}(E))$ is the kernel of the map

$$\partial : \Gamma(E^* \otimes \mathfrak{so}(E)) \to \Gamma(\Lambda^3 E^*)$$

defined by

$$(\partial\eta)(u,v,w) = \sum_{\mathfrak{S}} \langle \eta_u v, w\rangle, \quad u,v,w \in \Gamma(E).$$

Here, $\mathfrak{S}$ again indicates the sum over the cyclic permutations.

Lemma 2.12. There is a natural exact sequence

$$0 \longrightarrow \Gamma(S^3E^*) \xrightarrow{\alpha} \Gamma(S^2E^* \otimes E^*) \xrightarrow{\mathrm{alt}} \Gamma(E^* \otimes \Lambda^2 E^*) \xrightarrow{\partial} \Gamma(\Lambda^3 E^*) \longrightarrow 0$$

where α is the inclusion, alt is skew-symmetrization in the last two arguments and ∂ is the cyclic sum.

Proof. The exactness at $\Gamma(S^3E^*)$ and at $\Gamma(S^2E^* \otimes E^*)$ as well as $\partial \circ \mathrm{alt} = 0$ is immediate. If $\eta \in \Gamma(E^* \otimes \Lambda^2 E^*)$ such that $\partial\eta = 0$, then $\omega \in \Gamma(S^2E^* \otimes E^*)$, defined by $3\omega(u,v,w) := \eta(u,v,w) + \eta(v,u,w)$ for $u,v,w \in \Gamma(E)$, defines a preimage under alt, since

$$\begin{aligned} 3\mathrm{alt}(\omega)(u,v,w) &= 3\omega(u,v,w) - 3\omega(u,w,v) \\ &= \eta(u,v,w) + \eta(v,u,w) - \eta(u,w,v) - \eta(w,u,v) \\ &= \eta(u,v,w) - \eta(v,w,u) + \eta(u,v,w) - \eta(w,u,v) \\ &= 3\eta(u,v,w), \end{aligned}$$

where we used $\partial\eta = 0$ in the last line. This shows exactness at $\Gamma(E^* \otimes \Lambda^2 E^*)$. If $\eta \in \Gamma(E^* \otimes \Lambda^2 E^*)$, then the formula $\partial\eta(u,v,w) = \eta(u,v,w) - \eta(v,u,w) + \eta(w,u,v)$ shows skew-symmetry in the first two as well as the last two arguments, which in turn implies total skew-symmetry of $\partial\eta$. Finally, $\frac{1}{3}\xi$ defines a preimage under ∂ for $\xi \in \Gamma(\Lambda^3 E)$. □

We therefore have

$$\Sigma = \Gamma(\ker \partial) = \Gamma(\operatorname{im} \mathrm{alt}) \cong \Gamma((S^2E \otimes E)/S^3E).$$

Consider now the short exact sequence

$$0 \longrightarrow \ker r \longrightarrow \Gamma(S^2E \otimes E) \xrightarrow{r} \Gamma(E) \longrightarrow 0,$$

where $r((u \odot v) \otimes w) = \langle u, v\rangle w$. The map $\varphi : \Gamma(E) \to \gamma(S^2E \otimes E); e \mapsto \langle \cdot, \cdot\rangle^{-1} \otimes e$, where $\langle \cdot, \cdot\rangle^{-1}$ denotes the corresponding element of $\langle \cdot, \cdot\rangle \in S^2E^*$ in $\Gamma(S^2E)$, is a splitting of the sequence, that is $r \circ \varphi = id_{\Gamma(E)}$. Hence, $\Gamma(S^2E \otimes E) = \ker r \oplus \operatorname{im} \varphi$. Note that $\ker r = \langle (u \odot v) \otimes w : \langle u, v\rangle = 0\rangle =: \Gamma(S^2E \otimes E)_0$. We conclude that we have a direct sum decomposition $\Gamma(S^2E \otimes E) = \Gamma(S^2E \otimes E)_0 \oplus \Gamma(E)$, where we identify $\Gamma(E)$ with its image under φ.

Similarly, we obtain also a splitting of Σ by considering the sequence

$$0 \longrightarrow \ker \bar{r} \longrightarrow \Sigma \xrightarrow{\bar{r}} \Gamma(E) \longrightarrow 0.$$

Here $\bar{r}(\chi) = \frac{1}{rk(E)-1} \sum_{i=1}^{rk(E)} \chi_{e_i} \tilde{e}_i$ for an orthonormal local frame $(e_i)_i$ and sections $(\tilde{e}_i)_i$ of E defined so that $\langle e_i, \tilde{e}_j\rangle = \delta_{ij}$. As a splitting we choose $\bar{\varphi} : \Gamma(E) \to \Sigma; e \mapsto \chi^e$, where $\chi^e_u v = \langle u, v\rangle e - \langle e, v\rangle u$. As above, $\bar{r} \circ \bar{\varphi} = id_E$, and hence $\Sigma = \operatorname{im} \bar{\varphi} \oplus \ker \bar{r}$. We will denote

$$\Sigma_0 := \ker \bar{r} = \{\sigma \in \Sigma : \sum_{i=1}^{rk(E)} \sigma(e_i, \tilde{e}_i, \cdot) = 0\}. \tag{1}$$

We conclude again that we have a direct sum decomposition $\Sigma = \Sigma_0 \oplus \Gamma(E)$, where we identify $\Gamma(E)$ with its image under $\bar{\varphi}$. Here, we need the assumption that $rk(E) > 1$. Otherwise, Σ is trivial and there is at most one element in $\mathcal{D}^T$.

2.3 Divergence

We now introduce operators that will be necessary to define the generalized Ricci curvature invariantly.

Definition 2.13. [G] Let E be a Courant algebroid. A divergence operator on E is a differential operator $\delta : \Gamma(E) \to C^\infty(M)$ satisfying the following (anchored) Leibniz rule

$$\delta(fe) = \pi(e)f + f\delta(e)$$

for all $e \in \Gamma(E)$ and $f \in C^\infty(M)$.

The space of divergence operators is an affine space modeled on $\Gamma(E^*)$. The most important divergence operators are the following.

Definition 2.14. [AX] The divergence operator of a generalized connection D on E is

$$\delta_D(v) := \operatorname{tr} Dv = \operatorname{tr} (\Gamma(E) \ni u \mapsto D_u v \in \Gamma(E)) \in C^\infty(M).$$

Lemma 2.15. Let $T \in \Gamma(\Lambda^3E^*)$ and $\delta \colon \Gamma(E) \to C^\infty(M)$ be a divergence operator on E. Then

$$\mathcal{D}^T(\delta) := \{D \in \mathcal{D} \mid T_D = T,\ \delta_D = \delta\} \subset \mathcal{D}^T$$

is an affine space modeled on Σ_0 (see Equation (1)).

Proof. Given $D', D \in \mathcal{D}^T(\delta) \subset \mathcal{D}^T$, we denote $D' = D + \chi$ with $\chi \in \Sigma$. Then, by the definition of the divergence evaluated at $v \in \Gamma(E)$, we have

$$\delta_{D'}(v) = \delta_D(v) + \sum_{i=1}^{rk(E)} \langle \chi_{e_i} v, \tilde{e}_i \rangle = \delta_D(v) - \sum_{i=1}^{rk(E)} \langle \chi_{e_i} \tilde{e}_i, v \rangle.$$

Now $\delta_{D'} = \delta_D = \delta$ implies $\sum_{i=1}^{rk(E)} \chi_{e_i} \tilde{e}_i = 0$. Writing $\chi = \chi_0 + \chi^e$ in the decomposition $\Sigma = \Sigma_0 \oplus E$ for $e \in \Gamma(E)$, we obtain

$$e = \frac{1}{rk(E) - 1} \sum_{i=1}^{rk(E)} \chi_{e_i} \tilde{e}_i = 0.$$

This implies that $\chi^e = 0$ and hence $\chi \in \Sigma_0$. □

2.4 Generalized metrics and generalized Levi-Civita connections

Definition 2.16. Let E be a Courant algebroid with scalar product $\langle \cdot, \cdot \rangle$. A generalized (pseudo-Riemannian) metric on E is given by one of the following equivalent sets of data:

(i) An orthogonal decomposition $E = E_+ \oplus E_-$ such that the restriction of the scalar product $\langle \cdot, \cdot \rangle$ to E_+ defines a non-degenerate pairing.

(ii) An orthogonal endomorphism $\mathcal{G}^{\text{end}} \in \Gamma(\text{End}(E))$, that satisfies $\left(\mathcal{G}^{\text{end}}\right)^2 = id_E$, defining a non-degenerate, symmetric bilinear pairing $\mathcal{G}(u, v) := \langle \mathcal{G}^{\text{end}} u, v \rangle$ on E.

The pair $(E, \mathcal{G})$ will be called metric Courant algebroid. If $\mathcal{G}$ is positive definite, we omit the prefix pseudo and call it generalized Riemannian metric.

Remark 2.17. The equivalence of (i) and (ii) can be seen as follows. Given a decomposition $E = E_+ \oplus E_-$ as in (i) we obtain $\mathcal{G}^{\text{end}}$ as in (ii) by setting $\mathcal{G}^{\text{end}}|_{E_\pm} = \pm id_{E_\pm}$. Vice versa, an endomorphism as in (ii) defines a decomposition as in (i) by defining $E_\pm$ to be the ± 1-eigenbundles of $\mathcal{G}^{\text{end}}$.

We will denote by v^+ and v^- the orthogonal projection of $v \in E$ onto E_+ and E_-, respectively.

Definition 2.18. A generalized connection D is called compatible with a generalized metric, or just a metric generalized connection, if

$$D(\Gamma(E_\pm)) \subset \Gamma(E^* \otimes E_\pm),$$

or equivalently $D\mathcal{G}^{\text{end}} = \mathcal{G}^{\text{end}} D$, i.e. $D_u \mathcal{G}^{\text{end}} v = \mathcal{G}^{\text{end}} D_u v$ for all $u, v \in \Gamma(E)$.

The restrictions of such a generalized connection to the eigenbundles $E_\pm$ of $\mathcal{G}^{\text{end}}$ define four differential operators

$$D_+^\pm : \Gamma(E_\pm) \to \Gamma(E_+^* \otimes E_\pm), \qquad D_-^\pm : \Gamma(E_\pm) \to \Gamma(E_-^* \otimes E_\pm).$$

The operators $D^\pm_\pm$ are called pure-type operators while the operators $D^\mp_\pm$ are called mixed-type operators of D. Let D and $\tilde{D}$ be two connections compatible with a generalized metric and $\chi := D - \tilde{D} \in \Gamma(E \otimes \mathfrak{so}(E))$. Splitting χ into pure-type and mixed-type operators, we obtain

$$\chi = \chi^+_- + \chi^-_+ + \chi^+_+ + \chi^-_- = (\chi^+_+ + \chi^+_-) + (\chi^-_+ + \chi^-_-) \in \Gamma(E^* \otimes \mathfrak{so}(E_+)) \oplus \Gamma(E^* \otimes \mathfrak{so}(E_-)).$$

Hence, the space of connections compatible with a generalized metric, which will be denoted by $\mathcal{D}(\mathcal{G})$ or $\mathcal{D}(E_+)$, is an affine space modeled on $\Gamma(E^* \otimes \mathfrak{so}(E_+)) \oplus \Gamma(E^* \otimes \mathfrak{so}(E_-))$. As before, this justifies to assume from now on that $rk(E_+) > 1$ and $rk(E_-) > 1$, because otherwise either $\mathfrak{so}(E_+)$ or $\mathfrak{so}(E_-)$ are trivial.

Given a generalized connection D compatible with a generalized metric, its torsion T_D decomposes into eight components with respect to the splitting $E = E_+ \oplus E_-$. If $T_D \in \Gamma(\Lambda^3 E^*_+ \oplus \Lambda^3 E^*_-)$, we say that the torsion is of pure type.

Lemma 2.19. [G] Let $D, \tilde{D} \in \mathcal{D}(\mathcal{G})$ with $T_D = T_{\tilde{D}}$. Then their mixed-type operators coincide, $\tilde{D}^\pm_\mp = D^\pm_\mp$. Furthermore, T_D is of pure type, if and only if $D_{u^\mp} v^\pm = [u^\mp, v^\pm]^\pm$.

Proof. Let $u^- \in \Gamma(E_-)$ and $v^+, w^+ \in \Gamma(E_+)$. Then

$$\langle D_{u^-} v^+ - [u^-, v^+], w^+ \rangle = T_D(u^-, v^+, w^+) = T_{\tilde{D}}(u^-, v^+, w^+) = \langle \tilde{D}_{u^-} v^+ - [u^-, v^+], w^+ \rangle,$$

and hence $D_{u^-} v^+ = \tilde{D}_{u^-} v^+$. If T_D is of pure type, then in particular $0 = T_D(u^-, v^+, w^+)$, implying $D_{u^\mp} v^\pm = [u^\mp, v^\pm]^\pm$. The computation also shows immediately that T_D is of pure types if the mixed type operators of D are as required. □

We will denote the space of torsion-free E_+-compatible connections by $\mathcal{D}^0(E_+) := \mathcal{D}(E_+) \cap \mathcal{D}^0$ or alternatively by $\mathcal{D}^0(\mathcal{G}) := \mathcal{D}(\mathcal{G}) \cap \mathcal{D}^0$. Elements of this space will be called generalized Levi-Civita connections. To show that this space is always non-empty, we first prove the following lemma.

Lemma 2.20. Let D be a generalized metric connection with pure-type torsion. Then the connection

$$D^0 = D - \frac{1}{3} T_D$$

is a generalized Levi-Civita connection.

Proof. The generalized connection D^0 is compatible with the generalized metric, since we modified the metric generalized connection D by an element of $\Gamma(\Lambda^3 E^*_+ \oplus \Lambda^3 E^*_-) \subset \Gamma(E^* \otimes \mathfrak{so}(E_+)) \oplus \Gamma(E^* \otimes \mathfrak{so}(E_-))$. Its torsion is

$$\begin{aligned} T_{D^0}(u, v, w) &= \langle D^0_u v - D^0_v u - [u, v], w \rangle + \langle D^0_w u, v \rangle \\ &= T_D(u, v, w) - \frac{1}{3} T_D(u, v, w) + \frac{1}{3} T_D(v, u, w) - \frac{1}{3} T_D(w, u, v) \\ &= 0. \end{aligned}$$

□

Proposition 2.21. [G] Let E_+ be a generalized metric on a Courant algebroid E. Then there exists a generalized Levi-Civita connection for E.

Proof. First, we construct a generalized connection which is compatible with E_+ and has torsion of pure type. Its mixed-type operators are determined by Lemma 2.19. To define the pure type operators, we choose metric connections $\nabla^\pm$ on $E_\pm$ and set

$$D_{u^\pm} v^\pm = \nabla^\pm_{\pi(u^\pm)} v^\pm.$$

Applying now the previous Lemma proves the claim. □

We now will explore the space of generalized Levi-Civita connections with given divergence. Let $D, \tilde{D}$ be two such connections and $\eta = D - \tilde{D}$. Since the mixed type operators are fixed, we can write $\eta = \eta^+_+ + \eta^-_-$ with $\eta^\pm_\pm \in \Sigma^\pm = \Gamma(E_\pm^{\otimes 3}) \cap \Sigma$ (see Section 2.2). As in Section 2.2 we can split $\Sigma^\pm = \Sigma^\pm_0 \oplus \Gamma(E_\pm)$. More precisely, let (e^+_i) be an orthogonal frame for E_+ and $(\tilde{e}^+_i)$ such that $\langle e^+_i, \tilde{e}^+_j \rangle = \delta_{ij}$. If χ^+ is an element of Σ^+ we set $e^+ = \frac{1}{rk(E_+)-1} \sum_{i=1}^{rk(E_+)} \chi_{e^+_i} \tilde{e}^+_i$ and $(\chi^{e^+}_+)_{u^+} v^+ = \langle u^+, v^+ \rangle e^+ - \langle e^+, v^+ \rangle u^+$. Then $\chi^+_0 \coloneqq \chi^+ - \chi^{e^+}_+ \in \Sigma^+_0$, since it satisfies $\sum_{i=1}^{rk(E_+)} (\chi^+_0)_{e^+_i} \tilde{e}^+_i = 0$ and similarly for E_-. Here we needed the assumption that $rk(E_+) > 1$ and $rk(E_-) > 1$.

Summarizing, we have the following spaces of connections.

- Two elements of $\mathcal{D} = \{D \mid D \text{ is a generalized connection on E}\}$ differ by an element of $\Gamma(E^* \otimes \mathfrak{so}(E^*))$.
- Let $T \in \Gamma(\Lambda^3 E^*)$. Then two elements of $\mathcal{D}^T = \{D \in \mathcal{D} \mid T_D = D\}$ differ by an element of
$$\Sigma = \left\{ \sigma \in \Gamma(E^* \otimes \mathfrak{so}(E^*)) \;\middle|\; \sum_{\mathfrak{S}} \sigma(u, v, w) = 0 \right\}.$$
Here, $\mathfrak{S}$ again indicates the sum over the cyclic permutations.
- Let δ be a divergence operator. Then two elements of $\mathcal{D}^T(\delta) = \left\{ D \in \mathcal{D}^T \;\middle|\; \delta_D = \delta \right\}$ differ by an element of
$$\Sigma_0 = \left\{ \sigma \in \Sigma \;\middle|\; \sum_{i=1}^{rk(E)} \sigma(e_i, \tilde{e}_i, \cdot) = 0 \text{ for any orthonormal frame } (e_i)_i \right\}.$$
- Let $\mathcal{G}$ be a generalized metric on E and assume that T is of pure type. Then two elements of $\mathcal{D}^T(\mathcal{G}) = \left\{ D \in \mathcal{D}^T \;\middle|\; D \text{ is compatible with } \mathcal{G} \right\}$ differ by an element of $\Sigma^+ \oplus \Sigma^-$, where $\Sigma^\pm = \Sigma \cap \Gamma(E^*_\pm)^{\otimes 3}$.
- Two elements of $\mathcal{D}^T(\mathcal{G}, \delta) = \left\{ D \in \mathcal{D}^T(\mathcal{G}) \;\middle|\; \delta_D = \delta \right\}$ differ by an element of $\Sigma^+_0 \oplus \Sigma^-_0$, where $\Sigma^\pm_0 = \Sigma_0 \cap \Gamma(E^*_\pm)^{\otimes 3}$.

It was shown in [G] that all these spaces are non-empty.

Remark 2.22. The spaces of torsion-free generalized connections, compatible with a geometric structure that is given by a reduction of the structure group of $(E, \langle \cdot, \cdot \rangle)$, can also be described by the generalized first prolongation, defined in [CD] (see also Section 5.3). They are non-empty, if their so-called intrinsic torsion (also defined in [CD]) is zero.

2.5 Curvature of Courant algebroids

Definition 2.23. [G] Let $(E, \mathcal{G})$ be a metric Courant algebroid and D be a generalized connection on E compatible with $\mathcal{G}$. We define two curvature operators $R_D^{\pm} \in \Gamma(E_{\pm}^* \otimes E_{\mp}^* \otimes \mathfrak{so}(E_{\pm}))$, which we will sometimes call generalized curvature tensors, by

$$R_D^{\pm}(u, v)w = D_u D_v w - D_v D_u w - D_{[u,v]} w,$$

where $(u, v, w) \in \Gamma(E_{\pm}) \times \Gamma(E_{\mp}) \times \Gamma(E_{\pm})$.

Proposition 2.24. The curvature operators $R_D^{\pm}$ of a generalized metric connection are indeed sections of $E_{\pm}^* \otimes E_{\mp}^* \otimes \mathfrak{so}(E_{\pm})$. Furthermore, if D is torsion-free, we have

$$R_D^{\pm}(u, v)w = D_u D_v w - D_v D_u w - D_{D_u v} w + D_{D_v u} w$$

for all $(u, v, w) \in \Gamma(E_{\pm}) \times \Gamma(E_{\mp}) \times \Gamma(E_{\pm})$.

Proof. We first show tensoriality. Let u, v, w as above and $f \in C^{\infty}(M)$, then

$$\begin{aligned}
R_D^{\pm}(fu, v)w &= D_{fu} D_v w - D_v D_{fu} w - D_{[fu,v]} w \\
&= f D_u D_v w - D_v f D_u w + D_{[v,fu]} w \\
&= f D_u D_v w - \pi(v)(f) D_u w - f D_v D_u w + D_{\pi(v)(f)u} w + D_{f[v,u]} w \\
&= f D_u D_v w - \pi(v)(f) D_u w - f D_v D_u w + \pi(v)(f) D_u w - f D_{[u,v]} w \\
&= f R_D^{\pm}(u, v)w,
\end{aligned}$$

where in the second line we have used that u and v are orthogonal to each other and therefore $[u, v] = -[v, u]$. Similarly, one shows tensoriality in the second argument. Furthermore

$$\begin{aligned}
R_D^{\pm}(u, v) f w &= D_u D_v f w - D_v D_u f w - D_{[u,v]} f w \\
&= D_u (\pi(v)(f) w) + D_u f D_v w - D_v (\pi(u)(f) w) - D_v f D_u w \\
&\qquad - \pi([u, v])(f) w - f D_{[u,v]} w \\
&= \pi(u)(\pi(v)(f)) + \pi(v)(f) D_u w + \pi(u)(f) D_v w + f D_u D_v w \\
&\qquad - \pi(v)(\pi(u)(f)) - \pi(u)(f) D_v w - \pi(v)(f) D_u w - f D_v D_u w \\
&\qquad - [\pi(u), \pi(v)](f) w - f D_{[u,v]} w \\
&= f R_D^{\pm}(u, v) w.
\end{aligned}$$

Lastly we show $R_D^{\pm} \in \Gamma(E_{\pm}^* \otimes E_{\mp}^* \otimes \mathfrak{so}(E_{\pm}))$ by

$$\begin{aligned}
\left\langle R_D^{\pm}(u, v)w, e \right\rangle &= \langle D_u D_v w, e \rangle - \langle D_v D_u w, e \rangle - \left\langle D_{[u,v]} w, e \right\rangle \\
&= \pi(u) \langle D_v w, e \rangle - \langle D_v w, D_u e \rangle - \pi(v) \langle D_u w, e \rangle + \langle D_u w, D_v e \rangle \\
&\qquad - \pi([u, v]) \langle w, e \rangle + \left\langle w, D_{[u,v]} e \right\rangle \\
&= \pi(u) (\pi(v) \langle w, e \rangle) - \pi(u) \langle w, D_v e \rangle - \pi(v) \langle w, D_u e \rangle + \langle w, D_v D_u w \rangle
\end{aligned}$$

$$
\begin{aligned}
&- \pi(v) \left(\pi(u)\langle w, e\rangle\right) + \pi(v) \langle w, D_u e\rangle + \pi(u) \langle w, D_v e\rangle - \langle w, D_u D_v w\rangle \\
&\qquad - [\pi(u), \pi(v)] \langle w, e\rangle + \left\langle w, D_{[u,v]} e\right\rangle \\
&= \left\langle w, R^{\pm}_D(u, v)e\right\rangle ,
\end{aligned}
$$

where $e \in \Gamma(E_\pm)$. The formula for torsion-free generalized connections is true because $(Du)^* v = 0$ for $u, v \in \Gamma(E)$ orthogonal to each other. □

The generalized curvature is not an invariant of a metric Courant algebroid, but depends on the choice of generalized Levi-Civita connection (see also Section 3.2). Next, we will introduce curvature quantities, which do not depend on this choice after fixing the divergence.

Definition 2.25. Let D be a generalized connection on a metric Courant algebroid $(E, \mathcal{G})$ compatible with $\mathcal{G}$. Its Ricci tensors $Ric^{\pm}_D \in \Gamma(E^*_\mp \otimes E^*_\pm)$ are defined by

$$
Ric^{\pm}_D(v, w) = \mathrm{tr}(E_\pm \ni u \mapsto R^{\pm}_D(u, v)w \in E_\pm),
$$

for $(u, v, w) \in \Gamma(E_\pm) \times \Gamma(E_\mp) \times \Gamma(E_\pm)$.

The Ricci tensors have the following important property proved in [G].

Proposition 2.26. If D and D' are generalized connections compatible with the generalized metric with the same divergence and one of them is torsion-free and the other one has pure-type torsion, then their Ricci tensors coincide.

Corollary 2.27. Two generalized Levi-Civita connections with same divergence have the same Ricci tensors.

This allows us to meaningfully define the Ricci tensors of a generalized metric together with a divergence operator.

Definition 2.28. [G] We define the Ricci tensors of $(E, \mathcal{G}, \delta)$ as the Ricci tensors of any torsion-free generalized connection compatible with $\mathcal{G}$ and with divergence δ.

3 Exact Courant algebroids

We have seen the (twisted) generalized tangent bundle as the standard example of Courant algebroids (Proposition 2.4). They fall into the following class.

Definition 3.1. A Courant algebroid E is called exact if the sequence

$$0 \to T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \to 0 \tag{2}$$

is an exact sequence of vector bundles.

Note that by Lemma 2.5(i) we always have $\mathrm{im}(\pi^*) \subseteq \ker(\pi)$, hence the sequence (2) always defines a complex. Therefore a Courant algebroid is exact if and only if the anchor is surjective and $\ker(\pi) \subseteq \mathrm{im}(\pi^*)$.

3.1 Classification of exact Courant algebroids

In this section we show that any exact Courant algebroid is actually isomorphic to the Courant algebroid $\mathbb{T}M$ from Proposition 2.4. A consequence of this is that the isomorphism classes of exact Courant algebroids are classified by the third de Rham cohomology $H^3_{dR}(M)$ of M ([GSt],[S]).

Proposition 3.2. For any exact Courant algebroid $(E, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \pi)$ there exists an isotropic splitting $s : TM \to E$ of the exact sequence (2). The map

$$\begin{aligned} \Phi : TM \oplus T^*M &\to E \\ X + \xi &\mapsto s(X) + \frac{1}{2}\pi^*(\xi), \end{aligned}$$

defines an isomorphism of Courant algebroids between E and the Courant algebroid $\mathbb{T}M$ from Proposition 2.4, where the Dorfman bracket on it is induced by the three-form $H(X, Y, Z) \coloneqq 2\langle [s(X), s(Y)], s(Z)\rangle$.

Proof. First we show that an isotropic splitting always exists. Consider the isotropic subbundle $\ker\pi = \mathrm{im}\,\pi^*$ of E (see Lemma 2.5(ii)) and let F_0 be a complementary subbundle to it. The anchor π maps it isomorphically to TM. We denote its inverse by $s_0 : TM \to F_0$ and define another splitting $s : TM \to E$ of the exact sequence by

$$\langle s(X), v\rangle \coloneqq \langle s_0(X), v\rangle - \frac{1}{2}\langle s_0(X), s_0(\pi(v))\rangle,$$

for $X \in \Gamma(TM)$ and $v \in \Gamma(E)$. To check that it indeed defines a splitting of the exact sequence let $\xi \in \Gamma(T^*M)$ arbitrary. Then

$$\begin{aligned} \xi(\pi(s(X))) &= \langle \pi^*(\xi), s(X)\rangle \\ &= \langle \pi^*(\xi), s_0(X)\rangle - \frac{1}{2}\langle s_0(\pi(\pi^*(\xi))), s_0(X)\rangle \\ &= \langle \pi^*(\xi), s_0(X)\rangle \end{aligned}$$

$$
\begin{aligned}
&= \xi(\pi(s_0(X))) \\
&= \xi(X).
\end{aligned}
$$

Furthermore, the splitting s is an isotropic splitting, since

$$
\begin{aligned}
\langle s(X), s(Y)\rangle &= \langle s_0(X), s(Y)\rangle - \frac{1}{2}\langle s_0(X), s_0(\pi(s(Y)))\rangle \\
&= \langle s_0(X), s_0(Y)\rangle - \frac{1}{2}\langle s_0(\pi(s_0(X))), s_0(Y)\rangle - \frac{1}{2}\langle s_0(X), s_0(\pi(s(Y)))\rangle \\
&= 0,
\end{aligned}
$$

because $\pi \circ s = \pi \circ s_0 = id_{TM}$. Setting $F \coloneqq s(TM)$, we see that the map

$$
\begin{aligned}
\Phi : TM \oplus T^*M &\to F \oplus \ker\pi = E \\
X + \xi &\mapsto s(X) + \frac{1}{2}\pi^*(\xi)
\end{aligned}
$$

is an isomorphism of vector bundles. We want to show now that it also preserves the Courant algebroid structures, where $\mathbb{T}M$ is equipped with the bracket induced by $H(X,Y,Z) = 2\langle [s(X), s(Y)], s(Z)\rangle$. We denote the scalar product and the anchor of $\mathbb{T}M$ by $\langle \cdot,\cdot\rangle_{\mathbb{T}M}$ and $\pi_{\mathbb{T}M}$, respectively. First note that

$$
\pi(\Phi(X+\xi)) = \pi(s(X)) + \frac{1}{2}\pi(\pi^*(\xi)) = X = \pi_{\mathbb{T}M}(X+\xi)
$$

for all $X + \xi \in \mathbb{T}M$. Furthermore

$$
\begin{aligned}
&\langle \Phi(X+\xi), \Phi(Y+\eta)\rangle \\
=&\langle s(X), s(Y)\rangle + \frac{1}{2}\langle s(X), \pi^*\eta\rangle + \frac{1}{2}\langle \pi^*(X), s(Y)\rangle + \frac{1}{4}\langle \pi^*(X), \pi^*(Y)\rangle \\
=&\frac{1}{2}\left(\langle s(X), \pi^*\eta\rangle + \langle \pi^*(X), s(Y)\rangle\right) \\
=&\frac{1}{2}\left(\eta(X) + \xi(Y)\right) \\
=&\langle X+\xi, Y+\eta\rangle_{\mathbb{T}M}\,.
\end{aligned}
$$

In order to show that Φ is compatible with the Dorfman brackets, we evaluate it with respect to the splitting into vector fields and one-forms.

Let $\xi, \eta \in \Gamma(T^*M)$. Then

$$
[\Phi(\xi), \Phi(\eta)] = \frac{1}{4}[\pi^*(\xi), \pi^*(\eta)] = 0 = \Phi([\xi,\eta]_H),
$$

due to Lemma 2.5(iii), where $[\cdot,\cdot]_H$ is the bracket on $\mathbb{T}M$ induced by H (see Proposition 2.4).

Let now $X \in \Gamma(TM)$ and $\eta \in \Gamma(T^*M)$. We have

$$
\pi([\Phi(X), \Phi(\eta)]) = \frac{1}{2}\pi([s(X), \pi^*(\eta)]) = [X, \pi(\pi^*(0))] = 0,
$$

hence $[\Phi(X), \Phi(\eta)]$ lies in the isotropic subbundle $\ker \pi$. Therefore it suffices to compute the scalar product with $s(Y)$, $Y \in \Gamma(TM)$ arbitrary,

$$\begin{aligned}
\langle [\Phi(X), \Phi(\eta)], s(Y) \rangle &= \pi(s(X)) \langle \pi^*(\eta), s(Y) \rangle - \langle \pi^*(\eta), [s(X), s(Y)] \rangle \\
&= X(\eta(\pi(s(Y)))) - \eta(\pi([s(X), s(Y)])) \\
&= X(\eta(Y)) - \eta[X, Y] \\
&= (L_X \eta)(Y) \\
&= 2 \langle [X, \eta]_H, Y \rangle_{\mathbb{T}M} \\
&= 2 \langle \Phi([X, \eta]_H), \Phi(Y) \rangle_{\mathbb{T}M} \\
&= 2 \langle \Phi([X, \eta]_H), s(Y) \rangle_{\mathbb{T}M} .
\end{aligned}$$

This shows $[\Phi(X), \Phi(\eta)] = \Phi([X, \eta]_H)$.

Exchanging the order of vector field and one-form, $\xi \in \Gamma(T^*M)$, $Y \in \Gamma(TM)$, we see as before that $[\Phi(\xi), \Phi(Y)] = \frac{1}{2}[\pi^*(\xi), s(Y)]$ is a section of $\ker \pi$ and it suffices again to compute the scalar product with $s(X)$, $X \in \Gamma(TM)$ arbitrary,

$$\begin{aligned}
\langle [\pi^*(\xi), s(Y)], s(X) \rangle &= \pi(s(X)) \langle \pi^*(\xi), s(Y) \rangle - \langle [s(Y), \pi^*(\xi)], s(X))] \rangle \\
&= X \langle \pi^*(\xi), s(Y) \rangle - \pi(s(Y)) \langle \pi^*(\xi), s(X) \rangle + \langle \pi^*(\xi), [s(Y), s(X)] \rangle \\
&= X(\xi(Y)) - Y \langle \pi^*(\xi), s(X) \rangle + \xi(\pi([s(Y), s(X)])) \\
&= X(\xi(Y)) - Y(\xi(X)) + \xi([Y, X]) \\
&= d(\xi(Y))(X) - (L_Y \xi)(X) \\
&= 2 \langle -L_Y + d(\xi(Y)), X \rangle_{\mathbb{T}M} \\
&= 2 \langle [\xi, Y]_H, X \rangle_{\mathbb{T}M} \\
&= 2 \langle \Phi([\xi, Y]_H), \Phi(X) \rangle \\
&= \langle \Phi([\xi, Y]_H), s(X) \rangle .
\end{aligned}$$

Therefore $[\Phi(\xi), \Phi(Y)] = \Phi([\xi, Y]_H)$.

Finally, we consider two vector fields $X, Y \in \Gamma(TM)$ and show $[\Phi(X), \Phi(Y)] = [s(X), s(Y)]$ by pairing it with $s(Z) + \frac{1}{2}\pi^*(\zeta)$ for $Z \in \Gamma(TM), \zeta \in \Gamma(T^*M)$ arbitrary.

$$\begin{aligned}
\left\langle [s(X), s(Y)], s(Z) + \frac{1}{2}\pi^*(\zeta) \right\rangle &= \frac{1}{2}\zeta(\pi([s(X), s(Y)])) + \langle [s(X), s(Y)], s(Z) \rangle \\
&= \frac{1}{2}\zeta([X, Y]) + \frac{1}{2}H(X, Y, Z) \\
&= \langle [X, Y], \zeta \rangle_{\mathbb{T}M} + \langle H(X, Y, \cdot), Z \rangle_{\mathbb{T}M} \\
&= \langle [X, Y] + H(X, Y, \cdot), Z + \zeta \rangle_{\mathbb{T}M} \\
&= \langle [X, Y]_H, Z + \zeta \rangle_{\mathbb{T}M} \\
&= \langle \Phi([X, Y]_H), \Phi(Z + \zeta) \rangle \\
&= \left\langle \Phi([X, Y]_H), s(Z) + \frac{1}{2}\pi^*(\zeta) \right\rangle ,
\end{aligned}$$

which shows $[\Phi(X), \Phi(Y)] = \Phi([X, Y]_H)$.

It remains to prove that $H(X,Y,Z) = 2\langle [s(X), s(Y)], s(Z)\rangle$ indeed is a closed three-form. Skew-symmetry follows from

$$\begin{aligned}
\frac{1}{2}H(Y,X,Z) &= \langle [s(Y), s(X)], s(Z)\rangle \\
&= \pi(s(Z))\,\langle s(Y), s(X)\rangle - \langle [s(X), s(Y)], s(Z)\rangle \\
&= -\,\langle [s(X), s(Y)], s(Z)\rangle \\
&= -\frac{1}{2}H(X,Y,Z) \\
\frac{1}{2}H(X,Z,Y) &= \langle [s(X), s(Z)], s(Y)\rangle \\
&= \pi(s(X))\,\langle s(Z), s(Y)\rangle - \langle s(Z), [s(X), s(Y)]\rangle \\
&= -\,\langle s(Z), [s(X), s(Y)]\rangle \\
&= -\frac{1}{2}H(X,Y,Z),
\end{aligned}$$

for $X, Y, Z \in \Gamma(TM)$. Since H is apparently tensorial in the last argument, skew-symmetry yields tensoriality in the other arguments as well. The three-form is closed since the bracket $[\cdot,\cdot]_H$ defines a Courant algebroid, because Φ is an isomorphism with another Courant algebroid that preserves the bracket. Therefore H must be closed due to Proposition 2.4. □

We will now sketch how this yields a classification of exact Courant algebroids. We have seen that an exact Courant algebroid E together with an isotropic splitting s a splitting of the corresponding sequence (2) defines a closed three-form $H_{E,s}$, which defines a de Rham cohomology class $[H_{E,s}]$. In fact, Ševera [S] showed that the class only depends on the isomorphism class of E. Indeed, if E and $\tilde{E}$ are isomorphic exact Courant algebroids and s an isotropic splitting of the corresponding short exact sequence foe E, then the isotropic splitting $\tilde{s}$ of the sequence for $\tilde{E}$, defined by composing s with the isomorphism between E and $\tilde{E}$, defines the same three-form as the splitting s,

$$H_{\tilde{E},\tilde{s}} = 2\langle [\tilde{s}(X), \tilde{s}(Y)], \tilde{s}(Z)\rangle_{\tilde{E}} = 2\langle [s(X), s(Y)], s(Z)\rangle_E = H_{E,s}.$$

If on the other hand s and $\tilde{s}$ are two isotropic splittings of the same exact Courant algebroid E, then the corresponding three-forms differ by the exact three-form dB, where the two-form $B \in \Omega^2(M)$ is defined by $B(X,Y) = \langle s(X), \tilde{s}(Y)\rangle_E$, $X, Y \in \Gamma(TM)$. Therefore $[H_{E,s}]$ only depends on the isomorphism class of E.

Given now a de Rham cohomology class $[H] \in H_{dR}(M)$, we assign to it the isomorphism class of the Courant algebroid $(\mathbb{T}M, [\cdot,\cdot]_H)$. Since the map $X + \xi \mapsto X + B(X,\cdot) + \xi$, defines an isomorphism between the Courant algebroids $(\mathbb{T}M, [\cdot,\cdot]_H)$ and $(\mathbb{T}M, [\cdot,\cdot]_{H+dB})$, for any two-form $B \in \Omega^2(M)$, this does not depend on the choice of representative in $[H]$. Clearly, the assignments $E \mapsto [H_{E,s}]$ and $[H] \mapsto (\mathbb{T}M, [\cdot,\cdot]_H)$ are inverse to each other. This shows the following statement.

Proposition 3.3. [S] There is a bijection between the set of isomorphism classes of exact Courant algebroids and the de Rham cohomology $H^3_{dR}(M)$. The de Rham cohomology class corresponding to a given exact Courant algebroid is called its Ševera class.

3.2 Curvature of exact Courant algebroids

Let E be an exact Courant algebroid of rank $2n$, $n > 1$.[2] Let $E_+ \subset E$ be a positive definite generalized metric and δ an arbitrary divergence operator on E. In this section, following [G], we will construct a generalized Levi-Civita connection with the given divergence and with that compute the Ricci tensor of (E_+, δ).

The restrictions of the anchor π to $E_\pm$ are isomorphisms with TM. Indeed, $E_\pm$ and TM have both rank n, and $\ker(\pi|_{E_\pm}) = \ker \pi \cap E_\pm = 0$, since $\ker \pi$ is isotropic and $E_\pm$ are definite with respect to $\langle \cdot, \cdot \rangle$. Therefore the inverse maps of $\pi|_{E_\pm}$ define splittings $s_\pm$ of the exact sequence

$$0 \to T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \to 0$$

Their average $s := \frac{1}{2}(s_+ + s_-)$ defines an isotropic splitting of the sequence above. We define the closed three-form

$$H(X,Y,Z) := 2\langle [s(X), s(Y)], s(Z) \rangle \tag{3}$$

and a Riemannian metric

$$g(X,Y) := \langle s_+(X), s_+(Y) \rangle = -\langle s_-(X), s_-(Y) \rangle \tag{4}$$

on M. Note that the map

$$\phi : \mathbb{T}M \to E; \quad X + \xi \mapsto sX + \frac{1}{2}\pi^*\xi$$

defines an isomorphism between E and the generalized tangent bundle $\mathbb{T}M$, equipped with the bracket induced by the closed three-form H (see Proposition 3.2), such that the generalized metric E_+ gets transformed to $\{X + gX : X \in TM\} \subset \mathbb{T}M$. Indeed one can easily check that $\phi(\{X + gX : X \in TM\}) \subset E$ is the orthogonal complement of $E_- \subset E$ and therefore equal to E_+

$$\begin{aligned}
\langle \phi(X + gX), s_-(Y) \rangle &= \left\langle s(X) + \frac{1}{2}\pi^*(gX), s_-(Y) \right\rangle \\
&= \langle s(X), s_-(Y) \rangle + \frac{1}{2} g(X,Y) \\
&= \frac{1}{2}\langle s-(X), s_-(Y) \rangle + \frac{1}{2} g(X,Y) \\
&= -\frac{1}{2} g(X,Y) + \frac{1}{2} g(X,Y) \\
&= 0,
\end{aligned}$$

for any $X, Y \in TM$. The claim follows from dimensional reasons.

We set

$$\nabla^\pm := \nabla^g \pm \frac{1}{2} g^{-1} H,$$

where ∇^g denotes the Levi-Civita connection of g. They are sometimes called Bismut connections, introduced in [B].

[2] By Proposition 3.2, this is equivalent to $\dim(M) > 1$.

Lemma 3.4. $\nabla^{\pm}_X Y = \pi([s_{\mp}(X), s_{\pm}(Y)]^{\pm}) = 4\pi([s(X)^{\mp}, s(Y)^{\pm}]^{\pm})$

Proof. The first equality follows, because both expressions define metric connections on TM with skew torsion

$$g(T_{\nabla^{\pm}}(X,Y),Z) = \pm H(X,Y,Z).$$

Indeed, metricity of the first expression follows from

$$\begin{aligned}
&g(\nabla^{\pm}_X Y, Z) + g(Y, \nabla^{\pm}_X Z)\\
&\qquad = g(\nabla^g_X Y, Z) + g(Y, \nabla^g_X Z) \pm \frac{1}{2}(g(g^{-1}(H(X,Y,\cdot), Z) + g(Y, g^{-1}H(X,Z,\cdot))\\
&\qquad = X(g(Y,Z)) \pm (H(X,Y,Z) + H(X,Z,Y))\\
&\qquad = X(g(Y,Z)).
\end{aligned}$$

Its torsion computes as follows

$$\begin{aligned}
g(T_{\nabla^{\pm}}(X,Y),Z) &= g(\nabla^{\pm}_X Y - \nabla^{\pm}_Y X - [X,Y], Z)\\
&= g(\nabla^g_X Y \pm \frac{1}{2} g^{-1}\iota_Y\iota_X H - \nabla^g_Y X \mp \frac{1}{2} g^{-1}\iota_X\iota_Y H - [X,Y], Z)\\
&= g(\pm g^{-1}\iota_Y\iota_X H, Z)\\
&= \pm H(X,Y,Z).
\end{aligned}$$

The second expression is metric, because

$$\begin{aligned}
&g(\pi([s_-(X), s_+(Y)]^+), Z) + g(Y, \pi([s_-(x), s_+(Z)]^+))\\
=&\langle [s_-(X), s_+(Y)]^+, s_+(Z)\rangle + \langle s_+(Y), [s_-(x), s_+(Z)]^+\rangle\\
=&\langle [s_-(X), s_+(Y)], s_+(Z)\rangle + \langle s_+(Y), [s_-(x), s_+(Z)]\rangle\\
=&\pi(s_-(X))\langle s_+(Y), s_+(Z)\rangle\\
=&Xg(Y,Z).
\end{aligned}$$

To compute the torsion, first note that $[s(X), s(Y)] - s[X,Y] = \frac{1}{2}\pi^*(\iota_Y\iota_X H)$, since both sides of the equation are in the kernel of π, which is isotropic, and also

$$\begin{aligned}
\langle [s(X), s(Y)] - s[X,Y], s(Z)\rangle &= \langle [s(X), s(Y)], s(Z)\rangle\\
&= \frac{1}{2}H(X,Y,Z)\\
&= \langle \frac{1}{2}\pi^*(\iota_Y\iota_X H), s(Z)\rangle.
\end{aligned}$$

for all $Z \in \Gamma(TM)$. Hence

$$\begin{aligned}
&g(\pi([s_-(X), s_+(Y)]^+ - [s_-(Y), s_+(X)]^+) - [X,Y], Z)\\
=&\langle [s_-(X), s_+(Y)]^+ - [s_-(Y), s_+(X)]^+, s_+(Z)\rangle - \langle s_+([X,Y]), s_+(Z)\rangle\\
=&\langle [s_-(X), s_+(Y)] - [s_-(Y), s_+(X)], s_+(Z)\rangle - \langle 2s([X,Y]), s_+(Z)\rangle\\
=&\langle [s_-(X), s_+(Y)] + [s_+(X), s_-(Y)], s_+(Z)\rangle + \langle \pi^*(\iota_Y\iota_X H) - 2[s(X), s(Y)], s_+(Z)\rangle
\end{aligned}$$

$$
\begin{aligned}
=&\langle [s_-(X), s_+(Y)] + [s_+(X), s_-(Y)] - 2[s(X), s(Y)], s_+(Z)\rangle + \langle \pi^*(\iota_Y \iota_X H), s_+(Z)\rangle \\
=&\langle [s_-(X), s_+(Y)] + [s_+(X), s_-(Y)] \\
&\quad - \frac{1}{2}([s_+X, s_+Y] + [s_+X, s_-Y] + [s_-X, s_+Y] + [s_-X, s_-Y]), s_+(Z)\rangle + \\
&\qquad + \langle \pi^*(\iota_Y \iota_X H), s_+(Z)\rangle \\
=&-\frac{1}{2}\langle [s_+(X) - s_-(X), s_+(Y) - s_-(Y)], s_+(Z)\rangle + H(X,Y,Z) \\
=&H(X,Y,Z),
\end{aligned}
$$

where in the last step we used that $s_+(X) - s_-(X) \in \ker(\pi)$. Finally, the second equality follows from $s_\pm(X) = 2s(X)^\pm$. ☐

This implies that the canonical mixed type operators of the generalized metric $E_+ \subset E$ are given by $[sX^\mp, sY^\pm]^\pm = \frac{1}{2}s(\nabla^\pm_X Y)^\pm$. If we set the pure type operators to be $D^B_{sX^\pm} sY^\pm = \frac{1}{2}s(\nabla^\pm_X Y)^\pm$ as well, we obtain a generalized metric connection

$$
D^B_u v = [u^-, v^+]^+ + [u^+, v^-]^- + [C(u^-), v^-]^- + [C(u^+), v^+]^+, \tag{5}
$$

called the Gualtieri-Bismut connection [Gu3]. Here $C : E \to E$ denotes the reflection along $s(TM)$. It is given by $C(sX + \pi^*\xi) = sX - \pi^*\xi$ or equivalently by $C(s_\pm X) = s_\mp X$.

Lemma 3.5. The Gualtieri-Bismut connection has torsion $T_{D^B} = (\pi|_{E_+})^* H + (\pi|_{E_-})^* H$. In particular, the torsion is of pure type.

Proof. The torsion is of pure type due to Lemma 2.19. Furthermore, we compute

$$
\begin{aligned}
&T_{D^B}(s_+X, s_+Y, s_+Z) \\
=&8T_{D^B}(sX^+, sY^+, sZ^+) \\
=&8(\langle D^B_{sX^+} sY^+ - D^B_{sY^+} sX^- - [sX^+, sY^+], sZ^+\rangle + \langle D^B_{sZ^+} sX^+, sY^+\rangle) \\
=&8(\langle \frac{1}{2}s(\nabla^+_X Y - \nabla^+_Y X)^+ - [sX^+, sY^+]^+, sZ^+\rangle + \langle D^B_{sZ^+} sX^+, sY^+\rangle) \\
=&\langle s_+(\nabla^+_X Y - \nabla_Y X) - s_+\pi([s_+X, s_+Y]^+), s_+Z\rangle + 8\langle D^B_{sZ^+} sX^+, sY^+\rangle \\
=&\langle s_+(\nabla^+_X Y - \nabla_Y X - [X,Y]), s_+Z\rangle + \langle s_+\pi[s_+X, s_+Y]^-, s_+Z\rangle + 8\langle D^B_{sZ^+} sX^+, sY^+\rangle \\
=&g(\nabla^+_X Y - \nabla_Y X - [X,Y], Z) - \langle s_-\pi[s_+X, s_+Y]^-, s_-Z\rangle + 8\langle D^B_{sZ^+} sX^+, sY^+\rangle \\
=&g(T^{\nabla^+}(X,Y), Z) - 8\langle [sX^+, sY^+], sZ^-\rangle + 4\langle s(\nabla^+_Z X)^+, sY^+\rangle \\
=&g(T^{\nabla^+}(X,Y), Z) - 8(\pi(sX^+)\langle sY^+, sZ^-\rangle - \langle sY^+, [sX^+, sZ^-]\rangle) + 4\langle s(\nabla^+_Z X)^+, sY^+\rangle \\
=&g(T^{\nabla^+}(X,Y), Z) - 8\langle sY^+, [sZ^-, sX^+]^+\rangle + 4\langle s(\nabla^+_Z X)^+, sY^+\rangle \\
=&g(T^{\nabla^+}(X,Y), Z) - 4\langle sY^+, s(\nabla^+_Z X)^+\rangle + 4\langle s(\nabla^+_Z X)^+, sY^+\rangle \\
=&H(X,Y,Z).
\end{aligned}
$$

The same computation for s_- yields the claim. ☐

We construct a generalized Levi-Civita connection from D^B by (compare also Lemma 2.20)

$$D^0 = D^B - \frac{1}{3}T_{D^B}. \tag{6}$$

Lemma 3.6. If we introduce

$$\nabla^{\pm 1/3} = \nabla^g \pm \frac{1}{6}g^{-1}H,$$

then D^0 has pure type operators given by

$$D^0_{sX^\pm} sY^\pm = \frac{1}{2}s_\pm(\nabla^{\pm 1/3}_X Y).$$

Proof. W exemplarily do the computation for $D^0_{sX^+} sY^+$,

$$\begin{aligned}
&\langle D^0_{sX^+} sY^+, sZ^+\rangle \\
=&\langle D^B_{sX^+} sY^+ - \frac{1}{3}T_{D^B}(sX^+, sY^+), sZ^+\rangle \\
=&\langle D^B_{sX^+} sY^+, sZ^+\rangle - \frac{1}{24}\langle T_{D_B}(s_+X, s_+Y), s_+Z\rangle \\
=&\frac{1}{2}\langle s(\nabla^+_X Y)^+, sZ^+\rangle - \frac{1}{24}H(X,Y,Z) \\
=&\frac{1}{2}\langle s(\nabla^g_X Y)^+, sZ^+\rangle + \frac{1}{4}\langle sg^{-1}H(X,Y)^+ sZ^+\rangle - \frac{1}{24}H(X,Y,Z) \\
=&\frac{1}{2}\langle s(\nabla^g_X Y)^+, sZ^+\rangle + \frac{1}{4}\langle sg^{-1}H(X,Y)^+ sZ^+\rangle - \frac{1}{24}g(g^{-1}H(X,Y),Z) \\
=&\frac{1}{2}\langle s(\nabla^g_X Y)^+, sZ^+\rangle + \frac{1}{4}\langle sg^{-1}H(X,Y)^+ sZ^+\rangle - \frac{1}{24}\langle s_+g^{-1}H(X,Y), s_+Z\rangle \\
=&\frac{1}{2}\langle s(\nabla^g_X Y)^+, sZ^+\rangle + \frac{1}{4}\langle sg^{-1}H(X,Y)^+ sZ^+\rangle - \frac{1}{6}\langle sg^{-1}H(X,Y)^+, sZ^+\rangle \\
=&\langle\frac{1}{2}s(\nabla^g_X Y + \frac{1}{6}g^{-1}H(X,Y))^+, sZ^+\rangle \\
=&\langle\frac{1}{2}s(\nabla^{+1/3}_X Y), sZ^+\rangle
\end{aligned}$$

□

We can now write $\delta = \delta_{D^0} - \langle e, \cdot\rangle$ for some $e \in \Gamma(\mathbb{T}M)$[3]. Then there are one-forms $\varphi, \sigma \in \Gamma(T^*M)$ such that

$$e^+ = (\frac{1}{2}\pi^*\varphi)^+ \text{ and } e^- = (\frac{1}{2}\pi^*\sigma)^-. \tag{7}$$

We define a generalized connection

$$D = D^0 + \frac{1}{n-1}(\chi_+^{\varphi^+} + \chi_-^{\sigma^-}). \tag{8}$$

[3]The divergence operator δ_{D^0} is coincides with the so-called Riemannian divergence (see [GSt, Definition 2.46]).

Here $\chi_+^{\varphi^+} = \chi_+^{(\frac{1}{2}\pi^*\varphi)^+}$ and $(\chi_+^u)_v w = \langle v, w\rangle u - \langle u, w\rangle v$, whenever $u, v, w \in \Gamma(E_+)$, and similarly for σ. The pure type operators of D are given by

$$\begin{aligned}
D_{s(X)^+}s(Y)^+ &= D^0_{s(X)^+}s(Y)^+ + \frac{1}{n-1}(\langle s(X)^+, s(Y)^+\rangle \frac{1}{2}\pi^*\varphi^+ - \langle \frac{1}{2}\pi^*\varphi, s(Y)^+\rangle s(X)^+) \\
&= \frac{1}{2}s(\nabla^{1/3}_X Y)^+ + \frac{1}{n-1}(\frac{1}{4}g(X,Y)s(g^{-1}\varphi)^+ - \frac{1}{4}\langle \pi^*\varphi, s_+(Y)\rangle s(X)^+) \\
&= \frac{1}{2}s(\nabla^{1/3}_X Y + \frac{1}{2(n-1)}(g(X,Y)g^{-1}(\varphi) - \varphi(Y)X))^+
\end{aligned}$$

and similarly

$$D_{s(X)^-}s(Y)^- = \frac{1}{2}s(\nabla^{-1/3}_X Y + \frac{1}{2(n-1)}(g(X,Y)g^{-1}(\sigma) - \sigma(Y)X))^-.$$

Lemma 3.7. D is a torsion-free generalized connection, compatible with E_+ and satisfies $\delta_D = \delta$.

Proof. It is a generalized Levi-Civita connection, since we modified the generalized Levi-Civita connection D^0 by an element of $\Sigma^+ \oplus \Sigma^-$ (see Section 2.4). Let now $(u_i)_i$ be an orthogonal local frame for E_+ and $(w_i)_i$ be sections of E_+ defined so that $\langle u_i, w_j\rangle = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Let $v \in \Gamma(E_+)$, then

$$\begin{aligned}
\delta_D(v) &= \delta_{D^0}(v) + \frac{1}{n-1}\sum_{i=1}^n \langle (\chi_+^{\varphi^+})_{u_i} v, w_i\rangle \\
&= \delta_{D^0}(v) + \frac{1}{n-1}\sum_{i=1}^n \langle \langle u_i, v\rangle \frac{1}{2}\pi^*\varphi^+ - \langle \frac{1}{2}\pi^*\varphi^+, v\rangle u_i, w_i\rangle \\
&= \delta_{D^0}(v) + \frac{1}{n-1}\sum_{i=1}^n \langle u_i, v\rangle \langle \frac{1}{2}\pi^*\varphi^+, w_i\rangle - \langle \frac{1}{2}\pi^*\varphi^+, v\rangle \langle u_i, w_i\rangle \\
&= \delta_{D^0}(v) - \langle \frac{1}{2}\pi^*\varphi^+, v\rangle \\
&= \delta_{D^0}(v) - \langle \frac{1}{2}\pi^*\varphi, v\rangle.
\end{aligned}$$

Using the same computation for sections of E_-, we can compute for $v \in \Gamma(E)$ that

$$\begin{aligned}
\delta_D(v) &= \delta_D(v^+) + \delta_D(v^-) \\
&= \delta_{D^0}(v^+) - \langle \frac{1}{2}\pi^*\varphi, v^+\rangle + \delta_{D^0}(v^-) - \langle \frac{1}{2}\pi^*\sigma, v^-\rangle \\
&= \delta_{D^0}(v) - \langle e^+, v^+\rangle - \langle e^-, v^-\rangle \\
&= \delta_{D^0}(v) - \langle e, v\rangle \\
&= \delta(v),
\end{aligned}$$

as required. □

We now want to determine the Ricci tensor of the pair (G, δ). It is equal to the Ricci tensor of the generalized connection D from equation (8), which we will compute. First, we compute its generalized curvature tensor

$$R_D^{\pm}(u,v)w = D_u D_v w - D_v D_u w - D_{[u,v]} w$$

for $u \in \Gamma(E_\pm)$, $v \in \Gamma(E_\mp)$ and $w \in \Gamma(E_\pm)$. We exemplarily do the computation for R_D^+ in order to later compute Ric_D^+.

We will now denote by

$$R^{1/3}(X,Y)Z := \nabla_X^{1/3}\nabla_Y^+ Z - \nabla_Y^+\nabla_X^{1/3} Z + \nabla^{1/3}_{\nabla_Y^+ X} Z - \nabla^+_{\nabla_X^- Y} Z$$

to express the curvature tensors.

Proposition 3.8. The generalized curvature tensor R_D^+ of the generalized connection D constructed above are given by

$$\begin{aligned}&R_D^+(s(X)^+, s(Y)^-)s(Z)^+\\ =&\frac{1}{4}s(R^{1/3}(X,Y)Z - \frac{1}{2(n-1)}(g(X,Z)\nabla_Y^+ g^{-1}\varphi - \iota_Z(\nabla_Y^+\varphi)X))^+,\end{aligned}$$

where $X, Y, Z \in \Gamma(TM)$.

Proof. Let $X, Y, Z \in \Gamma(TM)$, then

$$\begin{aligned}
D_{s(X)^+}D_{s(Y)^-}s(Z)^+ &= D_{s(X)^+}[s(Y)^-, s(Z)^+]^+\\
&= \frac{1}{2}D_{s(X)^+}s(\nabla_Y^+ Z)^+\\
&= \frac{1}{4}s(\nabla_X^{1/3}\nabla_Y^+ Z + \frac{1}{2(n-1)}(g(X, \nabla_Y^+ Z)g^{-1}\varphi - \varphi(\nabla_Y^+ Z)X))^+,\\
D_{s(Y)^-}D_{s(X)^+}s(Z)^+ &= \frac{1}{2}D_{s(Y)^-}s(\nabla_X^{1/3} Z + \frac{1}{2(n-1)}(g(X,Z)g^{-1}\varphi - \varphi(Z)X))^+\\
&= \frac{1}{4}s(\nabla_Y^+\nabla_X^{1/3} Z + \frac{1}{2(n-1)}(\nabla_Y^+ g(X,Z)g^{-1}\varphi - \nabla_Y^+\varphi(Z)X))^+\\
&= \frac{1}{4}s(\nabla_Y^+\nabla_X^{1/3} Z + \frac{1}{2(n-1)}(Y g(X,Z)g^{-1}\varphi + g(X,Z)\nabla_Y^+ g^{-1}\varphi\\
&\qquad - Y\varphi(Z)X - \varphi(Z)\nabla_Y^+ X))^+\\
D_{[s(X)^+, s(Y)^-]}s(Z)^+ &= D_{[s(X)^+, s(Y)^-]^-}s(Z)^+ - D_{[s(Y)^-, s(X)^+]^+}s(Z)^+\\
&= \frac{1}{2}(D_{s(\nabla_X^- Y)^-}s(Z)^+ - D_{s(\nabla_Y^+ X)^+}s(Z)^+)\\
&= \frac{1}{4}s(\nabla^+_{\nabla_X^- Y} Z\\
&\qquad - (\nabla^{1/3}_{\nabla_Y^+ X} Z + \frac{1}{2(n-1)}(g(\nabla_Y^+ X, Z)g^{-1}\varphi - \varphi(Z)\nabla_Y^+ X)))^+.
\end{aligned}$$

Thus

$$
\begin{aligned}
&R_D^+(s(X)^+, s(Y)^-)s(Z)^+\\
=&\frac{1}{4}s(R^{1/3}(X,Y)Z + \frac{1}{2(n-1)}(g(X,\nabla_Y^+ Z)g^{-1}\varphi - \varphi(\nabla_Y^+ Z)X\\
&\quad - Yg(X,Z)g^{-1}\varphi - g(X,Z)\nabla_Y^+ g^{-1}\varphi + Y\varphi(Z)X + \varphi(Z)\nabla_Y^+ X\\
&\qquad + g(\nabla_Y^+ X, Z)g^{-1}\varphi - \varphi(Z)\nabla_Y^+ X))^+\\
=&\frac{1}{4}s(R^{1/3}(X,Y)Z - \frac{1}{2(n-1)}(g(X,Z)\nabla_Y^+ g^{-1}\varphi - \iota_Z(\nabla_Y^+\varphi)X))^+,
\end{aligned}
$$

using that ∇^+ is a metric connection. □

Remark 3.9. We remark that this formula differs from [G] by a relative factor $-\frac{1}{2}$.

For later purpose, we need another description of $R^{1/3}(X,Y)Z$

$$
\begin{aligned}
R^{1/3}(X,Y)Z &= \nabla_X^{1/3}\nabla_Y^+ Z - \nabla_Y^+\nabla_X^{1/3} Z + \nabla_{\nabla_Y^+ X}^{1/3} Z - \nabla_{\nabla_X Y}^+ Z\\
&= \nabla_X^g \nabla_Y^g Z + \frac{1}{2}\nabla_X^g g^{-1}H(Y,Z,\cdot) + \frac{1}{6}g^{-1}H(X,\nabla_Y^g Z,\cdot)\\
&+\frac{1}{12}g^{-1}H(X,g^{-1}H(Y,Z,\cdot),\cdot) - \nabla_Y^g\nabla_X^g Z - \frac{1}{6}\nabla_Y^g g^{-1}H(X,Z,\cdot)\\
&-\frac{1}{2}g^{-1}H(Y,\nabla_X^g Z,\cdot) - \frac{1}{12}g^{-1}H(Y,g^{-1}H(X,Z,\cdot),\cdot) + \nabla_{\nabla_Y^g X}^g Z\\
&+\frac{1}{2}\nabla_{g^{-1}H(Y,X,\cdot)}^g Z + \frac{1}{6}g^{-1}H(\nabla_Y^g X, Z,\cdot) + \frac{1}{12}g^{-1}H(g^{-1}H(Y,X,\cdot),Z,\cdot)\\
&-\nabla_{\nabla_X^g Y}^g Z + \frac{1}{2}\nabla_{g^{-1}H(X,Y,\cdot)}^g Z - \frac{1}{2}g^{-1}H(\nabla_X^g Y, Z,\cdot)\\
&+\frac{1}{4}g^{-1}H(g^{-1}H(X,Y,\cdot),Z,\cdot)\\
&= R^g(X,Y)Z + g^{-1}\Big(\frac{1}{2}((\nabla^g)_X^* H)(Y,Z,\cdot) - \frac{1}{6}((\nabla^g)_Y^* H)(X,Z,\cdot)\\
&+\frac{1}{12}H(X,g^{-1}H(Y,Z,\cdot),\cdot) - \frac{1}{12}H(Y,g^{-1}H(X,Z,\cdot),\cdot)\\
&-\frac{1}{6}H(Z,g^{-1}H(X,Y,\cdot),\cdot)\Big).
\end{aligned}
$$

We now can compute the Ricci tensor.

Proposition 3.10. With $s_\pm, g, H, \varphi$ and σ as above, the Ricci tensors of the pair $(\mathcal{G},\delta)$ are given by

$$
\begin{aligned}
Ric^+(s_-(Y), s_+(Z)) &= Ric^g(Y,Z) - \frac{1}{2}d^*H(Y,Z) - \frac{1}{4}H\circ H(Y,Z) + \frac{1}{2}(\nabla_Y^+\varphi)(Z)\\
Ric^-(s_+(Y), s_-(Z)) &= Ric^g(Y,Z) + \frac{1}{2}d^*H(Y,Z) - \frac{1}{4}H\circ H(Y,Z) + \frac{1}{2}(\nabla_Y^-\sigma)(Z),
\end{aligned}
$$

where $X,Y,Z \in \Gamma(TM)$.

Proof. Note first that the Ricci tensors of $(\mathcal{G},\delta)$ are by definition the Ricci tensors of the generalized connection D, since D is a generalized Levi-Civita connection of $\mathcal{G}$ with divergence δ. Let $X,Y,Z\in\Gamma(TM)$. Then

$$\begin{aligned}
Ric^+_D(s_-(Y),s_+(Z)) &= \operatorname{tr}\left(s_+(X)\mapsto R^+_D(s_+(X),s_+(Y))s_+(Z)\right)\\
&= 4\operatorname{tr}\left(s(X)^+\mapsto R^+_D(s(X)^+,s(Y)^-)s(Z)^+\right)\\
&= \operatorname{tr}\left(s(X)^+\mapsto s(R^{1/3}(X,Y)Z)^+\right)\\
&- \operatorname{tr}\left(s(X)^+\mapsto s(\frac{1}{2(n-1)}(g(X,Z)\nabla^+_Y g^{-1}\varphi-\iota_Z(\nabla^+_Y\varphi)X))^+\right)\\
&= \operatorname{tr}\left(s(X)^+\mapsto s(R^g(X,Y)Z)^+\right)\\
&+ \operatorname{tr}\Bigg(s(X)^+\mapsto s(g^{-1}\Big(\frac{1}{2}((\nabla^g)_X H)(Y,Z,\cdot)-\frac{1}{6}(\nabla^g_Y H)(X,Z,\cdot)\\
&+\frac{1}{12}H(X,g^{-1}H(Y,Z,\cdot),\cdot)-\frac{1}{12}H(Y,g^{-1}H(X,Z,\cdot),\cdot)\\
&-\frac{1}{6}H(Z,g^{-1}H(X,Y,\cdot),\cdot)\Big))^+\Bigg)\\
&- \operatorname{tr}\left(s(X)^+\mapsto s(\frac{1}{2(n-1)}(g(X,Z)\nabla^+_Y g^{-1}\varphi-\iota_Z(\nabla^+_Y\varphi)X))^+\right)\\
&= \operatorname{tr}\left(X\mapsto R_g(X,Y)Z\right)\\
&+ \operatorname{tr}\Bigg(X\mapsto g^{-1}\Big(\frac{1}{2}(\nabla^g_X H)(Y,Z,\cdot)-\frac{1}{6}(\nabla^g_Y H)(X,Z,\cdot)\\
&+\frac{1}{12}H(X,g^{-1}H(Y,Z,\cdot),\cdot)-\frac{1}{12}H(Y,g^{-1}H(X,Z,\cdot),\cdot)\\
&-\frac{1}{6}H(Z,g^{-1}H(X,Y,\cdot),\cdot)\Big)\Bigg)\\
&- \operatorname{tr}\left(X\mapsto \frac{1}{2(n-1)}(g(X,Z)\nabla^+_Y g^{-1}\varphi-\iota_Z(\nabla^+_Y\varphi)X)\right)
\end{aligned}$$

We will now treat the three traces individually. The first one equals the Ricci tensor Ric^g of the Levi-Civita connection ∇^g. For the second one we choose an orthonormal local frame $(X_i)_i$ of TM with respect to g and compute

$$\begin{aligned}
&\operatorname{tr}\Bigg(X\mapsto g^{-1}\Big(\frac{1}{2}(\nabla^g_X H)(Y,Z,\cdot)-\frac{1}{6}(\nabla^g_Y H)(X,Z,\cdot)+\frac{1}{12}H(X,g^{-1}H(Y,Z,\cdot),\cdot)\\
&\quad -\frac{1}{12}H(Y,g^{-1}H(X,Z,\cdot),\cdot)-\frac{1}{6}H(Z,g^{-1}H(X,Y,\cdot),\cdot)\Big)\Bigg)\\
&=\sum_{i=1}^n g\Bigg(g^{-1}(\frac{1}{2}(\nabla^g_{X_i}H)(Y,Z,\cdot)-\frac{1}{6}(\nabla^g_Y H)(X_i,Z,\cdot)+\frac{1}{12}H(X_i,g^{-1}H(Y,Z,\cdot),\cdot)\\
&\quad -\frac{1}{12}H(Y,g-1H(X_i,Z,\cdot),\cdot)-\frac{1}{6}H(Z,g^{-1}H(X_i,Y,\cdot),\cdot)),X_i\Bigg)
\end{aligned}$$

$$
\begin{aligned}
=&\sum_{i=1}^{n}\frac{1}{2}(\nabla^g_{X_i}H)(Y,Z,X_i)-\frac{1}{12}H(Y,g^{-1}H(X_i,Z,\cdot),X_i)-\frac{1}{6}H(Z,g^{-1}H(X_i,Y,\cdot),X_i)\\
=&\frac{1}{2}\sum_{i=1}^{n}\iota_{X_i}(\nabla^g_{X_i}H)(Y,Z)\\
&\quad-\frac{1}{12}\sum_{i=1}^{n}g(g^{-1}H(X_i,Y,\cdot),g^{-1}H(X_i,Z,\cdot))-\frac{1}{6}\sum_{i=1}^{n}g(g^{-1}H(X_i,Y,\cdot),g^{-1}H(X_i,Z,\cdot))\\
=&-\frac{1}{2}d^*H(Y,Z)-\frac{1}{4}H\circ H(Y,Z).
\end{aligned}
$$

For the last summand we compute

$$
\begin{aligned}
&\operatorname{tr}\left(X\mapsto\frac{1}{2(n-1)}(g(X,Z)\nabla^+_Y g^{-1}\varphi-\iota_Z(\nabla^+_Y\varphi)X)\right)\\
&\quad=\frac{1}{2(n-1)}\sum_{i=1}^{n}g\left(g(X_i,Z)\nabla^+_Y g^{-1}\varphi-\iota_Z(\nabla^+_Y\varphi)X_i,X_i\right)\\
&\quad=\frac{1}{2(n-1)}\left(\sum_{i=1}^{n}g(X_i,Z)g(\nabla^+_Y g^{-1}\varphi,X_i)-\sum_{i=1}^{n}\iota_Z(\nabla^+_Y\varphi)g(X_i,X_i)\right)\\
&\quad=\frac{1}{2(n-1)}\left(g(Z,\nabla^+_Y g^{-1}\varphi)-n(\nabla^+_Y\varphi)(Z)\right)\\
&\quad=\frac{1}{2(n-1)}\left(g(Z,g^{-1}((\nabla^+)_Y\varphi))-n(\nabla^+_Y\varphi)(Z)\right)\\
&\quad=-\frac{1}{2}\left(\nabla^+_Y\varphi\right)(Z).
\end{aligned}
$$

Putting this together we obtain the Ricci tensor Ric^+ of the pair $(\mathcal{G},\delta)$, as claimed. The term for Ric^- is obtained in a similar way. □

Remark 3.11. We note again that this formula differs from [G, Example 4.9] by a relative factor $-\frac{1}{2}$ (see Remark 3.9).

4 Computations in orthonormal frames

In this section we derive formulas for several objects on Courant algebroids, such as the Dorfman bracket and the Ricci curvature, if we have an orthonormal frame with respect to a generalized metric on it. In particular, we construct a generalized Levi-Civita connection.

Let $\mathcal{G}$ be a generalized metric on a Courant algebroid $(E, \pi, \langle\cdot,\cdot\rangle, [\cdot,\cdot])$, defining non-degenerate subbundles $E_\pm$. Denote by n_1 be the rank of E_+ and n_2 the rank of E_-. We assume that there is an orthonormal frame $(e_r)_r$ for E with respect to $\mathcal{G}$, such that $e_r \in \Gamma(E_+)$ for $r \leq n_1$ and $e_r \in \Gamma(E_-)$ for $r > n_1$. Note that the frame is orthonormal with respect to both, the scalar product $\langle\cdot,\cdot\rangle$ and the generalized metric $\mathcal{G}$, since they only differ by a sign on the subbundles E_+ and E_-. We will write $\epsilon_r := \langle e_r, e_r\rangle$ and $\eta_{rs} := \langle e_r, e_s\rangle$. Furthermore, $\eta^{rs} = \eta_{rs}$ and $\epsilon^r = \epsilon_r$ are the corresponding coefficients of the induced scalar product on E^*.

We will use the following notation for the indices. The indices $a, b, c, \dots$ will go from $1, \dots, n_1$, the indices $i, j, k, \dots$ from $n_1 + 1, \dots, n_1 + n_2$ and the indices $r, s, t, \dots$ from $1, \dots, n_1 + n_2 = rk(E)$.

To simplify notation we will use Einstein's summation convention, according to which the sum over an upper and a lower repeated index is understood.

4.1 Equations for the bracket

We define $\mathcal{B}^t_{rs} \in C^\infty(M)$ by

$$[e_r, e_s] = \sum_t \mathcal{B}^t_{rs} e_t,$$

called the Dorfman coefficients and will also write $\mathcal{B}_{rst} := \sum_u \eta_{tu}\mathcal{B}^u_{rs} = \epsilon_t \mathcal{B}^t_{rs}$ such that

$$\mathcal{B}_{rst} = \langle [e_r, e_s], e_t\rangle.$$ [4]

We start this section by investigating some properties of $\mathcal{B}^t_{rs}$ that follow from the axioms (C1) to (C5) from Definition 2.1.

Lemma 4.1. Let $f \in C^\infty(M)$ and $u \in \Gamma(E)$. Then

$$\langle [fe_r, e_s], u\rangle + \langle [e_s, fe_r], u\rangle = \pi(u)(f)\eta_{rs}.$$

Proof.

$$\begin{aligned}
&\langle [fe_r, e_s], u\rangle + \langle [e_s, fe_r], u\rangle \\
=&\frac{1}{2}\pi(u)\langle fe_r, fe_r\rangle + \langle [fe_r, e_s], u\rangle + \langle [e_s, fe_r], u\rangle + \frac{1}{2}\pi(u)\langle e_s, e_s\rangle - \frac{1}{2}\pi(u)\langle fe_r, fe_r\rangle \\
=&\langle [fe_r, fe_r], u\rangle + \langle [fe_r, e_s], u\rangle + \langle [e_s, fe_r], u\rangle + \langle [e_s, e_s], u\rangle - \frac{1}{2}\pi(u)\langle fe_r, fe_r\rangle
\end{aligned}$$

[4] Since (e_r) is an orthonormal frame and therefore $\eta_{rs} = \epsilon_r\delta_{rs}$, we can use $\epsilon_r = \langle e_r, e_r\rangle$ to lower the indices without summation.

$$
\begin{aligned}
&=\langle [fe_r + e_s, fe_r + e_s], u\rangle - \frac{1}{2}\pi(u)\langle fe_r, fe_r\rangle \\
&=\frac{1}{2}\pi(u)\langle fe_r + e_s, fe_r + e_s\rangle - \frac{1}{2}\pi(u)\langle fe_r, fe_r\rangle \\
&=\frac{1}{2}\pi(u)\left(\langle fe_r, fe_r\rangle + \langle fe_r, e_s\rangle + \langle e_s, fe_r\rangle + \langle e_s, e_s\rangle\right) - \frac{1}{2}\pi(u)\langle fe_r, fe_r\rangle \\
&=\pi(u)\left(f\langle e_r, e_s\rangle\right) \\
&=\pi(u)(f)\eta_{rs}
\end{aligned}
$$

□

Lemma 4.2. The expression $\mathcal{B}_{rst}$ is totally skew-symmetric in (r, s, t).

Proof. The property $\mathcal{B}_{rst} = -\mathcal{B}_{srt}$ follows form Lemma 4.1. Furthermore, we have

$$\mathcal{B}_{rst} = \langle [e_r, e_s], e_t\rangle = \pi(e_r)\langle e_s, e_t\rangle - \langle e_s, [e_r, e_t]\rangle = -\mathcal{B}_{rts}.$$

From this the skew-symmetry follows. □

Lemma 4.3. Let $f, g, h \in C^\infty(M)$. Then the following Leibniz rule holds

$$\langle [fe_r, ge_s], he_t\rangle = f\pi(e_r)(g)h\eta_{st} - \pi(e_s)(f)gh\eta_{rt} + \pi(e_t)(f)gh\eta_{rs} + fgh\langle [e_r, e_s], e_t\rangle.$$

Proof.

$$
\begin{aligned}
\langle [fe_r, ge_s], he_t\rangle =&\langle \pi(fe_r)(g)e_s, he_t\rangle + gh\langle [fe_r, e_s], e_t\rangle \\
=&f\pi(e_r)(g)h\eta_{st} + gh\pi(e_t)(f)\eta_{rs} - gh\langle [e_s, fe_r], e_t\rangle \\
=&f\pi(e_r)(g)h\eta_{st} + \pi(e_t)(f)gh\eta_{rs} - gh\langle \pi(e_s)(f)e_r, e_t\rangle - gh\langle f[e_s, e_r], e_t\rangle \\
=&f\pi(e_r)(g)h\eta_{st} + \pi(e_t)(f)gh\eta_{rs} - \pi(e_s)(f)gh\eta_{rs} + fgh\langle [e_r, e_s], e_t\rangle
\end{aligned}
$$

□

Lemma 4.4. The Jacobi identity for the bracket in the above coordinates reads as follows

$$\pi(e_r)(\mathcal{B}_{stv}) + \pi(e_s)(\mathcal{B}_{trv}) + \pi(e_t)(\mathcal{B}_{rsv}) - \pi(e_v)(\mathcal{B}_{rst}) = \sum_u \mathcal{B}^u_{rs}\mathcal{B}_{utv} + \mathcal{B}^u_{rt}\mathcal{B}_{suv} - \mathcal{B}^u_{st}\mathcal{B}_{ruv}.$$

Proof.

$$
\begin{aligned}
\pi(e_r)(\mathcal{B}_{stv}) + \sum_u \mathcal{B}^u_{st}\mathcal{B}_{ruv} =& \sum_u \left(\pi(e_r)(\mathcal{B}^u_{st})\eta_{uv} + \mathcal{B}^u_{st}\langle [e_r, e_u], e_v\rangle\right) \\
=&\left\langle \sum_u [e_r, \mathcal{B}^u_{st}e_u], e_v \right\rangle \\
=&\langle [e_r, [e_s, e_t]], e_v\rangle \\
=&\langle [[e_r, e_s], e_t] + [e_s, [e_r, e_t]], e_v\rangle \\
=&\sum_u \left(\langle [\mathcal{B}^u_{rs}e_u, e_t], e_v\rangle + \langle [e_s, \mathcal{B}^u_{rt}e_u], e_v\rangle\right) \\
=&\sum_u \left(-\pi(e_t)(\mathcal{B}^u_{rs})\eta_{uv} + \pi(e_v)(\mathcal{B}^u_{rs})\eta_{ut} + \mathcal{B}^u_{rs}\langle [e_u, e_t], e_v\rangle\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\pi(e_s)(\mathcal{B}^u_{rt})\eta_{uv} + \mathcal{B}^u_{rt}\langle[e_s,e_u],e_v\rangle) \\
=& -\pi(e_t)(\mathcal{B}_{rsv}) + \pi(e_v)(\mathcal{B}_{rst}) + \sum_u \mathcal{B}^u_{rs}\mathcal{B}_{utv} \\
&+ \pi(e_s)(\mathcal{B}_{rtv}) + \sum_u \mathcal{B}^u_{rt}\mathcal{B}_{suv} \qquad \square
\end{aligned}
$$

Remark 4.5. The equation in Lemma 4.4 can be written as

$$
\sum_{\mathfrak{S}} (\pi(e_r)(\mathcal{B}_{stv})) - \pi(e_v)(\mathcal{B}_{rst}) = \sum_{\mathfrak{S}} \left(\sum_u \mathcal{B}^u_{rs}\mathcal{B}_{utv} \right),
$$

where $\sum_{\mathfrak{S}}$ stands for the sum over cyclic permutations of (r,s,t).

4.2 Construction of a generalized Levi-Civita connection

We construct a generalized Levi-Civita connection of $\mathcal{G}$ solely from the bracket and the orthonormal frame $(e_r)_r$, that is from the coefficients $\mathcal{B}_{rst}$. We define functions

$$
\begin{aligned}
\omega_{abc} &= \frac{1}{3}\mathcal{B}_{abc}, \quad \omega_{ijk} = \frac{1}{3}\mathcal{B}_{ijk} \\
\omega_{ibc} &= \mathcal{B}_{ibc}, \qquad \omega_{ajk} = \mathcal{B}_{ajk}
\end{aligned}
\tag{9}
$$

and set $\omega_{rst} = 0$ for the remaining combinations of r, s, t. Recall here the agreed index ranges.

Lemma 4.6. Setting $\omega^t_{rs} \coloneqq \eta^{tu}\omega_{rsu}$ the expression

$$
D^0_{e_r} e_s \coloneqq \sum_t \omega^t_{rs} e_t,
$$

defines a generalized Levi-Civita connection of $\mathcal{G}$. Here, of course, this is extended to a map $D^0 : \Gamma(E) \to \Gamma(E^* \otimes E)$ by linearity and the Leibniz rule, that is

$$
D^0_{\sum_r X^r e_r} \sum_s Y^s e_s = \sum_{r,s} \left(X^r \pi(e_r)(Y^s) e_s + X^r D^0_{e_r} e_s \right),
$$

for $X^r, Y^s \in C^\infty(M)$.

Proof. Compatibility of the connection with the metric is clear, since D^0 preserves the decomposition $E = E_+ \oplus E_-$ into the eigenbundles of $\mathcal{G}^{\text{end}}$. For the torsion we compute

$$
\begin{aligned}
T_{D^0}(e_a,e_b,e_c) &= \left\langle D^0_{e_a} e_b - D^0_{e_b} e_a - [e_a,e_b], e_c \right\rangle + \left\langle D^0_{e_c} e_a, e_b \right\rangle \\
&= \frac{1}{3}\mathcal{B}_{abc} - \frac{1}{3}\mathcal{B}_{bac} - \mathcal{B}_{abc} + \frac{1}{3}\mathcal{B}_{cab} \\
&= 3\frac{1}{3}\mathcal{B}_{abc} - \mathcal{B}_{abc} \\
&= 0
\end{aligned}
$$

$$\begin{aligned} T_{D^0}(e_a, e_b, e_k) &= \left\langle D^0_{e_a} e_b - D^0_{e_b} e_a - [e_a, e_b], e_k \right\rangle + \left\langle D^0_{e_k} e_a, e_b \right\rangle \\ &= -\mathcal{B}_{abc} + \mathcal{B}_{cab} \\ &= 0. \end{aligned}$$

Similarly one shows $T_{D^0}(e_i, e_j, e_c) = T_{D^0}(e_i, e_j, e_k) = 0$. Using the skew-symmetry and tensoriality of the torsion, it follows that D^0 is torsion-free and therefore a generalized Levi-Civita connection. □

Lemma 4.7. Let $v = \sum_s X^s e_s \in \Gamma(E)$ be a section of E. The divergence of the connection D^0 constructed from $(e_r)_r$ at v is given by

$$\delta_{D^0}(v) = \sum_r \pi(e_r)(X^r).$$

In particular, we have that $\delta_{D^0}(e_s) = 0$.

Proof. One computes

$$\begin{aligned} \delta_{D^0}(v) &= \operatorname{tr}(u \mapsto D^0_u v) \\ &= \sum_r \epsilon^r \langle D^0_{e_r} v, e_r \rangle \\ &= \sum_r \epsilon^r \sum_s \langle D^0_{e_r} X^s e_s, e_r \rangle \\ &= \sum_r \epsilon^r \sum_s \left(\pi(e_r)(X^s) \langle e_s, e_r \rangle + X^s \langle D^0_{e_r} e_s, e_r \rangle \right) \\ &= \sum_r \pi(e_r)(X^r), \end{aligned}$$

where we used the definition of D^0 through the bracket and that $\mathcal{B}_{rst}$ is skew-symmetric in the last line. Since furthermore the coefficients of e_s are constant, also the equation $\delta_{D^0}(e_s) = 0$ follows. □

4.3 Generalized Ricci tensor

Recall from Section 2.5 that we defined two curvature operators $R^{\pm}_D \in \Gamma(E^*_{\pm} \otimes E^*_{\mp} \otimes \mathfrak{so}(E_{\pm}))$ of a metric generalized connection D by

$$R^{\pm}_D(u, v)w = D_u D_v w - D_v D_u w - D_{[u,v]} w,$$

where $(u, v, w) \in \Gamma(E_{\pm}) \times \Gamma(E_{\mp}) \times \Gamma(E_{\pm})$. Recall further that, if D is torsion-free, we have

$$R^{\pm}_D(u, v)w = D_u D_v w - D_v D_u w - D_{D_u v} w + D_{D_v u} w.$$

Lemma 4.8. If we write $R^{+}_{D^0}(e_a, e_j)e_c = \sum_d R^d_{ajc} e_d$ and $R^{-}_{D^0}(e_i, e_b)e_k = \sum_\ell R^\ell_{ibk} e_\ell$ then

$$R^d_{ajc} = \pi(e_a)(\omega^d_{jc}) - \pi(e_j)(\omega^d_{ac}) + \omega^e_{jc}\omega^d_{ae} - \omega^e_{ac}\omega^d_{je} - \omega^k_{aj}\omega^d_{kc} + \omega^e_{ja}\omega^d_{ec}$$

$$= \pi(e_a)(\mathcal{B}^d_{jc}) - \frac{1}{3}\pi(e_j)(\mathcal{B}^d_{ac}) + \frac{1}{3}\mathcal{B}^e_{jc}\mathcal{B}^d_{ae} - \frac{1}{3}\mathcal{B}^e_{ac}\mathcal{B}^d_{je} - \mathcal{B}^k_{aj}\mathcal{B}^d_{kc} + \frac{1}{3}\mathcal{B}^e_{ja}\mathcal{B}^d_{ec}$$

and

$$\begin{aligned} R^\ell_{ibk} &= \pi(e_i)(\omega^\ell_{bk}) - \pi(e_b)(\omega^\ell_{ik}) + \omega^m_{bk}\omega^\ell_{im} - \omega^m_{ik}\omega^\ell_{bm} - \omega^c_{ib}\omega^\ell_{ck} + \omega^m_{bi}\omega^\ell_{mk} \\ &= \pi(e_i)(\mathcal{B}^\ell_{bk}) - \frac{1}{3}\pi(e_b)(\mathcal{B}^\ell_{ik}) + \frac{1}{3}\mathcal{B}^m_{bk}\mathcal{B}^\ell_{im} - \frac{1}{3}\mathcal{B}^m_{ik}\mathcal{B}^\ell_{bm} - \mathcal{B}^c_{ib}\mathcal{B}^\ell_{ck} + \frac{1}{3}\mathcal{B}^m_{bi}\mathcal{B}^\ell_{mk} \end{aligned}$$

Proof. We only prove the formula for $R^+_{D^0}$, since the prove for $R^-_{D^0}$ works exactly the same way. For that we compute

$$\begin{aligned} R^+_{D^0}(e_a, e_j)e_c &= D^0_{e_a}D^0_{e_j}e_c - D^0_{e_j}D^0_{e_a}e_c - D^0_{D^0_{e_a}e_j}e_c + D^0_{D^0_{e_j}e_a}e_c \\ &= \sum_e D^0_{e_a}\omega^e_{jc}e_e - \sum_e D^0_{e_j}\omega^e_{ac}e_e - \sum_k D^0_{\omega^k_{aj}e_k}e_c + \sum_e D^0_{\omega^e_{ja}e_e}e_c \\ &= \sum_e \left(\pi(e_a)(\omega^e_{jc})e_e + \omega^e_{jc}D^0_{e_a}e_e\right) - \sum_e \left(\pi(e_j)(\omega^e_{ac})e_e + \omega^e_{ac}D^0_{e_j}e_e\right) \\ &\quad - \sum_k \omega^k_{aj}D^0_{e_k}e_c + \sum_e \omega^e_{ja}D^0_{e_e}e_c \\ &= \sum_e \left(\pi(e_a)(\omega^e_{jc})e_e + \sum_d \omega^e_{jc}\omega^d_{ae}e_d\right) \\ &\quad - \sum_e \left(\pi(e_j)(\omega^e_{ac})e_e + \sum_d \omega^e_{ac}\omega^d_{je}e_d\right) \\ &\qquad - \sum_k\sum_d \omega^k_{aj}\omega^d_{kc}e_d + \sum_e\sum_d \omega^e_{ja}\omega^d_{ec} \\ &= \sum_d \Big(\pi(e_a)(\omega^d_{jc})e_d + \sum_e \omega^e_{jc}\omega^d_{ae}e_d - \pi(e_j)(\omega^d_{ac})e_d - \sum_e \omega^e_{ac}\omega^d_{je}e_d \\ &\quad - \sum_k \omega^k_{aj}\omega^d_{kc}e_d + \sum_e \omega^e_{ja}\omega^d_{ec}e_d\Big). \end{aligned}$$

The claim then follows using the identities from the definition of D^0. □

Proposition 4.9. Lowering the indices $R_{rstu} \coloneqq \epsilon_u R^u_{rst}$ we obtain

$$R_{ajcd} = \frac{2}{3}\pi(e_a)(\mathcal{B}_{jcd}) + \frac{1}{3}\pi(e_c)(\mathcal{B}_{jad}) - \frac{1}{3}\pi(e_d)(\mathcal{B}_{jac}) - \frac{2}{3}\mathcal{B}^k_{aj}\mathcal{B}_{kcd} - \frac{1}{3}\mathcal{B}^k_{jc}\mathcal{B}_{akd} + \frac{1}{3}\mathcal{B}^k_{ac}\mathcal{B}_{jkd}$$

and

$$R_{ibk\ell} = \frac{2}{3}\pi(e_i)(\mathcal{B}_{bk\ell}) + \frac{1}{3}\pi(e_k)(\mathcal{B}_{bi\ell}) - \frac{1}{3}\pi(e_\ell)(\mathcal{B}_{bik}) - \frac{2}{3}\mathcal{B}^c_{ib}\mathcal{B}_{ck\ell} - \frac{1}{3}\mathcal{B}^c_{bk}\mathcal{B}_{ic\ell} + \frac{1}{3}\mathcal{B}^c_{ik}\mathcal{B}_{bc\ell}$$

Proof. Again we only prove the formula for R_{ajcd}. Using Lemma 4.4 and Lemma 4.8 we compute

$$R_{ajcd} = \pi(e_a)(\mathcal{B}_{jcd}) - \frac{1}{3}\pi(e_j)(\mathcal{B}_{acd}) + \frac{1}{3}\mathcal{B}^e_{jc}\mathcal{B}_{aed} - \frac{1}{3}\mathcal{B}^e_{ac}\mathcal{B}_{jed} - \mathcal{B}^k_{aj}\mathcal{B}_{kcd} + \frac{1}{3}\mathcal{B}^e_{ja}\mathcal{B}_{ecd}$$

$$
\begin{aligned}
&= \pi(e_a)(\mathcal{B}_{jcd}) - \frac{1}{3}\pi(e_j)(\mathcal{B}_{acd}) + \frac{1}{3}\mathcal{B}^e_{ja}\mathcal{B}_{ecd} + \frac{1}{3}\mathcal{B}^e_{jc}\mathcal{B}_{aed} - \frac{1}{3}\mathcal{B}^e_{ac}\mathcal{B}_{jed} - \mathcal{B}^k_{aj}\mathcal{B}_{kcd} \\
&= \pi(e_a)(\mathcal{B}_{jcd}) - \frac{1}{3}\pi(e_j)(\mathcal{B}_{acd}) + \frac{1}{3}\mathcal{B}^u_{ja}\mathcal{B}_{ucd} + \frac{1}{3}\mathcal{B}^u_{jc}\mathcal{B}_{aud} - \frac{1}{3}\mathcal{B}^u_{ac}\mathcal{B}_{jud} \\
&\qquad - \frac{1}{3}\mathcal{B}^k_{ja}\mathcal{B}_{kcd} - \frac{1}{3}\mathcal{B}^k_{jc}\mathcal{B}_{akd} + \frac{1}{3}\mathcal{B}^k_{ac}\mathcal{B}_{jkd} - \mathcal{B}^k_{aj}\mathcal{B}_{kcd} \\
&= \pi(e_a)(\mathcal{B}_{jcd}) - \frac{1}{3}\pi(e_j)(\mathcal{B}_{acd}) + \frac{1}{3}\pi(e_j)(\mathcal{B}_{acd}) - \frac{1}{3}\pi(e_a)(\mathcal{B}_{jcd}) + \frac{1}{3}\pi(e_c)(\mathcal{B}_{jad}) \\
&\qquad - \frac{1}{3}\pi(e_d)(\mathcal{B}_{jac}) + \frac{1}{3}\mathcal{B}^k_{aj}\mathcal{B}_{kcd} - \frac{1}{3}\mathcal{B}^k_{jc}\mathcal{B}_{akd} + \frac{1}{3}\mathcal{B}^k_{ac}\mathcal{B}_{jkd} - \mathcal{B}^k_{aj}\mathcal{B}_{kcd} \\
&= \frac{2}{3}\pi(e_a)(\mathcal{B}_{jcd}) + \frac{1}{3}\pi(e_c)(\mathcal{B}_{jad}) - \frac{1}{3}\pi(e_d)(\mathcal{B}_{jac}) - \frac{2}{3}\mathcal{B}^k_{aj}\mathcal{B}_{kcd} - \frac{1}{3}\mathcal{B}^k_{jc}\mathcal{B}_{akd} \\
&\qquad + \frac{1}{3}\mathcal{B}^k_{ac}\mathcal{B}_{jkd}. \square
\end{aligned}
$$

In Definition 2.25 we defined the Ricci tensors $Ric^{\pm}_D \in \Gamma(E^*_{\mp} \otimes E^*_{\pm})$ of a generalized Levi-Civita connection by

$$Ric^{\pm}_D(v, w) \coloneqq \mathrm{tr}(E_{\pm} \ni u \mapsto R^{\pm}_D(u, v)w \in E_{\pm})$$

for $(v, w) \in \Gamma(E_{\mp}) \times \Gamma(E_{\pm})$. In addition, we define now $R_{jc} \in C^{\infty}(M)$ by $Ric^{+}_{D^0}(e_j, e_c) = R_{jc} = R^a_{ajc}$ and $R_{bk} \in C^{\infty}(M)$ by $Ric^{+}_{D^0}(e_b, e_k) = R_{bk} = R^i_{ibk}$, where D^0 is the generalized connection constructed above from the orthonormal frame (e_r).

Proposition 4.10. We have the following formulas for the coefficients of the Ricci curvature

$$
\begin{aligned}
R_{jc} &= \pi(e_a)(\mathcal{B}^a_{jc}) - \mathcal{B}^k_{aj}\mathcal{B}^a_{kc} \\
&= \pi(e_a)(\epsilon^a\mathcal{B}_{jca}) - \epsilon^k\mathcal{B}_{ajk}\epsilon^a\mathcal{B}_{kca} \\
&= \epsilon^a\pi(e_a)(\langle [e_j, e_c], e_a\rangle) - \epsilon^a\epsilon^k\langle [e_a, e_j], e_k\rangle\langle [e_k, e_c], e_a\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
R_{bk} &= \pi(e_i)(\mathcal{B}^i_{bk}) - \mathcal{B}^c_{ib}\mathcal{B}^i_{ck} \\
&= \pi(e_i)(\epsilon^i\mathcal{B}_{bki}) - \epsilon^c\mathcal{B}_{ibc}\epsilon^i\mathcal{B}_{cki} \\
&= \epsilon^i\pi(e_i)(\langle [e_b, e_k], e_i\rangle) - \epsilon^i\epsilon^c\langle [e_i, e_b], e_c\rangle\langle [e_c, e_k], e_i\rangle
\end{aligned}
$$

Proof. Using Lemma 4.8, we see

$$
\begin{aligned}
R_{jc} = R^a_{ajc} &= \pi(e_a)(\mathcal{B}^a_{jc}) - \frac{1}{3}\pi(e_j)(\mathcal{B}^a_{ac}) + \frac{1}{3}\mathcal{B}^e_{jc}\mathcal{B}^a_{ae} - \frac{1}{3}\mathcal{B}^e_{ac}\mathcal{B}^a_{je} - \mathcal{B}^k_{aj}\mathcal{B}^a_{kc} + \frac{1}{3}\mathcal{B}^e_{ja}\mathcal{B}^a_{ec} \\
&= \pi(e_a)(\mathcal{B}^a_{jc}) - \frac{1}{3}\mathcal{B}^e_{ac}\mathcal{B}^a_{je} - \mathcal{B}^k_{aj}\mathcal{B}^a_{kc} + \frac{1}{3}\mathcal{B}^e_{ja}\mathcal{B}^a_{ec} \\
&= \pi(e_a)(\mathcal{B}^a_{jc}) - \frac{1}{3}\epsilon^e\mathcal{B}_{ace}\epsilon^a\mathcal{B}_{jea} - \mathcal{B}^k_{aj}\mathcal{B}^a_{kc} + \frac{1}{3}\epsilon^e\mathcal{B}_{jae}\epsilon^a\mathcal{B}_{eca} \\
&= \pi(e_a)(\mathcal{B}^a_{jc}) - \frac{1}{3}\epsilon^e\mathcal{B}_{eca}\epsilon^a\mathcal{B}_{jae} - \mathcal{B}^k_{aj}\mathcal{B}^a_{kc} + \frac{1}{3}\epsilon^e\mathcal{B}_{jae}\epsilon^a\mathcal{B}_{eca}
\end{aligned}
$$

$$
\begin{aligned}
&= \pi(e_a)(\mathcal{B}^a_{jc}) - \mathcal{B}^k_{aj}\mathcal{B}^a_{kc} \\
&= \pi(e_a)(\epsilon^a \mathcal{B}_{jca}) - \epsilon^k \mathcal{B}_{ajk}\epsilon^a \mathcal{B}_{kca} \\
&= \epsilon^a \pi(e_a)(\langle [e_j, e_c], e_a\rangle) - \epsilon^a\epsilon^k \langle [e_a, e_j], e_k\rangle\langle [e_k, e_c], e_a\rangle.
\end{aligned}
$$

As before, the computation for R_{bk} is similar. □

Remark 4.11. These are the coefficients of the Ricci tensor of the pair $(\mathcal{G}, \delta)$ for the divergence operator

$$
\delta\left(\sum_r X^r e_r\right) = \sum_r \pi(e_r)(X^r),
$$

due to Definition 2.28.

4.4 Dependence on the choice of orthonormal frame

We want to understand now how the Ricci tensors above depend on the choice of orthonormal frame. For that let $(\tilde{e}_r)$ be another orthonormal frame as above, such that $\tilde{e}_r \in \Gamma(E_+)$ for $r \leq n_1$ and $\tilde{e}_r \in \Gamma(E_-)$ otherwise. We can write $\tilde{e}_a = \sum_b S^b_a e_b$ and $\tilde{e}_i = \sum_j T^j_i e_j$ with $S^b_a, T^j_i \in C^\infty(M)$.

Lemma 4.12. We have the following equations

$$
\sum_a \epsilon^a S^b_a S^e_a = \eta^{be} \text{ and } \sum_k \epsilon^k T^m_k T^\ell_k = \eta^{m\ell}
$$

Proof. This follows because S and T are orthonormal transformations. Exemplarily, we do the computation for S

$$
\sum_a \epsilon^a S^b_a S^e_a = \sum_a \sum_c \eta^{ce} S^b_a (S^\top)^a_c = \sum_c \eta^{ce}\delta^b_c = \eta^{be},
$$

where $X^\top$ denotes the transpose of a matrix X □

Let $\tilde{D}^0$ be the generalized connection that is constructed from $(\tilde{e}_r)$ in the same way as D^0 was constructed from the frame (e_r). We first evaluate the Ricci tensors of $\tilde{D}^0$ and then of D^0 at the frame $(\tilde{e}_r)$.

Lemma 4.13. We have

$$
Ric^+_{\tilde{D}^0}(\tilde{e}_j, \tilde{e}_c) = T^i_j S^d_c \pi(e_b)(\mathcal{B}^b_{id}) - T^i_j S^d_c \mathcal{B}^m_{bi}\mathcal{B}^b_{md} + T^i_j \pi(e_i)\left(\pi(e_b)(S^b_c)\right)
$$

and

$$
Ric^-_{\tilde{D}^0}(\tilde{e}_b, \tilde{e}_k) = S^a_b T^\ell_k \pi(e_j)(\mathcal{B}^j_{a\ell}) - S^a_b STk^\ell \mathcal{B}^e_{ja}\mathcal{B}^j_{e\ell} + S^a_b \pi(e_a)\left(\pi(e_j)(T^\ell_k)\right).
$$

Proof. First note from Proposition 4.10 that

$$Ric^+_{\tilde{D}^0}(\tilde{e}_j, \tilde{e}_c) = \epsilon^a \pi(\tilde{e}_a)\langle[\tilde{e}_j, \tilde{e}_c], \tilde{e}_a\rangle - \epsilon^a \epsilon^k \langle[\tilde{e}_a, \tilde{e}_j], \tilde{e}_k\rangle\langle[\tilde{e}_k, \tilde{e}_c], \tilde{e}_a\rangle.$$

Plugging in the above representation of $\tilde{e}_r$ in terms of e_r, we get

$$\begin{aligned}
Ric^+_{\tilde{D}^0}(\tilde{e}_j, \tilde{e}_c) &= \epsilon^a \pi(\tilde{e}_a)\langle[\tilde{e}_j, \tilde{e}_c], \tilde{e}_a\rangle - \epsilon^a \epsilon^k \langle[\tilde{e}_a, \tilde{e}_j], \tilde{e}_k\rangle\langle[\tilde{e}_k, \tilde{e}_c], \tilde{e}_a\rangle \\
&= \epsilon^a \pi(S^b_a e_b)\langle[T^i_j e_i, S^d_c e_d], S^e_a e_e\rangle \\
&\quad - \epsilon^a \epsilon^k \langle[S^b_a e_b, T^i_j e_i], T^\ell_k e_\ell\rangle\langle[T^m_k e_m, S^d_c e_d], S^e_a e_e\rangle \\
&= \epsilon^a S^b_a \pi(e_b)\left(T^i_j \pi(e_i)(S^d_c) S^e_a \eta_{de} + T^i_j S^d_c S^e_a \mathcal{B}_{ide}\right) \\
&\quad - \epsilon^a \epsilon^k \left(S^b_a \pi(e_b)(T^i_j) T^\ell_k \eta_{i\ell} + S^b_a T^i_j T^\ell_k \mathcal{B}_{bi\ell}\right) \\
&\qquad\qquad \left(T^m_k \pi(e_m)(S^d_c) S^e_a \eta_{de} + T^m_k S^d_c S^e_a \mathcal{B}_{mde}\right) \\
&= \pi(e_b)\left(T^i_j \pi(e_i)(S^d_c)\epsilon^a S^b_a S^e_a \eta_{de} + T^i_j S^d_c \epsilon^a S^b_a S^e_a \mathcal{B}_{ide}\right) \\
&- \epsilon^a S^b_a S^e_a \epsilon^k T^m_k T^\ell_k \left(\pi(e_b)(T^i_j)\eta_{i\ell} + T^i_j \mathcal{B}_{bi\ell}\right)\left(\pi(e_m)(S^d_c)\eta_{de} + S^d_c \mathcal{B}_{mde}\right) \\
&= \pi(e_b)\left(T^i_j \pi(e_i)(S^d_c)\eta^{be}\eta_{de} + T^i_j S^d_c \eta^{be}\mathcal{B}_{ide}\right) \\
&\quad - \eta^{be}\eta^{\ell m}\left(\pi(e_b)(T^i_j)\eta_{i\ell} + T^i_j \mathcal{B}_{bi\ell}\right)\left(\pi(e_m)(S^d_c)\eta_{de} + S^d_c \mathcal{B}_{mde}\right) \\
&= \pi(e_b)\left(T^i_j \pi(e_i)(S^b_c) + T^i_j S^d_c \mathcal{B}^b_{id}\right) \\
&\quad - \left(\pi(e_b)(T^m_j) + T^i_j \mathcal{B}^m_{bi}\right)\left(\pi(e_m)(S^b_c) + S^d_c \mathcal{B}^b_{md}\right) \\
&= \pi(e_b)(T^i_j)\pi(e_i)(S^b_c) + T^i_j \pi(e_b)\left(\pi(e_i)(S^b_c)\right) \\
&\qquad + \pi(e_b)(T^i_j) S^d_c \mathcal{B}^b_{id} + T^i_j \pi(e_b)(S^d_c)\mathcal{B}^b_{id} + T^i_j S^d_c \pi(e_b)(\mathcal{B}^b_{id}) \\
&\quad - \pi(e_b)(T^m_j)\pi(e_m)(S^b_c) - \pi(e_b)(T^m_j) S^d_c \mathcal{B}^b_{md} \\
&\qquad\qquad - T^i_j \pi(e_m)(S^b_c)\mathcal{B}^m_{bi} - T^i_j S^d_c \mathcal{B}^m_{bi}\mathcal{B}^b_{md} \\
&= T^i_j \pi(e_b)\left(\pi(e_i)(S^b_c)\right) + T^i_j \pi(e_b)(S^d_c)\mathcal{B}^b_{id} + T^i_j S^d_c \pi(e_b)(\mathcal{B}^b_{id}) \\
&\quad - T^i_j \pi(e_m)(S^b_c)\mathcal{B}^m_{bi} - T^i_j S^d_c \mathcal{B}^m_{bi}\mathcal{B}^b_{md} \\
&= T^i_j [\pi(e_b), \pi(e_i)](S^b_c) + T^i_j \pi(e_i)\left(\pi(e_b)(S^b_c)\right) \\
&\qquad\qquad - T^i_j \pi(e_b)(S^d_c)\mathcal{B}^b_{di} + T^i_j S^d_c \pi(e_b)(\mathcal{B}^b_{id}) \\
&\quad - T^i_j \pi(e_m)(S^d_c)\mathcal{B}^m_{di} - T^i_j S^d_c \mathcal{B}^m_{bi}\mathcal{B}^b_{md} \\
&= T^i_j \pi([e_b, e_i])(S^b_c) + T^i_j \pi(e_i)\left(\pi(e_b)(S^b_c)\right) + T^i_j S^d_c \pi(e_b)(\mathcal{B}^b_{id}) \\
&\quad - T^i_j \pi(e_t)(S^d_c)\mathcal{B}^t_{di} - T^i_j S^d_c \mathcal{B}^m_{bi}\mathcal{B}^b_{md} \\
&= T^i_j S^d_c \pi(e_b)(\mathcal{B}^b_{id}) - T^i_j S^d_c \mathcal{B}^m_{bi}\mathcal{B}^b_{md} + T^i_j \pi(e_i)\left(\pi(e_b)(S^b_c)\right)
\end{aligned}$$

The proof for $Ric^-_{\tilde{D}^0}$ is again the same. □

On the other hand, we can compute $Ric^+_{D^0}(\tilde{e}_j, \tilde{e}_c)$.

Lemma 4.14. In the notation of above we have

$$Ric^+_{D^0}(\tilde{e}_j, \tilde{e}_c) = T^i_j S^d_c \pi(e_a)(\mathcal{B}^a_{id}) - T^i_j S^d_c \mathcal{B}^k_{ai}\mathcal{B}^a_{kd}$$

as well as

$$Ric^-_{D^0}(\tilde{e}_b, \tilde{e}_k) = S^a_b T^\ell_k \pi(e_i)(\mathcal{B}^i_{a\ell}) - S^a_b T^\ell_k \mathcal{B}^c_{ia}\mathcal{B}^i_{c\ell}.$$

Proof. Since, as before, the proofs for $Ric^+_{D^0}$ and $Ric^-_{D^0}$ work the same way, we only prove the formula for $Ric^+_{D^0}$. Using the tensoriality of the Ricci tensor we see

$$\begin{aligned}
Ric^+_{D^0}(\tilde{e}_j, \tilde{e}_c) &= Ric^+_{D^0}(T^i_j e_i, S^d_c e_d) \\
&= T^i_j S^d_c Ric^+_{D^0}(e_i, e_d) \\
&= T^i_j S^d_c \left(\pi(e_a)(\eta^{aa}\mathcal{B}_{ida}) - \eta^{kk}\mathcal{B}_{aik}\eta^{aa}\mathcal{B}_{kda}\right) \\
&= \eta^{aa} T^i_j S^d_c \pi(e_a)(\mathcal{B}_{ida}) - \eta^{aa}\eta^{kk} T^i_j S^d_c \mathcal{B}_{aik}\mathcal{B}_{kda} \\
&= T^i_j S^d_c \pi(e_a)(\mathcal{B}^a_{id}) - T^i_j S^d_c \mathcal{B}^k_{ai}\mathcal{B}^a_{kd}.
\end{aligned}$$

□

Since both are compatible with the generalized metric $\mathcal{G}$, the Ricci tensors of D^0 and $\tilde{D}^0$ only depend on their divergences. In order to see how, we want to compute their difference.

Lemma 4.15. Let $v = \sum_r X^r e_r = \sum_r \tilde{X}^r \tilde{e}_r \in \Gamma(E)$ and D^0 and $\tilde{D}^0$ as above. Then

$$\delta_{D^0}(v) - \delta_{\tilde{D}^0}(v) = \sum_a \sum_b \tilde{X}^a \pi(e_b)(S^b_a) + \sum_i \sum_j \tilde{X}^i \pi(e_j)(T^j_i).$$

In particular,

$$\delta_{D^0}(\tilde{e}_a) - \delta_{\tilde{D}^0}(\tilde{e}_a) = \sum_b \pi(e_b)(S^b_a) \text{ and } \delta_{D^0}(\tilde{e}_i) - \delta_{\tilde{D}^0}(\tilde{e}_i) = \sum_j \pi(e_j)(T^j_i).$$

Proof. Note first that $\sum_b X^b e_b = \sum_a \tilde{X}^a \tilde{e}_a = \sum_a \sum_b \tilde{X}^a S^b_a e_b$. Hence $X^b = \sum_a \tilde{X}^a S^b_a$ and similarly $X^j = \sum_i \tilde{X}^i T^j_i$. Now we can compute

$$\begin{aligned}
\delta_{D^0}(v) - \delta_{\tilde{D}^0}(v) &= \sum_r \pi(e_r)(X^r) - \sum_r \pi(\tilde{e}_r)(\tilde{X}^r) \\
&= \sum_b \pi(e_b)(X^b) + \sum_j \pi(e_j)(X^j) - \sum_a \pi(\tilde{e}_a)(\tilde{X}^a) - \sum_i \pi(\tilde{e}_i)(\tilde{X}^i) \\
&= \sum_a \sum_b \pi(e_b)(\tilde{X}^a S^b_a) + \sum_i \sum_j \pi(e_j)(\tilde{X}^i T^j_i) \\
&\qquad - \sum_a \sum_b \pi(S^b_a e_b)(\tilde{X}^a) - \sum_i \sum_j \pi(T^j_i e_j)(\tilde{X}^i) \\
&= \sum_a \sum_b \left(\pi(e_b)(\tilde{X}^a) S^b_a + \tilde{X}^a \pi(e_b)(S^b_a)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_i\sum_j\Big(\pi(e_j)(\tilde X^i)T_i^j+\tilde X^i\pi(e_j)(T_i^j)\Big)\\
&\qquad-\sum_a\sum_b S_a^b\pi(e_b)(\tilde X^a)-\sum_i\sum_j T_i^j\pi(e_j)(\tilde X^i)\\
&=\sum_a\sum_b \tilde X^a\pi(e_b)(S_a^b)+\sum_i\sum_j \tilde X^i\pi(e_j)(T_i^j)
\end{aligned}
\qquad\square
$$

In particular this shows, in accordance with [G], that the Ricci tensors are the same, if the connections D^0 and $\tilde D^0$ have the same divergence.

Corollary 4.16. If the divergences δ_{D^0} of D^0 and $\delta_{\tilde D^0}$ of $\tilde D^0$ coincide, then their Ricci tensors are the same.

Proof. According to Lemma 4.13 and Lemma 4.14 we have

$$
Ric^+_{\tilde D^0}(\tilde e_j,\tilde e_c)=T_j^iS_c^d\pi(e_b)(\mathcal B^b_{id})-T_j^iS_c^d\mathcal B^m_{bi}\mathcal B^b_{md}+T_j^i\pi(e_i)\Big(\pi(e_b)(S_c^b)\Big)
$$

and

$$
Ric^+_{D^0}(\tilde e_j,\tilde e_c)=T_j^iS_c^d\pi(e_a)(\mathcal B^a_{id})-T_j^iS_c^d\mathcal B^k_{ai}\mathcal B^a_{kd}.
$$

Noting that

$$
\sum_b\pi(e_b)(S_a^b)=\delta_{D^0}(\tilde e_a)-\delta_{\tilde D^0}(\tilde e_a)=0,
$$

the claim now follows easily. We skip again the proof for the Ric^- case. $\square$

The following proposition gives an explicit formula for the difference of the Ricci tensors of D^0 and $\tilde D^0$ in terms of the difference of the divergences of the connections.

Proposition 4.17. For $e\in\Gamma(E)$ we define $\Delta(e):=\delta_{D^0}(e)-\delta_{\tilde D^0}(e)$. Let $u^-\in\Gamma(E_-)$ and $v^+\in\Gamma(E_+)$, then

$$
Ric^+_{\tilde D^0}(u^-,v^+)-Ric^+_{D^0}(u^-,v^+)=(D^+_{u^-}\Delta)(v^+),
$$

where D^+ is any generalized connection satisfying $D^+_{u^-}\tilde e_a=0$. Similarly, if $u^+\in\Gamma(E_+)$ and $v^-\in\Gamma(E_-)$, then

$$
Ric^-_{\tilde D^0}(u^+,v^-)-Ric^-_{D^0}(u^+,v^-)=(D^-_{u^+}\Delta)(v^-),
$$

where D^- is any generalized connection satisfying $D^-_{u^+}\tilde e_i=0$

Proof. Again we only provide the proof for Ric^+. We write $v^+=\sum_c X^ce_c=\sum_c\tilde X^c\tilde e_c$ and $u^-=\sum_j Y^je_j=\sum_j\tilde Y^j\tilde e_j$. In particular, $X^d=\sum_c\tilde X^cS_c^d$ and $Y^k=\sum_j\tilde Y^jT_j^k$. Note that

$$
\begin{aligned}
Ric^+_{D^0}(u^-,v^+)&=Ric^+_{D^0}(Y^je_j,X^ce_c)\\
&=X^cY^jRic^+_D(e_j,e_c)
\end{aligned}
$$

$$= X^c Y^j \pi(e_a)\mathcal{B}^a_{jc} - X^c Y^j \mathcal{B}^k_{aj}\mathcal{B}^a_{kc}$$

and, using Lemma 4.13,

$$\begin{aligned}
Ric^+_{\tilde{D}^0}(u^-, v^+) &= Ric^+_{\tilde{D}^0}(\tilde{Y}^j \tilde{e}_j, \tilde{X}^c \tilde{e}_c) \\
&= \tilde{X}^c \tilde{Y}^j Ric^+_{\tilde{D}}(\tilde{e}_j, \tilde{e}_c) \\
&= \tilde{X}^c \tilde{Y}^j T^i_j S^d_c \pi(e_b)(\mathcal{B}^b_{id}) - \tilde{X}^c \tilde{Y}^j T^i_j S^d_c \mathcal{B}^m_{bi}\mathcal{B}^b_{md} + \tilde{X}^c \tilde{Y}^j T^i_j \pi(e_i)(\pi(e_b)(S^b_c)) \\
&= X^d Y^i \pi(e_b)(\mathcal{B}^b_{id}) - X^d Y^i \mathcal{B}^m_{bi}\mathcal{B}^b_{md} + \tilde{X}^c Y^i \pi(e_i)(\pi(e_b)(S^b_c)) \\
&= X^d Y^i \pi(e_b)(\mathcal{B}^b_{id}) - X^d Y^i \mathcal{B}^m_{bi}\mathcal{B}^b_{md} + \tilde{X}^c \pi(u^-)(\pi(e_b)(S^b_c))
\end{aligned}$$

Using that $\Delta(v^+) = \tilde{X}^c \pi(e_b)(S^b_c)$, we compute

$$\begin{aligned}
Ric^+_{\tilde{D}^0}(u^-, v^+) - Ric^+_{D^0}(u^-, v^+) &= \tilde{X}^c \pi(u^-)(\pi(e_b)(S^b_c)) \\
&= \pi(u^-)\left(\tilde{X}^c \pi(e_b)(S^b_c)\right) - \pi(u^-)(\tilde{X}^c)\pi(e_b)(S^b_c) \\
&= \pi(u^-)\left(\Delta(v^+)\right) - \pi(u^-)(\tilde{X}^c)\Delta(\tilde{e}_c) \\
&= \pi(u^-)\left(\Delta(v^+)\right) - \pi(u^-)(\tilde{X}^c)\Delta(\tilde{e}_c) - \Delta(\tilde{X}^c D^+_{u^-}\tilde{e}_c) \\
&= \pi(u^-)\left(\Delta(v^+)\right) - \Delta(D^+_{u^-} v^+) \\
&= (D^+_{u^-}\Delta)(v^+) \qquad \square
\end{aligned}$$

Remark 4.18. It would be interesting to determine what kind of divergence operators can appear as divergence operators of generalized Levi-Civita connections, which are constructed form an orthonormal frame as above. If one had a description of the space of such divergence operators, one would get an algorithm to compute the Ricci tensors of pairs $(\mathcal{G}, \delta)$, where δ is the divergence of a generalized connection that can be constructed from an orthonormal frame.

5 Generalized Einstein metrics on Lie groups

In this section we develop a general approach for the study of left-invariant generalized Einstein metrics on Lie groups. We review many results from the previous chapters and apply them in this special setting. All necessary proofs are given for this special case. We also note how to get the results from our previous theory directly. The following sections are based on joint work with Vicente Cortés [CK].

5.1 Twisted generalized tangent bundle of a Lie group

Recall from Proposition 2.4 that the generalized tangent bundle of a smooth manifold M is the sum

$$\mathbb{T}M := TM \oplus T^*M$$

of its tangent and its cotangent bundle and that any closed three-form H on M defines on $\mathbb{T}M$ the structure of a Courant algebroid, see also [G, Example 2.5]. We will write $\mathbb{T}_pM$ for the fiber at $p \in M$.

In what follows we consider only the special case when $M = G$ is a Lie group and the Courant algebroid structure is left-invariant.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and H a closed left-invariant three-form on G. The H-twisted generalized tangent bundle of G is the vector bundle $\mathbb{T}G \to G$ endowed with the Courant algebroid structure $(\pi, \langle\cdot,\cdot\rangle, [\cdot,\cdot]_H)$ given by (see Proposition 2.4)

1. the canonical projection $\pi : \mathbb{T}G \to TG$, called the anchor,
2. the canonical symmetric bilinear pairing $\langle\cdot,\cdot\rangle \in \Gamma(\mathrm{Sym}^2(\mathbb{T}G)^*)$, given by

   $$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y) + \eta(Y)),$$

   called the scalar product, and
3. the (H-twisted) Dorfman bracket $[\cdot,\cdot]_H : \Gamma(\mathbb{T}G) \times \Gamma(\mathbb{T}G) \to \Gamma(\mathbb{T}G)$, given by

   $$[X+\xi, Y+\eta]_H = L_X(Y+\eta) - \iota_Y d\xi + H(X,Y,\cdot), \tag{10}$$

   where $X, Y \in \Gamma(TG)$, $\xi, \eta \in \Gamma(T^*G)$, L denotes the Lie derivative and ι the interior product.

The above data satisfy the defining axioms of a Courant algebroid:

(C1) $[u,[v,w]_H]_H = [[u,v]_H, w]_H + [v,[u,w]_H]_H$,

(C2) the homomorphism of bundles π is a bracket-homomorphism, that is

$$\pi[u,v]_H = [\pi u, \pi v],$$

where $[\pi u, \pi v] = L_{\pi u}(\pi v)$ denotes the Lie bracket of $\pi u, \pi v \in \Gamma(TG)$, (C3) the map $[u,\cdot]_H : \Gamma(\mathbb{T}G) \to \Gamma(\mathbb{T}G)$ satisfies the Leibniz rule:

$$[u, fv]_H = (\pi u)(f)v + f[u,v]_H, \quad \forall f \in C^\infty(M),$$

(C4) $\pi(u)\langle v,w\rangle = \langle [u,v]_H, w\rangle + \langle v, [u,w]_H\rangle$ and

(C5) $\pi(u)\langle v,w\rangle = \langle u, [v,w]_H + [w,v]_H\rangle$

for all $u,v,w \in \Gamma(\mathbb{T}G)$ (see Definition 2.1).

For notational simplicity we will from now on write

$$u(f) := (\pi u)(f). \tag{11}$$

We will identify left-invariant sections of $\mathbb{T}G$ (by evaluation at the neutral element $e \in G$) with elements

$$X + \xi \in E = E(\mathfrak{g}) := \mathfrak{g} \oplus \mathfrak{g}^* \tag{12}$$

and use the same notation to denote them. Correspondingly, the three-form $H \in \Gamma(\bigwedge^3 T^*G)$ will be identified with an element $H \in \bigwedge^3 \mathfrak{g}^*$. With these identifications, $\langle \cdot,\cdot\rangle \in \mathrm{Sym}^2 E^*$ and the Dorfman bracket of $X+\xi$ and $Y+\eta \in \mathfrak{g}\oplus\mathfrak{g}^*$ is

$$[X+\xi, Y+\eta]_H = [X,Y] - \mathrm{ad}^*_X\eta - \iota_Y d\xi + H(X,Y,\cdot) \in \mathfrak{g}\oplus\mathfrak{g}^*, \tag{13}$$

where $[X,Y]$ is the Lie bracket in $\mathfrak{g}$, $\mathrm{ad}^*_X\eta = \eta\circ\mathrm{ad}_X$ and d denotes the restriction of the de Rham differential to left-invariant forms, such that $-\iota_Y d\xi = \mathrm{ad}^*_Y\xi$.

5.2 Generalized metrics on Lie groups

We adapt Definition 2.16 to generalized tangent bundle.

Definition 5.1. A generalized pseudo-Riemannian metric on a manifold M is a section $\mathcal{G} \in \Gamma(\mathrm{Sym}^2(\mathbb{T}M)^*)$ such that the endomorphism $\mathcal{G}^{\mathrm{end}} \in \Gamma(\mathrm{End}\,\mathbb{T}M)$ defined by

$$\langle \mathcal{G}^{\mathrm{end}}\cdot,\cdot\rangle = \mathcal{G} \tag{14}$$

is an involution and $\mathcal{G}|_{\mathrm{Sym}^2(T^*M)}$ is non-degenerate. The pair $(M,\mathcal{G})$ is called a generalized pseudo-Riemannian manifold. The prefix pseudo will be omitted when $\mathcal{G}$ is positive definite.

Remark 5.2. Note that $\mathcal{G}|_{\mathrm{Sym}^2(T^*M)}$ to be non-degenerate is an additional requirement. It does not follow from restricting Definition 2.16 to the generalized tangent bundle $\mathbb{T}M$. However, the non-degeneracy of $\mathcal{G}|_{\mathrm{Sym}^2(T^*M)}$ is automatic if $\mathcal{G}$ is positive or negative definite. Generalized metrics satisfying this additional assumption are sometimes called admissible in the literature [GS].

Recall that for a generalized metric the equation (14) is equivalent to $\mathcal{G}^{\mathrm{end}} = \mathcal{G}^{-1}\circ\langle\cdot,\cdot\rangle$, using the identification $(\mathbb{T}M)^*\otimes(\mathbb{T}M)^* = \mathrm{Hom}(\mathbb{T}M, (\mathbb{T}M)^*)$ given by evaluation in the first argument.

A left-invariant generalized metric on a Lie group G is identified (by evaluation at the neutral element $e \in G$) with a generalized metric on $\mathfrak{g} = \mathrm{Lie}\,G$ as defined in the following definition.

Definition 5.3. Let H be a left-invariant closed three-form on a Lie group G, which we identify (by evaluation at $e \in G$) with an element $H \in \bigwedge^3 \mathfrak{g}^*$. A generalized (pseudo-Riemannian) metric on its Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ is a symmetric bilinear form $\mathcal{G} \in \operatorname{Sym}^2 E^*$, compare (12), such that $\mathcal{G}^{\mathrm{end}} = \mathcal{G}^{-1} \circ \langle \cdot, \cdot \rangle$ is an involution and $\mathcal{G}|_{\operatorname{Sym}^2 \mathfrak{g}^*}$ is non-degenerate. The corresponding triple $(G, H, \mathcal{G})$ will be called a pseudo-Riemannian generalized Lie group and $(\mathfrak{g}, H, \mathcal{G})$ a pseudo-Riemannian generalized Lie algebra. The prefix pseudo will be omitted when $\mathcal{G}$ is positive definite.

Two pseudo-Riemannian generalized Lie groups $(G, H, \mathcal{G})$ and $(G', H', \mathcal{G}')$ are called isomorphic if there exists an isomorphism of Lie groups $\varphi : G \to G'$ and an isomorphism of bundles $\Phi : \mathbb{T}G \to \mathbb{T}G'$ covering φ such that Φ maps the Courant algebroid structure $(\pi, \langle \cdot, \cdot \rangle, [\cdot, \cdot]_H)$ on G determined by H to the Courant algebroid structure on G' determined by H' and the generalized metric $\mathcal{G}$ to the generalized metric $\mathcal{G}'$. The map Φ is called an isomorphism of pseudo-Riemannian generalized Lie groups.

Similarly, two pseudo-Riemannian generalized Lie algebras $(\mathfrak{g}, H, \mathcal{G})$ and $(\mathfrak{g}', H', \mathcal{G}')$ are called isomorphic if there exists an isomorphism of Lie algebras $\varphi : \mathfrak{g} \to \mathfrak{g}'$ and an isomorphism of vector spaces $\phi : E(\mathfrak{g}) \to E(\mathfrak{g}')$ covering φ which maps the data $(\langle \cdot, \cdot \rangle, [\cdot, \cdot]_H, \mathcal{G})$ on $\mathfrak{g}$, cf. (13), to the data $(\langle \cdot, \cdot \rangle', [\cdot, \cdot]_{H'}, \mathcal{G}')$ on $\mathfrak{g}'$. Here $\langle \cdot, \cdot \rangle'$ denotes the canonical symmetric pairing on $E(\mathfrak{g}')$ induced by the duality between $\mathfrak{g}'$ and $(\mathfrak{g}')^*$. The map ϕ is called an isomorphism of pseudo-Riemannian generalized Lie algebras.

Example 5.4. Let g be a left-invariant pseudo-Riemannian metric on G. We denote the corresponding bilinear form on the Lie algebra $\mathfrak{g}$ by the same symbol: $g \in \operatorname{Sym}^2 \mathfrak{g}^*$. It extends to a generalized metric $\mathcal{G}_g \in \operatorname{Sym}^2 E^*$ such that

$$\mathcal{G}_g(X + \xi, Y + \eta) = \frac{1}{2}(g(X, Y) + g^{-1}(\xi, \eta))$$

for all $X + \xi, Y + \eta \in E$. The corresponding endomorphism $\mathcal{G}^{\mathrm{end}}$ is

$$\mathcal{G}^{\mathrm{end}} = g \oplus g^{-1} : E = \mathfrak{g} \oplus \mathfrak{g}^* \to E^* = \mathfrak{g}^* \oplus \mathfrak{g}.$$

Proposition 5.5. Let $(G, H, \mathcal{G})$ be a pseudo-Riemannian generalized Lie group. Then there exist a left-invariant pseudo-Riemannian metric g on G and a closed left-invariant three-form $H' \in [H] \in H^3(\mathfrak{g})$ such that $(G, H, \mathcal{G})$ is isomorphic to $(G, H', \mathcal{G}_g)$, by an isomorphism Φ covering the identity map of G.

Remark 5.6. Recall that we have seen this statement for generalized Riemannian metrics on arbitrary manifolds M in Section 3.2. The following proof works the same way for generalized pseudo-Riemannian metrics on arbitrary manifolds M.

Proof. The decomposition $E = \mathfrak{g} \oplus \mathfrak{g}^*$ gives rise to the following block decomposition

$$2\mathcal{G} = \begin{pmatrix} h & A^* \\ A & \gamma \end{pmatrix},$$

where $h \in \operatorname{Sym}^2 \mathfrak{g}$, $A \in \operatorname{End}(\mathfrak{g})$ and $\gamma \in \operatorname{Sym}^2 \mathfrak{g}^*$ is non-degenerate, as follows from the symmetry of $\mathcal{G}$ and the non-degeneracy of $\mathcal{G}|_{\operatorname{Sym}^2 \mathfrak{g}^*}$. In terms of $g := \gamma^{-1} \in \operatorname{Sym}^2 \mathfrak{g}$ we can write the necessary and sufficient conditions for

$$\mathcal{G}^{\mathrm{end}} = \begin{pmatrix} A & g^{-1} \\ h & A^* \end{pmatrix} \tag{15}$$

to be an involution as:

$$A^2 + g^{-1}h = \mathbf{1}, \quad gA = -A^*g, \quad hA = -A^*h,$$

where the last two equations mean that A is skew-symmetric for g and h. In particular, we can write $A = -g^{-1}\beta$ for some $\beta \in \bigwedge^2 \mathfrak{g}^*$. Solving the first equation for h we obtain

$$h = g - gA^2 = g + \beta A = g - \beta g^{-1}\beta.$$

This implies that $\mathcal{G}^{\mathrm{end}} = \exp(B)(\mathcal{G}_g)^{\mathrm{end}} \exp(-B)$, where

$$B = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix},$$

or equivalently, $\mathcal{G} = \exp(-B)^*\mathcal{G}_g$. Now it suffices to check that the map

$$\phi = \exp(-B) : E \to E, \quad X + \xi \mapsto X + \xi - \beta X,$$

defines an isomorphism of pseudo-Riemannian generalized Lie algebras from $(\mathfrak{g}, H, \mathcal{G})$ to $(\mathfrak{g}, H', \mathcal{G}_g)$ covering the identity map of $\mathfrak{g}$, where $H' = H + d\beta$. The corresponding isomorphism Φ of pseudo-Riemannian generalized Lie groups is also given by $\exp(-B)$, now considered as an endomorphism of $\mathbb{T}G$. $\square$

Remark 5.7. Clearly, a decomposition of the form (15) holds for any generalized pseudo-Riemannian metric $\mathcal{G}$ on a manifold M. This shows that $\operatorname{tr} \mathcal{G}^{\mathrm{end}} = 0$, since A is skew-symmetric with respect to g.

5.3 Space of left-invariant Levi-Civita generalized connections

Let H be a closed three-form on a smooth manifold M and consider $\mathbb{T}M$ with the Courant algebroid structure defined by H. Recall Definition 2.6 of a generalized connection.

Definition 5.8. A generalized connection on M is a linear map

$$D : \Gamma(\mathbb{T}M) \to \Gamma((\mathbb{T}M)^* \otimes \mathbb{T}M), \quad v \mapsto Dv = (u \mapsto D_u v),$$

such that

(i) $D_u(fv) = u(f)v + fD_u v$ (anchored Leibniz rule), recall (11), and

(ii) $u\langle v, w\rangle = \langle D_u v, w\rangle + \langle v, D_u w\rangle$

for all $u, v, w \in \Gamma(\mathbb{T}M)$.

The torsion (see Definition 2.9) of a generalized connection D (with respect to the Dorfman bracket $[\cdot,\cdot]_H$) is the section $T \in \Gamma(\bigwedge^2(\mathbb{T}M)^* \otimes \mathbb{T}M)$ defined by

$$T_D(u,v) := D_u v - D_v u - [u,v]_H + (Du)^* v,$$

where $(Du)^*$ is the adjoint of Du with respect to the scalar product, compare [G]. The generalized connection D is called torsion-free if $T_D = 0$.

Given a generalized pseudo-Riemannian metric $\mathcal{G}$ on M, we say that a generalized connection D is metric if $D\mathcal{G} = 0$, where $D_u : \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ is extended to space of sections of the tensor algebra over $\mathbb{T}M$ as a tensor derivation for all $u \in \Gamma(\mathbb{T}M)$. More explicitly, the latter condition is

$$u\mathcal{G}(v,w) = \mathcal{G}(D_u v, w) + \mathcal{G}(v, D_u w), \quad \forall u,v,w \in \Gamma(\mathbb{T}M).$$

This condition is satisfied if and only if D preserves the eigenbundles of $\mathcal{G}^{\mathrm{end}}$.

Any metric and torsion-free generalized connection on a generalized pseudo-Riemannian manifold $(M,\mathcal{G})$ (endowed with the three-form H) is called a Levi-Civita generalized connection (see also Section 2.4).

We have seen in Lemma 2.10 that the torsion of a generalized connection is totally skew, that is $T_D \in \Gamma(\bigwedge^2(\mathbb{T}M)^* \otimes \mathbb{T}M)$ defines a section of $\bigwedge^3(\mathbb{T}M)^*$ upon identification $\mathbb{T}M \cong (\mathbb{T}M)^*$ using the scalar product.

Given a reduction of the structure group $\mathrm{O}(n,n)$ of $\mathbb{T}M$, $n = \dim M$, to a subgroup $L = \mathrm{O}(n,n)_S \subset \mathrm{O}(n,n)$ defined by tensor $S \in \bigoplus_{k=0}^{\infty} \bigotimes^k (\mathbb{R}^n \oplus (\mathbb{R}^n)^*)$, we consider the tensor field $\mathcal{S}$ which in any frame of the reduction has the same coefficients as S in the standard basis of $\mathbb{R}^n \oplus (\mathbb{R}^n)^*$. A generalized connection D is called compatible with the L-reduction if $D\mathcal{S} = 0$. It was shown in [CD] that a torsion-free generalized connection (on a Courant algebroid) compatible with an L-reduction exists if and only if its intrinsic torsion (defined in [CD, Definition 15]) vanishes. In that case, it was also shown there that the space of compatible torsion-free generalized connections is an affine space modeled on the space of sections of the generalized first prolongation $(\mathfrak{so}(\mathbb{T}M)_{\mathcal{S}})^{\langle 1\rangle}$ (defined in [CD, Definition 16]) of $\mathfrak{so}(\mathbb{T}M)_{\mathcal{S}}$. Note that the fiber of the bundle $\mathfrak{so}(\mathbb{T}M)_{\mathcal{S}}$ at a point $p \in M$ is $\mathfrak{so}(\mathbb{T}_pM)_{\mathcal{S}_p} \cong \mathfrak{so}(n,n)_S = \mathfrak{l} = \mathrm{Lie}\, L$, so that $(\mathfrak{so}(\mathbb{T}M)_{\mathcal{S}})^{\langle 1\rangle}|_p \cong \mathfrak{l}^{\langle 1\rangle}$.

As a special case, we can apply the above theory to the case when $\mathcal{S} = \mathcal{G}$ is a generalized pseudo-Riemannian metric. The existence of a Levi-Civita generalized connection shown in [G, Proposition 3.3] (see also Section 2.4) implies the following.

Proposition 5.9. Let $(M,\mathcal{G})$ be a generalized pseudo-Riemannian manifold and H a closed three-form on M. Then the space of Levi-Civita generalized connections (with respect to the H-twisted Dorfman bracket) is an affine space modeled on $(\mathfrak{so}(\mathbb{T}M)_{\mathcal{G}})^{\langle 1\rangle}$.

A generalized connection D on a Lie group G is called left-invariant if $D_u v \in \Gamma(\mathbb{T}G)$ is left-invariant for all left-invariant sections $u,v \in \Gamma(\mathbb{T}G)$. A left-invariant generalized connection on G can be identified with an element $D \in E^* \otimes \mathfrak{so}(E)$, where we recall that $E = \mathfrak{g} \oplus \mathfrak{g}^*$. Its torsion T_D is identified with an element $T_D \in (\bigwedge^2 E^* \otimes E) \cap (E^* \otimes \mathfrak{so}(E)) \cong \bigwedge^3 E^*$. We denote by E_+ and E_- the eigenspaces of $\mathcal{G}^{\mathrm{end}} \in \mathrm{End}(E)$ for the eigenvalues ± 1, respectively. Note that $\dim E_+ = \dim E_- = \dim G =: n$ by Remark 5.7.

Proposition 5.10. Let $(G, H, \mathcal{G})$ be a pseudo-Riemannian generalized Lie group. Then the space of left-invariant Levi-Civita generalized connections on G is an affine space modeled on $\mathfrak{so}(E)^{\langle 1 \rangle} = \Sigma_+ \oplus \Sigma_-$, where $\Sigma_+ \subset E_+^* \otimes \mathfrak{so}(E_+)$ is the kernel of the map

$$\partial : E_+^* \otimes \mathfrak{so}(E_+) \to \bigwedge^3 E^*$$

defined by

$$(\partial\alpha)(u, v, w) = \sum_{\mathfrak{S}} \langle \alpha_u v, w \rangle \quad u, v, w \in E, \tag{16}$$

and similarly for $\Sigma_- \subset E_-^* \otimes \mathfrak{so}(E_-)$. Here $\mathfrak{S}$ indicates the sum over the cyclic permutations and $\alpha_u \in \mathfrak{so}(E_+)$ stands for evaluation of $\alpha \in E_+^* \otimes \mathfrak{so}(E_+) = \mathrm{Hom}(E_+, \mathfrak{so}(E_+))$ at u.

Moreover,

$$\Sigma_+ = \mathrm{im}(\mathrm{alt}) \cong \frac{\mathrm{Sym}^2 E_+ \otimes E_+}{\mathrm{Sym}^3 E_+}$$

is the image of the map

$$\mathrm{alt} : \mathrm{Sym}^2 E_+^* \otimes E_+^* \to E_+^* \otimes \mathfrak{so}(E_+)$$

defined by

$$\langle \mathrm{alt}(\sigma)_u v, w \rangle = \sigma(u, v, w) - \sigma(u, w, v)$$

and similarly for Σ_-.

Proof. The first part of the proposition follows easily from the existence of a left-invariant Levi-Civita generalized connection (to be shown at the end of the proof), Proposition 5.9 and the definition of the generalized first prolongation [CD] as the kernel of the natural map

$$\partial : E^* \otimes \mathfrak{so}(E)_{\mathcal{G}} \to \bigwedge^3 E^*$$

given by the formula (16). To compute the kernel we can first observe that $\mathfrak{so}(E)_{\mathcal{G}} = \mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-) \cong \bigwedge^2 E_+^* \oplus \bigwedge^2 E_-^*$. Since ∂ maps $E_{\epsilon_1}^* \otimes \mathfrak{so}(E_{\epsilon_2})$ to $E_{\epsilon_1}^* \wedge E_{\epsilon_2}^* \wedge E_{\epsilon_2}^* \subset \bigwedge^3 E^*$, $\epsilon_1, \epsilon_2 \in \{-1, 1\}$, it suffices to consider the kernels of these four restrictions. On tensors of mixed type ∂ is injective, such that $\ker \partial = \Sigma_+ \oplus \Sigma_-$. The last part of the corollary follows from the exact sequence

$$0 \to \mathrm{Sym}^3 V \to \mathrm{Sym}^2 V \otimes V \xrightarrow{\mathrm{alt}_V} V \otimes \bigwedge^2 V \xrightarrow{\partial_V} \bigwedge^3 V \to 0 \tag{17}$$

that holds for any finite-dimensional vector space V and was used in [G] and we have explored in Lemma 2.12. Here alt_V is given by

$$(u \otimes v + v \otimes u) \otimes w \mapsto u \otimes v \wedge w + v \otimes w \wedge u$$

and ∂_V by

$$u \otimes v \wedge w \mapsto u \wedge v \wedge w.$$

We apply the sequence to $V = E_+$ (and similarly to $V = E_-$) using the metric identifications $E_+ \cong E_+^*$ and $\mathfrak{so}(E_+) \cong \bigwedge^2 E_+^* \cong \bigwedge^2 E_+$, which allow to identify the natural maps alt_V and ∂_V with $\mathrm{alt} : \mathrm{Sym}^2 E_+^* \otimes E_+^* \to E_+^* \otimes \mathfrak{so}(E_+)$ and $\partial : E_+^* \otimes \mathfrak{so}(E_+) \to \bigwedge^3 E_+^*$, respectively.

Now it suffices to show that there exists a left-invariant Levi-Civita generalized connection. We consider the tensor $\mathcal{B} \in \bigotimes^3 E^*$ defined by

$$\mathcal{B}(u, v, w) = \langle [u, v]_H, w\rangle, \quad u, v, w \in E. \tag{18}$$

Lemma 5.11. $\mathcal{B}$ is totally skew.

Proof. The skew-symmetry in (u, v) follows from axiom (C5) in Section 5.1:

$$\mathcal{B}(u, v, w) + \mathcal{B}(v, u, w) = \langle w, [u, v]_H + [v, u]_H\rangle = w\langle u, v\rangle = 0,$$

since $\langle u, v\rangle$ is a constant function. Using axiom (C4), we obtain

$$\mathcal{B}(u, v, w) = \langle [u, v]_H, w\rangle = u\langle v, w\rangle - \langle v, [u, w]_H\rangle = -\mathcal{B}(u, w, v).$$

Now it suffices to observe that skew-symmetry in (u, v) and (v, w) implies total skew-symmetry. $\square$

Remark 5.12. This follows also from Lemma 4.2 and Lemma 4.3, where we use that $\pi(e_r)(f) = 0$ for all left-invariant, hence constant, functions f and $r \in \{1, \ldots, 2n\}$.

Next we define as in Section 4.2

$$D^0 := \frac{1}{3}\mathcal{B}|_{\bigwedge^3 E_+} \oplus \frac{1}{3}\mathcal{B}|_{\bigwedge^3 E_-} \oplus \mathcal{B}|_{E_+\otimes\bigwedge^2 E_-} \oplus \mathcal{B}|_{E_-\otimes\bigwedge^2 E_+}.$$

As an element of $E^* \otimes \bigwedge^2 E^* \cong E^* \otimes \mathfrak{so}(E)$, it defines a left-invariant generalized connection. It is metric, since it takes values in the subalgebra $\mathfrak{so}(E_+) \oplus \mathfrak{so}(E_-) \subset \mathfrak{so}(E)$. Since $\partial\mathcal{B}|_{\bigwedge^3 E_\pm} = 3\mathcal{B}|_{\bigwedge^3 E_\pm}$ and $\partial\mathcal{B}|_{E_\mp\otimes\bigwedge^2 E_\pm} = \mathcal{B}|_{E_\mp\wedge E_\pm\wedge E_\pm}$, the torsion $T_{D^0} = \partial D^0 - \mathcal{B}$ of D^0 is given by

$$T_{D^0} = \left(\mathcal{B}|_{\bigwedge^3 E_+} \oplus \mathcal{B}|_{\bigwedge^3 E_-} \oplus \partial\mathcal{B}|_{E_+\otimes\bigwedge^2 E_-} \oplus \partial\mathcal{B}|_{E_-\otimes\bigwedge^2 E_+}\right) - \mathcal{B} = \mathcal{B} - \mathcal{B} = 0.$$

(Alternatively, one can use Lemma 4.6 to show that D^0 is a Levi-Civita generalized connection) $\square$

5.4 Levi-Civita generalized connections with prescribed divergence

In this subsection we show that every left-invariant divergence operator on the generalized tangent bundle of a generalized pseudo-Riemannian Lie group admits a compatible left-invariant Levi-Civita generalized connection. We then give an explicit construction of such a generalized connection in the case when $\mathcal{G}$ is associated with a left-invariant pseudo-Riemannian metric as in Example 5.4. In view of Proposition 5.5 there is no loss in generality by considering this special case. Recall Definitions 2.13 and 2.14.

Definition 5.13. A divergence operator on $\mathbb{T}M$ is a first order differential operator $\delta : \Gamma(\mathbb{T}M) \to C^\infty(M)$ which satisfies

$$\delta(fv) = v(f) + f\delta v,$$

for all $v \in \Gamma(\mathbb{T}M)$, $f \in C^\infty(M)$.

Example 5.14. Let D be a generalized connection on M. Then

$$\delta_D v = \operatorname{tr} Dv, \quad v \in \Gamma(\mathbb{T}M),$$

defines a divergence operator on $\mathbb{T}M$.

When $M = G$ is a Lie group we can ask for a divergence operator δ on $\mathbb{T}G$ to be left-invariant, that is for the function δv to be left-invariant (i.e. constant) for all left-invariant sections v of $\mathbb{T}G$. Such operators can can be identified with elements of $E^* = (\mathbb{T}_e G)^*$.

It was proved in [G] that there always exists a Levi-Civita generalized connection with a prescribed divergence. We now give a proof for this in our setting.

Proposition 5.15. Let $(G, H, \mathcal{G})$ be a generalized pseudo-Riemannian Lie group of dimension $\dim G \geq 2$ and $\delta \in E^*$. Then there exists a left-invariant Levi-Civita generalized connection D such that $\delta_D = \delta$.

Proof. Let $D \in E^* \otimes \mathfrak{so}(E)$ be a left-invariant Levi-Civita generalized connection. Any other left-invariant Levi-Civita generalized connection can be written as $D' = D + S$, where $S \in \mathfrak{so}(E)^{\langle 1 \rangle} \subset E^* \otimes \mathfrak{so}(E)$ (see Proposition 5.10). The divergence operators are related by

$$\delta_{D'} v - \delta_D v = \operatorname{tr} Sv = \operatorname{tr}(u \mapsto S_u v), \quad v \in E. \tag{19}$$

We consider the linear form $\lambda_S \in E^*$ defined by

$$\lambda_S(v) := \operatorname{tr} Sv. \tag{20}$$

It suffices to show that the linear map $S \mapsto \lambda_S$ is surjective. Given $\alpha, \beta \in E_+^* \cong (E_-)^0 \subset E^*$, the element $S = \operatorname{alt}(\alpha^2 \otimes \beta) \in \Sigma_+ \subset \mathfrak{so}(E)^{\langle 1 \rangle} = \Sigma_+ \oplus \Sigma_-$ has

$$\lambda_S = \langle \alpha, \beta \rangle \alpha - \langle \alpha, \alpha \rangle \beta. \tag{21}$$

Since $\dim E_+ = \dim G \geq 2$, this proves that $\operatorname{span}\{\lambda_S \mid S \in \Sigma_+\} = E_+^*$, and similarly $\operatorname{span}\{\lambda_S \mid S \in \Sigma_-\} = E_-^*$. □

Note that the condition $\dim G \geq 2$ is necessary. If $\dim G = 1$, then the Levi-Civita generalized connection D is unique and $\delta_D \in E^*$ is zero.

From now on we assume without loss of generality (see Proposition 5.5) that $\mathcal{G} = \mathcal{G}_g$ for some left-invariant pseudo-Riemannian metric g on G. We will first construct a particular left-invariant Levi-Civita generalized connection D with $\delta_D = 0 \in E^*$ and later prescribe an arbitrary divergence operator by adding a suitable element of the generalized first prolongation.

Adapted bases and notation. Let $(v_a) = (v_1, \ldots, v_n)$ be a g-orthonormal basis of $\mathfrak{g}$ and set $\varepsilon_a \coloneqq g(v_a, v_a)$. Then

$$e_a \coloneqq v_a + gv_a \tag{22}$$

defines a $\mathcal{G}$-orthonormal basis $(e_a)_{a=1,\ldots,n}$ of E_+ with $\mathcal{G}(e_a, e_a) = \varepsilon_a$ and

$$e_{n+a} \coloneqq v_a - gv_a \tag{23}$$

defines a $\mathcal{G}$-orthonormal basis $(e_i)_{i=n+1,\ldots,2n}$ of E_- with $\mathcal{G}(e_{n+a}, e_{n+a}) = \varepsilon_a$. Remember that $\langle \cdot, \cdot \rangle = \pm\mathcal{G}$ on the summands $E_\pm$ of the decomposition $E = E_+ \oplus E_-$, which is orthogonal for both the generalized metric $\mathcal{G}$ as well as the scalar product $\langle \cdot, \cdot \rangle$. Summarizing, we have an orthonormal basis $(e_r)_{r=1,\ldots,2n}$ of E adapted to the decomposition $E = E_+ \oplus E_-$. Note that $\langle e_r, e_s \rangle = \varepsilon_r \delta_{rs}$, where $\varepsilon_a = -\varepsilon_{n+a}$ for $a = 1, \ldots n$. From now on the indices $a, b, \ldots$ will always range from 1 to n, $i, j, \ldots$ will range from $n+1$ to $2n$ and $r, s, \ldots$ from 1 to $2n$. Note that this is in accordance with the notation in Section 4.

A left-invariant generalized connection D is completely determined by its coefficients ω_{rs}^t with respect to the basis (e_r):

$$D_{e_r} e_s = \omega_{rs}^t e_t,$$

where from now on we use Einstein's summation convention, according to which the sum over an upper and a lower repeated index is understood. Equivalently, we may use

$$\omega_{rst} \coloneqq \langle D_{e_r} e_s, e_t \rangle, \tag{24}$$

which has the advantage that it is skew-symmetric in (s, t). In fact, any tensor (ω_{rst}) skew-symmetric in (s, t) defines a left-invariant generalized connection D by the formula (24). We will say that (ω_{rst}) are the connection coefficients of D.

The next proposition follows from the fact that D is metric if and only if $DE_\pm \subset E_\pm$.

Proposition 5.16. A left-invariant generalized connection D is metric if and only if $\omega_{rst} = 0$ whenever $s \in \{1, \ldots, n\}$ and $t \in \{n+1, \ldots, 2n\}$.

Using the orthonormal basis (e_r) of E we define as in Section 4.1

$$\mathcal{B}_{rst} \coloneqq \mathcal{B}(e_r, e_s, e_t) = \langle [e_r, e_s]_H, e_t \rangle. \tag{25}$$

Proposition 5.17. Let $(G, H, \mathcal{G}_g)$ be a generalized pseudo-Riemannian Lie group. The following tensor (ω_{rst}) defines the connection coefficients of a left-invariant Levi-Civita generalized connection D^0 with zero divergence δ_{D^0}:

$$\omega_{abc} \coloneqq \frac{1}{3}\mathcal{B}_{abc}, \quad \omega_{ijk} \coloneqq \frac{1}{3}\mathcal{B}_{ijk}, \quad \omega_{ibc} \coloneqq \mathcal{B}_{ibc}, \quad \omega_{ajk} \coloneqq \mathcal{B}_{ajk}, \tag{26}$$

where $a, b, c \in \{1, \ldots, n\}$ and $i, j, k \in \{n+1, \ldots, 2n\}$ and the remaining components are zero. The connection D^0 does not depend on the choice of orthonormal basis (v_a) of $\mathfrak{g}$, from which the orthonormal basis (e_r) of $E = \mathfrak{g} \oplus \mathfrak{g}^*$ was constructed. It is therefore a canonical Levi-Civita generalized connection and will be called the canonical divergence-free Levi-Civita generalized connection.

Proof. The formulas (26) are precisely the connection coefficients of the left-invariant Levi-Civita generalized connection D^0 defined in the proof of Proposition 5.10. In particular, D^0 is independent of the basis (v_a). To show that the divergence δ of D^0 vanishes, it suffices to remark that $\delta(e_s) = \omega^r_{sr}$ vanishes due to $\omega_{ajc} = \omega_{ibk} = 0$ and the total skew-symmetry of ω_{abc} and ω_{ijk} (with the above index ranges), implied by Lemma 5.11. □

Remark 5.18. Another way to see that D^0 is to has zero divergence is directly from Lemma 4.7, noticing that again the derivatives of the left-invariant (i.e. constant) functions X^r are zero.

Proposition 5.19. Let $(G, H, \mathcal{G}_g)$ be a generalized pseudo-Riemannian Lie group endowed with the canonical divergence-free Levi-Civita generalized connection D^0 of Proposition 5.17. Fix an element $\delta \in E^*$. Then a left-invariant Levi-Civita generalized connection D with divergence $\delta_D = \delta$ can be obtained as follows. Choose, as above[5], a left-invariant orthonormal basis (e_r) of E associated with an orthonormal basis of $\mathfrak{g}$. Define the tensor $S := S_+ + S_-$ where

$$S_+ := -\mathrm{alt}\left(\delta_1\varepsilon_2(e^2)^2 \otimes e^1 + \sum_{a=2}^{n} \delta_a\varepsilon_1(e^1)^2 \otimes e^a\right) \in \Sigma_+,$$

and similarly for $S_- \in \Sigma_-$. Here (e^r) denotes the basis of E^* dual to (e_r) and $\delta_r = \delta(e_r)$. Then the left-invariant Levi-Civita generalized connection $D = D^0 + S$ has divergence δ.

Proof. From (19), (20) and (21) we see that $D = D^0 + S$ has divergence δ, since

$$\lambda_{S_+} = -\delta_1\varepsilon_2\lambda_{(e^2)^2\otimes e^1} - \sum_{a=2}^{n} \delta_a\varepsilon_1\lambda_{(e^1)^2\otimes e^a} = \sum_{a=1}^{n} \delta_a e^a = \delta|_{E_+}$$

and similarly $\lambda_{S_-} = \delta|_{E_-}$. □

We want to close this section by introducing a special divergence operator, the so-called Riemannian divergence, which is considered in the literature ([GSt, Definition 2.46]). If $(M, \mathcal{G})$ is a generalized pseudo-Riemannian manifold, one defines for all $v \in \Gamma(\mathbb{T}M)$

$$\delta^{\mathcal{G}}(v) = \mathrm{tr}\,(\nabla\pi v) = \mathrm{tr}\,(\Gamma(TM) \ni Y \mapsto \nabla_Y\pi(v) \in \Gamma(TM)),$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian metric g associated to $\mathcal{G}$ via Proposition 5.5. Denoting by μ the Riemannian density associated to g, we recall the well known fact that the divergence $\mathrm{tr}\,(\nabla X)$ of a vector field X can also be expressed by $\frac{L_X\mu}{\mu}$, since

$$L_X\mu = \nabla_X\mu - (\nabla X)\cdot\mu = \mathrm{tr}(\nabla X)\mu.$$

[5] Compare (22) and (23).

The divergence operator $\delta^{\mathcal{G}}$ can be recovered as the divergence of a generalized connection as in Example 5.14. For that one first extends the Levi-Civita connection to a connection on $\mathbb{T}M$ and then pulls it back to a generalized connection $\tilde{\tilde{\nabla}}$ via the anchor π. Then

$$\delta_{\tilde{\tilde{\nabla}}}(v) = \mathrm{tr}_{\mathbb{T}M}\left(\tilde{\nabla} v\right) = \mathrm{tr}_{TM}\left(\nabla \pi(v)\right) = \delta^{\mathcal{G}}(v),$$

since $\tilde{\nabla} v|_{T^*M} = 0$ and $\pi \circ \tilde{\nabla} v|_{TM} = \nabla \pi(v)$. Furthermore, note that $\tilde{\tilde{\nabla}}$ is a Levi-Civita generalized connection of $\mathcal{G}$, if $\mathcal{G} = \mathcal{G}^g$ and $H = 0$.

Proposition 5.20. Let $(G, H, \mathcal{G})$ be a generalized pseudo-Riemannian Lie group. Then the Riemannian divergence satisfies

$$\delta^{\mathcal{G}}(v) = -\tau(\pi(v)), \quad v \in E,$$

where $\tau \in \mathfrak{g}^*$ is the trace-form defined by $\tau(X) = \mathrm{tr}\,\mathrm{ad}_X$, $X \in \mathfrak{g}$. In particular, the Riemannian divergence is zero, if the Lie group G is unimodular.

Proof. Let $v = X + \xi \in E$ and (v_a) as usual a basis of $\mathfrak{g}$, which is orthonormal with respect to g. Furthermore, let ∇ be the Levi-Civita connection of g. It satisfies

$$g\left(\nabla_X Y, Z\right) = \frac{1}{2}\left(g\left([X,Y], Z\right) - g\left([Y,Z], X\right) + g\left([Z,X], Y\right)\right)$$

for $X, Y, Z \in \mathfrak{g}$. We can thus compute

$$\begin{aligned}
\delta^{\mathcal{G}}(X+\xi) &= \mathrm{tr}(\nabla X) \\
&= \sum_a \varepsilon_a g(\nabla_{v_a} X, v_a) \\
&= \frac{1}{2}\sum_a \varepsilon_a \left(g([v_a, X], v_a) - g([X, v_a], v_a) + g([v_a, v_a], X)\right) \\
&= -\sum_a \varepsilon_a g([X, v_a], v_a) \\
&= -\,\mathrm{tr}\,\mathrm{ad}_X \\
&= -\tau(\pi(X+\xi)).
\end{aligned}$$

□

5.5 Ricci curvatures and generalized Einstein metrics

After fixing a left-invariant section δ of $(\mathbb{T}G)^*$ over a generalized pseudo-Riemannian Lie group $(G, H, \mathcal{G})$ we define and compute two canonical Ricci curvature tensors $Ric^+ \in E_-^* \otimes E_+^*$ and $Ric^- \in E_+^* \otimes E_-^*$, which depend only on the data $(H, \mathcal{G}, \delta)$. A left-invariant solution $\mathcal{G}$ of the system $Ric^+ = 0, Ric^- = 0$ is what we will call a generalized Einstein metric on G with three-form H and dilaton δ.

Consider the generalized tangent bundle $\mathbb{T}M$ of a smooth manifold endowed with the Courant algebroid structure associated with a closed three-form H on M and a generalized pseudo-Riemannian metric $\mathcal{G}$. We denote by $(\mathbb{T}M)_\pm$ the eigenbundles of $\mathcal{G}^{\mathrm{end}}$.

Given a Levi-Civita generalized connection D on $\mathbb{T}M$ and two sections $u, v \in \Gamma(\mathbb{T}M)$, we consider the differential operator $R(u,v) : \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ defined by

$$R(u,v)w \coloneqq D_u D_v w - D_v D_u w - D_{[u,v]_H} w,$$

for all $w \in \Gamma(\mathbb{T}M)$. It was observed in [G] that R restricts to tensor fields (see Definition 2.23)

$$\begin{aligned} R_D^+ &\in \Gamma\left((\mathbb{T}M)_+^* \otimes (\mathbb{T}M)_-^* \otimes \mathfrak{so}((\mathbb{T}M)_+)\right), \\ R_D^- &\in \Gamma\left((\mathbb{T}M)_-^* \otimes (\mathbb{T}M)_+^* \otimes \mathfrak{so}((\mathbb{T}M)_-)\right). \end{aligned}$$

Hence there are tensor fields $Ric_D^+ \in \Gamma((\mathbb{T}M)_-^* \otimes (\mathbb{T}M)_+^*)$ and $Ric_D^- \in \Gamma((\mathbb{T}M)_+^* \otimes (\mathbb{T}M)_-^*)$ defined by

$$Ric_D^+(u,v) = \operatorname{tr} R_D^+(\cdot, u)v = \operatorname{tr}\left(\Gamma(\mathbb{T}M_+) \ni w \mapsto R(w,u)v \in \Gamma(\mathbb{T}M_+)\right)$$

for $u \in \Gamma(\mathbb{T}M_-)$, $v \in \Gamma(\mathbb{T}M_+)$, and

$$Ric_D^-(u,v) = \operatorname{tr} R_D^-(\cdot, u)v = \operatorname{tr}\left(\Gamma(\mathbb{T}M_-) \ni w \mapsto R(w,u)v \in \Gamma(\mathbb{T}M_-)\right)$$

for $u \in \Gamma(\mathbb{T}M_+)$, $v \in \Gamma(\mathbb{T}M_-)$ as in Definition 2.25.

It was also shown in [G] that the tensor fields $Ric_{D_1}^{\pm}$ and $Ric_{D_2}^{\pm}$ are the same for any pair of Levi-Civita generalized connections D_1, D_2 with the same divergence operator $\delta_{D_1} = \delta_{D_2}$.

As a consequence, the following definition is meaningful.

Definition 5.21. Let $(G, H, \mathcal{G})$ be a generalized pseudo-Riemannian Lie group and $\delta \in E^*$. Then the Ricci curvatures

$$Ric^+ = Ric_\delta^+ \in E_-^* \otimes E_+^* \quad \text{and} \quad Ric^- = Ric_\delta^- \in E_+^* \otimes E_-^*$$

of $(G, H, \mathcal{G}, \delta)$ (or of $(\mathfrak{g}, H, \mathcal{G}, \delta)$) are defined by evaluation of Ric_D^+ and Ric_D^- at $e \in G$, where D is any left-invariant Levi-Civita generalized connection D with divergence δ (compare Definition 2.28). $(G, H, \mathcal{G}, \delta)$ is called generalized Einstein if

$$Ric \coloneqq Ric^+ \oplus Ric^- = 0 \in E_-^* \otimes E_+^* \oplus E_+^* \otimes E_-^*.$$

We will consider Ric as a bilinear form on E vanishing on $E_+ \times E_+$ and $E_- \times E_-$.

Next we compute the Ricci curvatures in the case $\delta = 0$ using the canonical divergence-free Levi-Civita generalized connection of Proposition 5.17, which in the following we denote by D^0. We also refer to Section 4.3, from which many of the following statements can be deduced by again considering the left-invariance. The case of general divergence is then obtained by computing how the Ricci curvatures change under addition of an element of the generalized first prolongation. We denote by $R_{D^0}^{\pm} \in E_\pm^* \otimes E_\mp^* \otimes \mathfrak{so}(E_\pm)$ the tensors which correspond to the left-invariant tensor fields $R_{D^0}^{\pm} \in \Gamma\left((\mathbb{T}G)_\pm^* \otimes (\mathbb{T}G)_\mp^* \otimes \mathfrak{so}((\mathbb{T}G)_\pm)\right)$.

Proposition 5.22. Let D^0 be the canonical divergence-free Levi-Civita generalized connection of a generalized pseudo-Riemannian Lie group $(G, H, \mathcal{G}_g)$, defined in Proposition 5.17. The components $R_{rstu} := \langle R(e_r, e_s)e_t, e_u\rangle$, $r, s, t, u \in \{1, \dots, 2n\}$, of the tensors $R^{\pm}_{D^0}$ are given by

$$R_{ajcd} = \frac{2}{3}\mathcal{B}^{\ell}_{aj}\mathcal{B}_{c\ell d} + \frac{1}{3}\mathcal{B}^{\ell}_{jc}\mathcal{B}_{\ell ad} + \frac{1}{3}\mathcal{B}^{\ell}_{ca}\mathcal{B}_{\ell jd},$$
$$R_{ibk\ell} = \frac{2}{3}\mathcal{B}^{c}_{ib}\mathcal{B}_{kc\ell} + \frac{1}{3}\mathcal{B}^{c}_{bk}\mathcal{B}_{ci\ell} + \frac{1}{3}\mathcal{B}^{c}_{ki}\mathcal{B}_{cb\ell},$$

where $a, b, c, d \in \{1, \dots, n\}$ and $i, j, k, \ell \in \{n+1, \dots, 2n\}$.

Proof. This can be seen as a corollary of Proposition 4.9, using that the Dorfman coefficients $\mathcal{B}_{rst}$ are constant functions. We now give a direct proof where we use this fact. We denote by $\eta_{rs} = \langle e_r, e_s\rangle$ the coefficients of the scalar product with respect to the orthonormal basis (e_r) and by ω_{rst} and $\omega^t_{rs} = \sum_u \eta^{tu}\omega_{rsu}$ the connection coefficients of D^0. Here $\eta^{rs} = \eta_{rs}$ are the coefficients of the induced scalar product on E^*. Then (taking into account the agreed index ranges) we compute

$$\begin{aligned} R^{+}_{D^0}(e_a, e_j)e_c &= \left(\omega^d_{jc}\omega^f_{ad} - \omega^d_{ac}\omega^f_{jd} - \mathcal{B}^D_{aj}\omega^f_{Dc}\right) e_f, \\ &= \left(\omega^d_{jc}\omega^f_{ad} - \omega^d_{ac}\omega^f_{jd} - \mathcal{B}^d_{aj}\omega^f_{dc} - \mathcal{B}^{\ell}_{aj}\omega^f_{\ell c}\right) e_f \\ &= \left(\frac{1}{3}\mathcal{B}^d_{jc}\mathcal{B}^f_{ad} - \frac{1}{3}\mathcal{B}^d_{ac}\mathcal{B}^f_{jd} - \frac{1}{3}\mathcal{B}^d_{aj}\mathcal{B}^f_{dc} - \mathcal{B}^{\ell}_{aj}\mathcal{B}^f_{\ell c}\right) e_f, \end{aligned}$$

where the index f runs from 1 to n and $\mathcal{B}^t_{rs} = \mathcal{B}_{rsu}\eta^{ut}$. Next we observe that the axiom (C1), the Jacobi identity for the Dorfman bracket, can be written in components as

$$\sum_{\mathfrak{S}(r,s,t)} \mathcal{B}^w_{ru}\mathcal{B}^u_{st} = 0,$$

where the cyclic sum is over (r, s, t) (cf. Lemma 4.4). Specializing to $(r, s, t, w) = (a, j, c, f)$ we get

$$0 = \sum_{\mathfrak{S}(a,j,c)} \mathcal{B}^f_{au}\mathcal{B}^u_{jc} = \sum_{\mathfrak{S}(a,j,c)} (\mathcal{B}^f_{ad}\mathcal{B}^d_{jc} + \mathcal{B}^f_{a\ell}\mathcal{B}^{\ell}_{jc}).$$

So we obtain

$$\begin{aligned} R^{+}_{D^0}(e_a, e_j)e_c &= -\left(\mathcal{B}^{\ell}_{aj}\mathcal{B}^f_{\ell c} + \frac{1}{3}\sum_{\mathfrak{S}(a,j,c)} \mathcal{B}^{\ell}_{jc}\mathcal{B}^f_{a\ell}\right) e_f \\ &= \left(\frac{2}{3}\mathcal{B}^{\ell}_{aj}\mathcal{B}^f_{c\ell} + \frac{1}{3}\mathcal{B}^{\ell}_{jc}\mathcal{B}^f_{\ell a} + \frac{1}{3}\mathcal{B}^{\ell}_{ca}\mathcal{B}^f_{\ell j}\right) e_f \end{aligned}$$

Taking the scalar product with e_d gives the claimed formula for R_{ajcd}. The other formula is obtained similarly. □

Corollary 5.23. Let $(G, H, \mathcal{G}_g)$ be a generalized pseudo-Riemannian Lie group. Then the Ricci curvature of $(G, H, \mathcal{G}_g, \delta = 0)$ is symmetric, in the sense that $Ric^+(u, v) = Ric^-(v, u)$ for all $u \in E_-$, $v \in E_+$. The components $R_{ia} := Ric^+(e_i, e_a)$ of Ric^+ are given by

$$R_{ia} = \mathcal{B}^j_{bi}\mathcal{B}^b_{aj},$$

where $a, b \in \{1, \ldots, n\}$ and $i, j \in \{n+1, \ldots, 2n\}$.

Proof. From Proposition 5.22, by taking the trace using the complete skew-symmetry of $\mathcal{B}_{rst}$, see Lemma 5.11, we get:

$$\begin{aligned} R_{ia} = R_{ai} &= \frac{2}{3}\mathcal{B}^j_{bi}\mathcal{B}^b_{aj} + \frac{1}{3}\mathcal{B}^j_{ab}\mathcal{B}^b_{ji} \\ &= \eta^{bb'}\eta^{jj'}\left(\frac{2}{3}\mathcal{B}_{bij}\mathcal{B}_{aj'b'} + \frac{1}{3}\mathcal{B}_{ab'j'}\mathcal{B}_{jib}\right) \\ &= \eta^{bb'}\eta^{jj'}\mathcal{B}_{bij}\mathcal{B}_{aj'b'} = \mathcal{B}^j_{bi}\mathcal{B}^b_{aj}. \end{aligned}$$

Again, one could alternatively prove this Corollary as a direct consequence of Proposition 4.10. □

For $u_\pm \in E_\pm$ we define

$$\Gamma_{u_+} := \mathrm{pr}_{E_+} \circ [u_+, \cdot]_H\big|_{E_-} : E_- \to E_+, \quad \Gamma_{u_-} := \mathrm{pr}_{E_-} \circ [u_-, \cdot]_H\big|_{E_+} : E_+ \to E_-. \tag{27}$$

Corollary 5.24. A necessary and sufficient condition for $(G, H, \mathcal{G}_g, \delta = 0)$ to be generalized Einstein is that the subspace

$$\Gamma_{E_+} \subset \mathrm{Hom}(E_-, E_+) \quad \text{is perpendicular to} \quad \Gamma_{E_-} \subset \mathrm{Hom}(E_+, E_-),$$

with respect to the non-degenerate pairing $\mathrm{Hom}(E_-, E_+) \times \mathrm{Hom}(E_+, E_-) \to \mathbb{R}$ given by $(A, B) \mapsto \mathrm{tr}(AB) = \mathrm{tr}(BA)$. A sufficient condition in terms of the subspaces $\Gamma_{E_\pm}E_\mp \subset E_\pm$ is that

$$\Gamma_{E_+}E_- \perp [E_-, E_-]_H \quad \text{or} \quad \Gamma_{E_-}E_+ \perp [E_+, E_+]_H. \tag{28}$$

Proof. The necessary and sufficient condition follows immediately from

$$R_{ia} = R_{ai} = \mathcal{B}^j_{bi}\mathcal{B}^b_{aj} = -\,\mathrm{tr}(\Gamma_{e_a} \circ \Gamma_{e_i}).$$

Any of the two (non-equivalent) conditions $\Gamma_{E_+} \circ \Gamma_{E_-} = 0$ or $\Gamma_{E_-} \circ \Gamma_{E_+} = 0$ is clearly sufficient. These can be reformulated as (28), since by Lemma 5.11,

$$\langle \Gamma_{u_+}v_-, w_+\rangle = -\langle v_-, [u_+, w_+]_H\rangle \quad \text{and} \quad \langle \Gamma_{u_-}v_+, w_-\rangle = -\langle v_+, [u_-, w_-]_H\rangle,$$

for all $u_+, v_+, w_+ \in E_+$, $u_-, v_-, w_- \in E_-$. □

Next we will compute the Ricci curvature of an arbitrary left-invariant Levi-Civita generalized connection $D = D^0 + S$ on $(G, H, \mathcal{G}_g)$, where D^0 is the canonical divergence-free Levi-Civita generalized connection and S is an arbitrary element of the first generalized prolongation of $\mathfrak{so}(E)$.

Lemma 5.25. The curvature tensors $R_D^{\pm} \in \mathrm{Hom}(E_{\pm} \otimes E_{\mp} \otimes E_{\pm}, E_{\pm})$ of D are given by

$$R_D^{\pm} = R_{D^0}^{\pm} + d^{D^0} S|_{E_{\pm}\otimes E_{\mp}\otimes E_{\pm}}, \tag{29}$$

where

$$(d^{D^0} S)(u, v, w) = (d^{D^0} S)(u, v)w := D_u^0(S_v)w - D_v^0(S_u)w - S_{[u,v]_H} w, \quad u, v, w \in E.$$

Proof. A straightforward calculation shows that $R_D^{\pm} = R_{D^0}^{\pm} + (d^{D^0} S + [S, S])|_{E_{\pm}\otimes E_{\mp}\otimes E_{\pm}}$, where

$$[S, S](u, v, w) = [S, S](u, v)w := [S_u, S_v]w, \quad u, v, w \in E.$$

We observe that the map $[S, S] : (u, v, w) \mapsto [S_u, S_v]w$ vanishes on $E_+ \otimes E_- \otimes E_+$ and on $E_- \otimes E_+ \otimes E_-$, since $S_E E_{\pm} \subset E_{\pm}$ and $S_{E_{\pm}} E_{\mp} = 0$. This proves (29). ☐

In the following, we denote by $(d^{D^0} S)^{\pm}$ the restriction of $d^{D^0} S$ to an element

$$(d^{D^0} S)^{\pm} \in \mathrm{Hom}(E_{\pm} \otimes E_{\mp} \otimes E_{\pm}, E_{\pm}) \cong \mathrm{Hom}(E_{\pm} \otimes E_{\mp}, \mathrm{End}\, E_{\pm}).$$

Lemma 5.26. We have $R_D^{\pm} = R_{D^0}^{\pm} + (d^{D^0} S)^{\pm}$ and

$$(d^{D^0} S)^{\pm}(u, v)w = -(D_v^0 S)_u w,$$

for all $(u, v, w) \in E_{\pm} \times E_{\mp} \times E_{\pm}$.

Proof. The first formula is just (29). Since $D^0 E_{\pm} \subset E_{\pm}$ and $S_{E_{\pm}} E_{\mp} = 0$, we have

$$(d^{D^0} S)^{\pm}(u, v)w = -D_v^0(S_u)w - S_{[u,v]_H} w = -(D_v^0 S)_u w - S_{D_v^0 u} w - S_{[u,v]_H} w.$$

Using that D^0 is torsion-free we can write $[u, v]_H = D_u^0 v - D_v^0 u$, since $(D^0 u)^* v = 0$ for all $(u, v) \in E_{\pm} \times E_{\mp}$. Hence

$$-S_{D_v^0 u} w - S_{[u,v]_H} w = -S_{D_u^0 v} w = 0,$$

again because $D^0 E_{\pm} \subset E_{\pm}$ and $S_{E_{\pm}} E_{\mp} = 0$. This proves the lemma. ☐

Proposition 5.27. Let δ be a divergence operator on E and $S \in \mathfrak{so}(E)^{\langle 1 \rangle}$ such that the Levi-Civita generalized connection $D^0 + S$ has divergence δ. Then the Ricci curvatures $Ric_{\delta}^{\pm}$ of a generalized pseudo-Riemannian Lie group $(G, H, \mathcal{G}_g, \delta)$ with arbitrary divergence $\delta \in E^*$ are related to the Ricci curvatures $Ric_0^{\pm}$ of $(G, H, \mathcal{G}_g, 0)$ by

$$Ric_{\delta}^{\pm} = Ric_0^{\pm} + \mathrm{tr}_{E_{\pm}}(d^{D^0} S)^{\pm} = Ric_0^{\pm} - D^0\delta|_{E_{\mp}\otimes E_{\pm}}, \tag{30}$$

where

$$(\mathrm{tr}_{E_+} \alpha)(e_i, e_b) = \mathrm{tr}(u \mapsto \alpha(u, e_i)e_b),$$

for any $\alpha \in E_+^* \otimes E_-^* \otimes E_+^* \otimes E_+$ and, similarly,

$$(\mathrm{tr}_{E_-} \beta)(e_a, e_j) = \mathrm{tr}(u \mapsto \beta(u, e_a)e_j),$$

when $\beta \in E_-^* \otimes E_+^* \otimes E_-^* \otimes E_-$. Here we are assuming the usual index ranges for a, b and i, j.

Proof. An element S of the first generalized prolongation of $\mathfrak{so}(E)$ such that $D^0 + S$ has divergence δ exists due to Proposition 5.15. The first equation follows from Lemma 5.25 by taking traces. The formula

$$\mathrm{tr}_{E_\pm}(d^{D^0}S)^\pm = -D^0\delta|_{E_\mp \otimes E_\pm}$$

is a consequence of Lemma 5.26, since the trace maps tr_{E_+} and tr_{E_-} are parallel for any metric generalized connection. In fact, for instance,

$$\mathrm{tr}_{E_+}(d^{D^0}S)^+(e_i, e_b) = -\,\mathrm{tr}_{E_+}\left((D^0_{e_i}S)e_b\right) = -D^0_{e_i}\left(\mathrm{tr}_{E_+} S\right)e_b = -(D^0_{e_i}\delta)e_b,$$

where the $\mathrm{tr}_{E_+} S \in E^*$ is defined by $(\mathrm{tr}_{E_+} S)v := \mathrm{tr}_{E_+}(Sv) = \mathrm{tr}(E_+ \ni u \mapsto S_u v \in E_+)$ for all $v \in E$ and we have used that $\mathrm{tr}_{E_+}(Sv) = \mathrm{tr}(Sv) = \delta(v)$ for all $v \in E_+$. ☐

Summarizing we obtain the following theorem.

Theorem 5.28. The components $R^\delta_{ia} = Ric^+_\delta(e_i, e_a)$ and $R^\delta_{ai} = Ric^-_\delta(e_a, e_i)$ of the Ricci curvature tensors $Ric^\pm_\delta$ of a generalized pseudo-Riemannian Lie group $(G, H, \mathcal{G}_g, \delta)$ with arbitrary divergence $\delta \in E^*$ are given as follows:

$$\begin{aligned} R^\delta_{ia} &= \mathcal{B}^j_{bi}\mathcal{B}^b_{aj} + \mathcal{B}^c_{ia}\delta_c, \\ R^\delta_{ai} &= \mathcal{B}^j_{bi}\mathcal{B}^b_{aj} + \mathcal{B}^j_{ai}\delta_j. \end{aligned}$$

In particular, the Ricci tensor $Ric_\delta = Ric^+_\delta \oplus Ric^-_\delta$ is symmetric if and only if, δ satisfies the equation $\mathcal{B}^c_{ia}\delta_c = \mathcal{B}^j_{ai}\delta_j$. It is skew-symmetric if $(G, H, \mathcal{G}_g, 0)$ is generalized Einstein and $\mathcal{B}^c_{ia}\delta_c = -\mathcal{B}^j_{ai}\delta_j$. (Recall that we are always assuming the usual index ranges for a, b and i, j.)

In terms of the linear maps $\Gamma_{u_\pm} : E_\mp \to E_\pm$ defined in (27) for $u_\pm \in E_\pm$ we have

$$\begin{aligned} Ric^+_\delta(u_-, u_+) &= -\,\mathrm{tr}\left(\Gamma_{u_-} \circ \Gamma_{u_+}\right) + \delta(\mathrm{pr}_{E_+}[u_-, u_+]_H), \\ Ric^-_\delta(u_+, u_-) &= -\,\mathrm{tr}\left(\Gamma_{u_-} \circ \Gamma_{u_+}\right) + \delta(\mathrm{pr}_{E_-}[u_+, u_-]_H). \end{aligned}$$

The theorem shows that the Ricci curvature is completely determined by the one-form δ and the coefficients $\mathcal{B}_{ajk}$ and $\mathcal{B}_{ibc}$ of the Dorfman bracket in the orthonormal basis $(e_r) = (e_a, e_i)$. For future use we do now compute the latter coefficients in terms of the coefficients of the Lie bracket (the structure constants) and the coefficients of the three-form H using (13). Recall that (v_a) was a g-orthonormal basis of $\mathfrak{g}$. More precisely, we have $g_{ab} = g(v_a, v_b) = \langle e_a, e_b\rangle = \eta_{ab}$. We denote the corresponding structure constants of the Lie algebra $\mathfrak{g}$ by κ^c_{ab}, such that

$$[v_a, v_b] = \kappa^c_{ab} v_c.$$

Note that $\kappa_{abc} = \kappa^d_{ab} g_{dc} = \kappa^d_{ab}\eta_{dc}$ for $\kappa_{abc} := \langle [v_a, v_b], v_c\rangle$.

Proposition 5.29. The Dorfman coefficients $\mathcal{B}_{ajk}$, $\mathcal{B}_{ibc}$, $\mathcal{B}_{abc}$ and $\mathcal{B}_{ijk}$ (with $a, b, c \in \{1, \dots, n\}$, $i, j, k \in \{n+1, \dots, 2n\}$) are related to the structure constant κ_{abc} as follows:

$$\begin{aligned}
\mathcal{B}_{ajk} &= \frac{1}{2}\left(H_{aj'k'} - \kappa_{aj'k'} + \kappa_{j'k'a} - \kappa_{k'aj'}\right)\\
\mathcal{B}_{ibc} &= \frac{1}{2}\left(H_{i'bc} + \kappa_{i'bc} - \kappa_{bci'} + \kappa_{ci'b}\right)\\
\mathcal{B}_{abc} &= \frac{1}{2}\left(H_{abc} + (\partial\kappa)_{abc}\right)\\
\mathcal{B}_{ijk} &= \frac{1}{2}\left(H_{i'j'k'} - (\partial\kappa)_{i'j'k'}\right),
\end{aligned}$$

where $i' = i - n$, for $i \in \{n+1, \dots, 2n\}$ and $(\partial\kappa)_{abc} = \kappa_{abc} + \kappa_{bca} + \kappa_{cab}$.

Proof. Using (13) we compute

$$\begin{aligned}
[e_a, e_j]_H = [v_a + gv_a, v_{j'} - gv_{j'}]_H &= [v_a, v_{j'}]_H - [v_a, gv_{j'}]_H + [gv_a, v_{j'}]_H\\
&= [v_a, v_{j'}] + H(v_a, v_{j'}, \cdot) + ad^*_{v_a}(gv_{j'}) - \iota_{v_{j'}} d(gv_a)\\
&= [v_a, v_{j'}] + H(v_a, v_{j'}, \cdot) + g(v_{j'}, [v_a, \cdot]) + g(v_a, [v_{j'}, \cdot]).
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{B}_{ajk} &= \langle [e_a, e_j]_H, e_k\rangle = \langle [e_a, e_j]_H, v_{k'} - gv_{k'}\rangle\\
&= \frac{1}{2}\left(H(v_a, v_{j'}, v_{k'}) + g(v_{j'}, [v_a, v_{k'}]) + g(v_a, [v_{j'}, v_{k'}]) - g(v_{k'}, [v_a, v_{j'}])\right)\\
&= \frac{1}{2}\left(H_{aj'k'} + \kappa_{ak'j'} + \kappa_{j'k'a} - \kappa_{aj'k'}\right)\\
&= \frac{1}{2}\left(H_{aj'k'} - \kappa_{k'aj'} + \kappa_{j'k'a} - \kappa_{aj'k'}\right).
\end{aligned}$$

The proof of the second formula is similar, where now

$$[e_i, e_b]_H = [v_{i'}, v_b] + H(v_{i'}, v_b, \cdot) - g(v_b, [v_{i'}, \cdot]) - g(v_{i'}, [v_b, \cdot]).$$

The remaining equations are obtained in the same way. □

The next result shows that the underlying metric g of an Einstein generalized pseudo-Riemannian Lie group can be freely rescaled without changing the Einstein property, provided that the three-form and the divergence are appropriately rescaled.

Proposition 5.30. Let g be a left-invariant pseudo-Riemannian metric and H a closed left-invariant three-form on a Lie group G. Consider $g' = \varepsilon\mu^{-2}g$ and $H' = \varepsilon\mu^{-2}H$, where $\varepsilon \in \{\pm 1\}$ and $\mu > 0$. Then the generalized pseudo-Riemannian Lie group $(G, H, \mathcal{G}_g)$ is Einstein with divergence $\delta \in E^*$ if and only if $(G, H', \mathcal{G}_{g'})$ is Einstein with divergence $\delta' = \mu\delta$.

Proof. Let (v_a) be a g-orthonormal basis of $\mathfrak{g}$. Then $v'_a = \mu v_a$ defines a g'-orthonormal basis (v'_a). The corresponding basis (e'_r) of E, where $e'_a = v'_a + g'v'_a$ and $e_i = v_i - g'v'_i$, is still orthonormal with respect to the scalar product: $\langle e'_r, e'_s\rangle = \varepsilon\langle e_r, e_s\rangle$. The structure constants $\kappa'_{abc} := g'([v'_a, v'_b], v'_c)$ with respect to the basis (v'_a) are $\kappa'_{abc} = \varepsilon\mu\kappa_{abc}$. Similarly, $H'(v'_a, v'_b, v'_c) = \varepsilon\mu H(v_a, v_b, v_c)$. Finally, from these formulas and Proposition 5.29 we see that $\mathcal{B}'_{rst} := \langle [e'_r, e'_s]_{H'}, e'_t\rangle = \varepsilon\mu\mathcal{B}_{rst}$. Taking into account that $\langle e'_r, e'_s\rangle = \varepsilon\langle e_r, e_s\rangle$, we conclude that $(\mathcal{B}')^t_{rs} := (\eta')^{tu}\mathcal{B}'_{rsu} = \mu\mathcal{B}^t_{rs}$. Now Theorem 5.28 together with Proposition 5.17 shows that the coefficients of the Ricci curvatures Ric of $(G, H, \mathcal{G}_g, \delta)$ and Ric' of $(G, H', \mathcal{G}_{g'}, \delta')$ are related by $Ric'(e'_r, e'_s) = \mu^2 Ric(e_r, e_s)$. □

Remark 5.31. Denote by ∇ the Levi-Civita connection of the pseudo-Riemannian metric g and define its coefficients with respect to the orthonormal frame (v_a) as $\Gamma_{abc} := g(\nabla_{v_a} v_b, v_c)$. Then

$$\Gamma_{abc} = \frac{1}{2}\left(g([v_a, v_b], v_c) - g([v_b, v_c], v_a) + g([v_c, v_a], v_b)\right) = \frac{1}{2}(\kappa_{abc} - \kappa_{bca} + \kappa_{cab})$$

and hence the Dorfman coefficients $\mathcal{B}_{ajk}$ and $\mathcal{B}_{ibc}$ can be expressed by

$$\mathcal{B}_{ajk} = \frac{1}{2}\left(H_{aj'k'} - 2\Gamma_{aj'k'}\right) = \frac{1}{2}H_{aj'k'} - \Gamma_{aj'k'}$$
$$\mathcal{B}_{ibc} = \frac{1}{2}\left(H_{i'bc} + 2\Gamma_{i'bc}\right) = \frac{1}{2}H_{i'bc} + \Gamma_{i'bc}.$$

Proposition 5.32. Let g be a left-invariant pseudo-Riemannian metric on a Lie group G. Consider the generalized pseudo-Riemannian Lie group $(G, H = 0, \mathcal{G}_g)$. Then the Ricci curvature $Ric^+_0 = Ric^\pm_\delta|_{\delta=0}$ of the generalized metric $\mathcal{G}_g$ is related to the Ricci curvature Ric^g of the metric g by

$$Ric^+_0(v - gv, u + gu) = Ric^-_0(u + gu, v - gv) = Ric^g(u, v) + (\nabla_u\tau)(v), \quad u, v \in \mathfrak{g},$$

where $\tau \in \mathfrak{g}^*$ is the trace-form defined by $\tau(v) = \operatorname{tr}\operatorname{ad}_v$.

Proof. The symmetry of the Ricci tensor of $\mathcal{G}_g$ follows from $\delta = 0$. Therefore it suffices to compute $R_{ia} = Ric^+(e_i, e_a)$ from Theorem 5.28 and to compare with $R^g_{ai'} = Ric^g(v_a, v_{i'})$, $i' = i - n$. Note first that, by Remark 5.31 we have

$$\mathcal{B}^k_{aj} = \Gamma^{k'}_{aj'}, \quad \mathcal{B}^c_{ib} = \Gamma^c_{i'b},$$

since $H = 0$ and $\langle e_j, e_k\rangle = -\langle e_{j'}, e_{k'}\rangle = -g(v_{j'}, v_{k'})$. Hence, using Lemma 5.11 and the fact that the Levi-Civita connection has zero torsion, we obtain

$$R_{ia} = \mathcal{B}^j_{bi}\mathcal{B}^b_{aj} = -\Gamma^{j'}_{bi'}\Gamma^b_{j'a} = -\Gamma^{j'}_{bi'}(\Gamma^b_{aj'} + \kappa^b_{j'a}).$$

On the other hand, we have

$$R^g_{ai'} = \Gamma^d_{ai'}\Gamma^f_{fd} - \Gamma^d_{fi'}\Gamma^f_{ad} - \kappa^d_{fa}\Gamma^f_{di'} = \Gamma^d_{ai'}\Gamma^f_{fd} + R_{ia}.$$

To compute the first term we note that since the Levi-Civita connection is metric, we have

$$\Gamma^f_{fd} = \kappa^f_{fd} = -\tau_d = -\tau(v_d)$$

and, hence,

$$\Gamma^d_{ai'}\Gamma^f_{fd} = -\Gamma^d_{ai'}\tau_d = (\nabla\tau)_{ai'} = (\nabla_{v_a}\tau)v_{i'}. \qquad \square$$

Corollary 5.33. Let g be a left-invariant pseudo-Riemannian metric on a Lie group G. Then the generalized pseudo-Riemannian Lie group $(G, H = 0, \mathcal{G}_g)$ is Einstein with divergence $\delta = 0$ if and only if g satisfies the following Ricci soliton equation

$$Ric^g + \nabla\tau = 0, \tag{31}$$

where τ is the trace-form. The form τ is always closed and, hence, the solutions of the above equation are gradient Ricci solitons, if the first Betti number of the manifold G vanishes.

Proof. For all $u, v \in \mathfrak{g}$ we have

$$(d\tau)(u, v) = -\tau([u, v]) = -\mathrm{tr}\,\mathrm{ad}_{[u,v]} = -\mathrm{tr}\,[\mathrm{ad}_u, \mathrm{ad}_v] = 0. \qquad \square$$

Corollary 5.34. Let g be a left-invariant pseudo-Riemannian metric on a unimodular Lie group G. Then the generalized pseudo-Riemannian Lie group $(G, H = 0, \mathcal{G}_g)$ is Einstein with divergence $\delta = 0$ if and only if g is Ricci-flat.

6 Classification results in dimension 3

6.1 Preliminaries

We want to apply the results from the previous section in order to give a classification of all generalized Einstein Lie groups in dimension 3. Let G be a three-dimensional Lie group endowed with a left-invariant pseudo-Riemannian metric g and an orientation. We will identify g with a non-degenerate symmetric bilinear form $g \in \mathrm{Sym}^2\mathfrak{g}^*$. We begin by showing that the Lie bracket can be encoded in an endomorphism L of $\mathfrak{g}$ and study its properties.

Following Milnor [M], but allowing indefinite metrics, we denote by $L \in \mathrm{End}\,\mathfrak{g}$ the endomorphism such that

$$[u, v] = L(u \times v), \quad \forall u, v \in \mathfrak{g}, \tag{32}$$

where the cross-product $\times \in \bigwedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$ is defined by

$$g(u \times v, w) = \mathrm{vol}_g(u, v, w), \tag{33}$$

using the metric volume form vol_g. In terms of an oriented orthonormal basis (v_a), we have

$$v_a \times v_b = \varepsilon_c v_c, \quad \varepsilon_c = g(v_c, v_c),$$

for every cyclic permutation (a, b, c) of $\{1, 2, 3\}$. This implies that

$$[v_a, v_b] = \varepsilon_c L v_c, \quad \forall \quad \text{cyclic} \quad (a, b, c) \in \mathfrak{S}_3. \tag{34}$$

We denote by $(L^a{}_b)$ the matrix of L in the above basis,

$$L e_b = L^a{}_b e_a,$$

and by $L^{ab} = L^a{}_c g^{cb}$ the coefficients of the corresponding tensor $L \circ g^{-1} \in \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g}) \cong \mathfrak{g} \otimes \mathfrak{g}$.

From (34), we see that the structure constants κ_{ab}^c of $\mathfrak{g}$ with respect to the basis (v_a) can be written as

$$\kappa_{ab}^c = \varepsilon_{abd} L^{cd},$$

where $\varepsilon_{abd} = \mathrm{vol}_g(v_a, v_b, v_d)$ (in particular, $\varepsilon_{123} = 1$).

The following lemma is a straightforward generalization of [M, Lemma 4.1].

Lemma 6.1. The endomorphism L is symmetric with respect to g if and only if $\mathfrak{g}$ is unimodular.

Proof. Note first that L is symmetric with respect to g if and only if the matrix (L^{ab}) is symmetric. Therefore, the calculation

$$\mathrm{tr}\,\mathrm{ad}_{v_a} = \kappa_{ab}^b = \varepsilon_{abc} L^{bc}$$

shows that L is symmetric if and only if $\mathrm{tr}\,\mathrm{ad}_{v_a} = 0$ for all a, i.e. if and only if $\mathfrak{g}$ is unimodular. □

Proposition 6.2. Let g be a non-degenerate symmetric bilinear form on a three-dimensional unimodular Lie algebra $\mathfrak{g}$. Then there exists an orthonormal basis (v_a) of $(\mathfrak{g}, g)$ such that $g(v_1, v_1) = g(v_2, v_2)$ and such that the symmetric endomorphism L defined in Equation (32) is represented by one of the following matrices:

$$L_1(\alpha,\beta,\gamma) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \qquad L_2(\alpha,\beta,\gamma) = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \alpha & -\beta \\ 0 & \beta & \alpha \end{pmatrix},$$

$$L_3(\alpha,\beta) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \frac{1}{2}+\alpha & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2}+\alpha \end{pmatrix}, \qquad L_4(\alpha,\beta) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & -\frac{1}{2}+\alpha & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2}+\alpha \end{pmatrix},$$

$$L_5(\alpha) = \begin{pmatrix} \alpha & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \alpha & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \alpha \end{pmatrix},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $g(v_3, v_3) = -g(v_2, v_2)$ for the normal forms $L_2, \ldots, L_5$. If g is definite, then the orthonormal basis can be chosen such that L is represented by a diagonal matrix $L_1(\alpha,\beta,\gamma)$ and each diagonal matrix is realized in this way. If g is indefinite, then each of the above normal forms is realized by some unimodular Lie bracket.

Proof. It is well known that every symmetric endomorphism on a Euclidean vector space can be diagonalized. According to [CEHL, Lemma 2.2] and references therein, for an indefinite scalar product on a three-dimensional vector space there are the five normal forms of a symmetric bilinear form, from which one easily obtains the five normal forms $L_1(\alpha,\beta,\gamma), L_2(\alpha,\beta,\gamma), L_3(\alpha,\beta), L_4(\alpha,\beta)$ and $L_5(\alpha)$ for a symmetric endomorphism. It remains to check that for each of these normal forms $(L^a{}_b)$, the bracket with structure constants $\kappa^c_{ab} = \varepsilon_{abd} L^{cd}$ satisfies the Jacobi identity.

All the cases can be treated simultaneously by considering $(L^a{}_b)$ of the form

$$\begin{pmatrix} \alpha & \lambda & 0 \\ \lambda & \beta & \mu \\ 0 & \varepsilon_2\varepsilon_3\mu & \gamma \end{pmatrix},$$

where $\lambda, \mu \in \mathbb{R}$. For the corresponding endomorphism L we have

$$\begin{aligned} \mathrm{Jac}(v_1, v_2, v_3) &:= [v_1,[v_2,v_3]] + [v_2,[v_3,v_1]] + [v_3,[v_1,v_2]] = \sum [v_a, \varepsilon_a L v_a] \\ &= \varepsilon_1\lambda[v_1,v_2] + \varepsilon_2\lambda[v_2,v_1] + \varepsilon_3\mu[v_2,v_3] + \varepsilon_3\mu[v_3,v_2] = 0, \end{aligned}$$

where we have used that $\varepsilon_1 = \varepsilon_2$. □

6.2 Classification in the case of zero divergence

6.2.1 Unimodular Lie groups

Proposition 6.3. If $(H, \mathcal{G}_g, \delta = 0)$ is a divergence-free generalized Einstein structure on an oriented three-dimensional unimodular Lie group G, then there exists a g-orthonormal basis (v_a) of $\mathfrak{g}$ such that $g(v_1, v_1) = g(v_2, v_2)$ and such that the symmetric endomorphism L defined in Equation (32) is either of the form $L_1(\alpha, \beta, \gamma)$, that is L is diagonalizable by an orthonormal basis, or of one of the forms $L_3(0, 0)$ or $L_4(0, 0)$. In the non-diagonalizable case the three-form H is zero.

Proof. In the Euclidean case any symmetric endomorphism is always diagonalizable by an orthonormal basis. So we may assume that the scalar product is indefinite. By Proposition 6.2, there is an orthonormal basis (v_a), such that the endomorphism L takes one of the normal forms $L_1(\alpha, \beta, \gamma), L_2(\alpha, \beta, \gamma), L_3(\alpha, \beta), L_4(\alpha, \beta)$ or $L_5(\alpha)$ from said proposition. As in the proof of Proposition 6.2, we can treat all cases at once by considering the matrix

$$\begin{pmatrix} \alpha & \lambda & 0 \\ \lambda & \beta & \mu \\ 0 & -\mu & \gamma \end{pmatrix}.$$

Recall that we assume $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3$, where $\varepsilon_a = g(v_a, v_a)$. Using equation (34) we obtain the structure constants $\kappa_{abc} = \varepsilon_c \kappa^c_{ab}$ of the Lie algebra in the following way. The bracket is given by

$$\begin{aligned}
\kappa^a_{12} v_a &= [v_1, v_2] = \varepsilon_3 L v_3 = \varepsilon_3 \mu v_2 + \varepsilon_3 \gamma v_3 = -\varepsilon_2 \mu v_2 + \varepsilon_3 \gamma v_3 \\
\kappa^a_{23} v_a &= [v_2, v_3] = \varepsilon_1 L v_1 = \varepsilon_1 \alpha v_1 + \varepsilon_1 \lambda v_2 = \varepsilon_1 \alpha v_1 + \varepsilon_2 \lambda v_2 \\
\kappa^a_{31} v_a &= [v_2, v_1] = \varepsilon_2 L v_2 = \varepsilon_2 \lambda v_1 + \varepsilon_2 \beta v_2 - \varepsilon_2 \mu v_3 = \varepsilon_1 \lambda v_1 + \varepsilon_2 \beta v_2 + \varepsilon_3 \mu v_3,
\end{aligned}$$

and hence

$$\begin{array}{lll}
\kappa_{121} = 0, & \kappa_{122} = \varepsilon_2 \kappa^2_{12} = -\mu, & \kappa_{123} = \varepsilon_3 \kappa^3_{12} = \gamma \\
\kappa_{231} = \varepsilon_1 \kappa^1_{23} = \alpha, & \kappa_{232} = \varepsilon_2 \kappa^2_{23} = \lambda, & \kappa_{233} = 0 \\
\kappa_{311} = \varepsilon_1 \kappa^1_{31} = \lambda, & \kappa_{312} = \varepsilon_2 \kappa^2_{31} = \beta, & \kappa_{313} = \varepsilon_3 \kappa^3_{31} = \mu.
\end{array}$$

The remaining structure constants are determined by the skew-symmetry of κ_{abc} in the first two components.

By Proposition 5.29, the Dorfman coefficients are given as follows

$$\begin{aligned}
\mathcal{B}_{145} &= \frac{1}{2} (H_{112} - \kappa_{112} + \kappa_{121} - \kappa_{211}) = \kappa_{121} = 0 \\
\mathcal{B}_{146} &= \frac{1}{2} (H_{113} - \kappa_{113} + \kappa_{131} - \kappa_{311}) = -\kappa_{311} = -\lambda \\
\mathcal{B}_{156} &= \frac{1}{2} (H_{123} - \kappa_{123} + \kappa_{231} - \kappa_{312}) = \frac{1}{2} (h - \gamma + \alpha - \beta) \\
\mathcal{B}_{245} &= \frac{1}{2} (H_{212} - \kappa_{212} + \kappa_{122} - \kappa_{221}) = \kappa_{122} = -\mu \\
\mathcal{B}_{246} &= \frac{1}{2} (H_{213} - \kappa_{213} + \kappa_{132} - \kappa_{321}) = \frac{1}{2} (-h + \gamma - \beta + \alpha) \\
\mathcal{B}_{256} &= \frac{1}{2} (H_{223} - \kappa_{223} + \kappa_{232} - \kappa_{322}) = \kappa_{232} = \lambda \\
\mathcal{B}_{345} &= \frac{1}{2} (H_{312} - \kappa_{312} + \kappa_{123} - \kappa_{231}) = \frac{1}{2} (h - \beta + \gamma - \alpha)
\end{aligned}$$

$$\mathcal{B}_{346} = \frac{1}{2}(H_{313} - \kappa_{313} + \kappa_{133} - \kappa_{331}) = -\kappa_{313} = -\mu$$
$$\mathcal{B}_{356} = \frac{1}{2}(H_{323} - \kappa_{323} + \kappa_{233} - \kappa_{332}) = \kappa_{233} = 0$$
$$\mathcal{B}_{412} = \frac{1}{2}(H_{112} + \kappa_{112} - \kappa_{121} + \kappa_{211}) = -\kappa_{121} = 0$$
$$\mathcal{B}_{413} = \frac{1}{2}(H_{113} + \kappa_{113} - \kappa_{131} + \kappa_{311}) = \kappa_{311} = \lambda$$
$$\mathcal{B}_{423} = \frac{1}{2}(H_{123} + \kappa_{123} - \kappa_{231} + \kappa_{312}) = \frac{1}{2}(h + \gamma - \alpha + \beta)$$
$$\mathcal{B}_{512} = \frac{1}{2}(H_{212} + \kappa_{212} - \kappa_{122} + \kappa_{221}) = -\kappa_{122} = \mu$$
$$\mathcal{B}_{513} = \frac{1}{2}(H_{213} + \kappa_{213} - \kappa_{132} + \kappa_{321}) = \frac{1}{2}(-h - \gamma + \beta - \alpha)$$
$$\mathcal{B}_{523} = \frac{1}{2}(H_{223} + \kappa_{223} - \kappa_{232} + \kappa_{322}) = -\kappa_{232} = -\lambda$$
$$\mathcal{B}_{612} = \frac{1}{2}(H_{312} + \kappa_{312} - \kappa_{123} + \kappa_{231}) = \frac{1}{2}(h + \beta - \gamma + \alpha)$$
$$\mathcal{B}_{613} = \frac{1}{2}(H_{313} + \kappa_{313} - \kappa_{133} + \kappa_{331}) = \kappa_{313} = \mu$$
$$\mathcal{B}_{623} = \frac{1}{2}(H_{323} + \kappa_{323} - \kappa_{233} + \kappa_{332}) = -\kappa_{233} = 0$$

Now, Theorem 5.28 allows us to compute the Ricci curvature (for zero divergence δ) with respect to the orthonormal basis $(e_r) = (e_a, e_i)$, $e_a = v_a + gv_a$, $e_i = v_{i'} - gv_{i'}$, of $E = \mathfrak{g} \oplus \mathfrak{g}^*$ as

$$R_{ia} = \sum_{j,b} \mathcal{B}^j_{bi}\mathcal{B}^b_{aj} = \sum_{j,b} \mathcal{B}_{bij}\mathcal{B}_{ajb}(-\varepsilon_{j'})\varepsilon_b = \sum_{j,b} \mathcal{B}_{bij}\mathcal{B}_{jab}\varepsilon_{j'}\varepsilon_b \tag{35}$$

where we have used that $\langle e_i, e_i \rangle = -\langle e_{i'}, e_{i'} \rangle = -\varepsilon_{i'}$ and the standard index ranges $a, b \in \{1, 2, 3\}$, $i, j \in \{4, 5, 6\}$.

$$\begin{aligned}
R_{41} &= \mathcal{B}_{245}\mathcal{B}_{512}\varepsilon_2\varepsilon_2 + \mathcal{B}_{345}\mathcal{B}_{513}\varepsilon_2\varepsilon_3 + \mathcal{B}_{246}\mathcal{B}_{612}\varepsilon_3\varepsilon_2 + \mathcal{B}_{346}\mathcal{B}_{613}\varepsilon_3\varepsilon_3 \\
&= \mathcal{B}_{245}\mathcal{B}_{512} - \mathcal{B}_{345}\mathcal{B}_{513} - \mathcal{B}_{246}\mathcal{B}_{612} + \mathcal{B}_{346}\mathcal{B}_{613} \\
&= -\mu^2 - \frac{1}{4}(h - \beta + \gamma - \alpha)(-h - \gamma + \beta - \alpha) \\
&\qquad - \frac{1}{4}(-h + \gamma - \beta + \alpha)(h + \beta - \gamma + \alpha) - \mu^2 \\
&= -2\mu^2 - \frac{1}{4}\left(\alpha^2 - (h - (\beta - \gamma))^2 + \alpha^2 - (h + (\beta - \gamma))^2\right) \\
&= -2\mu^2 - \frac{1}{2}\alpha^2 + \frac{1}{2}h^2 + \frac{1}{2}(\beta - \gamma)^2 \\
R_{42} &= \mathcal{B}_{145}\mathcal{B}_{521}\varepsilon_2\varepsilon_1 + \mathcal{B}_{345}\mathcal{B}_{523}\varepsilon_2\varepsilon_3 + \mathcal{B}_{146}\mathcal{B}_{621}\varepsilon_3\varepsilon_1 + \mathcal{B}_{346}\mathcal{B}_{623}\varepsilon_3\varepsilon_3 \\
&= 0 - \mathcal{B}_{345}\mathcal{B}_{523} + \mathcal{B}_{146}\mathcal{B}_{612} + 0
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\lambda\left(h-\beta+\gamma-\alpha\right)-\frac{1}{2}\lambda\left(h+\beta-\gamma+\alpha\right)\\
&= -\lambda\left(\beta-\gamma+\alpha\right)\\
R_{43} &= \mathcal{B}_{145}\mathcal{B}_{531}\varepsilon_2\varepsilon_1+\mathcal{B}_{245}\mathcal{B}_{532}\varepsilon_2\varepsilon_2+\mathcal{B}_{146}\mathcal{B}_{631}\varepsilon_3\varepsilon_1+\mathcal{B}_{246}\mathcal{B}_{632}\varepsilon_3\varepsilon_2\\
&= 0-\mathcal{B}_{245}\mathcal{B}_{523}+\mathcal{B}_{146}\mathcal{B}_{613}+0\\
&= -2\mu\lambda\\
R_{51} &= \mathcal{B}_{254}\mathcal{B}_{412}\varepsilon_1\varepsilon_2+\mathcal{B}_{354}\mathcal{B}_{413}\varepsilon_1\varepsilon_3+\mathcal{B}_{256}\mathcal{B}_{612}\varepsilon_3\varepsilon_2+\mathcal{B}_{356}\mathcal{B}_{613}\varepsilon_3\varepsilon_3\\
&= 0+\mathcal{B}_{345}\mathcal{B}_{413}-\mathcal{B}_{256}\mathcal{B}_{612}+0\\
&= \frac{1}{2}\lambda\left(h-\beta+\gamma-\alpha\right)-\frac{1}{2}\lambda\left(h+\beta-\gamma+\alpha\right)\\
&= -\lambda\left(\beta-\gamma+\alpha\right)\\
R_{52} &= \mathcal{B}_{154}\mathcal{B}_{421}\varepsilon_1\varepsilon_1+\mathcal{B}_{354}\mathcal{B}_{423}\varepsilon_1\varepsilon_3+\mathcal{B}_{156}\mathcal{B}_{621}\varepsilon_3\varepsilon_1+\mathcal{B}_{356}\mathcal{B}_{623}\varepsilon_3\varepsilon_3\\
&= 0+\mathcal{B}_{345}\mathcal{B}_{423}+\mathcal{B}_{156}\mathcal{B}_{612}+0\\
&= \frac{1}{4}\left(h-\beta+\gamma-\alpha\right)\left(h+\gamma-\alpha+\beta\right)+\frac{1}{4}\left(h-\gamma+\alpha-\beta\right)\left(h+\beta-\gamma+\alpha\right)\\
&= \frac{1}{4}\left(\left(h+\left(\gamma-\alpha\right)\right)^2-\beta^2+\left(h-\left(\gamma-\alpha\right)\right)^2-\beta^2\right)\\
&= -\frac{1}{2}\beta^2+\frac{1}{2}h^2+\frac{1}{2}\left(\gamma-\alpha\right)^2\\
R_{53} &= \mathcal{B}_{154}\mathcal{B}_{431}\varepsilon_1\varepsilon_1+\mathcal{B}_{254}\mathcal{B}_{432}\varepsilon_1\varepsilon_2+\mathcal{B}_{156}\mathcal{B}_{631}\varepsilon_3\varepsilon_1+\mathcal{B}_{256}\mathcal{B}_{632}\varepsilon_3\varepsilon_2\\
&= 0+\mathcal{B}_{245}\mathcal{B}_{423}+\mathcal{B}_{156}\mathcal{B}_{613}+0\\
&= -\frac{1}{2}\mu\left(h+\gamma-\alpha+\beta\right)+\frac{1}{2}\mu\left(h-\gamma+\alpha-\beta\right)\\
&= -\mu\left(\gamma-\alpha+\beta\right)\\
R_{61} &= \mathcal{B}_{264}\mathcal{B}_{412}\varepsilon_1\varepsilon_2+\mathcal{B}_{364}\mathcal{B}_{413}\varepsilon_1\varepsilon_3+\mathcal{B}_{265}\mathcal{B}_{512}\varepsilon_2\varepsilon_2+\mathcal{B}_{365}\mathcal{B}_{513}\varepsilon_2\varepsilon_3\\
&= 0+\mathcal{B}_{346}\mathcal{B}_{413}-\mathcal{B}_{256}\mathcal{B}_{512}+0\\
&= -2\lambda\mu\\
R_{62} &= \mathcal{B}_{164}\mathcal{B}_{421}\varepsilon_1\varepsilon_1+\mathcal{B}_{364}\mathcal{B}_{423}\varepsilon_1\varepsilon_3+\mathcal{B}_{165}\mathcal{B}_{521}\varepsilon_2\varepsilon_1+\mathcal{B}_{365}\mathcal{B}_{523}\varepsilon_2\varepsilon_3\\
&= 0+\mathcal{B}_{346}\mathcal{B}_{423}+\mathcal{B}_{156}\mathcal{B}_{512}+0\\
&= -\frac{1}{2}\mu\left(h+\gamma-\alpha+\beta\right)+\frac{1}{2}\mu\left(h-\gamma+\alpha-\beta\right)\\
&= -\mu\left(\gamma-\alpha+\beta\right)\\
R_{63} &= \mathcal{B}_{164}\mathcal{B}_{431}\varepsilon_1\varepsilon_1+\mathcal{B}_{264}\mathcal{B}_{432}\varepsilon_1\varepsilon_2+\mathcal{B}_{165}\mathcal{B}_{531}\varepsilon_2\varepsilon_1+\mathcal{B}_{265}\mathcal{B}_{532}\varepsilon_2\varepsilon_2\\
&= \mathcal{B}_{146}\mathcal{B}_{413}+\mathcal{B}_{246}\mathcal{B}_{423}+\mathcal{B}_{156}\mathcal{B}_{513}+\mathcal{B}_{256}\mathcal{B}_{523}\\
&= -\lambda^2+\frac{1}{4}\left(-h+\gamma-\beta+\alpha\right)\left(h+\gamma-\alpha+\beta\right)\\
&\qquad+\frac{1}{4}\left(h-\gamma+\alpha-\beta\right)\left(-h-\gamma+\beta-\alpha\right)-\lambda^2\\
&= -2\lambda^2+\frac{1}{4}\left(\gamma^2-\left(h+\left(\beta-\alpha\right)\right)^2+\gamma^2-\left(h-\left(\beta-\alpha\right)\right)^2\right)
\end{aligned}
$$

$$= -2\lambda^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}h^2 - \frac{1}{2}(\beta - \alpha)^2$$

We see that the Einstein equations yield a system of homogeneous quadratic equations in the real variables $\alpha, \beta, \gamma, \lambda$ and μ.

The normal form $L_5(\alpha)$ is excluded by equation $R_{43} = 0$ for any $\alpha \in \mathbb{R}$.

Equation $R_{53} = 0$ for the normal form $L_2(\alpha, \beta, \gamma)$ reads as

$$0 = \beta (2\alpha - \gamma).$$

If $\beta = 0$, then the matrix is diagonal, so assume that $\gamma = 2\alpha$. Then $R_{52} = 0$ yields

$$\begin{aligned} 0 &= -\frac{1}{2}\alpha^2 + \frac{1}{2}h^2 + \frac{1}{2}(\alpha - \gamma)^2 \\ &= \frac{1}{2}h^2 - \alpha\gamma + \frac{1}{2}\gamma^2 \\ &= \frac{1}{2}h^2 \end{aligned}$$

and hence $h = 0$. Therefore, equation $R_{41} = 0$ is

$$0 = -2\beta^2 - \frac{1}{2}\gamma^2,$$

which gives $\beta = \gamma = 0$. Hence L is diagonalizable by an orthonormal basis.

If we consider the normal form $L_3(\alpha, \beta)$, the equation $R_{53} = 0$ is

$$0 = -\frac{1}{2}\left(\alpha - \frac{1}{2} - \beta + \alpha + \frac{1}{2}\right) = -\frac{1}{2}(2\alpha - \beta)$$

and hence $2\alpha = \beta$. Now, the equation for R_{41} yields

$$\begin{aligned} 0 &= -2\left(\frac{1}{2}\right)^2 - \frac{1}{2}\beta^2 + \frac{1}{2}h^2 + \frac{1}{2}\left(\alpha + \frac{1}{2} - \alpha + \frac{1}{2}\right)^2 \\ &= -\frac{1}{2} - \frac{1}{2}\beta^2 + \frac{1}{2}h^2 + \frac{1}{2} \\ &= -\frac{1}{2}\beta^2 + \frac{1}{2}h^2, \end{aligned}$$

hence $\beta^2 = h^2$. Applying this to the equation $R_{52} = 0$ gives

$$\begin{aligned} 0 &= -\frac{1}{2}\left(\frac{1}{2} + \alpha\right)^2 + \frac{1}{2}h^2 + \frac{1}{2}\left(\alpha - \frac{1}{2} - \beta\right)^2 \\ &= -\frac{1}{2}\left(\frac{1}{2} + \alpha\right)^2 + \frac{1}{2}h^2 + \frac{1}{2}\left(-\alpha - \frac{1}{2}\right)^2 \\ &= \frac{1}{2}h^2. \end{aligned}$$

From that see $h = 0$ and therefore $\alpha = \beta = 0$.

A similar computation for $L_4(\alpha, \beta)$, shows that the only possibility is $L_4(0, 0)$ with $h = 0$.

□

Theorem 6.4. Let $(H, \mathcal{G}_g)$ be a divergence-free generalized Einstein structure on an oriented three-dimensional unimodular Lie group G. If the endomorphism $L \in \operatorname{End}\mathfrak{g}$ defined in (33) is diagonalizable, then there exists an oriented g-orthonormal basis (v_a) of $\mathfrak{g} = \operatorname{Lie} G$ and $\alpha_1, \alpha_2, \alpha_3, h \in \mathbb{R}$ such that

$$[v_a, v_b] = \alpha_c \varepsilon_c v_c, \quad \forall \quad \text{cyclic} \quad (a, b, c) \in \mathfrak{S}_3, \quad H = h\mathrm{vol}_g, \tag{36}$$

where $\varepsilon_a = g(v_a, v_a)$ satisfies $\varepsilon_1 = \varepsilon_2$. The constants $(\alpha_1, \alpha_2, \alpha_3, h)$ can take the following values.

1. $\alpha_1 = \alpha_2 = \alpha_3 = \pm h$, in which case $\mathfrak{g}$ is either abelian and g is flat (the case $h = 0$) or $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(2, 1)$ or $\mathfrak{so}(3)$. The case $\mathfrak{so}(3)$ occurs precisely when g is definite (and $h \neq 0$).

2. There exists a cyclic permutation $\sigma \in \mathfrak{S}_3$ such that

$$\alpha_{\sigma(1)} = \alpha_{\sigma(2)} \neq 0 \quad \text{and} \quad h = \alpha_{\sigma(3)} = 0.$$

   In this case g is flat and $[\mathfrak{g}, \mathfrak{g}]$ is abelian of dimension 2, that is $\mathfrak{g}$ is metabelian. More precisely, $\mathfrak{g}$ is isomorphic to $\mathfrak{e}(2)$ (g definite on $[\mathfrak{g}, \mathfrak{g}]$) or $\mathfrak{e}(1, 1)$ (g indefinite on $[\mathfrak{g}, \mathfrak{g}]$), where $\mathfrak{e}(p, q)$ denotes the Lie algebra of the isometry group of the pseudo-Euclidean space $\mathbb{R}^{p,q}$.

If the endomorphism is not diagonalizable (g is necessarily indefinite in this case), then $h = 0$ and the Lie group G is isomorphic to the Heisenberg group.

Proof. Assume first that L is diagonalizable. Note, that the existence of $(\alpha_1, \alpha_2, \alpha_3, h)$ such that (36) is an immediate consequence of the diagonalizability of L. The corresponding structure constants κ_{abc} are given by

$$\kappa_{abc} = \alpha_c, \quad \forall \quad \text{cyclic} \quad (a, b, c) \in \mathfrak{S}_3.$$

In virtue of Proposition 5.29, this implies the following[6].

1. For all $a, b, c \in \{1, 2, 3\}$:

$$\mathcal{B}_{abc} = \frac{1}{2}(h + \alpha_1 + \alpha_2 + \alpha_3)\varepsilon_{abc},$$

   where $\varepsilon_{abc} = \mathrm{vol}_g(v_a, v_b, v_c)$.

2. For all $i, j, k \in \{4, 5, 6\}$:

$$\mathcal{B}_{ijk} = \frac{1}{2}(h - \alpha_1 - \alpha_2 - \alpha_3)\varepsilon_{i'j'k'},$$

   where $i' = i - 3$ for all $i \in \{4, 5, 6\}$.

[6] The first two formulas are not needed for the proof. They are only included for future use.

3. For $a \in \{1,2,3\}$ and $j,k \in \{4,5,6\}$ the coefficients

$$\mathcal{B}_{ajk} = \frac{1}{2}(H_{aj'k'} - \kappa_{aj'k'} + \kappa_{j'k'a} - \kappa_{k'aj'})$$

are given explicitly by

$$\begin{aligned} \mathcal{B}_{156} = -\mathcal{B}_{165} &= \frac{1}{2}(h - \alpha_3 + \alpha_1 - \alpha_2) =: \frac{1}{2}X_1 \\ \mathcal{B}_{264} = -\mathcal{B}_{246} &= \frac{1}{2}(h - \alpha_1 + \alpha_2 - \alpha_3) =: \frac{1}{2}X_2 \\ \mathcal{B}_{345} = -\mathcal{B}_{354} &= \frac{1}{2}(h - \alpha_2 + \alpha_3 - \alpha_1) =: \frac{1}{2}X_3, \end{aligned}$$

with all other components equal to zero.

4. For $i \in \{4,5,6\}$ and $b,c \in \{1,2,3\}$ the coefficients

$$\mathcal{B}_{ibc} = \frac{1}{2}(H_{i'bc} + \kappa_{i'bc} - \kappa_{bci'} + \kappa_{ci'b})$$

are given explicitly by

$$\begin{aligned} \mathcal{B}_{423} = -\mathcal{B}_{432} &= \frac{1}{2}(h + \alpha_3 - \alpha_1 + \alpha_2) =: \frac{1}{2}Y_1 \\ \mathcal{B}_{531} = -\mathcal{B}_{513} &= \frac{1}{2}(h + \alpha_1 - \alpha_2 + \alpha_3) =: \frac{1}{2}Y_2 \\ \mathcal{B}_{612} = -\mathcal{B}_{621} &= \frac{1}{2}(h + \alpha_2 - \alpha_3 + \alpha_1) =: \frac{1}{2}Y_3, \end{aligned}$$

with all other components equal to zero.

From these formulas and equation (35), we can now compute the components

$$R_{ia} = \sum_{j,b} \mathcal{B}_{bij}\mathcal{B}_{jab}\varepsilon_{j'}\varepsilon_b,$$

of the Ricci curvature (for zero divergence $\delta = 0$) with respect to the orthonormal basis $(e_r) = (e_a, e_i)$, $e_a = v_a + gv_a$, $e_i = v_{i'} - gv_{i'}$, of $E = \mathfrak{g} \oplus \mathfrak{g}^*$. Explicitly we obtain

$$\begin{aligned} R_{41} &= \mathcal{B}_{246}\mathcal{B}_{612}\varepsilon_3\varepsilon_2 + \mathcal{B}_{345}\mathcal{B}_{513}\varepsilon_2\varepsilon_3 = -\frac{\varepsilon_2\varepsilon_3}{4}(X_2Y_3 + X_3Y_2) \\ R_{52} &= \mathcal{B}_{156}\mathcal{B}_{621}\varepsilon_3\varepsilon_1 + \mathcal{B}_{354}\mathcal{B}_{423}\varepsilon_1\varepsilon_3 = -\frac{\varepsilon_1\varepsilon_3}{4}(X_1Y_3 + X_3Y_1) \\ R_{63} &= \mathcal{B}_{264}\mathcal{B}_{432}\varepsilon_1\varepsilon_2 + \mathcal{B}_{165}\mathcal{B}_{531}\varepsilon_2\varepsilon_1 = -\frac{\varepsilon_1\varepsilon_2}{4}(X_1Y_2 + X_2Y_1), \end{aligned}$$

with all other components equal to zero. We conclude that the generalized Einstein equations reduce to a system of three homogeneous quadratic equations in the variables X_a and Y_a:

$$X_1Y_2 + X_2Y_1 = X_1Y_3 + X_3Y_1 = X_2Y_3 + X_3Y_2 = 0.$$

A priori, we can distinguish four types of solutions depending on how many components of the vector (X_1, X_2, X_3) are equal to zero: 0,1,2 or 3.

Solutions of type 0: $X_1X_2X_3 \neq 0$ implies $Y_1 = Y_2 = Y_3 = 0$ and finally

$$\alpha_1 = \alpha_2 = \alpha_3 = -h \neq 0.$$

In this case the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(2,1)$ or $\mathfrak{so}(3)$. The latter case happens if and only if the metric g is definite.

Solutions of type 1: assume for example that $X_1X_2 \neq 0$, $X_3 = 0$. This implies that $Y_3 = 0$ and, hence, $\alpha_3 = \alpha_1 + \alpha_2$ and $h = 0$. But then the equation $X_1Y_2 + X_2Y_1 = 0$ reduces to $\alpha_1\alpha_2 = 0$, which is inconsistent with $X_1X_2 \neq 0$. This shows that solutions of type 1 do not exist.

Solutions of type 2: assume for example that $X_1 \neq 0$, $X_2 = X_3 = 0$. This implies $Y_2 = Y_3 = 0$ and finally $h = \alpha_1 = 0$, $\alpha_2 = \alpha_3 \neq 0$. So the solutions of type 2 are of the following form. There exists a cyclic permutation $\sigma \in \mathfrak{S}_3$ such that

$$\alpha_{\sigma(1)} = \alpha_{\sigma(2)} \neq 0 \quad \text{and} \quad h = \alpha_{\sigma(3)} = 0.$$

We conclude, for solutions of type 2, that g is flat (see Corollary 5.34) and $\mathfrak{g}$ is metabelian. $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{v_{\sigma(1)}, v_{\sigma(2)}\}$ is two-dimensional and $\mathrm{ad}_{v_{\sigma(3)}}$ acts on it by a non-zero g-skew-symmetric endomorphism. This implies that $\mathfrak{g}$ is isomorphic to $\mathfrak{e}(2)$ or $\mathfrak{e}(1,1)$.

Solutions of type 3: assume $X_1 = X_2 = X_3 = 0$. This implies

$$\alpha_1 = \alpha_2 = \alpha_3 = h.$$

In this case $\mathfrak{g}$ is either abelian and g is flat (the case $h = 0$ again by Corollary 5.34) or $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(2,1)$ or $\mathfrak{so}(3)$, as for type 0.

If L is not diagonalizable, then g is indefinite and there exists an orthonormal basis $(v_a)_a$ with $g(v_1, v_1) = g(v_2, v_2) = -g(v_3, v_3)$ such that L is either of the form $L_3(0,0)$ or $L_4(0,0)$, and $h = 0$ by Proposition 6.3. We consider first the case $L_3(0,0)$. To prove that G is isomorphic to the Heisenberg group, we show, using equation (34), that the generators $P := v_1, Q := v_2 + v_3$ and $R := \varepsilon_2(v_3 - v_2)$ of its Lie algebra $\mathfrak{g}$ satisfy the relations $[P,Q] = R$ and $[P,R] = [Q,R] = 0$:

$$\begin{aligned}
[P,Q] &= [v_1, v_2 + v_3] = [v_1, v_2] - [v_3, v_1] \\
&= \varepsilon_3 L v_3 - \varepsilon_2 L v_2 \\
&= \frac{1}{2}\varepsilon_3 v_2 - \frac{1}{2}\varepsilon_3 v_3 - \frac{1}{2}\varepsilon_2 v_2 + \frac{1}{2}\varepsilon_2 v_3 \\
&= -\frac{1}{2}\varepsilon_2 v_2 + \frac{1}{2}\varepsilon_2 v_3 - \frac{1}{2}\varepsilon_2 v_2 + \frac{1}{2}\varepsilon_2 v_3 \\
&= \varepsilon_2(v_3 - v_2) \\
&= R \\
[P,R] &= [v_1, \varepsilon_2(v_3 - v_2)] = -\varepsilon_2[v_3, v_1] - \varepsilon_2[v_1, v_2] \\
&= -\varepsilon_2\varepsilon_2 L v_2 - \varepsilon_2\varepsilon_3 L v_3
\end{aligned}$$

$$
\begin{aligned}
&= -Lv_2 + Lv_3 \\
&= -\frac{1}{2}v_2 + \frac{1}{2}v_3 + \frac{1}{2}v_2 - \frac{1}{2}v_3 \\
&= 0 \\
[Q, R] &= [v_2 + v_3, \varepsilon_2(v_3 - v_2)] \\
&= \varepsilon_2[v_2, v_3] - \varepsilon_2[v_3, v_2] \\
&= 2\varepsilon_2[v_2, v_3] \\
&= 2\varepsilon_2\varepsilon_1 L v_1 \\
&= 0.
\end{aligned}
$$

In the case that L takes the form $L_4(0,0)$, we see analogously that the generators $P = v_1, Q = v_2 + v_3$ and $R = \varepsilon_2(v_2 - v_3)$ satisfy the relations $[P, Q] = R$ and $[P, R] = [Q, R] = 0$. ☐

6.2.2 Non-unimodular Lie groups

We assume now that the Lie group G is not unimodular. Let $\mathfrak{u} := \{x \in \mathfrak{g} \mid \operatorname{tr} \operatorname{ad}_x = 0\}$ be the unimodular kernel of $\mathfrak{g}$. It can be easily checked that $\mathfrak{u}$ is a two-dimensional abelian ideal of $\mathfrak{g}$, containing the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$. This means that the Lie algebra $\mathfrak{g}$ is a semidirect product of $\mathbb{R}$ and $\mathbb{R}^2$, where $\mathbb{R}$ is acting on $\mathbb{R}^2$ by an endomorphism with non-zero trace. For details on the classification of non-unimodular, three-dimensional Lie algebras in terms of the Jordan normal form of this endomorphism we refer to [GOV, Chapter 7, Theorem 1.4].

We first treat the case that the restriction $g|_{\mathfrak{u}\times\mathfrak{u}}$ of the metric g to $\mathfrak{u}$ is non-degenerate.

Proposition 6.5. Let $(H, \mathcal{G}_g, \delta = 0)$ be a divergence-free generalized Einstein structure on an oriented three-dimensional non-unimodular Lie group G. Let $\mathfrak{u}$ be the unimodular kernel of the Lie algebra $\mathfrak{g}$ and assume that $g|_{\mathfrak{u}\times\mathfrak{u}}$ is non-degenerate. Then $H = 0$ and g is indefinite. Furthermore there exists an orthonormal basis (v_a) of $(\mathfrak{g}, g)$ such that $v_1, v_3 \in \mathfrak{u}$ and $g(v_1, v_1) = g(v_2, v_2) = -g(v_3, v_3)$ and a positive constant $\theta > 0$ such that

$$
\begin{aligned}
[v_1, v_3] &= 0 \\
[v_2, v_1] &= \theta v_1 - \theta v_3 \\
[v_2, v_3] &= \theta v_1 + \theta v_3.
\end{aligned}
$$

Proof. A g-orthonormal basis $(v_a)_a$ of $\mathfrak{g}$ such that $v_1, v_3 \in \mathfrak{u}$ exists, because $g|_{\mathfrak{u}\times\mathfrak{u}}$ is non-degenerate. Since $\mathfrak{u}$ is an abelian ideal, there are $\lambda, \mu, \nu, \rho \in \mathbb{R}$ such that

$$
\begin{aligned}
[v_3, v_1] &= 0 \\
[v_2, v_1] &= \varepsilon_1 \lambda v_1 + \varepsilon_3 \mu v_3 \\
[v_2, v_3] &= \varepsilon_1 \nu v_1 + \varepsilon_3 \rho v_3
\end{aligned}
$$

with $0 \neq \operatorname{tr} \operatorname{ad}_{v_2} = \varepsilon_1 \lambda + \varepsilon_3 \rho$. Using $\lambda = \kappa_{211}, \mu = \kappa_{213}, \nu = \kappa_{231}$ and $\rho = \kappa_{233}$, we can compute the Dorfman coefficients

$$
\mathcal{B}_{145} = \frac{1}{2}\left(H_{112} - \kappa_{112} + \kappa_{121} - \kappa_{211}\right) = -\kappa_{211} = -\lambda
$$

$$\mathcal{B}_{146} = \frac{1}{2}(H_{113} - \kappa_{113} + \kappa_{131} - \kappa_{311}) = -\kappa_{311} = 0$$
$$\mathcal{B}_{156} = \frac{1}{2}(H_{123} - \kappa_{123} + \kappa_{231} - \kappa_{312}) = \frac{1}{2}(h + \kappa_{213} + \kappa_{231}) = \frac{1}{2}(h + \mu + \nu)$$
$$\mathcal{B}_{245} = \frac{1}{2}(H_{212} - \kappa_{212} + \kappa_{122} - \kappa_{221}) = -\kappa_{212} = 0$$
$$\mathcal{B}_{246} = \frac{1}{2}(H_{213} - \kappa_{213} + \kappa_{132} - \kappa_{321}) = \frac{1}{2}(-h - \kappa_{213} + \kappa_{231}) = -\frac{1}{2}(h + \mu - \nu)$$
$$\mathcal{B}_{256} = \frac{1}{2}(H_{223} - \kappa_{223} + \kappa_{232} - \kappa_{322}) = \kappa_{232} = 0$$
$$\mathcal{B}_{345} = \frac{1}{2}(H_{312} - \kappa_{312} + \kappa_{123} - \kappa_{231}) = \frac{1}{2}(h - \kappa_{213} - \kappa_{231}) = \frac{1}{2}(h - \mu - \nu)$$
$$\mathcal{B}_{346} = \frac{1}{2}(H_{313} - \kappa_{313} + \kappa_{133} - \kappa_{331}) = -\kappa_{313} = 0$$
$$\mathcal{B}_{356} = \frac{1}{2}(H_{323} - \kappa_{323} + \kappa_{233} - \kappa_{332}) = \kappa_{233} = \rho$$
$$\mathcal{B}_{412} = \frac{1}{2}(H_{112} + \kappa_{112} - \kappa_{121} + \kappa_{211}) = \kappa_{211} = \lambda$$
$$\mathcal{B}_{413} = \frac{1}{2}(H_{113} + \kappa_{113} - \kappa_{131} + \kappa_{311}) = \kappa_{311} = 0$$
$$\mathcal{B}_{423} = \frac{1}{2}(H_{123} + \kappa_{123} - \kappa_{231} + \kappa_{312}) = \frac{1}{2}(h - \kappa_{213} - \kappa_{231}) = \frac{1}{2}(h - \mu - \nu)$$
$$\mathcal{B}_{512} = \frac{1}{2}(H_{212} + \kappa_{212} - \kappa_{122} + \kappa_{221}) = \kappa_{212} = 0$$
$$\mathcal{B}_{513} = \frac{1}{2}(H_{213} + \kappa_{213} - \kappa_{132} + \kappa_{321}) = \frac{1}{2}(-h + \kappa_{213} - \kappa_{231}) = -\frac{1}{2}(h - \mu + \nu)$$
$$\mathcal{B}_{523} = \frac{1}{2}(H_{223} + \kappa_{223} - \kappa_{232} + \kappa_{322}) = -\kappa_{232} = 0$$
$$\mathcal{B}_{612} = \frac{1}{2}(H_{312} + \kappa_{312} - \kappa_{123} + \kappa_{231}) = \frac{1}{2}(h + \kappa_{213} + \kappa_{231}) = \frac{1}{2}(h + \mu + \nu)$$
$$\mathcal{B}_{613} = \frac{1}{2}(H_{313} + \kappa_{313} - \kappa_{133} + \kappa_{331}) = \kappa_{313} = 0$$
$$\mathcal{B}_{623} = \frac{1}{2}(H_{323} + \kappa_{323} - \kappa_{233} + \kappa_{332}) = -\kappa_{233} = -\rho.$$

To prove that the case $\varepsilon_1 = \varepsilon_3$ cannot occur, we compute using equation (35)

$$\begin{aligned} R_{52} &= \mathcal{B}_{154}\mathcal{B}_{421}\varepsilon_1\varepsilon_1 + \mathcal{B}_{354}\mathcal{B}_{423}\varepsilon_1\varepsilon_3 + \mathcal{B}_{156}\mathcal{B}_{621}\varepsilon_3\varepsilon_1 + \mathcal{B}_{356}\mathcal{B}_{623}\varepsilon_3\varepsilon_3 \\ &= \mathcal{B}_{145}\mathcal{B}_{412} - \mathcal{B}_{345}\mathcal{B}_{423} - \mathcal{B}_{156}\mathcal{B}_{612} + \mathcal{B}_{356}\mathcal{B}_{623} \\ &= -\lambda^2 - \frac{1}{4}(h - \mu - \nu)^2 - \frac{1}{4}(h + \mu + \nu)^2 - \rho^2 \end{aligned}$$

where we have used that $\varepsilon_1 = \varepsilon_3$. But this can only be zero if $\lambda = \rho = 0$, which contradicts $0 \neq \operatorname{tr}\operatorname{ad}_{v_2} = \varepsilon_1\lambda + \varepsilon_3\rho$. This proves that $\varepsilon_1 = -\varepsilon_3$. Hence, we can assume that the basis is chosen such that $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3$.

In this case the components of the Ricci curvature are

$$R_{41} = \mathcal{B}_{245}\mathcal{B}_{512}\varepsilon_2\varepsilon_2 + \mathcal{B}_{345}\mathcal{B}_{513}\varepsilon_2\varepsilon_3 + \mathcal{B}_{246}\mathcal{B}_{612}\varepsilon_3\varepsilon_2 + \mathcal{B}_{346}\mathcal{B}_{613}\varepsilon_3\varepsilon_3$$

$$
\begin{aligned}
&= 0 - \mathcal{B}_{345}\mathcal{B}_{513} - \mathcal{B}_{246}\mathcal{B}_{612} + 0 \\
&= \frac{1}{4}(h-\mu-\nu)(h-\mu+\nu) + \frac{1}{4}(h+\mu-\nu)(h+\mu+\nu) \\
&= \frac{1}{4}\Big((h-\mu)^2 - \nu^2 + (h+\mu)^2 - \nu^2\Big) \\
&= \frac{1}{2}\left(h^2+\mu^2-\nu^2\right) \\
R_{42} &= \mathcal{B}_{145}\mathcal{B}_{521}\varepsilon_2\varepsilon_1 + \mathcal{B}_{345}\mathcal{B}_{523}\varepsilon_2\varepsilon_3 + \mathcal{B}_{146}\mathcal{B}_{621}\varepsilon_3\varepsilon_1 + \mathcal{B}_{346}\mathcal{B}_{623}\varepsilon_3\varepsilon_3 \\
&= 0 \\
R_{43} &= \mathcal{B}_{145}\mathcal{B}_{531}\varepsilon_2\varepsilon_1 + \mathcal{B}_{245}\mathcal{B}_{532}\varepsilon_2\varepsilon_2 + \mathcal{B}_{146}\mathcal{B}_{631}\varepsilon_3\varepsilon_1 + \mathcal{B}_{246}\mathcal{B}_{632}\varepsilon_3\varepsilon_2 \\
&= -\mathcal{B}_{145}\mathcal{B}_{513} + 0 + 0 + \mathcal{B}_{246}\mathcal{B}_{623} \\
&= -\frac{1}{2}\lambda(h-\mu+\nu) + \frac{1}{2}\rho(h+\mu-\nu) \\
R_{51} &= \mathcal{B}_{254}\mathcal{B}_{412}\varepsilon_1\varepsilon_2 + \mathcal{B}_{354}\mathcal{B}_{413}\varepsilon_1\varepsilon_3 + \mathcal{B}_{256}\mathcal{B}_{612}\varepsilon_3\varepsilon_2 + \mathcal{B}_{356}\mathcal{B}_{613}\varepsilon_3\varepsilon_3 \\
&= 0 \\
R_{52} &= \mathcal{B}_{154}\mathcal{B}_{421}\varepsilon_1\varepsilon_1 + \mathcal{B}_{354}\mathcal{B}_{423}\varepsilon_1\varepsilon_3 + \mathcal{B}_{156}\mathcal{B}_{621}\varepsilon_3\varepsilon_1 + \mathcal{B}_{356}\mathcal{B}_{623}\varepsilon_3\varepsilon_3 \\
&= \mathcal{B}_{145}\mathcal{B}_{412} + \mathcal{B}_{345}\mathcal{B}_{423} + \mathcal{B}_{156}\mathcal{B}_{612} + \mathcal{B}_{356}\mathcal{B}_{623} \\
&= -\lambda^2 + \frac{1}{4}(h-\mu-\nu)^2 + \frac{1}{4}(h+\mu+\nu)^2 - \rho^2 \\
&= -\lambda^2 + \frac{1}{2}h^2 + \frac{1}{2}(\mu+\nu)^2 - \rho^2 \\
R_{53} &= \mathcal{B}_{154}\mathcal{B}_{431}\varepsilon_1\varepsilon_1 + \mathcal{B}_{254}\mathcal{B}_{432}\varepsilon_1\varepsilon_2 + \mathcal{B}_{156}\mathcal{B}_{631}\varepsilon_3\varepsilon_1 + \mathcal{B}_{256}\mathcal{B}_{632}\varepsilon_3\varepsilon_2 \\
&= 0 \\
R_{61} &= \mathcal{B}_{264}\mathcal{B}_{412}\varepsilon_1\varepsilon_2 + \mathcal{B}_{364}\mathcal{B}_{413}\varepsilon_1\varepsilon_3 + \mathcal{B}_{265}\mathcal{B}_{512}\varepsilon_2\varepsilon_2 + \mathcal{B}_{365}\mathcal{B}_{513}\varepsilon_2\varepsilon_3 \\
&= -\mathcal{B}_{246}\mathcal{B}_{412} + 0 + 0 + \mathcal{B}_{356}\mathcal{B}_{513} \\
&= \frac{1}{2}\lambda(h+\mu-\nu) - \frac{1}{2}\rho(h-\mu+\nu) \\
R_{62} &= \mathcal{B}_{164}\mathcal{B}_{421}\varepsilon_1\varepsilon_1 + \mathcal{B}_{364}\mathcal{B}_{423}\varepsilon_1\varepsilon_3 + \mathcal{B}_{165}\mathcal{B}_{521}\varepsilon_2\varepsilon_1 + \mathcal{B}_{365}\mathcal{B}_{523}\varepsilon_2\varepsilon_3 \\
&= 0 \\
R_{63} &= \mathcal{B}_{164}\mathcal{B}_{431}\varepsilon_1\varepsilon_1 + \mathcal{B}_{264}\mathcal{B}_{432}\varepsilon_1\varepsilon_2 + \mathcal{B}_{165}\mathcal{B}_{531}\varepsilon_2\varepsilon_1 + \mathcal{B}_{265}\mathcal{B}_{532}\varepsilon_2\varepsilon_2 \\
&= 0 + \mathcal{B}_{246}\mathcal{B}_{423} + \mathcal{B}_{156}\mathcal{B}_{513} + 0 \\
&= -\frac{1}{4}(h+\mu-\nu)(h-\mu-\nu) - \frac{1}{4}(h+\mu+\nu)(h-\mu+\nu) \\
&= -\frac{1}{4}\Big((h-\nu)^2 - \mu^2 + (h+\nu)^2 - \mu^2\Big) \\
&= -\frac{1}{2}\left(h^2+\nu^2-\mu^2\right)
\end{aligned}
$$

Imposing the Einstein condition, we see from the equations $R_{41}+R_{63}=0$ and $R_{41}-R_{63}=0$, that $h^2=0$ and $\mu^2=\nu^2$. If $\mu=-\nu$, then $R_{52}=0$ reads as $0=-\lambda^2-\rho^2$, hence $\lambda=\rho=0$, which contradicts $0\neq\operatorname{tr}\operatorname{ad}_{v_2}=\varepsilon_1\lambda+\varepsilon_3\rho$. Therefore $\mu=\nu$ and, from

$R_{52} = 0$ we obtain,

$$2\mu^2 = \lambda^2 + \rho^2.$$

In particular $\mu \neq 0$, due to $0 \neq \operatorname{tr} \operatorname{ad}_{v_2} = \varepsilon_1 \lambda + \varepsilon_3 \rho$. Note now that $\mu = \nu$ implies that the endomorphism $M \in \operatorname{End}(\mathfrak{u})$, defined as the restriction of ad_{v_2} to $\mathfrak{u}$, is symmetric. A simple consequence of Proposition 6.2 is that there exists an orthonormal basis of $\mathfrak{u}$ such that M is represented by one of the matrices

$$M_1(\theta,\eta) = \begin{pmatrix} \theta & 0 \\ 0 & \eta \end{pmatrix}, \qquad M_2(\theta,\eta) = \begin{pmatrix} \theta & -\eta \\ \eta & \theta \end{pmatrix},$$
$$M_3(\theta) = \begin{pmatrix} \frac{1}{2}+\theta & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}+\theta \end{pmatrix}, \qquad M_4(\theta) = \begin{pmatrix} -\frac{1}{2}+\theta & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}+\theta \end{pmatrix}$$

in this basis. We may assume that the basis v_1, v_3 of $\mathfrak{u}$ is chosen such that M takes one of these normal forms with respect to v_1, v_3. We see that $M_1(\theta,\eta)$ is excluded by the condition $\mu \neq 0$. Applying $2\mu^2 = \lambda^2 + \rho^2$ to the normal form $M_3(\theta)$ yields

$$2\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}+\theta\right)^2 + \left(-\frac{1}{2}+\theta\right)^2 = 2\left(\frac{1}{2}\right)^2 + \theta^2.$$

Hence $\theta = 0$, which contradicts $\operatorname{tr} \operatorname{ad}_{v_2} \neq 0$. For the same reason M also cannot have the normal form $M_4(\theta)$. In the remaining case $M_2(\theta,\eta)$ the equation $2\mu^2 = \lambda^2 + \rho^2$ reads as $2\,(-\eta)^2 = \theta^2 + (-\theta)^2$. Therefore $\eta = \pm\theta$. Furthermore $\eta \neq 0$ because $\mu \neq 0$. Hence the only two normal forms are

$$M_2(\theta,\theta) = \begin{pmatrix} \theta & -\theta \\ \theta & \theta \end{pmatrix}, \quad M_2(\theta,-\theta) = \begin{pmatrix} \theta & \theta \\ -\theta & \theta \end{pmatrix}, \quad \theta \neq 0.$$

Replacing v_1 by $-v_1$ (exchanging $M_2(\theta,\theta)$ with $M_2(\theta,-\theta)$) and v_2 by $-v_2$ (replacing θ by $-\theta$), if necessary, we obtain the claimed equations for $\theta > 0$. □

Remark 6.6. Note that, while all the occurring Lie algebras in the previous Proposition are non-isomorphic as metric Lie algebras, they are isomorphic as Lie algebras. They are a semidirect product of $\mathbb{R}^2$ and $\mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ by the endomorphism $\operatorname{ad}_{v_2}|_{\mathfrak{u}}$, which has non-real and non-imaginary eigenvalues $(1+i)\theta$ and $(1-i)\theta$. This corresponds to the Lie algebra $\mathfrak{r}'_{3,1}(\mathbb{R})$ in the notation of [GOV, Chapter 7, Theorem 1.4].

Proposition 6.7. There is no divergence-free generalized Einstein structure $(H, \mathcal{G}_g, \delta = 0)$ on an oriented three-dimensional non-unimodular Lie group G such that g is degenerate on the unimodular kernel $\mathfrak{u}$ of $\mathfrak{g}$.

Proof. Note first that the metric g necessarily has to be indefinite. We define $\varepsilon := 1$ if the signature of g is $(2,1)$ and $\varepsilon := -1$ if it is $(1,2)$. Note that in both cases there is a two-dimensional subspace of $\mathfrak{g}$ on which εg is positive definite. Taking the intersection with $\mathfrak{u}$ we obtain a one-dimensional subspace generated by a vector w_1 such that $g(w_1, w_1) = \varepsilon$.

Next we choose a generator w_2 of the kernel of $g|_{\mathfrak{u}\times\mathfrak{u}}$ and a null vector w_3 orthogonal to w_1 such that $g(w_2, w_3) = \frac{\varepsilon}{2}$. Summarizing, we obtain a basis (w_a) of $\mathfrak{g}$ such that:

$$g(w_1,w_1)=\varepsilon,\ g(w_1,w_2)=g(w_1,w_3)=g(w_2,w_2)=g(w_3,w_3)=0,\ g(w_2,w_3)=\frac{\varepsilon}{2} \quad (37)$$

and $w_1, w_2 \in \mathfrak{u}$. Denote by θ^c_{ab} the structure constants of $\mathfrak{g}$ in the basis (w_a), $[w_a, w_b] = \theta^c_{ab} w_c$. Then

$$\begin{aligned}
[w_1, w_2] &= 0\\
[w_3, w_1] &= \theta^1_{31} w_1 + \theta^2_{31} w_2\\
[w_3, w_2] &= \theta^1_{32} w_1 + \theta^2_{32} w_2
\end{aligned}$$

with $0 \neq \operatorname{tr}\operatorname{ad}_{w_3} = \theta^1_{31} + \theta^2_{32}$. The basis $v_1 := w_1, v_2 := w_2 + w_3, v_3 := w_2 - w_3$ of $\mathfrak{g}$ is an orthonormal with respect to g satisfying $g(v_1, v_1) = g(v_2, v_2) = -g(v_3, v_3)$. If we define $\lambda := -\varepsilon_1 \theta^1_{31}, \mu := -\varepsilon_2 \frac{1}{2}\theta^2_{31}, \nu := -\varepsilon_1 2\theta^1_{32}$ and $\rho := -\varepsilon_2\theta^2_{32}$, where $\varepsilon_a = g(v_a, v_a)$, then

$$\begin{aligned}
\kappa^c_{12} v_c = [v_1, v_2] &= [w_1, w_2 + w_3] = -[w_3, w_1]\\
&= -\theta^1_{31} w_1 - \theta^2_{31} w_2\\
&= -\theta^1_{31} v_1 - \frac{1}{2}\theta^2_{31} v_2 - \frac{1}{2}\theta^2_{31} v_3\\
&= \varepsilon_1 \lambda v_1 + \varepsilon_2 \mu v_2 - \varepsilon_3 \mu v_3\\
\kappa^c_{23} v_c = [v_2, v_3] &= [w_2 + w_3, w_2 - w_3] = -2[w_3, w_2]\\
&= -2\theta^1_{32} w_1 - 2\theta^2_{32} w_2\\
&= -2\theta^1_{32} v_1 - \theta^2_{32} v_2 - \theta^2_{32} v_3\\
&= \varepsilon_1 \nu v_1 + \varepsilon_2 \rho v_2 - \varepsilon_3 \rho v_3\\
\kappa^c_{31} v_c = [v_3, v_1] &= [w_2 - w_3, w_1] = -[w_3, w_1]\\
&= -\theta^1_{31} w_1 - \theta^2_{31} w_2\\
&= -\theta^1_{31} v_1 - \frac{1}{2}\theta^2_{31} v_2 - \frac{1}{2}\theta^2_{31} v_3\\
&= \varepsilon_1 \lambda v_1 + \varepsilon_2 \mu v_2 - \varepsilon_3 \mu v_3
\end{aligned}$$

with $\lambda + \rho \neq 0$. Hence the structure constants κ_{abc} of $\mathfrak{g}$ with respect to $(v_a)_a$ are

$$\begin{aligned}
&\kappa_{121} = \lambda, \quad \kappa_{122} = \mu, \quad \kappa_{123} = -\mu\\
&\kappa_{231} = \nu, \quad \kappa_{232} = \rho, \quad \kappa_{233} = -\rho\\
&\kappa_{311} = \lambda, \quad \kappa_{312} = \mu, \quad \kappa_{313} = -\mu.
\end{aligned}$$

Now the Dorfman coefficients are

$$\begin{aligned}
\mathcal{B}_{145} &= \frac{1}{2}\left(H_{112} - \kappa_{112} + \kappa_{121} - \kappa_{211}\right) = \kappa_{121} = \lambda\\
\mathcal{B}_{146} &= \frac{1}{2}\left(H_{113} - \kappa_{113} + \kappa_{131} - \kappa_{311}\right) = -\kappa_{311} = -\lambda
\end{aligned}$$

$$
\begin{aligned}
\mathcal{B}_{156} &= \frac{1}{2}\left(H_{123} - \kappa_{123} + \kappa_{231} - \kappa_{312}\right) = \frac{1}{2}\left(h + \mu + \nu - \mu\right) = \frac{1}{2}\left(h + \nu\right) \\
\mathcal{B}_{245} &= \frac{1}{2}\left(H_{212} - \kappa_{212} + \kappa_{122} - \kappa_{221}\right) = \kappa_{122} = \mu \\
\mathcal{B}_{246} &= \frac{1}{2}\left(H_{213} - \kappa_{213} + \kappa_{132} - \kappa_{321}\right) = \frac{1}{2}\left(-h - \mu - \mu + \nu\right) = -\frac{1}{2}\left(h + 2\mu - \nu\right) \\
\mathcal{B}_{256} &= \frac{1}{2}\left(H_{223} - \kappa_{223} + \kappa_{232} - \kappa_{322}\right) = \kappa_{232} = \rho \\
\mathcal{B}_{345} &= \frac{1}{2}\left(H_{312} - \kappa_{312} + \kappa_{123} - \kappa_{231}\right) = \frac{1}{2}\left(h - \mu - \mu - \nu\right) = \frac{1}{2}\left(h - 2\mu - \nu\right) \\
\mathcal{B}_{346} &= \frac{1}{2}\left(H_{313} - \kappa_{313} + \kappa_{133} - \kappa_{331}\right) = -\kappa_{313} = \mu \\
\mathcal{B}_{356} &= \frac{1}{2}\left(H_{323} - \kappa_{323} + \kappa_{233} - \kappa_{332}\right) = \kappa_{233} = -\rho \\
\mathcal{B}_{412} &= \frac{1}{2}\left(H_{112} + \kappa_{112} - \kappa_{121} + \kappa_{211}\right) = -\kappa_{121} = -\lambda \\
\mathcal{B}_{413} &= \frac{1}{2}\left(H_{113} + \kappa_{113} - \kappa_{131} + \kappa_{311}\right) = \kappa_{311} = \lambda \\
\mathcal{B}_{423} &= \frac{1}{2}\left(H_{123} + \kappa_{123} - \kappa_{231} + \kappa_{312}\right) = \frac{1}{2}\left(h - \mu - \nu + \mu\right) = \frac{1}{2}\left(h - \nu\right) \\
\mathcal{B}_{512} &= \frac{1}{2}\left(H_{212} + \kappa_{212} - \kappa_{122} + \kappa_{221}\right) = -\kappa_{122} = -\mu \\
\mathcal{B}_{513} &= \frac{1}{2}\left(H_{213} + \kappa_{213} - \kappa_{132} + \kappa_{321}\right) = \frac{1}{2}\left(-h + \mu + \mu - \nu\right) = -\frac{1}{2}\left(h - 2\mu + \nu\right) \\
\mathcal{B}_{523} &= \frac{1}{2}\left(H_{223} + \kappa_{223} - \kappa_{232} + \kappa_{322}\right) = -\kappa_{232} = -\rho \\
\mathcal{B}_{612} &= \frac{1}{2}\left(H_{312} + \kappa_{312} - \kappa_{123} + \kappa_{231}\right) = \frac{1}{2}\left(h + \mu + \mu + \nu\right) = \frac{1}{2}\left(h + 2\mu + \nu\right) \\
\mathcal{B}_{613} &= \frac{1}{2}\left(H_{313} + \kappa_{313} - \kappa_{133} + \kappa_{331}\right) = \kappa_{313} = -\mu \\
\mathcal{B}_{623} &= \frac{1}{2}\left(H_{323} + \kappa_{323} - \kappa_{233} + \kappa_{332}\right) = -\kappa_{233} = \rho.
\end{aligned}
$$

By equation (35) the components of the generalized Ricci curvature are

$$
\begin{aligned}
R_{41} &= \mathcal{B}_{245}\mathcal{B}_{512}\varepsilon_2\varepsilon_2 + \mathcal{B}_{345}\mathcal{B}_{513}\varepsilon_2\varepsilon_3 + \mathcal{B}_{246}\mathcal{B}_{612}\varepsilon_3\varepsilon_2 + \mathcal{B}_{346}\mathcal{B}_{613}\varepsilon_3\varepsilon_3 \\
&= \mathcal{B}_{245}\mathcal{B}_{512} - \mathcal{B}_{345}\mathcal{B}_{513} - \mathcal{B}_{246}\mathcal{B}_{612} + \mathcal{B}_{346}\mathcal{B}_{613} \\
&= -\mu^2 + \frac{1}{4}\left(h - 2\mu - \nu\right)\left(h - 2\mu + \nu\right) + \frac{1}{4}\left(h + 2\mu - \nu\right)\left(h + 2\mu - \nu\right) - \mu^2 \\
&= -2\mu^2 + \frac{1}{4}\Big(\left(h - 2\mu\right)^2 - \nu^2 + \left(h + 2\mu\right)^2 - \nu^2\Big) \\
&= -2\mu^2 + \frac{1}{2}h^2 + \frac{1}{2}\left(2\mu\right)^2 - \frac{1}{2}\nu^2 \\
&= \frac{1}{2}h^2 - \frac{1}{2}\nu^2 \\
R_{42} &= \mathcal{B}_{145}\mathcal{B}_{521}\varepsilon_2\varepsilon_1 + \mathcal{B}_{345}\mathcal{B}_{523}\varepsilon_2\varepsilon_3 + \mathcal{B}_{146}\mathcal{B}_{621}\varepsilon_3\varepsilon_1 + \mathcal{B}_{346}\mathcal{B}_{623}\varepsilon_3\varepsilon_3
\end{aligned}
$$

$$
\begin{aligned}
&= -\mathcal{B}_{145}\mathcal{B}_{512} - \mathcal{B}_{345}\mathcal{B}_{523} + \mathcal{B}_{146}\mathcal{B}_{612} + \mathcal{B}_{346}\mathcal{B}_{623} \\
&= \lambda\mu + \frac{1}{2}\rho\left(h - 2\mu - \nu\right) - \frac{1}{2}\lambda\left(h + 2\mu + \nu\right) + \rho\mu \\
&= \frac{1}{2}\rho\left(h - \nu\right) - \frac{1}{2}\lambda\left(h + \nu\right) \\
R_{43} &= \mathcal{B}_{145}\mathcal{B}_{531}\varepsilon_2\varepsilon_1 + \mathcal{B}_{245}\mathcal{B}_{532}\varepsilon_2\varepsilon_2 + \mathcal{B}_{146}\mathcal{B}_{631}\varepsilon_3\varepsilon_1 + \mathcal{B}_{246}\mathcal{B}_{632}\varepsilon_3\varepsilon_2 \\
&= -\mathcal{B}_{145}\mathcal{B}_{513} - \mathcal{B}_{245}\mathcal{B}_{523} + \mathcal{B}_{146}\mathcal{B}_{613} + \mathcal{B}_{246}\mathcal{B}_{623} \\
&= \frac{1}{2}\lambda\left(h - 2\mu + \nu\right) + \mu\rho + \lambda\mu - \frac{1}{2}\rho\left(h + 2\mu - \nu\right) \\
&= \frac{1}{2}\lambda\left(h + \nu\right) - \frac{1}{2}\rho\left(h - \nu\right) \\
R_{51} &= \mathcal{B}_{254}\mathcal{B}_{412}\varepsilon_1\varepsilon_2 + \mathcal{B}_{354}\mathcal{B}_{413}\varepsilon_1\varepsilon_3 + \mathcal{B}_{256}\mathcal{B}_{612}\varepsilon_3\varepsilon_2 + \mathcal{B}_{356}\mathcal{B}_{613}\varepsilon_3\varepsilon_3 \\
&= -\mathcal{B}_{245}\mathcal{B}_{412} + \mathcal{B}_{345}\mathcal{B}_{413} - \mathcal{B}_{256}\mathcal{B}_{612} + \mathcal{B}_{356}\mathcal{B}_{613} \\
&= \mu\lambda + \frac{1}{2}\lambda\left(h - 2\mu - \nu\right) - \frac{1}{2}\rho\left(h + 2\mu + \nu\right) + \rho\mu \\
&= \frac{1}{2}\left(h - \nu\right) - \frac{1}{2}\left(h + \nu\right) \\
R_{52} &= \mathcal{B}_{154}\mathcal{B}_{421}\varepsilon_1\varepsilon_1 + \mathcal{B}_{354}\mathcal{B}_{423}\varepsilon_1\varepsilon_3 + \mathcal{B}_{156}\mathcal{B}_{621}\varepsilon_3\varepsilon_1 + \mathcal{B}_{356}\mathcal{B}_{623}\varepsilon_3\varepsilon_3 \\
&= \mathcal{B}_{145}\mathcal{B}_{412} + \mathcal{B}_{345}\mathcal{B}_{423} + \mathcal{B}_{156}\mathcal{B}_{612} + \mathcal{B}_{356}\mathcal{B}_{623} \\
&= -\lambda^2 + \frac{1}{4}\left(h - 2\mu - \nu\right)\left(h - \nu\right) + \frac{1}{4}\left(h + \nu\right)\left(h + 2\mu + \nu\right) - \rho^2 \\
&= -\lambda^2 + \frac{1}{4}\left(h - \nu\right)^2 - \frac{1}{2}\mu\left(h - \nu\right) + \frac{1}{4}\left(h + \nu\right)^2 + \frac{1}{2}\mu\left(h + \nu\right) - \rho^2 \\
&= -\lambda^2 + \frac{1}{2}h^2 + \frac{1}{2}\nu^2 + \mu\nu - \rho^2 \\
R_{53} &= \mathcal{B}_{154}\mathcal{B}_{431}\varepsilon_1\varepsilon_1 + \mathcal{B}_{254}\mathcal{B}_{432}\varepsilon_1\varepsilon_2 + \mathcal{B}_{156}\mathcal{B}_{631}\varepsilon_3\varepsilon_1 + \mathcal{B}_{256}\mathcal{B}_{632}\varepsilon_3\varepsilon_2 \\
&= \mathcal{B}_{145}\mathcal{B}_{413} + \mathcal{B}_{245}\mathcal{B}_{423} + \mathcal{B}_{156}\mathcal{B}_{613} + \mathcal{B}_{256}\mathcal{B}_{623} \\
&= \lambda^2 + \frac{1}{2}\mu\left(h - \nu\right) - \frac{1}{2}\mu\left(h + \nu\right) + \rho^2 \\
&= \lambda^2 - \mu\nu + \rho^2 \\
R_{61} &= \mathcal{B}_{264}\mathcal{B}_{412}\varepsilon_1\varepsilon_2 + \mathcal{B}_{364}\mathcal{B}_{413}\varepsilon_1\varepsilon_3 + \mathcal{B}_{265}\mathcal{B}_{512}\varepsilon_2\varepsilon_2 + \mathcal{B}_{365}\mathcal{B}_{513}\varepsilon_2\varepsilon_3 \\
&= -\mathcal{B}_{246}\mathcal{B}_{412} + \mathcal{B}_{346}\mathcal{B}_{413} - \mathcal{B}_{256}\mathcal{B}_{512} + \mathcal{B}_{356}\mathcal{B}_{513} \\
&= -\frac{1}{2}\lambda\left(h + 2\mu - \nu\right) + \mu\lambda + \mu\rho + \frac{1}{2}\rho\left(h - 2\mu + \nu\right) \\
&= -\frac{1}{2}\lambda\left(h - \nu\right) + \frac{1}{2}\rho\left(h + \nu\right) \\
R_{62} &= \mathcal{B}_{164}\mathcal{B}_{421}\varepsilon_1\varepsilon_1 + \mathcal{B}_{364}\mathcal{B}_{423}\varepsilon_1\varepsilon_3 + \mathcal{B}_{165}\mathcal{B}_{521}\varepsilon_2\varepsilon_1 + \mathcal{B}_{365}\mathcal{B}_{523}\varepsilon_2\varepsilon_3 \\
&= \mathcal{B}_{146}\mathcal{B}_{412} + \mathcal{B}_{346}\mathcal{B}_{423} + \mathcal{B}_{156}\mathcal{B}_{512} + \mathcal{B}_{356}\mathcal{B}_{523} \\
&= \lambda^2 + \frac{1}{2}\mu\left(h - \nu\right) - \frac{1}{2}\mu\left(h + \nu\right) + \rho^2 \\
&= \lambda^2 - \mu\nu + \rho^2
\end{aligned}
$$

$$\begin{aligned}
R_{63} &= \mathcal{B}_{164}\mathcal{B}_{431}\varepsilon_1\varepsilon_1 + \mathcal{B}_{264}\mathcal{B}_{432}\varepsilon_1\varepsilon_2 + \mathcal{B}_{165}\mathcal{B}_{531}\varepsilon_2\varepsilon_1 + \mathcal{B}_{265}\mathcal{B}_{532}\varepsilon_2\varepsilon_2 \\
&= \mathcal{B}_{146}\mathcal{B}_{413} + \mathcal{B}_{246}\mathcal{B}_{423} + \mathcal{B}_{156}\mathcal{B}_{513} + \mathcal{B}_{256}\mathcal{B}_{523} \\
&= -\lambda^2 - \frac{1}{4}(h + 2\mu - \nu)(h - \nu) - \frac{1}{4}(h + \nu)(h - 2\mu + \nu) - \rho^2 \\
&= -\lambda^2 - \frac{1}{4}(h - \nu)^2 - \frac{1}{2}\mu(h - \nu) - \frac{1}{4}(h + \nu)^2 + \frac{1}{2}\mu(h + \nu) - \rho^2 \\
&= -\lambda^2 - \frac{1}{2}h^2 - \frac{1}{2}\nu^2 + \mu\nu - \rho^2.
\end{aligned}$$

If we impose the Einstein condition, we see that $0 = R_{52} - R_{63} = h^2 + \nu^2$, hence $h = \nu = 0$. Therefore the equation $R_{63} = 0$ reads as $0 = -\lambda^2 - \rho^2$. This implies $\lambda = \rho = 0$, which is a contradiction to $\lambda + \rho \neq 0$. □

We summarize this by the following theorem.

Theorem 6.8. Let $(H, \mathcal{G}_g, \delta = 0)$ be a divergence-free generalized Einstein structure on an oriented three-dimensional non-unimodular Lie group G. Then $H = 0$ and g is indefinite. Furthermore there exists an orthonormal basis (v_a) of $(\mathfrak{g}, g)$ such that $v_1, v_3 \in \mathfrak{u}$ and $g(v_1, v_1) = g(v_2, v_2) = -g(v_3, v_3)$ as well as a positive constant $\theta > 0$ such that

$$\begin{aligned}
[v_1, v_3] &= 0 \\
[v_2, v_1] &= \theta v_1 - \theta v_3 \\
[v_2, v_3] &= \theta v_1 + \theta v_3.
\end{aligned}$$

The metric g is a Ricci-soliton which is not of constant curvature.

Proof. It remains to prove the last statement. The fact that g is a Ricci soliton is a direct consequence of Corollary 5.33. To see that the metric is non-flat, it suffices to check that $\nabla\tau \neq 0$. Since $\tau = 2\theta v_2^*$, where (v_a^*) denotes the dual basis, it suffices to compute ∇v_2:

$$g(\nabla_{v_1} v_2, v_1) = g([v_1, v_2], v_1) = -\theta\epsilon_1 \neq 0.$$

Similarly, $\nabla_{v_2} v_2 = 0$ shows that g is neither of non-zero constant curvature. □

Corollary 6.9. If the metric is definite there are no solutions to the Ricci soliton equation (31) in the non-unimodular case.

Remark 6.10. Note that in all our proofs in the unimodular and in the non-unimodular case we only used that the diagonal components $R_{ii'}$ are zero. In particular, the Ricci tensor is zero, if $R_{ii'} = 0$ for all $i \in \{4, 5, 6\}$, in the divergence free case.

6.3 Arbitrary divergence

Recall that $R^\delta_{ia} = Ric^+_\delta(e_i, e_a)$ and $R^\delta_{ai} = Ric^-_\delta(e_a, e_i)$ denote the components of the Ricci curvature tensors $Ric^\pm_\delta$ of a generalized pseudo-Riemannian Lie group $(G, H, \mathcal{G}_g, \delta)$ with arbitrary divergence $\delta \in E^*$. If $\delta = 0$, we often write $R_{ia} = R^0_{ia}$ and $R_{ai} = R^0_{ai}$. By Theorem 5.28 we have

$$
\begin{aligned}
R^\delta_{ia} &= R_{ia} + \sum_c B^c_{ia}\delta_c = R_{ia} + \sum_c \varepsilon_c B_{iac}\delta_c \\
R^\delta_{ai} &= R_{ai} + \sum_j B^j_{ai}\delta_j = R_{ia} - \sum_j \varepsilon_{j'} B_{aij}\delta_j.
\end{aligned}
$$

6.3.1 Unimodular Lie groups

Proposition 6.11. If $(H, \mathcal{G}_g, \delta)$ is a generalized Einstein structure on an oriented three-dimensional unimodular Lie group G, then there exists a g-orthonormal basis (v_a) of $\mathfrak{g}$ such that $g(v_1, v_1) = g(v_2, v_2)$ and such that the symmetric endomorphism L defined in equation (32) takes one of the following forms

1. $L_1(\alpha, \beta, \gamma)$, that is L is diagonalizable by an orthonormal basis;
2. $L_3(\alpha, 0)$ or $L_4(\alpha, 0)$, in both cases $-\varepsilon_1 \frac{1}{2}\delta_1 = -\varepsilon_1\frac{1}{2}\delta_4 = \alpha$ as well as $\delta_2 = \delta_3$ and $\delta_5 = \delta_6$. If $\alpha \neq 0$, then $\delta_2 = \delta_3 = \delta_5 = \delta_6 = 0$;
3. $L_5(0)$ with $\delta_2 = \delta_5 = 0$ and $\varepsilon_1\delta_1 = -\varepsilon_3\delta_3 = \varepsilon_1\delta_4 = -\varepsilon_5\delta_6 = -\sqrt{2}$, where $\delta_a = \delta(v_a + g(v_a))$ and $\delta_i = \delta(v_{i'} - g(v_{i'}))$.

Furthermore, in the non-diagonalizable case the three-form H is always zero (see Proposition 6.2 for the notation of the normal forms of L).

Proof. Since in the Euclidean case any symmetric endomorphism is always diagonalizable by an orthonormal basis, we may assume that the scalar product is indefinite. By Proposition 6.2, there is an orthonormal basis (v_a), such that the endomorphism L takes one of the normal forms $L_1(\alpha,\beta,\gamma), L_2(\alpha,\beta,\gamma), L_3(\alpha,\beta), L_4(\alpha,\beta)$ or $L_5(\alpha)$ from said Proposition. As in the proof of Proposition 6.3, we can treat all these cases at once by considering the matrix

$$
\begin{pmatrix} \alpha & \lambda & 0 \\ \lambda & \beta & \mu \\ 0 & -\mu & \gamma \end{pmatrix}.
$$

Recall that we assume $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3$, where $\varepsilon_a = g(v_a, v_a)$. Using the Dorfman coefficients and the coefficients of the Ricci curvature with divergence zero from the proof of Proposition 6.3, we can compute the components of the Ricci curvature with divergence δ as

$$
\begin{aligned}
R^\delta_{41} &= R_{41} + \varepsilon_2\mathcal{B}_{412}\delta_2 + \varepsilon_3\mathcal{B}_{413}\delta_3 \\
&= -2\mu^2 - \frac{1}{2}\alpha^2 + \frac{1}{2}h^2 + \frac{1}{2}(\beta-\gamma)^2 + \varepsilon_3\lambda\delta_3 \\
R^\delta_{42} &= R_{42} + \varepsilon_1\mathcal{B}_{421}\delta_1 + \varepsilon_3\mathcal{B}_{423}\delta_3 \\
&= -\lambda(\beta - \gamma + \alpha) + \varepsilon_3\frac{1}{2}(h + \gamma - \alpha + \beta)\delta_3
\end{aligned}
$$

$$
\begin{aligned}
R^{\delta}_{43} &= R_{43} + \varepsilon_1 \mathcal{B}_{431}\delta_1 + \varepsilon_2 \mathcal{B}_{432}\delta_2 \\
&= -2\mu\lambda - \varepsilon_1 \lambda\delta_1 - \varepsilon_2 \frac{1}{2}\left(h + \gamma - \alpha + \beta\right)\delta_2 \\
R^{\delta}_{51} &= R_{51} + \varepsilon_2 \mathcal{B}_{512}\delta_2 + \varepsilon_3 \mathcal{B}_{513}\delta_3 \\
&= -\lambda\left(\beta - \gamma + \alpha\right) + \varepsilon_2 \mu\delta_2 + \varepsilon_3 \frac{1}{2}\left(-h - \gamma + \beta - \alpha\right)\delta_3 \\
R^{\delta}_{52} &= R_{52} + \varepsilon_1 \mathcal{B}_{521}\delta_1 + \varepsilon_3 \mathcal{B}_{523}\delta_3 \\
&= -\frac{1}{2}\beta^2 + \frac{1}{2}h^2 + \frac{1}{2}\left(\gamma - \alpha\right)^2 - \varepsilon_1 \mu\delta_1 - \varepsilon_3 \lambda\delta_3 \\
R^{\delta}_{53} &= R_{53} + \varepsilon_1 \mathcal{B}_{531}\delta_1 + \varepsilon_2 \mathcal{B}_{532}\delta_2 \\
&= -\mu\left(\gamma - \alpha + \beta\right) - \varepsilon_1 \frac{1}{2}\left(-h - \gamma + \beta - \alpha\right)\delta_1 + \varepsilon_2 \lambda\delta_2 \\
R^{\delta}_{61} &= R_{61} + \varepsilon_2 \mathcal{B}_{612}\delta_2 + \varepsilon_3 \mathcal{B}_{613}\delta_3 \\
&= -2\lambda\mu + \varepsilon_2 \frac{1}{2}\left(h + \beta - \gamma + \alpha\right)\delta_2 + \varepsilon_3 \mu\delta_3 \\
R^{\delta}_{62} &= R_{62} + \varepsilon_1 \mathcal{B}_{621}\delta_1 + \varepsilon_3 \mathcal{B}_{623}\delta_3 \\
&= -\mu\left(\gamma - \alpha + \beta\right) - \varepsilon_1 \frac{1}{2}\left(h + \beta - \gamma + \alpha\right)\delta_1 \\
R^{\delta}_{63} &= R_{63} + \varepsilon_1 \mathcal{B}_{631}\delta_1 + \varepsilon_2 \mathcal{B}_{632}\delta_2 \\
&= -2\lambda^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}h^2 - \frac{1}{2}\left(\beta - \alpha\right)^2 - \varepsilon_1 \mu\delta_1 \\
R^{\delta}_{14} &= R_{41} - \varepsilon_2 \mathcal{B}_{145}\delta_5 - \varepsilon_3 \mathcal{B}_{146}\delta_6 \\
&= -2\mu^2 - \frac{1}{2}\alpha^2 + \frac{1}{2}h^2 + \frac{1}{2}(\beta - \gamma)^2 + \varepsilon_3 \lambda\delta_6 \\
R^{\delta}_{24} &= R_{42} - \varepsilon_2 \mathcal{B}_{245}\delta_5 - \varepsilon_3 \mathcal{B}_{246}\delta_6 \\
&= -\lambda\left(\beta - \gamma + \alpha\right) + \varepsilon_2 \mu\delta_5 - \varepsilon_3 \frac{1}{2}\left(-h + \gamma - \beta + \alpha\right)\delta_6 \\
R^{\delta}_{34} &= R_{43} - \varepsilon_2 \mathcal{B}_{345}\delta_5 - \varepsilon_3 \mathcal{B}_{346}\delta_6 \\
&= -2\mu\lambda - \varepsilon_2 \frac{1}{2}\left(h - \beta + \gamma - \alpha\right)\delta_5 + \varepsilon_3 \mu\delta_6 \\
R^{\delta}_{15} &= R_{51} - \varepsilon_1 \mathcal{B}_{154}\delta_4 - \varepsilon_3 \mathcal{B}_{156}\delta_6 \\
&= -\lambda\left(\beta - \gamma + \alpha\right) - \varepsilon_3 \frac{1}{2}\left(h - \gamma + \alpha - \beta\right)\delta_6 \\
R^{\delta}_{25} &= R_{52} - \varepsilon_1 \mathcal{B}_{254}\delta_4 - \varepsilon_3 \mathcal{B}_{256}\delta_6 \\
&= -\frac{1}{2}\beta^2 + \frac{1}{2}h^2 + \frac{1}{2}\left(\gamma - \alpha\right)^2 - \varepsilon_1 \mu\delta_4 - \varepsilon_3 \lambda\delta_6 \\
R^{\delta}_{35} &= R_{53} - \varepsilon_1 \mathcal{B}_{354}\delta_4 - \varepsilon_3 \mathcal{B}_{356}\delta_6 \\
&= -\mu\left(\gamma - \alpha + \beta\right) + \varepsilon_1 \frac{1}{2}\left(h - \beta + \gamma - \alpha\right)\delta_4 \\
R^{\delta}_{16} &= R_{61} - \varepsilon_1 \mathcal{B}_{164}\delta_4 - \varepsilon_2 \mathcal{B}_{165}\delta_5
\end{aligned}
$$

$$
\begin{aligned}
&= -2\lambda\mu - \varepsilon_1\lambda\delta_4 + \varepsilon_2\frac{1}{2}(h-\gamma+\alpha-\beta)\,\delta_5 \\
R^\delta_{26} &= R_{62} - \varepsilon_1\mathcal{B}_{264}\delta_4 - \varepsilon_2\mathcal{B}_{265}\delta_5 \\
&= -\mu(\gamma-\alpha+\beta) + \varepsilon_1\frac{1}{2}(-h+\gamma-\beta+\alpha)\,\delta_4 + \varepsilon_2\lambda\delta_5 \\
R^\delta_{36} &= R_{63} - \varepsilon_1\mathcal{B}_{364}\delta_4 - \varepsilon_2\mathcal{B}_{365}\delta_5 \\
&= -2\lambda^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}h^2 - \frac{1}{2}(\beta-\alpha)^2 - \varepsilon_1\mu\delta_4.
\end{aligned}
$$

For the normal form $L_2(\alpha,\beta,\gamma)$ the equations for Ric^+_δ read

$$
\begin{aligned}
R^\delta_{41} &= -2\beta^2 - \frac{1}{2}\gamma^2 + \frac{1}{2}h^2 \\
R^\delta_{42} &= \varepsilon_3\frac{1}{2}(h+2\alpha-\gamma)\,\delta_3 \\
R^\delta_{43} &= -\varepsilon_2\frac{1}{2}(h+2\alpha-\gamma)\,\delta_2 \\
R^\delta_{51} &= -\varepsilon_2\beta\delta_2 + \varepsilon_3\frac{1}{2}(-h-\gamma)\,\delta_3 \\
R^\delta_{52} &= -\frac{1}{2}\alpha^2 + \frac{1}{2}h^2 + \frac{1}{2}(\alpha-\gamma)^2 + \varepsilon_1\beta\delta_1 \\
R^\delta_{53} &= \beta(2\alpha-\gamma) - \varepsilon_1\frac{1}{2}(-h-\gamma)\,\delta_1 \\
R^\delta_{61} &= \varepsilon_2\frac{1}{2}(h+\gamma)\,\delta_2 - \varepsilon_3\beta\delta_3 \\
R^\delta_{62} &= \beta(2\alpha-\gamma) - \varepsilon_1\frac{1}{2}(h+\gamma)\,\delta_1 \\
R^\delta_{63} &= \frac{1}{2}\alpha^2 - \frac{1}{2}h^2 - \frac{1}{2}(\alpha-\gamma)^2 + \varepsilon_1\beta\delta_1.
\end{aligned}
$$

Imposing now the Einstein condition, we get $0 = R^\delta_{53} + R^\delta_{62} = 2\beta(2\alpha-\gamma)$. So either L is diagonalizable, if $\beta = 0$, or $2\alpha = \gamma$. But then the equation $0 = R^\delta_{52} - R^\delta_{63}$ is

$$
0 = -\alpha^2 + h^2 + (\alpha-\gamma)^2 = h^2,
$$

and hence $h = 0$. Applying this to the equation for R^δ_{41} yields $0 = -2\beta^2 - \frac{1}{2}\gamma^2$. Therefore $\beta = 0$ and the endomorphism L is diagonalizable by an orthonormal basis.

If L takes the normal form $L_3(\alpha,\beta)$, the components of the Ricci tensor are

$$
\begin{aligned}
R^\delta_{41} &= -\frac{1}{2}\beta^2 + \frac{1}{2}h^2 \\
R^\delta_{42} &= \varepsilon_3\frac{1}{2}(h+2\alpha-\beta)\,\delta_3 \\
R^\delta_{43} &= -\varepsilon_2\frac{1}{2}(h+2\alpha-\beta)\,\delta_2 \\
R^\delta_{51} &= \varepsilon_2\frac{1}{2}\delta_2 + \varepsilon_3\frac{1}{2}(-h+1-\beta)\,\delta_3
\end{aligned}
$$

$$
\begin{aligned}
R^{\delta}_{52} &= -\frac{1}{2}\left(\frac{1}{2}+\alpha\right)^2 + \frac{1}{2}h^2 + \frac{1}{2}\left(-\frac{1}{2}+\alpha-\beta\right)^2 - \varepsilon_1\frac{1}{2}\delta_1 \\
R^{\delta}_{53} &= -\frac{1}{2}\left(2\alpha-\beta\right) - \varepsilon_1\frac{1}{2}\left(-h+1-\beta\right)\delta_1 \\
R^{\delta}_{61} &= \varepsilon_2\frac{1}{2}\left(h+1+\beta\right)\delta_2 + \varepsilon_3\frac{1}{2}\delta_3 \\
R^{\delta}_{62} &= -\frac{1}{2}\left(2\alpha-\beta\right) - \varepsilon_1\frac{1}{2}\left(h+1+\beta\right)\delta_1 \\
R^{\delta}_{63} &= \frac{1}{2}\left(-\frac{1}{2}+\alpha\right)^2 - \frac{1}{2}h^2 - \frac{1}{2}\left(\frac{1}{2}+\alpha-\beta\right)^2 - \varepsilon_1\frac{1}{2}\delta_1 \\
R^{\delta}_{14} &= -\frac{1}{2}\beta^2 + \frac{1}{2}h^2 \\
R^{\delta}_{24} &= \varepsilon_2\frac{1}{2}\delta_5 - \varepsilon_3\frac{1}{2}\left(-h-1+\beta\right)\delta_6 \\
R^{\delta}_{34} &= -\varepsilon_2\frac{1}{2}(h-1-\beta)\delta_5 + \varepsilon_3\frac{1}{2}\delta_6 \\
R^{\delta}_{15} &= -\varepsilon_3\frac{1}{2}\left(h-2\alpha+\beta\right)\delta_6 \\
R^{\delta}_{25} &= -\frac{1}{2}\left(\frac{1}{2}+\alpha^2\right)^2 + \frac{1}{2}h^2 + \frac{1}{2}\left(-\frac{1}{2}+\alpha-\beta\right)^2 - \varepsilon_1\frac{1}{2}\delta_4 \\
R^{\delta}_{35} &= -\frac{1}{2}\left(2\alpha-\beta\right) + \varepsilon_1\frac{1}{2}\left(h-1-\beta\right)\delta_4 \\
R^{\delta}_{16} &= \varepsilon_2\frac{1}{2}\left(h-2\alpha+\beta\right)\delta_5 \\
R^{\delta}_{26} &= -\frac{1}{2}\left(2\alpha-\beta\right) + \varepsilon_1\frac{1}{2}\left(-h-1+\beta\right)\delta_4 \\
R^{\delta}_{36} &= \frac{1}{2}\left(-\frac{1}{2}+\alpha\right)^2 - \frac{1}{2}h^2 - \frac{1}{2}\left(\frac{1}{2}+\alpha-\beta\right)^2 - \varepsilon_1\frac{1}{2}\delta_4.
\end{aligned}
$$

First, equation $R^{\delta}_{63} - R^{\delta}_{36} = 0$ yields $\delta_1 = \delta_4$. Furthermore, due to $0 = R^{\delta}_{53} - R^{\delta}_{62} = \varepsilon_1\left(h+\beta\right)\delta_1$ and $0 = R^{\delta}_{35} - R^{\delta}_{26} = \varepsilon_1\left(h-\beta\right)\delta_4 = \varepsilon_1\left(h-\beta\right)\delta_1$, we have $\varepsilon_1\beta\delta_1 = 0$. If $\delta_1 = 0$, then we see from $0 = R^{\delta}_{53} = -\frac{1}{2}\left(2\alpha-\beta\right)$ that $2\alpha = \beta$. Then $0 = R^{\delta}_{63} = \frac{1}{2}\left(-\frac{1}{2}+\alpha\right)^2 - \frac{1}{2}h^2 - \frac{1}{2}\left(\frac{1}{2}-\alpha\right)^2 = -\frac{1}{2}h^2$ and $h = 0$. Equation $R^{\delta}_{41} = 0$ shows $\beta = 0$ and therefore $\alpha = 0$. Furthermore, $R^{\delta}_{51} = 0$ shows $\delta_2 = \delta_3$ and $R^{\delta}_{15} = 0$ shows $\delta_5 = \delta_6$, because $\varepsilon_2 = -\varepsilon_3$. If otherwise $\beta = 0$, we see again from $R^{\delta}_{41} = 0$ that $h = 0$ and also $\delta_2 = \delta_3$ and $\delta_5 = \delta_6$, because of $R^{\delta}_{51} = 0$ and $R^{\delta}_{15} = 0$, respectively. Finally, $0 = \alpha\delta_2 = \alpha\delta_3 = \alpha\delta_2 = \alpha\delta_3$ due to $0 = R^{\delta}_{42} = R^{\delta}_{43} = R^{\delta}_{15} = R^{\delta}_{16}$, as well as $\alpha = -\varepsilon_1\frac{1}{2}\delta_1 = -\varepsilon_1\frac{1}{2}\delta_4$ due to $R^{\delta}_{62} = R^{\delta}_{26} = 0$.

In a similar way we obtain the same equations for the normal form $L_4(\alpha,\beta)$.

Finally the equations for Ric^{+}_{δ} for the normal form $L_5(\alpha)$ are

$$
R^{\delta}_{41} = -1 - \frac{1}{2}\alpha^2 + \frac{1}{2}h^2 + \varepsilon_3\frac{1}{\sqrt{2}}\delta_3
$$

$$
\begin{aligned}
R^\delta_{42} &= -\frac{1}{\sqrt{2}}\alpha + \varepsilon_3 \frac{1}{2}\left(h+\alpha\right)\delta_3 \\
R^\delta_{43} &= -1 - \varepsilon_1 \frac{1}{\sqrt{2}}\delta_1 - \varepsilon_2 \frac{1}{2}\left(h+\alpha\right)\delta_2 \\
R^\delta_{51} &= -\frac{1}{\sqrt{2}}\alpha + \varepsilon_2 \frac{1}{\sqrt{2}}\delta_2 + \varepsilon_3 \frac{1}{2}\left(-h-\alpha\right)\delta_3 \\
R^\delta_{52} &= -\frac{1}{2}\alpha^2 + \frac{1}{2}h^2 - \varepsilon_1 \frac{1}{\sqrt{2}}\delta_1 - \varepsilon_3 \frac{1}{\sqrt{2}}\delta_3 \\
R^\delta_{53} &= -\frac{1}{\sqrt{2}}\alpha - \varepsilon_1 \frac{1}{2}\left(-h-\alpha\right)\delta_1 + \varepsilon_2 \frac{1}{\sqrt{2}}\delta_2 \\
R^\delta_{61} &= -1 + \varepsilon_2 \frac{1}{2}\left(h+\alpha\right)\delta_2 + \varepsilon_3 \frac{1}{\sqrt{2}}\delta_3 \\
R^\delta_{62} &= -\frac{1}{\sqrt{2}}\alpha - \varepsilon_1 \frac{1}{2}\left(h+\alpha\right)\delta_1 \\
R^\delta_{63} &= -1 + \frac{1}{2}\alpha^2 - \frac{1}{2}h^2 - \varepsilon_1 \frac{1}{\sqrt{2}}\delta_1.
\end{aligned}
$$

From $0 = R^\delta_{41} - R^\delta_{52} - R^\delta_{63} = -\frac{1}{2}\left(\alpha^2 - h^2\right)$ we see $\alpha^2 = h^2$. Therefore $\varepsilon_3\delta_3 = \sqrt{2}$ by $0 = R^\delta_{41} = -1 + \varepsilon_3 \frac{1}{\sqrt{2}}\delta_3$ as well as $\varepsilon_1\delta_1 = -\sqrt{2}$ by $0 = R^\delta_{63} = -1 - \varepsilon_1 \frac{1}{\sqrt{2}}\delta_1$. If now $\alpha = -h$, then $0 = R^\delta_{42} = -\frac{1}{\sqrt{2}}\alpha$ and $\alpha = h = 0$. By $0 = R^\delta_{53} = \varepsilon_2 \frac{1}{\sqrt{2}}\delta_2$ also $\delta_2 = 0$. If otherwise $\alpha = h$, we have $0 = R^\delta_{42} + R^\delta_{51} = -\sqrt{2}\alpha + \varepsilon_2 \frac{1}{\sqrt{2}}\delta_2$ and thus $2\alpha = \varepsilon_2\delta_2$. But at the same time $0 = R^\delta_{43} = -\varepsilon_2\alpha\delta_2$. This is only possible, if $\alpha = \delta_2 = 0$. The equations for Ric^-_δ are now

$$
\begin{aligned}
R^\delta_{14} &= -1 + \varepsilon_3 \frac{1}{\sqrt{2}}\delta_6 \\
R^\delta_{24} &= \varepsilon_2 \frac{1}{\sqrt{2}}\delta_5 \\
R^\delta_{34} &= -1 + \varepsilon_3 \frac{1}{\sqrt{2}}\delta_6 \\
R^\delta_{15} &= 0 \\
R^\delta_{25} &= -\varepsilon_1 \frac{1}{\sqrt{2}}\delta_4 - \varepsilon_3 \frac{1}{\sqrt{2}}\delta_6 \\
R^\delta_{35} &= 0 \\
R^\delta_{16} &= -1 - \varepsilon_1 \frac{1}{\sqrt{2}}\delta_4 \\
R^\delta_{26} &= \varepsilon_2 \frac{1}{\sqrt{2}}\delta_5 \\
R^\delta_{36} &= -1 - \varepsilon_1 \frac{1}{\sqrt{2}}\delta_4.
\end{aligned}
$$

This finally yields $\delta_5 = 0$ and $\varepsilon_1\delta_4 = -\varepsilon_3\delta_6 = -\sqrt{2}$. □

Theorem 6.12. Let $(H, \mathcal{G}_g, \delta)$ be a generalized Einstein structure on an oriented three-dimensional unimodular Lie group G. If the endomorphism $L \in \operatorname{End}\mathfrak{g}$ defined in (33) is diagonalizable, then there exists an oriented g-orthonormal basis (v_a) of $\mathfrak{g} = \operatorname{Lie} G$ and $\alpha_1, \alpha_2, \alpha_3, h \in \mathbb{R}$ such that

$$[v_a, v_b] = \alpha_c \varepsilon_c v_c, \quad \forall \quad \text{cyclic} \quad (a, b, c) \in \mathfrak{S}_3, \quad H = h\mathrm{vol}_g,$$

where $\varepsilon_a = g(v_a, v_a)$ satisfies $\varepsilon_1 = \varepsilon_2$. The constants $(\alpha_1, \alpha_2, \alpha_3, h)$ can take the following values.

1. $\alpha_1 = \alpha_2 = \alpha_3 = h = 0$, in which case $\mathfrak{g}$ is abelian. The divergence can take an arbitrary value in E^*.

2. $\alpha_1 = \alpha_2 = \alpha_3 = \pm h \neq 0$, and $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(2,1)$ or $\mathfrak{so}(3)$. The case $\mathfrak{so}(3)$ occurs precisely when g is definite. Furthermore $\delta|_{E_\pm} = 0$.

3. There exists a cyclic permutation $\sigma \in \mathfrak{S}_3$ such that such that

$$\alpha_{\sigma(1)} = \alpha_{\sigma(2)} \neq 0 \quad \text{and} \quad h = \alpha_{\sigma(3)} = 0.$$

   In this case $[\mathfrak{g}, \mathfrak{g}]$ is abelian of dimension 2, that is $\mathfrak{g}$ is metabelian. More precisely, $\mathfrak{g}$ is isomorphic to $\mathfrak{e}(2)$ (g definite on $[\mathfrak{g}, \mathfrak{g}]$) or $\mathfrak{e}(1,1)$ (g indefinite on $[\mathfrak{g}, \mathfrak{g}]$). The components of the divergence δ satisfy $\delta_{\sigma(1)} = \delta_{\sigma(2)} = \delta_{\sigma(1)+3} = \delta_{\sigma(2)+3} = 0$

If L is not diagonalizable, then $h = 0$.

1. If L takes the normal form $L_3(0,0)$ or $L_4(0,0)$, then the Lie algebra $\mathfrak{g}$ is isomorphic to the Heisenberg algebra $\mathfrak{heis}$. In this case $\delta_1 = \delta_4 = 0$, $\delta_2 = \delta_3$ and $\delta_5 = \delta_6$.

2. If L takes the normal form $L_3(\alpha, 0)$ or $L_4(\alpha, 0)$, $\alpha \neq 0$, then $\mathfrak{g}$ is isomorphic to $\mathfrak{e}(1,1)$. In these cases $-\varepsilon_1 \frac{1}{2}\delta_1 = -\varepsilon_1 \frac{1}{2}\delta_4 = \alpha$ as well as $\delta_2 = \delta_3 = \delta_5 = \delta_6 = 0$.

3. If L takes the normal form $L_5(0)$, then $\mathfrak{g}$ is isomorphic to $\mathfrak{e}(1,1)$. In this case $\varepsilon_1\delta_1 = -\varepsilon_3\delta_3 = \varepsilon_1\delta_4 = -\varepsilon_3\delta_6 = -\sqrt{2}$ and $\delta_2 = \delta_5 = 0$.

Proof. Assume first L is diagonalizable. To compute the components of the Ricci curvature, we use the formulas for the Dorfman coefficients and the notation for variables X_a and Y_a from the proof of Theorem 6.4.

$$\begin{aligned}
R^\delta_{41} &= R_{41} + \varepsilon_2 \mathcal{B}_{412}\delta_2 + \varepsilon_3 \mathcal{B}_{413}\delta_3 \\
&= R_{41} \\
R^\delta_{42} &= R_{42} + \varepsilon_1 \mathcal{B}_{421}\delta_1 + \varepsilon_3 \mathcal{B}_{423}\delta_3 \\
&= \frac{1}{2}\varepsilon_3\delta_3 Y_1 \\
R^\delta_{43} &= R_{43} + \varepsilon_1 \mathcal{B}_{431}\delta_1 + \varepsilon_2 \mathcal{B}_{432}\delta_2 \\
&= -\frac{1}{2}\varepsilon_2\delta_2 Y_1
\end{aligned}$$

$$
\begin{aligned}
R^{\delta}_{51} &= R_{51} + \varepsilon_2 \mathcal{B}_{512}\delta_2 + \varepsilon_3 \mathcal{B}_{513}\delta_3 \\
&= -\frac{1}{2}\varepsilon_3\delta_3 Y_2 \\
R^{\delta}_{52} &= R_{52} + \varepsilon_1 \mathcal{B}_{521}\delta_1 + \varepsilon_3 \mathcal{B}_{523}\delta_3 \\
&= R_{52} \\
R^{\delta}_{53} &= R_{53} + \varepsilon_1 \mathcal{B}_{531}\delta_1 + \varepsilon_2 \mathcal{B}_{532}\delta_2 \\
&= \frac{1}{2}\varepsilon_1\delta_1 Y_2 \\
R^{\delta}_{61} &= R_{61} + \varepsilon_2 \mathcal{B}_{612}\delta_2 + \varepsilon_3 \mathcal{B}_{613}\delta_3 \\
&= \frac{1}{2}\varepsilon_2\delta_2 Y_3 \\
R^{\delta}_{62} &= R_{62} + \varepsilon_1 \mathcal{B}_{621}\delta_1 + \varepsilon_3 \mathcal{B}_{623}\delta_3 \\
&= -\frac{1}{2}\varepsilon_1\delta_1 Y_3 \\
R^{\delta}_{63} &= R_{63} + \varepsilon_1 \mathcal{B}_{631}\delta_1 + \varepsilon_2 \mathcal{B}_{632}\delta_2 \\
&= R_{63} \\
R^{\delta}_{14} &= R_{41} - \varepsilon_2 \mathcal{B}_{145}\delta_5 - \varepsilon_3 \mathcal{B}_{146}\delta_6 \\
&= R_{41} \\
R^{\delta}_{24} &= R_{42} - \varepsilon_2 \mathcal{B}_{245}\delta_5 - \varepsilon_3 \mathcal{B}_{246}\delta_6 \\
&= \frac{1}{2}\varepsilon_3\delta_6 X_2 \\
R^{\delta}_{34} &= R_{43} - \varepsilon_2 \mathcal{B}_{345}\delta_5 - \varepsilon_3 \mathcal{B}_{346}\delta_6 \\
&= -\frac{1}{2}\varepsilon_2\delta_5 X_3 \\
R^{\delta}_{15} &= R_{51} - \varepsilon_1 \mathcal{B}_{154}\delta_4 - \varepsilon_3 \mathcal{B}_{156}\delta_6 \\
&= -\frac{1}{2}\varepsilon_3\delta_6 X_1 \\
R^{\delta}_{25} &= R_{52} - \varepsilon_1 \mathcal{B}_{254}\delta_4 - \varepsilon_3 \mathcal{B}_{256}\delta_6 \\
&= R_{52} \\
R^{\delta}_{35} &= R_{53} - \varepsilon_1 \mathcal{B}_{354}\delta_4 - \varepsilon_3 \mathcal{B}_{356}\delta_6 \\
&= \frac{1}{2}\varepsilon_1\delta_4 X_3 \\
R^{\delta}_{16} &= R_{61} - \varepsilon_1 \mathcal{B}_{164}\delta_4 - \varepsilon_2 \mathcal{B}_{165}\delta_5 \\
&= \frac{1}{2}\varepsilon_2\delta_5 X_1 \\
R^{\delta}_{26} &= R_{62} - \varepsilon_1 \mathcal{B}_{264}\delta_4 - \varepsilon_2 \mathcal{B}_{265}\delta_5 \\
&= -\frac{1}{2}\varepsilon_1\delta_4 X_2 \\
R^{\delta}_{36} &= R_{63} - \varepsilon_1 \mathcal{B}_{364}\delta_4 - \varepsilon_2 \mathcal{B}_{365}\delta_5
\end{aligned}
$$

$$= R_{63}.$$

Note that if $(H, \mathcal{G}_g, \delta)$ is a generalized Einstein structure, also $(H, \mathcal{G}_g, 0)$ is. Therefore, as in the proof of Theorem 6.4, we can distinguish cases depending on how many components of the vector (X_1, X_2, X_3) are equal to zero.

Solutions of type 0: $X_1X_2X_3 \neq 0$ implies $\delta_4 = \delta_5 = \delta_6 = 0$. Furthermore recall that $Y_1 = Y_2 = Y_3 = 0$ and

$$\alpha_1 = \alpha_2 = \alpha_3 = -h \neq 0.$$

In this case the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(2,1)$ (g indefinite) or $\mathfrak{so}(3)$ (g definite).

We have seen that solutions of type 1 do not exist.

Solutions of type 2: assume for example that $X_1 \neq 0$, $X_2 = X_3 = 0$. This implies $\delta_5 = \delta_6 = 0$. Moreover we have seen that $Y_2 = Y_3 = 0$, $h = \alpha_1 = 0$ and $\alpha_2 = \alpha_3 \neq 0$. This shows $Y_1 \neq 0$ and thus $\delta_2 = \delta_3 = 0$. So the solutions of type 2 are of the following form. There exists a cyclic permutation $\sigma \in \mathfrak{S}_3$ such that

$$\alpha_{\sigma(1)} = \alpha_{\sigma(2)} \neq 0 \quad \text{and} \quad h = \alpha_{\sigma(3)} = \delta_{\sigma(1)} = \delta_{\sigma(2)} = \delta_{\sigma(1)+3} = \delta_{\sigma(2)+3} = 0.$$

As in the divergence-free case, we conclude that $\mathfrak{g}$ is metabelian. The commutator ideal $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{v_{\sigma(1)}, v_{\sigma(2)}\}$ is two-dimensional and $\mathrm{ad}_{v_{\sigma(3)}}$ acts on it by a non-zero g-skew-symmetric endomorphism. This implies that $\mathfrak{g}$ is isomorphic to $\mathfrak{e}(2)$ or $\mathfrak{e}(1,1)$.

Solutions of type 3: assume $X_1 = X_2 = X_3 = 0$. This implies

$$\alpha_1 = \alpha_2 = \alpha_3 = h.$$

If $h = 0$, then $Y_1 = Y_2 = Y_3 = 0$ and $\delta \in E^*$ arbitrary, and if $h \neq 0$, then $Y_1 = Y_2 = Y_3 = 2h \neq 0$ and therefore $\delta_1 = \delta_2 = \delta_3 = 0$.

By Proposition 6.11, if L is not diagonalizable, it is of the forms $L_3(\alpha, 0)$, $L_4(\alpha, 0)$ or $L_5(0)$ and the divergence has the claimed properties. From Theorem 6.4 we know that G is the Heisenberg group if L takes the normal for $L_3(0,0)$ or $L_4(0,0)$. If $\alpha \neq 0$, ad_{v_1} acts on $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{v_2, v_3\}$ by a symmetric endomorphism with eigenvalues α and $-\alpha$. Therefore $\mathfrak{g} \cong \mathfrak{e}(1,1)$.

If L takes the normal form $L_5(0)$, one can show that the only unimodular Lie algebra whose Killing form has the same signature as the one of $\mathfrak{g}$, is the Lie algebra $\mathfrak{e}(1,1)$. Alternatively, one can check that $\mathrm{ad}_{v_1+v_3}$ acts on $\mathrm{span}\{v_2, v_1 - v_3\}$ a symmetric endomorphism with eigenvalues $\sqrt{2}$ and $-\sqrt{2}$. Therefore again $\mathfrak{g} \cong \mathfrak{e}(1,1)$. □

Remark 6.13. Except for the cases that the endomorphism L takes the normal form $L_3(\alpha, 0)$, $L_4(\alpha, 0)$ ($\alpha \neq 0$) and $L_5(0)$, the solutions are such that the Ricci tensor for zero divergence and the contribution of the divergence to the Ricci tensor vanish simultaneously.

6.3.2 Non-unimodular Lie groups

Proposition 6.14. Let $(H, \mathcal{G}_g, \delta)$ be a generalized Einstein structure on an oriented three-dimensional non-unimodular Lie group G. Let $\mathfrak{u}$ be the unimodular kernel of the

Lie algebra $\mathfrak{g}$ and assume that $g|_{\mathfrak{u}\times\mathfrak{u}}$ is non-degenerate. Then there exists an orthonormal basis (v_a) of $(\mathfrak{g}, g)$ such that $v_1, v_3 \in \mathfrak{u}$ and $g(v_1, v_1) = g(v_2, v_2) = -g(v_3, v_3)$. Furthermore $\delta_i = \delta_{i'}$. If $\delta_2 = \delta_5 = 0$, then, as in the divergence free case, $h = 0$ and one can choose v_1 and v_3 such that there is a positive constant $\theta > 0$ such that

$$[v_2, v_1] = \theta v_1 - \theta v_3$$
$$[v_2, v_3] = \theta v_1 + \theta v_3.$$

If $\delta_2 = \delta_5 \neq 0$, $M := \mathrm{ad}_{v_2}|_{\mathfrak{u}}$ is diagonalizable. We have $h^2 = (\mathrm{tr}\, M)^2 \neq 0$ and $\delta_2 = \delta_5 = -\mathrm{tr}\, M \neq 0$. In the special case that M has a double eigenvalue, it is diagonalizable by an orthonormal basis. That is, one can choose v_1 and v_3 such that there exists a positive constant $\theta > 0$ such that

$$[v_2, v_1] = \theta v_1$$
$$[v_2, v_3] = \theta v_3.$$

In this case $h^2 = (2\theta)^2 \neq 0$ and $\delta_2 = \delta_5 = -2\theta \neq 0$. Furthermore $\delta_1 = \delta_3 = \delta_4 = \delta_6 = 0$.

Proof. As in the proof of Proposition 6.5 there exists a g-orthonormal basis $(v_a)_a$ of $\mathfrak{g}$ such that $v_1, v_3 \in \mathfrak{u}$ and $\lambda, \mu, \nu, \rho \in \mathbb{R}$ such that

$$[v_3, v_1] = 0$$
$$[v_2, v_1] = \varepsilon_1 \lambda v_1 + \varepsilon_3 \mu v_3$$
$$[v_2, v_3] = \varepsilon_1 \nu v_1 + \varepsilon_3 \rho v_3$$

with $0 \neq \mathrm{tr}\, \mathrm{ad}_{v_2} = \varepsilon_1 \lambda + \varepsilon_3 \rho$. Using the Dorfman coefficients, that were computed in the proof of Proposition 6.5, we obtain the components of the Ricci tensor.

In the case $\varepsilon_1 = \varepsilon_3$, we have

$$\begin{aligned} R^{\delta}_{52} &= R_{52} + \varepsilon_1 \mathcal{B}_{521}\delta_1 + \varepsilon_3 \mathcal{B}_{523}\delta_3 \\ &= -\lambda^2 - \frac{1}{4}(h - \mu - \nu)^2 - \frac{1}{4}(h + \mu + \nu)^2 - \rho^2, \end{aligned}$$

which is always non-zero due to $0 \neq \varepsilon_1 \lambda + \varepsilon_3 \rho$. Hence, we can assume that the basis is chosen such that $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3$.

In this case the components of the Ricci tensor are

$$\begin{aligned} R^{\delta}_{41} &= R_{41} + \varepsilon_2 \mathcal{B}_{412}\delta_2 + \varepsilon_3 \mathcal{B}_{413}\delta_3 \\ &= \frac{1}{2}\left(h^2 + \mu^2 - \nu^2\right) + \varepsilon_2 \lambda \delta_2 \\ R^{\delta}_{42} &= R_{42} + \varepsilon_1 \mathcal{B}_{421}\delta_1 + \varepsilon_3 \mathcal{B}_{423}\delta_3 \\ &= -\varepsilon_1 \lambda \delta_1 + \varepsilon_3 \frac{1}{2}(h - \mu - \nu)\delta_3 \\ R^{\delta}_{43} &= R_{43} + \varepsilon_1 \mathcal{B}_{431}\delta_1 + \varepsilon_2 \mathcal{B}_{432}\delta_2 \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\lambda\left(h-\mu+\nu\right)+\frac{1}{2}\rho\left(h+\mu-\nu\right)-\varepsilon_2\frac{1}{2}\left(h-\mu-\nu\right)\delta_2 \\
R^{\delta}_{51} &= R_{51}+\varepsilon_2\mathcal{B}_{512}\delta_2+\varepsilon_3\mathcal{B}_{513}\delta_3 \\
&= -\varepsilon_3\frac{1}{2}\left(h-\mu+\nu\right)\delta_3 \\
R^{\delta}_{52} &= R_{52}+\varepsilon_1\mathcal{B}_{521}\delta_1+\varepsilon_3\mathcal{B}_{523}\delta_3 \\
&= -\lambda^2+\frac{1}{2}h^2+\frac{1}{2}\left(\mu+\nu\right)^2-\rho^2 \\
R^{\delta}_{53} &= R_{53}+\varepsilon_1\mathcal{B}_{531}\delta_1+\varepsilon_2\mathcal{B}_{532}\delta_2 \\
&= \varepsilon_1\frac{1}{2}\left(h-\mu+\nu\right)\delta_1 \\
R^{\delta}_{61} &= R_{61}+\varepsilon_2\mathcal{B}_{612}\delta_2+\varepsilon_3\mathcal{B}_{613}\delta_3 \\
&= \frac{1}{2}\lambda\left(h+\mu-\nu\right)-\frac{1}{2}\rho\left(h-\mu+\nu\right)+\varepsilon_2\frac{1}{2}\left(h+\mu+\nu\right)\delta_2 \\
R^{\delta}_{62} &= R_{62}+\varepsilon_1\mathcal{B}_{621}\delta_1+\varepsilon_3\mathcal{B}_{623}\delta_3 \\
&= -\varepsilon_1\frac{1}{2}\left(h+\mu+\nu\right)\delta_1-\varepsilon_3\rho\delta_3 \\
R^{\delta}_{63} &= R_{63}+\varepsilon_1\mathcal{B}_{631}\delta_1+\varepsilon_2\mathcal{B}_{632}\delta_2 \\
&= -\frac{1}{2}\left(h^2-\mu^2+\nu^2\right)+\varepsilon_2\rho\delta_2 \\
R^{\delta}_{14} &= R_{41}-\varepsilon_2\mathcal{B}_{145}\delta_5-\varepsilon_3\mathcal{B}_{146}\delta_6 \\
&= \frac{1}{2}\left(h^2+\mu^2-\nu^2\right)+\varepsilon_2\lambda\delta_5 \\
R^{\delta}_{24} &= R_{42}-\varepsilon_2\mathcal{B}_{245}\delta_5-\varepsilon_3\mathcal{B}_{246}\delta_6 \\
&= \varepsilon_3\frac{1}{2}\left(h+\mu-\nu\right)\delta_6 \\
R^{\delta}_{34} &= R_{43}-\varepsilon_2\mathcal{B}_{345}\delta_5-\varepsilon_3\mathcal{B}_{346}\delta_6 \\
&= -\frac{1}{2}\lambda\left(h-\mu+\nu\right)+\frac{1}{2}\rho\left(h+\mu-\nu\right)-\varepsilon_2\frac{1}{2}\left(h-\mu-\nu\right)\delta_5 \\
R^{\delta}_{15} &= R_{51}-\varepsilon_1\mathcal{B}_{154}\delta_4-\varepsilon_3\mathcal{B}_{156}\delta_6 \\
&= -\varepsilon_1\lambda\delta_4-\varepsilon_3\frac{1}{2}\left(h+\mu+\nu\right)\delta_6 \\
R^{\delta}_{25} &= R_{52}-\varepsilon_1\mathcal{B}_{254}\delta_4-\varepsilon_3\mathcal{B}_{256}\delta_6 \\
&= -\lambda^2+\frac{1}{2}h^2+\frac{1}{2}\left(\mu+\nu\right)^2-\rho^2 \\
R^{\delta}_{35} &= R_{53}-\varepsilon_1\mathcal{B}_{354}\delta_4-\varepsilon_3\mathcal{B}_{356}\delta_6 \\
&= \varepsilon_1\frac{1}{2}\left(h-\mu-\nu\right)\delta_4-\varepsilon_3\rho\delta_6 \\
R^{\delta}_{16} &= R_{61}-\varepsilon_1\mathcal{B}_{164}\delta_4-\varepsilon_2\mathcal{B}_{165}\delta_5 \\
&= \frac{1}{2}\lambda\left(h+\mu-\nu\right)-\frac{1}{2}\rho\left(h-\mu+\nu\right)+\varepsilon_2\frac{1}{2}\left(h+\mu+\nu\right)\delta_5
\end{aligned}
$$

$$
\begin{aligned}
R^{\delta}_{26} &= R_{62} - \varepsilon_1 \mathcal{B}_{264}\delta_4 - \varepsilon_2 \mathcal{B}_{265}\delta_5 \\
&= -\varepsilon_1 \frac{1}{2}\left(h + \mu - \nu\right)\delta_4 \\
R^{\delta}_{36} &= R_{63} - \varepsilon_1 \mathcal{B}_{364}\delta_4 - \varepsilon_2 \mathcal{B}_{365}\delta_5 \\
&= -\frac{1}{2}\left(h^2 - \mu^2 + \nu^2\right) + \varepsilon_2 \rho \delta_5 .
\end{aligned}
$$

Note first that $R^{\delta}_{41} = R^{\delta}_{14}$ and $R^{\delta}_{63} = R^{\delta}_{36}$ yield $\varepsilon_2 \lambda \left(\delta_2 - \delta_5\right) = 0$ and $\varepsilon_2 \rho \left(\delta_2 - \delta_5\right) = 0$. Therefore $\delta_2 = \delta_5$, because $0 \neq \varepsilon_1 \lambda + \varepsilon_3 \rho$. If $\delta_2 = \delta_5 = 0$, we see that $R^{\delta}_{ii'} = R^{\delta}_{i'i} = R_{ii'}$ for all $i \in \{4, 5, 6\}$. So we can deduce the same way as in the proof of Proposition 6.5, that $h = 0$ and there is a positive constant $\theta > 0$ such that $\varepsilon_1 \lambda = -\varepsilon_3 \mu = \varepsilon_1 \nu = \varepsilon_3 \rho = \theta$. Then we see that $0 = R^{\delta}_{42} = -\theta\delta_1 + \theta\delta_3$ and $R^{\delta}_{15} = -\theta\delta_4 + \theta\delta_6$ imply $\delta_1 = \delta_3$ and $\delta_4 = \delta_6$. Note that the endomorphism $M \in \mathrm{End}(\mathfrak{u})$, defined as the restriction of ad_{v_2} to $\mathfrak{u}$, has the two complex eigenvalues $\theta \pm i\theta$. Assume now $\delta_2 = \delta_5 \neq 0$. Then

$$
0 = R^{\delta}_{41} - R^{\delta}_{63} = h^2 + \varepsilon_2 \delta_2 \left(\lambda - \rho\right). \tag{38}
$$

Using $\lambda - \rho = \varepsilon_1 \left(\varepsilon_1 \lambda + \varepsilon_3 \rho\right) \neq 0$, we see $h \neq 0$. From $0 = R^{\delta}_{61} - R^{\delta}_{43} = h\left(\lambda - \rho + \varepsilon_2 \delta_2\right)$ we see $\varepsilon_2 \delta_2 = -\lambda + \rho = -\varepsilon_2 \operatorname{tr} M$. Then equation (38) is equivalent to $h^2 = \left(\lambda - \rho\right)^2 = \left(\operatorname{tr} M\right)^2$. Since

$$
\begin{aligned}
R^{\delta}_{52} &= -\lambda^2 + \frac{1}{2}h^2 + \frac{1}{2}\left(\mu + \nu\right)^2 - \rho^2 \\
&= -\lambda^2 - \rho^2 + \frac{1}{2}\left(\lambda - \rho\right)^2 + \frac{1}{2}\left(\mu + \nu\right)^2 \\
&= -\frac{1}{2}\left(\lambda + \rho\right)^2 + \frac{1}{2}\left(\mu + \nu\right)^2 ,
\end{aligned}
$$

the equation $R^{\delta}_{52} = 0$ is equivalent to

$$
\left(\lambda + \rho\right)^2 = \left(\mu + \nu\right)^2 .
$$

Note now that the discriminant Δ of the characteristic polynomial $X^2 - \varepsilon_1 \left(\lambda - \rho\right) X - \lambda\rho + \mu\nu$ of M is

$$
\begin{aligned}
\Delta &= \left(\lambda - \rho\right)^2 + 4\lambda\rho - 4\mu\nu \\
&= \left(\lambda + \rho\right)^2 - 4\mu\nu \\
&= \left(\mu + \nu\right)^2 - 4\mu\nu \\
&= \left(\mu - \nu\right)^2 ,
\end{aligned}
$$

which is never negative. Therefore M has real eigenvalues. Hence M is either diagonalizable with two distinct eigenvalues, or it has a double eigenvalue. The latter happens precisely if the discriminant is zero, that is if $\mu = \nu$. But then M is a symmetric matrix, and hence takes one of the normal forms

$$
M_1(\theta, \eta) = \begin{pmatrix} \theta & 0 \\ 0 & \eta \end{pmatrix}, \qquad M_2(\theta, \eta) = \begin{pmatrix} \theta & -\eta \\ \eta & \theta \end{pmatrix},
$$

$$
M_3(\theta) = \begin{pmatrix} \frac{1}{2} + \theta & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} + \theta \end{pmatrix}, \qquad M_4(\theta) = \begin{pmatrix} -\frac{1}{2} + \theta & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} + \theta \end{pmatrix}
$$

with respect to an orthonormal basis v_1, v_3 of $\mathfrak{u}$, as in the proof of Proposition 6.5. If it takes the normal form $M_1(\theta, \eta)$, then $\theta = \eta$, since M has a double eigenvalue. Hence that M is of the form

$$M_1(\theta,\theta) = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}.$$

We may assume θ is positive by replacing v_2 with $-v_2$. If it takes the normal form $M_2(\theta, \eta)$, then we have $R^\delta_{43} + R^\delta_{61} = -2\theta\eta$. But $\theta \neq 0$, since $\operatorname{tr} M \neq 0$. Therefore $\eta = 0$ and we get the same normal form as before. The normal forms $M_3(\theta)$ and $M_4(\theta)$ are excluded, because in both cases $R^\delta_{43} + R^\delta_{61} = 2\theta$, which cannot be zero because $\operatorname{tr} M \neq 0$. Note, that in the case that M takes the normal form $M_1(\theta,\theta)$, we have $\delta_1 = \delta_3 = \delta_4 = \delta_6 = 0$, due to $R^\delta_{53} = R^\delta_{51} = R^\delta_{26} = R^\delta_{24} = 0$. □

Proposition 6.15. Let G be an oriented three-dimensional non-unimodular Lie group. As any three-dimensional non-unimodular Lie algebra, its Lie algebra $\mathfrak{g}$ is isomorphic to a semidirect product of $\mathbb{R}$ and $\mathbb{R}^2$, with $\mathbb{R}$ acting on $\mathbb{R}^2$ by a 2 by 2 matrix M. Then there exists a generalized Einstein structure $(H, \mathcal{G}_g, \delta)$ on G, such that the restriction of g to the unimodular kernel $\mathfrak{u}$ is degenerate, if and only if $H = 0$ and M has real eigenvalues.

Proof. Note first that the metric g necessarily has to be indefinite. As in the proof of Proposition 6.7, there exists an orthonormal basis $(v_a)_a$ of $(\mathfrak{g}, g)$ such that $g(v_1, v_1) = g(v_2, v_2)$ and $\lambda, \mu, \nu, \rho \in \mathbb{R}$ such that

$$\begin{aligned}
[v_1, v_2] &= \varepsilon_1 \lambda v_1 + \varepsilon_2 \mu v_2 - \varepsilon_3 \mu v_3 \\
[v_2, v_3] &= \varepsilon_1 \nu v_1 + \varepsilon_2 \rho v_2 - \varepsilon_3 \rho v_3 \\
[v_3, v_1] &= \varepsilon_1 \lambda v_1 + \varepsilon_2 \mu v_2 - \varepsilon_3 \mu v_3
\end{aligned}$$

with $\lambda + \rho \neq 0$. Using the Dorfman coefficients, that were computed in the proof of Proposition 6.7, we obtain the components of the Ricci tensor

$$\begin{aligned}
R^\delta_{41} &= R_{41} + \varepsilon_2 \mathcal{B}_{412}\delta_2 + \varepsilon_3 \mathcal{B}_{413}\delta_3 \\
&= \frac{1}{2}h^2 - \frac{1}{2}\nu^2 - \varepsilon_2 \lambda \delta_2 + \varepsilon_3 \lambda \delta_3 \\
R^\delta_{42} &= R_{42} + \varepsilon_1 \mathcal{B}_{421}\delta_1 + \varepsilon_3 \mathcal{B}_{423}\delta_3 \\
&= \frac{1}{2}\rho\,(h-\nu) - \frac{1}{2}\lambda\,(h+\nu) + \varepsilon_1 \lambda \delta_1 + \varepsilon_3 \frac{1}{2}\,(h-\nu)\,\delta_3 \\
R^\delta_{43} &= R_{43} + \varepsilon_1 \mathcal{B}_{431}\delta_1 + \varepsilon_2 \mathcal{B}_{432}\delta_2 \\
&= \frac{1}{2}\lambda\,(h+\nu) - \frac{1}{2}\rho\,(h-\nu) - \varepsilon_1 \lambda \delta_1 - \varepsilon_2 \frac{1}{2}\,(h-\nu)\,\delta_2 \\
R^\delta_{51} &= R_{51} + \varepsilon_2 \mathcal{B}_{512}\delta_2 + \varepsilon_3 \mathcal{B}_{513}\delta_3 \\
&= \frac{1}{2}\lambda\,(h-\nu) - \frac{1}{2}\rho\,(h+\nu) - \varepsilon_2 \mu \delta_2 - \varepsilon_3 \frac{1}{2}\,(h - 2\mu + \nu)\,\delta_3 \\
R^\delta_{52} &= R_{52} + \varepsilon_1 \mathcal{B}_{521}\delta_1 + \varepsilon_3 \mathcal{B}_{523}\delta_3
\end{aligned}$$

$$
\begin{aligned}
&= -\lambda^2 + \frac{1}{2}h^2 + \frac{1}{2}\nu^2 + \mu\nu - \rho^2 + \varepsilon_1\mu\delta_1 - \varepsilon_3\rho\delta_3 \\
R^{\delta}_{53} &= R_{53} + \varepsilon_1\mathcal{B}_{531}\delta_1 + \varepsilon_2\mathcal{B}_{532}\delta_2 \\
&= \lambda^2 - \mu\nu + \rho^2 + \varepsilon_1\frac{1}{2}\left(h - 2\mu + \nu\right)\delta_1 + \varepsilon_2\rho\delta_2 \\
R^{\delta}_{61} &= R_{61} + \varepsilon_2\mathcal{B}_{612}\delta_2 + \varepsilon_3\mathcal{B}_{613}\delta_3 \\
&= -\frac{1}{2}\lambda\left(h - \nu\right) + \frac{1}{2}\rho\left(h + \nu\right) + \varepsilon_2\frac{1}{2}\left(h + 2\mu + \nu\right)\delta_2 - \varepsilon_3\mu\delta_3 \\
R^{\delta}_{62} &= R_{62} + \varepsilon_1\mathcal{B}_{621}\delta_1 + \varepsilon_3\mathcal{B}_{623}\delta_3 \\
&= \lambda^2 - \mu\nu + \rho^2 - \varepsilon_1\frac{1}{2}\left(h + 2\mu + \nu\right)\delta_1 + \varepsilon_3\rho\delta_3 \\
R^{\delta}_{63} &= R_{63} + \varepsilon_1\mathcal{B}_{631}\delta_1 + \varepsilon_2\mathcal{B}_{632}\delta_2 \\
&= -\lambda^2 - \frac{1}{2}h^2 - \frac{1}{2}\nu^2 + \mu\nu - \rho^2 + \varepsilon_1\mu\delta_1 - \varepsilon_2\rho\delta_2 \\
R^{\delta}_{14} &= R_{41} - \varepsilon_2\mathcal{B}_{145}\delta_5 - \varepsilon_3\mathcal{B}_{146}\delta_6 \\
&= \frac{1}{2}h^2 - \frac{1}{2}\nu^2 - \varepsilon_2\lambda\delta_5 + \varepsilon_3\lambda\delta_6 \\
R^{\delta}_{24} &= R_{42} - \varepsilon_2\mathcal{B}_{245}\delta_5 - \varepsilon_3\mathcal{B}_{246}\delta_6 \\
&= \frac{1}{2}\rho\left(h - \nu\right) - \frac{1}{2}\lambda\left(h + \nu\right) - \varepsilon_2\mu\delta_5 + \varepsilon_3\frac{1}{2}\left(h + 2\mu - \nu\right)\delta_6 \\
R^{\delta}_{34} &= R_{43} - \varepsilon_2\mathcal{B}_{345}\delta_5 - \varepsilon_3\mathcal{B}_{346}\delta_6 \\
&= \frac{1}{2}\lambda\left(h + \nu\right) - \frac{1}{2}\rho\left(h - \nu\right) - \varepsilon_2\frac{1}{2}\left(h - 2\mu - \nu\right)\delta_5 - \varepsilon_3\mu\delta_6 \\
R^{\delta}_{15} &= R_{51} - \varepsilon_1\mathcal{B}_{154}\delta_4 - \varepsilon_3\mathcal{B}_{156}\delta_6 \\
&= \frac{1}{2}\lambda\left(h - \nu\right) - \frac{1}{2}\rho\left(h + \nu\right) + \varepsilon_1\lambda\delta_4 - \varepsilon_3\frac{1}{2}\left(h + \nu\right)\delta_6 \\
R^{\delta}_{25} &= R_{52} - \varepsilon_1\mathcal{B}_{254}\delta_4 - \varepsilon_3\mathcal{B}_{256}\delta_6 \\
&= -\lambda^2 + \frac{1}{2}h^2 + \frac{1}{2}\nu^2 + \mu\nu - \rho^2 + \varepsilon_1\mu\delta_4 - \varepsilon_3\rho\delta_6 \\
R^{\delta}_{35} &= R_{53} - \varepsilon_1\mathcal{B}_{354}\delta_4 - \varepsilon_3\mathcal{B}_{356}\delta_6 \\
&= \lambda^2 - \mu\nu + \rho^2 + \varepsilon_1\frac{1}{2}\left(h - 2\mu - \nu\right)\delta_4 + \varepsilon_3\rho\delta_6 \\
R^{\delta}_{16} &= R_{61} - \varepsilon_1\mathcal{B}_{164}\delta_4 - \varepsilon_2\mathcal{B}_{165}\delta_5 \\
&= -\frac{1}{2}\lambda\left(h - \nu\right) + \frac{1}{2}\rho\left(h + \nu\right) - \varepsilon_1\lambda\delta_4 + \varepsilon_2\frac{1}{2}\left(h + \nu\right)\delta_5 \\
R^{\delta}_{26} &= R_{62} - \varepsilon_1\mathcal{B}_{264}\delta_4 - \varepsilon_2\mathcal{B}_{265}\delta_5 \\
&= \lambda^2 - \mu\nu + \rho^2 - \varepsilon_1\frac{1}{2}\left(h + 2\mu - \nu\right)\delta_4 + \varepsilon_2\rho\delta_5 \\
R^{\delta}_{36} &= R_{63} - \varepsilon_1\mathcal{B}_{364}\delta_4 - \varepsilon_2\mathcal{B}_{365}\delta_5 \\
&= -\lambda^2 - \frac{1}{2}h^2 - \frac{1}{2}\nu^2 + \mu\nu - \rho^2 + \varepsilon_1\mu\delta_4 - \varepsilon_2\rho\delta_5.
\end{aligned}
$$

Assume now that $(H, \mathcal{G}_g, \delta)$ is generalized Einstein. We first want to show that $h = \nu = 0$ and $\varepsilon_2\delta_2 = \varepsilon_3\delta_3$. For this, consider the system of equations $0 = R^\delta_{42} + R^\delta_{43} = -\frac{1}{2}(h-\nu)(\varepsilon_2\delta_2 - \varepsilon_3\delta_3)$ and $0 = R^\delta_{51} + R^\delta_{61} = \frac{1}{2}(h+\nu)(\varepsilon_2\delta_2 - \varepsilon_3\delta_3)$. This implies that either $h = \nu = 0$ or $\varepsilon_2\delta_2 = \varepsilon_3\delta_3$. If $h = \nu = 0$, then $0 = R^\delta_{41} = -\lambda(\varepsilon_2\delta_2 - \varepsilon_3\delta_3)$ and $0 = R^\delta_{63} - R^\delta_{52} = -\rho(\varepsilon_2\delta_2 - \varepsilon_3\delta_3)$, which van only be the case if $\varepsilon_2\delta_2 = \varepsilon_3\delta_3$, since $\lambda + \rho \neq 0$. If we otherwise assume, that $\varepsilon_2\delta_2 = \varepsilon_3\delta_3$, then $0 = R^\delta_{63} - R^\delta_{52} = -h^2 - \nu^2$ and therefore $h = \nu = 0$. Similarly one can also show that $\varepsilon_2\delta_5 = \varepsilon_3\delta_6$. Hence, the Einstein condition is equivalent to the set of equations

$$\begin{aligned} h = \nu &= 0 \\ \lambda\varepsilon_1\delta_1 = \lambda\varepsilon_1\delta_4 &= 0 \\ \varepsilon_2\delta_2 &= \varepsilon_3\delta_3 \\ \varepsilon_2\delta_5 &= \varepsilon_3\delta_6 \\ \lambda^2 + \rho^2 - \varepsilon_1\mu\delta_1 + \varepsilon_2\rho\delta_2 &= 0 \\ \lambda^2 + \rho^2 - \varepsilon_1\mu\delta_4 + \varepsilon_2\rho\delta_5 &= 0. \end{aligned} \tag{39}$$

Now, as in the proof of Proposition 6.7, there exists a basis (w_a) of $\mathfrak{g}$, such that $w_1, w_2 \in \mathfrak{u}$,

$$\begin{aligned} [w_1, w_2] &= 0 \\ [w_3, w_1] &= -\varepsilon_1\lambda w_1 - 2\varepsilon_2\mu w_2 \\ [w_3, w_2] &= -\frac{1}{2}\varepsilon_1\nu w_1 - \varepsilon_2\rho w_2 \end{aligned}$$

and $g(w_a, w_b)$ satisfies Equation (37). Hence $\mathfrak{g}$ is a semidirect product of $\mathbb{R} \cong span(w_3)$ and $\mathbb{R}^2 \cong span(w_1, w_2)$, the former acting on the latter with the matrix

$$M = \begin{pmatrix} -\varepsilon_1\lambda & -2\varepsilon_2\mu \\ -\frac{1}{2}\varepsilon_1\nu & -\varepsilon_2\rho \end{pmatrix}.$$

Since in the Einstein case $\nu = 0$, its eigenvalues $-\varepsilon_1\lambda$ and $-\varepsilon_2\rho$ are real. Furthermore, for any such matrix, with $\nu = 0$, one can find $\delta \in E^*$, such that $(H = 0, \mathcal{G}_g, \delta)$ is generalized Einstein. □

6.4 Riemannian divergence

In this section we want to determine those solutions $(G, H, \mathcal{G}, \delta)$ to the generalized Einstein equation for which the divergence δ coincides with the Riemannian divergence $\delta^{\mathcal{G}} = -\tau \circ \pi \in E^*$ (see Proposition 5.20). If the Lie group is unimodular, the trace-form τ, and therefore the Riemannian divergence, is zero. This was covered in Theorem 6.4. It remains to specify the results of Proposition 6.14 and Proposition 6.15 to the case $\delta = \delta^{\mathcal{G}}$.

In the case that g is non-degenerate on the unimodular kernel $\mathfrak{u}$, the components of δ in the basis (v_a) of $\mathfrak{g}$ from Proposition 6.14 are

$$\begin{aligned}\delta_1 = \delta_4 &= -\operatorname{tr}\operatorname{ad}_{v_1} = 0\\ \delta_2 = \delta_5 &= -\operatorname{tr}\operatorname{ad}_{v_2} \neq 0\\ \delta_3 = \delta_6 &= -\operatorname{tr}\operatorname{ad}_{v_3} = 0.\end{aligned}$$

Therefore $M = \operatorname{ad}_{v_2}|_{\mathfrak{u}}$ is diagonalizable, in virtue of Proposition 6.14.

In the case that g is degenerate on the unimodular kernel $\mathfrak{u}$, we compute the components of δ in the basis (v_a) of $\mathfrak{g}$ from Proposition 6.15 as

$$\begin{aligned}\delta_1 = \delta_4 &= -\operatorname{tr}\operatorname{ad}_{v_1}\\ &= \kappa_{12}^2 + \kappa_{13}^3\\ &= 0\\ \delta_2 = \delta_5 &= -\operatorname{tr}\operatorname{ad}_{v_2}\\ &= \kappa_{21}^1 + \kappa_{23}^3\\ &= \varepsilon_1(\lambda - \rho)\\ \delta_3 = \delta_6 &= -\operatorname{tr}\operatorname{ad}_{v_3}\\ &= \kappa_{31}^1 + \kappa_{32}^2\\ &= -\varepsilon_1(\lambda - \rho).\end{aligned}$$

From the system of equations (39), which is equivalent to the Einstein condition, we see now that $\lambda^2 + \lambda\rho = 0$. Hence $\lambda = 0$, since $\lambda + \rho \neq 0$. Finally we conclude that $\mathfrak{g} \cong \mathbb{R} \ltimes_A \mathbb{R}^2$, where A has one eigenvalue equal to zero and one non-zero eigenvalue.

Proposition 6.16. Let $(H, \mathcal{G}_g, \delta)$ be a generalized Einstein structure on an oriented three-dimensional non-unimodular Lie group G, with $\delta = \delta^{\mathcal{G}_g}$ the Riemannian divergence of $\mathcal{G}_g$. Let $\mathfrak{u}$ be the unimodular kernel of the Lie algebra $\mathfrak{g}$. If the pseudo-Riemannian metric g is non-degenerate on $\mathfrak{u}$, then $\mathfrak{g} \cong \mathbb{R} \ltimes_A \mathbb{R}^2$ for a diagonalizable matrix A, with $\operatorname{tr} A \neq 0$. If g is degenerate on $\mathfrak{u}$, then $\mathfrak{g} \cong \mathbb{R} \ltimes_A \mathbb{R}^2$ for a matrix A, whose kernel is one-dimensional. In both cases, the matrix A can be brought to the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad s \in (-1, 1],$$

by an automorphism of $\mathfrak{g}$, where $s = 0$ if $\mathfrak{u}$ is degenerate. (The precise tensors H, g and δ are specified in Proposition 6.14, Proposition 6.15 by specializing to the formulas for $\delta = \delta^{\mathcal{G}_g}$ given in this section.)

7 Tables

With the following two tables we want to summarize the results from Section 6, to where we refer for further details. Here $L\mathsf{D}$ and $L\neg\mathsf{D}$ mean that the endomorphism L defined in Equation (32) is diagonalizable and not diagonalizable, respectively. Furthermore we write def, indef, deg and non-deg instead of definite, indefinite, degenerate and non-degenerate. For the notations of the isomorphism classes of Lie algebras we refer to [GOV, Chapter 7, Theorem 1.4].

class of Lie algebras	H	g	
$\mathbb{R}^3$	$= 0$	flat	$L\mathsf{D}$
$\mathfrak{so}(3)$	$\neq 0$	def	$L\mathsf{D}$
$\mathfrak{so}(2,1)$	$\neq 0$	indef	$L\mathsf{D}$
$\mathfrak{e}(2)$	$= 0$	flat, def on $[\mathfrak{g},\mathfrak{g}]$	$L\mathsf{D}$
$\mathfrak{e}(1,1)$	$= 0$	flat, indef on $[\mathfrak{g},\mathfrak{g}]$	$L\mathsf{D}$
$\mathfrak{heis}$	$= 0$	flat, indef	$L\neg\mathsf{D}$
$\mathfrak{r}'_{3,1}(\mathbb{R})$	$= 0$	indef	$g\|_{\mathfrak{u}\times\mathfrak{u}}$ non-deg

Table 1: Divergence-free solutions to the generalized Einstein equation

class of Lie algebra	H	g	δ	
$\mathbb{R}^3$	$=0$		$\delta \in E^*$ arbitrary	$L\mathsf{D}$
$\mathfrak{so}(3)$	$\neq 0$	def	$\delta\|_{E_+} = 0$ or $\delta\|_{E_-} = 0$	$L\mathsf{D}$
$\mathfrak{so}(2,1)$	$\neq 0$	indef	$\delta\|_{E_+} = 0$ or $\delta\|_{E_-} = 0$	$L\mathsf{D}$
$\mathfrak{e}(2)$	$=0$	def on $[\mathfrak{g},\mathfrak{g}]$	$\delta_{\sigma(1)} = \delta_{\sigma(2)} = \delta_{\sigma(1)+3} = \delta_{\sigma(2)+3} = 0$	$L\mathsf{D}$
$\mathfrak{e}(1,1)$	$=0$	indef on $[\mathfrak{g},\mathfrak{g}]$	$\delta_{\sigma(1)} = \delta_{\sigma(2)} = \delta_{\sigma(1)+3} = \delta_{\sigma(2)+3} = 0$	$L\mathsf{D}$
$\mathfrak{heis}$	$=0$	indef	$\delta_1 = \delta_4 = 0, \delta_2 = \delta_3, \delta_5 = \delta_6$	$L\neg\mathsf{D}$
$\mathfrak{e}(1,1)$	$=0$	indef	$\delta_1 = \delta_4 \neq 0, \delta_2 = \delta_3 = \delta_5 = \delta_6 = 0$	$L\neg\mathsf{D}$
$\mathfrak{e}(1,1)$	$=0$	indef	$\delta_1 = -\delta_4 = -\delta_3 = \delta_6 = -\sqrt{2}, \delta_2 = \delta_5$	$L\neg\mathsf{D}$
$\mathfrak{r}'_{3,1}(\mathbb{R})$	$=0$	indef	$\delta_i = \delta_{i'}, \delta_2 = \delta_5 = 0$	$g\|_{\mathfrak{u}\times\mathfrak{u}}$ non-deg
$\mathfrak{r}_2(\mathbb{R}) \oplus \mathbb{R}$	$\neq 0$	indef	$\delta_i = \delta_{i'}, \delta_2 = \delta_5 = -\operatorname{tr}\operatorname{ad}_{v_2} \neq 0$	$g\|_{\mathfrak{u}\times\mathfrak{u}}$ non-deg
$\mathfrak{r}_{3,\lambda}(\mathbb{R}), \quad \lambda \neq 1$	$\neq 0$	indef	$\delta_i = \delta_{i'}, \delta_2 = \delta_5 = -\operatorname{tr}\operatorname{ad}_{v_2} \neq 0$	$g\|_{\mathfrak{u}\times\mathfrak{u}}$ non-deg
$\mathfrak{r}_{3,1}(\mathbb{R})$	$\neq 0$	indef	$\delta_r = 0, r = 1,3,4,6, \delta_2 = \delta_5 = -\operatorname{tr}\operatorname{ad}_{v_2} \neq 0$	$g\|_{\mathfrak{u}\times\mathfrak{u}}$ non-deg
$\mathfrak{r}_2(\mathbb{R}) \oplus \mathbb{R}$	$=0$	indef		$g\|_{\mathfrak{u}\times\mathfrak{u}}$ deg
$\mathfrak{r}_3(\mathbb{R})$	$=0$	indef		$g\|_{\mathfrak{u}\times\mathfrak{u}}$ deg
$\mathfrak{r}_{3,\lambda}(\mathbb{R})$,	$=0$	indef		$g\|_{\mathfrak{u}\times\mathfrak{u}}$ deg

Table 2: Solutions to the generalized Einstein equation with arbitrary divergence

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 1. Dezember 2022

David Krusche

Publications

Sections 5, 6 and 7 of this thesis are based on the following pre-print. My own contribution to the publication is 50%.

V. Cortés and D. Krusche, *Classification of generalized Einstein metrics on 3-dimensional Lie groups*, arXiv:2206.01157 [math.DG].